Topics in Algebraic K-theory

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This is to certify that this dissertation entitled **Topics in Algebraic K-theory** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents original research carried out by K. Arun Kumar at Indian Institute of Science Education and Research under the supervision of Dr. Amit Hogadi, Associate Professor, Department of Mathematics, during the academic year 2014-2015.

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To my family

Declaration

I hereby declare that the matter embodied in the report entitled **Topics in Algebraic K-theory** are the results of the investigations carried out by me at the Department of Mathematics, the Indian Institute of Science Education and Research Pune, under the supervision of Dr. Amit Hogadi and the same has not been submitted elsewhere for any other degree.

K. Arun Kumar

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Abstract

The objective of this thesis is to study the algebraic K-theory of exact categories. In algebraic K-theory we construct a sequence of groups, called K_n, which are invariants of a given exact category. We look at two different constructions of K_n, Quillen's Q-construction of the K-groups of an exact category as the homotopy groups of a topological space and Wladhausen's S-construction of the K-groups as the stable homotopy groups of a spectrum, and show that they are equivalent. The S-construction is then used to prove the main aim of this thesis, the additivity theorem. The additivity theorem then helps us prove fundamental results about the K-groups. The main results considered are, the cofinality theorem and resolution theorem for exact categories, and the devissage theorem and localisation theorem for abelian categories.

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Chapter 1

Preliminaries

In this chapter we look at all the topological preliminaries that are required for this thesis. The main reference for this section is [4].

1.1 CW complexes

Let D^n denote the closed unit ball in $\mathbb{R}^n$, e^n denote the open unit ball in $\mathbb{R}^n$ and S^n the n-sphere. We define ∂D^n, called the boundary of D^n, to be the boundary of D^n as a subset of R^n, then $\partial D^n \cong S^{n-1}$ and $D^n - \partial D^n \cong e^n$. In this section we define CW-complexes and state some of their properties that will be needed in this paper, these are topological spaces constructed by gluing together D^ns along their boundaries. The following definition and theorems are from Hatcher's book [4].

Definition 1.1.1. A topological space X is called a CW complex if there exist subspaces X^n for all $n \in \mathbb{N} \cup \{0\}$ satisfying the following,

(1) X^0 is a discrete subspace of X,

(2) For all n, there exists a collection of continuous functions $\{\Phi^n_\alpha : S^{n-1}_\alpha \to X^{n-1}\}$ such that, $X^n = X^{n-1} \coprod_\alpha D^n_\alpha / \sim$, where the equivalence relation is, if $x \in \partial D^n_\alpha$, then $x \sim \Phi^n_\alpha(x)$ under the identification $\partial D^n_\alpha \cong S^{n-1}_\alpha$.

(3) $X = \bigcup_n X^n$ with the weak topology, i.e, $A \subseteq X$ is open iff $A \cap X^n$ is open in X^n for all n.

Note that the third condition does not follow from the first two in general, unless $X = X^n$ for some n. Given the maps $\Phi^n_\alpha : S^{n-1}_\alpha \to X^{n-1}$, we can use the identification

$\partial D_\alpha^n \cong S_\alpha^{n-1}$ to extend this to a map $\Phi_\alpha^n : D_\alpha^n \to X$ such that restricted to $D_\alpha^n - \partial D_\alpha^n \cong e_\alpha^n$ it is a homeomorphism onto the image. The images of e_α^n under these maps are called the $n-$cells of the CW complex, and $\overline{e_\alpha^n} = \Phi_\alpha^n(D_\alpha^n)$. Also note that given a topological space there maybe more than one way give it a CW complex structure, for example a 2-sphere can be realised as a CW complex with a single 0-cell(a point) and a single 2-cell, with its boundary collapsed to the single 0-cell, or it can be realised with two 2-cells with their boundaries (which are the 1-cells) identified homeomorphically. We call a subspace A of a CW complex X a **sub-complex** if for all cells e_α^n of X, $e_\alpha^n \cap A \neq \emptyset$, implies $\overline{e_\alpha^n} \subseteq A$. We note the following important facts about CW-complexes [4, A.1],

Theorem 1.1.1. (1) Every CW complex is compactly generated, that is a subset is open (or closed), iff its intersection with each compact subset is open(or closed).

(2) A subspace of a CW complex X is compact iff it is contained in a union of finitely many cells

Product of CW complexes

The product of two CW complexes may not in general be a CW complex, similar to how the product of two compactly generated spaces may not be compactly generated. But given any topological space X we can define a new topology on X, where a subset is said to be open if and only if its intersection with every compact set, under the previous topology, is open. We denote this space by X_c, it has the same compact subspaces as X and is compactly generated.

