Motives of Algebraic Stacks

A thesis

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by

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To My Father,

in whose footsteps I have walked this far.

Certificate

Certified that the work incorporated in the thesis entitled "*Motives of Algebraic Stacks*", submitted by *Neeraj Deshmukh* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: September 14, 2021 *Dr. Amit Hogadi*

Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Contents

1

Introduction

In [Toë], Bertrand Toën defined and studied the notion of a motive of Deligne-Mumford stacks initiating the study of theory of motives and motivic cohomology of algebraic stacks. The general theory has been subsequently developed and extended by various authors. In [Jos1, Jos2], a theory of étale motives for algebraic stacks is developed from the hypercohomology point of view, while in [Cho] the motive is defined via the unstable $\mathbb{A}^1$-homotopy category of Morel-Voevodsky using the description of a stack as a simplicial presheaf. And in [Hoy], a six functor formalism is developed for motives of tame quotient stacks in line with the six functor formalism for motives of schemes developed by Ayoub and others (see [Ayo1, Ayo2], [CD]).

In [Tot1], Totaro defines motives for quotient stacks using a "finite dimensional approximation" via vector bundles. This technique is further extended to stacks admitting filtrations by quotient substacks in [HL].

In this thesis, we will further study the theory of motives for algebraic stacks.

While there already exists sufficient literature on motives of algebraic stacks, there are often two limitations to the treatments found in literature. Firstly, that they almost exclusively work with the étale topology as opposed to the Nisnevich topology. And secondly, that they primarily focus on Deligne-Mumford stacks or quotient stacks.

While we are unable to make sweeping generalisations about the latter, we will make some important observations about the former. In order to understand why one should look at the Nisnevich topology for defining the motive of an algebraic stack, it is instructive to first recall the theory of motives for schemes.

1.1 Motives of schemes

The idea of motives of schemes dates back to Grothendieck. Grothendieck envisioned motives as a universal cohomology theory together realisation functors that produce all Weil cohomology theories. Examples of Weil cohomology theories are Betti cohomology, algebraic De Rham cohomology, étale cohomology, etc. Grothendieck also constructed an abelian category of motives for smooth projective varieties with such realisation functors for weil cohomology theories. These are called *Pure motives*.

Just as pure motives describe the cohomology of smooth projective varieties, we should also have motives for arbitrary varieties which describe cohomology. These objects are called *mixed motives*.

For the category of smooth separated schemes over a field Sm/k, one expects the following:

Claim 1.1.1. There exists an abelian category $MM(k)$ of mixed motives over a field k with a functor

$$M : Sm/k \to MM(k)$$

such that the canonical inclusion into the associated derived category (denoted as $\mathbf{DM}^{eff}(k,\mathbb{Z})$) $Sm/k \xrightarrow{M} MM(k) \hookrightarrow \mathbf{DM}^{eff}(k,\mathbb{Z})$ satisfies the following properties:

1. we have realisation functors from the derived category $\mathbf{DM}^{eff}(k,\mathbb{Z})$ to derived category of abelian groups corresponding to each Weil cohomology theory of smooth schemes, and

2. Given a smooth separated scheme X, Chow groups of X appear as the cohomology groups of $M(X)$ in $\mathbf{DM}^{eff}(k,\mathbb{Z})$.

The construction of the abelian category $MM(k)$ is not yet known. Beilinson and Deligne have suggested that one can try to directly construct the triangulated category $\mathbf{DM}^{eff}(k,\mathbb{Z})$ with the expected properties, and then attempt to find a t-structure on $\mathbf{DM}^{eff}(k,\mathbb{Z})$ that recovers the abelian category $MM(k)$.

There are many constructions of such a triangulated category with realisation functors due to Hanamura, Levine, Voevodsky.

Voevodsky also managed to prove that Chow groups appear as cohomology groups (see Theorem 4.2.10) in $\mathbf{DM}^{eff}(k,\mathbb{Z})$. The proof works in the Nisnevich topology on Sm/k.

Also, most techniques of motivic cohomology theory for Sm/k have been developed for the Nisnevich topology.

Furthermore, Grothendieck's category of Chow motives embeds contravariantly into $\mathbf{DM}^{eff}(k, \mathbb{Z})$.

To reiterate, the Nisnevich topology is better suited for the study of motives for certain purposes. For example, to analyse Chow groups of smooth schemes from the motivic perspective, we need to use the Nisnevich topology as is evident from the comparison isomorphisms of Voevodsky (Theorem 4.2.10).

Motivic cohomology with $\mathbb{Q}$-coefficents, on the other hand, is insensitive to the choice of topology. That is, we get the same information whether we work with the Nisnevich or the étale topology on Sm/k. Hence, for discussing motives of stacks many authors work with $\mathbb{Q}$-coefficients.

However, working with $\mathbb{Q}$-coefficents leads to a loss of all torsion information in motivic cohomology, and therefore, only gives the non-torsion Chow groups for schemes and stacks. For example, current literature only relates motivic cohomology of a quotient stack with its rational intersection theory (see [Jos2] or [RS]). Thus, in order to obtain a more refined relation between Chow groups and motivic cohomology of stacks, having a theory of motives in the Nisnevich topology with integral coefficients is desirable. We will eventually relate the motivic cohomology of a quotient stack with its intersection theory integrally.

1.2 Results

The main thrust of this thesis is to show that a reasonable theory of motivic cohomology exists in Nisnevich topology for algebraic stacks.

The construction of Totaro in [Tot1] and its extension in [HL] does indeed work in the Nisnevich topology. However, it is restricted to stacks admitting filtrations by quotient substacks.

The construction of the motive of a Deligne-Mumford stack given in [Cho] for the étale topology can be carried out in the Nisnevich topology for any algebraic stack locally of finite type over a field k. This gives us a canonical definition of the motive of an algebraic stack which has many functorial properties. We will, henceforth, refer to it as the *Nisnevich motive*.

While this construction defines the motive of an algebraic stack in great generality, it is ill-suited for the purpose of computations. For example, it is not clear how to prove any of the standard exact triangles of motivic cohomology (see §4.2.1) for smooth stacks using this definition.

To be able to prove such exact triangles for stacks, we study algebraic stacks that are Nisnevich locally quotient stacks or what we will call *cd-quotient stacks* (Defintion 5.1.1).

The property of being a cd-quotient is enjoyed by a large class of algebraic stacks. In particular, this includes the class of Deligne-Mumford stacks and quotient stacks. Moreover, various result about the motives of smooth schemes also hold for smooth cd-quotient stacks. This is possible because we can construct a presentation of a cd-quotient stack in terms of certain simplicial schemes in the (unstable) $\mathbb{A}^1$-homotopy category $\mathcal{H}_\bullet(k)$. More precisely, we have the following:

Theorem 1.2.1. *Let $\mathcal{Y} \to \mathcal{X}$ be a representable Nisnevich cover of algebraic stacks over a field k. Assume that $\mathcal{Y}$ is of the form $[Y/GL_n]$ for some algebraic space Y, and let $p : Y \to \mathcal{Y} \to \mathcal{X}$ be the composite. Then for the Čech hypercover $Y_\bullet$ associated to p, $p_\bullet : Y_\bullet \to \mathcal{X}$ is a Nisnevich local weak equivalence, i.e, $Y_\bullet \simeq \mathcal{X}$ in $\mathcal{H}_\bullet(k)$.*

Recall that a morphism $f : X \to Y$ of schemes (or, more generally, algebraic spaces) is called a *Nisnevich covering* if it is étale, surjective, and for any $y \in Y$, there exists an $x \in f^{-1}(y)$ such that $k(x) = k(y)$.

There exists a natural functor $M : \mathcal{H}_\bullet(k) \to \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$ which assigns to any smooth scheme its motive in $\mathbf{DM}^{\mathit{eff}}(k, Z)$ (see Section 4.3). The above theorem, then, gives us a nice description of the motive of a cd-quotient stack in terms of the motive of the simplicial scheme $Y_\bullet$ in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$. This allows us to reduce many computations about motives to stacks to motives of (simplicial) schemes.

Theorem 1.2.1 allows us to conclude that many of the properties of motives of smooth schemes listed in §4.2.1 continue to hold for cd-quotient stacks. In particular, we establish Nisnevich descent (Proposition 5.3.3), projective bundle formula (Theorem 5.3.4), Blow-up triangle (Proposition 5.3.5) and Gysin triangle (Theorem 5.3.10) for cd-quotient stacks.

Furthermore, we will use Theorem 1.2.1 to show that for quotient stacks, the cohomology groups of the Nisnevich motive agree with the Chow groups of Edidin-Graham-Totaro (see [EG], [Kre]).

Theorem 1.2.2. *Let $\mathfrak{X} := [X/GL_r]$ be a quotient stack, where X is a smooth scheme with a smooth action of GL_r. Then the Edidin-Graham-Totaro (higher) Chow groups and the motivic cohomology groups agree integrally, i.e,*

$$CH^i(\mathfrak{X}, 2i-n) \simeq H^{n,i}(\mathfrak{X}, \mathbb{Z}).$$

A precursor of this result appears in [CDH], where X is taken to be smooth and quasi-projective over k.

The results in this thesis are either joint work with Amit Hogadi and Utsav Choudhury and appear in [CDH]; or are a culmination of various discussions on the subject with either of them.

1.3 Outline

This thesis is organised into two parts: in Part I, we collect all the preliminaries required to discuss the Nisnevich motive of an algebraic stack; Part II is dedicated to the study of the Nisnevich motive of an algebraic stack and its applications.

Chapter 2 is a primer on algebraic stacks and deals with the general theory of stacks.

In Chapter 3, we discuss the basics of Quillen's homotopical algebra which uses the theory of Model categories and their localisations. This culminates into the construction of the $\mathbb{A}^1$-homotopy category of Morel-Voevodsky, as well as Voevodsky's triangulated category of motives in Chapter 4.

All the material in Part I is taken from standard sources. Chapter-wise references are mentioned at the beginning of every chapter.

In Part II, we present our results on the Nisnevich motives of algebraic stacks and its applications in Chapters 5 and 6, respectively.

We first convince ourselves that cd-quotient stacks is indeed a sufficently large class of stacks in Section 5.1. Then, in Section 5.2, we define the Nisnevich motive for an algebraic stack following [Cho] and also prove the presentation theorem for cd-quotient stacks (Theorem 1.2.1). The argument involves two key ideas: that every principal

GL_n-bundle over a (Henselian) local ring is trivial, and that fibre products of stacks are a model for their homotopy fibre products in the model category of presheaves of groupoids (see [Hol1, Remark 2.3]).

In Section 5.3, we convince ourselves that various exact triangles for motives continue to hold for our construction. This section adapts the corresponding results for the étale topology in [Cho].

We then show, in Section 6.1, that for quotients of smooth schemes by GL_n, the integral motivic cohomology of an algebraic stack agrees with the Edidin-Graham-Totaro Chow groups (Theorem 1.2.2).

Finally, in Section 6.2, we compare our construction of the motive with the one in [HL]. We show that the two constructions agree in $\mathbf{DM}^{eff}(k, \mathbb{Z})$. A similar comparison is proved for étale motives in [HL, Appendix A].

Part I

Preliminaries

2

Algebraic Stacks

In this section we will review the theory of algebraic stacks. The standard references are [LMB], [Ols], and, of course, [Sta].

2.1 Schemes as functors

The construction of stacks requires a shift to a more category theoretic viewpoint. It is, therefore, instructive to first observe how the functorial point of view works in the case of schemes, i.e, we will describe the definition of a scheme in terms of functors (Definition 2.1.7). This exposition follows [Ols], although we take a slightly different approach.

Let F, G be two functors on a category $\mathcal{C}$ with values in *Sets*. Note that by Yoneda Lemma, an element $\xi \in G(U)$ is the same as a morphism of functors $\xi : h_U \to G$.

Definition 2.1.1. Let $\mathcal{C}$ be a category. Let $F, G : \mathcal{C}^{opp} \to Sets$ be functors. We say a morphism $a : F \to G$ is *representable*, or that F *is relatively representable over* G, if for every $U \in Ob(\mathcal{C})$ and any $\xi \in G(U)$ the functor $h_U \times_G F$ is representable. That is, there is a $W \in \mathcal{C}$ and a natural isomorphism $h_W \simeq h_U \times_G F$

Remark 2.1.2. *If $\mathcal{C}$ admits fibre products then any morphism $h_U \to h_V$ between representable functors is representable.*

We have the following lemma about representable morphisms.

Lemma 2.1.3. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

1. *the diagonal $\Delta : F \to F \times F$ is representable,*
2. *for every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable,*
3. *for every pair U, V in $\mathcal{C}$ and any $\xi \in F(U)$, $\xi' \in F(V)$ the fibre product $h_U \times_{\xi,F,\xi'} h_V$ is representable.*

Proof. see [Sta, Tag 0024]. □

Representable morphisms of functors is a useful notion because many properties of morphisms of schemes can be extended to representable morphisms.

Definition 2.1.4. Let $\mathcal{P}$ be a property of morphisms of schemes that is stable under base change. Let $a : F \to G$ be a representable morphism of functors. We say that a has the property $\mathcal{P}$, if for every scheme U and every $\xi \in G(U)$ the morphism of schemes $a_U : h_U \times_G F \to h_U$ has $\mathcal{P}$.

To illustrate what the above means consider the property that a morphism of schemes is affine. Then a morphism of functors $F \to G$ is said to be affine, if for any scheme U and any element $\xi \in G(U)$, the fibre product $F \times_G U$ is representable by a scheme, and the projection morphism $F \times_G U \to U$ is an affine morphism of schemes.

Many commonly used properties of morphisms of schemes are stable under base change. For example, immersions, proper morphisms, separated morphisms, Nisnevich coverings, étale morphisms, morphisms of finite presentation, etc.

We will often confuse U with its functor of points h_U.

We now describe the functors affine schemes which are schemes.

We will first define what schemes with affine diagonal are. Let Aff_{Zar} denote the category of affine schemes with the Zariski topology.

Definition 2.1.5. Let F be a functor on the category of affine schemes Aff. We say that F is a *scheme with affine diagonal* if the following conditions are satisfied:

1. F is a sheaf on $(\mathit{Aff})_{Zar}$.
2. The diagonal $\Delta : F \to F \times F$ is a representable.

3. There exist affine schemes $\{U_i\}_{i\in I}$ and morphisms $\xi_i : U_i \to F$ which are representable open immersions, such that the map from the disjoint union $\amalg_i U_i \to F$ is an epimorphism of sheaves in the Zariski topology.

By gluing the affine schemes U_i, it is easy to see that the functor F as in Defintion 2.1.5 does indeed define a scheme (in the usual sense) whose diagonal is affine.

Definition 2.1.6. We will say $a : F \to G$ is *representable by schemes with affine diagonal* if the fibre product $h_U \times_G F$ is a scheme with affine diagonal in the sense of Definition 2.1.5.

Replacing condition (2) in Definition 2.1.5 with the condition that diagonal is representable by schemes with affine diagonal gives us an alternate definition of a scheme.

Definition 2.1.7. Let F be a functor on the category of affine schemes *Aff*. We say that F is a *scheme* if the following conditions are satisfied:

1. F is a sheaf on $(Aff)_{Zar}$.

2. The diagonal $\Delta : F \to F \times F$ is a representable by separated schemes.

3. There exist affine schemes $\{U_i\}_{i\in I}$ and morphisms $\xi_i : U_i \to F$ which are representable open immersions, such that the map from the disjoint union $\amalg_i U_i \to F$ is an epimorphisms of sheaves in the Zariski topology.

Remark 2.1.8. *It is not possible to directly define schemes using representable morphisms of functor, since the intersection of affines need not be affine for a non-separated scheme. However, since the diagonal of scheme is always a monomorphism, it is representable by schemes with affine diagonal.*

Remark 2.1.9. *Such a functorial point of view, was first explored by Grothendieck and Artin for constructing quotients of étale equivalence relations of schemes. A formal thoery of such quotients appears in [Knu], where he uses the funcotrial formalism to define* algebraic spaces *as functors which are which are locally representable by schemes in the étale topology.*

2.2 Categories fibred in groupoids

We will now define the notion of categories fibred in groupoids {CFG}. Let *Sch* denote the category of schemes. The following diagram describes the interrelation between the various extensions of the category of schemes.

$$\begin{array}{ccccccc}
 & & \{\text{Algebraic Spaces}\} & \hookrightarrow & \{\text{Sheaves}\} & \hookrightarrow & \{\text{Presheaves}\} \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow \\
\{\text{Sch}\} & & & & & & \\
 & \searrow & & & & & \\
 & & \{\text{Algebraic Stacks}\} & \hookrightarrow & \{\text{Stacks}\} & \hookrightarrow & \{\text{CFG}\}
\end{array}$$

Definition 2.2.1. Let $p : \mathcal{S} \to Sch$ be a functor. We say that $\mathcal{S}$ is *fibred in groupoids* over *Sch* if the following two conditions hold:

1. For every morphism $f : V \to U$ in *Sch* and every lift x of U there is a lift $\phi : y \to x$ of f with target x.

2. For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

For any scheme U, we denote by $\mathcal{S}_U$ the cateogory of objects and morphisms lying over the identity morphism of U.

In plain English, the above conditions say that (1) pullbacks of objects exist and; (2) that they are unique upto unique isomorphism.

Example 2.2.2 (Presheaves as categories fibred in groupoids)**.** *Let* $F : Sch^{op} \to Sets$ *be a presheaf. We can define a category fibred in groupoid* $\mathcal{F}$ *associated to* F *as follows: objects of* $\mathcal{F}$ *are pairs,* (U, ξ) *where* U *is a scheme and* $\xi \in F(U)$*. Morphisms are given by* $f : (U, \xi) \to (V, \eta)$*, where* $f : U \to V$ *is morphism of schemes such that the induced map* $f^* : F(V) \to F(U)$ *sends* $\eta \mapsto \xi$*.*
Thus, any scheme can be thought of as a category fibred in groupoids via its functor of points.

