Black Holes in Large Number of Dimensions

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Black Holes in Large Dimensions" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by " Anandita De" at "Tata Institute of Fundamental Research", under the supervision of "Prof Shiraz Minwalla" during the academic year 2015-2016.

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Declaration

I hereby declare that the matter embodied in the report entitled " Black Holes in Large Dimensions " are the results of the investigations carried out by me at the Department of Theoretical Physics, Tata Institute of Fundamental Research, under the supervision of Prof. Shiraz Minwalla and the same has not been submitted elsewhere for any other degree.

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Acknowledgements

I would like to thank my supervisor Prof Shiraz Minwalla for giving me an opportunity to work on this project. I am really grateful to him for his patience and kindness during the entire project. I have learnt a great deal from him not only about this project but physics in general. This project was carried out in collaboration with Arunabha Saha, Yogesh Dandekar and Subhajit Mazumdar. This project would not have been successful without their tireless efforts. A special thanks to them for all their help and support. I would like to thank Sayantani Bhattacharya who helped me a lot during the initial stages to understand the problem. I also thank Mangesh Mandlik for all the discussions and things he helped me understand.

I am grateful to Prof. Sunil Mukhi without whose help this project would not have been possible. The reading project I did under him inspired me to take up theoretical physics seriously. I cannot thank Dr. Arjun Bagchi enough for all his help and support through out. He has been a great mentor even outside physics.

I would like to thank everyone at TIFR theory room whose warmth and friendliness made me feel at home. I would also like to thank the staff at TIFR guest house for their warmth and hospitality. I am especially indebted to my friends Shruti, Sharvaree and Harsha for being there through the difficult times. I would like to thank Sourav for pushing me to do better all the time and never letting me give up. Finally, I would like to thank my family without whose love and support I would not have reached here.

Abstract

We study the effective horizon dynamics of black holes in large number of dimensions(D). To do this,we construct $SO(D-p-2)$ invariant solutions to Einstein's equations in large number of dimensions D in a power series expansion in $\frac{1}{D-3}$ holding p fixed and finite. We find that the horizon dynamics of black holes in large D can be recast into a well-posed initial value problem of dynamics of a non gravitational co-dimension one membrane propagating in flat space. The dynamical degrees of freedom of this membrane are its shape function and a divergence free velocity field. We find the equation of motion governing the dynamics of this membrane upto first subleading order in $\frac{1}{D-3}$.

Contents

Chapter 1

Introduction

Einstein's equations

$$R_{\mu\nu} = 0 \tag{1.1}$$

capture all space times dynamics. One of the most interesting solutions of these equations are black hole solutions. Black holes feature a space-time singularity (a point where the curvature becomes infinite). However this singularity is always behind an event horizon. The event horizon causally separates the interior of the black hole from the exterior. Black holes have been of great interest since Schwarzschild found the solution to (1.1). We study black hole space-times in large number of dimensions.

We find that the Einstein's equations simplify to ordinary differential equations in large D [1, 2, 3, 4, 5, 6, 7, 8] Let us try to see why this is the case. Consider a Schwarzschild black hole in D dimensions

$$ds^2 = -(1 - (\frac{r_0}{r})^{D-3})dt^2 + \frac{1}{(1 - (\frac{r_0}{r})^{D-3})}dr^2 + r^2 d\Omega_{d-2}^2 \tag{1.2}$$

When $D \to \infty$ with r held fixed at a greater value than r_0 then $(\frac{r_0}{r})^{D-3} \to 0$ and the space becomes flat. To see what happens near the horizon let us set $r = r_0(1 + \frac{R}{D-3})$. Now with R held fixed we see that

$$lim_{D\to\infty}(\frac{r_0}{r})^{D-3} = lim_{D\to\infty}\frac{1}{(1 + \frac{R}{D-3})^{D-3}} = e^{-R} \tag{1.3}$$

Thus the spacetime is not flat in a very thin region of $O(\frac{r_0}{D})$ from the horizon. We call this the membrane region.

We see that there are two length scales in the problem of black holes in large D. A length scale of $O(r_0)$, the radius of the black hole and another length $O(\frac{r_0}{D})$, the thickness of the membrane region. Emparan, Suzuki and Tanabe have done the quasinormal modes analysis of Schwarzschild large D

[6]. They found that a few light modes with frequencies of $O(\frac{1}{r_0})$ decouple from an infinite tower of heavy modes with frequencies of $O(\frac{D}{r_0})$ at each angular momentum. The heavy modes are supported all the way to infinity but the light modes are supported only in the membrane region and decay exponentially fast.

Whenever low energy modes decouple from high energy modes in physics we expect to find an effective theory of the low energy by integrating out the high energy modes. Here also we find an effective theory for the dynamics of the membrane governed by these light modes. We call this theory of the light modes confined in the membrane region the "Membrane paradigm".

We find that the problem of finding the horizon dynamics of black holes in large D can be recast into a well posed initial value problem of the dynamics a non gravitational codimension one membrane propagating in flat space time. In other words solutions to Einstein's equations in large D are in one to one correspondence to the solutions of the equation of motion for the auxiliary membrane in flat space time. The details of this duality will become clear as we describe our solutions.

The thesis is divided in the following way. We start with an ansatz metric having a $SO(D-p-2)$ isometry which solves the Einstein's equations at leading order in $\frac{1}{D}$. I will discuss the details of this ansatz in chapter 2. In chapter 3, I will review the Einstein's equations in large D where the Einstein-Hilbert action has a $SO(D-p-2)$ isometry. I will discuss the perturbation procedure that we adopt to correct our leading order ansatz to subsequent order in $\frac{1}{D}$. I will review the first order calculation in this section. I will also discuss how we get the equation of motion for the membrane in the auxiliary space in this chapter. In chapter 4, I will give details of the second order calculation. In section 5, I will discuss how the quasinormal modes of the Schwarzschild black hole can be obtained from our membrane equation. In chapter 6, I will summarise the final results of this work and discuss future directions.

This thesis is based on work contained in the original research paper [9] and the preprint [10]. This thesis mainly follows the structure of the paper [11]. The first of these papers written in collaboration with Sayantani Bhattacharya, Ravi Mohan, Shiraz Minwalla and Arunabha Saha [9] will not appear in any other thesis. Chapters 2, 3 ,4 and parts chapter 5 are based on the material contained in this paper. The preprint is being written in collaboration with Yogesh Dandekar, Subhajit Mazumdar, Shiraz Minwalla, Arunabha Saha and is the basis for the material contained in chapter 4 and second order corrections to quasinormal modes in chapter 5. The content of chapter 4 may also appear in the PhD theses of Subhajit and Yogesh.

Chapter 2

The Collective Co-ordinate Ansatz

2.1 Schwarzschild Metric in Kerr-Schild Co-ordinates

The metric for Schwarzschild black holes in D dimensions is given by

$$ds^2 = -\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right) dt^2 + \frac{dr^2}{\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right)} + r^2 d\Omega_{D-2} \tag{2.1}$$

The above metric can be written in an ingoing Eddington-Finklestein coordinate system as

$$ds^2 = 2dvdr - \left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right) dv^2 + r^2 d\Omega_{D-2} \tag{2.2}$$

using the coordinate transformation

$$dv = dt + \frac{dr}{\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right)}$$

We can now go to the Kerr-Schild form of the metric by using the coordinate transformation $dv = dT + dr$ to get the metric

$$\begin{aligned} ds^2 &= -dT^2 + dr^2 + r^2 d\Omega_{D-2} + \left(\frac{r_0}{r}\right)^{D-3} (dT + dr)^2 \\ &= ds^2_{flat} + \left(\frac{r_0}{r}\right)^{D-3} (dT + dr)^2 \end{aligned} \tag{2.3}$$

The metric (2.3) is that of a Schwarzschild black hole at rest with velocity $u_\mu = (-1, 0, ..)$. We can boost the black hole with velocity u to get the metric

$$ds^2 = ds^2_{flat} + \frac{O_M O_N}{\psi^{D-3}} dx^M dx^N$$
$$O = n - u, \quad u = \text{constant}, \quad u \cdot u = -1, \quad \psi = \frac{r}{r_0},$$
$$r^2 = (\eta_{MN} + u_M u_N) x^M x^N, \quad n = r_0 d\psi, \quad \text{and} \quad n.u = 0 \qquad (2.4)$$

All the dot products above are with respect to flat space. The velocity field also obviously satisfies

$$\nabla.u = 0 \qquad (2.5)$$

The advantage of writing the metric in this form is that it gives us a way to view the one form fields in the flat space , ie $\eta_{\mu\nu}$ of (2.3).

2.2 Collective Co-ordinate Spacetimes from Boosted Black Holes

We guess an ansatz metric from the form of the boosted Schwarzschild metric in the Kerr-Schild co-ordinates above. Our metric is in terms of

- A function B in the D dimensional space-time. The surface $B = 0$ plays a special role in our ansatz as it corresponds to the horizon in the black hole space time.
- The normal to this surface, given by $n = dB$, the extrinsic curvature of this surface given by $K_{AB} = \nabla_A n_B$ and the trace of the extrinsic curvature given by $\mathcal{K} = \nabla_A n^A$.
- A velocity field u on this surface $B = 0$.

To get the zeroth order ansatz we generalise the vector fields, u and n to be arbitrary functions . Our metric is then

$$ds^2 = ds^2_{flat} + \frac{O_M O_N}{\psi^{D-3}} dx^M dx^N$$
$$O = n - u, \quad \psi = 1 + \frac{\mathcal{K}}{D-3} B, \quad u.u = -1, \quad u.n = 0 \qquad (2.6)$$

We demand that u and n satisfy the same constraints on the surface $B = 0$ as they do for the Schwarzschild black hole. The velocity field is constrained by $\nabla.u = 0$ with the covariant derivative on the surface $B = 0$. Our definition for ψ is satisfied by the ψ defined in (2.4) for the Schwarzschild black hole at leading order. For Schwarzschild black holes

$$\begin{aligned} & B = r - r_0, \quad \mathcal{K} = \frac{D-2}{r_0} \\ & \psi = 1 + \frac{D-2}{(D-3)r_0}(r - r_0) = \frac{r}{r_0} (\text{at leading order in D}) \end{aligned} \tag{2.7}$$

The surface $B = 0$ is assumed to be a smooth timelike submanifold of the D dimensional Minkowski space. The surface $B = 0$ or equivalently $\psi = 1$ will be referred to as the membrane henceforth. The membrane separates its interior ie $B < 0$ from its exterior ie $B > 0$. The function B is chosen so that $B > 0$ is a connected spacetime and includes spatial infinity as well as $\mathcal{I}^+$ and $\mathcal{I}^-$. The world volume $B = 0$ need not be connected.
The spacetimes (2.6) have the following properties.

- 1. The static black holes (2.3) are special cases of (2.6), upto corrections of order $1/D$ with the ψ and u functions given as in (2.4). In these special cases $\psi = 1$ is the black hole event horizon.

- 2.The membrane surface $\psi = 1$ is a null submanifold of the metric (2.6) for a general spacetime of this form. This can be easily verified.This submanifold may be identified with the spacetime event horizon when (2.6) settles down to a stationary black hole at late times (as we will assume). [1]

- 3. Consider a point x_0^μ on the membrane ($\psi = 1$) of the spacetime (2.6). Let u_0^μ and $\mathcal{K}_0$ denote the velocity and trace of membrane extrinsic curvature at that point. Comparing with (2.4), we will see in subsection 3.4.2 below that a patch of size of order $\frac{1}{D}$ centered about x_0^μ is identical, at leading order in D, to the metric of a patch centered about the membrane of a Schwarzschild hole of radius $(D-2)/\mathcal{K}$ and boost velocity u_0^μ.

- 4. It seems plausible from point (3) above that every patch centered about the membrane of the configuration (2.6) obeys the Einstein equations at leading order in $\frac{1}{D}$. In subsection 3.4.2 below we demonstrate that this is the case provided the spacetime (2.6) enjoys an SO($D-p-2$) isometry for any p that is held fixed as D is taken to infinity.

[1]The dissipative nature of the membrane equations of motion we derive below suggests that all solutions reduce to stationary solutions at late times.

- 5. The deviation of the metric (2.6) from ds^2_{flat} scales like $e^{-D(\psi-1)}$. It follows (2.6) approaches flat space exponentially fast for $\psi - 1 \gg 1/D$.
- 6. Combining (4) and (5) above it follows that (2.6) also obeys the Einstein equations at leading order in $1/D$ (or better) everywhere outside its event horizon.
- 7.For $1 - \psi \gg 1$ the equations of motion do not admit solutions. But since this lies inside the event horizon which is causally disconnected from the outside we do not care about this region.

The metric (2.6) is made by stitching patches of event horizon of of length $1/D$ of the Schwarzschild black hole with arbitrary radius and boost velocity. The only constraint is that the radius and the velocity field varies smoothly over the horizon. Thus the metric (2.6) solves the Einstein's equations at leading order in $1/D$ everywhere outside the horizon.

2.3 Subsidiary constraints on ψ and u

We use the metric (2.6) as the starting point for perturbative expansion of the solutions to Einstein's equations in a power series in $\frac{1}{D-3}$.[2] These spacetimes are parameterised by two functions ψ and u. As we have seen the space time becomes flat for $\psi - 1 \gg 1$, the functions which agree on the surface $\psi = 1$ and differ only at $O(\frac{1}{D})$ serve as equivalent starting point for perturbation theory. Different evolutions of these functions away from the surface $\psi = 1$ gives different perturbation expansions.

We demand that on the surface u and n satisfies the same condition as those satisfied by the u and n of the Schwarzschild metric.

$$n.n|_{B=0} = 0, \quad u.u|_{B=0} = 0 \quad u.n|_{B=0} = 0, \quad \nabla^M.u = 0 \tag{2.8}$$

where ∇^M is the covariant derivative taken on the membrane. To determine how they evolve off the surface we choose a set of completely geometric rules which makes the result of our perturbation theory simpler.

$$n.\nabla n = 0, \quad n.\nabla u = 0 \tag{2.9}$$

The constraints (2.9) tell us that n and u are parallely transported along n off the surface. This makes sure that the constraints (2.8)(except $\nabla^M.u = 0$) are satisfied everywhere outside the membrane.

[2]We find it convenient to use $\frac{1}{D-3}$ rather than $\frac{1}{D}$ as this makes our metric corrections simpler.

