Market Making in High Frequency Trading via Mathematical Modelling

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Certificate

This is to certify that this dissertation entitled Market Making in High Frequency Trading via Mathematical Modelling towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Srishti Gupta partly in a collaboration with Ms. Garima Agrawal, a senior research fellow, at Indian Institute of Science Education and Research under the supervision of Dr. Anindya Goswami, Associate Professor, Department of Mathematics, during the academic year 2020-2021.

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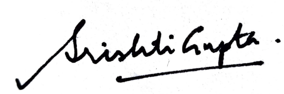
This thesis is dedicated to my parents.

Declaration

I hereby declare that the matter embodied in the report entitled Market Making in High Frequency Trading via Mathematical Modelling are the results of the work carried out by me partly in a collaboration with Ms. Garima Agrawal, a senior research fellow, at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anindya Goswami and the same has not been submitted elsewhere for any other degree.



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Abstract

This project attempts to understand microstructure modelling of tick-by-tick asset price via a semi-Markov model. It has been observed in the literature that such models are capable of reproducing various stylized facts of market microstructure, such as mean reversion and volatility clustering. We perform mathematical analyses of certain functionals of the stock price dynamics. In particular, these functionals are expressed using the conditional expectation of stock price. As an application of the mathematical analyses of the functional, we investigate the market making problem of the agent. Typically an agent optimally submits limit orders at the best ask and best bid prices. It has been shown in the literature that this problem can be solved using a Hamilton-Jacobi-Bellman equation, and a viscosity solution to such HJB equations has been obtained. However, we have obtained a classical solution to a related linear PDE, and this indicates that one can obtain a classical solution to the HJB equation with further investigation.

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Chapter 1

Introduction

"Market microstructure is the study of the process and outcomes of exchanging assets under explicit trading rules" [9]. Microstructure literature aims to analyze the effect of different trading mechanisms on the price formation process. It primarily focuses on studying the structure of the electronic exchanges and trading venues, the price discovery process, transaction costs, and intraday trading behaviour. Due to the advent of faster and more reliable computers, the financial market today is different in fundamental ways. High-frequency trading (HFT) has not only made transactions faster but also revolutionized the market structure. The way liquidity and price discovery arises, the structure of the market, and traders' trading behaviour are different in the HFT scenario. At such fast speeds, studying market microstructure becomes more relevant.

In an electronic market, various types of financial contracts are traded, such as, shares, bonds, mutual funds, etc. Market participants which include the market makers, informed traders and the liquidity takers place orders in the exchange to trade these commodities. An electronic market receives mainly two types of orders: market orders and limit orders. Market orders (MO) are aggressive orders which takes liquidity from the LOB and receives the best prices currently available. These orders are put in by the trader to either buy or sell stocks shares, bonds, or other available assets at the best price obtainable in the current financial market and are executed immediately. On the other hand, limit orders (LO) are passive orders to buy or sell assets at a pre-decided price. They are registered in the Limit Order Book along with the volume put up for trading. Limit orders usually offer prices worse than the prevailing market price, that is, they are placed with higher price than the best buy price for sell limit orders, and at lower price than the best ask price for buy limit orders. The market orders that arrive "walk the LOB" and gets matched with the posted limit orders according to the rule of the book. Two important order matching frameworks are price-time priority and pro-rata.

Market makers is one class of the participants of trading. They are professional traders who profit from their expertise in facilitating exchange in a particular asset. They provide liquidity to the market by quoting buy and sell prices via submitting limit orders on both sides of the LOB. The market maker faces a risk due to a jump in the asset price. In particular he faces the following risks: *(i)Market risk*: Due to sudden jump in the price, the inventory is re-evaluated and his portfolio wealth alters immediately. Thus, he faces a finite amount of risk in no time. *(ii) Adverse Selection risk*: When trading with informed traders who have private or better information than the market maker, he exposes himself to the adverse selection risk. He risks placing a sell limit order that can be fulfilled just before the price jumps upwards, or a buy limit order getting filled just before a drop in the price.

The existing literature on high frequency trading roughly deals with two types of problems. One stream encompasses the description of tick-by-tick asset prices in the Limit Order Book. These models can be broadly categorised into two units, according to their guiding philosophy. First, modelling via the macro-to-microscopic approach, or the latent process approach which describes the observed price process as a noisy representation of the underlying unobserved process. It is usually characterized by a continuous Itô semi-martingale. But, this framework is not suitable for the high frequency data. Second is modelling via the micro-to-macroscopic approach that directly models the observed stock price by a point process. These are not dependent on the existence of the fundamental price and are able to produce various microstructure stylized facts such as volatility clustering and microstructure noise. The other stream of microstructure literature is devoted to solving the high frequency trading problems such as stock liquidation and the market making problem. Typically, stochastic control methods are employed to determine optimal trading strategy. Such studies are mostly based on classical models of asset price, usually, arithmetic or geometric Brownian motion. They model the market order flow via Poisson process and independent of the continuous price process.

In this project, we seek to understand the modelling of tick-by-tick asset price presented in [1]. We, then, perform mathematical analyses of certain functionals of the stock price dynamics via studying an integral equation. As an application of the analyses of the functional, we investigate the market making problem of the agent.

This thesis is structured as follows. Preliminaries including definitions and basic concepts of market microstructure are mentioned in chapter 2. Description of semi-Markov model of financial assets is provided in chapter 3. Chapter 4 includes the study of mathematical properties of the stock price model previously described. Chapter 5 discusses the market making problem of the agent as an application of the mathematical analyses performed in chapter 4. Conclusions and a few remarks are made in chapter 6.

1.1 Original Contribution

In this thesis, we revisit the microstructure modelling of tick-by-tick asset price as proposed in [1]. We add details to the proofs of Lemmas 1 and 2 of [1]. A very short proof of Lemma 1 appears in [1] and Lemma 2 is not accompanied with any proof. Lemma 1 gives the correlation coefficient between two consecutive price jumps and Lemma 2 provides an estimator for the same. The statistical estimates obtained in [1] are explained. These details can be found in Chapter 3.

The mathematical properties of the model described are studied in Chapter 4. It includes the following original work. We study the conditional expectation of stock price at terminal time. We show that the mean stock price satisfies a first order, linear PDE with terminal condition. Then we show that the mean stock price solves this problem in the classical sense. Although this PDE appears in [2], only a viscosity solution is established. Chapter 5 discusses the modelling of market order flow and the market making problem of the agent as presented in [2]. We provide original proof for Lemma 7 which appears as Lemma 4.1 of [2]. We derive the infinitesimal generator of an augmented Markov process which comprises of price process, switching process, age process, wealth process and inventory process. This derivation does not appear in the literature. The content of Chapters 4 and 5 are results of a collaboration with the supervisor and Ms. Garima Agrawal, a senior research fellow of the department.

Chapter 2

Preliminaries

We borrow definitions from [5], [7], [8], [9], [12], [13] and [14].

Definition 2.0.1. For an ordered set T, a filtration of σ-algebras $(\mathcal{F}_t)_{t\in T}$ on an underlying set Ω satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t \ \forall \ s \leq t \ in \ T$.

Definition 2.0.2. A filtered probability space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\in T}$ contained in $\mathcal{F}$.

Definition 2.0.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process is a collection $\{X_t, t \in T\}$ of random variables X_t defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where T is a set, called the index set of the process $\{X_t\}_{t\in T}$.

Definition 2.0.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space and $\{X_t\}_{t\in T}$ be a stochastic process such that X_t is integrable $\forall \ t \in T$. Then X_t is called a $\mathcal{F}_t$-martingale if $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$, for every $s \leq t$, $\mathbb{P}$ a.s.

Definition 2.0.5. Let $(E, \mathcal{E})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random measure on $(E, \mathcal{E})$ is a mapping $\mu : \Omega \times \mathcal{E} \mapsto \mathbb{R}_+$ such that $\omega \mapsto \mu(\omega, A)$ is a random variable for each A in $\mathcal{E}$ and $A \mapsto \mu(\omega, A)$ is a measure on $(E, \mathcal{E})$ for each $\omega \in \Omega$.

Definition 2.0.6. Let $(E, \mathcal{E})$ be a measurable space and let ν be a measure on it. A random measure $\wp$ on $(E, \mathcal{E})$ is said to be Poisson with mean ν if

1. for every $A \in \mathcal{E}$, the random variable $\wp(A)$ has the Poisson distribution with mean $\nu(A)$, and

2. for any $n \geq 2$, whenever $A_1, \cdots, A_n$ are in $\mathcal{E}$ and pairwise disjoint, the random variables $\wp(A_1), \cdots, \wp(A_n)$ are independent.

Definition 2.0.7. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space and $(S, \mathcal{S})$ be a metric space. An S- valued stochastic process $X = \{X_t\}_{t\in T}$ adapted to the filtration is said to satisfy the Markov property with respect to the given filtration if, for each $A \in \mathcal{S}$ and each $s, t \in T$ with $s < t$,

$$\mathbb{P}(X_t \in A|\mathcal{F}_s) \;=\; \mathbb{P}(X_t \in A|X_s).$$

A Markov process is a stochastic process which satisfies Markov property with respect to its natural filtration.

Definition 2.0.8. A C_0- semigroup of operators $\{S(t)\}_{t\geq 0}$ on a Banach space V is a map $S \;:\; \mathbb{R}_+ \to BL(V)$, such that

1. $S_o f \;=\; f$
2. $S_{t+s} \;=\; S_t o S_s \;\forall t, s \geq 0$, and
3. $\|S_t f - f\| \to 0$ as $t \downarrow 0, \forall f \in V$.

Definition 2.0.9. Let $\{S(t)\}_{t\geq 0}$ be a C_0- semigroup of operators. The domain of semigroup is defined as

$$\mathcal{D} \coloneqq \left\{ f \in V | \lim_{t\to 0} \frac{S_t f \;-\; f}{f} exists \right\}$$

and the infinitesimal generator of f is the operator $\mathcal{A}$, defined such that

$$\mathcal{A}f \coloneqq \lim_{t\to 0} \frac{S_t f \;-\; f}{f}$$

$\forall f \in \mathcal{D}$.

Definition 2.0.10. A semi-Markov process is a process $\{X_t\}_{t\geq 0}$ that satisfies the following properties:

1. X_t is a piecewise constant rcll process with discontinuities at a discrete set $\{T_n\}n \geq 1$.
2. The transition probabilities satisfy

$$\begin{aligned} & \mathbb{P}[X_{T_{n+1}} = j, Tn + 1 - T_n \leq y|(X_0, T_0), (X_1, T_1), \cdots, (X_{T_n}, T_n)] \\ = \;& \mathbb{P}[X_{T_{n+1}} = j, Tn + 1 - T_n \leq y|X_{T_n}]. \end{aligned}$$

Definition 2.0.11. The hazard function is the ratio of probability density function and the survival function given by

$$h(x) = \frac{P(x)}{1 - F(x)}$$

where $F(x)$ is the distribution function.

Definition 2.0.12 (Itô's formula)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}_c^\infty$ and let $Z = \{Z_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a rcll process with decomposition

$$Z_t = Z_t^c + Z_t^d;\ 0 \le t < \infty$$

where Z_t^c and Z_t^d are the continuous and discontinuous parts in the decomposition. Then,

$$\begin{aligned} f(Z_t) \quad &= \quad f(Z_0) + \int_0^t f^{'}(Z_{s^-})dZ_s^c + \frac{1}{2}\int_0^t f^{"}(Z_{s^-})d\langle Z^c, Z^c\rangle_s \\ &\quad + \sum_{s\le t}\{f(Z_s) - f(Z_{s^-}) \end{aligned} \tag{2.1}$$

Definition 2.0.13. $\{X_t\}_{t\ge1}$ is $(\mathcal{F}_t)_{t\ge0}$-predictable if for every $t \ge 0, X_{t+1}$ is $\mathcal{F}_t$-measurable.

Definition 2.0.14 (Kernel Density Estimator)**.** Let $x_1, x_2, \ldots, x_n$ be sample points of an independent and identically distributed random variable X with an unknown density f. Let h_n be smoothing parameter called bandwidth. Then, the kernel density estimator is given by

$$f_n(x) \quad = \quad \frac{1}{nh_n}\sum_{i=1}^n K\left(\frac{x - x_i}{h_n}\right) \tag{2.2}$$

where $h_n \to 0$ as $n \to \infty$, and K is the kernel function which satisfies the following properties:

1. $\sup_{-\infty<x<\infty} K(x) \le M$;
2. $|x|K(x) \to 0$ as $|x| \to \infty$;
3. $\int_{-\infty}^{\infty} x^2 K(x)dx < \infty$.

Definition 2.0.15. Market microstructure is the study of the process and outcomes of exchanging assets under explicit trading rules.

Definition 2.0.16. Liquidity is defined as the efficiency with which assets and securities can be converted into ready cash without changing its market price.

Definition 2.0.17. Bid-ask spread is the difference between the highest price a buyer is willing to pay for an asset (best-bid price) and the lowest price a seller is willing to accept (best-ask price).

Definition 2.0.18. Mid price is the mean between the best-bid and best-ask price.

Definition 2.0.19. Tick size is the minimum price change in the bid and ask prices of an asset traded on an exchange platform.

Definition 2.0.20. A Limit order book is the collection of currently available buy and sell orders, their available prices and their available volumes.

Definition 2.0.21. A market order an aggressive order which takes liquidity from the LOB and receives the best prices currently available.

Definition 2.0.22. A limit order is a passive order posted at a fixed price which is executed at the cap or better price, if offered.