Definition 2.2.3. Let $p : \mathcal{S} \to Sch$, $q : \mathcal{S}' \to Sch$ be categories fibred in groupoids. A morphism of categories fibred in groupoids is a functor $F : \mathcal{S} \to \mathcal{S}'$ such that $q \circ F = p$. Given two categories fibred in groupoids $\mathcal{S}, \mathcal{S}'$, we denote by $HOM(\mathcal{S}, \mathcal{S}')$ the category of morphisms from $\mathcal{S}$ to $\mathcal{S}'$. The objects of $HOM(\mathcal{S}, \mathcal{S}')$ are functors $\mathcal{S} \to \mathcal{S}'$ that are morphisms of fibred categories over Sch and morphisms are given by natural transformations of functors.

Note that, we have a version of Yoneda Lemma in this setup as well.

Lemma 2.2.4 (2-Yoneda lemma)**.** *Let $\mathcal{S} \to Sch$ be fibred in groupoids. Let $U \in Ob(Sch)$. The functor*

$$HOM(U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(id_U)$ is an equivalence.

Example 2.2.5. *We will now define the moduli of elliptic curves $\mathcal{M}_{1,1}$ as a category fibred in groupoids. For any scheme U, the fibre is defined as,*

$$Ob(\mathcal{M}_{1,1})_U := \{(E, \sigma) \mid \textit{elliptic curves over } U\}$$

and morphisms are given by morphisms over U. This defines a category fibred in groupods. Moreover, by 2-Yoneda it also has the universal property which we desire.
However, this is not good enough resolution. Any reasonable notion of moduli should have some sheaf theory. We will eventually show that $\mathcal{M}_{1,1}$ constructed above, indeed, has such a feature.

2.2.1 Fibre products

Fibre products of categories fibred in groupoids exists. We will define them in brief (for details see [Sta, Tag 0040]).

Let $F : \mathcal{X} \to \mathcal{Z}$ and $G : \mathcal{Y} \to \mathcal{Z}$ be morphisms of categories fibred in groupoids. Their fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is constructed as follows. For any scheme U, we define the category $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_U$ over it as,

$$Ob(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_U := \{(x, y, \phi) \mid \phi : F(x) \to G(y) \in \mathcal{Z}_U\}$$

and morphisms given by $(f, g) : (x_1, y_1, \phi_1) \to (x_2, y_2, \phi_2)$ such that $f_1 : x_1 \to x_2$ in $\mathcal{X}_U$ and $g : y_1 \to y_2$ in $\mathcal{Y}_U$ satisfying

$$\begin{array}{ccc} F(x_1) & \xrightarrow{F(f)} & F(x_2) \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ G(y_1) & \xrightarrow{G(g)} & G(y_2) \end{array}$$

2.2.2 Representable morphisms

Just as we defined representable morphisms of functors (Definition 2.1.1), we can define representable morphisms of categories fibred in groupoids in an analogous way.

Definition 2.2.6. A morphism $\mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids is said to be representable by schemes if for any scheme U and any object $x \in \mathcal{Y}_U$, the fibre product $\mathcal{X} \times_{\mathcal{Y},x} U$ is representable by a scheme.

Remark 2.2.7. *Instead of defining categories fibred in groupoids, one can instead consider functors to the category of groupoids such that pullbacks are defined upto equivalence. These are called lax-2-functors. the theory of lax-2-functors is equivalently to categories fibred in groupoids (see [FGI+, Part 1].*

2.3 Stacks

Just as sheaves are presheaves satisfying gluing axioms, stacks are categories fibred in groupoids satisfying certain gluing axioms. We will now work with the category $Sch_{ét}$ of scheme with the étale topology.

Definition 2.3.1. Let $p : \mathcal{X} \to Sch$ be a category fibred in groupoids. We say that $\mathcal{X}$ is a prestack if it satisfies the following

1. for all $U \in Ob(Sch)$, for all $x, y \in Ob(\mathcal{X}_U)$ the presheaf

$$Isom(x, y) : Sch/U \to Sets$$
$$V \to Isom(x_V, y_V)$$

is a sheaf on the site $(Sch/U)_{ét}$.

We say that a prestack is a stack if, additionally, the following condition holds

2. for all coverings $\mathcal{U} = \{U_i \to U\}$ in *Sch*, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

A descent datum (x_i, ϕ_{ij}) with respect to a covering $\{U_i \to U\}$ is a collection of objects $x_i \in \mathcal{X}_{U_i}$ together with isomorphisms $\phi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ji}}$ which satisfy the cocycle condition $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$. A descent datum is said to be effective if there exists an $x \in \mathcal{X}_U$ which pulls back to this descent datum.

Lemma 2.3.2. *Let F be a presheaf. Then F is a sheaf if and only if its the associated category fibred in groupoids $\mathcal{F}$ is a stack.*

Proof. Note that for $\mathcal{F}$, the $Isom$ presheaves define the condition of when two objects become equal, i.e, for any $V \in Sch/U$ and any $x, y \in \mathcal{F}_U$

$$Isom(x,y)(V) = \begin{cases} \emptyset & \text{if } x_V \neq y_V \\ 1 & \text{if } x_V = y_V \end{cases}$$

Furthermore, the descent condition is just that $x_i = x_j$ in $\mathcal{F}_{U_{ij}}$. From these two observation, the lemma is clear.

□

Just as any presheaf can be turned into a sheaf in the étale topology, similary any category fibred in groupoids can be turned into a stack. This is called *stackification* and it satisfies a universal property similar to the sheaf case.

Proposition 2.3.3 (Stackification)**.** *Let $\mathcal{X}$ be a prestack. There exists a stack $\widetilde{\mathcal{X}}$ and a 1-morphism*

$$\iota : \mathcal{X} \to \widetilde{\mathcal{X}}$$

such that for any stack $\mathcal{Y}$ there is an equivalence of categories

$$HOM(\widetilde{\mathcal{X}}, \mathcal{Y}) \to HOM(\mathcal{X}, \mathcal{Y})$$

Proof. See [LMB, Lemme 3.2] □

2.4 Algebraic stacks

In this section we will define algebraic stacks and describe examples of stacks.

Definition 2.4.1 (Deligne-Mumford stack)**.** Let $p : \mathcal{X} \to Sch$ be a category fibred in groupoids. We say that $\mathcal{X}$ is a *Deligne-Mumford stack* if it satisfies the following conditions

1. $\mathcal{X}$ is a stack in the étale toplogy.
2. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a representable, separated and quasi-compact.
3. There exists a scheme X and a morphism $X \to \mathcal{X}$ which is étale and surjective. The map $X \to \mathcal{X}$ is called a *presentation* of $\mathcal{X}$.

If $\mathcal{X}$ is a sheaf on *Sch*, then we say that $\mathcal{X}$ is an *algebraic space*.

Deligne-Mumford stacks are nice, and are as close to being schemes as possible in the presence of automorphisms. However, there are important examples which are not Deligne-Mumford stacks: for example, the moduli of vector bundles, or the classifying stack of G-bundles for an infinite group G. The following definition due to Artin, takes care of these examples:

Definition 2.4.2. Let $p : \mathcal{X} \to Sch$ be a category fibred in groupoids. We say that $\mathcal{X}$ is an *algebraic stack* (or *Artin stack*) if it satisfies the following conditions

1. $\mathcal{X}$ is a stack in the étale toplogy.
2. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a representable *by algebraic spaces*, separated and quasi-compact.
3. There exists an algebraic space X and a morphism $X \to \mathcal{X}$ which is smooth and surjective. The map $X \to \mathcal{X}$ is called a *smooth presentation* of $\mathcal{X}$.

Remark 2.4.3. *Consider the diagonal* $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$*. For any scheme* T*, a* T*-point* $(x_1, x_2) : T \to \mathcal{X} \times \mathcal{X}$ *corresponds to two objects* x_1, x_2 *over* T*. Then the fibre product* $T \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ *parametrises triples* $(f, y, \phi) \in (T \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X})_U$ *where* $y \in \mathcal{X}_U$ *and* $f : U \to T$ *is a morphism such that* $\phi : (y, y) \xrightarrow{\simeq} (f^*x_1, f^*x_2) \in (\mathcal{X} \times \mathcal{X})_U$*. This shows that* $T \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$

is, in fact, equivalent to the sheaf $Isom(x_1, x_2)$. Thus, for an algebraic stack, the $Isom$ sheaves are representable by algebraic spaces.

Example 2.4.4 (Inertia stack)**.** *For an algebraic stack $\mathcal{X}$, we define the inertia stack of $\mathcal{X}$ as $\mathcal{I}_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$. From the above remark, it is easy to see that $\mathcal{I}_{\mathcal{X}}$ parametrises the sheaves $Aut(x)$ for any object $x \in \mathcal{X}_T$.*

Lemma 2.4.5. *Let $\mathcal{X}$ be a Deligne-Mumford stack. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified.*

Proof. Since being unramified can be checked étale locally, we reduce the question to an étale cover $X \times X \to \mathcal{X} \times \mathcal{X}$. By base changing the diagonal, we have a morphism of algebraic spaces, $X \times_{\mathcal{X}} X \to X \times X$ such that the composite $X \times_{\mathcal{X}} X \to X \times X \to X$ is étale (therefore, unramfied). This implies that $X \times_{\mathcal{X}} X \to X \times X$ is unramified (for a composite of ring maps $A \to B \to C$, if C is unramified over A, then is also unramified over B). □

For an algebraic stack, a similar argument proves that the diagonal is of finite type.

The above result implies that for Deligne-Mumford stacks the automorphisms of geometric points are given by finite étale group schemes. Moreover, it turns out that this condition is also sufficient for an algebraic stack to be Deligne-Mumford.

Theorem 2.4.6. *Let $\mathcal{X}$ be an algebraic stack. Then $\mathcal{X}$ is Deligne-Mumford if and only if the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified.*

Remark 2.4.7. *In nice situation, algebraicity of a stack is almost automatic from the defining moduli problem. However, this is not always the case (for example, the Hilbert functor of a proper scheme). Proving representability of the diagonal and the existence of a smooth presentation can be quite hard in general. The standard technique for addressing these question involves the use of formal deformation theory developed by Artin (see [Art]).*

The following bootstrap result shows that for the diagonal to be representable by algebraic spaces, it is actually sufficient to find a smooth presentation of the stack by an algebraic space.

Lemma 2.4.8 ([Sta, Tag 05XW]). *Let S be a scheme. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks over (Sch/S). If*

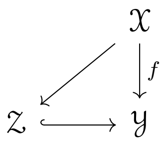
1. *$\mathcal{U}$ is representable by an algebraic space, and*

2. *u is representable by algebraic spaces, surjective, and smooth,*

then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ representable by algebraic spaces.

The following definitions will be used in Section 6.2.

Definition 2.4.9. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *scheme theoretic image* of f is the smallest closed substack $\mathcal{Z} \subset \mathcal{Y}$ such that we have a factorisation



The scheme theoretic image always exists (see [Sta, Tag 0CPU] for a proof).

Definition 2.4.10. Let $\mathcal{X}$ be an algebraic stack and $X \to \mathcal{X}$ be a smooth atlas. For a closed substack $\mathcal{Z} \subset \mathcal{X}$, we define the codimension of $\mathcal{Z}$ in $\mathcal{X}$ to be the codimension $\mathrm{codim}(\mathcal{Z} \times_{\mathcal{X}} X, X)$. Since codimension is preserved under flat maps (see [Gro2]), this notion is independent of the choice of atlas.

2.4.1 Chow groups of stacks

We now recall Totaro's construction of Chow groups for quotient stacks (see [EG, Section 2.2]). The definition is via a “Borel type construction".

Let X be an n-dimensional smooth algebraic space with an action of GL_r and let $[X/GL_r]$ be quotient of this action. Choose an l-dimensional representation V of GL_r such that V has an open subset U on which GL_r acts freely and whose complement has codimension greater than $n - i$. Then, we have a principal GL_r-bundle $X \times U \to (X \times U)/GL_r$, and we denote the quotient (which exists as an algebraic space) as $X_{GL_r} := (X \times U)/GL_r$.

Definition 2.4.11. With set-up as above, we define the i-th Chow group of $[X/GL_r]$ as

$$CH^i([X/GL_r]) := CH^i(X_{GL_r}).$$

This definition is independent of the choice of V and U so long as $\text{codim}(V, U) > i$ (see [EG, section 2.2] for details).

Using Bloch's cycle complex (see Definition 4.2.9), we can extend this definition to higher Chow groups in a similar manner.

2.4.2 Examples

We will now see examples of many different moduli problems that are algebraic stacks.

Example 2.4.12 (Moduli of Elliptic Curves)**.** *(See [Ols] for details) Consider the moduli stack of elliptic curves $\mathcal{M}_{1,1}$. Recall that for any scheme U the object over U are defined as*

$$Ob(\mathcal{M}_{1,1})_U := \left\{ (E, \sigma) \,\middle|\, \begin{array}{ll} 1. & f : E \to U \textit{ is a smooth proper genus one curve} \\ 2. & \sigma \textit{ is a section} \end{array} \right\}$$

and morphisms of elliptic curves are given by cartesian diagrams

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

First of all, we need to check that the stack condition holds. For this note that the sheaf $\mathcal{O}_E(3\sigma)$ is very ample. This follows from cohomology and base change, and the theory of elliptic curves over a field. Thus, we have an embedding $E \hookrightarrow \mathbb{P}(f_*\mathcal{O}_E(3\sigma))$. Since closed immersions descend, this shows that $\mathcal{M}_{1,1}$ is a stack.

To show that the diagonal is representable, consider $Isom((E, \sigma), (E', \sigma')) \to U$ for two elliptic curves (E, σ) and (E', σ') over U. Note that we have an inclusion

$$Isom_U(E, E') \subset Hilb_U(E \times_U E')$$

which sends any isomorphism to its graph, and a morphism is an isomorphism if and only if both the projections of the graph $\Gamma_f \subset E \times_U E'$ *onto* E *and* E' *are isomorphisms. Now, by the theory of Hilbert functors for projective scheme (see [FGI+, Part 2]),* $Hilb_U(E \times_U E')$ *is representable as a projective scheme. Since being an isomorphism is an open condition,* $Isom_U(E, E')$ *is an open subscheme of the Hilbert scheme* $Hilb(E \times_U E')$*. Similarly, the condition that* σ *maps to* σ' *is open. Thus, the diagonal of* $\mathcal{M}_{1,1}$ *is representable.*

To give a presentation of $\mathcal{M}_{1,1}$*, consider* $\mathbb{A}^5$ *in the coordinates* a_1, a_2, a_3, a_4, a_5*. Consider the equation*

$$E : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4X^2Z + a_6Z^3.$$

inverting the discriminant, we get an open set $U \subset \mathbb{A}^5$ *on which* $E_U \to U$ *is a family of elliptic curves. This gives us a map*

$$\pi : U \to \mathcal{M}_{1,1}.$$

This gives a smooth presentation of $\mathcal{M}_{1,1}$*.*

Example 2.4.13 (Moduli of Vector bundles)**.** *Let* X *be projective over a field* k*. Define* $\mathcal{V}_{X/k}$ *to be category fibred in groupoids whose fibre over any* k*-scheme* U *is given by*

$$(\mathcal{V}_{X/k})_U := \{ \text{ locally free sheaves } E \text{ of finite rank on } X \times_k U \ \}.$$

Note that the stack condition follows from descent for quasi-coherent sheaves. This an algebraic stack over k*, locally of finite presentation (see [LMB, Théoremè 4.6.2.1] for details).*

Example 2.4.14 (Quotient Stacks)**.** *Let* G *be an algebraic group. Let* X *be a scheme with an action of* G*. Define the stack* $[X/G]$ *which classifies principal* G*-bundles with an equivariant map to* X*, as follows*

$$[X/G]_V = \left\{ \begin{array}{l} P \xrightarrow{\phi} X \\ \downarrow p \\ V \end{array} \right\}$$

where $p : P \to V$ is a principal G-bundle and ϕ is a G-equivariant map. Note that this means that there exists an étale morphism $U \to V$ such that $P_U \simeq U \times G$.

$[X/G]$ is a stack in the étale topology. Further, we have a cartesian diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & [X/G] \end{array}$$

where σ is the action and pr_1 is the projection onto X.

It follows from descent that the map $\pi : X \to [X/G]$ is a representable flat, finitely presented and surjective. A theorem of Artin ([LMB, Théoremè 10.1]) then implies that $[X/G]$ admits a smooth presentation.

Example 2.4.15 (Classifying stacks)**.** *Consider the situation when $X = \operatorname{Spec} k$ with the trivial action of G in the above example. We use a special notation for this, viz., $[\operatorname{Spec} k/G] := BG$. It classifies all G-bundles. That is for any k-scheme U, the category $HOM(U, BG)$ is equivalent to the category of principal G-bundles over U.*

Example 2.4.16 (Stacky projective space)**.** *Let us consider one last example of a quotient stack. Consider the action of the multiplicative group $\mathbb{G}_m$ on $\mathbb{A}^1_k$ over a field k. Let $[\mathbb{A}^1/\mathbb{G}_m]$ be the assoicated quotient stack. We have a smooth presentation $\mathbb{A}^1 \to [A^1/\mathbb{G}_m]$. Note that $(\mathbb{A}^1 \setminus \{0\})/\mathbb{G}_m = \mathbb{P}^0$. Thus, we have an open embedding $\mathbb{P}^0 \subset [\mathbb{A}^1/\mathbb{G}_m]$. Also, since the origin is a fixed point of this action, its image is the closed substack $B\mathbb{G}_m \subset [\mathbb{A}^1/\mathbb{G}_m]$.*

Example 2.4.17 (Hilbert stack of points)**.** *Let $\mathfrak{X}$ be an algebraic stack which is locally of finite type over a field k. Let $\pi : V \to \mathfrak{X}$ be a finitely presented quasi-finite flat cover by a separated scheme V (This ensures that $\pi : V \to \mathfrak{X}$ is represented by schemes). We define the Hilbert stack of points $Hilb_{V/\mathfrak{X}}$ associated to $\pi : V \to \mathfrak{X}$ by the assignment for any $T \in Sch/\mathfrak{X}$,*

$$(Hilb_{V/\mathfrak{X}})_T := \{Z \subset V \times_{\mathfrak{X}} T \text{ such that } Z \to T \text{ is finite locally free}\}.$$

This comes with a projection map $p : Hilb_{V/\mathfrak{X}} \to \mathfrak{X}$. For any morphism $T \to \mathfrak{X}$ from a scheme T, consider the fibre product $Hilb_{V/\mathfrak{X}} \times_{\mathfrak{X}} T$. A U-point of this fibre product is a closed subscheme $Z \subset V \times_{\mathfrak{X}} U$ finite locally free over U and an isomorphism $V \times_{\mathfrak{X}} U \simeq (V \times_{\mathfrak{X}} T) \times_T U$. Thus, $Hilb_{V/\mathfrak{X}} \times_{\mathfrak{X}} T$ is the Hilbert functor of points $Hilb_{(V \times_{\mathfrak{X}} T)/T}$ associated *to the map* $V \times_{\mathcal{X}} T \to T$*. As,* $V \times_{\mathcal{X}} T \to T$ *is quasi-finite and finitely presented. Since this Hilbert functor is representable,* $p : Hilb_{V/\mathcal{X}} \to \mathcal{X}$ *is a representable morphism, and* $Hilb_{V/\mathcal{X}}$ *is an algebraic stack which is locally of finite presentation over* $\mathcal{X}$*.*

3

Homotopical Algebra

In this chapter, we recall the basics of Quillen's approach to abstract homotopy theory. Standard references include [Hov], [GJ], [Hir].