Theorem 1.1.2. Let X and Y be CW complexes with cell maps Φ_α and Ψ_β, then $(X \times Y)_c$ is a CW complex with cell maps $\Phi_\alpha \times \Psi_\beta$ and it is the product of X and Y in the category of compactly generated spaces. If either X or Y is compact or locally compact then $(X \times Y)_c = X \times Y$

1.2 Higher homotopy groups

Given a pointed topological space (X, x_0) we can assign to it the fundamental group $\pi_1(X, x_0)$ comprising of homotopy classes of maps $f : (I, 0) \to (X, x_0)$, where I is the unit interval, such that $f(0) = f(1)$. These can also be seen as continuous functions

$f : (S^1, (1,0)) \to (X, x_0)$. Higher homotopy groups generalise this notion to classes of maps from $I^n \to X$. First we need some notation, by (X, A) we mean a topological space X along with a subspace $A \subseteq X$ and map $f : (X, A) \to (Y, B)$ is a continuous function $X \to Y$ such that $f(A) \subseteq B$.

Let I^n denote the product of n copies of I and ∂I^n its boundary as a subset of $\mathbb{R}^n$. The n^{th} homotopy group $\pi_n(X, x_0)$ of a pointed topological space (X, x_0), as a set is set of all homotopy classes of functions $f : (I^n, \partial I) \to (X, x_0)$, where f and g are homotopic if there exists $F : (I^n \times I, \partial I^n \times I) \to (X, x_0)$ such that $F(\vec{s}, 0) = f(\vec{s})$ and $F(\vec{s}, 1) = g(\vec{s})$. The group structure is defined as follows:

$$f + g(x_1, x_2, \ldots, x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \leq 1/2 \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \geq 1/2 \end{cases} \tag{1.1}$$

Just like for the case of π_1, π_n can be described as the set of maps $f : (S^n, s_0) \to (X, x_0)$, as $I^n/\partial I^n$ is homeomorphic to S^n. The group operation is then given by first considering the wedge sum $f \vee g$ of the given two maps defined on the wedge of two copies of (S^n, s_0), and then composing it with the map $S^n \to S^n \vee S^n$ obtained by collapsing the equator to a point. We have used the '+' symbol for the group law here because for $n \geq 2$, $\pi_n(X, x_0)$ is abelian for any space (X, x_0). A proof of this fact is given in [4, 4.1], we give a different proof here using the Eckmann-Hilton argument.

Theorem 1.2.1 (Eckmann-Hilton argument)**.** Let X be a set equipped with two binary operations '$\circ$' and '*', such that,

1. Both operations have a two sided identity say $1_\circ$ and 1_*
2. $\forall a, b, c, d \in X (a * b) \circ (c * d) = (a \circ c) * (b \circ d)$

Then * and $\circ$ are the same binary operation and both of them are commutative.

Proof. First we show that the identities of the two operations are the same.

$$1_* = 1_* * 1_* = (1_\circ \circ 1_*) * (1_* \circ 1_\circ) = (1_\circ * 1_*) \circ (1_* * 1_\circ) = 1_\circ \circ 1_\circ = 1_\circ$$

Therefore we denote the identity by just 1. Now we have

$$f * g = (f \circ 1) * (1 \circ g) = (f * 1) \circ (1 * g) = f \circ g$$

and

$$f * g = (1 \circ f) * (g \circ 1) = (1 * g) \circ (f * 1) = g \circ f$$

Therefore both the operations are the same and the unique operation is commutative.

□

We will use the above theorem and the fact that, for $n \geq 2$ there is more than one coordinate along which we can concatenate two functions, to show the group structure on $\pi_n(X, x_0)$ is abelian.