It supplies liquidity to the limit order book It receives a guaranteed price, but does not guarantee execution.

Definition 2.0.23. A market maker is a professional trader who profits from facilitating exchange in a particular asset and exploits his skills in executing trades.

Definition 2.0.24 (Adverse Selection)**.** In the market, there are various participants who trade for varying reasons and have different level of information. Adverse selection is a situation where one party possesses information and exploits the asymmetry for his benefit. The market maker faces the adverse selection risk in the following manner. When trading with informed traders, he is exposed to a filled sell limit order just before the price jumps up, or a filled buy limit order just before a downward jump in price.

Definition 2.0.25. Mean-reversion is a stylized fact where the consecutive high-frequency asset price returns are anti-correlated.

Definition 2.0.26. Volatility clustering refers to the fact that high volatility events tend to cluster in time. Large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes.

Chapter 3

Semi-Markov Model for Financial Asset Microstructure

The models of tick-by-tick asset prices are broadly categorised into the following two units, according to their guiding philosophy:

- Macro-to-microscopic approach, or the latent process approach describes the observed price process as a noisy representation of the underlying unobserved process which is usually characterized by a continuous Itô semi-martingale. But, this framework is not suitable for the high frequency data.
- Micro-to-macroscopic approach directly models the observed stock price via point process. These are not dependent on the existence of the fundamental price and are able to produce various microstructure stylized facts such as volatility clustering and microstructure noise.

In this chapter, we attempt to understand the microstructure modelling of tick-by-tick price for liquid assets in a Limit Order Book with a constant bid-ask spread. A model-free description of the piecewise constant mid price is presented. It is characterized by a marked point process $(T_n, J_n)_n$ where, (T_n) are the timestamps representing jump times of the asset price and (J_n) indicate the price increments. It is modelled by a Markov renewal process.

It has been observed in the literature that such modelling can reproduce stylized facts of market microstructure. In particular, mean-reversion of price returns has been reproduced by modelling (J_n) with a suitable Markov chain. The counting process (N_t) associated with the asset prices models volatility clustering.

The results and discussion in the chapter are adapted from [1]. In particular, the results obtained in lemma 1 and lemma 2 are quoted from [1]. However, detailed proofs are missing in [1]. We incorporate them here in this chapter.

3.1 Price return modelling

Model of the tick-by-tick asset price for liquid assets in the limit order book with a constant bid-ask spread is described via a marked point process $(T_n, J_n)_{n\in\mathbb{N}}$. $(T_n)_n$ is an increasing sequence which represents the jump times and $(J_n)_n$ represents the price increments. It takes values in $E = \{-m, ..., -1, 1, ..., m\} \subset \mathbb{Z}\backslash\{0\}$. The continuous-time price process is a piece-wise constant, pure jump process given by

$$P_t = P_0 + \sum_{n=1}^{N_t} J_n, t \geq 0 \tag{3.1}$$

where (N_t) is the counting process associated with the jump times $(T_n)_n$.

The tick size has been normalized to 1, and the asset price P is considered as the mean price between the best-bid and best-ask price. The continuous time dynamics (3.1) is a model-free description of piecewise constant price process in a market microstructure.

The price return is written as

$$J_n = \hat{J}_n \xi_n, n \geq 1 \tag{3.2}$$

where $\hat{J}_n \coloneqq \text{sign}(J_n)$ and $\xi_n \coloneqq |J_n|$. The following assumptions are in effect throughout:

1. Only the current price direction will affect the next price jump.
2. ξ_i are independent and also independent of the direction of the jump.

$\hat{J}_n$ is valued in $\{+1, -1\}$. A realization of 1 indicates an upward price jump. Similarly, a downwards price jump is indicated by -1. $(\hat{J}_n)_n$ is an irreducible Markov chain with probability transition matrix given by

$$\begin{pmatrix} \frac{1+\alpha_+}{2} & \frac{1-\alpha_+}{2} \\ \frac{1-\alpha_-}{2} & \frac{1+\alpha_-}{2} \end{pmatrix}$$

with $\alpha \in [-1, 1)$. To be more precise, the conditional probability $P(\hat{J}_{n+1} = 1 \mid \hat{J}_n = 1) = \frac{1+\alpha_+}{2}$. $(\xi_n)_n$ is an iid sequence valued in $\{1, \ldots, m\}$ with distribution law $p_i = \mathbb{P}[\xi_n = i]$. It is independent of $\hat{J}_n$. Therefore, (J_n) is an irreducible Markov chain with the following transition probability matrix

$$\begin{pmatrix} p_1\hat{Q} & \cdots & p_m\hat{Q} \\ \vdots & \ddots & \vdots \\ p_1\hat{Q} & \cdots & p_m\hat{Q} \end{pmatrix} \tag{3.3}$$

where the states are arranged as $+1, -1, \ldots, +m, -m$. The symmetric case is considered, where $\alpha_- = \alpha_+ \coloneqq \alpha$.

Lemma 1. *In the symmetric case, the invariant distribution of the Markov chain $(\hat{J}_n)_n$ is $\hat{\pi} = (\frac{1}{2}, \frac{1}{2})$, and the invariant distribution of $(J_n)_n$ is $\pi = (p_1\hat{\pi}, \ldots, p_m\hat{\pi})$. Moreover, we have $\alpha = corr_\pi(\hat{J}_n, \hat{J_{n-1}}), \forall\ n \geq 1$, where $corr_\pi$ denotes the correlation under the stationary probability P_π starting from the initial distribution π ([1]).*

Proof. Given probability transition matrix $\hat{Q}$

$$\begin{pmatrix} \frac{1+\alpha_+}{2} & \frac{1-\alpha_+}{2} \\ \frac{1-\alpha_-}{2} & \frac{1+\alpha_-}{2} \end{pmatrix},$$

we take any (x, y). Then,

$$x\left(\frac{1+\alpha}{2}\right) + y\left(\frac{1-\alpha}{2}\right) = x; \tag{3.4}$$

$$x\left(\frac{1-\alpha}{2}\right) + y\left(\frac{1+\alpha}{2}\right) = y. \tag{3.5}$$

Above two equations imply that $x = y$. Also, $x + y = 1$. Therefore, $x = y = \frac{1}{2}$.

For Q given by

$$\begin{pmatrix} p_1\hat{Q} & \cdots & p_m\hat{Q} \\ \vdots & \ddots & \vdots \\ p_1\hat{Q} & \cdots & p_m\hat{Q} \end{pmatrix},$$

we check that $(\frac{p_1}{2}, \frac{p_1}{2}, \frac{p_2}{2}, \frac{p_2}{2}, \cdots, \frac{p_m}{2}, \frac{p_m}{2})$ is a stationary distribution of Q.

$$\begin{aligned} \text{LHS} &= p_1\left(\frac{1+\alpha}{2}\right)\left(\sum_{i=1}^{m}\frac{p_i}{2}\right) + p_1\left(\frac{1-\alpha}{2}\right)\left(\sum_{i=1}^{m}\frac{p_i}{2}\right) \\ &= p_1\left(\sum_{i=1}^{m}\frac{p_i}{2}\right)\times 1 \\ &= \frac{p_1}{2} \end{aligned}$$

since $2\sum_{i=1}^{m}\frac{p_i}{2} = 1$. So, $\hat{\pi} = (\frac{1}{2}, \frac{1}{2})$ satisfies $\hat{\pi}\hat{Q} = \hat{\pi}$ and hence is the stationary distribution of $(\hat{J}_n)_n$. Similarly, $\pi = (p_1\hat{\pi}, p_2\hat{\pi}, \cdots, p_m\hat{\pi})$ is the stationary distribution of $(J_n)_n$.

Now, $\hat{J}_n$ takes values +1 and −1 with probability 0.5 each. Therefore, $E_\pi[\hat{J}_n] = 0\ \forall n$. Also, $\forall n, E_\pi[\hat{J}_n^2] = 1$. This gives the variance as 1. Then, we write the correlation coefficient of $\hat{J}_n$ and $\hat{J}_{n-1}$ as

$$\begin{aligned} corr_\pi(\hat{J}_n, \hat{J}_{n-1}) &= \frac{cov_\pi(\hat{J}_n, \hat{J}_{n-1})}{\sigma_\pi(\hat{J}_n)\sigma_\pi(\hat{J}_{n-1})} \\ &= \frac{E_\pi((\hat{J}_n - E[\hat{J}_n])(\hat{J}_{n-1} - E[\hat{J}_{n-1}]))}{1\times 1} \\ &= E_\pi(\hat{J}_n, \hat{J}_{n-1}). \end{aligned}$$

Now, $\hat{J}_n$ takes value $\hat{J}_{n-1}$ w.p. $(\frac{1+\alpha}{2})$ and $-(\hat{J}_{n-1})$ w.p. $(\frac{1-\alpha}{2})$.

Therefore, $\hat{J}_n\hat{J}_{n-1}$ takes values $(\hat{J}_{n-1})^2$ with probability $(\frac{1+\alpha}{2})$ and $-(\hat{J}_{n-1})^2$ with probability $(\frac{1-\alpha}{2})$. $E_\pi[\hat{J}_n, \hat{J}_{n-1}] = (\frac{1+\alpha}{2})E[(\hat{J}_{n-1})^2] - (\frac{1-\alpha}{2})E[(\hat{J}_{n-1})^2] = \alpha$.

□

Lemma 2. *In the symmetric case, the Markov chain $(\hat{J}_n)_n$ can be written as:*

$$\hat{J}_n = \hat{J}_{n-1}B_n, n \geq 1$$

where (B_n) is a sequence of i.i.d. random variables with Bernoulli distribution on $\{+1,-1\}$, and parameter $(\frac{1+\alpha}{2})$, i.e. of mean $E[B_n]=\alpha$. The price increment Markov chain can also be written in an explicit form as:

$$J_n = \hat{J}_{n-1}\zeta_n$$

where $(\zeta_n)_n$ is a sequence of i.i.d. random variables valued in $E=\{+1,-1,...,+m,-m\}$, and with distribution $\mathbb{P}[\zeta_n = k] = p_k(1+sign(k)\alpha)/2$ ([1]).

Proof. $\hat{J}_n = \hat{J}_{n-1}B_n$. So, $J_n = \hat{J}_{n-1}B_n\xi_n$ or $J_n = \hat{J}_{n-1}\zeta_n$.

Therefore, $\zeta_n = B_n\xi_n$. Since both B_n and ξ_n are i.i.d. sequences therefore, ζ_n is also an i.i.d. sequence and the state space is given by $\{+1,-1,...,+m,-m\}$ as the state space of $\xi_n = \{1,2,...,m\}$. Now, $\mathbb{P}[\zeta_n = k] = \mathbb{P}[\xi_n = k]\cdot\mathbb{P}[sign(k)] = p_k(1+sign(k)\alpha)/2$. □

Lemma 2 can be used to estimate α. The following consistent estimator $\hat{\alpha}^{(n)}$ is considered:

$$\hat{\alpha}^{(n)} = \frac{1}{n}\sum_{k=1}^{n}\frac{\hat{J}_k}{\hat{J_{k-1}}} = \frac{1}{n}\sum_{k=1}^{n}(\hat{J}_k\hat{J}_{k-1}).$$

By strong law of large numbers, $\hat{\alpha}^{(n)} = \frac{\sum_{k=1}^{n}B_k}{n} \to \alpha$.

And,

$$\begin{aligned} E[\hat{\alpha}^{(n)}] &= \frac{1}{n}\sum_{k=1}^{n}E(\hat{J}_k\hat{J}_{k-1}) \\ &= \frac{1}{n}\sum_{k=1}^{n}\alpha \\ &= \alpha. \end{aligned}$$

Also, $Var(\hat{\alpha}^{(n)}) = \frac{1}{n^2}nVar(B_k) = \frac{1}{n}$. Thus, from central limit theorem,

$$\sqrt{n}(\hat{\alpha}^{(n)} - \alpha) \xrightarrow{(d)} N(0,1), \ as\ n \to \infty.$$

From Lemma 1, we can interpret the parameter α as the correlation coefficient between two consecutive price return directions. When $\alpha = 0$, the price returns are independent. $\alpha < 0$ corresponds to mean-reversion of price returns whereas $\alpha > 0$ indicates a trend. The estimated parameter $\tilde{\alpha} < 0$ gives the anticorrelation of the direction of the price returns.

Authors in [1] have found the estimated parameter to be $\hat{\alpha} = -87.5\%$ for Euribor future data. This is consistent with the anticipated anti-correlation of price returns.