3.1 Model categories

3.1.1 Model structures

All categories are assumed to be locally small. Given a category $\mathcal{C}$ we can construct the arrow category $Arr(\mathcal{C})$ whose objects are arrows in $\mathcal{C}$ and morphisms are commutative squares.

Definition 3.1.1. Let $\mathcal{C}$ be a category.

1. A morphism f in $\mathcal{C}$ is said to be a *retract* of a map g in $\mathcal{C}$ if there is a commutative diagram of the form,

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

2. A *functorial factorisation* is an ordered pair (α, β) of functors $Arr(\mathcal{C}) \to Arr(\mathcal{C})$ such that $f = \beta(f) \circ \alpha(f)$ for all f in $Arr(\mathcal{C})$. That is, any $A \xrightarrow{f} B$ in $Arr(\mathcal{C})$ can be decomposed as $A \xrightarrow{\beta(f)} C \xrightarrow{\alpha(f)} B$, where C is some object in $\mathcal{C}$.

3. Let $i : A \to B$ and $p : X \to Y$ be morphism in $\mathcal{C}$. We say that i has *left lifting property with respect to* p and that p has *right lifting property with respect to* i if for every commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

there is a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Definition 3.1.2. A *model structure* on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, cofibration, and fibrations, and two functorial factorisations (α, β) and (γ, δ) satifying the following properties:

1. (2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g, and gf are weak equivalences, then so is the third.

2. (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

3. (Lifting) Define a map to be a *trivial cofibration* if it is both a cofibration and a weak equivalence. Similarly, define a map to be a *trivial fibration* if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations the left lifting property with respect to trivial fibrations.

4. (Factorisation) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

Essentially, (4) says that any morphism in $\mathcal{C}$ can be factorised as a cofibration followed by a trivial fibration, or a trivial cofibration followed by a fibration:

$$\begin{aligned} f &= \underset{\text{Trivial Fibration}}{\beta(f)} \circ \underset{\text{Cofibration}}{\alpha(f)} \\ &= \underset{\text{Fibration}}{\delta(f)} \circ \underset{\text{Trivial Cofibration}}{\gamma(f)} \end{aligned}$$

Definition 3.1.3. A category $\mathcal{C}$ with a model structure and in which all small limits and colimits exist is called a *model category*.

Lemma 3.1.4. *(The Retract Argument). Let f be a morphism in $\mathcal{C}$ such that $f = p \circ i$, and f has the left lifting property with respect p. Then, f is a retract of i.*

Proof. Since f has the left lifting property with respect to p, we have the following commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & B \\ \Big\downarrow{\scriptstyle f} & \overset{r}{\nearrow} & \Big\downarrow{\scriptstyle p} \\ C & = & C \end{array}$$

Then, the following diagram finishes the proof,

$$\begin{array}{ccccc} A & = & A & = & A \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle f} \\ C & \xrightarrow{\;r\;} & B & \xrightarrow{\;p\;} & C \end{array}$$

□

Lemma 3.1.5. *A map f is a cofibration (trivial cofibrations) if and only if f has the left lifting property with respect to trivial fibrations (fibrations).*

Proof. Clearly, cofibration have the left lifting property with respect to trivial fibrations. For the converse, let f be morphism which has the left lifting property with respect to trivial fibrations. Then, using the functorial factorisation, $f = \beta(f) \circ \alpha(f)$. Since, f has the left lifting property with respect to $\beta(f)$, by the retract argument f is a retract of

$\alpha(f)$ and hence, a cofibration.
The trivial cofibration part is proved similarly, using the (2-OUT-OF-3) property.
The fibrations and trivial fibrations parts are duals to the above arguments. □

Remark 3.1.6. *The above lemma implies that isomorphisms in $\mathcal{C}$ are trivial fibrations as well as trivial cofibrations. In particular, an isomorphism is a weak equivalence. Note that this is not apriori assumed in the definition of a model category.*

Corollary 3.1.7. *Let $\mathcal{C}$ be a model category. Cofibration (trivial cofibrations) are closed under pushouts. Dually, fibrations (trivial fibrations) are closed under pullbacks.*

Proof. Follows from the universal property of pushouts (pullbacks). □

Remark 3.1.8. *If $\mathcal{C}$ is a model category, it has both an initial object (the colimit of the empty diagram) and a final object (the limit of the empty diagram). An object A of $\mathcal{C}$ is called cofibrant if the map from the inital object 0 to it $0 \to A$ is a cofibration. Dually, an object B is called fibrant if the map to the final object $*$ from it $B \to *$ is a fibration.*

Moreover, if $0 \stackrel{f}{\to} B$ is any object in $\mathcal{C}$, we have a functorial factorisation $0 \stackrel{\alpha(f)}{\to} B' \stackrel{\beta(f)}{\to} B$, with $\alpha(f)$ a cofibration. Then B' is called the cofibrant replacement of B. The notion of a fibrant replacement is defined similarly by considering the map to final object and the functorial factorisation (γ, δ).

Lemma 3.1.9. *(Ken Brown's lemma). Suppose $\mathcal{C}$ is a model category and $\mathcal{D}$ is a category with weak equivalences (that satisfy the two out of three property). Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor which takes trivial cofibrations between cofibrant objects to weak equivalences. Then, F takes all weak equivalences between cofibrant objects to weak equivalences.*

Proof. Let $f : A \to B$ be a weak equivalence between cofibrant objects. We wish to show that $F(f)$ is also a weak equivalence.

Let $A \amalg B$ denote the pushout of A, B over the initial object. We have the map $(f, Id_B) : A \amalg B \to B$. Using the functorial factorisation, this can be factored as a cofibration q followed by a trivial fibration p. This gives us the following commutative diagram,

Note that i_1, i_2, being pushouts of cofibrations, are cofibrations themselves. By the 2-out-of-3 property, as Id_B, p are weak equivalences, so is qi_2. Similarly, qi_1 is a weak equivalence as f, p are weak equivalences. Thus, qi_1 and qi_2 are trivial cofibrations. So, $F(qi_1)$ and $F(qi_2)$ are weak equivalences. As, $F(pqi_2) = F(Id_B)$, we see that $F(p)$ is a weak equivalence. Hence, $F(f) = F(pqi_1)$ is also a weak equivalence □

Definition 3.1.10. Suppose $\mathcal{C}$ is a category with a subcategory of equivalences $\mathcal{W}$. We define the homotopy category $\operatorname{Ho}\mathcal{C}$ as follows. Form the free category $F(\mathcal{C}, \mathcal{W}^{-1})$ on the arrows of $\mathcal{C}$ and the reversals of arrows in $\mathcal{W}$. An object of $F(\mathcal{C}, \mathcal{W}^{-1})$ is an object of $\mathcal{C}$, and a morphism is a finite string of composable arrows $(f_1, f_2, \ldots, f_n)$, where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i of $\mathcal{W}$. The empty string at a particular object is the identity at that object, and composition is defined by concatenation of strings. Now, define $\operatorname{Ho}\mathcal{C}$ to be the quotient category of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations $1_A = (1_A)$ for all object A, $(f, g) = (g \circ f)$ for all composable arrows f, g of $\mathcal{C}$, and $1_{\mathrm{dom} w} = (w, w^{-1})$ and $1_{\mathrm{codom} w} = (w^{-1}, w)$ for all $w \in \mathcal{W}$.

Remark 3.1.11. *It is not clear from the above construction whether Ho $\mathcal{C}$ is locally small. In general, Ho $\mathcal{C}$ as a category will only make sense after passing to a higher universe. For a model category, however, we will show that Ho $\mathcal{C}$ is indeed an honest category in itself.*

Note that there is a functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ which is identity on objects and takes morphisms of $\mathcal{W}$ to isomorphisms.

The category $\operatorname{Ho}\mathcal{C}$ has the following universal property.

Lemma 3.1.12. *Let $\mathcal{C}$ be a category with a subcategory $\mathcal{W}$.*

1. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor that sends maps of $\mathcal{W}$ to isomorphisms, then there is a unique functor $\operatorname{Ho} F : \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ such that $(\operatorname{Ho} F) \circ \gamma = F$.*

2. *Suppose $\delta : \mathcal{C} \to \mathcal{E}$ is a functor that takes maps of $\mathcal{W}$ to isomorphisms and enjoys the universal property of part(i). Then there is a unique isomorphism $F : \operatorname{Ho}\mathcal{C} \to \mathcal{E}$ such that $F \circ \gamma = \delta$.*

3. *The correspondence of part (1) induces an isomorphism of categories between the category of functors $\operatorname{Ho}\mathcal{C} \to \mathcal{D}$ and natural transformations and the category of functors $\mathcal{C} \to \mathcal{D}$ that take maps of $\mathcal{W}$ to isomorphisms and natural transformations.*

Proof. The functor $\mathrm{Ho}\, F$ is defined to be F on objects and morphisms of $\mathcal{C}$, and $F(w^{-1}) := F(w)^{-1}$, for the reversal of any arrow in $\mathcal{W}$. The rest of conditions follow from standard category theory arguments (see [Hov] for details). □

Suppose $\mathcal{C}$ is a model category. We have the following three subcategories:

$$\begin{pmatrix} \mathcal{C}_c & = & \text{cofibrant objects of } \mathcal{C} \\ \mathcal{C}_f & = & \text{fibrant objects of } \mathcal{C} \\ \mathcal{C}_{cf} & = & \text{simultaneously cofibrant and fibrant objects of } \mathcal{C}. \end{pmatrix}$$

together with inclusion functors $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_c \hookrightarrow \mathcal{C}$ and $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_f \hookrightarrow \mathcal{C}$.

Lemma 3.1.13. *The inclusion functors* $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_c \hookrightarrow \mathcal{C}$ *and* $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_f \hookrightarrow \mathcal{C}$ *induce equivalences of categories,*

$$\begin{array}{ccccc} & & \mathrm{Ho}\,\mathcal{C}_c & & \\ & \nearrow & & \searrow & \\ \mathrm{Ho}\,\mathcal{C}_{cf} & & & & \mathrm{Ho}\,\mathcal{C} \\ & \searrow & & \nearrow & \\ & & \mathrm{Ho}\,\mathcal{C}_f & & \end{array}$$

Proof. Consider the inclusion functor $i_c : \mathcal{C}_c \hookrightarrow \mathcal{C}$. It certainly takes weak equivalences to weak equivalences. Thus, we have a functor $\mathrm{Ho}\, i_c : \mathrm{Ho}\,\mathcal{C}_c \to \mathrm{Ho}\,\mathcal{C}$. We will show that its inverse is given by the cofibrant replacement $Q : \mathcal{C} \to \mathcal{C}_c$.

Given a map $f : X \to Y$ in $\mathcal{C}$, we get map $Qf : QX \to QY$ between their cofibrant replacements. If $f : X \to Y$ is a weak equivalence, then so is the composite $QX \to X \to Y$ (since it is a trivial fibration followed by a weak equivalence). Thus, by (2-OUT-OF-3) property $Qf : QX \to QY$ is also a weak equivalence. So, we have a functor $\mathrm{Ho}\, Q : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}_c$. Further, the maps $q_X : QX \to X$ give rise to natural transformations $Q \circ i_{\mathcal{C}_c} \to 1_{\mathcal{C}_c}$ and $i_{\mathcal{C}_c} \circ Q \to 1_{\mathcal{C}}$, which induce isomorphisms on the homotopy categories (since the maps q_X are trivial fibrations).

The other cases are proved similarly. □

3.1.2 The homotopy category $\operatorname{Ho}\mathcal{C}_{cf}$

In this subsection, we will give an alternative description of the homotopy caegory $\operatorname{Ho}\mathcal{C}_{cf}$ via homotopy equivalence relations. The construction reminiscent of (and, in fact, generalises) the homotopy equivalnce relation in Topology. This will show that the homotopy category of a model cateogory defined in Definition 3.1.10 is actually a locally small category (see Theorem 3.1.19).

Definition 3.1.14. Let $\mathcal{C}$ be a model category, and $f, g : B \to X$ be two maps.

1. A *cylinder object* for B is a factorisation of the fold map $\nabla : B \amalg B \to B$ into a cofibration $B \amalg B \overset{i_0+i_1}{\to} B'$ followed by a weak equivalence $B' \overset{s}{\to} B$.

2. A *path object* for X is a factorisation of the diagonal map $\Delta : X \to X \times X$ into a weak equivalence $X \overset{r}{\to} X'$ followed by a fibration $X' \overset{p_0,p_1}{\to} X \times X$.

3. A *left homotopy* from f to g is a map $H : B' \to X$ for some cylinder object B' for B such that $Hi_0 = f$ and $Hi_1 = g$. We write $f \overset{l}{\sim} g$, if a left homotopy exists.

4. A *right homotopy* from f to g is a map $K : B \to X'$ for some path object X' for X such that $p_0K = f$ and $p_1K = g$. We write $f \overset{r}{\sim} g$, if a right homotopy exists.

5. We say that f and g are *homotopic*, and write $f \sim g$ is they are both left and right homotopic.

6. f is a *homotopy equivalence* if there is a map $h : X \to B$ such that $hf \sim 1_B$ and $fh \sim 1_X$.

Proposition 3.1.15. *Let $\mathcal{C}$ be a model category and $f, g : B \to X$ be two maps.*

1. *If $f \overset{l}{\sim} g$ and $h : X \to Y$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h : A \to B$, then $fh \overset{r}{\sim} gh$.*

2. *If X is fibrant, $f \overset{l}{\sim} g$, and $h : A \to B$, then $fh \overset{l}{\sim} gh$. Dually, if B is cofibrant, $f \overset{r}{\sim} g$, and $h : X \to Y$, then $hf \overset{r}{\sim} hg$.*

3. *If B is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(B, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation on $\mathcal{C}(B, X)$.*

4. *If B is cofibrant and $h : X \to Y$ is a trivial fibration or a weak equivalence of fibrant objects, then h induces an isomorphism,*

$$\mathcal{C}(B,X)/\overset{l}{\sim} \overset{\cong}{\to} \mathcal{C}(B,Y)/\overset{l}{\sim}.$$

Dually, if X is fibrant and $h : A \to B$ is a trivial cofibration or a weak equivalence of cofibrant objects, then h induces an isomorphism,

$$\mathcal{C}(B,X)/\overset{r}{\sim} \overset{\cong}{\to} \mathcal{C}(B,Y)/\overset{r}{\sim}.$$

5. *If B is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, if X' is any path object for X, then there is a right homotopy $K : B \to X'$ from f to g. Dually, if X is a fibrant object, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$, and there is a left homotopy from f to g using any cylinder object for B.*

Proof. See [Hov]. □

Corollary 3.1.16. *If $\mathcal{C}$ is a model category, B is a cofibrant object, and X is a fibrant object, then left homotopy and right homotopy relations coincide on $\mathcal{C}(B,X)$ and are equivalence relations on it.*

Proof. Follows from Proposition 3.1.15 above. □

Corollary 3.1.17. *The homotopy relation in $\mathcal{C}_{cf}$ is an equivalence relation. Hence the category $Ho(C_{cf})$ is locally small.*

Proof. Follows from Proposition 3.1.15 above. □

Proposition 3.1.18. *A map in $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. See [Hov]. □

Theorem 3.1.19. *Let $\mathcal{C}$ be a model category. Let $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ denote the canonical functor, Q denote the cofibrant replacement functor of $\mathcal{C}$ and R denote the fibrant replacement functor.*

1. *The inclusion* $\mathcal{C}_{cf} \to \mathcal{C}$ *induces an equivalence of categories*

$$\mathcal{C}_{cf}/\sim \xrightarrow{\cong} \operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}.$$

2. *There are natural isomorphisms*

$$\operatorname{Ho}\mathcal{C}(X,Y) \cong \mathcal{C}(QRX, QRY)/\sim \cong \mathcal{C}(QX, RY)/\sim .$$

3. *if* $f : A \to B$ *is a map in* $\mathcal{C}$ *such that* $\gamma(f)$ *is an isomorphism, then* f *is a weak equivalence.*

Proof. Part (1) follows from Lemma 3.1.13 and Proposition 3.1.18. Part (2) follows from Proposition 3.1.15

For part (3), take a map $f : A \to B$ such that $\gamma(f)$ is an isomorphism. Then QRf is an isomorphism in $\mathcal{C}_{cf}/\sim$. Thus, it is a homotopy equivalence in $\mathcal{C}_{cf}$, and so is a weak equivalence by Proposition 3.1.18. Since $QA \to A$ and $A \to RA$ are weak equivalence, we have that f is also a weak equivalence. □

3.1.3 Quillen functors, derived functors and Quillen equivalences

We will finish this section by defining the notions of derived functors in the model category setting.

Definition 3.1.20. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. $F : \mathcal{C} \to \mathcal{D}$ is called a *left Quillen functor* if F is a left adjoint and preserves trivial cofibrations and cofibrations.

2. $U : \mathcal{D} \to \mathcal{C}$ is called a *right Quillen functor* if U is a right adjoint and preserves trivial fibrations and fibrations.

