Studies of Asymptotic Symmetries at Null and Spatial Infinity

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Declaration

I hereby declare that the matter embodied in the report entitled "Studies of Asymptotic Symmetries at Null and Spatial Infinity" are the results of the investigations carried out by me at the Department of Physics, Chennai Mathematical Institute, under the supervision of Dr. Alok Laddha and the same has not been submitted elsewhere for any other degree.

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Certificate

This is to certify that this thesis entitled "Studies of Asymptotic Symmetries at Null and Spatial Infinity" submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by "KHADSE AKSHAY VIJAY" at "CHENNAI MATHEMATICAL INSTITUTE", under the supervision of "DR. ALOK LADDHA" during the academic year 2015-2016.

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Abstract

In this thesis we study the relationship between Asymptotic symmetry group of (Asymptotically flat) space-times at Null and Spatial Infinity. We first review the basic framework concerning Null and Spatial Infinity and then discuss the notion of asymptotic symmetry groups in these two regimes. Former is known as the BMS group and the latter is known as SPI group developed by Ashtekar and Hansen [1]. Based on the formalism developed by Campiglia et al. [2] we then develop an approach to extend the BMS group at Null infinity to Spatial infinity. For so-called super translation subgroup of the BMS, we see how this extension precisely matches the corresponding supertranslation subgroup of the SPI group. As both the BMS and SPI groups are (semi-direct) products of super-translation groups with Lorentz group, our work shows how the SPI group is just a manifestation of BMS at spatial infinity.

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Chapter 1

Introduction

The study of symmetries in a theoretical models has been one of the most important aspect in developing a consistent theory describing nature. This is especially important because symmetries are associated to conserved quantities which are observable in nature, thus allowing to not only validate the correctness of a theory but also build constrains to develop a theoretical model for a physical phenomenon.

Asymptotic structure of solutions to Einsteins equations with zero cosmological constant has been well studies in last 50 years with seminal work of Bondi et al. [4] who defined the space of all asymptotically flat geometries and showed how in a suitably defined system of co-ordinates (which are amenable to the notion of Null infinity) Einsteins equations could be solved recursively off null-infinity as a boundary value problem with Boundary data given by radiative part of gravitational field at Null infinity. Following this work, Sachs characterized the group of asymptotic symmetries which map one solution of Einstein's equations to another solution of Einstein's equation by transforming the radiative data at Null infinity. This group, to the surprise of many was an infinite dimensional group (as opposed to 10 dimensional Poincare group). The group is now known as BMS group and has a key role in mathematical aspects of Classical and Quantum gravity. Ashtekar and Hansen [1] analyzed the structure of asymptotically flat space-times at spatial infinity as opposed to Null infinity as done by Bondi et al earlier. Their motivation was to understand Spatial infinity on the same footing as Null Infinity and compute charges associated for isolated systems in gravity. Spatial infinity of an asymptotically flat space-time unlike that of Null Infinity has an entirely different structure being just a singular point. Ashtekar et al. [1] could characterize symmetry group of Asymptotically flat space-time at spatial infinity (known as the SPI group) and just as in the case of BMS the group it turned out to be infinite dimensional which contained Poincare group as a finite dimensional subgroup. They showed that charges associated to this finite dimensional subgroup are precisely Energy, Momentum and angular momentum. However it was still not clear about what was the relationship between the BMS group and the SPI group. If they were different manifestations of same group or were they really two distinct groups of asymptotic symmetries in gravity. As far as Poincare subgroups are concerned, it has been known for a long time that these subgroups are the same in the sense that the conserved charges associated to them at null or spatial infinity give rise to well known quantities like energy and angular momentum. Situation has never been so clear for the so-called infinite dimensional super-translations however. Building on a recent work of Campiglia et al. [2] who showed how BMS could be extended to an infinite dimensional group at time-like infinity, in this thesis we show that the SPI group and BMS group are isomorphic.

The organization of the thesis is as follows. In the next chapter we give the basic motivations for studying Asymptotics and build the foundation to discuss Asymptotic symmetries. In the next two chapters we briefly discuss Null Infinity and Spatial Infinity. Here we review the standard definitions of Asymptotic flatness of space-times at null and spatial infinities and discuss the associated universal structure which is expected to be common for all asymptotically flat space-times. Then the associated Asymptotic symmetries are reviewed which keeps this universal structure invariant. In the next chapter we develop the formulation based on [2] to arrive at a procedure to obtain the asymptotic symmetry vector fields at spatial infinity starting from the Asymptotic symmetry elements of null infinity, i.e, the BMS vector fields. We will then show that we could successfully use this approach to map the supertranslation vector fields of BMS to the supertranslation vector fields at Spatial infinity as had been previously defined by Ashtekar et al. [1]. In the next chapter we discuss the scope of future work and the problems to be addressed in the current approach.

Chapter 2

Introduction to Asymptotics

Physical theories which describe nature can often become very complicated mathematically. To overcome this we use a smaller and simplified model retaining some key features. In other words instead of considering all the minute details of the entire system we can isolate some part of it and neglect the remaining system which seems to have a small effect on physical quantities. For example while studying the dynamics of planets in our solar system we may neglect to a very good approximation the presence of other stars and galaxies in the universe. But the recognition of these isolated bodies may not always be very obvious and therefore we would need a standard working definition of isolated systems whose conditions will be motivated by observing the properties displayed by physically obvious isolated systems in nature. For example in Newtonian gravity we could call a system to be isolated if [5] (i)the mass density vanishes outside some compact set in the Euclidean 3-sphere and (ii) the Newtonian gravitational potential approaches zero in the limit far from that compact set.

The advantage of isolated systems is not only that it would, to a great extent simplify the mathematics related to within the the system, but also we will be able to attach to this isolated system as a whole some physical quantities like mass, angular momentum, etc.

Similar motivations encourage us to define isolated bodies in the General theory of relativity. But things here are not so obvious as it was in Newtonian gravity or in Electrodynamics considered in Special theory of relativity. The central reason for this is that the background geometry in the General theory of relativity is itself a dynamical quantity. This can be better appreciated if we understand the advantage electromagnetic theory has. In Electromagnetism we could call a system to be isolated if [6] the charge-current density, j^a, vanishes outside a "world tube" of compact spatial support, $F_{\mu\nu} = O(1/r^2)$ as $r \to \infty$ at fixed t, and that $F_{\mu\nu} = O(1/r)$ as $r \to \infty$ along any null geodesics. But we could define such conditions assuming that physical changes are occurring in the background of a fixed Space-time(Minkowski). We lack such fixed background advantage in general theory of relativity with respect to which we could formulate such conditions.

Thus we could classify the fields into two broad categories [5], first the universal or geometric fields and second, the physical fields. The universal fields are those that provide a background or an arena for the physical fields to live in, where as physical fields as those which are responsible for the properties described by the system. For example, in the electrodynamics in special relativity, Minkowski space-time behaves as the universal field and the electromagnetic field as the physical field. In the light of this classification we can now understand that the basic difficulty in general relativity is that the geometric and the physical fields are not independent. The metric behaves as both the geometric and the physical field at the same time.

From physical intuition we would call a system isolated if at very far distances the local properties would tend to behave as though the original body was not present. Using this argument and the observation that Minkowski space-time with Lorentz metric and zero stress energy tensor is a solution to Einstein's equation gives us the basic condition that any isolated body should tend to approach Minkowski space-time in the asymptotic limits. Thus we could call such isolated systems to be asymptotically flat (As Minkowski space-time is referred to as flat space-time).

Time being a coordinate in general relativity, given a asymptotically flat space-time we could travel far in asymptotic limits not just in spatial direction but also in null and time-like directions. Therefore while defining an isolated system to be asymptotically flat we need to distinguish the flatness is in respect to what direction. While trajectories to go "far" away from the system can be easily seen in spatial and null directions, one should be careful in selecting the trajectory for time-like infinity. For example, a person just rotating around a star at some small distance is going far away in "space-time" but he in no way has approached a asymptotic limit to apply conditions of asymptotic flatness.

In the next two chapters we briefly describe the definitions of Asymptotic flatness in Null and Spatial infinity. It should be noted that there are multiple definitions but we only discuss the one which seemed relevant for the thesis.