2.4 Fixing Co-ordinate Redefinition Invariance

In the next section we will describe the perturbative procedure that we used to obtain the solutions to Einstein's equations upto $O(\frac{1}{D})^2$. We need to fix co-ordinate redefinition invariance to obtain unambiguous solutions to Einstein's equations. We can write our solution as

$$g_{MN} = \eta_{MN} + h_{MN} \tag{2.10}$$

We fix the co-ordinate redefinition invariance by demanding

$$O^M h_{MN} = 0 \tag{2.11}$$

where $O = n - u$. Note that the raising and lowering of indices are done using η_{MN}. For our leading order ansatz (2.6),

$$h_{MN} = \frac{O_M O_N}{\psi^{D-3}} \tag{2.12}$$

Since $O.O = 0$, where the dot is with respect to the flat metric the above condition (2.11) is automatically satisfied at the leading order.

2.5 Perturbation Theory

We want our metric to solve Einstein equations not only at leading order in $\frac{1}{D-3}$ but also at subsequent orders. We expand our metric in a power series in $\frac{1}{D-3}$ so that at each order they solve the Einstein's equations. Our metric can be written as

$$\begin{aligned} g_{MN} &= \eta_{MN} + h_{MN} \\ h_{MN} &= \sum_{n=0}^{\infty} \frac{h^{(n)}_{MN}}{(D-3)^n} \\ h^{(0)}_{MN} &= \frac{O_M O_N}{\psi^{D-3}} \end{aligned} \tag{2.13}$$

Chapter 3

Perturbation Theory Assuming SO$(D-p-2)$ isometry

3.1 Einstein's Equations in SO$(D-p-2)$ Sector

We take the large D limit assuming a $SO(D-p-2)$ isometry of the metric where $D \to \infty$ while holding p fixed. We do this to find out how different quantities scale when we take the $D \to \infty$ limit. Though we assume this large isometry our final answers turn out to be independent of p.

In our intermediate calculations we assume our metric to be of the following form

$$\begin{aligned} ds^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu + e^{\phi(x^\mu)}d\Omega_d^2 \\ d = D-p-3, \quad \mu = 1,2,3\ldots p+3 \end{aligned} \tag{3.1}$$

We derive the equations of motion for the above $g_{\mu\nu}(x^\mu)$ and the scalar field $\phi(x^\mu)$ from the Einstein-Hilbert action

$$\mathcal{S} = \frac{1}{16\pi G}\int d^D x\sqrt{\tilde{g}}\tilde{R} \tag{3.2}$$

where
$\tilde{g}$= Determinant of the metric in the full D dimensional spacetime
$\tilde{R}$= Ricci Scalar in the full D dimensional spacetime
Substituting (3.1) in (3.2) we get the effective Lagrangian

$$\begin{aligned} \mathcal{S} &= \int d^{p+3}x\sqrt{g}e^{\frac{d\phi}{2}}\left(R + d(d-1)e^{-\phi} + \frac{d(d-1)}{4}(\partial\phi)^2\right) \\ (\partial\phi)^2 &= g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) \end{aligned} \tag{3.3}$$

Varying the above action with respect to ϕ and $R_{\mu\nu}$ we get the following equations of motion.

$$
\begin{aligned}
e^{-\phi}(d-1) - \frac{d}{4}(\partial\phi)^2 - \frac{1}{2}\nabla^2\phi = 0 \\
R_{\mu\nu} = \frac{d}{2}\nabla_\mu\nabla_\nu\phi + \frac{d}{4}\nabla_\mu\phi\nabla_\nu\phi
\end{aligned}
\tag{3.4}
$$

3.2 Setting up the Perturbative Computation

3.2.1 Convenient Co-ordinates for Flat Space

The metric (2.6) is completely determined in terms of the vector field u and the function ψ. These functions live in D dimensional flat space. To study SO$(D-p-2)$ invariant configurations, the following co-ordinates for flat space are useful

$$
\begin{aligned}
ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta + dS^2 + S^2 d\Omega_d^2 \\
i = 1, 2 \ldots p+1, \quad d = D-p-3
\end{aligned}
\tag{3.5}
$$

The SO(D-p-2) isometry in these co-ordinates imply that the functions u and ψ which determine the metric are functions of $(w^a, S) = x^\mu$ only.

3.2.2 Auxiliary Embedding Space

The metric (3.5) describes flat R^D as 'fibration' of S^d over $p+3$ dimensional base metric of the form

$$
ds^2_{flat} = \eta_{\alpha\beta}dx^\alpha dx^\beta + dS^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{3.6}
$$

The membrane world volume with SO$(D-p-2)$ symmetry can be thought of as (p+2 dimensional) co-dimension one surface in the base space((3.6)) with each point fibred over a d dimensional sphere. Consequently the functions u and ψ can be thought of as vector fields and functions in this base space which are then extended to the full D dimensional space with a SO(d+1) symmetry in the obvious way. The auxiliary space (3.6) does not have D. When we formulate the perturbation theory with u and ψ moving in the auxiliary space, the factors of D are manifest. This allows us to have a clean formulation of the perturbation theory in this language.

The covariant derivatives of the fields in the auxiliary space (3.6) do *not* agree with the covariant derivatives in the D dimensional embedding space

(3.5).[1] We have given a dictionary between the covariant derivatives in (3.6) and the covariant derivatives in (3.5) in (fill up whereever you are giving the dictionary). Though we do our calculation in this auxiliary space we can recast our final result in the full D dimensional space.

3.2.3 Zooming into patches

In this section we discuss how we can take in interesting $D \to \infty$ limit for our SO(D-p-2) configurations. To do this, we look at the Einstein's equations (3.4) for our configurations. We see that the derivatives of ϕ are weighted with an extra factor of d as compared to the derivatives of $g_{\mu}\nu$. [2] Thus there are two length scales in the problem as discussed in the introduction. One is of $O(\frac{1}{D})$ over which the metric $g_{\mu\nu}$ varies and the other is of $O(1)$ over which ϕ varies.

We want to employ a co-ordinate system in which both $g_{\mu\nu}$ and ϕ are of $O(1)$ but derivatives of g are of order d while the derivatives of ϕ are of order unity. To do this, we zoom into patches of length $\frac{1}{D-3}$ on the surface $B = 0$. We view our manifold$B = 0$ as the union of such patches. We zoom into a point x_0^μ, and use the following co-ordinates and the rescaled metric in the patch centered about x_0^μ,

$$
\begin{aligned}
x_\mu &= x_0^\mu + \alpha_a^\mu \frac{y^a}{D-3} \\
G_{ab} &= D^2 g_{ab} \qquad (3.7) \\
g_{\mu\nu} &= (D-3)^2 \alpha_\mu^a \alpha_\nu^b g_{ab} = \alpha_\mu^a \alpha_\nu^b G_{ab} \\
\chi_a &= (D-3)\nabla_a \phi = \alpha_a^\mu \nabla_\mu \phi \qquad (3.8)
\end{aligned}
$$

In these scaled co-ordinates the Einstein's equations are

$$
\begin{aligned}
\frac{1}{2}\nabla_a \chi^a &= e^{-\phi}\frac{d}{D-3} - \frac{d}{4(D-3)}\chi^2 \\
R_{ab} &= \frac{d}{2(D-3)}\nabla_a \chi_b + \frac{d}{4(D-3)^2}\chi_a \chi_b \qquad (3.9)
\end{aligned}
$$

All quantities(Christoffel symbols, curvatures) are constructed out of metric G_{ab}. We look for solutions of the equations (3.9) in a perturbative expansion in $\frac{1}{D-3}$. The solutions of the Einstein's equations in each patch can be joined smoothly to give the metric for the entire manifold.

[1] This is because of the contribution from the Christoffel symbol Γ^S_{AB} where A,B are in the angular directions of the d-sphere in the space (3.5) which is not there in the space (3.6).

[2] The term $d\nabla_\mu \nabla_\nu \phi$ has a term which is $\Gamma^\alpha_{\mu\nu}\partial_\alpha \phi$, so at leading order this term should be thought of as one ϕ derivative and one metric derivative.

3.3 Data at First Order

The different quantities B and u are expanded in Taylor expansion in the patch centred about the point x_0^μ. This expansion is in the auxiliary space discussed in 3.2.2 where covariant derivatives are replaced by partial derivatives.

$$\begin{aligned} B &= B(x_0) + n_\mu \delta X^\mu + \frac{1}{2}\partial_\mu n_\nu \delta X^\mu \delta X^\nu + \frac{1}{6}\partial_\alpha \partial_\mu \partial_\nu B \delta X^\mu \delta X^\nu \delta X^\alpha + \ldots \\ u_\mu &= u_\mu(x_0) + \partial_\nu u_\mu \delta X^\nu + \frac{1}{2}\partial_\nu \partial_\alpha u_\mu \delta X^\nu \delta X^\alpha + \ldots \end{aligned} \tag{3.10}$$

where

$$\delta X^\mu = \frac{y^\mu}{D}, \quad \mu = 1, 2, 3 \ldots p + 3 \tag{3.11}$$

are the patch-co-ordinates in the auxiliary space.

The data at zeroth order is $u_\mu(x_0)$ and $n_\mu(x_0)$. [3]. The data at first order are $\partial_\nu u_\mu$ and $\partial_\mu n_\nu$. Since the evolution of the n is given by $n.\nabla n = 0$, $\nabla_\nu n_\mu = K_{\mu\nu}$ the extrinsic curvature of the surface $B = 0$.

There are three special directions n_μ,u_μ and dS. The rest of the directions are equivalent, so we can classify our data according to how they transform under SO(p). However all this data is not independent since they are subject to the constraints (2.8).

3.3.1 Independent Data in the Scalar Sector

From $K_{\mu\nu}$ we can have scalars if both the indices are in the scalar directions and from the trace of the tensor when both μ and ν are in the p directions. However, one of the scalar directions do not exist since $n.K = 0$. So we get $\frac{2.3}{2} + 1 = 4$ scalars from $K_{\mu\nu}$ since it is symmetric. The indices of $\partial_\mu u_\nu$ are not symmetric. From the constraints $u.n = 0$ and $u^\mu \partial_\nu u_\mu = 0$ we see that when ν index is along n or u, it does not give us independent data. The constraint $\nabla.u = 0$ gives $u_s = O(\frac{1}{D}$ as shown in , so $\partial_\mu u_\nu = O(\frac{1}{D^2})$ and contributes to the next order. The only scalar we get is from the trace of the tensor when both the indices are in the p directions.

[3]Though n_μ appears with a factor of $\frac{1}{D}$ in the expansion (3.10), in the metric B appears in $\frac{1}{\psi^{D-3}}$ where $\psi = 1 + \frac{\mathcal{K}}{D-3}B$. $\frac{\mathcal{K}}{D-3}$ is $O(1)$ at leading order as is shown in . $lim_{D\to\infty}\frac{1}{\psi^{D-3}} = e^{-R}$ where $R = \frac{\mathcal{K}}{D-3}n_\mu y^\mu$ and y^μ are the patch co-ordinates. So n_μ contributes to the zeroth order

Table 3.1: Independent Data in 3 Symmetry Channels

Scalars	Vectors	Tensors
$S_1 = dS.K.dS$	$V_1 = u^\alpha K_{\alpha\beta} P^\beta_\mu$	$T_1 = P^\mu_\alpha P^\nu_\beta (K_{\mu\nu} - \frac{\eta_{\mu\nu}}{p} P_{\gamma\theta} K^{\gamma\theta})$
$S_2 = u.K.u$	$V_2 = (dS)^\alpha K_{\alpha\beta} P^\beta_\mu$	$T_2 = P^\mu_\alpha P^\nu_\beta (\partial_{(\mu} u_{\nu)} - \frac{\eta_{\mu\nu}}{p} P_{\gamma\theta} \partial^\gamma u^\theta)$
$S_3 = u.K.dS$	$V_3 = u^\alpha \partial_\alpha u_\beta P^\beta_\mu$	$T_3 = \partial_{[\mu} u_{\nu]}$
$S_4 = P^{\mu\nu} K_{\mu\nu}$	$V_4 = dS^\alpha \partial_\alpha u_\beta P^\beta_\mu$	
$S_5 = P^{\mu\nu} \partial_\mu u_\nu$		

3.3.2 Independent Data in the Vector Sector

We will get a vector from $K_{\mu\nu}$ if one of the indices is along the scalar directions. Since $n.K = 0$, the scalar indices can only be along u and dS which gives us two vectors from $K_{\mu\nu}$. As discussed in 3.3.1, in $\partial_\mu u_\nu$ the ν index cannot be in the scalar directions, so they are always vector index. The μ index in the scalar direction will give us vectors but $n.\nabla u = 0$. So μ can only be along u or s which gives us 2 vectors from $\partial_\mu u_\nu$.

3.3.3 Independent Data in the Tensor Sector

There are no constraints in the tensor sector so we get one symmetric traceless tensor from $K_{\mu\nu}$, one symmetric traceless tensor from $\partial_\mu u_\nu$ and one anti-symmetric tensor from $\partial_\mu u_\nu$.

The table 3.1 lists all the independent data in the 3 symmetry channels and $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu - n^\mu n^\nu - \frac{(dS - n_s n)^\mu (dS - n_s n)^\nu}{1 - n_s^2}$ is the projector orthogonal to u, n and dS.

3.4 Solving the Einstein's Equations

3.4.1 Choice of Patch Co-ordinates

As mentioned in 3.2.3, we solve the Einstein's equations in each patch of length $\frac{1}{D}$ and then sew them together to get a smooth solution. We need to make a local choice of co-ordinates to write our Einstein's equations in each patch. We have already noted that there are three distinguished one form fields, n_μ, u_μ and dS. From these, we construct two one form fields which we are going to use as the basis for our metric,

$$\begin{aligned} O_\mu &= n_\mu - u_\mu \\ dX &= dS - n_S n_\mu \end{aligned} \tag{3.12}$$

Let Y^i be p- one form fields such that

$$Y^i.dX = Y^i.n = Y^i.O = 0, \quad Y^i.Y^j = \delta^{ij} \tag{3.13}$$

Let $\{x_0^\mu, S_0\}$ be a point on the surface $B = 0$ about which the patch is centered. We solve the Einstein's equations in the following co-ordinates in this patch

$$\begin{aligned} R &= (D-3)(\psi - 1) \\ V &= (D-3)(x^\mu - x_0^\mu)O_\mu(x_0) \\ X &= (D-3)(x^\mu - x_0^\mu)X_\mu(x_0) \\ y^i &= (D-3)(x^\mu - x_0^\mu)Y^i(x_0) \end{aligned} \tag{3.14}$$

3.4.2 The Perturbative Metric in the Patch

In the co-ordinates (3.14), the metric (2.6) at zeroth order looks like

$$ds^2 = 2\frac{S_0}{n_S^0}dVdR - (1 - e^{-R})dV^2 + \frac{dX^2}{1-(n_S^0)^2} + \sum_{i=i}^{p} dy^i dy^i \tag{3.15}$$

where $n_S^0 = dS.n|_{x^\mu = x_0^\mu}$. We call this the black brane metric. The Schwarzschild metric (2.1) with radius $\frac{n_s^0}{S_0}$ when expanded in a patch about a point $\{x_0^\mu, S_0\}$ on the membrane looks the same. It is easily verified that (3.15) satisfies the equations of motion (3.4) at leading order. Also each little patch of the horizon is looks like the Schwarzschild metric with radius $\frac{n_s^0}{S_0}$ and boost velocity $u(x_0)$ at leading order.