3.2 Jump times modelling

The counting process (N_t) is modelled via Markov renewal process. The inter-arrival times are denoted by $S_n = T_n - T_{n-1},\ n \geq 1$. The conditional distribution is given by

$$F_{ij}(s) = \mathbb{P}[S_{n+1} \leq s | J_n = i, J_{n+1} = j]$$

where $(i,j) \in E$. Then, $(T_n, J_n)_n$ is a Markov renewal process with the following transition kernel:

$$\mathbb{P}[J_{n+1} = j, S_{n+1} \leq t | J_n = i] = q_{ij} F_{ij},\ (i,j) \in E.$$

The transition rate function is written as

$$h_{ij}(t) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{P}[t \leq S_{n+1} \leq t + \delta, J_{n+1} = j | S_{n+1} > t, J_n = i] \tag{3.6}$$

for $i, j \in E$. It represents the instantaneous probability that there will be a jump with mark j, given that the current mark is i and no jump took place in the elapsed time t.

$$\begin{aligned}
h_{ij}(t) &= \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{P}[t \leq S_{n+1} \leq t + \delta, J_{n+1} = j | S_{n+1} \geq t, J_n = i] \\
&= \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{P}[t \leq S_{n+1} \leq t + \delta | S_{n+1} \geq t, J_n = i, J_{n+1} = j] \times \mathbb{P}[J_{n+1} = j | J_n = i] \\
&= \lambda_{ij} q_{ij}
\end{aligned}$$

where $\lambda_{ij} \coloneqq \frac{h_{ij}(t)}{q_{ij}}$ is the jump intensity. By assuming that the distribution of the tick times S_n admit a density f_{ij} corresponding to the cdf F_{ij}, h_{ij} can be written as

$$h_{ij} \;=\; q_{ij}\left(\frac{f_{ij}(t)}{1 - H_i(t)}\right)$$

where,

$$H_i(t) \;=\; \mathbb{P}[S_{n+1} \le t | J_n = i] \;=\; \sum_{j \in E} q_{ij} F_{ij}(t).$$

$H_i(t)$ is the conditional distribution of the renewal time in state i.

The symmetric case is considered when F_{ij} depends only on the sign of ij and assume that F_{ij} admits a density f_{ij}. Following notations are used. $F_+(s) = F_{ij}(s)$, if $ij > 0$ as the distribution function of inter-arrival times given two consecutive jumps in the same direction. Similarly, $F_-(s) = F_{ij}(s)$, if $ij < 0$ gives the distribution function in mean-reverting case.

$$F_\pm = \mathbb{P}[S_{n+1} \le s | \hat{J}_n \hat{J}_{n+1} = \pm 1], \;\; n \ge 1, \;\; s \ge 0$$

(S_n) has an iid distribution given by

$$\begin{aligned} F &= \mathbb{E}[\mathbb{P}(S_n \le s | \hat{J}_n, \hat{J}_{n-1}] \\ &= \mathbb{P}(S_n \le s | \hat{J}_n \hat{J}_{n-1} = +1)\mathbb{P}(B_n = +1) + \mathbb{P}(S_n \le s | \hat{J}_n \hat{J}_{n-1} = -1)\mathbb{P}(B_n = -1) \\ &= F_+\left(\frac{1+\alpha}{2}\right) + F_-\left(\frac{1-\alpha}{2}\right) \end{aligned}$$

The transition rate function function in the symmetric case is given by:

$$\lim_{\Delta s \to 0^+} \frac{1}{\Delta s} \mathbb{P}[s \le S_{n+1} \le s + \Delta s, \hat{J}_{n+1} = \pm \hat{J}_n | S_{n+1} > s, \hat{J}_n]$$

$$
\begin{aligned}
&= \lim_{\Delta s \to 0^+} \frac{1}{\Delta s} \mathbb{P}[s \le S_{n+1} \le s + \Delta s, B_{n+1} = \pm 1 | S_{n+1} > s, \hat{J}_n] \\
&= \lim_{\Delta s \to 0^+} \frac{1}{\Delta s} \mathbb{P}[s \le S_{n+1} \le s + \Delta s | B_{n+1} = \pm 1, S_{n+1} > s, \hat{J}_n] \\
&\quad \times \mathbb{P}[B_{n+1} = \pm 1 | S_{n+1} > s, \hat{J}_n] \\
&= \lim_{\Delta s \to 0^+} \frac{1}{\Delta s} \frac{\mathbb{P}[s \le S_{n+1} \le s + \Delta s | B_{n+1} = \pm 1]}{\mathbb{P}[S_{n+1} > s | B_{n+1} = \pm 1]} \times \left(\frac{1 \pm \alpha}{2} \right) \\
&= \lim_{\Delta s \to 0^+} \frac{1}{\Delta s} \frac{F_{\pm}(s + \Delta s) - F_{\pm}(s)}{1 - F_{\pm}(s)} \times \left(\frac{1 \pm \alpha}{2} \right) \\
&= \frac{f_{\pm}(s)}{1 - F_{\pm}(s)} \times \left(\frac{1 \pm \alpha}{2} \right) =: h_{\pm}(s) \text{ (say)}. \qquad (3.7)
\end{aligned}
$$

The renewal distribution functions may be modelled by the Gamma and Weibull distribution.

3.3 Statistical Procedures

The parameters of distribution function F_{ij} of the inter-arrival times and the transition rate function are estimated by both parametric and non parametric methods.

Following notations would be used.

For a subsample of i.i.d. data $\{S_k = T_k - T_{k-1} : k \text{such that} J_{k-1} = i, J_k = j\}$, set

$$
\begin{aligned}
I_{ij} &= \#\{k \text{ such that } J_{k-1} = i, J_k = j\} \\
I_i &= \#\{k \text{ such that } J_{k-1} = i\}
\end{aligned}
$$

with cardinality n_{ij} and n_i respectively.

3.3.1 Parametric Estimation

Gamma and Weibull distributions are considered for the estimation of the distribution function F_{ij} of the renewal times with shape and scale parameters β_{ij} and θ_{ij} respectively. By considering the Maximum Likelihood method of estimation, the MLE $(\hat{\beta}_{ij}, \hat{\theta}_{ij})$ can be obtained as a solution to

$$ln\hat{\beta}_{ij} - \frac{\Gamma'(\hat{\beta}_{ij})}{\Gamma(\hat{\beta}_{ij})} = ln\left(\frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}} S_k\right) - \frac{1}{N}\sum_{k=1}^{n_{ij}} lnS_k$$

and,

$$\hat{\theta}_{ij} = \frac{1}{\hat{\beta}_{ij}}\overline{S}_{n_{ij}}$$

where $\overline{S}_{n_{ij}} := \frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}} S_k$.

However, a closed form solution for $\hat{\beta}_{ij}$ cannot be obtained numerically. Instead, the moment matching method is employed to estimate the parameters. Considering the Gamma distribution, we know the shape and scale parameters satisfy

$$\beta = \frac{(E[S])^2}{Var[S]}, \qquad \frac{1}{\theta} = \frac{(E[S])}{Var[S]}. \tag{3.8}$$

By putting the emperical estimators of the mean and variance, β_{ij} and θ_{ij} are estimated as follows:

$$\overline{\beta_{ij}} = \frac{n_{ij}\overline{S_{n_{ij}}}^2}{\sum_{k=1}^{n_{ij}}(S_k - \overline{S_{n_{ij}}})^2}, \qquad \frac{1}{\theta_{ij}} = \frac{n_{ij}\overline{S_{n_{ij}}}}{\sum_{k=1}^{n_{ij}}(S_k - \overline{S_{n_{ij}}})^2}. \tag{3.9}$$

In [1], it has been observed from the estimates that distribution of inter-arrival times is symmetric, that is, they depend on the product ij rather than the individual values if i and j separately. They perform parametric estimation for the Euribor on the year 2010 from 10h to 14h, with one tick. Figure 3.1 provides the goodness-of-fit results for the estimation. It also gives the histogram of the estimated density function in the symmetric case.

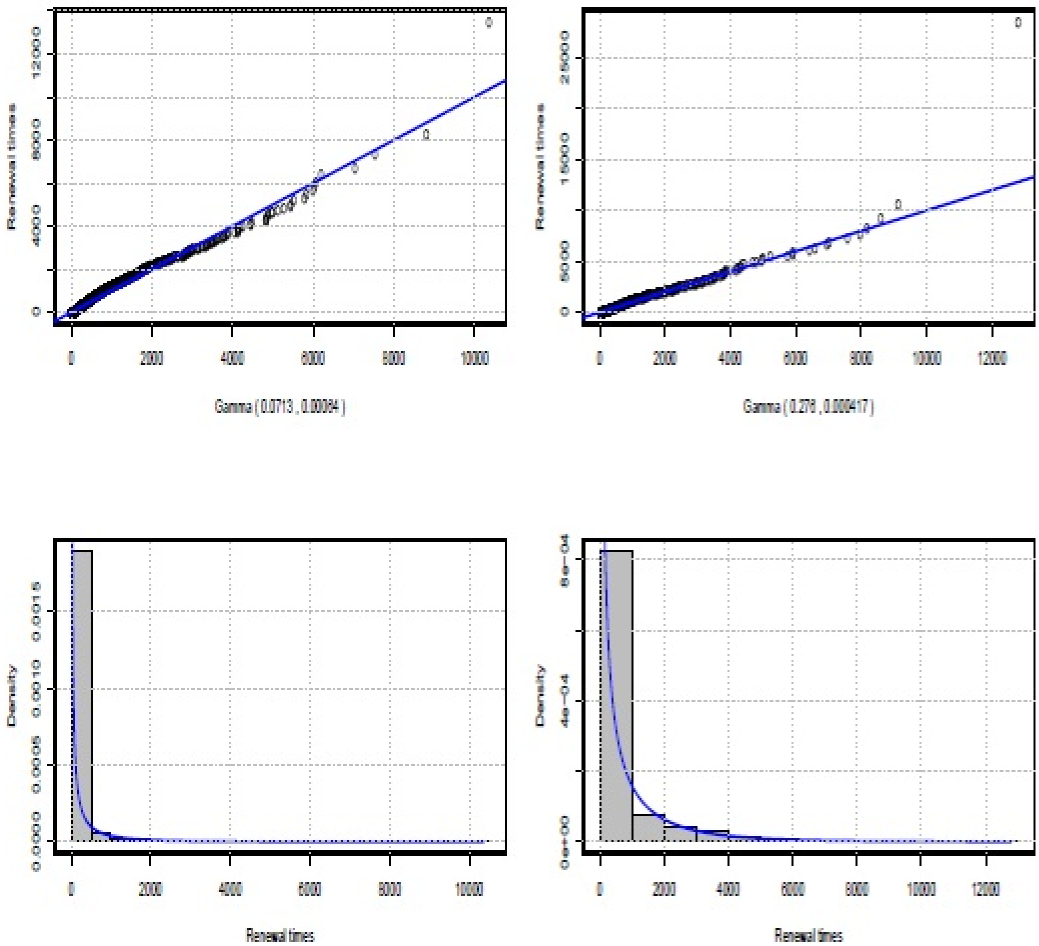

Figure 3.1: QQ plot and histogram of F_{-}(left) and F_{+}(right). Euribor3m, 2010, 10h-14h. Image adapted from: Pietro Fodra and Huyen Pham (2015), Semi Markov Model for Market Microstructure, Applied Mathematical Finance, 22(3):261-295

For gamma distribution $X \sim \Gamma(\beta, \theta)$, the hazard function is given by

$$\hat{h}_{Gam}(t) \;=\; \frac{1}{\theta}\frac{(\frac{t}{\theta}^{\beta-1})e^{-\frac{t}{\theta}}}{\Gamma(\beta) - \Gamma_{\frac{t}{\theta}}(\beta)}.$$

It is decreasing in t if and only if $\beta < 1$, that is, the more the time passes, the less likely is the occurrence of an event. The estimated values of the shape parameter $\tilde{\beta}_{+}$ and $\tilde{\beta}_{-}$ for the trend and mean-reverting case respectively are less than 1 [1]. This is consistent with the phenomenon of volatility clustering which states that large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes. When the price jumps, the probability another price jump in a short period is high, but if the jump does not occur, then the price is likely to stabilize.

3.3.2 Non parametric estimation

The empirical histogram of the density is described as follows. For every collection of time steps $\{0 < t_1 < \cdots < t_M \leq \infty\}$, $\delta_r \coloneqq t_{r+1} - t_r$, we bin the sample $(S_k)_{\{}k = 1, \cdots, n\}$. The empirical histogram is then given by

$$f_{ij}^{hist}(t_r) = \frac{1}{\delta_r} \frac{\#\{k \in I_{ij} | t_r \leq S_k < t_{r+1}\}}{n_{ij}}.$$

This estimator depends on the size of the bins. The density can be estimated by another method namely the smooth kernel method. A Gaussian kernel with density given by normal law of mean 0 and variance b^2 is chosen.

$$\begin{aligned} f_{ij}^{np}(t) &= \frac{1}{n_{ij}} \sum_{k \in I_{ij}} K_b(t - S_k) \\ &= \frac{1}{n_{ij}\delta_r} \sum_{k \in I_{ij}} K\left(\frac{t - S_k}{\delta_r}\right) \qquad \text{since } K_b(x) \coloneqq \frac{1}{b} K\left(\frac{x}{b}\right). \end{aligned}$$

For the symmetric case,

$$f_{\pm}^{hist}(t_r) = \frac{1}{\delta_r} \frac{\#\{k \in I_{\pm} | t_r \leq S_k < t_{r+1}\}}{n_{\pm}}$$

and,

$$f_{\pm}^{np}(t) = \frac{1}{n_{\pm}} \sum_{k \in I_{\pm}} K_b(t - S_k).$$

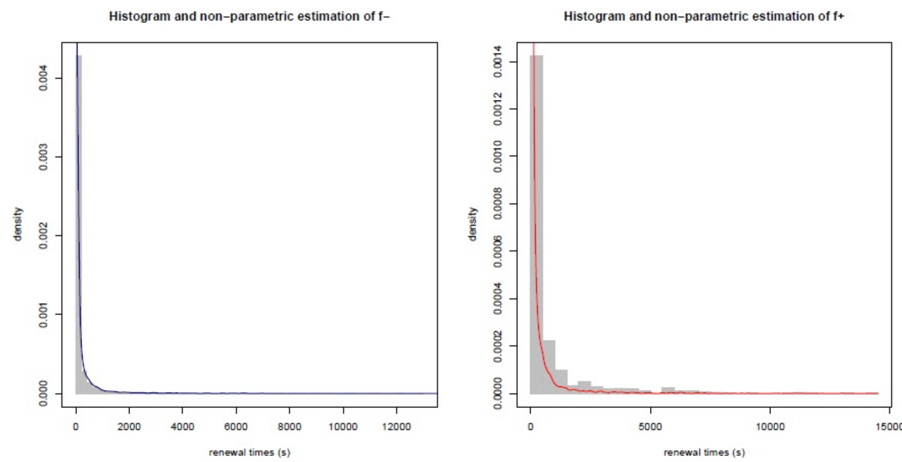

Figure 3.2: Non parametric estimation of the densities f_- and f_+. Image adapted from: Pietro Fodra and Huyen Pham (2015), Semi Markov Model for Market Microstructure, Applied Mathematical Finance, 22(3):261-295

The kernel estimation of $f_\pm(t)$ performed in [1] and the corresponding histogram is given in figure 3.2. From the decreasing nature of the curve of both the densities $f_\pm(t)$, we interpret that the most of the jumps in the stock price takes place in a short duration, even though some renewal times can take values in hours.