Chapter 3

Null Infinity

3.1 Introduction

The key idea in proceeding for an approach to define asymptotic flatness in null direction is the use of conformal compactification and by observing the fact that null geodesics remain null in conformally related space-times. Therefore by going to a conformally related unphysical space-time that brings infinity of the physical space-time to a finite distance in the unphysical one, we can now take well defined limits while still remaining on the null geodesics.

The methodology is to attach to the physical space-time manifold M, some additional points which represent infinity in the physical space-time but are at a finite distance in the conformally related unphysical space-time such that certain conditions motivated from simple space-times are satisfied. In other words, asymptotic flatness of a space-time at null infinity can be defined as the possibility of attaching such additional points to a physical space-time.

3.2 Definition [3]

A space-time $(\hat{M}, \hat{g_{ab}})$ will be said to be asymptotically flat at null infinity if there exists a manifold M with boundary $\Im$ equipped with a metric g_{ab} and a diffeomorphism from $\hat{M}$ onto M$\backslash\Im$ (with which we identify $\hat{M}$ and M$\backslash\Im$)such that :

(i) there exists a smooth function Ω on M with $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$; $\Omega = 0$ on $\Im$; and $n_a := \nabla_a \Omega$ is nowhere vanishing on $\Im$;

(ii) $\Im$ is topologically $\mathbb{S}^2 \times \mathbb{R}$;

(iii) $\hat{g}_{ab}$ satisfies Einstein's equations $\hat{R}_{ab} - \frac{1}{2}\hat{R}\hat{g}_{ab} = 8\pi G\hat{T}_{ab}$, where $\Omega^{-2}\hat{T}_{ab}$ has a smooth limit to $\Im$.

The first condition ensures that the boundary $\Im$ is at an infinite distance and that Ω falls off as $\frac{1}{r}$. The second condition makes sure that the topology of $\Im$ is the same as that in the Minkowski space and that null infinity can be reached starting from any angular direction. The fall off condition of $\hat{T}_{ab}$ is motivated by observing the fall of field in Minkowski and Schwarzschild space-time.

The definition implies that :
. i) $\Im$ is null 3-Dimensional manifold
ii) $n^a = \nabla^a\Omega$ is null
iii) The pullback $q_{ab} := g_{\underleftarrow{ab}}$ has signature (0,+,+) where g_{ab} is the metric of the unphysical space-time.

We can now obtain various unphysical space-times by choosing different Ω. Choose two unphysical spacetimes (M, g_{ab}) and (M', g_{ab}) with conformal factors Ω and $\Omega' = \omega\Omega$ where $\omega \neq 0$ on $\Im$ smooth on M. With this conformal rescaling freedom it can be shown that we can choose an ω such that $\nabla_a n^a = 0$ on $\Im$. Such a frame is called divergence free conformal frame. Also it can be shown that $\nabla_a n_b := \nabla_a\nabla_b\Omega \hat{=} 0$. There still remains conformal freedom such that $\pounds_n\omega \hat{=} 0$

The above definition however does not demand for completeness of $\Im$ in $\mathbb{R}$ direction. Therefore for the global structure of $\Im$ for a general space-time to be similar to that of Minkowski space-time we demand that $\Im$ is complete in any divergence free conformal frame. Such asymptotically flat space-times are said to be asymptotically Minkowskian [3].

3.3 Asymptotic Symmetries at Null Infinity

As we have seen in the previous section, null infinity for an asymptotically Minkowski space-time is the manifold with topology $\mathbb{R} \times \mathbb{S}^2$ and is equipped with the pair of fields (q_{ab}, n^a) where q_{ab} is a degenerate metric with signature (0,+,+) and $q_{ab}n^b = 0$.

We can call $(\Im, q_{ab}, n^a)$ as the universal structure, i.e, the structure shared by all the asymptotically Minkowskian space-times at null infinity. The asymptotic symmetry group $\mathfrak{B}$ at null Infinity (also called BMS Group [4] [7]) should therefore be the group of diffeomorphisms that preserve this universal structure.

The BMS group, $\mathfrak{B}$ is therefore, $\mathrm{Diff}_\infty(\mathrm{M})/\mathrm{Diff}^\circ_\infty(\mathrm{M})$ i.e the group of diffeomorphism which do not die down at infinity. Let the infinitesimal asymptotic vector fields ξ^a be the elements of lie algebra $\mathfrak{b}$ of $\mathfrak{B}$ which represents the vector field along which the universal structure is left invariant.

This is true if (i) $\pounds_\xi q_{ab} = 2\alpha q_{ab}$ (ii) $\pounds_\xi n^a = -\alpha n^a$ where α is some function on $\Im$ and satisfies $\pounds_n \alpha = 0$. As opposed to the 10-parameter Poincare group which is the symmetry group for Minkowski space-time the BMS group comes out to be an infinite dimensional group.

The vector field given by $\xi^a = f n^a$ with $\pounds_n f = 0$ satisfies the above conditions and therefore are infinitesimal asymptotic symmetries. The subgroup of such symmetries is called the BMS Supertranslations. The group of BMS Supertranslation is such that : (i) It is infinite dimensional (ii) Abelian normal subgroup of BMS group (iii) The factor obtained by quotienting the BMS group by the supertranslation subgroup is isomorphic to the Lorentz group.

Also it can be shown that there is a unique subgroup of the supertranslation group such that it is a normal subgroup of the BMS group and in the Minkowski space-time, this subgroup corresponds exactly to the translational symmetries of Minkowski spacetime. A similar approach for rotations and boosts fails and therefore at null Infinity for any general asymptotically flat space-time there exists "pure translation" but we do not have any analogous notion for "pure rotation" or "pure boosts".

Concluding we see that the BMS group, $\mathfrak{B}$, is the semi-direct product , $\mathfrak{B} = \mathcal{S} \ltimes \mathcal{L}$ where $\mathcal{S}$= group of supertranslations and $\mathcal{L}$= Lorentz group, and that the only difference between BMS group and the Poincare group is that the 4-D Abelian group of translations is replaced by infinite dimensional Abelian group of Supertranslations.

Chapter 4

Spatial Infinity

4.1 Ashtekar-Hansen Approach- Introduction

Previous approaches(like ADM formalism and [8]) to define spatial infinity had a disadvantage that spatial infinity was seen as a boundary of space-like 3-surfaces and the associated fields were "3-dimensional". This was in contrast to the way null infinity was defined. Null Infinity was seen as the boundary of the space-time manifold as a whole and the associated fields at null infinity were all "4-dimensional". This difference between the two proved to be a major hurdle in trying to bring about a unification between null infinity and spatial infinity.

The approach by Ashtekar-Hansen [1] was therefore aimed at reformulating the structure of spatial infinity without the splitting of space and time and thereby to bring spatial infinity in the same footing as null infinity.

The resulting structure of spatial infinity came out to be a 4-manifold called "Spi" (Spatial Infinity) and was seen to have a principal fiber bundle structure. This structure is very similar to Null Infinity ,$\Im$, in the Penrose formulation [9]. Also the group of asymptotic symmetries at Spatial infinity was seen to be very analogous to the group of asymptotic symmetries at null infinity, i.e, the BMS group.

This new approach - defines a space-time that is asymptotically flat in null direction as asymptotically flat also in the spatial direction if we are able to attach a single point, i°, to its null boundary, $\Im$, such that $\Im$ is the null cone of i°. This was motivated from the observation that in the Penrose diagram of Minkowski space-time, $\Im$ comes as the null cone of i°.

Now appropriate differential conditions are to be applied at i° which are strong enough to allow physically interesting notions to be developed and weak enough to allow some space-time structures. The main difficulty, in providing an easy differentiable structure at i°, is that the physical fields at i° admit direction dependent limits, i.e, moving in different directions from the source, the physical field can be expected to attain different limits, but in any direction the spatial infinity has to meet at the single point i° in the completed structure.