3. Let (F, U, ϕ) be an adjunction from $\mathcal{C}$ to $\mathcal{D}$. It is called a *Quillen adjunction* if F is a left Quillen functor.

Definition 3.1.21. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. Let $F : \mathcal{C} \to \mathcal{D}$ be a left Quillen functor. The *total left derived functor* LF of F is the composite
$$\operatorname{Ho}\mathcal{C} \xrightarrow{\operatorname{Ho}Q} \operatorname{Ho}\mathcal{C}_c \xrightarrow{\operatorname{Ho}F} \operatorname{Ho}\mathcal{D}.$$

2. Similarly, the *the total right derived functor* RU of a right Quillen functor $U : \mathcal{D} \to \mathcal{C}$ is the composite
$$\operatorname{Ho}\mathcal{D} \xrightarrow{\operatorname{Ho}R} \operatorname{Ho}\mathcal{D}_f \xrightarrow{\operatorname{Ho}U} \operatorname{Ho}\mathcal{C}.$$

Definition 3.1.22. A Quillen adjunction $(F, U, \phi) : \mathcal{C} \to \mathcal{D}$ is called a *Quillen equivalence* if for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f : FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $\phi(f) : X \to UY$ is a weak equivalence in $\mathcal{C}$.

3.2 Simplicial sets

In this section we will recall some basics about simplicial sets. Standard reference is [GJ].

Definition 3.2.1. Let Δ be the category of finite ordered sets
$$[n] := \{0 < 1 < \ldots < n\}$$
and order-preserving maps between them.

For $0 \geq i \geq n$, we have the following maps in the category Δ

$$d^i : [n-1] \to [n], \qquad d^i(j) = \begin{cases} j & j < i \\ j+1 & i \geq j \end{cases}$$

$$s^i : [n+1] \to [n], \qquad s^i(j) = \begin{cases} j & j \leq i \\ j-1 & i < j \end{cases}$$

Every map in Δ can be factored uniquely as a composition of the maps d^i and s^i. They satisfy the following *cosimplicial identities*:

$$\begin{aligned}
d^j d^i &= d^i d^{j-1} && i < j \\
s^j d^i &= d^i s^{j-1} && i < j \\
s^j d^j &= 1 = s^j d^{j+1} && \\
s^j d^i &= d^{i-1} s^j && i > j+1 \\
s^j s^i &= s^i s^{j+1} && i \leq j
\end{aligned}$$

Definition 3.2.2. A simplicial set is a contravariant functor $X : \Delta^{op} \to Sets$. Thus, a simplicial set may be thought of as a collection of sets X_n, together with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the following *simplicial identities*:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && i < j \\
d_i s_j &= s_{j-1} d_i && i < j \\
d_j s_j &= 1 = d_{j+1} s_j && \\
d_i s_j &= s_j d_{i-1} && i > j+1 \\
s_i s_j &= s_{j+1} s_i && i \leq j
\end{aligned} \quad .$$

For each $[n] \in \Delta$, the corresponding set X_n is called the set of n-simplices of X. The maps d_i and s_i are called the face and degeneracy maps, respectively.
We denote the category of simplicial sets as $SSets$.

A morphism of simplicial sets $f : X \to Y$ is a natural transformation of $Sets$-valued contravariant functors on Δ^{op}. For every $[n] \in \Delta$, we have a contravariant functor $\Delta^n : \Delta^{op} \to Sets$, defined as $\Delta^n := Hom_\Delta(-, [n])$. This is called as the standard n-simplex. By Yoneda Lemma, we have a natural bijection for any simplicial set Y,

$$Hom(\Delta^n, Y) \cong Y_n$$

between the set of simplical maps $\Delta^n \to Y$ and the set of n-simplices of Y.

Given a simplicial sets Y, we can construct a simplicial abelian group $\mathbb{Z}Y$, whose set of n-simplices $\mathbb{Z}Y_n$ is given by the free abelian groups on Y_n. We can associated to $\mathbb{Z}Y$, a chain complex

$$\mathbb{Z}Y_0 \xleftarrow{\partial} \mathbb{Z}Y_1 \xleftarrow{\partial} \mathbb{Z}Y_2 \ldots,$$

where $\partial := \sum_{i=0}^{n} (-1)^i d_i$ in degree n. This is called the *Moore complex* of the simplcial set Y.

3.2.1 Geometric realisation

We have a functor

$$\begin{aligned} |-|:\Delta &\to \mathbf{K} \\ [n] &\mapsto |\Delta^n| \end{aligned}$$

from the category of finite ordered set to the category of compact Hausdorf spaces $\mathbf{K}$. Here, $|\Delta^n|$ is the standard n-simplex in $\mathbb{R}^{n+1}$ defined by

$$|\Delta^n| := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}.$$

This is called the geometric realisation functor.

Since, any simplicial set can be constructed as a colimit of the standard n-simplices, we can extend the geometric realisation functor to the category of simplicial set by defining for any $X \in SSets$,

$$|X| := \operatorname*{colim}_{\Delta^n \to X} |\Delta^n|.$$

The functor $|-| : SSets \to \mathbf{K}$, thus constructed admits a right adjoint $S : \mathbf{K} \to SSets$ which sends any compact Hausdorf space Y to the simplicial set $S(Y)$ whose n-simplices can be described as,

$$S(Y)_n := Hom_{\mathbf{K}}(|\Delta^n|, Y)$$

Note that by definition we have

$$Hom_{SSets}(\Delta^n, S(Y)) \simeq S(Y)_n := Hom_{\mathbf{K}}(|\Delta^n|, Y).$$

The functor S is called the singular functor and the simplicial set $S(Y)$ computes the singular homology of Y.

Proposition 3.2.3. *The functor $S : \mathbf{K} \to SSets$ is a right adjoint of $|-|$.*

Proof. This follows from the fact that Hom commutes with colimits. Let X be a simplicial set. Then we have a series of tautological identifications,

$$\begin{aligned} Hom_{\mathbf{K}}(|X|, Y) &\simeq Hom_{\mathbf{K}}(\operatorname*{colim}_{\Delta^n \to X} |\Delta^n|, Y) \\ &\simeq \lim_{\Delta^n \to X} Hom_{\mathbf{K}}(|\Delta^n|, Y) \\ &\simeq \lim_{\Delta^n \to X} Hom_{SSets}(\Delta^n, S(Y)) \\ &\simeq Hom_{SSets}(\operatorname*{colim}_{\Delta^n \to X} \Delta^n, S(Y) \\ &\simeq Hom_{SSets}(X, S(Y)). \end{aligned}$$

□

Definition 3.2.4. A map $f : X \to Y$ of simplicial sets is said to be a *weak equivalence* if its geometric realisation $|f| : |X| \to |Y|$ is a weak equivalence of topological spaces.

Remark 3.2.5 (Model structure on $SSets$)**.** *The category of simplicial sets admits a model structure where weak equivalences are as defined above, cofibrations are monomorphisms, and fibrations have the right lifting property with respect to trivial cofibrations. Fibrations of simplicial sets are called* Kan fibrations, *in honour of Daniel Kan who pioneered the subject.*

3.3 Bousfield localisation

Given a model category $\mathcal{C}$, it is often desirable to increase the class of weak equivalences in such a way that the resulting structure is still a model category. Such problems were first explored in the work of Bousfield, and hence the technique is now referred to as Bousfield localisation. We will only deal with Bousfield localisation in the context of simplicial model categories. Standard reference for this section is [Hir].

The starting point of our discussion is a *category enriched over simplicial sets* (see [Hir, Definition 9.1.2]).

Definition 3.3.1. Let $\mathcal{C}$ be a category enriched over simplicial sets. We say that $\mathcal{C}$ is a *simplicial model category* if it is a model category satisfying the following:

1. For any object X, Y and any simplicial set K there exist objects $X \otimes K$ and Y^K

in $\mathcal{C}$ with natural isomorphisms of simplicial sets

$$Map(X \otimes K, Y) \simeq Map(K, Map(X, Y)) \simeq Map(X, Y^K).$$

2. Let $i : A \to B$ be a cofibration and $p : X \to Y$ be a fibration. Then the induced map of simplicial sets

$$Map(B, X) \to Map(A, X) \times_{Map(A,Y)} Map(B, Y)$$

is a trivial fibration if either i or p is a weak equivalence.

Definition 3.3.2 (I-local objects and equivalences)**.** Let $\mathcal{C}$ be a simplicial model category and I be a set of maps.

1. An object W in $\mathcal{C}$ is said to be *I-local* if it is fibrant and if for any map $A \to B$ in $\mathcal{C}$, the induced map on mapping spaces

$$Map(B, W) \to Map(A, W)$$

is a weak equivalence of simplicial sets.

2. A map $A \to B$ in $\mathcal{C}$ is said to be an *I-local equivalence* if for every I-local object W, the induced map on mapping spaces

$$Map(B, W) \to Map(A, W)$$

is a weak equivalence of simplicial sets.

Remark 3.3.3. *It is possible to define I-local objects and equivalences for any model category. One just needs a good notion of mapping spaces for this. For example, given objects X and Y, one can take a cosimplicial resolution $X'_\bullet \to X$ and a fibrant replacement Y' of Y in the category of simplcial diagrams $\mathcal{C}^\Delta$ with the Reedy model structure. Now the simplicial set given by $Map(X'_n, Y')$ serves as a mapping space for X and Y.*
The details of this construction will require us to take a detour into cosimplicial resolutions and Reedy model structures. As most model categories we will encounter will be enriched over simplicial sets, we do not pursue this here. The details can be found in [Hir] or [DK].

Definition 3.3.4. Let $\mathcal{C}$ be a model category and I be a set of maps in $\mathcal{C}$. The *left Bousfield localisation* of $\mathcal{C}$ with respect to I is a model category structure $L_I\mathcal{C}$ on the underlying category of $\mathcal{C}$ such that

1. Weak equivalences in $L_I\mathcal{C}$ are the same as I-local equivalences in $\mathcal{C}$.
2. Cofibrations in $L_I\mathcal{C}$ are the same as cofibrations in $\mathcal{C}$.
3. Fibrations in $L_I\mathcal{C}$ are maps which have the right lifting property with respect to cofibrations and I-local equivalences.

Left Bousfield localisation is a technique to enlarge the class of weak equivalences while preserving the cofibrations. In similar spirit, we can define a *right Bousfield localisation* which changes the weak equivalences while preserving the fibrations. We will use this in Chapter 4 to invert $\mathbb{A}^1$.

In general, a Bousfield localisation may not exist. However, when the model category at hand is sufficiently well-behaved, we have the following result:

Theorem 3.3.5. *Let $\mathcal{C}$ be a left proper cellular model category and I be a set of maps. Then the left Bousfield localisation $L_I\mathcal{C}$ of $\mathcal{C}$ exists as a left proper cellular model category.*

Proof. See [Hir] for details. □

3.4 Homotopy (co)limits and derived functors

In this section, we explore the relation between homotopy (co)limits and derived functors of Quillen functors.

Let I be a small category, and let M^I denote the category of I-diagrams in M, i.e, the category of functors $Fun(I, M)$. We have a constant functor $c : M \to M^I$, taking every object A of M to the constant I-diagram whose every object is A and all maps are id_A. The left adjoint of c (if it exists) is the colimit functor colim $: M^I \to M$.
Let $F : M \rightleftarrows N : G$ be an adjunction. Then we also have an induced adjunction between the category of I-diagrams,

$$F^I : M^I \rightleftarrows N^I : G^I.$$

The following lemma shows that just as left adjoints commute with colimits, derived functors of left adjoints commute with *homotopy colimits*.

Lemma 3.4.1. *Let $F : M \rightleftarrows N : G$ be a Quillen adjunction of model categories. Let I be an index category such that the projective model structure is defined on M^I and N^I. Then, for an I-diagram E in M, we have*

$$LF(\operatorname{hocolim} E) \simeq \operatorname{hocolim} LF^I(E),$$

where LF and LF^I are the derived functors of F and F^I, repectively.

Remark 3.4.2. *The projective model structure on M^I is given by defining fibrations and weak equivalences objectwise, and cofibration are maps that have the left lifting property with respect to trivial fibrations. The injective model structure is defined dually. Note that, in general, injective and projective model structures need not exist.*

Proof of Lemma 3.4.1. Note that LF is defined as the composite

$$Ho(M) \overset{Q}{\to} Ho(M_c) \overset{F}{\to} Ho(N)$$

where Q is the cofibrant replacement functor.

Let M^I denote the category of I-diagrams in M. Note that the fibrations and weak equivalences in the projective model structure on M^I are defined object-wise. The homotopy colimit functor is the derived functor of the colimit functor $\operatorname{colim} : M^I \to M$ which is the left adjoint of the constant functor $M \to M^I$. More precisely,

$$\operatorname{hocolim} : Ho(M^I) \overset{Q}{\to} Ho(M^I) \xrightarrow{\operatorname{colim}} Ho(M).$$

Note that here Q is the cofibrant replacement functor in the *projective model structure* of M^I. This says that for any I-diagram E the homotopy colimit can be computed by taking the ordinary colimit of its cofibrant replacement QE, i.e,

$$\operatorname{hocolim} E \simeq \operatorname{colim} QE$$

is a weak equivalence in the homotopy category. Since, M^I has the projective model structure, colim is left Quillen. Thus, $\operatorname{colim} QE$ is, in fact, a cofibrant object in M and

$$\begin{aligned} LF(\operatorname{hocolim} E) &\simeq LF(\operatorname{colim} QE) \\ &\simeq F(\operatorname{colim} QE). \end{aligned}$$

Now, observe that the adjoint pair (F, G) induces an adjunction on the diagram categories associated to I,

$$F^I : M^I \rightleftarrows N^I : G^I.$$

As fibrations are defined object-wise and G is right Quillen, G^I preserves fibrations. Hence, F^I is left Quillen, and preserves cofibrations. This means that the image $F^I(QE)$ is cofibrant in N^I. Then, we have

$$\begin{aligned} \operatorname{hocolim} LF^I(E) &\simeq \operatorname{hocolim} F^I(QE) \\ &\simeq \operatorname{colim} F^I(QE). \end{aligned}$$

As F is a left adjoint, it commutes with ordinary colimits. That is, we have a commutative diagram,

$$\begin{array}{ccc} M^I & \xrightarrow{F^I} & N^I \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array}$$

where the vertical arrows are the colimit functor. Thus,

$$F(\operatorname{colim} QE) = \operatorname{colim} F^I(QE),$$

as required. □

Lemma 3.4.3. *Let $F : M \rightleftarrows N : G$ be a Quillen adjunction of model categories. Let I be an index category such that the injective model structure is defined on M^I and N^I. Then, for an I-diagram E in M, we have*

$$RG(\operatorname{holim} E) \simeq \operatorname{holim} RG^I(E),$$

where RG and RG^I are the derived functors of G and G^I, repectively.

Proof. Dual of the proof of Lemma 3.4.1. □

4

The Triangulated Category of Motives

In this chapter, we will describe the homotopy theory of smooth schemes over a field. In particular, we will recall the notions of the (unstable) $\mathbb{A}^1$-homotopy category over a field, as well as the construction of Voevodsky's triangulated category of motives over a field. The standard references are [MV], [VSF], [MVW].

Fix a base field k. Let Sm/k denote the category of smooth separated schemes over k.

We will denote by $PSh(Sm/k)$ the category of presheaves on Sm/k.

4.1 The $\mathbb{A}^1$-homotopy category

Definition 4.1.1. A simplicial presheaf on Sm/k is a contravariant functor $X : (Sm/k)^{op} \to SSets$. A morphism $f : X \to Y$ of simplicial presheaves is simply a natural transformation of $SSets$-valued functors on Sm/k.
We denote by $\Delta^{op}PSh(Sm/k)$ be the category of simplicial presheaves on Sm/k.

Remark 4.1.2. *Just as for simplicial sets, a simplicial presheaf on Sm/k may be thought of as a collection of presheaves on Sm/k together with face and degeneracy maps. Alternatively, a simplicial presheaf may also be considered as a functor $X : \Delta^{op} \to PSh(Sm/k)$, i.e, a* simplicial object *in $PSh(Sm/k)$.*

$\Delta^{op}PSh(Sm/k)$ has a local model structure with respect to the Nisnevich topology (see [Jar]). A morphism $f : X \to Y$ in $\Delta^{op}PSh(Sm/k)$ is a weak equivalence if the induced morphisms on stalks (for the Nisnevich topology) are weak equivalences of simplicial sets. Cofibrations are monomorphisms, and fibrations are characterised by the right lifting property. The resulting homotopy category will be denoted as $\mathcal{H}_{Nis}(k)$.

Consider the class of maps $X \times \mathbb{A}^1 \to X$. The Bousfield localisation with respect to this class of maps exists and the resulting model structure is called the Nisnevich motivic model structure. We denote the resulting homotopy category by $\mathcal{H}_\bullet(k)$. This is the (unstable) $\mathbb{A}^1$-homotopy category for smooth schemes over k. See [MV] for details.

4.2 Triangulated category of motives

Definition 4.2.1. Let X, Y be smooth separated schemes over a field k. Assume that X is connected. An *elementary correspondence* from X to Y is a reduced and irreducible closed subscheme $W \subset X \times Y$ that is finite and surjective over X.
For X non-connected, an elementary correspondence from X to Y is an elemenatary correspondence from a connected compenent of X to Y.

Definition 4.2.2. Let $Cor(X, Y)$ denote the free abelian group generated by elementary correspondences from X to Y. Elements of $Cor(X, Y)$ are called *finite correspondences.*

Using basic facts about algebraic cycles it is easy to show that finite correspondences behave well with respect to composition.

Definition 4.2.3. Let Cor_k denote the additive category of finite correspondences over a field k. Objects of Cor_k are smooth separated schemes over k, while morphisms are given by finite correspondences from X to Y.

Definition 4.2.4. An additive functor $F : Cor_k^{op} \to \mathbf{Ab}$ is called a presheaf with transfers. Here, **Ab** denotes the category of abelian groups. we will denote by $PST(k, \mathbb{Z})$ the category presheaves with transfers.