Our ansatz metric (2.6) no longer satisfies the Einstein's equations at first order when the fields B and u are Taylor expanded. In order to satisfy the equations (3.4), we add first order corrections to (2.6) that are allowed by the guage condition (2.11)

$$\begin{aligned} ds^2 = \quad & (ds^2)^0 + \frac{1}{D-3}(H_{VV}dV^2 + 2H_{VX}dVdX + H_{XX}dX^2 + H_{ii}dy^i dy^i \\ & + 2H_{Vi}dVdy^i + 2H_{Xi}dXdy^i + H_{ij}dy^i dy^j) \\ & \phi = 2lnS_0 + \frac{1}{(D-3)^2}\delta\phi_2 \end{aligned} \tag{3.16}$$

In the Einstein's equations (3.4), $\chi = D - 3(\partial\phi)$ appears; if ϕ had a first order correction it would enter the equations of motion at zeroth order and our ansatz (2.6) would no longer be a solution of (3.4) at zeroth order. So the correction to ϕ starts at $\frac{1}{(D-3)^2}$ and enters the equations of motion at the first order. In addition to the four scalars in the metric we also have to determine $\delta\phi_2$.

The equations that we have to solve schematically look like

$$Hv^{(1)} = s^{(1)} \tag{3.17}$$

where H is some differential operator, $h^{(1)}$ represents the metric corrections and the correction to ϕ in (3.16) and $s^{(1)}$ represents the sources. The independent data listed in 3.1 forms a basis for the sources. The sources arise from the Taylor expansion of the fields n, u and ψ in (2.6). The equations (3.4) also has an $\frac{1}{D-3}$ expansion and the black brane metric (3.16) does not solve the equations at the first order. As discussed in the introduction, the normal to the membrane is the direction in which the metric varies very fast. The derivative of the metric in the normal direction(which is captured by the R co-ordinate should) should be of D times the derivatives in the directions tangent to the membrane. In other words when we move distances of $O(\frac{1}{D})$ from the point x_0^μ in the patch, the metric remains constant in the directions tangent to the membrane and only changes in the directions normal to the membrane. This implies that

$$v^{(1)} = v(R, \frac{V}{D}, \frac{X}{D}, \frac{y^i}{D}) \tag{3.18}$$

where R and the other scaled co-ordinates are as defined in (3.14). Since these corrections are already at order $\frac{1}{D-3}$, the derivatives in the V, X, y^i contribute at the next order. So differential operator H is a operator only in the variable R. The equations (3.4) become ordinary differential equations in the variable R which can be easily solved.

Though the source functions $s^{(1)}$ arise from the Taylor expansion of n, u and ψ in (2.6), they are not explicit functions of V, X, y^i. There is a simple reason for this. the locality of the Einstein's equations imply that the $s^{(1)}$ contains the derivatives of the functions n, u and ψ only at order $\frac{1}{D}$. The derivatives of n, u and ψ come from the Taylor expansion of these about the point x_0^μ and the terms proportional to V, X, y^i are only order $\frac{1}{D^2}$ or smaller.

3.4.3 Equations in the Three symmetry channels

As we have seen that the metric corrections (3.16) contains 5 unknown scalar functions, 2 unknown vector functions and 1 unknown tensor function.(The scalar, vector and tensor are determined by their transformation under $SO(p)$ rotations). Since the black brane metric (3.15) is invariant under $SO(p)$ isometry we expect the first order corrections to preserve this isometry. the equations in these three sectors decouple.

Tensor

The equation in the tensor sector is turns out to be a single differential equation for variable $\mathcal{T}_{ij}(R)$

$$\partial_R((1-e^{-R})\partial_R\mathcal{T}_{ij}) = 0 \tag{3.19}$$

This equation is solved easily and $\mathcal{T}_{ij}$ is 0 from boundary conditions that are discussed later.

Vector Sector

There are 3 coupled vector equations

$$\mathcal{E}_{Ri} = 0, \quad \mathcal{E}_{Vi} = 0$$
$$\mathcal{E}_{Xi} = 0$$

for two variables H_{Vi} and H_{Xi}. The combination of equations

$$\partial_R\left[\left(\frac{S_0}{n_S^0}\right)\mathcal{E}_{Vi} + f_0(R)\mathcal{E}_{Ri}\right] + \left[\left(\frac{S_0}{n_S^0}\right)\mathcal{E}_{Vi} + f_0(R)\mathcal{E}_{Ri}\right] + \left[\frac{1-(n_S^0)^2}{S_0}\right]\mathcal{E}_{Xi} = 0,$$
$$\text{where} \quad f_0(R) = 1 - e^{-R} \tag{3.20}$$

vanishes identically at first order. So we have two independent equations for the two variables H_{Vi} and H_{Xi} which are easily solved.

Scalar Sector

There are 8 scalar equations for 5 unknowns

$$\begin{aligned} &\mathcal{E}_{RR} = 0, \quad \mathcal{E}_{RV} = 0, \quad \mathcal{E}_{RX} = 0, \\ &\mathcal{E}_{VV} = 0, \quad \mathcal{E}_{VX} = 0, \quad \mathcal{E}_{XX} = 0, \\ &\sum_{i=1}^{p}\mathcal{E}_{ii} = 0, \quad \mathcal{E}_{\phi} = 0, \end{aligned} \tag{3.21}$$

At first order the following three combinations vanish identically.

$$\begin{aligned}
\text{Combination-1:}\quad & \partial_R\left[\mathcal{E}_{VV}+\left(\frac{n_S^0}{S_0}\right)f_0(R)\mathcal{E}_{RV}\right]+\left[\mathcal{E}_{VV}+\left(\frac{n_S^0}{S_0}\right)f_0(R)\mathcal{E}_{RV}\right]\\
&+\left(n_S^0-\frac{1}{n_S^0}\right)\mathcal{E}_{VX}=0,\\
\text{Combination-2:}\quad & \partial_R\left[\left(\frac{n_S^0}{S_0}\right)f_0(R)\mathcal{E}_{RX}+\mathcal{E}_{VX}\right]+\left[\left(\frac{n_S^0}{S_0}\right)f_0(R)\mathcal{E}_{RX}+\mathcal{E}_{VX}\right]\\
&-\left(n_S^0-\frac{1}{n_S^0}\right)\left(\frac{n_S^0}{S_0}\right)\mathcal{E}_{XX}=0,\\
\text{Combination-2:}\quad & \partial_R\left[\mathcal{E}_\phi+2\left(\frac{n_S^0}{S_0}\right)^2 f_0(R)\ \mathcal{E}_{RR}-2[1-(n_S^0)^2]\ \mathcal{E}_{XX}-\mathcal{E}_{ii}\right]\\
&+2\left(\frac{n_S^0}{S_0}\right)^2[\partial_R f_0(R)+2f_0(R)]\ \mathcal{E}_{RR}+4\left(\frac{n_S^0}{S_0}\right)\mathcal{E}_{RV}=0.
\end{aligned}\tag{3.22}$$

Thus we have exactly 5 independent equations to solve for the 5 unknowns $H_{VV}, H_{VX}, H_{XX} H_{ii}$ and $\delta\phi_2$.

3.4.4 Equation of Motion from Regularity at horizon

We want the functions $H_{\mu\nu}$ in (3.16) to be regular for all $R \geq 0$. Our source functions $s^{(1)}$ are regular at $R = 0$ but this does not guarantee that the functions $H_{\mu\nu}$ are regular at $R = 0$. This fact plays a very important in our work. We study this fact and its consequences more closely in this section.

Let E^{MN} denote the Einstein's equations obtained by varying the Einstein-Hilbert action. One of aims is to determine the metric as a function of ψ. To do this, we look at the 'evolution' of the Einstein's equations along $d\psi$. The equations

$$C^M_{Ein} = E^{MN} d\psi_M = E^{M\psi} \tag{3.23}$$

gives us the Einstein constraint equation on constant ψ slices. These equations do not give us the evolution of the metric along $d\psi$ but imposes constraints on it on constant ψ slices.

The dot product of C^M_{Ein} with n and u does not play any role in the discussions of this section. So we can just consider C^M_{Ein} projected orthogonal to n and O. From the geometrical point of view(see below for more discussion), C^M_{Ein} is a vector equation but so far our perturbative procedure has not been geometrical. We treat the isometry directions as special. In our current point of view, C^M_{Ein} can be decomposed into a SO(p) scalar $C^M_{Ein}.X$ and a vector C^M_{Ein} projected orthogonal to X.

It can be easily verified in the scalar sector

$$
\begin{aligned}
(C_{Ein} \cdot X) &\propto \left[\left(\frac{S_0}{n_S^0} \right) \mathcal{E}_{VX} + f_0(R)\mathcal{E}_{RX} \right], \\
&\propto f_0(R)^2 \frac{d}{dR} \left[\frac{H_{(VX)}(R)}{f_0(R)} \right] + \Sigma_{(VX)}(R) = 0,
\end{aligned} \tag{3.24}
$$

Here $\Sigma_{(VX)}(R)$ is the full source for the combination $\frac{S_0}{n_S^0}\mathcal{E}_{VX} + f_0(R)\mathcal{E}_{RX}$. From (3.24), we can see that H_{VX} has a solution regular at $R = 0$ if the term in $\frac{\Sigma_{(VX)}(R)}{f_0(R)^2}$ having a simple pole at $R = 0$ vanishes. We demand that this happens which gives us the following scalar equation of motion

$$
(\frac{dS}{n_S} - u) \cdot K \cdot (\frac{dS}{n_S} - u) = \left(\frac{1 - n_S^2}{S n_S} \right) \tag{3.25}
$$

H_{VX} is regular at $R = 0$ if and only if (3.25) is satisfied.

In the vector sector, C_{Ein}^M projected orthogonal to n, u and dS can be shown to be proportional to

$$
\left[\left(\frac{S^0}{n_S^0} \right) \mathcal{E}_{Vi} + f_0(R)\mathcal{E}_{Ri} \right] \propto f_0(R) \frac{d}{dR} \left[V_i^{(X)}(R) \right] + \mathcal{V}_i^{(X)}(R) = 0. \tag{3.26}
$$

where $\mathcal{V}_i^{(X)}(R)$ is the full source of the combination $\left(\frac{S^0}{n_S^0} \right) \mathcal{E}_{Vi} + f_0(R)\mathcal{E}_{Ri}$. We can see that $V_i^{(X)}(R)$ has regular at $R = 0$ if $\mathcal{V}_i^{(X)}(R)$ vanishes at $R = 0$. This gives us the vector equation of motion

$$
P_j^i \left[\left(\frac{dS}{n_S} - u \right) \cdot \partial(n - u)_i \right] = 0 \tag{3.27}
$$

It can be verified that once the equations (3.25) and (3.27) exhaust the conditions for regularity at $R = 0$; once these equations are satisfied the first order corrections to the black brane metric is regular everywhere on and outside the horizon.

3.4.5 Conditions to Fix Integration Constants

As we have discussed above that we obtain the first order corrections to (2.6) which solves the Einstein's equations (3.4) that turn out to be a set of ordinary differential equations in each patch. In section 3.4.4 we discussed the equations that has to be satisfied so that $H_{\mu\nu}$ in (3.16) is regular everywhere. But these conditions do not give us unique solutions for (3.4). They are unspecified upto integration constants. Some of these constants are fixed by the regularity condition at $R = 0$. However this condition is not enough to fix all the integration constants. We will impose additional physically motivated constraints to obtain unique solutions to our equations.

Asymptotic Flatness

We want the corrections $H_{\mu\nu}$ to vanish exponentially as $R \to \infty$. This is not unreasonable since we want to recover the flat space metric as we move large distance(in units of $\frac{1}{D}$) away from the horizon. This condition fixes many of the integration constants.

Normalization Conditions

We are still left with two integration constants even after ensuring asymptotic flatness; one in scalar sector and one in vector sector. This is what should be expected on physical grounds. The metric (2.6) was parameterised by a scalar function B and a vector function u. The metric is left invariant by the transformation of the form $B \to B + \frac{1}{D}B_1$. Such a redefinition will change the first order correction. We are left with a two parameter ambiguity of our first order solutions exactly as we expect. This is because we have not given a precise all orders definition of our shape function B and velocity field u.

We fix this ambiguity of field redefinition for B and u by providing additional constraints on all subsequent order corrections to the metric. We demand that $H_{VV}, V_i^{(V)}$ vanish at $R = 0$. This constraint written invariantly amounts to

$$H_{MN}n^N|_{B=0} = 0 \tag{3.28}$$

We refer to these additional constraints which are effectively constraints on B and u as normalisation conditions. These conditions together with asymptotic flatness fixes all integration constants and determines the first order corrections to (2.6) uniquely.

3.5 Results for the first order metric corrections in the patch

$$\begin{aligned}
H_{ij}(R) =& 0 \\
H_{iX}(R) =& 0 \\
H_{iV}(R) =& Re^{-R}\left(\frac{S}{n_S}\right)\left(\frac{V_2}{n_S} - V_3\right) \\
H_{VX}(R) =& Re^{-R}\left(\frac{S}{n_S}\right)\left(\frac{S_1}{n_S^2} - S_2\right) \\
H_{VV}(R) =& Re^{-R}\left(R\left(-1 + \frac{S_3}{n_S} + \frac{S}{2n_S}S_2\right) - 1\right)
\end{aligned} \tag{3.29}$$

3.6 Geometrical Form of the First Order Corrections

3.6.1 Geometrical Form and Redistribution Invariance

The membrane equations of motion (3.25) and (3.27) are make special reference to S and n_S and X. The expressions involving S, n_S and X are well defined only for configurations preserving SO(D-p-2) symmetry. Also the definition of $S^2 = e^{\phi(x^\mu)}$ depends on the details of the isometry.