4.2 Definition [1]

A Space-time (M, g_{ab}) will be said to be asymptotically empty and flat at null and spatial infinity(AEFANSI) if :

(i) There exists a manifold $\bar{M}$ with boundary $(\partial\bar{M} =: \Im)$ equipped with a (C^3) conformal structure, and, an embedding of M into $\bar{M}$ which displays (M, g_{ab}) as a weakly asymptotically simple space-time,

(ii) There exists a manifold $\hat{M}$ with a (Lorentz) metric $\hat{g_{ab}}$ and a conformal-structure-preserving imbedding ψ of $\bar{M}$ into $\hat{M}$ (which is C^4 on $\bar{M}$),

(iii) There exists a point i° in $\hat{M}$ with the following properties :

(a) $\hat{M}$ has a $C^{>1}$ differential structure at i°, and $\hat{g}_{ab}$ is $C^{>0}$ at i°,

(b) In $\hat{M}$, $\psi(\Im)$ is a null cone of i°,

(c) The function Ω defined on $\psi(M)$ via $\psi_*(\hat{g_{ab}}) = \Omega^2 g_{ab}$ admits a C^2 extension at i°, with $\Omega \mid_{i^\circ}= 0$, $\hat{\nabla}_a\hat{\Omega} \mid_{i^\circ}= 0$, $(\hat{\nabla}_a\hat{\nabla}\Omega - 2\hat{g}_{ab}) \mid_{i^\circ}= 0$; and finally

(iv) The Ricci tensor R_{ab} of g_{ab} vanishes in the intersection in $\hat{M}$ of the image of the physical space-time with some neighborhood of $\Im \cup i^\circ$.

Condition (i) makes sure that the space-time is asymptotically flat in the null direction and conditions (ii) and (iii) are required for the asymptotic flatness in the space-like directions. Condition (iii)(a) is required for the metric to admit direction dependent limit at i°. Condition (iii)(b) is just the extension of observation from Minkowski space-time. Similarly various other requirements in the definitions have been motivated from observing simple space-times. As is seen that the definition is not formulated using initial data sets and all the aspects of it consider only 4-dimensional space-time fields which was the primary aim of this approach.

4.3 Universal Structure at Spatial Infinity

By definition, Spatial Infinity represented by i° is just a single point. Therefore the study of various physical fields and conserved quantities at spatial infinity becomes very complicated. It would therefore be convenient to introduce an appropriate "blown up" structure at i°, i.e, to take some sub-manifold of the physical space-time and to attach i° to this sub-manifold such that appropriate limits can be taken along it to reach i°.

Because of the differentiability requirement for the completed manifold to be $C^{>1}$ at i°, only first and second order tangent spaces can be constructed. Therefore the blown up structure at i° is obtained using the space-like curves in $(\hat{M}, \hat{g}_{ab})$. 3-manifolds can also be used but space-like curves have the advantage that the direction dependent limits at i° would appear smooth along space-like curves as they would reach i° along a fixed direction while that on the space-like 3-surfaces the limits would still remain direction dependent . Also the space-like trajectories are in spirit with the "4-dimensional" approach which is the central theme while the space-like Cauchy surfaces will split the space-time into space and time. Thus each of these space-like trajectory would represent a distinct way of approaching spatial infinity and together they form the required blown up structure of i°. This is similar to $\Im$ where each null geodesic represents a distinct way of approaching $\Im$. Simply taking all the space-like trajectories will make the blown up structure too large and not useful. Therefore some conditions are to be put on the space-like trajectories as in the null case the condition was to consider only null geodesics that like any null curve. However for space-like curves the concept of geodesics is not conformally invariant. So instead, space-like geodesics were considered in the physical space-time, expressed the condition for geodesic in terms of the unphysical space-time fields and then the limit to i° was taken. Let $(\hat{M}, \hat{g}_{ab})$ be the physical space-time and let η^a be the tangent vector to the geodesic in M. Therefore $\eta^{[a}A^{b]} = 0$ where $A^b = \eta^a \nabla_a \eta^b$ is the acceleration of the curve relative to g_{ab}. Let this curve be parameterized by $p(\lambda)$ in the unphysical space-time $(\hat{M}, \hat{g}_{ab})$. In terms of $\hat{g}_{ab}$ the condition for geodesics in M becomes, $\eta^{[a}\hat{A}^{b]} + \Omega^{-1}\eta^{[a}\hat{\nabla}^{b]}\Omega = 0$, i.e, $\hat{h}_{ab}(\hat{A}^b + \Omega^{-1}\hat{\nabla}^b\Omega) = 0$ where $\hat{A}^b = \eta^a \hat{\nabla}_a \eta^b$ is the acceleration of the curve relative to$\hat{g}_{ab}$ and$\hat{h}_{ab} = \hat{g}_{ab} - (\hat{g}_{pq}\eta^p\eta^q)^{-1}\eta_a\eta_b$ is the projection operator in the 3-surface orthogonal to η^a. Now on taking the limit the condition becomes,

$$lim_{\rightarrow i^\circ} \hat{h}_{ab}(\hat{A}^b + \Omega^{-1}\hat{\nabla}^b\Omega) = 0$$

Here, along with this condition on the space-like curve $p(\lambda)$ as above, two additional constraints are put. First one being, $p(0) = i^\circ$ and the second one is that the tangent vector at i° is unit. The second condition is possible because the metric at i° is conformal invariant, i.e, even if we rescale the conformal factor Ω to $\omega\Omega$ it is required that $\omega = 1$ at i°.

Therefore from above condition it is seen that the component of acceleration that is orthogonal to the tangent vector is completely fixed while that along the tangent vector is completely arbitrary. These three conditions form the regulatory conditions for the space-like curves that would form the blown up structure at i°. Such curves are said to be regular curves.

Definition [1]: A space-like curve $p(\lambda)$ in $(\hat{M}, \hat{g}_{ab})$, passing through i° will be said to be regular if and only if (i) it is $C^{>1}$ at i° and C^3 elsewhere; (ii) it is parameterized so that p(0) is i° and the tangent vector to the curve,η, is unit at i° ; and (iii) η^a satisfies

$$lim_{\rightarrow i^\circ} \hat{h}_{ab}(\hat{A}^b + \Omega^{-1}\hat{\nabla}\Omega) = 0$$

Two regular curves are said to be equivalent if the have the same tangent vector and acceleration at i°. Let S be the collection of these equivalence classes of regular curves. A point on S will therefore be given by $(\eta^a, \hat{g}_{ab}\eta^a\hat{A}^b)$. Here each point of S can be seen as a distinct way of approaching i° just as each point on $\Im$ can be seen as a distinct way of approaching null infinity. S is therefore the blown up structure of i°.

S is seen to have a principal fibre bundle structure. This can be seen because there is natural projection mapping π from S onto the unit time-like hyperboloid κ in the tangent space of i°. The hyperboloid κ is the time-like 3-surface perpendicular to the space-like curves we have chosen. Therefore we can construct fibres over each point on κ such that at that point the fibre and the tangent vector,η^a, coincide. Now each point on this fibre can be labeled by some possible value of acceleration in the equivalence class of regular curves such that they have the same tangent vector, η^a. As the tangential acceleration can have any real value we can parameterize the fibre by the tangential component of the acceleration. Therefore F is homeomorphic to the real line. It can also be shown that under conformal rescaling the coordinization along the fibre does not remain invariant but shifts by some amount, i.e, under a conformal transformation $\tilde{g}_{ab} = \omega^2\hat{g}_{ab}$ for some function ω on $\hat{M}$ which is $C^{>0}$ at i°, C^2 elsewhere and such that $\omega = 1$ at i°, $\tilde{a} = \hat{a} + [\eta^a\hat{\nabla}_a\omega]_{i^\circ}$.

But it is seen that $\hat{a}_1 - \hat{a}_2 = \tilde{a}_1 - \tilde{a}_2$. Therefore there is a mapping from F×F to the reals. Therefore the blowing up of i° is seen to have the structure of a principal fiber bundle where the base space is the unit time-like hyperboloid and the structure group is the additive group of reals. The motion along the fibres on conformal rescaling is seen to correspond to the supertanslations at spatial infinity defined in the next section.