For any smooth scheme X, let $\mathbb{Z}_{tr}(X)$ denote the presheaf with tranfers which on any smooth scheme Y is defined as

$$\mathbb{Z}_{tr}(X)(Y) := Cor(X, Y)$$

Definition 4.2.5. Let $\{(X_i, x_i)\}_{i=1}^n$ be a collection of pointed schemes. We define $\mathbb{Z}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ or simply $\mathbb{Z}_{tr}(X_1 \wedge \ldots \wedge X_n)$ as the cokernel

$$\operatorname{coker}\left(\oplus_i \mathbb{Z}_{tr}(X_1 \times \ldots \times \hat{X}_i \times \ldots \times X_n) \xrightarrow{id \times \ldots \times x \times \ldots \times id} \mathbb{Z}_{tr}(X_1 \times \ldots \times X_n)\right)$$

For any $n \geq 0$, consider the affine schemes defined by

$$\Delta^n := \operatorname{Spec} k[x_0, \ldots, x_n] / \left(\sum x_i = 1\right).$$

These schemes are the algebraic analogues of the standard n-simplices in topology. We have obvious face maps $\partial_i : \Delta^n \to \Delta^{n+1}$ given by setting $x_i = 0$. Thus, we get a cosimplicial object $\Delta^\bullet$ in the category Sm/k.

Definition 4.2.6. Let F be a presheaf with transfers on Sm/k. We will write $C_\bullet F$ for the simplical presheaf with transfers given by the assignment

$$U \mapsto F(U \times \Delta^\bullet).$$

Taking the associated Moore complex we get a chain complex of abelian groups associated to F which we will denote by $C_* F$.

Definition 4.2.7 (Motivic complexes)**.** We define the motivic complexes $\mathbb{Z}(q)$ as

$$\mathbb{Z}(q) := C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]$$

We consider $\mathbb{Z}(q)$ to be a cochain complex. From this point on, we will deal with cochain complexes.

We will now explain the contruction of the triangulated category of motives. Voevodsky's original contruction is via the classical theory of derived categories due to Grothendieck-Verdier (for details, see [MVW]). We will use the language of model categories for this construction, the details of which can be found in [CD].

Let $K(PST(k, \mathbb{Z}))$ denote the category of complexes of presheaves with transfers. The category $K(PST(k, \mathbb{Z}))$ also has Nisnevich local and motivic model structures which are defined analogously as in the case of $\Delta^{op} PSh(Sm/k)$. A morphism $K \to L$ in $K(PST(k, \mathbb{Z}))$ is a Nisnevich local weak equivalence if it induces a quasi-isomorphism on the stalks in the Nisnevich topology. Cofibrations are monomorphisms and fibrations are defined by the right lifting property. The associated homotopy category is just the derived category $\mathbf{D}(PST(k,\mathbb{Z}))$.

The motivic model structure is the left Bousfield localisation of the local model structure with respect to the maps

$$\mathbb{Z}_{tr}(X \times \mathbb{A}^1)[n] \to \mathbb{Z}_{tr}(X)[n],$$

for any $X \in Sm/k$ and any $n \in \mathbb{Z}$. The resulting homotopy category is denoted by $\mathbf{DM}^{eff}(k,\mathbb{Z})$. This is Voevodsky's triangulated category of mixed motives in the Nisnevich topology.

For a smooth scheme X, the image of $\mathbb{Z}_{tr}(X)$ in $\mathbf{DM}^{eff}(k,\mathbb{Z})$ is defined to be the motive $M(X)$ of X. Note that, by construction $C_*\mathbb{Z}_{tr}(X) \simeq \mathbb{Z}_{tr}(X)$ in $\mathbf{DM}^{eff}(k,\mathbb{Z})$

4.2.1 Properties of $\mathbf{DM}^{eff}(k,\mathbb{Z})$

We will now list some important properties of $\mathbf{DM}^{eff}(k,\mathbb{Z})$.

Motivic cohomology satisfies the following properties (see [MVW] for details):

1. (Künneth isomorphism) $M(X) \otimes M(Y) \cong M(X \times Y)$;

2. ($\mathbb{A}^1$-invariance) $M(X \times \mathbb{A}^1) \cong M(X)$;

3. (Mayer-Vietoris) For an open cover $X = U \cup V$ of a smooth scheme X, we have an exact triangle in $\mathbf{DM}^{eff}(k,\mathbb{Z})$,
$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1];$$

4. If $E \to X$ is a vector bundle, then $M(E) \cong M(X)$;

5. (Projective bundle formula) If $\mathbb{P}(\mathcal{E}) \to X$ is a projective bundle associated to a locally free sheaf $\mathcal{E}$ of rank $n+1$, we have a canonical isomorphism:
$$M(\mathbb{P}(\mathcal{E})) \cong \bigoplus_{i=0}^{n} \left(M(X) \otimes \mathbb{Z}(i)[2i]\right);$$

6. (Blow-up triangle) Let X be a smooth scheme and $Z \subset X$ be a smooth closed subscheme of pure codimension c. Let $X' \to X$ be the blow-up of X in Z and let Z' denote the exceptional divisor over Z. Then we have an exact triangle

$$M(Z') \to M(X') \oplus M(Z) \to M(X) \to M(Z')[1].$$

In fact, this triangle splits and we have

$$M(X') \simeq M(X) \oplus \left(\oplus_{i=1}^{c-1} M(Z)(i)[2i]\right).$$

7. (Gysin triangle) Let X be a smooth scheme with a smooth closed subscheme Z of codimension c. Then we have an exact triangle

$$M(X \setminus Z) \to M(X) \to M(Z)(c)[2c] \to M(X \setminus Z)[1].$$

In the sequel, we will generalise some of these properties to cd-quotient stacks.

Definition 4.2.8 (Motivic cohomology)**.** The motivic cohomology of a smooth scheme X over k is defined as

$$H^{p,q}(X, \mathbb{Z}) := Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}(M(X), \mathbb{Z}(q)[p])$$

4.2.2 Comparison with higher Chow groups

Higher Chow groups were first defined by Bloch in [Blo] to study the relation between algebraic cycles and algebraic K-theory.

Definition 4.2.9. Let X be an equidimensional scheme. We write $z^i(X, m)$ be the free abelian group generated by all codimension i subvarieties of $X \times \Delta^m$ which intersect all faces $X \times \Delta^j$ properly for all $j \leq m$.
This defines a chain complex of abelian groups $z^i(X, *)$.
The *higher Chow groups* of X are defined as the cohomologies of this complex

$$CH^i(X, m) := H_m(z^i(X, *)).$$

The complexes are called Bloch's cycle complexes for higher Chow groups.
Note that $CH^i(X, 0) = CH^i(X)$.

The following deep theorem of Voevodsky gives an isomorphism between motivic cohomology and higher Chow groups (See [VSF] or [MVW] for a proof).

Theorem 4.2.10 (Voevodsky)**.** *Let X be a smooth scheme over a perfect field. We have natural isomorphisms between motivic cohomology and higher Chow groups,*

$$H^{n,i}(X, \mathbb{Z}) \cong CH^i(X, 2i - n)$$

4.2.3 Motivic cohomology as hypercohomology

Let X be a smooth scheme over a perfect field k. For each motivic complex $\mathbb{Z}(q)$, we can consider its hypercohomology $\mathbb{H}^p_{Zar}(X, \mathbb{Z}(q))$ in the category of complexes of abelian sheaves on X in the Zariski topology. Similarly, we can compute hypercohomolgy of $\mathbb{Z}(q)$ in the Nisnevich topology, denoted by $\mathbb{H}^p_{Nis}(X, \mathbb{Z}(q))$.

The following result of Voevodsky shows that these hypercohomology groups compute the motivic cohomology of X.

Theorem 4.2.11. *Let X be a smooth scheme over a perfect field k. Then we have canonical isomorphism,*

$$H^{p,q}(X, \mathbb{Z}) \simeq \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q)) \simeq \mathbb{H}^p_{Nis}(X, \mathbb{Z}(q))$$

Proof. See [MVW, Proposition 14.16]. □

4.3 The motive functor

The motive of a smooth scheme defined in the previous section in fact has a more functorial description using the $\mathbb{A}^1$-homotopy category. We will describe it now.

We have a functor $\mathbb{Z}_{tr}(-) : Sm/k \to PST(k, \mathbb{Z})$ which sends any scheme to its associated presheaf with transfers. This can be extended to a functor

$$\mathbb{Z}_{tr}(-) : Psh(Sm/k) \to PST(k, \mathbb{Z})$$

by defining $\mathbb{Z}_{tr}(F) := \operatorname{colim}_{X \to F} \mathbb{Z}_{tr}(X)$. We can extend this further to a functor

$$N\mathbb{Z}_{tr}(-) : \Delta^{op} PSh(Sm/k) \to K(PST(k, \mathbb{Z}))$$

from simplicial presheaves to chain complexes of presheaves with transfers, which sends a simplicial scheme $X_\bullet$ to its normalised chain complex $N\mathbb{Z}_{tr}(X_\bullet)$. The i-th degree term of the chain complex $N\mathbb{Z}_{tr}(X_\bullet)$ is given by $\mathbb{Z}_{tr}(X_i)$. Since every simplicial presheaf is weakly equivalent to a simplicial scheme (see [DHI]), this determines the derived functor of $N\mathbb{Z}_{tr}(-)$ completely. In fact, $N\mathbb{Z}_{tr}(-)$ is a left Quillen functor.

Proposition 4.3.1. *The functor $N\mathbb{Z}_{tr}(-) : \Delta^{op} PSh(Sm/k) \to K(PST(k, \mathbb{Z}))$ is left Quillen. Thus, it admits a left derived functor M on the homotopy categories*

$$M : \mathcal{H}_{Nis}(k) \to \mathbf{D}(PST(k, \mathbb{Z})).$$

Proof. The proof is standard. See, for instance, [Cho, Proposition 2.2]. □

Remark 4.3.2. *Composing $M : \mathcal{H}_{Nis}(k) \to \mathbf{D}(PST(k, \mathbb{Z}))$ with the canonical map $\mathbf{D}(PST(k, \mathbb{Z})) \to \mathbf{DM}^{eff}(k, \mathbb{Z})$ gives us a functor into the triangulated category of motives. In fact, [Cho, Proposition 2.2] shows that M also respects $\mathbb{A}^1$-localisation. That is, it also induces a functor*

$$M : \mathcal{H}_\bullet(k) \to \mathbf{DM}^{eff}(k, \mathbb{Z}).$$

from the $\mathbb{A}^1$-homotopy category to the triangulated category of motives.

Remark 4.3.3. *A smooth scheme X can be described as a constant simplicial object in $\Delta^{op} PSh(Sm/k)$. The functor M then takes X to the chain complex*

$$\ldots \to \mathbb{Z}_{tr} \xrightarrow{id} \mathbb{Z}_{tr}(X) \xrightarrow{0} \ldots \to \mathbb{Z}_{tr} \xrightarrow{id} \mathbb{Z}_{tr}(X) \xrightarrow{0} \mathbb{Z}_{tr}(X) \to 0$$

which is quasi-isomorphic to $\mathbb{Z}_{tr}(X)$.
Thus, $M(X)$ maps to the image of $\mathbb{Z}_{tr}(X)$ in $\mathbf{DM}^{eff}(k, \mathbb{Z})$.

Remark 4.3.4. $N\mathbb{Z}_{tr}(-)$ *is a left Quillen functor with respect to both the Nisnevich local and Nisnevich motivic model structures, on* $\mathcal{H}_\bullet(k)$ *and* $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})$. *Thus, Lemma 3.4.1 implies that* M *commutes with homotopy colimits.*

Part II

Results and Applications

5

The Nisnevich Motive

We now discuss our results on Nisnevich motives of algebraic stacks.

5.1 Cd-quotient stacks

In this section, we define the notion of a cd-quotient stack and also show that several interesting classes of algebraic stacks are cd-quotient stacks.

Definition 5.1.1. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a field k. We say that $\mathcal{X}$ is a *global quotient stack* if $\mathcal{X} = [X/GL_n]$ for an algebraic space X. We say that $\mathcal{X}$ is a *cd-quotient stack* if it admits a representable Nisnevich covering $[X/GL_n] \to \mathcal{X}$ by a global quotient stack[1].

Two important classes of stacks that are cd-quotient stacks are global quotient stacks, and stacks with quasi-finite diagonal (see Proposition 5.1.4).

For us, a Nisnevich covering of an algebraic stack will always mean a representable morphism (representable by *algebraic spaces*) whose base change to any scheme is a Nisnevich covering. Note that we do not apriori assume that the morphism is representable by schemes.

[1]This nomenclature is inspired by [Ryd, Definition 2.1]. Cd stands for *completely decomposable*. Cd-topology is the old name for Nisnevich topology.

A morphism $f : X \to Y$, with X an algebraic space and Y a scheme, is Nisnevich covering if it is étale, surjective and such that given any field valued point $x : \operatorname{Spec} k \to Y$, there exists a lift $y : \operatorname{Spec} k \to X$ such that $f \circ y = x$,

$$\begin{array}{ccc} & & \operatorname{Spec} k \\ & \overset{y}{\swarrow} & \downarrow x \\ X & \xrightarrow{f} & Y. \end{array}$$

5.1.1 Quotients of linear algebraic groups

Every global quotient stack is a cd-quotient stack. Further, for any quotient stack $[X/G]$ with G a linear algebraic group, there is a canonical GL_n-torsor $X \times_G GL_n \to [X/G]$, realising $[X/G]$ as a global quotient stack. This follows from the following proposition.

Proposition 5.1.2. *Let G be an algebraic group acting on an algebraic space X over a field k. Let $H \subset G$ be an algebraic subgroup of G. Then we have an isomorphism of stacks,*

$$[X/H] \simeq [X \times_H G/G]$$

Proof. For any scheme U, the category $[X/H]_U$ classifies principal H-bundles over U with H equivariant maps to X (see Example 2.4.14). We will construct two functors $[X/H]_U \rightleftarrows [X \times_H G/G]_U$ which induce an equivalence of categories.

$$F : [X/H]_U \to [X \times_H G/G]_U$$

$$\left\{ \begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \downarrow p & & \\ U & & \end{array} \right\} \mapsto \left\{ \begin{array}{ccc} P \times_H G & \xrightarrow{\phi} & X \times_H G \\ \downarrow p & & \\ U & & \end{array} \right\}$$

That $P \times_H G/G \simeq U$ follows from Lemma 5.1.3.

To construct a map in the other direction, we observe that given a G-bundle $p : Q \to U$ with a G-equivariant map to $X \times_H G$, we have a cartesian diagram

$$\begin{array}{ccc} Q' & \longrightarrow & X \times G \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\phi} & X \times_H G \end{array}$$

where the vertical arrows are H-bundles.

We can now define a functor in the other direction as

$$G : [X \times_H G/G]_U \to [X/H]_U$$
$$\left\{\begin{array}{ccc} Q & \xrightarrow{\phi} & X \times_H G \\ \downarrow p & & \\ U & & \end{array}\right\} \mapsto \left\{\begin{array}{ccc} Q'/G & \xrightarrow{\phi} & X \\ \downarrow p & & \\ U & & \end{array}\right\}$$

To see that this is well-defined, note that $Q' = Q \times_{X \times_H G} X \times G$. Then a similar argument as in Lemma 5.1.3 shows that $Q'/G \simeq Q \times_{X \times_H G} X \to U$ is a principal H-bundle.

These operations are equivalences since

$$GF(P) = G(P \times_H G) = (P \times G)/G = P,$$

and

$$FG(Q) = F(Q \times_{X \times_H G} X) = Q \times_{X \times_H G} X \times_H G = Q$$

The same arguments also give us the required equivariant maps. □

Lemma 5.1.3. *Let $P \to V$ be a principal G-bundle. Let $H \subset G$ a subgroup of G. Then, $(P \times_H G)/G \simeq P/H$.*

Proof. The key idea is to observe that for a principal G-bundle $p : P \to V$, the product $P \times G$ admits both an H-action and a G-action which commute with each other. The G-action is given by multiplication on the right $(p, g, g') \overset{\nu_G}{\mapsto} (p, gg')$. While the H-action is given by the map $(p, g, h) \overset{\nu_H}{\mapsto} (ph, h^{-1}g)$. We write the quotient $(P \times G)/H$ as $P \times_H G$.

Now, we have diagram

$$\begin{array}{ccc} P \times G & \xrightarrow{pr_1} & P \\ \downarrow p & & \\ (P \times_H G) & & \end{array}$$

Here, pr_1 is the projection onto P. Note that pr_1 is H-equivariant, and p is a G-equivariant map. Thus, we get the following cartesian square in which the horizontal maps are principal G-bundle, whereas the vertical maps are principal H-bundle.

$$\begin{array}{ccc} P \times G & \xrightarrow{pr_1} & P \\ \downarrow{\scriptstyle p} & & \downarrow \\ P \times_H G & \longrightarrow & P/H \end{array}$$

This shows that $(P \times_H G)/G \simeq P/H$. □

5.1.2 Stacks with quasi-finite diagonal

Proposition 5.1.4. *Let $\mathcal{X}$ be an algebraic stack of finite type over a perfect field k. Assume that the diagonal of $\mathcal{X}$ is quasi-finite. Then $\mathcal{X}$ is a cd-quotient stack.*

Proof. By [Con, Lemma 2.1], Zariski locally $\mathcal{X}$ admits a quasi-finite, flat and finitely-presented covering by a quasi-projective scheme V. Thus, working Zariski locally we may assume that we have a quasi-finite, flat and finitely-presented covering by a quasi-projective scheme $V \to \mathcal{X}$.

Consider the Hilbert stack of finite flat cover $\mathcal{H} := Hilb_{V/\mathcal{X}}$ defined in Example 2.4.17 and let $\mathcal{W}$ denote the étale locus of the structure map $\mathcal{H} \to \mathcal{X}$. The substack $\mathcal{W}$ parametrises families that are finite and étale.

We will show that $\mathcal{W}$ is Zariski locally a global quotient stack and that $\mathcal{W} \to \mathcal{X}$ is a Nisnevich covering.