Unconstrained dependence on S, n_S and X are not acceptable for the following reason. A configuration preserving SO$(D-p-2)$ symmetry must also preserve SO$(D-p'-2)$ symmetry for all $p' > p$. Any solution to the equations for a particular p must also be a solution for all $p' > p$. We refer to this property of the equations as redistribution invariance.

Our equations can be manifestly redistribution invariant if they can be written down in terms of quantities of the full D dimensions. One might wonder how this can be possible with explicit appearances of S, n_S and X in our membrane equations and metric corrections. However this is possible in the large D limit. Consider the extrinsic curvature of the surface $B = 0$, $\nabla_A n^A = \mathcal{K}$ which appears in the metric (2.6). This is a manifestly geometrical quantity as A runs over all D dimensions. We compute this quantity explicitly with the metric of the embedding D dimensional Minkowski space written in the following co-ordinates

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta + dS^2 + d\Omega_d^2 \tag{3.30}$$

Then $\mathcal{K}$ is [4]

$$\nabla_A n^A = \frac{1}{S^d}\partial_\mu(S^d n^\mu) = d\frac{n_S}{S} + \partial_\mu n^\mu \tag{3.31}$$

We see that this manifestly geometric quantity is given by $d\frac{n_S}{S}$ at leading order. So whenever we encounter this quantity we can replace it by $\mathcal{K}$. A full list of geometrical quantities and what they correspond to in terms of S and n_S is given in Appendix A. It turns out that first order metric corrections $H_{\mu\nu}$ and the equations of motion (3.25) and (3.27) can be geometrised.

3.6.2 Geometrised metric correction

Though we expect our metric to be geometrical on physical grounds, this is non-trivial at algebraic level. The basis for one-forms in which we write our

[4] μ is not in the large angle directions as we are looking at configurations having SO(D-p-2) symmetry, so n is a function of the p co-ordinates and S.

metric has $X_\mu = (ds - n_S n)^\mu$ which is not intrinsically geometrical because n_S and S explicitly depends on the details of the split. The only two geometrical vector fields we have in the set-up are n and u. Also the correction $\delta\phi_2$ which is the correction to the radius of the large sphere is not geometrical by itself. Let us consider the part that looks ungeometrical in (3.16)

$$H_{ii}dy^i dy^i + H_{XX}dX^2 + 2H_{VX}dVdX + 2H_{Vi}dVdy^i + 2H_{Xi}dXdy^i + 2H_{ij}dy^i dy^j \tag{3.32}$$

dy^i in a covariant form can be written as $dy^\mu = P^\mu_\alpha dx^\alpha$ where $\mu = 1, 2, 3 \ldots D$, $P^{\mu\nu} = \eta^{\mu\nu} - n^\mu n^\nu + u^\mu u^\nu - \frac{X^\mu X^\nu}{1-n_S^2}$ and x^μ are the Cartesian co-ordinates in D dimensions. Rewriting the ungeometrical part again in terms of the above quantities we get

$$\begin{aligned} &H_{ii}dy^i dyi + H_{XX}dX^2 + 2H_{VX}dVdX + 2H_{V\mu}dVP^\mu_\alpha dx^\alpha + \\ &2H_{X\mu}dXP^\mu_\alpha dx^\alpha + 2H_{\mu\nu}P^\mu_\alpha dx^\alpha P^\nu_\beta dx^\beta \end{aligned} \tag{3.33}$$

We know that the only geometrical part in $P^{\mu\nu}$ is $\eta^{\mu\nu} - n^\mu n^\nu + u^\mu u^\nu$. The above ungeometrical part can be geometrised if $H_{VX}dVdX$ cancels with $H_{V\mu}\frac{X^\mu X_\alpha}{1-n_S^2}dx^\alpha$ part of P^μ_α and $2H_{X\mu}dXP^\mu_\alpha dx^\alpha$ cancels with $2H_{\mu\nu}\frac{X^\mu X_\alpha}{1-n_S^2}dx^\alpha P^\nu_\beta dx^\beta$ part of $2H_{\mu\nu}P^\mu_\alpha dx^\alpha P^\nu_\beta dx^\beta$. We see that indeed this is what happens. This is a very impressive and non trivial check of our metric corrections.

Now we come to the trace part. Note that now we are looking at the full D dimensional metric where e^ϕ is the radius of the d-sphere. So the correction to ϕ also enters the $\mathcal{P}^{\mu\nu}H_{\mu\nu}$ where $\mathcal{P}^{\mu\nu}$ is now the projector orthogonal to n and u. The η_{MN} in (2.6) can be split in the following way.

$$\begin{aligned} \eta_{\mu\nu}dx^\mu dx^\nu + e^\phi d\Omega_d^2 \\ \mu = 1, 2, \ldots p+2 \end{aligned} \tag{3.34}$$

The correction to ϕ changes the radius of the d-sphere and the metric in the isometry directions look like

$$e^\phi(1 + \frac{1}{(D-3)^2}\delta\phi)d\Omega_d^2 = e^\phi d\Omega_d^2 + \frac{\delta\phi}{(D-3)^2}d\Omega_d^2 \tag{3.35}$$

The trace over the d sphere part of the metric gives a correction $\frac{\delta\phi}{(D-3)^2} \times d$ which is equal to $\frac{\delta\phi}{(D-3)}$ at leading order. The full geometrical form of the correction is therefore $H_{\mu\nu}\mathcal{P}^{\mu\nu} = H_{ii} + (1-n_S^2)H_{XX} + \delta\phi$. We find a separate Einstein's equation for this whole combination which gives us $H_{\mu\nu}\mathcal{P}^{\mu\nu}$. Now we are ready to write (3.16) in a geometric basis(we replace dV by $O_M dx^M$)

$$H_{MN} = H_{VV}O_M O_N + H_{VA}\mathcal{P}^A_M O_N + H^{(T)}_{MN} + H^{(Tr)} \tag{3.36}$$

where

$$\begin{aligned}\mathcal{P}^{AB} &= \eta^{AB} - n^A n^B + u^A u^B \\ H^{(T)}_{MN}\mathcal{P}^{MN} &= 0 \\ H^{(Tr)} &= H_{MN}\mathcal{P}^{MN}\end{aligned} \tag{3.37}$$

The metric corrections (3.36) are now manifestly geometric as the indices run over all D dimensions. The metric written has two scalars H_{VV} and $H^{(Tr)}$, one vector $\mathcal{P}^M_N H_{VM}$, and one traceless symmetric tensor, $H^{(T)}_{MN}$. The metric in geometrical form upto first order is

$$\begin{aligned}&\eta_{MN} + \frac{O_M O_N}{\psi^{D-3}} - 2(\psi-1)(D-3)\psi^{-(D-3)}\left(\frac{D}{\mathcal{K}}\left(\frac{\nabla_N\mathcal{K}}{\mathcal{K}} - (u.\nabla)u_N\right)\right)P^N_M O_N \\ &- (\psi-1)(D-3)\psi^{-(D-3)}\left(\frac{1}{\mathcal{K}}\left((\psi-1)(D-3)\left(\frac{\mathcal{K}}{D} - u.\nabla\mathcal{K}\frac{1}{2}u.K.u\right) + \mathcal{K}\right)\right)O_M O_N\end{aligned} \tag{3.38}$$

3.6.3 Geometrised equations of motion

The equations (3.25) and (3.27) can be combined into one geometrical vector equation. The geometrical vector equation is

$$\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{\nabla_A\mathcal{K}}{\mathcal{K}} + u_C K^C_A - u.\nabla u_A\right)\mathcal{P}^A_B = 0 \tag{3.39}$$

The geometrised divergence of the above equation (3.39) to leading order is

$$\frac{\nabla^2\mathcal{K}}{\mathcal{K}^2} - \frac{u.\nabla\mathcal{K}}{\mathcal{K}} + u.K.u - \frac{u.\nabla\mathcal{K}}{\mathcal{K}} = 0 \tag{3.40}$$

gives the scalar equation (3.25) at leading order. Thus the equations (3.25) and (3.27) can be geometrised in to single equation (3.27).

(3.39) gives us an equation of the membrane dynamics which can be solved independently of the gravitational problem of horizon dynamics in large D. Note that the equation (3.39) has $D-2$ components. We have $D-3$ independent components of u owing to the three constraints $u.u = -1$, $u.n = 0$ and $\nabla^M.u = 0$ and a shape function B. This gives us a total of $D-2$ unknown functions. Since we have $D-2$ equations for $D-2$ functions they can be solved for in principle. The equation (3.39) gives us a closed system for the membrane dynamics.

Chapter 4

Second Order Calculation

We follow the same perturbative procedure that has been set up in the previous chapter to calculate the $\frac{1}{(D-3)^2}$ corrections to the metric. For this order we use the geometric form of the metric (3.36) as our starting point for our perturbation theory. We expand all the quantities in this metric to order $\frac{1}{(D-3)^2}$. At this order the relevant data is the second derivatives of n and u can be seen from there Taylor expansion in the patch (3.10). In addition to the pure second order derivative data we also have bilinears of the first order data appearing at this order. For our calulation at this order, we demand in addition to the constraints (2.8) and eqrefsubcond the derivative of the leading order equation of motions (3.25) and (3.27) are 0 along the membrane. More precisely we demand that

$$(\partial_i - n_i n.\partial)\left(\left(\frac{dS}{n_S} - u\right) \cdot K \cdot \left(\frac{dS}{n_S} - u\right) - \left(\frac{1-n_S^2}{Sn_S}\right)\right) = 0$$
$$(\partial_k - n_k n.\partial)\, P^i_j \left[\left(\frac{dS}{n_S} - u\right) \cdot \partial(n-u)_i\right] \tag{4.1}$$

The first one gives us 2 scalar constraint and 1 vector constraint equation. The second one gives us 2 vector constraints, 1 scalar constraint , 1 symmetric tensor constraint and 1 antisymmetric constraint equation.

The Einstein's equations schematically look the same as (3.17).

$$Hv^{(2)} = s^{(2)} \tag{4.2}$$

The argument that the sources and thus in turn the corrections $v^{(2)}$ are function of the fast varying direction R remain the same as given in 3.4.2.The homogenous differential operator H acting on the corrections remain the same, but the sources $s^{(2)}$ are more complicated and have many more analytical structures. The same combinations of Einstein's equations as given in (3.20) and (3.22) become identically 0. The first order correction to the equation is determined in the same manner as discussed in 3.4.4.

Before beginning the discussion of the second order corrections to the metric and the first order correction to the equation we list the pure second order data which consists of $\partial_\mu \partial_\nu n_\alpha$ and $\partial_\mu \partial_\nu u_\alpha$.

4.1 Independent Data at Second Order

Recall that we can divide the data into three symmetry channel in the reduced $p+3$ dimensional space depending on how they transform under SO(p) rotations.

4.1.1 Independent Data in the Scalar Sector

Let us first consider the scalar data from $\partial_\mu \partial_\nu n_\alpha$. Recall that there are three scalar directions n, u and dS. We can get a scalar if all the three indices are in the scalar directions. But the pieces of data where one of the indices are along n are not free because of the constraint $n.K = 0$. So the indices along two scalar directions u and dS gives us independent data. There are $\frac{2\times 3 \times \times 4}{6} = 4$ such pieces of data. We also get a scalar data if one of the indices is in the scalar direction and the other two indices are traced over. Again there are 2 ways to choose the scalar index, which gives us 2 more scalars from $\partial_\mu \partial_\nu n_\alpha$. So there are a total of 6 scalar data two derivative shape data.

Let us consider the scalar data from $\partial_\mu \partial_\nu u_\alpha$ now. As discussed in **??** u is only in the vector directions. So the u index is can only be in the vector direction. We can get get a scalar if one of the derivative indices is traced over with the u index and the other derivative index is in the scalar direction. Again due to the constraint $n.\nabla u = 0$, the derivative index can only be in two scalar directions. Thus we have two scalar data from $\partial_\mu \partial_\nu u_\alpha$.

From (4.1) there are 3 scalar constraint equations. So there are total $6 + 2 - 3 = 5$ independent scalar data.

4.1.2 Independent Data in Vector Sector

From $\partial_\mu \partial_\nu n_\alpha$ we get a vector if one of the indices is a vector index and the other two are in the scalar directions. Again there are two free scalar directions because of the constraint $n.K = 0$. Thus we get $\frac{2\times 3}{2} = 3$ vectors. We get another vector when two of the indices in the p-direction is traced over and the remaining one is also in the p-direction. In total we get 4 vectors from $\partial_\mu \partial_\nu n_\alpha$.

Table 4.1: Independent Data at Second Order in 3 Symmetry Channels

Scalars	Vectors	Tensors
$S_1^{(2)} = C_{SSS}$	$V_1^{(2)} = C_{SS\beta} P_\mu^\beta$	$T_1^{(2)} = P_\alpha^\mu P_\beta^\nu (C_{S\mu\nu} - \frac{\eta_{\mu\nu}}{p} P_{\gamma\theta} C^{S\gamma\theta})$
$S_2^{(2)} = u^\alpha C_{\alpha SS}$	$V_2^{(2)} = P^{\alpha\beta} K_{\alpha\beta\delta} P_\mu^\delta$	$T_2^{(2)} = P_\alpha^\mu P_\beta^\nu (u^\gamma C_{\gamma\mu\nu} - \frac{\eta_{\mu\nu}}{p} P^{\gamma\theta} u^\alpha C_{\alpha\gamma\theta})$
$S_3^{(2)} = u^\alpha u^\beta C_{\alpha\beta S}$	$V_3 = \partial_S \partial_S u_\beta P_\mu^\beta$	$T_3^{(2)} = P_\alpha^\mu P_\beta^\nu (\partial_S \partial_\mu u_\nu - \frac{\eta_{\mu\nu}}{p} P^{\gamma\theta} \partial_S \partial_\gamma u_\theta)$
$S_4^{(2)} = P^{\mu\nu} C_{S\mu\nu}$	$V_4 = P^{\alpha\beta} \partial_\beta \partial_\alpha u_\delta P_\mu^\delta$	
$S_5^{(2)} = P^{\mu\nu} \partial_\mu \partial_S u_\nu$	$V_5^{(2)} = P_\mu^\alpha \partial_\alpha \partial_\beta u_\delta P^{\beta\delta}$	
	$V_6^{(2)} = u^\alpha u^\beta \partial_\alpha \partial_\beta u_\delta P_\mu^\delta$	

As we have discussed the index on u in $\partial_\mu \partial_\nu u_\alpha$ is always in the vector direction. We get a vector if the derivatives are in the scalar directions. This gives us $\frac{2\times 3}{2} = 3$ vectors. We also get vectors two of the derivative indices are traced over the p directions or if one derivative index in the p-direction is traced over with the index on u and one of the derivative index is a free derivative index. So we have a total of 5 vectors from $\partial_\mu \partial_\nu u_\alpha$.