4.4 Asymptotic Symmetries at Spatial Infinity

The "universal structure" of Spi or the structure which is common for all the asymptotically flat space-times satisfying the above definition is the fibre bundle character of S, the tensor field h_{ab} and the vertical vector field V^a. The asymptotic symmetries of Spi are therefore nothing but the group of diffeomorphisms of S that keeps this universal structure invariant. Let η^a denote the generator for this diffeomorphisms then,

(1) $\pounds_{\eta} h_{ab} = 0$

(2) $\pounds_{\eta} v^a = 0$

The collection of such vector fields has the structure of lie algebra denoted by $\pounds_{\mathcal{G}}$. For $\eta^a \epsilon \pounds_{\mathcal{G}}$, let $\bar{\eta}^a$ be the projection of η^a on the base space κ. The condition (1) and (2) are equivalent to

(3) $\pounds_{\bar{\eta}} \bar{h}_{ab} = 0$ on κ

(4) $\pounds_{\eta} v^a = 0$ on S

where $\hat{h}_{ab}$ is the projection of h_{ab} on κ.

Supertranslations : Considering the special case where $\bar{\eta}^a = 0$, i.e the projection of η^a on κ is zero, η^a can be written as $\eta^a = f_{\eta} v^a$ where f_{η} is some scalar function on S. Condition (4) becomes $\pounds_v f_{\eta} = 0$ and holds true if and only if f_{η} is the pull back on S for some $\hat{f}_{\eta}$ on κ. Therefore there exists a 1-1 correspondence from scalar fields on κ to the elements of $\mathcal{L}_{\mathcal{G}}$ satisfying the condition that their projection on κ is zero. Such elements are defined as infinitesimal Spi supertranslations. It can be shown that the subgroup of these supertransations,$\mathcal{L}_{\mathcal{S}}$, is Abelian lie sub-algebra of $\mathcal{L}_{\mathcal{G}}$ and also is an ideal of $\mathcal{L}_{\mathcal{G}}$. $\mathcal{L}_{\mathcal{G}}$ is very similar to the group of supertranslations in BMS with the only difference being that the BMS supertranslations are in 1-1 correspondence with the 2-sphere whereas Spi supertranslations are to that of the functions on 3-manifold κ. Also it is seen that like in BMS the quotient group of Spi supertranslations, i.e, $\mathcal{L}_{\mathcal{S}}$ is the Lorentz lie algebra.

It can be shown that for a sub-group of supertranslations of Spi, denoted by $\mathcal{L}_{\mathcal{T}}$, of the form $\eta^a = f(k)v^a$ where $f(k) = k_a\eta^a$ where k_a is some covector at i° and η^a is the position vector of points on κ, in the tangent space of i°. The subgroup $\mathcal{L}_{\mathcal{T}}$ is a lie ideal of $\mathcal{L}_{\mathcal{G}}$ and that for Minkowski space-time $\mathcal{L}_{\mathcal{G}}$ corresponds exactly to the asymptotic symmetry group of space-time translations in Minkowski space-time. This subgroup $\mathcal{L}_{\mathcal{T}}$ is therefore called Spi translations and this adds to the similarity of the BMS group and $\mathcal{L}_{\mathcal{G}}$, the asymptotic symmetry group at Spatial infinity.

Chapter 5

New approach for Asymptotic symmetries at Spatial infinty

5.1 Introduction

The approach here is based on the previous work by Campiglia et al. [2]. In their paper they showed that we could map the BMS vector fields at Null Infinity to that of the asymptotic symmetry vector fields at time-like infinity using certain green's functions. Here we take a similar approach but we aim to map the BMS vector fields to the asymptotic symmetry vector fields at Spatial Infinity. Here we only do the calculations starting from BMS super-translations. We then compare these symmetry vector fields obtained from BMS super-translations to the already known supertransation vector fields obtained by Ashtekar et al. [1] and we see that both the approaches give the same result for supertranslations at spatial infinity.

The main difference between this approach and Ashtekar-Hansen approach is to use in the asymptotic limit the linearized metric rather than working with the generalized metric. Here we work in the de Donder guage where it is known that the gauge transformations that do not die down at infinity puts the constraint on the asymptotic vector fields given by $\Box\xi^a = 0$.

5.2 Main Equations

We start by considering the region in space-time for which $r > t$. Let the co-ordinate system $(\rho, \tau, \theta, \phi)$ be such that :

$$\rho = \sqrt{r^2 - t^2} \qquad\qquad for \;\; r > t \tag{5.1}$$

$$\tau = \frac{r}{\sqrt{r^2 - t^2}} \tag{5.2}$$

Therefore writing Minkowski Space-time with signature(−+++) in the above co-ordinates we have,

$$ds^2 = d\rho^2 + \rho^2 d\sigma^2 \tag{5.3}$$

where

$$d\sigma^2 = \frac{-d\tau^2}{\tau^2 - 1} + \tau^2 \gamma_{AB} dx^A dx^B := h_{\alpha\beta} dx^\alpha dx^\beta \tag{5.4}$$

Therefore, $$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-\rho^2}{\tau^2-1} & 0 & 0 \\ 0 & 0 & \rho^2\tau^2 & 0 \\ 0 & 0 & 0 & \rho^2\tau^2 sin^2\theta \end{pmatrix}$$

where γ_{AB} is the metric on the unit sphere and $h_{\alpha\beta}$ is the metric on the induced metric on the hyperboliod $\rho = constant$ We always denote capital letters A,B,... for the coordinates on unit sphere and $\alpha, \beta, ..$ for the coordinates on the hyperboloid $\mathcal{H}$.

We can write,

$$t = \rho\sqrt{\tau^2 - 1} \tag{5.5}$$

$$\hat{x} = \frac{\bar{x}}{r} \tag{5.6}$$

$$\hat{x} = \tau\rho\hat{x} \tag{5.7}$$

We can show that, $\rho =$ constant surface is a space-like co-ordinate and τ is a time-like co-ordinate and that ρ is a unit time-like hyperboloid. Here we denote it by $\mathcal{H}$.

Here we note that as τ is always greater that 1, the intrinsic metric h_{ab} has signature (-++) as expected, as $\mathcal{H}$ is a time-like surface.

Now if we move on this unit hyperboloid $\mathcal{H}$, i.e, on $\rho = constant$ surface, to null infinity, i.e, for u=constant and $r \to \infty$, we have,

As

$$u = t - r \tag{5.8}$$

$$\rho = \sqrt{r^2 - t^2} = constant =: c \tag{5.9}$$

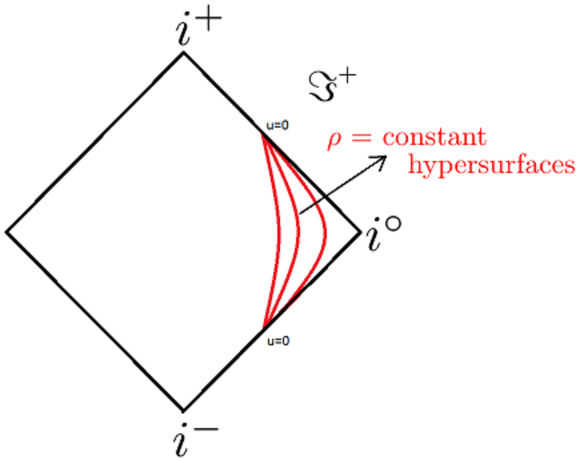
from equations (5.8) and (5.9) we get

$$u = \sqrt{r^2 - c^2} - r = O(r^{-1}) \tag{5.10}$$

Similarly,

$$u = t - \sqrt{c^2 + t^2} = O(t^{-1}) \tag{5.11}$$

In the Penrose diagram we draw the hypersurfaces $\mathcal{H}$ as below :



Let ∇ be the derivative operator corresponding to g_{ab} and let D be the derivative operator corresponding to h_{ab} on the hyperboloid $\mathcal{H}$.Then the non-zero Christoffel symbols Γ^a_{bc} are :

$$\Gamma^\rho_{\alpha\beta} = -\rho h_{\alpha\beta} \qquad \Gamma^\alpha_{\beta\rho} = \rho^{-1}\delta_{\alpha\beta} \qquad \Gamma^\tau_{\tau\tau} = \frac{\tau}{1-\tau^2}$$

$$\Gamma^\tau_{\theta\theta} = \tau(\tau^2 - 1) \qquad \Gamma^\tau_{\phi\phi} = \tau(\tau^2 - 1)sin^2\theta \qquad \Gamma^\theta_{\tau\theta} = \frac{1}{\tau}$$

$$\Gamma^\theta_{\tau\phi} = \frac{1}{\tau} \qquad \Gamma^\theta_{\phi\phi} = -sin\theta cos\theta \qquad \Gamma^\phi_{\theta\phi} = cot\theta$$

Now let ξ^a be the vector field in the space-time. We assume that in the limit $\rho \to \infty$ we could write ξ^a as a series expansion in the inverse power of ρ.