Similarly, the transition rate function is estimated.

$$h_{ij} \;=\; \lim_{\delta\to 0}\frac{1}{\delta}\frac{\mathbb{P}[t \le S_k < t+\delta, J_k = j | J_{k-1} = i]}{\mathbb{P}[S_k \ge t | J_{k-1} = i]}.$$

Then, the empirical histogram is given by

$$h_{ij}^{hist}(t_r) = \frac{1}{\delta}\frac{\#\{k \in I_{ij} | t_r \le S_k < t_r + \delta\}}{\#\{k \in I_i | S_k \ge t_r\}}$$

and the associated smooth kernel estimator is given by

$$h_{ij}^{np}(t) = \sum_{k \in I_{ij}} K_b(t - S_k)\frac{1}{\#\{k \in I_i | S_k \ge t\}}.$$

From the estimation results, we see that immediately after a price jump the price is unstable and another jump is likely to occur. If it does not happen, the price stabilizes with time and the probability of another jump is small. Also, due to mean-reversion of the price returns, the intensity of consecutive jumps in the opposite direction is larger than in the same direction.

Chapter 4

Mathematical Properties of the Stock Price Model

In this chapter, we study the properties of the model of the stock price described previously. We are interested in studying the conditional expectation of the stock price at terminal time. We first derive the infinitesimal generator of the augmented process (P_t, I_t, S_t). Then we show that the mean stock price satisfies the PDE (4.8) in the classical sense. We establish some regularities of the functional via studying an integral equation.

For simplicity, we consider a constant bid-ask spread of 2δ. We define the following pure jump process representing the last price jump.

$$I_t = J_{N_t} \tag{4.1}$$

I_t is a semi-Markov process such that (I_t, S_t) is a Markov process, where

$$S_t = t - T_n, \; T_n \le t < T_{n+1}$$

is the age process.

4.1 The infinitesimal generator

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and χ be a finite state space. For each $i, j \in \chi$ and $i \neq j$, we define

$$h_{ij} : (0, \infty) \to [0, \infty) \tag{4.2}$$

to be a measurable function with

$$\sup_{s\in(0,\infty)} \sum_{j\in i} h_{ij}(s) < \infty; \tag{4.3}$$

and

$$\lim_{s\to\infty} H_i(s) = \infty \qquad \text{where, } H_i(s) = \int_0^s \sum_{j\neq i} h_{ij}(v)dv. \tag{4.4}$$

For $i \neq j, s \geq 0$, let $H_{ij}(s)$ be consecutive right-open, left-closed intervals with respect to the lexicographical ordering. The lengths of these intervals are given by $h_{ij}(s)$.

Also,

$$q_{ij}(s) \ = \ p_{ij}(s)F_i(s)$$

We consider the following system of stochastic integral equations:

$$\begin{aligned}
I_t &= I_0 + \int_0^t \int_{\mathbb{R}} \sum_{j\neq I_{t^-}} (j - I_{t^-}) \cdot 1_{H_{I_{t^-}j}(S_{t^-})}(z)\wp(dt, dz) \\
P_t &= P_0 + \int_0^t \int_{\mathbb{R}} \sum_{j\neq I_{t^-}} 2\delta I_{t^-}j \cdot 1_{H_{I_{t^-}j}(S_{t^-})}(z)\wp(dt, dz)
\end{aligned}$$

$$S_t = S_0 + t - \int_0^t \int_{\mathbb{R}} S_{t^-} \sum_{j \neq I_{t^-}} 1_{H_{I_{t^-}j}(S_{t^-})}(z) \wp(dt, dz) \tag{4.5}$$

where $\wp(dt, dz)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dtdz$.

We will derive the expression for the infinitesimal generator of the augmented age-dependent process given by (4.5). Let $\varphi : E \times \chi \times \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function. Then, by Itô's formula,

$$\begin{aligned}
d\varphi(P_t, I_t, S_t) &= \frac{\partial \varphi(P_{t^-}, I_{t^-}, S_{t^-})}{\partial s} dS_t^c + \Big\{ \varphi(P_t, I_t, S_t) - \varphi(P_{t^-}, I_{t^-}, S_{t^-}) \Big\} \\
&= \frac{\partial \varphi}{\partial s} dt + \Bigg\{ \varphi\Bigg[\Bigg(P_{t^-} + \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} 2\delta I_{t^-} j \cdot 1_{H_{I_{t^-}j}(S_{t^-})}(z) \wp(dt, dz), \\
&\quad I_{t^-}) + \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} (j - I_{t^-}) \cdot 1_{H_{I_{t^-}j}(S_{t^-})}(z) \wp(dt, dz) \Bigg), \\
&\quad \Bigg(S_{t^-} - \int_{\mathbb{R}} S_{t^-} \sum_{j \neq I_{t^-}} 1_{H_{I_{t^-}j}(S_{t^-})}(z) \wp(dt, dz) \Bigg) \Bigg] - \varphi(P_{t^-}, I_{t^-}, S_{t^-}) \Bigg\} \\
&= \frac{\partial \varphi}{\partial s} dt + \int_{\mathbb{R}} \Bigg\{ \varphi\Bigg[\Bigg(P_{t^-} + \sum_{j \neq I_{t^-}} 2\delta I_{t^-} j \cdot 1_{H_{I_{t^-}j}(S_{t^-})}(z) \wp(dt, dz), \\
&\quad I_{t^-}) + \sum_{j \neq I_{t^-}} (j - I_{t^-}) \cdot 1_{H_{I_{t^-}j}(S_{t^-})}(z) \wp(dt, dz) \Bigg), \\
&\quad \Bigg(S_{t^-} - S_{t^-} \sum_{j \neq I_{t^-}} 1_{H_{I_{t^-}j}(S_{t^-})}(z) \wp(dt, dz) \Bigg) \Bigg] - \varphi(P_{t^-}, I_{t^-}, S_{t^-}) \Bigg\} dtdz + dM_t \\
&= \frac{\partial \varphi}{\partial s} dt + \sum_{j \neq i} \Bigg[\varphi(P_{t^-} + 2\delta I_{t^-} j, j, 0) - \varphi(P_{t^-}, I_{t^-}, S_{t^-}) \Bigg] h_{ij}(s) dt + dM_t.
\end{aligned}$$

Therefore,

$$\mathcal{L}\varphi(p, i, s) = \frac{\partial \varphi}{\partial s} + \sum_{j \neq i} \Bigg[\varphi(p + 2\delta i j, j, 0) - \varphi(p, i, s) \Bigg] h_{ij}(s). \tag{4.6}$$

4.2 The stock price conditional mean

We define the mean value of the stock price at horizon by

$$\pi(t,p,i,s) := \mathbb{E}[P_T | t, P_t = p, I_t = i, S_t = s] \tag{4.7}$$

where $(t,p,i,s) \in [0,T] \times 2\delta\mathbb{Z} \times \chi \times \mathbb{R}_+$.

Later in the section we will establish that π solves the following PDE:

$$\begin{aligned} \frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial s} + \frac{f_i(s)}{1-F_i(s)} \sum_{j \neq i} p_{ij}(s) \Big(\pi(t, p+\delta j, j, 0) - \pi(t,p,i,s) \Big) &= 0 \\ \text{with } \pi(T,p,i,s) &= p. \end{aligned} \tag{4.8}$$

The authors in [2] consider $\chi = \{+1, -1\}$, that is, the price jumps by only one tick, either upwards or downwards. In the symmetric case, they show that the conditional mean price solves the following PDE in the viscosity sense:

$$\begin{aligned} \frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial s} + \sum_{\nu \in \chi} h_\nu(s) [\pi(t, p + 2\delta\nu i, \nu i, 0) - \pi(t,p,i,s)] &= 0 \\ \pi(T,p,i,s) &= p \end{aligned} \tag{4.9}$$

where $\nu \in \{-1, +1\}$ and h_ν is given by equation (3.7). However, we provide an original proof that π satisfies the above PDE classically. Further in the chapter, we study the functional π and establish properties such as differentiability. In order to show that π solves the system (4.8) classically, we take a two step approach. First, we consider an integral equation, establish its existence and uniqueness, and study some regularity features. Then, we go on to show that the PDE (4.8) and the considered integral equation are equivalent hence establishing the existence and uniqueness of the solution to the system (4.8).

We make the following assumptions which will be in effect throughout:

1. F_i is differentiable. Let the derivative, say, f_i be bounded;

2. f_i is differentiable and f_i' is bounded and continuous;

3. $p_{ij}(s)$ is continuously differentiable.

Let $\mathcal{D} := \{\ (t,p,i,s) : t \in [0,T],\ p \in \{-m,\cdots,m\},\ i \in \chi,\ s \in [0,t]\ \}$.

Lemma 3. *Consider the following integral equation*

$$\begin{aligned}\varphi(t,p,i,s) &= p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) + \int_0^{T-t} \frac{f_i(s+v)}{1-F_i(s)} \times \\ &\quad \sum_{j\neq i} p_{ij}(s+v)\varphi(t+v,p+2\delta j,j,0)dv \qquad (4.10)\\ \textit{with}\ \ \varphi(T,p,i,s) &= p. \qquad (4.11)\end{aligned}$$

$\forall (t,p,i,s) \in \mathcal{D}$. *Then (i) the problem* (4.10)-(4.11) *has unique solution in B, and (ii) the solution of the integral equation is in $C^{2,2}(\mathcal{D})$.*

Proof. (i) We first note that a solution of (4.10)-(4.11) is a fixed point of the operator $\mathcal{A}$ and vice versa, where

$$\begin{aligned}\mathcal{A}\varphi(t,p,i,s) &:= p\frac{1-F_i(T-t+s)}{1-F_i(s)} + \int_0^{T-t} \frac{f_i(s+v)}{1-F_i(s)} \sum_{j\neq i} p_{ij}(s+v) \\ &\quad \varphi(t+v,p+2\delta j,j,0)dv,\end{aligned}$$

Let $\mathcal{B}$ be the space of continuous functions on $\mathcal{D}$. Then, $\mathcal{B}$ is a Banach space. In order to show existence and uniqueness in the prescribed class, it is sufficient to show that A is a contraction in $\mathcal{B}$. As $A : B \to B$ is also a contraction, Banach fixed point theorem ensures existence and uniqueness of the fixed point in $\mathcal{B}$. To this end, we need to show, for $\varphi_1, \varphi_2 \in \mathcal{B}$, $\|A\varphi_1 - A\varphi_2\| \leq k\|\varphi_1 - \varphi_2\|$ where $k < 1$.

$$\begin{aligned}\|\mathcal{A}\varphi_1 - \mathcal{A}\varphi_2\| &= \left\| \int_0^{T-t} \frac{f_i(s+v)}{1-F_i(s)} \sum_{j\neq i} p_{ij}(s+v)\varphi_1(t+v,p+2\delta j,j,0)dv \right. \\ &\quad \left. - \int_0^{T-t} \frac{f_i(s+v)}{1-F_i(s)} \sum_{j\neq i} p_{ij}(s+v)\varphi_2(t+v,p+2\delta j,j,0)dv \right\| \\ &= \left\| \int_0^{T-t} \frac{f_i(s+v)}{1-F_i(s)} \sum_{j\neq i} p_{ij}(s+v)(\varphi_1-\varphi_2)(t+v,p+2\delta j,j,0)dv \right\|\end{aligned}$$

$$
\begin{aligned}
&\leq \; \|\varphi_1 - \varphi_2\| \left\| \int_0^{T-t} \frac{f_i(s+v)}{1-F_i(s)} \sum_{j \neq i} p_{ij}(s+v) dv \right\| \\
&= \; \|\varphi_1 - \varphi_2\| \left\| \frac{1}{1-F_i(s)} \int_0^{T-t} f_i(v+s) dv \right\| \\
&= \; \|\varphi_1 - \varphi_2\| \left\| \frac{F_i(T-t+s) - F_i(s)}{1-F_i(s)} \right\| \\
&= \; \|\varphi_1 - \varphi_2\| \; \sup_{\mathcal{D}} \left| \frac{F_i(T-t+s) - F_i(s)}{1-F_i(s)} \right|
\end{aligned}
$$

Thus, $\|\mathcal{A}\varphi_1 - \mathcal{A}\varphi_2\| \leq k\|\varphi_1 - \varphi_2\|$ where,

$$
\begin{aligned}
k &= \sup_{\mathcal{D}} \left| \frac{F_i(T-t+s) - F_i(s)}{1-F_i(s)} \right| \\
&< \frac{1-F_i(s)}{1-F_i(s)} = 1
\end{aligned}
$$

using (A1).