The (4.1) gives 3 vector constraint equations. So there are a total of $4 + 5 - 3 = 6$ independent vector data at second order.

4.1.3 Independent Data in Tensor Sector

From $\partial_\mu \partial_\nu n_\alpha$ we get two traceless symmetric two-tensors when one of the indices in the scalar direction and the other two are in the vector direction. We also get a 3-tensor when all the three indices are in the p-directions. From $\partial_\mu \partial_\nu u_\alpha$ also we get two symmetric traceless 2-tensor and two anti-symmetric 2-tensor data when one of the derivative index is in the scalar direction. We also get a 3-tensor when all the indices are in the p-directions.

The (4.1) gives one symmetric tensor constraint. So there are $2+2-1 = 3$ independent symmetric traceless 2 tensors.

In 4.1 we list the independent data at the second order. We denote $\partial_\alpha K_{\beta\gamma} = C_{\alpha\beta\gamma}$

4.2 Correction to Membrane Equation of Motion

We obtained our membrane equation of motion at first order by demanding regularity at the horizon ie $R = 0$ for the corrections $H_{VX}^{(1)}$ and $H_{Vi}^{(1)}$. This regularity condition was got from the Einstein constraint equations $E^{MN} \cdot d\psi_N$ as discussed in 3.4.4. We look at the same combination as (3.24) and (3.26) at second order. Since the homogenous differential operator does not change at each order the form of the equations remain the same. The regularity condition for the solutions $H^{(2)}_{VX}$ and $H^{(2)}_{Vi}$ give us constraint on our second order data. This is the first order correction to the zeroth order equations of motion (3.27) and (3.25).

To geometrise this first order correction, we first expand the geometric form of the equation of motion (3.39) and its divergence (3.40) to first subleading order explicitly in terms of our data given in 3.1 and 4.1. We subtract this subleading part from the equations that regularity at second order gives us, as this is the part that gives us the new constraints on our data arising purely in second order. We see that we are able to geometrise this equation. Also the equation obtained from the regularity of $H^{(2)}_{VX}$ turns out to be the divergence $H^{(2)}_{Vi}$ when geometrised. We present the final geometrised form of the equation with first order correction over here.

$$\begin{aligned}
&\left(\frac{\nabla^2 u}{\mathcal{K}} - \frac{\nabla\mathcal{K}}{\mathcal{K}} + u\cdot K - (u\cdot\nabla)u + \frac{1}{D}\left(D\frac{\nabla^2\nabla^2 u}{\mathcal{K}^3} - D\frac{\nabla(\nabla^2\mathcal{K})}{\mathcal{K}^3}\right.\right.\\
+\quad & 3D\frac{(u\cdot K\cdot u)(u\cdot\nabla u)}{\mathcal{K}} - 3D\frac{(u\cdot K\cdot u)(u\cdot\nabla n)}{\mathcal{K}} - 6D\frac{(u\cdot(\nabla^2 n))(u\cdot\nabla u)}{\mathcal{K}^2}\\
+\quad & \left.\left. 6D\frac{(u\cdot(\nabla^2 n))(u\cdot\nabla n)}{\mathcal{K}^2} + 3u\cdot\nabla u - 3u\cdot\nabla n\right)\right)P = 0
\end{aligned}$$

where

$$P^{AB} = \eta^{AB} - n^A n^B + u^A u^B \tag{4.3}$$

where the ∇ is space-time covariant derivative in D dimensions taken with respect to the metric (3.5). We can convert the equation into an equation with quantities on the membrane world-volume using the dictionary given in Appendix B. The equation in membrane world volume quantities is given by

$$\begin{aligned}
&\left[\left(\frac{\nabla^2 u_A}{\mathcal{K}} - \frac{u^C K_{CB} K^B_A}{\mathcal{K}}\right) - \frac{\nabla_A\mathcal{K}}{\mathcal{K}} + u^B K_{BA} - u\cdot\nabla u_A\right.\\
+\quad & \left(\frac{\nabla^2\nabla^2 u_A}{\mathcal{K}^3} - \frac{u\cdot\nabla\mathcal{K}\nabla_A\mathcal{K}}{\mathcal{K}^3} - \frac{\nabla^B\mathcal{K}\nabla_B u_A}{\mathcal{K}^2} - 2\frac{K^{CD}\nabla_C\nabla_D u_A}{\mathcal{K}^2}\right)\\
+\quad & \left(-\frac{\nabla_A\nabla^2\mathcal{K}}{\mathcal{K}^3} + \frac{\nabla_A\left(K_{BC}K^{BC}\mathcal{K}\right)}{\mathcal{K}^3}\right) + 3\frac{(u\cdot K\cdot u)(u\cdot\nabla u_A)}{\mathcal{K}} - 3\frac{(u\cdot K\cdot u)(u^B K_{BA})}{\mathcal{K}}\\
-\quad & \left. 6\frac{(u\cdot\nabla\mathcal{K})(u\cdot\nabla u_A)}{\mathcal{K}^2} + 6\frac{(u\cdot\nabla\mathcal{K})(u^B K_{BA})}{\mathcal{K}^2} + \frac{3}{D}u\cdot\nabla u_A - \frac{3}{D}u^B K_{BA}\right]P^{AC} = 0
\end{aligned} \tag{4.4}$$

4.3 Solving Einstein's Equations at Second Order

The Einstein equation for the tensor sector is

$$
\begin{aligned}
& e^{-R}\frac{d}{dR}((e^R-1)\frac{d}{dR}(P_A^M P_B^N H_{MN})) = \\
& e^{-R}T_{AB}^{(1)} - Re^{-R}T_{AB}^{(2)} - e^{-2R}T_{AB}^{(2)} + Re^{-2R}T_{AB}^{(3)} - R^2 e^{-2R}\frac{T_{AB}^{(2)}}{2} \\
& \Rightarrow \frac{d}{dR}(P_A^M P_B^N H_{MN}) = \\
& \frac{1}{e^R-1}\int_0^R \left(T_{AB}^{(1)} - xT_{AB}^{(2)} - e^{-x}T_{AB}^{(2)} + xe^{-x}T_{AB}^{(3)} - x^2 e^{-x}\frac{T_{AB}^{(2)}}{2}\right) dx
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
T_{AB}^{(1)} \quad &= \frac{2D^2}{\mathcal{K}^2} P_A^M P_B^N \left(\left(\frac{\mathcal{K}}{D}(K_{MN} - \nabla_{(M}u_{N)})\right) - (K_{MC} - \nabla_C u_M)P^{CD}(K_{DN} - \nabla_D u_N) + V_M V_N \right) \\
&- \frac{2D^2}{\mathcal{K}^2}\frac{\eta_{AB}}{D} P^{MN} \left(\left(\frac{\mathcal{K}}{D}(K_{MN} - \nabla_{(M}u_{N)})\right) - (K_{MC} - \nabla_C u_M)P^{CD}(K_{DN} - \nabla_D u_N) + V_M V_N \right)
\end{aligned}
$$

and

$$
T_{AB}^{(2)} = \frac{2D^2}{\mathcal{K}^2}\left(P_A^M P_B^N V_M V_N - \frac{P^{MN} V_M V_N \eta_{AB}}{D}\right)
$$

and

$$
T_{AB}^{(3)} = \frac{4D^2}{\mathcal{K}^2}\left(P_A^M P_B^N V_M V_N - \frac{P^{MN} V_M V_N \eta_{AB}}{D}\right)
$$

where,

$$
V_A = u_C K_A^C - u.\nabla u_A
$$

The RHS above is regular at $R \to 0$ provided the integral vanishes at $R = 0$, which is implemented manifestly above as the integrand is regular at $R = 0$. Integrating once again the above solution and then implementing the boundary condition that $P_A^M P_B^N H_{MN} = 0$ at $R \to \infty$ we get

$$
\begin{aligned}
P_A^M P_B^N H_{MN} = & - e^{-R}(3T_{AB}^{(2)} - T_{AB}^{(3)}) + Re^{-R}(T_{AB}^{(3)} + T_{AB}^{(2)}) - \\
& \int_R^\infty \frac{R\left(T_{AB}^{(1)} + T_{AB}^{(3)} + T_{AB}^{(2)}\right)}{e^R - 1}
\end{aligned}
\tag{4.6}
$$

The ϕ equation is given by

$$\begin{aligned} \frac{1}{2}\frac{d^2}{dR^2}\left(P^{MN}H_{MN}\right) &= e^{-R}S_\phi - e^{-R}RS_\phi \\ P^{MN}H_{MN} &= \phi + 2H_{ii} + (1-ns^2)H_{yy} \end{aligned} \tag{4.7}$$

where

$$S_\phi = \frac{D^2}{\mathcal{K}^2}\left(u\cdot K - u\cdot\nabla u\right)\cdot P\cdot\left(u\cdot K - u\cdot\nabla u\right)$$

Now, the boundary condition on ϕ is that both ϕ and its derivative vanish at $R\to\infty$.The $P^{\mu\nu}H_{\mu\nu}$ solution written above satisfies this condition. Integrating the above equation once we get

$$\frac{1}{2}\frac{d}{dR}\left(P^{MN}H_{MN}\right) = -\int_R^\infty\left(e^{-x}S_\phi - e^{-x}xS_\phi\right)dx \tag{4.8}$$

From the above expression we see that the boundary condition on ϕ' is satisfied. Integrating once more the solution for ϕ satisfying the above mentioned boundary condition can be written as

$$P^{MN}H_{MN} = 2\int_R^\infty\left(\int_y^\infty\left(S_\phi e^{-x} - S_\phi xe^{-x}\right)dx\right)dy \tag{4.9}$$

The geometrised sources for Hvi and Hvy give

$$\begin{aligned} &e^{-2R}(-1+e^R)\frac{d}{dR}(e^R\frac{d}{dR}P_A^M H_{VM}) = \\ &R^2e^{-2R}Sv^{(1)} + Re^{-2R}Sv^{(2)} + e^{-2R}\left(-Sv^{(2)} - 2Sv^{(1)}\right) + R^2e^{-R}\left(-Sv^{(1)}\right) \\ &+ Re^{-R}\left(-Sv^{(2)} + Sv^{(3)}\right) + e^{-R}\left(Sv^{(2)} + 2Sv^{(1)}\right) \end{aligned} \tag{4.10}$$

where

$$Sv^{(1)} = \frac{3}{2}\frac{D}{\mathcal{K}}\left(1 + 2\frac{u\cdot\nabla\mathcal{K}\ D}{\mathcal{K}^2} - \frac{u\cdot K\cdot u\ D}{\mathcal{K}}\right)\left(u\cdot\nabla u - u\cdot K\right).P$$

$$Sv^{(2)} = \frac{D}{\mathcal{K}}\left(\frac{D}{\mathcal{K}^3}\left(\nabla\nabla^2\mathcal{K} - \nabla^2\nabla^2 u\right) + 8(u\cdot K - u\cdot\nabla u) + u\cdot K + \frac{\nabla^2 u}{\mathcal{K}}\right)\cdot P$$

$$Sv^{(3)} = \left(-2\frac{D^2}{\mathcal{K}^2}\left(\frac{\nabla_M\mathcal{K}}{\mathcal{K}} - u^A K_{AM}\right)P^{MN}\left(\nabla_N u_C - K_{NC}\right) + \frac{D}{\mathcal{K}}\left(u\cdot K - \frac{\nabla^2 u}{\mathcal{K}}\right)\right).P$$

The solution to the above equation can be written as

$$
\begin{aligned}
& e^R \frac{d}{dR} P_A^M H_{VM} = \\
& \int \frac{1}{1-e^{-R}} \Big(R^2 e^{-R} Sv^{(1)} + R e^{-R} Sv^{(2)} + e^{-R} \left(-Sv^{(2)} - 2Sv^{(1)}\right) \\
& + R^2 \left(-Sv^{(1)}\right) + R\left(-Sv^{(2)} + Sv^{(3)}\right) + \left(Sv^{(2)} + 2Sv^{(1)}\right) \Big) dR + C1
\end{aligned}
\tag{4.11}
$$

For regularity we need the numerator of RHS above to vanish at $R \to 0$, which is manifestly true. The final solutions is

$$
P_A^M H_{VM} = -\int_R^\infty e^{-R} \int \frac{P1(R)e^{-R} + P2(R)e^{-2R}}{1-e^{-R}} dR - C1 e^{-R} \tag{4.12}
$$

where,

$$
P1(R) = R^2 Sv^{(1)} + R Sv^{(2)} - (Sv^{(2)} + 2Sv^{(1)})
$$

and,

$$
P2(R) = -R^2 Sv^{(1)} + R(Sv^{(3)} - Sv^{(2)}) + 2Sv^{(1)} + Sv^{(2)}
$$

We can fix the integration constant $C1$ with the boundary condition that $P_\alpha^\mu H_{\nu\mu} = 0$ at $R = 0$. This gives

$$
C1 = -\frac{1}{12}\left(9Sv^{(1)} + 3Sv^{(2)} + \left(3 - 4\pi^2\right) Sv^{(3)}\right)
$$