Therefore in the limit $\rho \to \infty$,

$$\xi^\rho(\rho, \tau, \hat{x}) = \mathring{\xi}^\rho(\tau, \hat{x}) + O(\rho^{-1}) \tag{5.12}$$

$$\xi^\alpha(\rho, \tau, \hat{x}) = \mathring{\xi}^\alpha(\tau, \hat{x}) + \rho^{-1}\xi^{(1)\alpha}(\tau, \hat{x}) + O(\rho^{-2}) \tag{5.13}$$

Now the residual guage symmetry in de Donder gauge is given by

$$\Box\xi^a = 0 \tag{5.14}$$

We have ,

$$\Box = \nabla^\mu\nabla_\mu$$

$$\Box = \nabla^2_\rho + \rho^{-2}h^{\alpha\beta}\nabla_\alpha\nabla_\beta \tag{5.15}$$

Solving the de Donder guage condition, i.e, condition (5.14) using the asymptotic vector fields at spatial infinity given by (5.12) and (5.13) gives (Calculations are given in appendix):

$$\Box\xi^{\rho} = (\rho^{-1})(-2D_{\alpha}\mathring{\xi}^{\alpha}) + (\rho^{-2})(\bar{\Box}\mathring{\xi}^{\rho} - 2D_{\alpha}\xi^{(1)\alpha} - 3\mathring{\xi}^{\rho}) + O(\rho^{-3}) \quad (5.16)$$

$$\Box\xi^{\alpha} = (\rho^{-2})(\bar{\Box}\mathring{\xi}^{\alpha} + 2\mathring{\xi}^{\alpha}) + O(\rho^{-3}) \quad (5.17)$$

here $\bar{\Box} = h^{\alpha\beta}D_{\alpha}D_{\beta}$

from (5.14),(5.16),(5.17) we have,

$$D_{\alpha}\mathring{\xi}^{\alpha} = 0 \quad (5.18)$$

$$\bar{\Box}\mathring{\xi}^{\rho} - 2D_{\alpha}\xi^{(1)\alpha} - 3\mathring{\xi}^{\rho} = 0 \quad (5.19)$$

and

$$\bar{\Box}\mathring{\xi}^{\alpha} + 2\mathring{\xi}^{\alpha} = 0 \quad (5.20)$$

Consider the divergence of vector field ξ^{a}:

$$\psi := \nabla_{a}\xi^{a} = D_{\alpha}\mathring{\xi}^{\alpha} + (\rho)(3\mathring{\xi}\rho + D\alpha\xi^{(1)\alpha}) + O(\rho^{(-2)}) \quad (5.21)$$

It is known that the fall off of ψ at null $O(r^{-1})$.

Here we make an assumption that at spatial infinity the fall-off condition of ψ will be $O(\rho^{-2})$.

Now to satisfy the assumed fall-off condition, from (5.18) we already have the first term in (5.21) zero. For the second term to be zero we should have,

$$3\mathring{\xi}^{\rho} + D_{\alpha}\xi^{(1)\alpha} = 0 \quad (5.22)$$

Substituting (5.22) in (5.19) we have:

$$\bar{\Box}\mathring{\xi}^{\rho} = -3\mathring{\xi}^{\rho} \quad (5.23)$$

Therefore the conditions on the vector field components to be satisfied are given by Equations (5.18),(5.20) and (5.23).

5.3 Boundary Conditions

The basic idea of this approach is to map the asymptotic symmetry vector fields at null infinity, i.e, the BMS vector fields, to the bulk space-time which in the limit of spatial infinity be the asymptotic symmetry vector field of the defined Spatial infinity, i.e the hyperboloid $\mathcal{H}$. We have expressed the vector field in the bulk in the $(\rho, \tau, \theta, \phi)$co-ordinates but the BMS vector fields are expressed in the (u, r, θ, ϕ) co-ordinates. Therefore we re-express the vector fields of bulk in the BMS co-ordinates and take the limit to null infinity, i.e u=constant and $r \to \infty$

We have,

$$u = t - r \tag{5.24}$$

Using (5.1) and (5.2),

$$u = \rho(\sqrt{\tau^2 - 1} - \tau) \tag{5.25}$$

$$r = \tau\rho \tag{5.26}$$

Now we see that in the limit going to Null Infinity we have $\rho \to \infty$ $\tau \to \infty$ with the condition that

$$\frac{-\tau}{2\rho} = u + O(r^{-1}) = constant \tag{5.27}$$

Note that as we are in the region $r > t$, so we always have $u < 0$.

BMS vector field is given by,

$$\xi^a(r, u, \hat{x}) = f\partial_u + V^A\partial_A + u\alpha\partial_u - r\alpha\partial_+ ... \tag{5.28}$$

where 2α is the 2-D divergence of V^A and the dots represent higher order terms in the (1/r) expansion.

The sphere components are given by

$$\xi^A = V^A + O(r^{-1}) \tag{5.29}$$

from this and eq.(5.18),i.e, $D_\alpha \mathring{\xi}^\alpha = 0$ we get,

$$\mathring{\xi}^\tau = -\tau\alpha + O(1) \tag{5.30}$$

From Eq.(5.27) we get the radial component as,

$$\xi^r = -r\alpha + O(1) \tag{5.31}$$

Now,

$$\xi^r = (\partial_\rho r)\xi^\rho + (\partial_\tau r)\xi^\tau \tag{5.32}$$

$$\partial_r \rho = \tau; \qquad\qquad \partial_r \tau = \rho \tag{5.33}$$

From above equations,

$$-r\alpha = \tau \xi^\rho + \rho \tau \tag{5.34}$$

$$-r\alpha = \tau \mathring{\xi}^\rho + \frac{\tau}{\rho} \xi^{(1)\rho} + \rho \mathring{\xi}^\tau + \xi^{(1)\tau} \tag{5.35}$$

From Eqs.(5.30) and (5.35),

$$-r\alpha = \tau \mathring{\xi}^\rho + \frac{\tau}{\rho} \xi^{(1)\rho} + \rho(-\tau\alpha) + \xi^{(1)\tau} \tag{5.36}$$

$$\tau \mathring{\xi}^\rho + \xi^{(1)\tau} = O(\rho^{-1}) \tag{5.37}$$

For the null coordinate u we have to first order,

$$\xi^u = f + u\alpha \tag{5.38}$$

Now,

$$\xi^u = (\partial_u \rho) \xi^\rho + (\partial_u \tau) \xi^\tau \tag{5.39}$$

We know,

$$u = \rho(\sqrt{\tau^2 - 1} - \tau) \tag{5.40}$$

Therefore,

$$\partial_u \rho = \sqrt{\tau^2 - 1} - \tau = \frac{-1}{2\tau} + O(\tau^{-3}) \tag{5.41}$$

and

$$\partial_u \tau = \frac{\rho \tau}{\sqrt{\tau^2 - 1}} - \rho = \frac{\rho}{2\tau^2} + O(\tau^{-4}) \tag{5.42}$$

Therefore,

$$\xi^u = (\frac{-1}{2\tau}) \xi^\rho + (\frac{\rho}{2\tau^2}) \xi^\tau \tag{5.43}$$

$$\xi^u = (\frac{-1}{2\tau}) \mathring{\xi}^\rho + (\frac{\rho}{2\tau^2}) \mathring{\xi}^\tau - \frac{\xi^{(1)\rho}}{2\rho\tau} + (\frac{1}{2\tau^2}) \xi^{(1)\tau} + ... \tag{5.44}$$