(ii) Now, we establish the required regularity. We deal with the two terms of (4.10) separately. Using (A1), first term is in $\mathcal{C}^{2,2}(\mathcal{D})$.

$$
\frac{\partial}{\partial t}\left[p \left(\frac{1-F_i(T-t+s)}{1-F_i(s)} \right) \right] = p \left(\frac{f_i(T-t+s)}{1-F_i(s)} \right)
$$

and,

$$
\begin{aligned}
\frac{\partial}{\partial s}\left[p \left(\frac{1-F_i(T-t+s)}{1-F_i(s)} \right) \right] &= p \left\{ \frac{-f_i(T-t+s)(1-F_i(s)) + f_i(s)(1-F_i(T-t+s))}{(1-F_i(s))^2} \right\} \\
&= -p \left(\frac{f_i(T-t+s)}{1-F_i(s)} \right) + \frac{f_i(s)}{1-F_i(s)} \; p \left(\frac{(1-F_i(T-t+s))}{1-F_i(s)} \right)
\end{aligned}
$$

The second term is

$$
\mathcal{T}\varphi \coloneqq \int_0^{T-t} \frac{f_i(s+v)}{1-F_i(s)} \sum_{j \neq i} p_{ij}(s+v) \; \varphi(t+v, p+2\delta j, j, 0) \; dv.
$$

$$
\begin{aligned}
&\frac{1}{h}\Big(\mathcal{T}\varphi(t+h,p,i,s)-\mathcal{T}\varphi(t,p,i,s)\Big)\\
=\ &\frac{1}{h}\Bigg\{\int_0^{T-t-h}\frac{f_i(v+s)}{1-F_i(s)}\Big[\sum_{j\neq i}p_{ij}(s+v)\ \varphi(t+v+h,p+2\delta j,j,0)\Big]\,dv-\mathcal{T}\varphi(t,p,i,s)\Bigg\}\\
=\ &\frac{1}{h}\Bigg\{\int_0^{T-t}\frac{f_i(v-h+s)}{1-F_i(s)}\Big[\sum_{j\neq i}p_{ij}(s+v-h)\ \varphi(t+v,p+2\delta j,j,0)\Big]\,dv\\
&-\int_0^{h}\frac{f_i(v-h+s)}{1-F_i(s)}\Big[\sum_{j\neq i}p_{ij}(s+v-h)\ \varphi(t+v,p+2\delta j,j,0)\Big]\,dv\\
&-\int_0^{T-t}\frac{f_i(v+s)}{1-F_i(s)}\Big[\sum_{j\neq i}p_{ij}(s+v)\ \varphi(t+v,p+2\delta j,j,0)\Big]\,dv\Bigg\}\\
=\ &\frac{1}{h}\Bigg\{\int_0^{T-t}\Bigg[\frac{f_i(v-h+s)}{1-F_i(s)}\Big(\sum_{j\neq i}p_{ij}(s+v-h)\ \varphi(t+v,p+2\delta j,j,0)\Big)\\
&\qquad-\frac{f_i(v+s)}{1-F_i(s)}\Big(\sum_{j\neq i}p_{ij}(s+v)\ \varphi(t+v,p+2\delta j,j,0)\Big)\Bigg]\,dv\\
&-\int_0^{h}\frac{f_i(v-h+s)}{1-F_i(s)}\Big[\sum_{j\neq i}p_{ij}(s+v-h)\ \varphi(t+v,p+2\delta j,j,0)\Big]\,dv\Bigg\}\\
=\ &\frac{1}{1-F_i(s)}\sum_{j\neq i}\int_0^{T-t}\Bigg[\frac{p_{ij}(v-h+s)f_i(v-h+s)-p_{ij}(v+s)f_i(v+s)}{h}\Bigg]\varphi(t+v,p+2\delta j,j,0)dv\\
&-\frac{1}{1-F_i(s)}\sum_{j\neq i}\frac{1}{h}\int_0^{h}f_i(v-h+s)\ p_{ij}(v-h+s)\ \varphi(t+v,p+2\delta j,j,0)\ dv
\end{aligned}
$$

As $h\to 0$, the integral of the first term of right hand side goes to $-\int_0^{T-t}(p_{ij}f_i)'(v+s)\ \varphi(t+v,p+2\delta j,j,0)$ and the one in the second term goes to $f_i(s)p_{ij}(s)\ \varphi(t,p+2\delta j,j,0)$.

$$
\begin{aligned}
\partial\mathcal{T}\varphi(t,p,i,s)\ =\ &-\frac{1}{1-F_i(s)}\sum_{j\neq i}\Bigg[\int_0^{T-t}(p_{ij}f_i)'(v+s)\ \varphi(t+v,p+2\delta j,j,0)\ dv\\
&-f_i(s)p_{ij}(s)\ \varphi(t,p+2\delta j,j,0)\ dv\Bigg]
\end{aligned}\tag{4.12}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{h}\Bigg(\mathcal{T}\varphi(t,p,i,s+h) \;-\; \mathcal{T}\varphi(t,p,i,s)\Bigg)\\
=\;& \frac{1}{h}\Bigg\{\int_0^{T-t}\frac{f_i(v+s+h)}{1-F_i(s+h)}\sum_{j\neq i}p_{ij}(v+s+h)\;\varphi(t+v,p+2\delta j,j,0)\;dv\\
&-\int_0^{T-t}\frac{f_i(v+s)}{1-F_i(s)}\sum_{j\neq i}p_{ij}(v+s)\;\varphi(t+v,p+2\delta j,j,0)\;dv\Bigg\}\\
=\;& \sum_{j\neq i}\Bigg\{\int_0^{T-t}\frac{1}{h}\Bigg(\frac{f_i(v+s+h)\;p_{ij}(v+s+h)}{1-F_i(s+h)} \;-\; \frac{f_i(v+s)\;p_{ij}(v+s)}{1-F_i(s)}\Bigg)\varphi(t+v,p+2\delta j,j,0)\Bigg\}
\end{aligned}
$$

Let $\overline{f}_i(s) := f_i(v+s)$ and $\overline{p}_{ij}(s) := p_{ij}(v+s)$.

The integral above goes to $\int_0^{T-t}\left(\frac{\overline{f}_i(s)\overline{p}_{ij}(s)}{1-F_i}\right)'(s)\;\varphi(t+v,p+\delta j,j,0)\;dv$ as $h\to 0$.

$$
\frac{\partial}{\partial s}\mathcal{T}\varphi(t,p,i,s) \;=\; \sum_{j\neq i}\int_0^{T-t}\left(\frac{\overline{f}_i(s)\overline{p}_{ij}(s)}{1-F_i}\right)'(s)\;\varphi(t+v,p+\delta j,j,0)\;dv \tag{4.13}
$$

$\mathcal{T}\varphi$ is in $\mathcal{C}^{2,2}$ using (A1). Hence, $\varphi(t,p,i,s)$ is in $\mathcal{C}^{2,2}$. $\square$

Proposition 4. *The unique solution of* (4.10)-(4.11) *also solves the following initial boundary value problem.*

$$
\frac{\partial}{\partial t}\varphi+\frac{\partial}{\partial s}\varphi+\frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i}p_{ij}(s)\Bigg(\varphi(t,p+2\delta j,j,0)-\varphi(t,p,i,s)\Bigg) \;=\; 0 \tag{4.14}
$$

$$
\textit{with } \varphi(T,p,i,s) \;=\; p \tag{4.15}
$$

Proof. Let φ be the solution of (4.10)-(4.11). From Lemma 3(ii), using (4.13) and (4.13),

$$
\begin{aligned}
&\frac{\partial}{\partial t}\varphi+\frac{\partial}{\partial s}\varphi\\
=\;& p\frac{f_i(T-t+s)}{1-F_i(s)}-\int_0^{T-t}\sum_{j\neq i}\frac{(\overline{f}_i\overline{p}_{ij})'(s)}{1-F_i(s)}\;\varphi(t+v,p+2\delta j,j,0)\;dv
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(s)\ \varphi(t,p+2\delta j,j,0) - p\frac{f_i(T-t+s)}{1-F_i(s)} \\
& +\frac{f_i(s)}{1-F_i(s)}\ p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) + \int_0^{T-t}\left(\frac{\overline{f}_i\overline{p}_{ij}}{1-F_i}\right)'(s)\ \varphi(t+v,p+\delta j,j,0)\ dv \\
= \ & \frac{f_i(s)}{1-F_i(s)}\ p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) - \frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(s)\ \varphi(t,p+2\delta j,j,0) \\
& +\int_0^{T-t}\sum_{j\neq i}\left(\left(\frac{\overline{f}_i\overline{p}_{ij}}{1-F_i}\right)'(s) - \frac{(\overline{f}_i\overline{p}_{ij})'(s)}{1-F_i(s)}\right)\varphi(t+v,p+2\delta j,j,0)\ dv \\
= \ & \frac{f_i(s)}{1-F_i(s)}\ p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) - \frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(s)\ \varphi(t,p+2\delta j,j,0) \\
& +\int_0^{T-t}\sum_{j\neq i}\left(\frac{(\overline{f}_i\overline{p}_{ij})'(1-F_i)(s)+f_i(s)(\overline{f}_i\overline{p}_{ij})(s)}{(1-F_i(s))^2} - \frac{(\overline{f}_i\overline{p}_{ij})'(s)}{1-F_i(s)}\right)\varphi(t+v,p+2\delta j,j,0)\ dv \\
= \ & \frac{f_i(s)}{1-F_i(s)}\ p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) - \frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(s)\ \varphi(t,p+2\delta j,j,0) \\
& +\int_0^{T-t}\sum_{j\neq i}\frac{(\overline{f}_i\overline{p}_{ij})(s)}{(1-F_i(s))^2}\ f_i(s)\ \varphi(t+v,p+2\delta j,j,0)\ dv \\
= \ & \frac{f_i(s)}{1-F_i(s)}\left\{p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) + \int_0^{T-t}\sum_{j\neq i}\frac{(\overline{f}_i\overline{p}_{ij})(s)}{1-F_i(s)}\ \varphi(t+v,p+2\delta j,j,0)\ dv\right\} \\
& -\frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(s)\ \varphi(t,p+2\delta j,j,0) \\
= \ & \frac{f_i(s)}{1-F_i(s)}\left\{p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) + \int_0^{T-t}\frac{f_i(v+s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(v+s)\ \varphi(t+v,p+2\delta j,j,0)\ dv\right\} \\
& -\frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(s)\ \varphi(t,p+2\delta j,j,0) \\
= \ & \frac{f_i(s)}{1-F_i(s)}\ \varphi(t,p,i,s) - \frac{f_i(s)}{1-F_i(s)}\sum_{j\neq i} p_{ij}(s)\ \varphi(t,p+2\delta j,j,0)
\end{aligned}
$$

From Lemma 3 and Proposition 4 it follows that (4.14)-(4.15) has a classical solution. □

Proposition 5. *A classical solution of* (4.14)-(4.15) *also solves the considered integral equation* (4.10)-(4.11).

Proof. Let ψ be the classical solution of (4.14)-(4.15). Then, we have

$$\frac{\partial}{\partial t}\psi + \frac{\partial}{\partial s}\psi + \frac{f_i(s)}{1-F_{(i)}}\sum_{j\neq i} p_{ij}(s)\Big(\psi(t,p+2\delta j,j,0) - \psi(t,p,i,s)\Big) \;=\; 0 \tag{4.16}$$

We define

$$N_t \coloneqq \psi(t,P_t,I_t,S_t).$$

Then, $dN_t = d\psi(t,P_t,I_t,S_t)$. We apply Itô's lemma to get the expression for $d\psi(t,P_t,I_t,S_t)$.

$$\begin{aligned}
\psi(t,P_t,I_t,S_t) &= \psi(0,P_0,I_0,S_0) + \int_0^t \psi'(s,P_s,I_s,S_s)dS_s^c \\
&\quad + \sum_{s\leq t}[\psi(t,P_t,I_t,S_t) - \psi(t,P_{t^-},I_{t^-},S_{t^-})] \\
&= \psi(0,P_0,I_0,S_0) + \int_0^t \psi'(s,P_s,I_s,S_s)dt \\
&\quad + \int_0^t \sum_{j\neq i}\frac{f_i(s)}{1-F_i(s)} p_{ij}[\psi(t,p+2\delta j,j,0) - \psi(t,p,i,s)] + M_t
\end{aligned}$$

where M_t is $\{\mathcal{F}\}-t$ martingale. Then,

$$\begin{aligned}
d\psi(t,P_t,I_t,S_t) &= \frac{\partial\psi}{\partial t}(t,P_{t^-},I_{t^-},S_{t^-})dt + \frac{\partial\psi}{\partial s}(t,P_{t^-},I_{t^-},S_{t^-})dt \\
&\quad + \sum_{j\neq i}\frac{f_i(s)}{1-F_i(s)} p_{ij}[\psi(t,p+2\delta j,j,0) - \psi(t,p,i,s)]dt + dM_t.
\end{aligned}$$