The Hvv equation is

$$
\begin{aligned}
& e^{-R}\big(\frac{d}{dR}\left(e^R \frac{d}{dR} H_{vv}\right) - \frac{1}{2} e^{-R}\left(\phi' + 2H'_{ii} + (1-n_s^2)H'_{yy}\right) = \\
& e^{-R} Svv^{(1)} + R e^{-R} Svv^{(2)} + R^2 e^{-R} Svv^{(3)} + R^3 e^{-R} Svv^{(4)} + R e^{-2R} Svv^{(5)} \\
& + R^2 e^{-2R} Svv^{(6)} + e^{-R} Svv^{(7)} \int \frac{R dR}{1-e^{-R}}
\end{aligned}
\tag{4.13}
$$

where,

$$
\begin{aligned}
Svv^{(1)} \;=\;& 2 + \frac{1}{2}\frac{D^2}{\mathcal{K}^2}\Big(\nabla_{[M}u_{N]}\nabla_{[P}u_{Q]}P^{NP}P^{MQ}\Big) - 2\frac{D^2}{\mathcal{K}^2}\Big(K_{MN}K_{PQ}P^{NP}P^{MQ} \\
-\;& \frac{\mathcal{K}^2}{D-3}\Big) + \frac{D^2}{\mathcal{K}^2}\Big(u\cdot\nabla u_M u\cdot\nabla u_N P^{MN}\Big) + 2\frac{D}{\mathcal{K}}\left(\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right) + 2\frac{D}{\mathcal{K}} u.K.u \\
+\;& 2\frac{D^2}{\mathcal{K}^2}\Big(u^M K_{MN} u^P K_{PQ} P^{NQ}\Big) - \frac{D}{\mathcal{K}}\nabla.C1
\end{aligned}
$$

$$
\begin{aligned}
Svv^{(2)} &= -2 + (\frac{D^2}{\mathcal{K}^2}\left(\frac{\nabla^2 u_M}{\mathcal{K}}\frac{\nabla^2 u_N}{\mathcal{K}}P^{MN}\right) - 2\frac{D^2}{\mathcal{K}^2}\left(\frac{\nabla^2 u_M}{\mathcal{K}}u^P\nabla_P u_N P^{MN}\right) \\
&-\frac{D^2}{\mathcal{K}^2}\left(\nabla_M u_N \nabla_P u_Q P^{NP}P^{MQ}\right) + 2\frac{D^2}{\mathcal{K}^2}\left(\nabla_M u_N K_{PQ}P^{NP}P^{MQ}\right) \\
&+\frac{D^2}{\mathcal{K}^2}\left(K_{MN}K_{PQ}P^{NP}P^{MQ} - \frac{\mathcal{K}^2}{D-3}\right) - \frac{D^2\nabla^2\nabla^2\mathcal{K}}{\mathcal{K}^5} \\
&-\frac{D^2}{\mathcal{K}^2}\left(u\cdot\nabla u_M u\cdot\nabla u_N P^{MN}\right) - 28\frac{D}{\mathcal{K}}\left(\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right) + 9\frac{D^2}{\mathcal{K}^2}\left(\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right)^2 \\
&+7\frac{D}{\mathcal{K}}\left(u\cdot K\cdot u\right) - 12\frac{D^2}{\mathcal{K}^2}\left((u.K.u)\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right) + \\
&3\frac{D^2}{\mathcal{K}^2}\left(u.K.u\right)^2 - 2\frac{D^2}{\mathcal{K}^2}\left(u^M K_{MN}u^P K_{PQ}P^{NQ}\right) \qquad (4.14)
\end{aligned}
$$

$$
\begin{aligned}
Svv^{(3)} &= -\frac{1}{2}\frac{D^2}{\mathcal{K}^2}\left(\frac{\nabla^2 u_M}{\mathcal{K}}\frac{\nabla^2 u_N}{\mathcal{K}}P^{MN}\right) + \frac{D^2}{\mathcal{K}^2}\left(\frac{\nabla^2 u_M}{\mathcal{K}}u^P\nabla_P u_N P^{MN}\right) \\
&+\frac{D^2\nabla^2\nabla^2\mathcal{K}}{\mathcal{K}^5} - \frac{D^2}{\mathcal{K}^2}\left(u\cdot\nabla u_M u\cdot\nabla u_N P^{MN}\right) + 19\frac{D}{\mathcal{K}}\left(\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right) \\
&+\frac{3}{2}\frac{D^2}{\mathcal{K}^2}\left(u.K.u\right)^2 - 7\frac{D}{\mathcal{K}}\left(u\cdot K\cdot u\right) \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
Svv^{(4)} &= -2\frac{D}{\mathcal{K}}\left(\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right) - 2\frac{D^2}{\mathcal{K}^2}\left(\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right)^2 + \frac{D}{\mathcal{K}}\left(u\cdot K\cdot u\right) \\
&+2\frac{D^2}{\mathcal{K}^2}\left((u.K.u)\frac{u.\nabla\mathcal{K}}{\mathcal{K}}\right) - \frac{1}{2}\frac{D^2}{\mathcal{K}^2}\left(u.K.u\right)^2 \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
Svv^{(5)} &= -2\frac{D^2}{\mathcal{K}^2}\left(u\cdot\nabla u_M u\cdot\nabla u_N P^{MN}\right) + 4\frac{D^2}{\mathcal{K}^2}\left(u\cdot\nabla u_M u^P K_{PQ}P^{MQ}\right) \\
&-2\frac{D^2}{\mathcal{K}^2}\left(u^M K_{MN}u^P K_{PQ}P^{NQ}\right) \qquad (4.17)
\end{aligned}
$$

$$
Svv^{(6)} = -\frac{1}{2}Svv^{(5)}
$$

$$
Svv^{(7)} = \frac{2}{\mathcal{K}}\left(\frac{D}{\mathcal{K}}\left(\frac{\nabla_M\mathcal{K}}{\mathcal{K}} - u^A K_{AM}\right)P^{MN}\left(\frac{\nabla_N\mathcal{K}}{\mathcal{K}} - u^A K_{AN}\right) + \frac{u.\nabla\mathcal{K}}{\mathcal{K}} - u.K.u\right) \qquad (4.18)
$$

Substituting (4.8) in (4.13) we get

$$e^{-R}(\frac{d}{dR}\left(e^{R}\frac{d}{dR}H_{vv}\right) =$$
$$e^{-R}Svv^{(1)} + Re^{-R}Svv^{(2)} + R^2e^{-R}Svv^{(3)} + R^3e^{-R}Svv^{(4)} + Re^{-2R}Svv^{(5)} + R^2e^{-2R}Svv^{(6)} +$$
$$e^{-R}Svv^{(7)}\int\frac{RdR}{1-e^{-R}} + Re^{-2R}S_{\phi} \tag{4.19}$$

Integrating the above equation once we get

$$\begin{aligned}\frac{d}{dR}H_{vv} = \quad & e^{-R}\Bigg(\int dR\Bigg(Svv^{(1)} + RSvv^{(2)} + R^2Svv^{(3)} + R^3Svv^{(4)} + Re^{-R}(Svv^{(5)} + S_{\phi}) \\ & +R^2e^{-R}Svv^{(6)} + Svv^{(7)}\int\frac{RdR}{1-e^{-R}}\Bigg)\Bigg) + C_{vv}e^{-R}\end{aligned} \tag{4.20}$$

The right hand side of the above equation is regular at $R = 0$ and 0 at $R = \infty$. We integrate the above equations with limits such that H_{vv} is 0 at infinity. We set the integration constant C_{vv} such that H_{vv} is 0 at $R = 0$.

$$\begin{aligned}H_{vv} = \quad & -\int_R^{\infty}\Bigg(e^{-x}\Bigg(\int dx\Bigg(Svv^{(1)} + xSvv^{(2)} + x^2Svv^{(3)} + x^3Svv^{(4)} + xe^{-x}(Svv^{(5)} + S_{\phi}) \\ & +x^2e^{-x}Svv^{(6)} + Svv^{(7)}\int\frac{xdx}{1-e^{-x}}\Bigg)\Bigg)\Bigg) - \int_R^{\infty} dxC_{vv}e^{-x}\end{aligned} \tag{4.21}$$

where

$$C_{vv} = Svv^{(1)}+Svv^{(2)}+2Svv^{(3)}+6Svv^{(4)}-\frac{3Svv^{(5)}}{4}-\frac{7Svv^{(6)}}{4}+\frac{\pi^2Svv^{(7)}}{3}+2Svv^{(7)}\zeta(3) \tag{4.22}$$

Chapter 5

Spectrum of Light Quasinormal Modes about Schwarschild Black Holes

Our membrane equations (4.4) should describe all SO(D-p-2) invariant black hole dynamics at large D. As an application of our membrane equations of motion we obtain the light quasinormal modes of the Schwarzschild black hole(not the ones whose frequencies are $O(D)$). The Schwarzschild black hole is dual to a perfectly spherical stationary membrane. To calculate the quasinormal modes from the membrane we look at linear fluctuations about this spherical membrane.

We find it convenient to work in spherical polar co-ordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2 \quad . \tag{5.1}$$

The exact solution to our leading order equation of motion (3.39) dual to Schwarzschild black hole in the co-ordinates (5.1) is given by

$$r = 1 \quad u = dt \tag{5.2}$$

We have chosen the size of the membrane to be unity. [1]

[1]We do not loose generality by making this choice. The classical Einstein equations studied in this paper enjoy invariance under the following 'scaling' symmetry:

$$\tilde{g}_{MN} = \alpha^2 g_{MN}.$$

This scale transformation together with the coordinate change $\tilde{x}^M = \alpha x^M$ transforms a Schwarzschild black hole with Schwarzschild radius r_0 into a Schwarzschild black hole with Schwarzschild radius αr_0. It follows that the quasinormal mode frequencies of the black hole parameterized by (r_0) are simply $\frac{1}{r_0}$ times those for the black hole parameterized by (1). For this reason we will perform all computations in this section with black holes of radius unity, and simply reinsert factors of r_0 in the final answer.

We consider the most general linearised perturbations around (5.2).

$$\begin{aligned} r &= 1 + \epsilon \ \delta r(t, \theta), \\ u &= -dt + \epsilon \ \delta u_\mu(t, \theta) dx^\mu. \end{aligned} \tag{5.3}$$

The light modes correspond to the functions f and u which live on the membrane world volume. To get them we simply plug in (5.3) into the membrane equation of motion (4.4) and expanding it to linear order in ϵ. This gives us an effective equation for f and u. We write the induced metric on the membrane world volume in the co-ordinates θ^a on Ω_{D-2} and time. The induced metric on the world volume can be found by substituting (5.3) into (5.1). The induced metric on the membrane to linear order in ϵ is given by

$$ds^2 = -dt^2 + (1 + 2\epsilon\delta r) \, d\Omega_{D-2}^2 \quad . \tag{5.4}$$

We develop a dictionary to go between vectors and one forms in the space time and on the membrane world volume. A vector field on the membrane worldvolume can be uplifted to the whole space time. The different components of this vector field will be given by

$$A^a_{(ST)} = A^a, \quad A^t_{(ST)} = A^t, \quad A^r_{(ST)} = \epsilon \left(A^t \partial_t f + A^a \partial_a f \right). \tag{5.5}$$

Similarly a one-form defined in the space time can be pulled back to the membrane easily.

$$B_a = B_a^{(ST)} + \epsilon B_r^{(ST)} \partial_a f, \quad B_t = B_t^{(ST)} + \epsilon B_r^{(ST)} \partial_t f. \tag{5.6}$$

We will treat u_μ as a one form field on the membrane world volume. Recall that $\nabla . u = 0$ where ∇ is the covariant derivative taken with respect to the metric (5.4). To evaluate the equation of motion (4.4), we need to evaluate the extrinsic curvature, its trace, its derivatives and the derivatives of u on the membrane. We list the different quantities that we will require in our computation. We find it convenient to define the following notations for the different metrics that appear in our computation.

$$\begin{aligned} \overline{g}_{\mu\nu} dx^\mu dx^\nu &= -dt^2 + (1 + 2\epsilon f(t, \theta)) d\Omega_{D-2}^2 \\ \hat{g}_{\mu\nu} dx^\mu dx^\nu &= -dt^2 + d\Omega_{D-2}^2 \\ \overline{g}_{ab} dx^a dx^b &= (1 + 2\epsilon f(t, \theta)) d\Omega_{D-2}^2 \\ \hat{g}_{ab} dx^a dx^b &= d\Omega_{D-2}^2 \end{aligned} \tag{5.7}$$

The index μ runs over time and angular co-ordinates , ie ($\mu = (t, a)$) and the a, b runs over angular co-ordinates on a sphere Ω_{D-2}. We list the quantites that we require to compute the equation of motion below.

$$
\begin{aligned}
u_t^M &= -1 \quad u.u = -1 \Rightarrow \delta u_t = 0 \\
u_a &= \epsilon \delta u_a \\
n_t &= -\epsilon \partial_t f \\
n_a &= -\epsilon \hat{\nabla}_a f \\
\\
K_{tt} &= -\epsilon \partial_t^2 f \\
K_{ta} &= -\epsilon \partial_t \hat{\nabla}_a f \\
K_{ab} &= -\epsilon \hat{\nabla}_a \hat{\nabla}_b f + (1+\epsilon f)\hat{g}_{ab} \\
\\
(\nabla_t u_t) &= 0 \\
(\nabla_a u_t) &= 0 \\
(\nabla_t u_a) &= \epsilon \partial_t \delta u_a \\
(\nabla_a u_b) &= \epsilon \hat{\nabla}_a \delta u_b - \epsilon \partial_t f \hat{g}_{ab} \\
(\hat{\nabla}_a \delta u^a) &= -(D-2)\partial_t f \\
\mathcal{K} &= (D-2) - \epsilon \overline{\nabla}^2 f - \epsilon (D-2) f
\end{aligned}
\tag{5.8}
$$

All covariant derivatives in (5.8) are taken with respect to the background metric $\hat{g}_{\mu\nu}$ as defined in (5.7). The equation of motion (4.4) evaluates to linear order in ϵ evaluates to

$$
\begin{aligned}
&\frac{\nabla^2 \delta u_a}{D-2} - \frac{\partial_t \partial_a f}{D-2} + \frac{\nabla_a \nabla^2 f}{D-2} + \nabla_a f - \partial_a \partial_t f + \delta u_a - \partial_t \delta u_a \\
&+ \frac{\overline{\nabla}^2 \overline{\nabla}^2 \delta u_a}{(D-2)^3} - \frac{2(\hat{\nabla}^2 \delta u_a - \partial_a \partial_t f)}{(D-2)^2} - \frac{\nabla_a \overline{\nabla}^2 (\nabla^2 f + f(D-2))}{(D-2)^3} \\
&- \frac{9\nabla_a (f(D-2)^2 - (D-2)(9\hat{\nabla}^2 f - \partial_t^2 f))}{3(D-2)^3} + \frac{3\partial_t \delta u_a}{D} - \frac{3(-\partial_t \partial_a f + \delta u_a)}{D} = 0
\end{aligned}
\tag{5.9}
$$

The first line in (5.9) comes from the leading order equation of motion and the last two lines come from the $\frac{1}{D-3}$ correction to it. ∇ is with respect to the back ground metric $\hat{g}_{\mu\nu}$, $\overline{\nabla}$ is with respect to the full metric on the membrane $\overline{g}_{\mu\nu}$ and $\hat{\nabla}$ is with respect to the metric on Ω_{D-2}.