Using Eq.(5.30),

$$\xi^u = (\frac{-1}{2\tau}) \mathring{\xi}^\rho + (\frac{\rho\alpha}{2\tau}) - \frac{\xi^{(1)\rho}}{2\rho\tau} + (\frac{1}{2\tau^2}) \xi^{(1)\tau} + ... \tag{5.45}$$

Using Eq.(5.27)

$$\xi^u = (\frac{-1}{2\tau}) \mathring{\xi}^\rho + u\alpha - \frac{\xi^{(1)\rho}}{2r} + (\frac{1}{2\tau^2}) \xi^{(1)\tau} \tag{5.46}$$

Now from Eqn.(5.28), we know

$$\lim_{r\to\infty} \xi^u = f + u\alpha \tag{5.47}$$

From Eqs.(5.46) and (5.47)

$$\lim_{\tau\to\infty} \frac{-\mathring{\xi}^\rho}{2\rho} + \frac{1}{2\rho^2}\xi^{(1)\tau} = f \tag{5.48}$$

To satisfy(5.48) and (5.30) simultaneously we should have,

$$\mathring{\xi}^\rho = -\tau f + O(1) \tag{5.49}$$

$$\xi^{(1)\tau} = \tau^2 f + O(\tau) \tag{5.50}$$

Equations (5.29),(5.49)and (5.50) are the required boundary conditions.

5.4 The Supertranslation subgroup

From the above two sections we see that the vector field ξ^a should satisfy the following conditions :

$$\bar{\Box}\xi^\rho = -3\xi^\rho \qquad \lim_{\tau\to\infty} -\tau^{-1}\mathring{\xi}^\rho(\tau,\hat{x}) = f(\hat{x}) \tag{5.51}$$

$$\bar{\Box}\mathring{\xi}^\alpha = -2\mathring{\xi}^\alpha \qquad D_\alpha\mathring{\xi}^\alpha = 0 \qquad \lim_{\tau\to\infty} \mathring{\xi}^A(\tau,\hat{x}) = V^A(\hat{x}) \tag{5.52}$$

Here we recall that $\bar{\Box} = h^{\alpha\beta}\nabla_\alpha\nabla_\beta$ where $h_{\alpha\beta}$ is the metric on the time-like hyperboloid $\mathcal{H}$.

These equations can be solved using Green's functions techniques. Let $\mathcal{G}_{ST}(\tau,\hat{x};\hat{q})$ and $\mathcal{G}^\alpha_A(\tau,\hat{x};\hat{q})$ be the Green's functions associated to the equations (5.51) and (5.52) respectively, where $\hat{q}$ is a unit vector on 2-sphere.

Therefore we have,

$$\mathring{\xi}^\rho(\tau,\hat{x}) = \int_{S^2} d^2q\,\mathcal{G}_{ST}(\tau,\hat{x};\hat{q})f(\hat{q}) \tag{5.53}$$

$$\mathring{\xi}^\alpha(\tau,\hat{x}) = \int_{S^2} d^2q\,\mathcal{G}^\alpha_A(\tau,\hat{x};\hat{q})V^A(\hat{q}) \tag{5.54}$$

The Green's function should satisfy,

$$\bar{\Box}\mathcal{G}_{ST} = -3\mathcal{G}_{ST} \qquad \lim_{\tau\to\infty} -\tau^{-1}\mathcal{G}_{ST} = \delta^{(2)}_{(\hat{x},\hat{q})} \tag{5.55}$$

$$\bar{\Box}\mathcal{G}^{\alpha}_{B} = -2\mathcal{G}^{\alpha}_{B} \qquad D_\alpha \mathcal{G}^{\alpha}_{B} = 0 \qquad \lim_{\tau\to\infty} \mathcal{G}^{A}_{B} = \delta^{A}_{B}\delta^2(\hat{x},\hat{q}) \tag{5.56}$$

Here we calculate only $\mathcal{G}_{ST}(\tau,\hat{x};\hat{q})$, while $\mathcal{G}^{\alpha}_{A}(\tau,\hat{x};\hat{q})$ will be calculated in future. We take the ansatz of the form of $\mathcal{G}(\tau,\hat{x};\hat{q})$ motivated from [10] :

$$\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = \frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\Big(q\cdot\frac{x}{\rho}\Big)^{-n} \tag{5.57}$$

where $q^\mu = (1,\hat{q})$ is a null 4-vector and $x^\mu = (t,r)$ is 4-vector in space-time.

Now in spherical coordinates, $x^\mu = (t,r) = \rho(\sqrt{t^2-1},\tau\hat{x})$ where $\hat{x} = \frac{\bar{x}}{\rho\tau}$
Here $\hat{x}$ is a unit vector on a 2-sphere which is common to both the coordinate systems (t,r,θ,ϕ) and (ρ,τ,θ,ϕ).

Now to check for the first part of equation (5.55),

$$\bar{\Box}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = \bar{\Box}\Big[\frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\big(q\cdot\frac{x}{\rho}\big)^{-n}\Big] \tag{5.58}$$

$$\bar{\Box}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = \frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\bar{\Box}\Big[\big(q\cdot\frac{x}{\rho}\big)^{-n}\Big] \tag{5.59}$$

We know,

$$\begin{aligned}\Box &= \nabla^\mu\nabla_\mu = g^{\mu\nu}\nabla_\mu\nabla_\nu \\ &= \nabla^2_\rho + \rho^{-2}h^{\alpha\beta}\nabla_\alpha\nabla_\beta \\ &= \nabla^2_\rho + \rho^{-2}\bar{\Box}\end{aligned} \tag{5.60}$$

Therefore,

$$\bar{\Box} = \rho^2(\Box - \nabla^2_\rho) \tag{5.61}$$

From equations (5.59) and (5.61) :

$$\bar{\Box}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = \frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\rho^2(\Box - \nabla^2_\rho)\Big(q\cdot\frac{x}{\rho}\Big)^{-n} \tag{5.62}$$

Now, $\Big(q\cdot\frac{x}{\rho}\Big) = (-\sqrt{\tau^2-1} + \tau\hat{q}\cdot\hat{x})$ is a scalar and not a function of ρ and therefore , $\nabla^2_\rho\Big(q\cdot\frac{x}{\rho}\Big) = 0$.

Therefore,

$$\bar{\Box}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = \frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\rho^2\Box\left(q\cdot\frac{x}{\rho}\right)^{-n} \tag{5.63}$$

As shown in the appendix this form of $\mathcal{G}$ satisfies the required Equation (5.55) for n=3.

Therefore,

$$\bar{\Box}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = -3\mathcal{G}(\tau,\hat{x};\hat{q}) \tag{5.64}$$

Now to get the second condition of Eq. (5.55);

We see that

$$\lim_{\tau\to\infty}(q\cdot\frac{x}{\rho}) = \begin{cases} O(\tau^{-4}) & \hat{q}\neq\hat{x} \\ O(\tau^{2}) & \hat{q}=\hat{x} \end{cases} \tag{5.65}$$

Now we calculate the integral of $\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q})$ over the unit sphere $\hat{q}$, i.e,

$$\int d^2\hat{q}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = \int d^2\hat{q}\frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\left(q\cdot\frac{x}{\rho}\right)^{-n} \tag{5.66}$$

We change the co-ordinates to $\tau = i\rho$ and we use the result of integration already given in [cite campiglia paper] to get,

$$\int d^2\hat{q}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = \frac{1}{2^{n-1}\tau}\left((\sqrt{\tau^2-1}-\tau)^{n-1} - (\sqrt{\tau^2-1}+\tau)^{n-1}\right) \tag{5.67}$$

and therefore we see that

$$\lim_{\tau\to\infty}(-\tau^{2-n})\int d^2\hat{q}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = 1 \tag{5.68}$$

as expected. Therefore from (5.55),(5.64),(5.65) and (5.68)we see that

$$\mathcal{G}^{(3)}(\tau,\hat{x};\hat{q}) = \mathcal{G}_{ST}(\tau,\hat{x};\hat{q}) \tag{5.69}$$

and from 5.53

$$\mathring{\xi}^{\rho}(\tau,\hat{x}) = \int_{S^2} d^2\hat{q}\frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\left(q\cdot\frac{x}{\rho}\right)^{-3}f(\hat{q}) \tag{5.70}$$

Thus the asymptotic vector field component $\xi^{\rho}(\tau,\hat{x})$ at spatial infinity upto first order is given by Equation (5.70)

5.5 Comparison with the Supertranslations in Ashtekar-Hansen approach

We first briefly recall here the way in which Ashtekar-Hansen defined supertranslation vector fields at Spatial Infinity.