By [Con, Lemma 2.2], $\mathcal{W} \to \mathcal{X}$ is a representable étale covering of $\mathcal{X}$ such that there exits a finite flat map $Z \to \mathcal{W}$ from a scheme Z. From *loc. cit.*, it follows that Zariski locally Z can be chosen to be quasi-projective. In particular, that Z has the resolution property. As the resolution property descends along finite flat finitely presented maps [Gro1, Proposition 2.13], we see that $\mathcal{W}$ also has the resolution property. Thus, by the Totaro-Gross theorem (see [Tot2,Gro1]), $\mathcal{W} \simeq [U/GL_n]$ with U a quasi-affine scheme.

To see that $\mathcal{W} \to \mathcal{X}$ is a Nisnevich cover, note that since k is perfect, every field extension of k is separable. Let L be a field and $\operatorname{Spec} L \to \mathcal{X}$ be any point. Then the base change $V \times_{\mathcal{X}} \operatorname{Spec} L \to \operatorname{Spec} L$ is a finite flat cover of $\operatorname{Spec} L$. Since k is perfect, it is, in fact, finite étale. Thus, it defines a point $\operatorname{Spec} L \to \mathcal{W}$ lifting the point $\operatorname{Spec} L \to \mathcal{X}$. This implies that $\mathcal{W} \to \mathcal{X}$ is a Nisnevich covering. □

Corollary 5.1.5. *Let $\mathcal{X}$ be a Deligne-Mumford stack of finite type over a perfect field k. Then $\mathcal{X}$ is a cd-quotient stack.*

Proof. As $\mathcal{X}$ is a Deligne-Mumford stack, its diagonal is quasi-compact and unramified. By [Sta, 02V5, 01TJ], this implies that the diagonal is quasi-finite. The result now follows from Proposition 5.1.4. □

Remark 5.1.6. *Let $\mathcal{X}$ be a separated Deligne-Mumford stack (over $\mathbb{Z}$). In [AV, Lemma 2.2.3] it shown that étale locally on its coarse space, $\mathcal{X}$ is of the form $[U/\Gamma]$ where Γ is a finite group acting on U.*

5.1.3 More examples of cd-quotient stacks

Example 5.1.7. *Let $\mathcal{X}$ be an algebraic stack that admits a good moduli space X (in the sense of [Alp]), then by [AHR, Theorem 13.1], Nisnevich locally on the good moduli space, we have a cartesian diagram,*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

where $W \to X$ is a Nisnevich covering and $\mathcal{W} \simeq [\mathrm{Spec}\,(A)/GL_n]$. Hence, $\mathcal{X}$ is a cd-quotient stack.

The following theorem was communicated to us by David Rydh. The proof will appear in the paper *Artin algebraization for pairs and applications to the local structure of stacks and Ferrand pushouts* ([AHLHR]) by Jarod Alper, Daniel Halpern-Leistner, Jack Hall, and David Rydh.

Theorem 5.1.8 ([AHLHR])**.** *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack such that every point of $\mathcal{X}$ has stabiliser group that is an extension of a finite linearly reductive group scheme by an algebraic group of multiplicative type. Then $\mathcal{X}$ is a cd-quotient stack.*

In Section 6.2, we will show that *exhaustive stacks* of [HL] are also cd-quotient stacks.

5.2 Nisnevich motive of an algebraic stack

In this section we will define the motive of an algebraic stack as well as prove Theorem 1.2.1. In order to do this, the following observation will be crucial.

Remark 5.2.1 (Stacks as simplicial sheaves)**.** *Given an algebraic stack $\mathcal{X}$, one associates a simplicial sheaf to $\mathcal{X}$ as follows: $\mathcal{X}$ defines a (strict) sheaf of groupoids $\mathcal{X} \in Fun((Sch/k)^{op}, Grpds)$, which sends any k-scheme $U \mapsto \mathcal{X}_U$ in the category of groupoids. Applying the nerve functor objectwise, we get a sheaf of simplicial sets. Let us briefly recall this procedure.*
Given a groupoid $\mathcal{X}_U$ its nerve is a simplicial set $N(\mathcal{X}_U)$ whose k-simplices are given by k-tuples of composable arrows,

$$N(\mathcal{X}_U)_k = \{A_0 \xrightarrow{f_1} \dots \xrightarrow{f_k} A_k \mid A_i\text{'s are objects and } f_i \text{ are morphisms in } \mathcal{X}_U\}$$

Note that 0-simplices are just objects of $\mathcal{X}_U$, while 1-simplices are morphisms between them.
The face maps $d_i : N(\mathcal{X}_U)_k \to N(\mathcal{X}_U)_{k-1}$ are given by composition of morphism at the i-th object (or deleting the i-th object for $i = 0, k$). Similarly, the degeneracy maps $s_i : N(\mathcal{X}_U)_k \to N(\mathcal{X}_U)_{k+1}$ are given by inserting the identity morphism at the i-th object (see [Hol2] for more details).
If $\mathcal{X} \to \mathcal{Z}$, $\mathcal{Y} \to \mathcal{Z}$ are two morphisms of algebraic stacks, then viewing them as simplicial sheaves, one can form their homotopy fibre product $\mathcal{X} \times^h_{\mathcal{Z}} \mathcal{Y}$ in the homotopy category of simplicial presheaves. By [Hol1, Remark 2.3], the usual fibre product in the category of stacks $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ serves as a model for this homotopy fibre product.

In the remaining article, we will abuse notation by denoting the simplicial sheaf associated to a stack $\mathcal{X}$ by $\mathcal{X}$ itself.

Definition 5.2.2 (Motive of an algebraic stack)**.** Let $\mathcal{X}$ be an algebraic stack over k thought of as a simplicial presheaf by the nerve construction outlined in Remark 5.2.1. The motive of an algebraic stack $\mathcal{X}$ is defined to be the image $M(\mathcal{X})$ in $\mathbf{DM}^{eff}(k, \mathbb{Z})$. Note that when $\mathcal{X}$ is representable by a scheme X, $M(X)$ is the image of $\mathbb{Z}_{tr}(X)$ in $\mathbf{DM}^{eff}(k, \mathbb{Z})$.

Remark 5.2.3. *The above construction was first done in [Cho] for the étale model structure. However, it still goes through if we use the Nisnevich model structure instead. See also [Jos1] for an alternative approach.*

We will now show that cd-quotient stacks admit presentations by simplicial schemes in $\mathcal{H}_\bullet(k)$. This is the content of Theorem 1.2.1. Having such a presentation will allow us to use homotopical descent techniques in the sequel in order to reduce various problems to the case of (simplicial) schemes.

For the sake of clarity we will first prove Theorem 1.2.1 in the case when $\mathcal{Y} = \mathcal{X}$, i.e, when $\mathcal{X}$ is a global quotient stack. Theorem 1.2.1 is a minor extension of this case.

Lemma 5.2.4. *Let X be an algebraic space with an action of GL_n and $\mathcal{X} := [X/GL_n]$ be the corresponding quotient stack. Let $X_\bullet$ denote the Čech hypercover associated to $p : X \to \mathcal{X}$. Then the map of simplicial presheaves $p_\bullet : X_\bullet \to \mathcal{X}$ is a Nisnevich local weak equivalence.*

Proof. It suffices to check that given a hensel local ring $\mathcal{O}$, the induced map on $\mathcal{O}$-points $p_\bullet : X_\bullet(\operatorname{Spec}\mathcal{O}) \to \mathcal{X}(\operatorname{Spec}\mathcal{O})$ is a weak equivalence of simplicial sets. Further, as a stack is a 1-truncated simplicial set (being a groupoid valued functor), $\pi_i = 0$ for $i \geq 2$. Thus, we only need to verify that p induces an isomorphism of homotopy groups for $i = 0, 1$.

<u>$i = 0$:</u> Any map $\operatorname{Spec}\mathcal{O} \to \mathcal{X}$ gives rise to a GL_n-torsor $p' : X \times_{\mathcal{X}} \operatorname{Spec}\mathcal{O} \to \operatorname{Spec}\mathcal{O}$. As $\mathcal{O}$ is a Hensel local ring, any GL_n-torsor over it is trivial and hence admits a section. This implies surjectivity on π_0.

For injectivity, let $f_1, f_2 : \operatorname{Spec}\mathcal{O} \to X$ be two $\mathcal{O}$-points of X such that $p(f_1) = p(f_2)$, then there exists a map $F : \operatorname{Spec}\mathcal{O} \to X \times_{\mathcal{X}} X$ which after composing with each of the projection maps becomes f_1 and f_2, respectively. The map F may be thought of as a 1-simplex in the simplicial set $X_\bullet(\operatorname{Spec}\mathcal{O})$, and therefore, corresponds to a map $\Delta^1 \to X_\bullet(\operatorname{Spec}\mathcal{O})$, which gives a homotopy between the points f_1 and f_2 implying injectivity.

<u>$i = 1$:</u> In this case, we need to show that for any $\operatorname{Spec}\mathcal{O}$-valued point of $\mathcal{X}$, the homotopy fibre product with $p_\bullet$ is contractible. Then, the long exact sequence of homotopy sheaves gives the required isomorphism. By [Hol1, Remark 2.3], the homotopy fibre product is precisely the fibre product in the category of stacks. This is a (Čech nerve of a) trivial GL_n-torsor over $\operatorname{Spec}\mathcal{O}$, i.e, given a point $\operatorname{Spec}\mathcal{O} \to \mathcal{X}$, the homotopy fibre of $p_\bullet$ is precisely the Čech hypercover $X_\bullet \times_{\mathcal{X}} \operatorname{Spec}\mathcal{O}$ corresponding to the GL_n-torsor p' :

$X \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathcal{O}$. Since p' admits a section, the augmentation map $X_\bullet \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathcal{O}$ is a Nisnevich local weak equivalence. As, $\pi_i(\operatorname{Spec} \mathcal{O}) = 0$ for $i > 0$, we get the desired result. □

Remark 5.2.5. *In fact, since a GL_n-bundle on any local ring is trivial, Lemma 5.2.4 also holds in the Zariski topology.*

Any Nisnevich cover $\mathcal{Y} \to \mathcal{X}$ admits sections Nisnevich locally. Theorem 1.2.1 now follows easily from this fact and Lemma 5.2.4.

Proof of Theorem 1.2.1. We need to check that $p_\bullet : Y_\bullet \to \mathcal{X}$ induces an isomorphism on all homotopy sheaves, π_i. Further, it suffices to check this on all Hensel local schemes.
$\underline{i = 0}$: Let $\operatorname{Spec} \mathcal{O} \to \mathcal{X}$ be a point of $\mathcal{X}$. Base changing, we get maps $Y \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O} \to \mathcal{Y} \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathcal{O}$, where the first map is a principal GL_n-bundle and the second is a Nisnevich cover. So, the second map admits a Nisnevich local section. By the previous lemma, so does the first. This proves surjectivity.
For injectivity, let $f_1, f_2 : \operatorname{Spec} \mathcal{O} \to Y$ be two points which map to the same point in $\mathcal{X}$. This implies that there exists a section $\operatorname{Spec} \mathcal{O} \to Y \times_{\mathcal{X}} Y$ which after composing with each of the projection maps becomes f_1 and f_2, respectively.
$\underline{i > 0}$: To show this we need to show that for any $\operatorname{Spec} \mathcal{O}$-valued point of $\mathcal{X}$, the homotopy fibre product with $p_\bullet$ is contractible. Then, the long exact sequence of homotopy sheaves gives us the required isomorphism.
As noted in the previous lemma, the homotopy fibre product is equal to the stacky fibre product [Hol1, Remark 2.3]. Hence, for a point $\operatorname{Spec} \mathcal{O} \to \mathcal{X}$, the homotopy fibre of $p_\bullet$ is precisely the Čech hypercover $Y_\bullet \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O}$ of $\operatorname{Spec} \mathcal{O}$. Since it admits a section, the augmentation map $Y_\bullet \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathcal{O}$ is a Nisnevich local weak equivalence. As, $\pi_i(\operatorname{Spec} \mathcal{O}) = 0$ for $i > 0$, we get the desired result. □

Remark 5.2.6. *For a cd-quotient stack, we have a Čech hypercover $Y_\bullet \to \mathcal{X}$ which is a Nisnevich local weak equivalence, by Theorem 1.2.1. Applying the functor $M : \mathcal{H}_\bullet(k) \to \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$, we see that the motive of a cd-quotient stack $\mathcal{X}$ is given by the normalised chain complex $N\mathbb{Z}_{tr}(Y_\bullet)$.*

Definition 5.2.7. For a cd-quotient stack $\mathcal{X}$, let $p : X \to \mathcal{X}$ be the presentation obtained from the composition $X \to [X/GL_n] \to \mathcal{X}$ (as in the hypothesis of Theorem 1.2.1), and let $p_\bullet : X_\bullet \to \mathcal{X}$ denote the associated Čech hypercover. Motivated by the content of Theorem 1.2.1, we will call $p_\bullet : X_\bullet \to \mathcal{X}$ a *GL_n-presentation* of $\mathcal{X}$.

Remark 5.2.8. *The category generated by motives of stacks as constructed above is larger than the category of geometric motives (motives generated by smooth quasi-projective schemes). In fact, if G is a finite group, BG is not a geometric motive. To see this note that any realization of a geometric motive must have bounded cohomology. Further, for a finite group, the cohomology of BG is the same as the group cohomology of the group G. But if G is finite cyclic, then the latter is periodic in odd degrees, showing that BG does not have bounded cohomology for a cyclic group.*

5.2.1 Relation with étale motives

As stated earlier, Definition 5.2.2 was first used in [Cho] for the étale model structure (see also [Jos1] for an alternative approach). For any stack $\mathcal{X}$, this produces a motive $M(\mathcal{X})$ in $\mathbf{DM}^{eff}_{\acute{e}t}(k, \mathbb{Z})$. We will call this the *étale motive* of $\mathcal{X}$. We will now compare Definition 5.2.2 with its étale motive.

Proposition 5.2.9. *Let $\mathcal{X}$ be a cd-quotient stack. The Nisnevich motive $M(Y_\bullet)$ of $\mathcal{X}$ agrees with its étale motive in $\mathbf{DM}^{eff}_{\acute{e}t}(k, \mathbb{Z})$ after étale sheafification.*

Proof. Let $Y_\bullet \to \mathcal{X}$ be a GL_n-presentation so that $M(Y_\bullet) \simeq M(\mathcal{X})$ in $\mathbf{DM}^{eff}(k, \mathbb{Z})$. Since sheafification is an exact functor, we have a functor $\sigma : \mathbf{DM}^{eff}(k, \mathbb{Z}) \to \mathbf{DM}^{eff}_{\acute{e}t}(k, \mathbb{Z})$ which takes Nisnevich local weak equivalences to étale local weak equivalences (see [MVW, Remark 14.3]). Thus, the Nisnevich local weak equivalence $M(Y_\bullet) \to M(\mathcal{X})$ becomes an étale local weak equivalence after étale sheafification. Hence, $M(Y_\bullet) \simeq M(\mathcal{X})$ in $\mathbf{DM}^{eff}_{\acute{e}t}(k, \mathbb{Z})$.

In fact, more is true. We can show that any GL_n-presentation is equivalent to the Čech hypercovers associated to a smooth presentation of $\mathcal{X}$.

Let $p : U \to \mathcal{X}$ be a smooth presentation, and $p_\bullet : U_\bullet \to \mathcal{X}$ be the associated Čech hypercover. Given any strict Hensel local point $\operatorname{Spec} \mathcal{O} \to \mathcal{X}$ consider the base change $p_\mathcal{O} : U_\mathcal{O} \to \operatorname{Spec} \mathcal{O}$. This is a smooth morphism and so admits étale local sections. In fact, since $\mathcal{O}$ is a Hensel local ring, we have a section $\operatorname{Spec} \mathcal{O} \to U_\mathcal{O}$ of $p_\mathcal{O}$. This implies that the induced map of simplicial sets $p_\bullet(\mathcal{O}) : U_\bullet(\mathcal{O}) \to \mathcal{X}(\mathcal{O})$ is a weak equivalence of simplicial sets (proof is similar to Lemma 5.2.4). Thus, $U_\bullet \to \mathcal{X}$ is an étale local weak equivalence. So the induced map on étale motives $M(U_\bullet) \to M(\mathcal{X})$ is also an étale local weak equivalence in $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbb{Z})$ (see [Cho, Corollary 2.14]). Hence, $M(Y_\bullet) \simeq M(U_\bullet)$ in $\mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbb{Z})$. □

Remark 5.2.10. *In Proposition 5.2.9, the GL_n-presentation $Y_\bullet$ and the Čech hypercover $U_\bullet$ are simplicial objects in the category of algebraic spaces. This is because a smooth presentation $p : U \to \mathcal{X}$ of an algebraic stack need not be representable by schemes, but only algebraic spaces. However, as any algebraic space admits a Nisnevich presentation by a scheme [Knu, Theorem II.6.4], we can refine the Čech hypercover $U_\bullet$ to a generalised hypercovering $V_\bullet$ such that each V_i is a scheme. Then $M(V_\bullet)$ computes the motive $M(\mathcal{X})$ (see [DHI] for details).*

Remark 5.2.11 (Étale motives with $\mathbb{Q}$-coefficients)**.** *Tensoring with $\mathbb{Q}$ gives us a functor $- \otimes \mathbb{Q} : \mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbb{Z}) \to \mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbb{Q})$ which is just change of coefficients (this also works in the Nisnevich topology). We write $M(Y_\bullet) \otimes \mathbb{Q} := M(Y_\bullet)_\mathbb{Q}$. By [MVW, Theorem 14.30], the étale sheafification functor $\sigma : \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Q}) \to \mathbf{DM}^{\mathit{eff}}_{\acute{e}t}(k, \mathbb{Q})$ is an equivalence of categories.*

By [Cho, Theorem 4.6], for a smooth separated Deligne-Mumford stack $\mathcal{X}$, $M(\mathcal{X})_\mathbb{Q}$ is a geometric motive. This does not contradict Remark 5.2.8, since the cohomology groups of a cyclic group are torsion and will vanish after tensoring with $\mathbb{Q}$.