The condition $\nabla . u = 0$ where ∇ is with respect to the full metric on the membrane to linear order in ϵ is

$$
\nabla_a \delta u^a = -(D-2)\partial_t f \tag{5.10}
$$

We split u into a gradient of a scalar and a divergence free vector to solve (5.9).

$$\delta u_a = \delta_a \Phi + v_a \tag{5.11}$$

where

$$\nabla . v = 0 \tag{5.12}$$

It follows from (5.10) that

$$\nabla^2 \Phi = -(D-2)\partial_t f \tag{5.13}$$

We will eliminate Φ in favour of f in (5.9). Note that the LHS of (5.13) is always 0 if Φ lies in the kernel of ∇^2, ie if Φ is a constant function on the sphere. It follows that for (5.13) to be solved consistently the spatially constant part of f (the $l = 0$ mode) has to be time independent. Once this condition is obeyed, Φ can be solved in terms of f. Plugging in the expansion (5.11) into (5.9)

$$\begin{aligned}
&\left(\frac{\nabla^2}{D-2} + 1 - \partial_t + \frac{\overline{\nabla}^2 \overline{\nabla}^2}{(D-2)^3} - \frac{2(\hat{\nabla}^2)}{(D-2)^2} + \frac{3\partial_t}{D} - \frac{3}{D} \right) v_a = \\
&- \left(- \frac{\partial_t \partial_a}{D-2} + \frac{\nabla_a \nabla^2}{D-2} + \nabla_a - \partial_a \partial_t + \frac{2\partial_a \partial_t}{(D-2)^2} - \frac{\nabla_a \overline{\nabla}^2 (\nabla^2 + (D-2))}{(D-2)^3} \right. \\
&- \left. \frac{9\nabla_a((D-2)^2 - (D-2)(9\hat{\nabla}^2 - \partial_t^2))}{3(D-2)^3} + \frac{3\partial_t \partial_a}{D} \right) f \\
&- \left(\frac{\nabla^2}{D-2} + 1 - \partial_t + \frac{\overline{\nabla}^2 \overline{\nabla}^2}{(D-2)^3} - \frac{2(\hat{\nabla}^2)}{(D-2)^2} + \frac{3\partial_t}{D} - \frac{3}{D} \right) \nabla_a \Phi
\end{aligned} \tag{5.14}$$

5.1 Spectrum of shape fluctuations

Taking the divergence of (5.14) and plugging in (5.12) and (5.13) we get the following equation for f

$$\begin{aligned}
- \quad & (\nabla^2 + D - 3)\partial_t f - \frac{\partial_t \hat{\nabla}^2 f}{D-2} + \frac{\hat{\nabla}^2 \bar{\nabla}^2 f}{(D-2)} + \hat{\nabla}^2 f - \partial_t \hat{\nabla}^2 f - (D-2)\partial_t f + (D-2)\partial_t^2 f \\
+ \quad & \frac{\hat{\nabla}^2 \partial_t f + (D-2)\partial_t f}{D-2} - \frac{(\nabla^2 + D - 3)^2 (D-2)\partial_t f + (\nabla^2 + D - 3)\hat{\nabla}^2 \partial_t f}{(D-2)^3} \\
+ \quad & 2\frac{(\nabla^2 + D - 3)(D-2)\partial_t f + \hat{\nabla}^2 \partial_t f}{(D-2)^2} + \frac{\hat{\nabla}^2 \nabla^2 (\nabla^2 f + f(D-2))}{(D-2)^3} - \frac{\hat{\nabla}^2 (3\hat{\nabla}^2 f - \partial_t^2 f + 3f(D-2))}{(D-2)^2} \\
- \quad & 3\frac{D-2}{D}\partial_t^2 f + \frac{3}{D}(\partial_t \hat{\nabla}^2 f + (D-2)\partial_t f) = 0
\end{aligned}$$

[2] The most general function linear fluctuation f on a sphere can be expanded as

$$\delta r = \sum_{l,m} a_{lm} Y_{lm} e^{-i\omega_l^r t} \quad . \tag{5.17}$$

where Y_{lm} are spherical harmonics on S^{D-2}, l labels the spherical harmonic representation, m is a collective label for all the internal quantum numbers within a given spherical harmonic representation.

To understand the solutions better we give a more complete description of scalar spherical harmonics in arbitrary dimensions, and in particular the computation the eigenvalue under ∇^2 acting on the l^{th} spherical harmonic. The l^{th} spherical harmonic, Y_{lm}, are composed of the collection of functions on S^{D-2} obtained by restricting homogeneous degree l polynomials in R^{D-1} to the unit sphere. The polynomials in questions are linear combinations of monomials of the form $a_{\mu_1\mu_2\mu_3\ldots\mu_l} x^{\mu_1} x^{\mu_2} \ldots x^{\mu_l}$ where $a_{\mu_1\mu_2\mu_3\ldots\mu_l}$ are symmetric and traceless tensors. It is easily shown that

$$-\nabla^2_{S^{D-2}} Y_{lm} = l(D+l-3) Y_{lm}. \tag{5.18}$$

[3] So we have

$$\begin{aligned} \nabla^2 f &= \omega^2 f - l(D+l-3) f \\ \hat{\nabla}^2 f &= -l(D+l-3) f \end{aligned} \tag{5.20}$$

We plug in (5.20) into (5.17) to get a relation between ω and l. To solve this equation we expand ω as $\omega_0 + \frac{1}{D}\omega_1$. This is easily solved for ω_0 and ω_1 to get

$$\omega_l^r = i(l-1) \pm \sqrt{l-1} + \frac{1}{D}\left(-i(l-1)(l-2) \pm \sqrt{l-1}\left(\frac{3l}{2} - 2\right)\right) \tag{5.21}$$

[2] In order to obtain (5.15) we have used and

$$\begin{aligned} \nabla^a \nabla^2 \delta u_a &= \nabla^2 \nabla_a \delta u^a + R^{ab} \nabla_a \delta u_b, \\ &= \nabla^2 \nabla_a \delta u^a + (D-3)\, g^{ab} \nabla_a \delta u_b, \end{aligned} \tag{5.15}$$

Using the above formula twice

$$\nabla^a \nabla^2 \nabla^2 \delta u_a = -(D-2)\nabla^2\nabla^2 \partial_t f - 2(D-3)(D-2)\nabla^2 \partial_t f - (D-3)^2 (D-2)\partial_t f \tag{5.16}$$

[3] This may be demonstrated as follows. The condition of tracelessness ensures that the degree l polynomials described above obey the equation $\nabla^2 \Phi = 0$, where ∇^2 is evaluated in R^{D-1}. But

$$0 = \nabla^2_{R^{D-1}} \Phi = \frac{1}{r^{D-2}} \partial_r \left(r^{D-2} \partial_r r^l\right) + \frac{\nabla^2_{S^{D-2}} \Phi}{r^2}. \tag{5.19}$$

(the RHS of this equation is ∇^2 of the function in R^{D-1} evaluated in polar coordinates). Here $\nabla^2_{S^{D-2}}$ is the Laplacian evaluated on the unit sphere. (5.18) follows from (5.19).

Reinserting the factor of r_0 as discussed in the beginning of this section we find that

$$r_0\omega_l^r = i(l-1) \pm \sqrt{l-1} + \frac{1}{D}\left(-i(l-1)(l-2) \pm \sqrt{l-1}\left(\frac{3l}{2}-2\right)\right) \quad (5.22)$$

This result matches with the frequencies obtained by Emparan and collaborators in (5.21) gives us the formula for ω_l^r for all $l > 1$. But it needs clarification for the modes in which $l = 0$ and $l = 1$. Let us first consider the case $l = 0$. For this case, (5.22) gives $r_0\omega_l^r = 0$ and $r_0\omega_l^r = 2i - \frac{1}{D}2i$. As discussed below (5.13), that equation is consistent only if $l = 0$ mode is does not have time dependence. Thus we have only one mode at $l = 0$ ie $r_0\omega_l^r = 0$. This mode has a simple physical interpretation, it corresponds to a rescaling of the black hole radius. [4]

Now let us look at the $l = 1$. In this case we have a degeneracy of the quasinormal mode frequencies; $r_0\omega_l^r = 0$. The formula (5.22) was obtained by assuming harmonic time dependence of the function f and solving for the harmonic frequencies but it is well known that this procedure has to be modified when the frequencies are degenerate. To see how this works let us look at (5.15) for $l = 1$. It turns out to be

$$\partial_t^2 f + \frac{1}{D}\partial_t^2 f(-4 + \partial_t f) \quad (5.23)$$

The solutions of the above equations has the same form $f = Y_1^m = (a_m + tb_m)$ where a_m and b_m are arbitrary constants at zeroth order and first order. These two zero modes have a very simple physical interpretation. The mode multiplying a_m corresponds to an infinitesimal translation and the mode multiplying b_m corresponds to a boost.

5.2 Spectrum of velocity fluctuations

The divergence of the LHS of (5.14) is 0. This ensures that the LHS itself is 0 whixh gives us the equation for vector fluctuations.

The fluctuation field δv may be expanded in vector spherical harmonics

$$\delta v_a = \sum_{l,m} b_{lm} Y_a^{lm} e^{-i\omega_l^v t} \quad (5.24)$$

[4] If $r_0\omega_l^r = 2i - \frac{1}{D}2i$ was an acceptable frequency it would imply that the there is a mode growing exponentially with time which would mean that the Schwarzschild black hole is unstable. But it is a well-known result that Schwarzschild black holes are stable

We pause here to describe vector spherical harmonics in arbitrary dimension in more detail. A vector field on R^{D-1} restricted to a unit sphere S^{D-2} gives us a vector spherical harmonic. The lth vector spherical harmonic is a polynomial valued vector field of degree l which is made up of linear sum of monomials of the form $V_{\mu\mu_1\mu_2\ldots\mu_l}x^{\mu_1}x^{\mu_2}\ldots x^{\mu_l}$ where $V_{\mu\mu_1\mu_2\ldots\mu_l}$ is traceless, symmetric in all of its indices except the first one, and it is zero when it's first index is symmetrized with any of the others. In particular, tracing the first index of V with any of the others gives zero.

Each of the vector valued monomials listed above obeys the equations

$$\nabla.V = 0, \qquad \nabla^2 V = 0 \tag{5.25}$$

where the covariant derivatives are taken in the flat space R^{D-1}. The above vector field in R^{D-1} when restricted to S^{D-2} lies only in the tangent space (note that the r component of these vector field is 0. This vector field dotted with $\hat{r}$ is same is dotting the first index with x^μ which vanishes. Let this vector field be denoted by V. It is easily verified that $\nabla.V = 0$ (where the covariant derivative is now taken on the unit sphere). We demonstrate in Appendix C that

$$\nabla^2 V = -[(D+l-3)l-1]V \tag{5.26}$$

where, in this equation, V is a vector field on S^{D-2} and ∇ is the covariant derivative on the unit sphere.

We plug the expansion of v_a into vector spherical harmonics (5.24) into (5.14). The coefficient of each independent vector spherical harmonic has to be set to 0 which gives us to first subleading order in $\frac{1}{D}$ the frequencies

$$\omega_l^v = -i(l-1) - \frac{1}{D}i(l-1)^2. \tag{5.27}$$

This formula agrees with the formula for the spectrum of vector quasinormal modes presented in obtained in Reinstating factors of r_0 we have

$$r_0\omega_l^v = -i(l-1) \quad (l = 1, 2, 3 \ldots) \tag{5.28}$$

The pure (negative) imaginary part of the velocity quasinormal modes represent that they decay without any oscillation. Vector harmonics with $l = 1$ are zero modes. These modes transform in the representation $(1, 1, 0, 0, \ldots 0)$ - i.e. the adjoint representation - of $SO(D-1)$ and have a simple physical interpretation. These zero modes correspond to on an infinitesimal rotation of the black hole which gives us a version of the Kerr black holes in large D.

Chapter 6

Conclusion and Future Directions

We see that the effective dynamics of the horizon of black holes in large D governed by the light modes can be recast into a problem of dynamics of a membrane propagating in flat space. The dynamics of the membrane in flat space is given by an equation of motion. The dynamics of the membrane can be determined fully from these equations of motions since there are as many equations as there are unknowns. Thus this forms a well posed initial value problem which can be solved independent of the black hole problem in principle. For every solution of the membrane equation of motion there is a corresponding well defined solution to Einstein's equations in large D in a $\frac{1}{D}$ expansion. We have found a class of metrics which solves the Einstein's equations till second order in $\frac{1}{D}$ which are **non-singular** if and only if our **membrane equation is satisfied**.

Quasinormal modes are linearised solutions about black hole back grounds. We have reproduced the quasinormal modes frequencies of Schwarzschild black holes in large D from our membrane equation of motion upto the first subleading order. This serves as a check for our equations of motion. We have also found that our equations of motions admit solutions of rotating membranes. The velocity field u for such membranes are given by the angular velocity. We see that the shape function for such a configuration turns out to be the horizon shape we would expect for a rotating black hole.

Our results have been generalised to the charged case in [11]. We would like to understand our equations of motion as the conservation of some stress tensor on the membrane. Some of my collaborators are working on finding this stress tensor. We would also like to find the gravitational radiation from the fluctuating membrane. We find that our equations of motion to be dissipative since linearised solutions of the membrane equations give decaying frequencies for the quasinormal modes. We would like to calculate entropy production to understand dissipation better.

We would like to see if the problem of black hole collision can be set up in large D. If the process can be analytically solvable then we can make some approximations for 4 dimensions. We could then compare our predictions with LIGO's data to check if the large D is really a good approximation for real processes.

Appendix A

Geometrical quantities in terms of data in auxiliary space

We divide count the quantities as the number of derivatives on zeroth order geomterical quantities n and u.