Hence we get,

$$\begin{aligned}
dN_t &= \frac{\partial\psi}{\partial t}(t,P_{t^-},I_{t^-},S_{t^-})dt + \frac{\partial\psi}{\partial s}(t,P_{t^-},I_{t^-},S_{t^-})dt \\
&\quad \sum_{j\neq i}\frac{f_i(s)}{1-F_i(s)} p_{ij}[\psi(t,p+2\delta j,j,0) - \psi(t,p,i,s)]dt + dM_t \\
&= dM_t.
\end{aligned}$$

Thus, N_t is a $\{\mathcal{F}\}_t$ martingale.

$$\begin{aligned}\psi(t, P_t, I_t, S_t) &= N_t \\ &= \mathbb{E}[N_T|P_t, I_t, S_t] \\ &= \mathbb{E}[P_T|P_t, I_t, S_t].\end{aligned}$$

By conditioning on transition times,

$$\begin{aligned}\psi(t, P_t, I_t, S_t) &= N_t \\ &= \mathbb{E}[\mathbb{E}[P_T|P_t, I_t, S_t, T_{N(t)+1}]|P_t, I_t, S_t] \\ &= \mathbb{E}[P_T \cdot 1_{T_{N(t)+1}>T}|P_t = p, I_t = i, S_t = s] \\ &\quad +\mathbb{E}[P_T \cdot 1_{T_{N(t)+1}\leq T}|P_t = p, I_t = i, S_t = s].\end{aligned} \tag{4.17}$$

The above expression has two terms in the RHS. We deal with them separately as follows.

$$\begin{aligned}&\mathbb{E}[P_T \cdot 1_{T_{N(t)+1}>T}|P_t = p, I_t = i, S_t = s] \\ =\ &\mathbb{E}[P_T] \cdot \mathbb{P}[T_{N(t)+1} > T|P_t = p, I_t = i, S_t = s] \\ =\ &p \cdot \mathbb{P}[T_{N(t)+1} > T|P_t = p, I_t = i, S_t = s].\end{aligned} \tag{4.18}$$

We calculate the probability that no transition takes place in the interval $(0, t)$ given I_0 and S_0 in the following manner. Since, $\{(I_t, S_t)\}_{t\geq 0}$ is a Markov process, we have

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}\mathbb{E}][f(I_t, S_t)|I_0 = i, S_0 = s] &= \lim_{t\to 0} T_t f(i, s) \\ &= \frac{\partial}{\partial s} f(i, s) + \sum_{j\neq i} \lambda_{ij}(s)\bigg(f(j, 0) - f(i, s)\bigg).\end{aligned}$$

We put $f(i,s) = 1_{(\chi,\{i\})\times\mathbb{R}}(i,s)$. Then,

$$\lim_{t\to 0}\frac{1}{t}\mathbb{P}[I_t \neq i|I_0 = i, S_0 = s] = 0 + \sum_{j\neq i}\lambda_{ij}(s)(1-0) = |\lambda_{ii}(s)| = \lambda_i(s)$$

$$\mathbb{P}[\text{No transition in}(0,\epsilon)|I_0 = i, S_0 = s] = 1 - \lambda_i(s)\epsilon + o(\epsilon)$$

$$\mathbb{P}[\text{No transition in }(n\epsilon,(n+1)\epsilon)|I_{n\epsilon} = i, S_{n\epsilon} = s + n\epsilon] = 1 - \lambda_i(s+n\epsilon)\epsilon + o(\epsilon)$$

$$\mathbb{P}[\text{No transition in}(0,t)|I_0 = i, S_0 = s] = \prod_{i=1}^{N} 1 - \lambda_i(s+n\epsilon)\epsilon + o(\epsilon)$$

where $N\epsilon \leq t < (N+1)\epsilon$. We take natural logarithm on both the sides,

$$\begin{aligned} ln\Big\{\mathbb{P}[\text{No transition in}(0,t)|\ I_0 = i, S_0 = s]\Big\} &= \sum_{i=1}^{N} ln(1-\lambda_i(s+n\epsilon)\epsilon) + \mathcal{O}(\epsilon) \\ &= \sum_{i=1}^{N} -\lambda_i(s+n\epsilon)\epsilon + \mathcal{O}(\epsilon) \\ &= -\int_s^{s+t}\lambda_i(x)dx. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{P}[\text{No transition in}(0,t)|I_0 = i, S_0 = s] &= \exp\left(-\int_s^{s+t}\lambda_i(x)dx\right) \\ &= \frac{\exp(-\int_0^{s+t}\lambda_i(x)dx)}{\exp(-\int_0^{s}\lambda_i(x)dx)} \\ &= \frac{1-F_i(s+t)}{1-F_i(s)}. \end{aligned} \tag{4.19}$$

Using (4.19), we can write (4.18) as

$$\mathbb{E}[P_T \cdot 1_{T_{N(t)+1}>T}|P_t = p, I_t = i, S_t = s] = p\left(\frac{1 - F_i(T - t + s)}{1 - F_i(s)}\right) \tag{4.20}$$

Now, we simplify the second term of the RHS of (4.18). Let $Z = T_{N(t)+1} - T_{N(t)} - S_t$.

$$\begin{aligned}
& \mathbb{E}[P_T \cdot 1_{\{T_{N(t)+1}\le T\}}|P_t = p, I_t = i, S_t = s] \\
= \; & \mathbb{E}\Bigg[\mathbb{E}[P_T \cdot 1_{Z\le T-t}||P_t = p, I_t = i, S_t = s, Z]|P_t = p, I_t = i, S_t = s\Bigg] \\
= \; & \mathbb{E}\Bigg[1_{Z\le T-t}\mathbb{E}[P_T|P_t = p, I_t = i, S_t = s, Z]|P_t = p, I_t = i, S_t = s\Bigg]
\end{aligned} \tag{4.21}$$

Now, $\forall\ T' \ge t$

$$\begin{aligned}
\mathbb{P}(T_{N(t)+1} > T'|I_t = i, S_t = s) &= \frac{1 - F_i(T' - t + s)}{1 - F_i(s)} \\
or, \mathbb{P}(T_{N(t)+1} \le T'|I_t = i, S_t = s) &= \frac{F_i(T' - t + s) - F_i(s)}{1 - F_i(s)}.
\end{aligned}$$

Then,

$$\lim_{\delta\to 0}\frac{1}{\delta}\mathbb{P}\Bigg(T_{N(t)+1} \in (T', T' + \delta)|I_t = i, S_t = s\Bigg) = \frac{f_i(T' - t + s)}{1 - F_i(s)},\ T' \in (t, \infty). \tag{4.22}$$

We let $v = T' - t$.

$$\begin{aligned}
& \mathbb{E}[P_T|T_{N(t)+1} = t + v, I_t = i, S_t = s] \\
= \; & \sum_{j\ne i}\mathbb{E}[P_T \cdot 1_{I_{t+v}=j}|T_{N(t)+1} = t + v, I_t = i, S_t = s] \\
= \; & \sum_{j\ne i}\mathbb{E}[P_T|I_{t+v} = j, S_{t+v} = 0, I_t = i, S_t = s] \cdot \mathbb{P}[I_{t+v} = j|S_{t+v} = 0, I_t = i, S_t = s] \\
= \; & \sum_{j\ne i}p_{ij}(s)\mathbb{E}[P_T|I_{t+v} = j, S_{t+v} = 0, P_{t+v} = p + 2\delta j]
\end{aligned}$$

$$= \sum_{j \neq i} p_{ij}(s)\phi(t+v, p+2\delta j, j, 0). \tag{4.23}$$

Hence, from (4.22) and (4.23) we get the simplified expression for the second term of the RHS as

$$\begin{aligned} & \int_0^{T-t} \mathbb{E}[P_T | T_{N(t)+1} = t+v, I_t = i, S_t = s] \frac{f_i(v+s)}{1-F_i(s)} \\ = & \int_0^{T-t} \sum_{j \neq i} p_{ij}(s)\phi(t+v, p+2\delta j, j, 0). \end{aligned} \tag{4.24}$$

We put (4.20) and (4.24) in (4.14) to get

$$\begin{aligned} \psi(t,p,i,s) &= p\left(\frac{1-F_i(T-t+s)}{1-F_i(s)}\right) \\ &+ \int_0^{T-t} \sum_{j \neq i} p_{ij}(s)\phi(t+v, p+2\delta j, j, 0). \end{aligned}$$

Hence, we conclude that ψ is a solution to (4.10)-(4.11). □

Theorem 6. *The boundary value problem* (4.8) *has a unique solution.*

Proof. From lemma 3 and proposition 4, we see that a solution to (4.8) exists. To prove uniqueness, we assume that ψ_1 and ψ_2 are two classical solutions. Then, according to lemma 3, both ψ_1 and ψ_2 solve (4.10)-(4.11). But, from lemma 3, only one such function exists. Hence, $\psi_1 = \psi_2$. □

Thus, we have established that PDE (4.8) has a classical solution. We also discussed the properties of the solution π. This study has been carried out in a collaboration with Ms. Garima Agrawal. A similar study would be relevant in the investigation of control and optimization, and the corresponding Hamilton-Jacobi-Bellman equation.

Chapter 5

Optimal Market-making Strategy

An electronic market receives mainly two types of orders: market orders and limit orders. Market orders (MO) are aggressive orders which takes liquidity from the LOB and receives the best prices currently available. These orders are put in by the trader to either buy or sell stocks shares, bonds, or other available assets at the best price obtainable in the current financial market and are executed immediately. On the other hand, limit orders (LO) are passive orders to buy or sell assets at a pre-decided price. They are registered in the Limit Order Book along with the volume put up for trading. Limit orders usually offer prices worse than the prevailing market price, that is, they are placed with higher price than the best buy price for sell limit orders, and at lower price than the best ask price for buy limit orders. The market orders that arrive "walk the LOB" and gets matched with the posted limit orders according to the rule of the book. Following are two important order matching frameworks:

- Price time priority: In this order book, orders are matched according to their price and time. The orders placed earliest and closer to the mid-price are preferred. The matching algorithm selects the oldest limit orders placed at the best price and executes them in order until the entire market order is executed. If the order is not completely executed at the best price, the algorithm matches it with limit orders placed at second best price and so on.

- Pro-rata: In this order book, price priority is given but not time priority. Market orders are matched with the limit orders posted at the best price and in proportion to the quantities posted.

Market makers is one class of the participants of trading. They are professional traders who profit from their expertise in facilitating exchange in a particular asset. They provide liquidity to the market by quoting buy and sell prices via submitting limit orders on both sides of the LOB. The market maker faces a risk due to a jump in the asset price. In particular he faces the following risks:

- Market risk: Due to sudden jump in the price, the inventory is re-evaluated and his portfolio wealth alters immediately. Thus, he faces a finite amount of risk in no time.
- Adverse Selection risk: When trading with informed traders who have private or better information than the market maker, he exposes himself to the adverse selection risk. He risks placing a sell limit order that can be fulfilled just before the price jumps upwards, or a buy limit order getting filled just before a drop in the price.

In this chapter, we understand the modelling of market order flow via a marked point process. We then discuss the market making problem of the agent of submitting optimal limit orders at the best bid and best ask prices. The wealth and inventory process of the agent and the associated value function is described. We derive the infinitesimal generator of the process $(P_t, I_t, S_t, X_t, Y_t)$. The results and discussion in the chapter are adapted from [2].

5.1 Market Order Flow Model

The small market order flow is modelled by a marked point process (θ_k, Z_k). The increasing sequence (θ_k) represents the time stamps of the arrival of the small market orders and $(Z_k) \in \{-1, +1\}$ represent the side of the exchange. It follows the convention that the trade is exchanged at best bid price when $Z_k = -1$, that is, a market sell order has arrived. Similarly, when $Z_k = +1$, the trade is exchanged at best ask price, that is, a buy limit order has arrived. The size of the market orders is not considered.

Let (M_t) be a Cox process with conditional intensity $\lambda(S_t)$, where λ is a bounded continuous function on $\mathbb{R}_+$. It is the counting process associated to (θ_k). λ is estimated using MLE algorithm for point processes. It is observed that multiple trades arrive in the LOB when the price is unstable. On the other hand, there is a weaker trading activity present upon stabilization of the price.

The correlation between the market order trade and stock prices in the LOB is establsihed as follows. Define

$$Z_k \coloneqq \Gamma_k I_{\theta_{k^-}}, \tag{5.1}$$

where (Γ_k) is an i.i.d. sequence with Bernoulli distribution on $\{-1, +1\}$ with parameter $(\frac{1+\rho}{2})$, where $\rho \in (-1, 1)$. ρ can be interpreted as the correlation coefficient between Z_k and $I_{\theta_{k^-}}$. Following observations are made:

1. when $\rho > 0$, market orders arrive more frequently in the strong side of the LOB, that is, at best ask when price jumped upwards and at best bid when price jumped downwards.
2. when $\rho = 0$, the trade sides do not depend on the stock price and arrive independently.
3. when $\rho < 0$, market orders arrive more often in the weak side of the LOB, that is, at best ask when the price jumped downwards whearas at the best bid in the opposite case.

The authors in [2] estimated the value of ρ to be around -50%. This implies that three out of four trades arrive in the weak side of the LOB. This means that buy market orders arrive at best bid price when the price jumped upwards. Alternatively, sell market orders arrive at best ask prices after a downward price jump.

We recall the correlation coefficient α of the price increments. [1] estimates its value to be negative. Estimation of $\rho < 0$ in [2] is consistent with execution dynamics. On the contrary, let us assume that α and ρ are of opposite signs. Let $\alpha < 0$ and $\rho > 0$. Also, let the last price jump be downwards. Since $\alpha < 0$, the current market is a bull market. As $\rho > 0$, the limit orders posted by the agent at the best bid price will be executed. Thus, the agent ends up buying stocks in a bull market which creates a low risk profitable position for him. The quantity $\alpha\rho$ gives the probability of building a profitable position via a limit order. The bigger the value, the smaller is the probability. This phenomenon is called weak adverse selection.