Further, if $\pi : \mathcal{X} \to X$ is the coarse space map, then $M(\pi)_\mathbb{Q} : M(\mathcal{X})_\mathbb{Q} \to M(X)_\mathbb{Q}$ is an isomorphism by [Cho, Theorem 3.3]. This is clearly false integrally, since for a finite group G over a field k, the structure map $BG \to \operatorname{Spec} k$ is a coarse space map.

5.3 Various triangles

We now establish Nisnevich descent and blow-up sequence for the Nisnevich motive and also prove the projective bundle formula. As a consequence of the projective bundle formula, we get a Gysin triangle for cd-quotient stacks. These result are already known for étale motives (see [Cho]). All the arguments in this section are directly adapted from their étale counterparts in [Cho] − except for the projective bundle formula. The argument for the projective bundle formula in the étale case relies on the identification of the Picard group with $H^2_{\acute{e}t}(\mathcal{X}, \mathbb{Z}(1))$. This identification fails for stacks if étale topology is replaced by Nisnevich topology. So we adopt a different approach using homotopical descent.

In this section, we work exclusively with cd-quotient stacks.

Remark 5.3.1. *Let $\mathcal{Z}$ be a cd-quotient stack. If $\mathcal{Y} \to \mathcal{Z}$ is a representable morphism, then $\mathcal{Y}$ is also a cd-quotient stack. To see this, let $Z \to [Z/GL_n] \to \mathcal{Z}$ be a Nisnevich covering of $\mathcal{Z}$ by a quotient stack. Now, observe that we have a cartesian diagram by base change,*

$$\begin{array}{ccc} \mathcal{Y} \times_{\mathcal{Z}} Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \mathcal{Y} \times_{\mathcal{Z}} [Z/GL_n] & \longrightarrow & [Z/GL_n] \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

where $\mathcal{Y} \times_{\mathcal{Z}} Z \to \mathcal{Y} \times_{\mathcal{Z}} [Z/GL_n]$ is a GL_n-torsor and $\mathcal{Y} \times_{\mathcal{Z}} [Z/GL_n] \to \mathcal{Y}$ is a Nisnevich cover. Denote the algebraic space $\mathcal{Y} \times_{\mathcal{Z}} Z$ by Y. Thus, $\mathcal{Y}$ is a cd-quotient stack. Hence, by Theorem 1.2.1, $Y_\bullet \to \mathcal{Y}$ is a Nisnevich local weak equivalence.
This tells us that GL_n-presentations respect base change with respect to representable morphisms.

Definition 5.3.2. A cartesian square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow p \\ \mathcal{X} & \xrightarrow{i} & \mathcal{Z} \end{array}$$

of algebraic stacks is said to be a *distinguished Nisnevich square* if i is an open immersion, and p is an étale morphism which induces an isomorphism $p^{-1}(\mathcal{Z} \setminus \mathcal{X})_{red} \simeq (\mathcal{Z} \setminus \mathcal{X})_{red}$.

Proposition 5.3.3. *For a distinguished Nisnevich square,*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow p \\ \mathcal{X} & \stackrel{j}{\hookrightarrow} & \mathcal{Z} \end{array}$$

where j is an open immersion and p is étale representable, the induced diagram on motives is homotopy cartesian.

Proof. For a distinguished Nisnevich square of stacks

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{X} & \overset{j}{\hookrightarrow} & \mathcal{Z} \end{array}$$

we need to show that the induced diagram

$$\begin{array}{ccc} M(\mathcal{W}) & \longrightarrow & M(\mathcal{Y}) \\ \downarrow & & \downarrow \\ M(\mathcal{X}) & \longrightarrow & M(\mathcal{Z}) \end{array}$$

of motives is homotopy cartesian. If $Z \to [Z/GL_n] \to \mathcal{Z}$ is a Nisnevich covering of $\mathcal{Z}$ by a GL_n-torsor, then by Remark 5.3.1, we can base change this covering to $\mathcal{Y}, \mathcal{X}$ and $\mathcal{W}$. This gives us a cartesian diagram of hypercovers:

$$\begin{array}{ccc} W_\bullet & \longrightarrow & Y_\bullet \\ \downarrow & & \downarrow{\scriptstyle p_\bullet} \\ X_\bullet & \overset{j_\bullet}{\hookrightarrow} & Z_\bullet \end{array}$$

For each i, the above the diagram is a distinguished Nisnevich square of schemes. Thus, for each i, the following diagram of motives is homotopy (co)cartesian,

$$\begin{array}{ccc} M(W_i) & \longrightarrow & M(Y_i) \\ \downarrow & & \downarrow \\ M(X_i) & \longrightarrow & M(Z_i) \end{array}$$

By Remark 4.3.4, $M(Z_\bullet) \simeq \operatorname{hocolim} M(Z_i)$. As homotopy colimits commute with homotopy colimits, the following diagram is again homotopy (co)cartesian,

$$\begin{array}{ccc} M(W_\bullet) & \longrightarrow & M(Y_\bullet) \\ \downarrow & & \downarrow \\ M(X_\bullet) & \longrightarrow & M(Z_\bullet) \end{array}$$

By Theorem 1.2.1, we get the required result. $\square$

Theorem 5.3.4 (Projective Bundle Formula). *Let $\mathcal{E}$ be a vector bundle of rank $n+1$ on a stack $\mathcal{X}$. There exists a canonical isomorphism in $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})$:*

$$M(\mathbb{P}roj(\mathcal{E})) \to \bigoplus_{i=0}^{n} M(\mathcal{X})(i)[2i]$$

Proof. As projective bundle formula is known for smooth schemes by [MVW, Theorem 15.12], we will deduce the result for stacks by a homotopical descent argument. To make such a homotopical descent argument, we need to ensure that homotopy colimits commute with M and derived tensor. But both M and derived tensor are derived functors of left Quillen functors, so Lemma 3.4.1 ensures that this is true.

Let $p : \mathbb{P}roj(\mathcal{E}) \to \mathcal{X}$ be the projective bundle, and $\mathcal{O}(1)$ the canonical line bundle on it. This construction behaves well with respect to base change. If $U_\bullet \to \mathcal{X}$ is a GL_n-presentation, then by base change we get projective bundles $p_i : V_i \to U_i$ for every i, and line bundles $\mathcal{O}(1)_{V_i}$ on V_i by pullback. Moreover, by Remark 5.3.1, the Čech hypercover $V_\bullet \to \mathbb{P}roj(\mathcal{E})$ is a GL_n-presentation of $\mathbb{P}roj(\mathcal{E})$. As each $p_i : V_i \to U_i$ is a projective bundle, by the projective bundle formula (see [MVW, Theorem 15.12]), we have:

$$M(V_i) \simeq \oplus_{j=0}^{n} M(U_i) \otimes \mathbb{Z}(j)[2j], \tag{5.1}$$

in $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})$.

Since M and derived tensor are left Quillen, using Lemma 3.4.1, we have

$$\begin{aligned} M(\mathbb{P}roj(\mathcal{E})) \simeq M(\operatorname{hocolim} V_i) &\simeq \operatorname{hocolim}\,(M(V_i)) \\ &\simeq \operatorname{hocolim}\,(\oplus_{j=0}^{n} M(U_i) \otimes \mathbb{Z}(j)[2j]) \\ &\simeq \oplus_{j=0}^{n}(M(\operatorname{hocolim} U_i) \otimes \mathbb{Z}(j)[2j]) \\ &\simeq \oplus_{j=0}^{n} M(\mathcal{X}) \otimes \mathbb{Z}(j)[2j], \end{aligned}$$

as required. □

Proposition 5.3.5. *Let $\mathcal{Z} \subset \mathcal{X}$ be a smooth closed substack of $\mathcal{X}$. Let $Bl_{\mathcal{Z}}(\mathcal{X})$ denote the blow-up of $\mathcal{X}$ in the centre $\mathcal{Z}$, and $\mathcal{E}$ be the exceptional divisor. Then we have a canonical* *distinguised triangle:*

$$M(\mathcal{E}) \to M(\mathcal{Z}) \oplus M(Bl_{\mathcal{Z}}(\mathcal{X})) \to M(\mathcal{X}) \to M(\mathcal{E})[1]$$

Proof. Let $X \to [X/GL_n] \to \mathcal{X}$ be a Nisnevich covering of $\mathcal{X}$ by a GL_n-torsor. Since the morphism $Bl_{\mathcal{Z}}(\mathcal{X}) \to \mathcal{X}$ is projective, it is representable. Then, we can base change X to $Bl_{\mathcal{Z}}(\mathcal{X}), \mathcal{Z}$ and $\mathcal{E}$. The rest of the proof is the same as Proposition 5.3.3. □

Theorem 5.3.6. *Let $\mathcal{X}$ be a smooth stack and $\mathcal{Z} \subset \mathcal{X}$ be a smooth closed substack of pure codimension c. Then,*

$$M(Bl_{\mathcal{Z}}(\mathcal{X})) \simeq M(\mathcal{X}) \oplus_{i=0}^{c-1} M(\mathcal{Z})(i)[2i]$$

Proof. Using the previous result, we have a canonical distinguished triangle:

$$M(\mathcal{E}) \to M(\mathcal{Z}) \oplus M(Bl_{\mathcal{Z}}(\mathcal{X})) \to M(\mathcal{X}) \to M(\mathcal{E})[1],$$

where $p : Bl_{\mathcal{Z}}(\mathcal{X}) \to \mathcal{X}$ is the blow-up. The exceptional divisor is the projectivisation of the normal bundle $N_{\mathcal{Z}}(\mathcal{X})$ of Z in $\mathcal{X}$, i.e, $\mathcal{E} \simeq \mathbb{P}roj(N_{\mathcal{Z}}(\mathcal{X}))$. If $M(\mathcal{X}) \to M(\mathcal{E})[1]$ is zero, then the projective bundle formula for $\mathbb{P}roj(N_{\mathcal{Z}}(\mathcal{X}))$ gives us the result.

To prove that $M(\mathcal{X}) \to M(\mathcal{E})[1]$ is zero, we argue exactly as in [Cho, Theorem 3.7] (see also [VSF, Chapter 5, Proposition 3.5.3]). Take $\mathcal{X} \times \mathbb{A}^1$ and consider the blow-up along $\mathcal{Z} \times \{0\}$. We have a map $q : Bl_{\mathcal{Z}\times\{0\}}(\mathcal{X} \times \mathbb{A}^1) \to \mathcal{X} \times \mathbb{A}^1$. Consider the morphism of exact triangles,

$$\begin{array}{ccc}
M(\mathcal{E}) & \longrightarrow & M(q^{-1}(\mathcal{Z} \times \{0\})) \\
\downarrow & & \downarrow \\
M(\mathcal{Z}) \oplus M(Bl_{\mathcal{Z}}(\mathcal{X})) & \longrightarrow & M(\mathcal{Z} \times \{0\}) \oplus M(Bl_{\mathcal{Z}\times\{0\}}(\mathcal{X} \times \mathbb{A}^1)) \\
\downarrow & & \downarrow f \\
M(\mathcal{X}) & \xrightarrow{s_0} & M(\mathcal{X} \times \mathbb{A}^1) \\
\downarrow g & & \downarrow h \\
M(\mathcal{E})[1] & \xrightarrow{a} & M(q^{-1}(\mathcal{Z} \times \{0\}))[1]
\end{array}$$

By the projective bundle formula, the morphism a is split injective, and s_0 is an isomorphism. Hence, to show that g is zero, it suffices to show that h is zero. To see this, note that the composition

$$M(\mathcal{X} \times \{1\}) \to M(Bl_{\mathcal{Z}\times\{0\}}(\mathcal{X} \times \mathbb{A}^1)) \to M(\mathcal{X} \times \mathbb{A}^1)$$

is an isomorphism. This implies that f admits a section so h must be zero. □

Definition 5.3.7. For a map $M(X) \to M(Y)$ of motives of stacks (or simplicial schemes), we denote the cone by

$$M\Big(\frac{X}{Y}\Big) := cone(M(X) \to M(Y)) \text{ in } \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z}).$$

Lemma 5.3.8. *Let $f : \mathcal{X}' \to \mathcal{X}$ be an étale representable morphism of algebraic stacks, and let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack such that f induces an isomorphism $f^{-1}(\mathcal{Z}) \simeq \mathcal{Z}$. Then the canonical morphism*

$$M\Big(\frac{\mathcal{X}'}{\mathcal{X}' \setminus \mathcal{Z}}\Big) \to M\Big(\frac{\mathcal{X}}{\mathcal{X} \setminus \mathcal{Z}}\Big)$$

is an isomorphism.

Proof. Let $U_\bullet \to \mathcal{X}$ be a GL_n-presentation. Let $V_\bullet := U_\bullet \times_{\mathcal{X}} \mathcal{X}'$ be a GL_n-presentation of $\mathcal{X}'$ obtained by base change. We have a map of simplicial sets $f_\bullet : V_\bullet \to U_\bullet$ induced by the map $f : \mathcal{X}' \to \mathcal{X}$. Note that for every simplicial degree, f_i is étale and that $f_i^{-1}(\mathcal{Z} \times_{\mathcal{X}} U_i) \simeq \mathcal{Z} \times_{\mathcal{X}} U_i$. Let Z_i denote the base change $\mathcal{Z} \times_{\mathcal{X}} U_i$. Then, by [VSF, Chapter 3, Proposition 5.18], the canonical morphism

$$M\Big(\frac{V_\bullet}{V_\bullet \setminus Z_\bullet}\Big) \to M\Big(\frac{U_\bullet}{U_\bullet \setminus Z_\bullet}\Big).$$

is an isomorphism. Now, Theorem 1.2.1 gives us the required result. □

Corollary 5.3.9. *Let $p : V \to \mathcal{X}$ be a vector bundle of rank n over an algebraic stack. Denote by $s : \mathcal{X} \to V$ the zero section. Then*

$$M\Big(\frac{V}{V \setminus s}\Big) \simeq M(\mathcal{X})(d)[2d].$$

Proof. From the previous lemma we have an isomorphism

$$M\Big(\frac{V}{V\setminus s}\Big) \simeq M\Big(\frac{\mathbb{P}roj(V\oplus \mathcal{O})}{\mathbb{P}roj(V\oplus \mathcal{O}\setminus s)}\Big),$$

and we have

$$M\Big(\frac{\mathbb{P}roj(V\oplus \mathcal{O})}{\mathbb{P}roj(V\oplus \mathcal{O}\setminus s)}\Big) \simeq M(\mathcal{X})(d)[2d]$$

from the projective bundle formula (see [Cho, Lemma 3.9] for details). □

Theorem 5.3.10 (Gysin Triangle)**.** *Let $\mathcal{Z}\subset\mathcal{X}$ be a smooth closed substack of codimension c. Then there exists a Gysin triangle:*

$$M(\mathcal{X}\setminus\mathcal{Z}) \to M(\mathcal{X}) \to M(\mathcal{Z})(c)[2c] \to M(\mathcal{X}\setminus\mathcal{Z})[1].$$

Proof. Note that we have an exact triangle

$$M(\mathcal{X}\setminus\mathcal{Z}) \to M(\mathcal{X}) \to M\Big(\frac{\mathcal{X}}{\mathcal{X}\setminus\mathcal{Z}}\Big) \to M(\mathcal{X}\setminus\mathcal{Z})[1].$$

So it suffices to show that $M\big(\frac{\mathcal{X}}{\mathcal{X}\setminus\mathcal{Z}}\big) \simeq M(Z)(c)[2c]$ in $\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})$. The argument is exactly as in [Cho, Theorem 3.10]. □

6

Applications

In this chapter, we will discuss applications of Theorem 1.2.1 to Chow groups of algebraic stacks (Section 6.1) and motives of exhaustive stacks (Section 6.2).

6.1 Comparison with Edidin-Graham-Totaro Chow groups

We will now proceed to show that the motivic cohomology groups of the motive defined by Theorem 1.2.1 agree with the (higher) Chow groups defined by Edidin-Graham-Totaro for quotients of smooth schemes. This is Theorem 1.2.2. For smooth quasi-projective schemes, this implicitly follows from the following result.

Lemma 6.1.1. *[Kri, Proposition 3.2] Let $\rho = (V_i, U_i)_{i\geq 1}$ be an admissible gadget for a linear algebraic group G over k. For any quasi-projective G-scheme X, there is a canonical isomorphism $X_G(\rho) \cong X_G^\bullet$ in $\mathcal{H}_\bullet(k)$.*

The simplicial presheaf $X_G(\rho)$ is defined in Section 6.1.1 below. We denote by $X_G^\bullet$ the simplicial scheme:

$$\dots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \times G \times X \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; G \times X \; \substack{\longrightarrow\\\longrightarrow} \; X$$

To extend the result to all smooth schemes, we need a version of [Kri, Proposition 3.2] for smooth schemes. In fact, in Lemma 6.1.3, we prove such a statement for all algebraic spaces.

In what follows we will consider the action of $G := GL_r$ on an algebraic space X. We denote the quotient stack $[X/G]$ or $[X/GL_r]$.

6.1.1 Admissible gadgets

We will now construction an object which approximates all the Edidin-Graham-Totaro Chow groups of $[X/GL_r]$ (see Definition 2.4.11). The following definition is a special case of the one in [MV, Section 4.2]:

Definition 6.1.2. [Kri, Definition 2.1] A pair (V, U) of smooth schemes over k is said to be a *good pair* for G if V is a k-rational representation of G and $U \subset V$ is a G-invariant open subset on which G acts freely and the quotient U/G is a smooth quasi-projective scheme.
A sequence of pairs $\rho = (V_i, U_i)_{i \geq 1}$ is said to be an *admissible gadget* for G if there exists a good pair (V, U) for G such that $V_i = V^{\oplus i}$ and $U_i \subset V_i$ is a G-invariant open subscheme such that the following hold:

- $(U_i \oplus V) \cup (V \oplus U_i) \subseteq U_{i+1}$ as G-invariant open subsets.
- $codim_{U_{i+2}}(U_{i+2} \setminus (U_{i+1} \oplus V)) > codim_{U_{i+1}}(U_{i+1} \setminus (U_i \oplus V))$.
- $codim_{V_{i+1}}(V_{i+1} \setminus U_{i+1}) > codim_{V_i}(V_i \setminus U_i)$.
- The action of G on U_i is free, and the quotient is a quasi-projective scheme.