A.1 Zero derivative quantities

$$\begin{aligned} &(1)n_A \\ &(2)u_A \end{aligned} \tag{A.1}$$

A.2 One derivative quantities

$$\begin{aligned}
(1)(\nabla_A n_D)^{\text{projected traceless}} &= P^{AB}\nabla_B n_C P^{CD} - \frac{1}{D-2}P^{AD}P^{CE}\nabla_C n_E \\
(2)u^A\nabla_A n_B P^{BC} &= (\boldsymbol{K_{ti}})^C + \boldsymbol{K_{ts}}\frac{(dS-n_s n)^C}{1-n_s^2} \\
P^{CA}(\nabla_A u_B)n^B &\Rightarrow -P^{CA}(\nabla_A n_B)u^B \\
(3)u^A\nabla_A n_B u^B &= \boldsymbol{K_{tt}} \\
u^A(\nabla_A u_B)n^B &\Rightarrow -u^A(\nabla_A n_B)u^B \\
(4)(\nabla_A u_D)^{\text{projected traceless}} &= P^{AB}\nabla_B u_C P^{CD} - \frac{1}{D-2}P^{AD}P^{CE}\nabla_C u_E \\
(5)u^A\nabla_A u_B P^{BC} &= (\boldsymbol{\partial_t u_i})^C + \frac{(dS-n_s n)^C}{1-n_s^2}n_s\boldsymbol{K_{tt}} \\
(6)\frac{\mathcal{K}}{D} &= \frac{1}{D}\left(\frac{dn_s}{S} + K^i_i - \boldsymbol{K_{tt}} + \frac{\boldsymbol{K_{ss}}}{1-n_s^2}\right)
\end{aligned}$$

A.3 Two space-time derivatives

Here $P^{AB} = \eta^{AB} - n^A n^B + u^A u^B$, the geometrical projector, $K_{tt} = u.K.u$ at leading order, $K_{abc} = \partial_a K_{bc}$, $a_{pqr} = \partial_q \partial_r u_p$. Let us define ungeometrical projector as $P'^{AB} = P^{AB} + \frac{X^A X^B}{1-n_S^2}$. Then $K_{ij} = P'^A_i P^B_j K_{AB}$. In other words i, j, k denote the isometry directions p in the auxiliary space.

$$
\begin{aligned}
(1)P^{AB}\nabla^2 u_A &= (\partial_\alpha \partial^\alpha u_i)^B + \frac{d}{S}(\partial_s u_i)^B + \frac{X^B}{1-n_s^2}\partial_\alpha\partial^\alpha u_s + \frac{n_s X^B}{1-n_s^2}\Bigg(2\partial^\alpha u^\beta \partial_\alpha n_\beta \\
&+ K^\alpha_{t\alpha}\Bigg) + \frac{d}{S}\frac{X^B}{1-n_s^2} n_s \boldsymbol{K_{ts}} - \frac{d}{S^2} X^B u_s \\
\text{where} & \\
(\partial_\alpha\partial^\alpha u_i)^B &= ({a_{ij}}^j)^B - (a_{itt})^B + \frac{(a_{iss})^B}{1-n_s^2} + \frac{2n_s}{1-n_s^2}(\partial_s n^\beta)(\partial_\beta u_i)^B \\
(2)u_A\nabla^2 u^A &= -\nabla_B u_A \nabla^B u^A \\
(3)n_A\nabla^2 u^A &= u^A\nabla^2 n_A - 2\nabla_A n^B \nabla^A u_B \\
(4)P^{AB}\nabla^2 n_A &= (K^\alpha_{i\alpha})^\mu + \frac{d}{S}(\boldsymbol{K_{si}})^\mu + \frac{X^\mu}{1-n_s^2}k^\alpha_{s\alpha} + \frac{X^\mu n_s}{1-n_s^2}K_{\alpha\mu}K^{\alpha\mu} \\
&+ \frac{d}{S}\frac{X^\mu}{1-n_s^2}\boldsymbol{K_{ss}} - \frac{d}{S^2}X^\mu n_s \\
\text{where} & \\
K^\alpha_{i\alpha} &= K_{ijj} - K_{itt} + \frac{K_{iss}}{1-n_s^2} + \frac{2n_s}{1-n_s^2}K_{i\alpha}K^\alpha_s \\
\text{and} & \\
K_{i\alpha}K^\alpha_s &= \boldsymbol{K_{ij}K_{sj} - K_{ts}K_{ti}} + \frac{\boldsymbol{K_{ss}K_{si}}}{1-n_s^2} \\
(5)u_A\nabla^2 n^A &= K^\alpha_{t\alpha} + \frac{d}{S}\boldsymbol{K_{ts}} - \frac{d}{S^2}n_s u_s \\
(6))n_A\nabla^2 n^A &= -\nabla^B n^A \nabla_B n_A = -\partial_\mu n_\nu \partial^\mu n^\nu - \frac{d}{S^2}n_s^2 \\
(7)u^\alpha u^\beta K_{\alpha\beta\gamma} &= u^A u^B \partial_A \partial_B n_C P^{CD} = (\boldsymbol{b_{tti}})^D + \frac{X^D}{1-n_s^2}\left(\boldsymbol{K_{tts}} + n_s K^\alpha_t K_{t\alpha}\right) \\
(8)u^A u^B \nabla_A \nabla_B u_C P^{CD} &= (\boldsymbol{a_{itt}})^D + \frac{X^D n_s}{1-n_s^2}\left(\boldsymbol{K_{ttt} + K^\alpha_t a_{\alpha t}}\right) \\
(9)n^\gamma \nabla_\alpha \nabla_\beta u_\gamma &= u^\alpha K_{\alpha\beta\gamma} \quad \text{(already calculated)}
\end{aligned}
$$

A.4 Three space-time derivatives

$$
\begin{aligned}
(1)\nabla^2\mathcal{K} &= d^2\left(\frac{K_{ss}}{S^2}-\frac{n_s}{S^3}\right)+d\left(\frac{2K_{\mu\ \ s}^{\mu}}{S}-\frac{2K_{ss}}{S^2}+\frac{2n_s}{S^3}\right)\\
\text{where} K_{s\ \ \beta}^{\beta} &= K_{sii}-K_{stt}+\frac{K_{sss}}{1-n_s^2}+\frac{2n_s}{1-n_s^2}P^{\mu\nu}K_{s\mu}K_{s\nu}\\
&- \frac{2n_s}{1-n_s^2}K_{ts}K_{ts}+\frac{2n_s}{(1-n_s^2)^2}K_{ss}^2\\
(2)n^A\nabla_A\nabla_B\mathcal{K}P^{BC} &= -\frac{d}{S}(K_A^i)^C b_S^A-\frac{dn_s}{S^2}(\boldsymbol{K_{si}})^C-\frac{d}{S}(K_s^A b_{AS})\frac{X^C}{1-n_s^2}\\
&- \frac{dn_s}{S^2(1-n_s^2)}\boldsymbol{K_{ss}}X^C+\frac{2dn_s^2X^C}{S^3}\\
\text{where} &\\
(K_A^i)^C K_S^A &= K_j^i K_{js}-K_t^i\boldsymbol{K_{ts}}+\frac{K_s^i\boldsymbol{K_{ss}}}{1-n_s^2}\\
(3)u^A\nabla_A\nabla_B\mathcal{K}P^{BC} &= \frac{d}{S}(\boldsymbol{K_{tsi}})^C-\frac{n_s d}{S}\frac{X^C}{1-n_s^2}b_t^B K_{sB}-\frac{dX^C}{S(1-n_s^2)}\boldsymbol{K_{tss}}-\frac{d}{S^2}\boldsymbol{K_{ts}}X^C\\
(4)n^D\nabla_C(\nabla^2u_D)P^{BC} &= -\frac{d}{S}\Bigg((K_i^D a_{DS})^B+(K_S^D a_{Di})^B+(\boldsymbol{K_{tsi}})^B+\frac{X^B}{1-n_s^2}K_S^D a_{DS}\\
&+ \frac{X^B}{1-n_s^2}K_S^D a_{DC}X^C+\boldsymbol{b_{tsC}}\frac{X^CX^B}{1-n_s^2}-\frac{X^B\boldsymbol{K_{ts}}}{S}\Bigg)\\
(5)u^D\nabla_C(\nabla^2u_D)P^{BC} &= -\frac{d}{S}(a_D^i K_{SD})^B-\frac{d}{S}\frac{X^B}{1-n_s^2}a_S^D K_{SD}\\
(6)u^D\nabla_D(\nabla^2u_C)P^{BC} &= \frac{d}{S}\left((\boldsymbol{a_{its}})^B+\frac{X^Bn_su^D}{1-n_s^2}\left(\partial_s u^C\partial_D n_C+u^C\partial_S\partial_D n_C+\partial_s n^C\partial_D u_C\right)\right)\\
(7)n^D\nabla_D(\nabla^2u_C)P^{BC} &= -\frac{d}{S}\left((\partial_S n^D\partial_D u^i)^B+n_s(\partial_S u^i)^B+\frac{X^Bn_s}{1-n_s^2}(K_t^D K_{SD})+\frac{X^Bn_s^2}{1-n_s^2}\boldsymbol{K_{ts}}\right)\\
(8)(u.\nabla)(u.\nabla\mathcal{K}) &= \frac{d}{S}\boldsymbol{K_{tst}}+\frac{2d}{S}\left(\boldsymbol{a_{is}}\boldsymbol{K_{is}}+\frac{n_s}{1-n_s^2}\boldsymbol{K_{tt}}\boldsymbol{K_{ss}}\right)\\
(9)\nabla^2(u\cdot K\cdot u) &= \frac{d}{S}\boldsymbol{K_{tst}}+\frac{2d}{S}\left(\boldsymbol{a_{is}}\boldsymbol{K_{it}}+\frac{n_s}{1-n_s^2}\boldsymbol{K_{ts}^2}\right)\\
(10)u\cdot\nabla(\nabla^2\mathcal{K}) &= d^2\left(\frac{\boldsymbol{K_{tss}}}{S^2}-\frac{\boldsymbol{K_{ts}}}{S^3}\right)
\end{aligned}
$$

A.5 Four space-time derivatives

$$(1)\nabla^2\nabla^2\mathcal{K} \quad = \quad \frac{d^3}{S^3}\left(\boldsymbol{K_{sss}} - 3\frac{\boldsymbol{K_{ss}}}{S} + 3\frac{n_s}{S^2}\right) \tag{A.2}$$

Appendix B

Converting Spacetime quantities to Geometric Quantities

The membrane quantities are denoted by overhead bar. Here we list the conversion from the relevant spacetime quantities to membrane quantities. We also neglect subleading terms while calculating corresponding membrane quantities. Note that any free indices inside bar are supposed to be projected on the worldvolume.

$$
\begin{aligned}
\mathcal{K} &= \overline{\mathcal{K}} \\
K_{AB} &= \overline{K_{AB}} \\
\nabla_C u_A &= \overline{\nabla_C u_A} \\
u^C K_{CA} &= \overline{u^C K_{CA}} \\
u^C \nabla_C u_A &= \overline{u^C \nabla_C u_A} \\
\nabla_A \mathcal{K} &= \overline{\nabla_A \mathcal{K}} \\
\nabla^2 u_A &= \overline{\nabla^2 u_A} - \overline{u^B K_{BC} K^C_A} \\
u^C K_{CD} u^D &= \overline{u^C K_{CD} u^D} \\
u^A \nabla_A \mathcal{K} &= \overline{u^A \nabla_A \mathcal{K}} \\
\nabla^2 \mathcal{K} &= \overline{\nabla^2 \mathcal{K}} - \overline{\mathcal{K} K_{AB} K^{AB}} \\
\nabla_C \nabla^2 \mathcal{K} &= \overline{\nabla_C \nabla^2 \mathcal{K}} - \overline{\nabla_C (\mathcal{K} K_{AB} K^{AB})} \\
\nabla^2 \nabla^2 u_A &= \overline{\nabla^2 \nabla^2 u_A} - \overline{u \cdot \nabla \mathcal{K} \nabla_A \mathcal{K}} - \overline{\mathcal{K} \nabla_B \mathcal{K} \nabla^B u_A} - \overline{2\mathcal{K} K_{BC} \nabla^B \nabla^C u_A}
\end{aligned}
\tag{B.1}
$$

Appendix C

Eigenvalues of the Laplacian for Vector Spherical Harmonics

In this Appendix we calculate the eigenvalue of the Laplacian acting on the l^{th} vector spherical harmonic. This spherical harmonic was defined in terms of the restriction of a collection of vector valued monomials to the unit sphere in subsection 5.2.

We evaluate the Laplacian of $V_{\mu\mu_1\mu_2\ldots\mu_l}x^1x^2\ldots x^l$ in spherical polar coordinates in R^{D-1}. The Laplacian of this vector valued monomial vanishes (see subsection 5.2). We use this fact to evaluate the Laplacian of the same vector field restricted to unit sphere.

Consider any divergenceless vector field on R^{D-1} with vanishing radial component, i.e. $V_r = 0$. Using explicit expressions for the Christoffel symbols for flat space in polar coordinates we find

$$\begin{aligned}
\nabla_r V_r &= 0, \\
\nabla_r V_a &= \partial_r V_a - \frac{V_a}{r}, \\
\nabla_a V_r &= \frac{V_a}{r}, \\
\nabla_a V_b &= \hat{\nabla}_a V_b,
\end{aligned} \tag{C.1}$$

where $\hat{\nabla}$ denotes the covariant derivative taken on a unit sphere.
We will now use these results to evaluate $\nabla^2 V$ on R^{D-1} in spherical polar coordinates. The result of this computation depends on the free index in this equation. Let us first consider the case with the free index equal to r. In

this case

$$\begin{aligned}\nabla^2 V_r &= \nabla_r(\nabla_r V_r) + \frac{1}{r^2} g^{ab}\nabla_a\nabla_b V_r,\\ &= \frac{1}{r^2}\hat{\nabla}_a\hat{\nabla}^a V_r - \frac{1}{r^2}\hat{\nabla}_a V^a,\\ &= 0.\end{aligned} \tag{C.2}$$

In other words the vanishing of the r component of $\nabla^2 V$ is just a triviality - it follows as an identity upon assuming $V_r = 0$ and $\nabla.V = 0$.

Let us now turn to the more interesting case of the free index being an angular direction on the unit sphere. In this case

$$\begin{aligned}\nabla^2 V_c =&\nabla_r(\nabla_r V_c) + \frac{1}{r^2} g^{ab}\nabla_a\nabla_b V_c,\\ =&\partial_r\left(\partial_r V_c - \frac{V_c}{r}\right) - \Gamma^a_{rc}\left(\partial_r V_a - \frac{V_a}{r}\right) + \frac{1}{r^2}\hat{\nabla}_a\hat{\nabla}^a V_c\\ &+ \Gamma^a_{ar}\left(\partial_r V_c - \frac{V_c}{r}\right) + \frac{1}{r^2}\Gamma^r_{ac}\frac{A^a}{r},\\ =&\partial_r\left(\partial_r V_c - \frac{V_c}{r}\right) - \frac{1}{r}\left(\partial_r V_c - \frac{V_c}{r}\right) + \frac{1}{r^2}\hat{\nabla}_a\hat{\nabla}^a V_c\\ &+ \frac{D-2}{r}\left(\partial_r V_c - \frac{V_c}{r}\right) - \frac{V_c}{r^2}\quad .\end{aligned} \tag{C.3}$$

Let us now specialize to V_c is the vector field corresponding to the l^{th} vector spherical harmonic. In this case $V_c \propto r^{l+1}$. Using this fact and $\nabla^2 V_c = 0$ we get

$$-\frac{1}{r^2}\hat{\nabla}^2 V_c = (l(l+1) - l - l + (D-2)l - 1)V_c = [(D+l-3)l - 1]V_c. \tag{C.4}$$

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