5.1.1 Adverse Selection

The market orders are categorized into two types based on their size with respect to the liquidity available in the market: big market orders and small market orders. Arrival of big market orders that are influential in nature causes the mid-price to jump. They move the ask and bid prices. Arrival of small market orders, on the other hand, do not affect the price.

Let us assume that an upward jump at time t corresponds to a big market order arrival which aims at clearing the market of all available liquidity at the best ask price. Also, assume that our agent is a small agent, that is, he posts limit orders of a small size compared to the available liquidity and hence does not affect the market. Then, the case where he has posted a small limit order on the ask side is considered. In the presence of a big market order, the limit order gets executed as the goal of the former is to clear the market of all liquidity rather than consume a fixed amount of it. The arrival of big MO changes the price at time t whereas the agent sells his assets at a price prevalent at t^-. In particular, since the tick size is constantly one, the current price (after an upward jump) is given by $P_t = P_{t^-} + 2\delta$. The limit orders are posted at best ask given by $P_{t^-} + \delta$, Since the LO gets immediately executed without giving the agent a chance to update his quote, he loses δ due to the disadvantageous transaction.

He faces a similar adverse selection risk in the case of submitting limit orders on the bid side. Here, the current price (after a downward jump) is given by $P_t = P_{t^-} - 2\delta$ whereas the agent has submitted a limit order at the best bid price given by $P_{t^-} - \delta$. If this LO gets executed, he again loses δ amount.

5.2 The market making problem

Let us assume that the agent is small and has to continuously place limit orders of constant small size $L \in \mathbb{N}\backslash\{0\}$ on both sides at the best price available. The market making strategy of the agent is then described by a pair of predictable processes (ℓ^+, ℓ^-) valued in $\{0, 1\}$. When $\ell_t^\pm = 0$, no limit order is submitted by the agent on either side. In the opposite case, when $\ell_t^+ = 1$, a limit order of size L is posted at time t on the strong side of the limit order book. Similarly, for $\ell_t^- = 1$ the limit order is submitted in the weak side of the LOB.

Let the set of market making controls $\ell = (\ell^+, \ell^-)$ be denoted by $\mathcal{A}$. An arriving small market order is matched and executed with the small limit order posted by the agent in the corresponding side of the LOB. The LO is executed according to a random variable K whose distribution is given by $\vartheta_\pm(dk, L)$ on $\{0, \cdots, L\}$. $\vartheta_+(dk, L)$ is the distribution of the executed quantity of limit orders of size L in the strong side of the LOB. Similarly, $\vartheta_-(dk, L)$ is the distribution of the executed quantity of limit orders of size L in the weak side of the LOB.

5.2.1 The Wealth and Inventory process

The wealth and the inventory of the agent are denoted by the processes (X_t) and (Y_t) respectively. Also, let us assume a fixed cost of $\epsilon \geq 0$ for each transaction.

Lemma 7. *For a market making strategy $l \in \mathcal{A}$, the dynamics of the portfolio value processes (X_t) and (Y_t) are given by*

$$dX_t = \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} k\ell^{\nu}_{t^-}(\nu P_{t^-} I_{t^-} + \delta - \epsilon)\left(\sum_k (k \wedge L) 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z) \right) \wp(dt, dz), \quad (5.2)$$

$$dY_t = -\int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} \nu \ell^{\nu}_{t^-} I_{t^-} \left(\sum_k (k \wedge L) 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z) \right) \wp(dt, dz), \quad (5.3)$$

where $H_{\nu I_{t^-} jk}(S_{t^-})$ is left close right open consecutive intervals placed on the positive and negative side of the real axis for $\nu = +1$ and $\nu = -1$ respectively, and

$$|H_{\nu ijk}(s)| = \begin{cases} \int_{k'} \lambda_{\nu ij}(s) \vartheta_\nu(dk', L) & k \in \{0, 1, \cdots, L\} \\ h_{\nu ij}(s) & k = L + 1. \end{cases}$$

$\wp(dt, dz)$ is a Poisson random measure on Borel σ- algebra of $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dtdz$.

Proof. We prove this lemma by constructing a truth table. K_1 and K_2 are the realizations of the random variable K. (5.2) is written as

$$dX_t = \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} k\ell^{\nu}_{t^-}(\nu P_{t^-} I_{t^-} + \delta - \epsilon)\left(\sum_k (k \wedge L) 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z) \right) \wp(dt, dz)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} k\ell^+_{t^-}(P_{t^-}I_{t^-} + \delta - \epsilon)\left(\sum_k (k \wedge L) 1_{H_{I_{t^-}jk}(S_{t^-})}(z) \right) \wp(dt, dz) \\
&\quad + \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} k\ell^-_{t^-}(-P_{t^-}I_{t^-} + \delta - \epsilon)\left(\sum_k (k \wedge L) 1_{H_{-I_{t^-}jk}(S_{t^-})}(z) \right) \wp(dt, dz) \\
&= \text{Term } 1 + \text{Term } 2;
\end{aligned}
$$

and

$$
\begin{aligned}
dY_t &= -\int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} \nu \ell^\nu_{t^-} I_{t^-} \left(\sum_k (k \wedge L) 1_{H_{\nu I_{t^-}jk}(S_{t^-})}(z) \right) \wp(du, dz) \\
&= -\int_{\mathbb{R}} \sum_{j \neq I_{t^-}} \ell^+_{t^-} I_{t^-} \left(\sum_k (k \wedge L) 1_{H_{I_{t^-}jk}(S_{t^-})}(z) \right) \wp(du, dz) \\
&\quad - \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} -\ell^-_{t^-} I_{t^-} \left(\sum_k (k \wedge L) 1_{H_{-I_{t^-}jk}(S_{t^-})}(z) \right) \wp(du, dz) \\
&= \text{Term } 3 + \text{Term } 4.
\end{aligned}
$$

We consider the case where $I_{t^-} = +1$, that is, the price has jumped upwards at time t. When the agent has posted limit orders on both the sides, that is, $\ell^{\pm}_{t^-} = +1$, then following two scenarios are plausible.

- The limit order gets partially executed by an incoming market order of size K with trade intensity $\lambda_{\nu ij}(S_{t^-})$, where K is a random variable with distribution given by $\vartheta_\nu(dk, L)$
- The entire limit order gets executed by a big market order with rate $h_{\nu ij}(s)$. In this case, $K = L$.

In either case, the wealth of the agent jumps by $\pm k \times P_{t^-}$. If a limit order on the strong side is executed due to a small market order of size $K = k(\neq L)$, the agent gains kP_{t^-} along with the half-spread, and loses the transaction cost ϵ, whereas if the limit order executed is on the weak side, she loses kP_{t^-}. Correspondingly, her inventory decreases or increases by k units. On the other hand, if the incoming matching market order is big, that is $K = L$, the agent gains LP_{t^-} (loses LP_{t^-}) for an execution in the strong (weak, respectively) side of the LOB. The inventory jumps by L in this case.

Similar arguments hold for the other cases mentioned in Table 5.1 and Table 5.2.

I_{t^-}	$\ell^+_{t^-}$	$\ell^-_{t^-}$	$1_{\{k_1=L\}}$	$1_{\{k_2=L\}}$	Term 1	Term 2	Term 1+Term 2	dX_t
1	0	0	1	1	0	0	0	0
1	0	0	0	1	0	0	0	0
1	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0
1	1	0	1	1	$L(P_{t^-}+\delta-\epsilon)$	0	$L(P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$
1	1	0	0	1	$k_1(P_{t^-}+\delta-\epsilon)$	0	$k_1(P_{t^-}+\delta-\epsilon)$	$k_1(P_{t^-}+\delta-\epsilon)$
1	1	0	1	0	$L(P_{t^-}+\delta-\epsilon)$	0	$L(P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$
1	1	0	0	0	$k_1(P_{t^-}+\delta-\epsilon)$	0	$k_1(P_{t^-}+\delta-\epsilon)$	$k_1(P_{t^-}+\delta-\epsilon)$
1	0	1	1	1	0	$L(-P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$
1	0	1	0	1	0	$L(-P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$
1	0	1	1	0	0	$k_2(-P_{t^-}+\delta-\epsilon)$	$k_2(-P_{t^-}+\delta-\epsilon)$	$k_2(-P_{t^-}+\delta-\epsilon)$
1	0	1	0	0	0	$k_2(-P_{t^-}+\delta-\epsilon)$	$k_2(-P_{t^-}+\delta-\epsilon)$	$k_2(-P_{t^-}+\delta-\epsilon)$
1	1	1	1	1	$L(P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)+L(-P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)+L(-P_{t^-}+\delta-\epsilon)$
1	1	1	0	1	$k_1(P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$	$k_1(P_{t^-}+\delta-\epsilon)+L(-P_{t^-}+\delta-\epsilon)$	$k_1(P_{t^-}+\delta-\epsilon)+L(-P_{t^-}+\delta-\epsilon)$
1	1	1	1	0	$L(P_{t^-}+\delta-\epsilon)$	$k_2(-P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)+k_2(-P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)+k_2(-P_{t^-}+\delta-\epsilon)$
1	1	1	0	0	$k_1(P_{t^-}+\delta-\epsilon)$	$k_2(-P_{t^-}+\delta-\epsilon)$	$k_1(P_{t^-}+\delta-\epsilon)+k_2(-P_{t^-}+\delta-\epsilon)$	$k_1(P_{t^-}+\delta-\epsilon)+k_2(-P_{t^-}+\delta-\epsilon)$
-1	0	0	1	1	0	0	0	0
-1	0	0	0	1	0	0	0	0
-1	0	0	1	0	0	0	0	0
-1	0	0	0	0	0	0	0	0
-1	1	0	1	1	$L(-P_{t^-}+\delta-\epsilon)$	0	$L(-P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$
-1	1	0	0	1	$k_1(-P_{t^-}+\delta-\epsilon)$	0	$k_1(-P_{t^-}+\delta-\epsilon)$	$k_1(-P_{t^-}+\delta-\epsilon)$
-1	1	0	1	0	$L(-P_{t^-}+\delta-\epsilon)$	0	$L(-P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)$
-1	1	0	0	0	$k_1(-P_{t^-}+\delta-\epsilon)$	0	$k_1(-P_{t^-}+\delta-\epsilon)$	$k_1(-P_{t^-}+\delta-\epsilon)$
-1	0	1	1	1	0	$L(P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$
-1	0	1	0	1	0	$L(P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$
-1	0	1	1	0	0	$k_2(P_{t^-}+\delta-\epsilon)$	$k_2(P_{t^-}+\delta-\epsilon)$	$k_2(P_{t^-}+\delta-\epsilon)$
-1	0	1	0	0	0	$k_2(P_{t^-}+\delta-\epsilon)$	$k_2(P_{t^-}+\delta-\epsilon)$	$k_2(P_{t^-}+\delta-\epsilon)$
-1	1	1	1	1	$L(-P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)+L(P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)+L(P_{t^-}+\delta-\epsilon)$
-1	1	1	0	1	$k_1(-P_{t^-}+\delta-\epsilon)$	$L(P_{t^-}+\delta-\epsilon)$	$k_1(-P_{t^-}+\delta-\epsilon)+L(P_{t^-}+\delta-\epsilon)$	$k_1(-P_{t^-}+\delta-\epsilon)+L(P_{t^-}+\delta-\epsilon)$
-1	1	1	1	0	$L(-P_{t^-}+\delta-\epsilon)$	$k_2(P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)+k_2(P_{t^-}+\delta-\epsilon)$	$L(-P_{t^-}+\delta-\epsilon)+k_2(P_{t^-}+\delta-\epsilon)$
-1	1	1	0	0	$k_1(-P_{t^-}+\delta-\epsilon)$	$k_2(P_{t^-}+\delta-\epsilon)$	$k_1(-P_{t^-}+\delta-\epsilon)+k_2(P_{t^-}+\delta-\epsilon)$	$k_1(-P_{t^-}+\delta-\epsilon)+k_2(P_{t^-}+\delta-\epsilon)$

Table 5.1: X_t

I_{t^-}	$\ell^+_{t^-}$	$\ell^-_{t^-}$	$1_{\{k_1=L\}}$	$1_{\{k_2=L\}}$	Term 3	Term 4	Term 3+Term 4	dY_t
1	0	0	1	1	0	0	0	0
1	0	0	0	1	0	0	0	0
1	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0
1	1	0	1	1	$-L$	0	$-L$	$-L$
1	1	0	0	1	$-k_1$	0	$-k_1$	$-k_1$
1	1	0	1	0	$-L$	0	$-L$	$-L$
1	1	0	0	0	$-k_1$	0	$-k_1$	$-k_1$
1	0	1	1	1	0	L	L	L
1	0	1	0	1	0	L	L	L
1	0	1	1	0	0	k_2	k_2	k_2
1	0	1	0	0	0	k_2	k_2	k_2
1	1	1	1	1	$-L$	L	0	0
1	1	1	0	1	$-k_1$	L	$L-k_1$	$L-k_1$
1	1	1	1	0	$-L$	k_2	k_2-L	k_2-L
1	1	1	0	0	$-k_1$	k_2	k_2-k_1	k_2-k_1
-1	0	0	1	1	0	0	0	0
-1	0	0	0	1	0	0	0	0
-1	0	0	1	0	0	0	0	0
-1	0	0	0	0	0	0	0	0
-1	1	0	1	1	L	0	L	L
-1	1	0	0	1	k_1	0	k_1	k_1
-1	1	0	1	0	L	0	L	L
-1	1	0	0	0	k_1	0	k_1	k_1
-1	0	1	1	1	0	$-L$	$-L$	$-L$
-1	0	1	0	1	0	$-L$	$-L$	$-L$
-1	0	1	1	0	0	$-k_2$	$-k_2$	$-k_2$
-1	0	1	0	0	0	$-k_2$	$-k_2$	$-k_2$
-1	1	1	1	1	L	$-L$	0	0
-1	1	1	0	1	k_1	$-L$	k_1-L	k_1-L
-1	1	1	1	0	L	$-k_2$	$L-k_2$	$L-k_2$
-1	1	1	0	0	k_1	$-k_2$	k_1-k_2	k_1-k_2

Table 5.2: Y_t

5.3 Generator of $(P_t, I_t, S_t, X_t, Y_t)$

We derive the generator of the augmented process $(P_t, I_t, S_t, X_t, Y_t)$ as follows.