Spatial Infinity, Spi, was seen as the blown up structure at i°. It is a 4-manifold having a fibre bundle structure with base space as the unit timelike hyperboloid κ and vertical vector field v^a as its fibres. The fibres were parametrized by the tangential component of the acceleration along the regular curves and each fibre corresponded to the tangent vectors of regular curves at i°.

The universal structure was the fibre bundle character of S, the tensor field h_{ab}, and vertical vector field v^a. If ξ^a denoted the generators of diffeomorphism which kept this universal structure constant it was shown that they satisfied $(i)\pounds_{\bar{\xi}}\bar{h}_{ab} = 0$ on κ and $(ii)\pounds_{\xi}v^a = 0$ on $\mathcal{S}$, where $\bar{h}_{ab}$ and $\bar{\xi}^a$ are the projections of h_{ab} and ξ^a on κ.

Supertranslation vector fields, ξ^a, were defined as the one for which $\bar{\xi}^a$ vanishes, i.e, $\xi^a = f_{\xi}v^a$ where f_{ξ} is the pull back to S of some scalar $\bar{f}_{\xi}$ on κ. the elements of $\mathcal{L}_G$ are thus in one-one correspondence between scalar fields on κ.

As we can see from Equation(5.12) and (5.13) the symmetry vector fields at spatial infinity were given by:

$$\xi^{\rho}(\rho,\tau,\hat{x}) = \mathring{\xi}^{\rho}(\tau,\hat{x}) + O(\rho^{-1}) \tag{5.71}$$

$$\xi^{\alpha}(\rho,\tau,\hat{x}) = \mathring{\xi}^{\alpha}(\tau,\hat{x}) + O(\rho^{-1}) \tag{5.72}$$

and from equation (5.70) and (5.54) we know,

$$\mathring{\xi}^{\rho}(\tau,\hat{x}) = \int_{S^2} d^2\hat{q}\frac{(n-1)}{2^{n-1}}\frac{\sqrt{\gamma(\hat{q})}}{2\pi}\Big(q\cdot\frac{x}{\rho}\Big)^{-3} f(\hat{q}) \tag{5.73}$$

$$\mathring{\xi}^{\alpha}(\rho,\hat{x}) = \int_{S^2} d^2q\mathcal{G}^{\alpha}_{A}(\tau,\hat{x};\hat{q})V^A(\hat{q}) \tag{5.74}$$

Now we consider the case where we start with only pure supertranslation vector fields at null infinity, i.e as seen from equation (5.28),

$$\xi^a(r,u,\hat{x}) = f\partial_u \tag{5.75}$$

which we get when we take $V^a = 0$. We then see from the above equations that starting from pure supertranslations at null infinity the only non-zero component of the asymptotic vector fields we get at spatial infinity is $\xi^\rho(\rho, \tau, \hat{x})$ given by equation (5.71) and (5.73).

We also note that from (5.73), that up to first order the $\xi^a(\rho, \tau, \hat{x})$ can be completely specified by the functions on the ρ =constant hyperboloid κ. Thus we can write $\xi^a(\rho, \tau, \hat{x}) = f\eta^a$ where η^a is any tangent vector to the space-like curves orthogonal to the hyperboloid κ and f is some function on κ. But this is exactly the form of supertranslations defined by Ashtekar-Hansen as discussed above.

The other part of the SPI group is just the Lorentz group which is the same as that in the BMS group. Thus we see that we could get all the elements in the symmetry group at spatial infinity (SPI group) stating from elements in the BMS group at Null infinity using the above procedure.

Chapter 6

Discussion and future work

In this thesis we hope to have provided some evidence that there really is one symmetry group in gravity. The symmetry group at spatial infinity (SPI) is isomorphic to the BMS group. However in recent work [11], the BMS group itself was extended to a larger group which unlike the BMS which is a semi-direct product of Super translations with Lorentz group is a semi-direct product of super translations with an infinite dimensional group $Diff(S^2)$ which is the group of diffeomorphisms of the conformal sphere. Hence we would like to see if the SPI group also admits the same extension. This is important due to the fact that unlike super-translations whose corresponding charge at Null infinity is known to contain information about gravitational radiation, the charges associated to $Diff(S^2)$ admit no clear physical interpretation in classical gravity. Perhaps by studying these charges at spatial infinity, one could hope to understand their physical significance better. The calculation of the charges associated to super translations of the SPI group has already been done by compere et al. We would in the future like to extend this computation to all the symmetries of above mentioned extension of BMS.

Chapter 7

Appendix

7.1 Calculations for main Equations

(1) Calculation for ξ^ρ

$$\begin{aligned}
\Box\xi^\rho &= h^{\alpha\beta}\nabla_\alpha\nabla_\beta \\
&= [\nabla_\rho^2 + \rho^{-2}h^{\alpha\beta}\nabla_\alpha\nabla_\beta] \\
&= -\partial_\rho^2\rho^2 + \frac{1}{\rho^2}h^{\alpha\beta}\Big(D_\alpha\nabla_\beta\xi^\rho - \Gamma^\rho_{\alpha\beta}\partial_\rho\xi^\rho + \Gamma^\rho_{\alpha_\mu}\nabla_\beta\xi^\rho\Big) \\
&= \frac{1}{\rho^2}h^{\alpha\beta}\Big(D_\alpha(D_\beta\xi^\rho + \Gamma^\rho_{\beta\mu}\xi^\mu) - \Gamma^\rho_{\alpha\beta}\partial_\rho\xi^\rho + \Gamma^\rho_{\alpha\mu}(D_\beta\xi^\mu + \Gamma^\mu_{\beta\rho}\xi^\rho\Big) \\
&= \frac{h^{\alpha\beta}}{\rho^2}D_\alpha D_\beta + \frac{h^{\alpha\beta}}{\rho^2}(-\rho h_{\beta\mu})\Big(D_\alpha\mathring{\xi}^\mu + \frac{D_\alpha\xi^{(1)\mu}}{\rho}\Big) + \\
&\frac{h^{\alpha\beta}}{\rho^2}(-\rho h_{\alpha\mu})\Big(D_\beta\mathring{\xi}^\mu + \frac{D_\beta\xi^{(1)\mu}}{\rho}\Big) + \frac{h^{\alpha\beta}}{\rho^2}(-\rho h_{\alpha\mu})\Big(\frac{\delta_{\mu\beta}\mathring{\xi}^\rho}{\rho}\Big) \\
&= \frac{\bar{\Box}\mathring{\xi}^\rho}{\tau^2} - \frac{D_\alpha\mathring{\xi}^\alpha}{\rho} - \frac{D_\alpha\xi^{(1)\alpha}}{\rho^2} - \frac{D_\beta\mathring{\xi}^\beta}{\rho^2} - \frac{D_\beta\xi^{(1)\beta}}{\rho} - \frac{3\mathring{\xi}^\rho}{\rho^2} \\
\Box\xi^\rho &= (\rho^{-1})(-2D_\alpha\mathring{\xi}^\alpha) + (\rho^{-2})(\bar{\Box}\mathring{\xi}^\rho - 2D_\alpha\xi^{(1)\alpha} - 3\mathring{\xi}^\rho) + O(\rho^{-3})
\end{aligned} \tag{7.1}$$

(2) Calculation for ξ^γ

$$\begin{aligned}
\Box\xi^\gamma &= h^{\alpha\beta}\nabla_\alpha\nabla_\beta \\
&= [\nabla_\rho^2 + \rho^{-2}h^{\alpha\beta}\nabla_\alpha\nabla_\beta]\xi^\gamma
\end{aligned} \tag{7.2}$$