An example of such an admissible gadget can be given as follows. Let V be a k-rational representation of GL_r, and let U be a GL_r-invariant open subset on which GL_r acts freely, and the quotient U/GL_r is a quasi-projective scheme. Then (V, U) is a good pair, and we define an admissible gadget $\rho = (V_i, U_i)_{i \geq 1}$ by taking $U_{i+1} := (U_i \oplus V) \cup (V \oplus U_i)$.
For an algebraic space X with an action G, consider the mixed quotients $X^i(\rho) := X \times_G U_i$. For every pair $(i, i+1)$, we have maps $X \times_G U_i \to X \times_G U_{i+1}$. We define the colimit as

$$X_G(\rho) := \operatorname{colim} X \times_G U_i.$$

The sheaf $X_G(\rho)$ computes all the Edidin-Graham-Totaro Chow groups of the stack $[X/GL_r]$.

6.1.2 Chow comparison theorem

The following lemma relates the motives of $X_G(\rho)$ and $X_\bullet$.

Lemma 6.1.3. *Let $\rho = (V_i, U_i)_{i\geq 1}$ be an admissible gadget for $G = GL_n$ over k. For any algebraic space X with a G-action, there is an isomorphism $M(X_G(\rho)) \cong M(X_\bullet)$ in $\mathbf{DM}^{eff}(k, \mathbb{Z})$.*

Proof. For any i, we have a principal G-bundle, $X \times U_i \to X \times^G U$. By Theorem 1.2.1, we have a Nisnevich local weak equivalence $(X \times U_i)_\bullet \to X \times^G U_i$. Now observe that for $j \geq 0$, we have equalities

$$(X \times U_i)_j = X \times G^j \times U_i.$$

Here, G^j denotes the j-fold product $G \times \ldots \times G$.

We now compute the homotopy colimits over the indices i, j,

$$\mathrm{colim}_i\, \mathrm{hocolim}_j\, X \times G^j \times U_i \simeq \mathrm{colim}_i\, X \times^G U_i \simeq X_G(\rho)$$

Similarly,

$$\mathrm{hocolim}_j\, \mathrm{colim}_i X \times G^j \times U_i \simeq \mathrm{hocolim}_j X \times G^j \times \mathrm{colim}_i U_i.$$

As i becomes larger, the codimension of $V_i \setminus U_i$ tends to infinity. Thus, by Lemma 6.1.4,

$$\mathrm{colim}\, M(U_i) \simeq \mathrm{colim}\, M(V_i).$$

is an $\mathbb{A}^1$-weak equivalence. But as V_i are affine spaces, $\mathrm{colim}\, M(U_i)$ is $\mathbb{A}^1$-contractible. Hence,

$$\mathrm{hocolim}_j M(X \times G^j \times \mathrm{colim}_i U_i) \simeq \mathrm{hocolim}_j M(X \times G^j) \simeq M(X_G).$$

Thus, the result follows after applying the motive to $X_G(\rho)$ and by noting that homotopy colimits commute with filtered colimits. □

Lemma 6.1.4 ([HL, Proposition 2.13]). *Let $U_n \hookrightarrow X_n$ be an inductive system of open immersions of smooth finite type schemes over k. Let c_n be the codimension of the complement $X_n \setminus U_n$ in X_n. If $c_n \to \infty$, then the morphism*

$$\mathrm{colim}_n M(U_n) \to \mathrm{colim}_n M(X_n)$$

is an $\mathbb{A}^1$-weak equivalence.

Lemma 6.1.3 immediately gives us a proof of Theorem 1.2.2.

Proof of Theorem 1.2.2. Note that $GL_r \times X$ is isomorphic to $X \times_{\mathfrak{X}} X$. Thus, the simplicial scheme $X^{\bullet}_{GL_r}$ is isomorphic to the Čech hypercover $X_{\bullet} \to \mathfrak{X}$. By Theorem 1.2.1, $X_{\bullet} \to \mathfrak{X}$ is a Nisnevich local equivalence.

Let $\rho = (V_i, U_i)$ be an admissible gadget for GL_r, with $\dim(V) = l$, $m := r^2$. By definition,

$$CH^i(\mathfrak{X}, 2i-n) = CH^i(X^N(\rho), 2i-n) \simeq Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}(M(X^N(\rho)), \mathbb{Z}(i)[n])$$

for N sufficiently large. Here, the second equivalence follows from Theorem 4.2.10 and the fact that $H^{n,i}(X, \mathbb{Z}) = Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}(M(X), \mathbb{Z}(i)[n])$. Note that this is well-defined since the maps $X^s(\rho) \to X^t(\rho)$ are vector bundles and so have the same Chow groups by $\mathbb{A}^1$-invariance.

Now, as $X_{GL_r}(\rho) = \operatorname*{colim}_N X^N(\rho)$ is a filtered colimit, by [BK, Example 12.3.5] it is also a homotopy colimit. Then, by Remark 4.3.4,

$$\begin{aligned} Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}(M(X_{GL_r}(\rho)), \mathbb{Z}(i)[n]) &\simeq Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}(M\big(\operatorname*{hocolim}_N X^N(\rho)\big), \mathbb{Z}(i)[n]) \\ &\simeq Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}\big(\operatorname*{hocolim}_N M(X^N(\rho)), \mathbb{Z}(i)[n]\big) \\ &\simeq \operatorname*{hocolim}_N Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}\big(M(X^N(\rho)), \mathbb{Z}(i)[n]\big). \end{aligned}$$

Since the maps $X^n(\rho) \to X^m(\rho)$ are $\mathbb{A}^1$-invariant, the groups $Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}\big(M(X^N(\rho)), \mathbb{Z}(i)[n]\big)$ stabilise for all $m \geq N$. Thus,

$$Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}(M(X_G(\rho)), \mathbb{Z}(i)[n]) \simeq Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}\big(M(X^N(\rho)), \mathbb{Z}(i)[n]\big)$$

whenever N is large enough. Further, we also have the relations,

$$H^{n,i}(\mathcal{X},\mathbb{Z}) = Hom_{\mathbf{DM}^{eff}(k,\mathbb{Z})}(M(\mathcal{X}),\mathbb{Z}(i)[n]) \simeq Hom_{\mathbf{DM}^{eff}(k,\mathbb{Z})}(M(X_\bullet),\mathbb{Z}(i)[n])$$

in $\mathbf{DM}^{eff}(k,\mathbb{Z})$. By Lemma 6.1.3, $M(X_G(\rho)) \simeq M(X_\bullet)$ in $\mathcal{H}_\bullet(k)$. Putting these together, we get the required isomorphism. □

Remark 6.1.5. *In [Jos2, Theorem 3.5] an analogous result is proved for quasi-projective X in $\mathbf{DM}^{eff}_{ét}(k,\mathbb{Q})$.*

Remark 6.1.6. *Equivariant algebraic cobordisms (see [HML], [Kri]) are defined by a Borel type construction analogous to definition of equivariant (higher) Chow groups. By the above considerations, one can think of equivariant algebraic cobordism as an algebraic cobordism of the associated quotient stack.*

6.1.3 Quotients of algebraic spaces

We will now show that Theorem 1.2.2 can be extended to GL_n-actions on algebraic spaces that are themselves quotient stacks.

Remark 6.1.7. *Let X be a smooth algebraic space over k that is the quotient of a smooth scheme by the action of GL_n, i.e, $X \simeq U/GL_n$ (in particular, this includes any smooth scheme or any smooth algebraic space with the resolution property). This gives us a GL_n-presentation $U_\bullet \to X$.*
Furthermore, since GL_n-action on U has trivial stabilisers the following lemma implies that the GL_n-equivariant Chow groups and ordinary Chow groups agree for X.

Lemma 6.1.8 ([EG, Proposition 8(a)])**.** *Let $Y \to X$ be a principal GL_n-bundle of algebraic spaces then*

$$CH^i(Y,m)^{GL_n} \simeq CH^i(X,m).$$

The above lemma together with Theorem 1.2.2 has the following corollary

Corollary 6.1.9. *Let X be a smooth algebraic space over k as in Remark 6.1.7. Then* *the motivic cohomology of X agrees with its the ordinary higher Chow groups*

$$CH^i(X, 2i-n) \simeq H^{n,i}(X, \mathbb{Z}).$$

Proof. By Theorem 1.2.2, motivic cohomology groups of X agree with the Edidin-Graham-Totaro Chow groups of X. Lemma 6.1.8 now implies the required isomorphisms. □

The above discussion gives us the following extension of Theorem 1.2.2 for GL_n-quotients of smooth algebraic spaces.

Proposition 6.1.10. *Let X be a smooth algebraic space over k as in Remark 6.1.7. Assume that X admits a smooth action of GL_r and denote by $\mathfrak{X} := [X/GL_r]$ the quotient stack. Then the Edidin-Graham-Totaro (higher) Chow groups and the motivic cohomology groups agree integrally, i.e,*

$$CH^i(\mathfrak{X}, 2i-n) \simeq H^{n,i}(\mathfrak{X}, \mathbb{Z}).$$

Proof. Same as the proof of Theorem 1.2.2 using Corollary 6.1.9. □

6.2 Application to exhaustive stacks

In [HL], a motive M_{exh} is defined for a class of stacks which they call as *exhaustive stacks*. They do this by using an idea similar to Totaro's "finite-dimensional approximation" technique in [Tot1]. Examples of exhaustive stacks are quotient stacks and the moduli stack of vector bundles on a curve of fixed rank and degree. In fact, exhaustive stacks turn out to be special cases of cd-quotient stacks (see Lemma 6.2.2). We will compare the motive M_{exh} with the motive of Definition 5.2.2.

In this section, we adopt the conventions used in [HL] for algebraic stacks. In particular this means that we work with stacks $\mathfrak{X}$ which admit a smooth atlas $p : U \to \mathfrak{X}$ by a locally finite type k-scheme U such that p is schematic (representable by *schemes*)[1].

[1]This is only to maintain consistency with the conventions in [HL]. It does not particularly affect the arguments that we present, which work for any stack locally of finite type over k.

Definition 6.2.1 ([HL, Definition 2.15])**.** Let $\mathcal{X}$ be an algebraic stack locally of finite type over a field k. Let $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \ldots$ be an increasing filtration of $\mathcal{X}$ such $\mathcal{X}_i \subset \mathcal{X}$ are quasi-compact open substacks and their union covers $\mathcal{X}$, i.e, $\mathcal{X} = \cup_i \mathcal{X}_i$. Then an *exhaustive sequence of vector bundles* with respect to this filtration is a sequence of pairs $\{(V_i, W_i)\}_{i\geq 0}$ where V_i is a vector bundle on $\mathcal{X}_i$ and $W_i \subset V_i$ is a closed substack such that

1. the complement $U_i := V_i \setminus W_i$ is a separated k-scheme of finite type,
2. we have injective maps of vector bundles $f_{i,i+1} : V_i \to V_{i+1} \times_{\mathcal{X}_{i+1}} \mathcal{X}_i$ such that $f^{-1}_{i,i+1}(W_{i+1} \times_{\mathcal{X}_{i+1}} \mathcal{X}_i) \subset W_i$ and,
3. the codimension of W_i in V_i tends to infinity as i increases.

A stack admitting an exhaustive sequence with respect to some filtration is said to be *exhaustive.*

In fact, one can show that every exhaustive stack admits a filtration by global quotient stacks. This implies that it is a cd-quotient stack.

Lemma 6.2.2. *Let $\mathcal{X}$ be an exhaustive stack. Let $\mathcal{X} = \cup_i \mathcal{X}_i$ be an increasing filtration with an exhaustive sequence of vector bundles. Then there exists an increasing filtration $\mathcal{X} = \cup_i \mathcal{Y}_i$ with $\mathcal{Y}_i \subseteq \mathcal{X}_i$ and each $\mathcal{Y}_i$ is a global quotient stack. In particular, it is a cd-quotient stack.*

Proof. Let $\{(V_i, W_i)\}_{i\geq 0}$ denote the exhaustive sequence of vector bundles corresponding to the filtration $\{\mathcal{X}_i\}_{i\geq 0}$. Let $p_i : V_i \to \mathcal{X}_i$ denote the structure map of the vector bundle V_i. Now by definition the complement $U_i = V_i \setminus W_i$ is a separated finite type k-scheme. Since p_i is smooth, the image $p_i(U_i) \subset \mathcal{X}_i$ is an open substack which is of finite type over k. Set $\mathcal{Y}_i := p_i(U_i)$. Consider the restriction $V_i \times_{\mathcal{X}_i} \mathcal{Y}_i \to \mathcal{Y}_i$ of V_i to this substack. This is a vector bundle on $\mathcal{Y}_i$ which contains an open representable substack $U_i \subset V_i \times_{\mathcal{X}_i} \mathcal{Y}_i$ that surjects onto $\mathcal{Y}_i$. Thus, $\mathcal{Y}_i$ is a global quotient stack, by [EHKV, Lemma 2.12].

Thus, we have an increasing filtration $\{\mathcal{Y}_i\}_{i\geq 0}$ such that $\mathcal{Y}_i \subseteq \mathcal{X}_i$. The only thing left to check is that this filtration covers $\mathcal{X}$. This follows from the following topological argument.

Take a point $x \in \mathcal{X}$. Then as the filtration $\{\mathcal{X}_i\}_{i\geq 0}$ covers $\mathcal{X}$, there exists an i such that $x \in \mathcal{X}_i$. We will show that there exists an $N \geq i$ such that $x \in \mathcal{Y}_N$.

If $x \in \mathcal{Y}_i$, there is nothing to prove. So assume that $x \notin \mathcal{Y}_i$. This means that $p_i^{-1}(x) \subset W_i$ in the vector bundle V_i. Let $Z := \overline{p_i^{-1}\{x\}}$ be the closure of the fibre in V_i. Since $Z \subseteq W_i$ we see that

$$n := \operatorname{codim} Z \geq \operatorname{codim} W_i.$$

As $\{(V_i, W_i)\}_{i\geq 0}$ is an exhaustive sequence, there exists an N such that $\operatorname{codim} W_N > n$. Further, we have a map $f_{i,N} : V_i \to V_N$ such that $f_{i,N}^{-1}(W_N) \subset W_i$. If Z was contained in $f_{i,N}^{-1}(W_N)$, we would have

$$n = \operatorname{codim} Z \geq \operatorname{codim} f_{i,N}^{-1}(W_N) > n,$$

a contradiction. Thus, there exists $y \in p_i^{-1}(x)$ such that $f_{i,N}(y) \in U_N$ implying that $x \in \mathcal{Y}_N$. □

Definition 6.2.3 ([HL, Definition 2.17])**.** Let $\mathcal{X}$ be an exhaustive stack with an exhaustive sequence of vector bundles $\{(V_i, W_i)\}_{i\geq 0}$. The motive $M_{exh}(\mathcal{X})$ is defined in as the colimit of the motives of the schemes U_i. That is,

$$M_{exh}(\mathcal{X}) = \operatorname{colim} M(U_i).$$

Since exhaustive stacks are cd-quotient stacks, we would like to compare the motive $M_{exh}(\mathcal{X})$ with the motive $M(\mathcal{X})$ in Definition 5.2.2. The following proposition shows that they are isomorphic in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$.

Proposition 6.2.4. *Let $\mathcal{X}$ be a smooth exhaustive stack. Then*

$$M(\mathcal{X}) \simeq M_{exh}(\mathcal{X}) \ \ \text{in} \ \ \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$$

Proof. Let $X \to \mathcal{X}$ be the 0-skeleton of a GL_n-presentation. By [Knu, Theorem II.6.4], there exists a Nisnevich covering $Y \to X$ with Y a scheme. This gives us a presentation $Y \to \mathcal{X}$. Let $Y_\bullet \to \mathcal{X}$ be the associated Čech hypercover. By similar argument as in Theorem 1.2.1, we get that $Y_\bullet \simeq \mathcal{X}$ in $\mathcal{H}_\bullet(k)$, and so we have $M(Y_\bullet) \simeq M(\mathcal{X})$ in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$. Hence, it suffices to show that $M(Y_\bullet) \simeq M_{exh}(\mathcal{X})$ in $\mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$. The proof is exactly the same as [HL, Proposition A.7] using the atlas $Y_\bullet \to \mathcal{X}$. □

Example 6.2.5. *Let C be a smooth projective geometrically connected curve of genus g*

over a field k. In [HL, Section 3], it is proved that the moduli stack $Bun_{n,d}$ of vector bundles on a C of fixed rank n and degree d is exhaustive. To show this, they observe that it admits a filtration by the maximal slope of all vector bundles. Thus, $Bun_{n,d}$ is a filtered colimit of open substacks $Bun_{n,d}^{\geq \mu_l}$ where $\{\mu_l\}$ is an increasing sequence of rational number representing the maximal slope. Also, each of these open substacks is a global quotient stack and can be written as $Bun_{n,d}^{\geq \mu_l} := [Q^{\geq \mu_l}/GL_N]$ where $Q^{\geq \mu_l}$ is an open subscheme of a Quot scheme (see [LMB, Théorème 4.6.2.1] for futher details). Then, by [HL, Lemma 2.26], we have

$$M_{exh}(Bun_{n,d}) = \operatorname{hocolim}_l M_{exh}(Bun_{n,d}^{\geq \mu_l})$$

and from Proposition 6.2.4 we get that

$$M(Bun_{n,d}) = \operatorname{hocolim}_l M(Bun_{n,d}^{\geq \mu_l}) = \operatorname{hocolim}_l M(Q_\bullet^{\geq \mu_l}).$$

Thus, for $Bun_{n,d}$, the motive M_{exh} of [HL] can be computed as a homotopy colimit of the motive of Definition 5.2.2. Since these homotopy colimits are being taken over filtered categories, they can actually be computed by their ordinary colimits (see [BK, Example 12.3.5]).

Remark 6.2.6. *Proposition 6.2.4 shows that for exhaustive stacks the motive defined using the nerve construction in Definition 5.2.2 agrees with M_{exh} defined in [HL] in the Nisnevich topology. This potentially simplifies many of the functoriality arguments in [HL, §2]. For example, it is now immediate that M_{exh} is independent of the choices involved in its construction (see also [HL, Lemma 2.20]). In [HL, Appendix A], such a comparison is proved for étale motives.*

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