For $\nu \in \{-1, +1\}, i, j \in \chi, s \in \mathbb{R}_+, k \in \{0, 1, \cdots, L+1\}$, let $H_{\nu ijk}(s)$ be left close right open consecutive intervals on the real line placed on positive and negative side for $\nu = +1$ and $\nu = -1$ respectively. The length of these intervals is given by

$$|H_{\nu ijk}(s)| = \begin{cases} \int_{k'} \lambda_{\nu ij}(s)\vartheta_\nu(dk', L) & k \in \{0, 1, \cdots, L\} \\ h_{\nu ij}(s) & k = L+1. \end{cases}$$

Let

$$H_{ij}(s) = \biguplus_{\nu} H_{\nu ijL+1}(s).$$

We consider the process $(P_t, I_t, S_t, X_t, Y_t)$ described the following stochastic integrals:

$$P_t = P_0 + \int_0^t \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} 2\delta I_{t^-} j \cdot 1_{H_{\nu I_{t^-} j}(S_{t^-})}(z) \wp(dt, dz)$$

$$I_t = I_0 + \int_0^t \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} (j - I_{t^-}) \cdot 1_{H_{\nu I_{t^-} j}(S_{t^-})}(z) \wp(dt, dz)$$

$$S_t = S_0 + t - \int_0^t \int_{\mathbb{R}} S_{t^-} \sum_{j \neq I_{t^-}} 1_{H_{\nu I_{t^-} j}(S_{t^-})}(z) \wp(dt, dz)$$

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} k\ell^{\nu}_{t^-}(\nu P_{t^-} I_{t^-} + \delta - \epsilon)(\sum_k (k \wedge L) 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z))\wp(dt, dz)$$

$$Y_t = Y_0 - \int_0^t \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} \nu\ell^{\nu}_{t^-} I_{t^-}(\sum_k (k \wedge L) 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z))\wp(dt, dz)$$

where $\wp(dt, dz)$ is a Poisson random measure on Borel σ- algebra of $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dtdz$.

Applying Itô's formula, for a rcll process on ϕ, we get

$$\begin{aligned}
d\phi(P_t, I_t, S_t, X_t, Y_t) &= \frac{\partial \phi}{\partial s}(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) dS^c_t + 0 \\
&\quad + \sum_{s \leq t} \{\phi(P_t, I_t, S_t, X_t, Y_t) - \phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-})\} \\
&= \frac{\partial \phi}{\partial s}(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) dt \\
&\quad + \{\phi(P_t, I_t, S_t, X_t, Y_t) - \phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-})\}
\end{aligned}$$

$$\begin{aligned}
&\phi(P_t, I_t, S_t, X_t, Y_t) - \phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) \\
= \; &\phi(P_{t^-} + dP_t, I_{t^-} + dI_t, S_{t^-} + dS_t, X_{t^-} + dX_t, Y_{t^-} + dY_t) - \phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) \\
= \; &\Bigg\{\phi\Bigg(P_{t^-} + \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} 2\delta I_{t^-} j \cdot 1_{H_{\nu I_{t^-} j}(S_{t^-})}(z)\wp(dt, dz), \\
&\qquad I_{t^-} + \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} (j - I_{t^-}) \cdot 1_{H_{\nu I_{t^-} j}(S_{t^-})}(z)\wp(dt, dz) \\
&\qquad S_{t^-} - \int_{\mathbb{R}} S_{t^-} \sum_{j \neq I_{t^-}} 1_{H_{\nu I_{t^-} j}(S_{t^-})}(z)\wp(dt, dz), \\
&\qquad X_{t^-} + \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} k\ell^{\nu}_{t^-}(\nu P_{t^-} I_{t^-} + \delta - \epsilon)\Big(\sum_k (k \wedge L) 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z)\Big)\wp(dt, dz)
\end{aligned}$$

$$
\begin{aligned}
&\qquad Y_{t^-} - \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} \nu \ell^{\nu}_{t^-} I_{t^-} (\sum_{k} (k \wedge L) 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z)) \wp(dt, dz) \Big) \Big\} \\
&-\phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) \\
= &\Big\{ \phi \Big(P_{t^-} + \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} 2\delta I_{t^-} j \cdot 1_{H_\nu I_{t^-} j(S_{t^-})}(z), I_{t^-} + \int_{\mathbb{R}} \sum_{j \neq I_{t^-}} (j - I_{t^-}) \cdot 1_{H_\nu I_{t^-} j(S_{t^-})}(z) \\
&\qquad S_{t^-} - \int_{\mathbb{R}} s \sum_{j \neq I_{t^-}} 1_{H_\nu I_{t^-} j(S_{t^-})}(z), \\
&\qquad X_{t^-} + \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} k \ell^{\nu}_{t^-} (\nu P_{t^-} I_{t^-} + \delta - \epsilon) (\sum_{k=1}^{L} k 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z)) \\
&\qquad Y_{t^-} - \int_{\mathbb{R}} \sum_{\nu} \sum_{j \neq I_{t^-}} \nu \ell^{\nu}_{t^-} I_{t^-} (\sum_{k=1}^{L} k 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z)) \Big) \\
&-\phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) \Big\} \wp(dt, dz) \\
= &\int_{\mathbb{R}} \Big\{ \phi \Big(P_{t^-} + \sum_{j \neq I_{t^-}} 2\delta I_{t^-} j \cdot 1_{H_\nu I_{t^-} j(S_{t^-})}(z), I_{t^-} + \sum_{j \neq I_{t^-}} (j - I_{t^-}) \cdot 1_{H_\nu I_{t^-} j(S_{t^-})}(z) \\
&\qquad S_{t^-} - S_{t^-} \sum_{j \neq I_{t^-}} 1_{H_\nu I_{t^-} j(S_{t^-})}(z), \\
&\qquad X_{t^-} + \sum_{\nu} \sum_{j \neq I_{t^-}} k \ell^{\nu}_{t^-} (\nu P_{t^-} I_{t^-} + \delta - \epsilon) (\sum_{k=1}^{L} k 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z)) \\
&\qquad Y_{t^-} - \sum_{\nu} \sum_{j \neq I_{t^-}} \nu \ell^{\nu}_{t^-} I_{t^-} (\sum_{k=1}^{L} k 1_{H_{\nu I_{t^-} jk}(S_{t^-})}(z)) \Big) - \phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) \Big\} dt dz \\
&+dM_t \\
= &\sum_{\nu} \sum_{j \neq I_{I_{t^-}}} \int_{H_{\nu I_{t^-} jL+1}(S_{t^-})} \Big\{ \phi \Big(P_{t^-} + 2\delta j I_{t^-}, j, 0, \\
&\qquad X_{t^-} + L \ell^{\nu}_{t^-} (\nu P_{t^-} I_{t^-} + \delta - \epsilon), Y_{t^-} - L \nu \ell^{\nu}_{t^-} I_{t^-} \Big) - \phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) \Big\} ds dz \\
&+ \sum_{\nu} \sum_{j \neq I_{I_{t^-}}} \sum_{k} \int_{H_{\nu I_{t^-} jk}(S_{t^-})} \Big\{ \phi \Big(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-} + k \nu \ell^{\nu}_{t^-} (\nu P_{t^-} I_{t^-} + \delta - \epsilon), \\
&\qquad Y_{t^-} - k \nu \ell^{\nu}_{t^-} I_{t^-} \Big) - \phi(P_{t^-}, I_{t^-}, S_{t^-}, X_{t^-}, Y_{t^-}) \Big\} ds dz \\
= &\sum_{\nu} \sum_{j \neq i} h_{\nu ij}(s) \Big[\phi(t, p + 2\delta ij, j, 0, x + L\ell(\nu ip + \delta - \epsilon), y - L\ell\nu i) - \phi(t, p, i, s, x, y) \Big] \\
&+ \sum_{\nu} \sum_{j \neq i} \lambda_{\nu ij}(s) \int_{k} \Big[\phi(t, p, i, s, x + k\ell(\nu ip + \delta - \epsilon), y - k\ell\nu i) - \phi(t, p, i, s, x, y) \Big].
\end{aligned}
$$

Therefore the generator of the process $(P_t, I_t, S_t, X_t, Y_t)$ is given by

$$
\begin{aligned}
\mathcal{L}\phi(p,i,s,x,y) \quad = \quad & \frac{\partial \phi}{\partial s}(p,i,s,x,y) \\
& + \sum_{\nu}\sum_{j \neq i} h_{\nu ij}(s)\Big[\phi(t, p+2\delta ij, j, 0, x + L\ell(\nu ip + \delta - \epsilon), y - L\ell\nu i) \\
& \qquad\qquad - \phi(t,p,i,s,x,y)\Big] \\
& + \sum_{\nu}\sum_{j \neq i} \lambda_{\nu ij}(s) \int_k \Big[\phi(t,p,i,s,x + k\ell(\nu ip + \delta - \epsilon), y - k\ell\nu i) \\
& \qquad\qquad - \phi(t,p,i,s,x,y)\Big]\vartheta_\nu(dk, L).
\end{aligned}
$$

5.4 Value function

The optimal market making strategy for the agent maximizes her expected wealth at the terminal time T, evaluated at mid-price and penalizes for inventory stock. We define the value function associated to the market making problem as follows:

$$
v(t,p,i,s,x,y) \;\coloneqq \max_{\ell in \mathcal{A}} \mathbb{E}[X_T + Y_T P_T - \eta Y_T^2 | P_t = p, I_t = i, S_t = s, X_t = x, Y_t = y] \tag{5.4}
$$

where $(t,p,i,s,x,y) \in [0,T] \times 2\delta\mathbb{Z} \times \chi \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{Z}$ and $\eta \geq 0$ is risk aversion parameter. The strategy controls the agent's final inventory by the quadratic penalization term ηY_T^2. At the end of trading day, a large inventory position will let the agent to execute it all by placing a large order. This will impact the market, contrary to the small agent assumption. Thus, she does not want to hold a large inventory at terminal time and the penalization term controls her inventory risk.

Using the dynamic programming principle heuristic, we write the following Hamilton-Jacobi-Bellman equation corresponding to the control problem given in (5.4).

$$
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} + \sum_{\nu}\sum_{j\neq i}\max_{\ell\in\{0,1\}}\Bigg\{ h_{\nu ij}(s)\Big(v(t,p+2\delta ij, j, 0, x+L\ell(\nu ip+\delta-\epsilon), y-\nu iL\ell) - v(t,p,i,s,x,y)\Big) \\
+ \int_k \lambda_{\nu ij}(s)\Big(v(t,p,i,s,x+k\ell(\nu ip+\delta-\epsilon), y-\nu ik\ell) - v(t,p,i,s,x,y)\Big)\vartheta_\nu(dk,L)\Bigg\} &= 0 \\
v(T,p,i,s,x,y) &= x + yp - \eta y^2
\end{aligned}
$$

We remark that if we restrict the state space χ to be a two-state space ($\chi = \{-1,+1\}$), then we can obtain results published in [2].

In the literature, viscosity solution of the above HJB equation has been obtained [2]. However, we have obtained the classical solution of a related linear PDE. This study has been carried out in a collaboration with Ms. Garima Agrawal. This study indicates that one may obtain classical solution of (5.5) with further investigation.

Chapter 6

Conclusion

In this project, we studied the microstructure modelling of the financial asset price. We investigated the semi-Markov model described in [1] and the market making problem studied in [2]. We performed mathematical analyses on certain functionals of the stock price. In particular, these are expressed using the conditional expectation of stock price. Authors in [2] show the existence of viscosity solution of (4.8) on a restricted state space. However, we produced an original proof that conditional mean price given by (4.7) solves the PDE (4.8). We showed this in two steps. First, we considered an integral equation, and established the existence and uniqueness of its solution. We then studied some regularity features and went on to show that the PDE (4.8) and the considered integral equation are equivalent. Thus, we established existence and uniqueness of the solution to the system (4.8). We also described the wealth and inventory process of the agent and discussed the market making problem. Using dynamic programming principle heuristic, we wrote the Hamilton-Jacobi-Bellman equation corresponding to the control problem given in (5.4).

Proving existence of classical solution of the HJB equation (5.5) requires further investigation. A viscosity solution to (5.5) has been obtained in [2] on a restricted state sapce. However, we have obtained a classical solution to a related linear PDE which indicates that a classical solution to (5.5) can also be established.

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