-

$$\begin{aligned}\nabla_\rho(\nabla_\rho \xi^\gamma) &= \partial(\nabla_\rho \xi^\gamma) + \Gamma^\gamma_{\rho\alpha}(\nabla_\rho \xi^\alpha) \\ &= \partial^2_\rho \xi^\gamma + \partial_\rho(\Gamma^\gamma_{\rho\alpha}\xi^\alpha) + \Gamma^\gamma_{\rho\alpha}\partial_\rho \xi^\alpha + \Gamma^\gamma_{\rho\alpha}\Gamma^\alpha_{\rho\beta}\xi^\beta \\ &= \partial^2_\rho \xi^\gamma - \frac{1}{\rho^2}\xi^\gamma + \frac{1}{\rho}\partial_\rho \xi^\gamma + \frac{1}{\rho}\partial_\rho \xi^\gamma + \frac{1}{\rho^2}\xi^\beta \\ &= O(\rho^{-3})\end{aligned} \tag{7.3}$$

-

$$\begin{aligned}\frac{1}{\rho^2}h^{\alpha\beta}\nabla_\alpha\nabla_\beta\xi^\gamma &= \frac{1}{\rho^2}h^{\alpha\beta}\nabla_\alpha(\nabla_\beta\xi^\gamma) \\ &= \frac{1}{\rho^2}h^{\alpha\beta}\Big(D_\alpha(\nabla_\beta\xi^\gamma) - \Gamma^\rho_{\alpha\beta}\nabla_\rho\xi^\gamma + \Gamma^\gamma_{\alpha\rho}\nabla_\beta\xi^\rho\Big) \\ &= \frac{1}{\rho^2}h^{\alpha\beta}\Big(D_\alpha D_\beta\xi^\gamma + D_\alpha(\Gamma^\gamma_{\beta\rho}\xi^\rho)\Big) \\ &= \frac{1}{\rho^2}h^{\alpha\beta}\Big(D_\alpha D_\beta\xi^\gamma + (D_\alpha\Gamma^\gamma_{\beta\rho})\xi^\rho - \Gamma^\rho_{\alpha\beta}\partial_\rho\xi^\gamma - \Gamma^\rho_{\alpha\beta}\Gamma^\gamma_{\rho\mu}\xi^\mu + \\ &\Gamma^\gamma_{\alpha\rho}\partial_\beta\xi^\rho + \Gamma^\gamma_{\alpha\rho}\Gamma^\rho_{\beta\mu}\xi^\mu\Big) \\ &= \frac{1}{\rho^2}\Big(\bar\Box\xi^\gamma + \frac{1}{\rho}D^\gamma\xi^\rho + 3\rho\partial_\rho\xi^\gamma + 3\xi^\gamma + \frac{1}{\rho}D^\gamma\xi^\rho + \xi^\gamma\Big) \\ &= \frac{1}{\rho^2}\Big(\bar\Box\xi^\gamma + 4\xi^\gamma\Big)\end{aligned} \tag{7.4}$$

Therefore,

$$\Box\xi^\gamma = \frac{1}{\rho^2}\Big(\bar\Box\xi^\gamma + 4\xi^\gamma\Big) + O(\rho^{-3}) \tag{7.5}$$

7.2 Green's function calculations

Here we show that for Green's function of the form :

$$\bar\Box\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = N(\hat{q})\rho^2\Box\Big(q\cdot\frac{x}{\rho}\Big)^{-n} \tag{7.6}$$

satisfies the required Equation (5.51) for n=3, i.e,

$$\bar\Box\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = -3\mathcal{G}(\tau,\hat{x};\hat{q}) \tag{7.7}$$

Consider first,

$$
\begin{aligned}
\Box\left(q\cdot\frac{x}{\rho}\right)^{-n} &= \partial^{\mu}\partial_{\mu}\left(q\cdot\frac{x}{\rho}\right)^{-n} \\
&= \partial^{\mu}\left[(-n)\left(q\cdot\frac{x}{\rho}\right)^{-n-1}\partial_{\mu}\left(q\cdot\frac{x}{p}\right)\right] \\
&= \partial^{\mu}\left[(-n)\left(q\cdot\frac{x}{\rho}\right)^{-n-1}\left\{\frac{q_{\mu}}{\rho}-\frac{1}{\rho^{2}}\frac{1}{\rho}x_{\mu}(q\cdot x)\right\}\right] \\
&= \partial^{\mu}\left[(-n)\left(q\cdot\frac{x}{\rho}\right)^{-n-1}\left\{\frac{q_{\mu}}{\rho}-\frac{1}{\rho^{3}}x_{\mu}(q\cdot x)\right\}\right] \\
&= g^{\mu\nu}(-n)\left[(-n-1)\left(q\cdot\frac{x}{\rho}\right)^{-n-2}\partial_{\nu}\left(q\cdot\frac{x}{\rho}\right)\left\{\frac{q_{\mu}}{\rho}-\frac{(q\cdot x)}{\rho^{3}}x_{\mu}\right\}+\right. \\
&\left.\left(q\cdot\frac{x}{\rho}\right)^{-n-1}\partial_{\nu}\left\{\frac{q_{\mu}}{\rho}-\frac{1}{\rho^{3}}x_{\mu}(q\cdot x)\right\}\right] \\
&= g^{\mu\nu}\left[(-n-1)\left(q\cdot\frac{x}{\rho}\right)^{-n-2}\left(\frac{q_{\nu}}{\rho}-\frac{(q\cdot x)x_{\nu}}{\rho^{3}}\right)\left(\frac{q_{\mu}}{\rho}-\frac{(q\cdot x)}{\rho^{3}}x_{\mu}\right)+\right. \\
&\left.\left(q\cdot\frac{x}{\rho}\right)^{-n-1}\left\{-\frac{q_{\mu}}{\rho^{3}}x_{\nu}+\frac{3}{\rho^{5}}x_{\nu}x_{\mu}(q\cdot x)-\frac{x_{\mu}}{\rho^{3}}q_{\nu}-\frac{(q\cdot x)}{\rho^{3}}\delta_{\mu\nu}\right\}\right] \\
&= -(n)\left[(-n-1)\left(q\cdot\frac{x}{\rho}\right)^{(-n-2)}\left(\frac{0}{\rho^{2}}-\frac{(q\cdot x)^{2}}{\rho^{4}}-\frac{(q\cdot x)^{2}}{\rho^{4}}+\frac{(q\cdot x)}{\rho^{6}}(x\cdot x)\right)\right. \\
&\left.\left(q\cdot\frac{x}{\rho}\right)^{-n-1}\left\{-\frac{(q\cdot x)}{\rho^{3}}+\frac{3}{\rho^{5}}(x\cdot x)(q\cdot x)-\frac{(q\cdot x)}{\rho^{3}}-\frac{(q\cdot x)}{\rho^{3}}(2)\right\}\right] \\
&= (-n)\left(q\cdot\frac{x}{\rho}\right)^{-n}\left[(-n-1)\left(-\frac{1}{\rho^{2}}\right)-\frac{3}{\rho^{2}}\right] \\
&= \left(q\cdot\frac{x}{\rho}\right)^{-n}(n)\left[(n+1)\left(\frac{-1}{\rho^{2}}\right)+\frac{3}{\rho^{2}}\right] \\
&= \frac{(-n)(n-2)}{\rho^{2}}\left(q\cdot\frac{x}{\rho}\right)^{-n}
\end{aligned}
\tag{7.8}
$$

Therefore,

$$
\rho^{2}\Box\left(q\cdot\frac{x}{\rho}\right)^{-n}=(-n)(n-2)\left(q\cdot\frac{x}{\rho}\right)^{-n} \tag{7.9}
$$

We have,

$$\bar{\Box}\,\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = (-n)(n-2)\,\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) \tag{7.10}$$

$$\bar{\Box}\,\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = (-n)(n-2)\,\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) \tag{7.11}$$

We see that the above equation reduces to the required form (5.51) for n=3.

Therefore for n=3 we get,

$$\bar{\Box}\mathcal{G}^{(n)}(\tau,\hat{x};\hat{q}) = -3\mathcal{G}(\tau,\hat{x};\hat{q}) \tag{7.12}$$

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