Theorem 1.2.2. Let $+$ be the operation on $\pi_n(X.x_0)$ as above for $n \geq 2$ and $+'$ be the operation as below

$$f +' g(x_1, x_2, \ldots, x_n) = \begin{cases} f(x_1, 2x_2, \ldots, x_n) & x_2 \leq 1/2 \\ g(x_1, 2x_2 - 1, \ldots, x_n) & x_2 \geq 1/2 \end{cases} \tag{1.2}$$

Then $f + g = f +' g$ for all $f, g \in \pi_n(X, x_0)$ and the operation is commutative

Proof. Let $f, g, h, i : (I^n, \partial I^n) \to (X, x_0)$, then

$$(f + g)(x_1, x_2, \ldots, s_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \leq 1/2 \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \geq 1/2 \end{cases}$$

and,

$$(f +' g)(x_1, x_2, \ldots, x_n) = \begin{cases} f(x_1, 2x_2, \ldots, x_n) & x_2 \leq 1/2 \\ g(x_1, 2x_2 - 1, \ldots, x_n) & x_2 \geq 1/2 \end{cases}$$

Therefore we have,

$$(f + g) +' (h + i)(x_1, x_2, \ldots, x_n) = \begin{cases} f(2x_1, 2x_2, \ldots, x_n) & x_1 \leq 1/2, x_2 \leq 1/2 \\ g(2x_1 - 1, 2x_2, \ldots, x_n) & x_1 \geq 1/2, x_2 \leq 1/2 \\ h(2x_1, 2x_2 - 1, \ldots, x_n) & x_1 \leq 1/2, x_2 \geq 1/2 \\ i(2x_1 - 1, 2x_2 - 1, \ldots, x_n) & x_1 \geq 1/2, x_2 \geq 1/2 \end{cases}$$

Expanding out $(f +' h) + (g +' i)$ we get the same expression as the RHS above. Therefore by the theorem above $[f] + [g] = [f] +' [g]$ for all f and g. □

Note that the equality of the operations is only true for the homotopy classes, even though the equality above was equality as functions, as the Eckmann-Hilton argument only holds for operations with an identity and this is true only on homotopy classes. Also, generalising the above argument we can show that on $\pi_n(X, x_0)$, gluing along any coordinate gives the same group structure. We state two important theorems that are easy to verify.

Theorem 1.2.3. Let $p : (E, e_0) \to (B, b_0)$ be a covering map, then for all $n \geq 2$, $p_* : \pi_n(E, e_0) \to \pi_n(B, b_0)$ is an isomorphism.

The above theorem follows from the universal property of covering maps and the fact that S^n is simply connected for $n \geq 2$. This implies that if a space has a contractible universal cover than its higher homotopy groups are trivial. The next theorem is about homotopy groups of products of spaces.

Theorem 1.2.4. All homotopy groups commute with products, that is, $\pi_n(\prod_\alpha X_\alpha, (x_\alpha)) \cong \prod_\alpha \pi_n(X_\alpha, x_\alpha)$ for all n and all (X_α, x_α)

1.2.1 Relative Homotopy groups

Given a pair (X, A) of a topological space and its subspace, we want to assign a group structure to the homotopy class of maps $(I^n, \partial I^n) \to (X, A)$. But just like in the case of homotopy groups of (X, x_0), we need to fix a point so that we can glue two such functions. To do this fist we define $J^{n-1} \subset \partial I^n$ be the subset consisting of all the faces of I^n except for the interior of one face (we choose it to be $x_n = 0$ face). For example J^1 consists of three sides of a square with the endpoints of the fourth side, J^0 is just the endpoint 1. Now we look at $\pi_n(X, A, x_0)$, the set of homotopy classes of maps between the triples of spaces $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, where the homotopies considered are maps of the kind, $(I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (X, A, x_0)$. The group operation can be defined similar to the case of homotopy groups, but unlike the previous case, where we could concatenate along any coordinate, here we cannot concatenate along x_n as by definition, fixing $x_n = 0$ does not guarantee that $f(x_1, \ldots, x_n = 0) = x_0$. Therefore $\pi_1(X, A, x_0)$ is the set of homotopy classes of paths beginning at x_0 and ending at a point in A, and for $n \geq 0$ we have a group structure which can be shown to be abelian for $n \geq 3$ the same way as for $\pi_n(X, x_0)$. If we take $A = x_0$, then $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$.

We have two inclusions $j : (X, x_0, x_0) \to (X, A, x_0)$ and $i : (A, x_0) \to (X, x_0)$ which induce maps $j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0)$ and $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ for all n. Along with these we have the boundary maps $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ which is the restriction of the map $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to the $x_n = 0$ face giving a map $f : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$, therefore we have a sequence

$$\begin{aligned} \dots \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \dots \\ \dots \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0) \end{aligned} \tag{1.3}$$

Note that the last two terms are singleton sets. Now we state the most important theorem we will need about relative homotopy groups. This is Th.4.3 in [4, Pg.344].

Theorem 1.2.5. The above sequence is exact.

Where at the end of the sequence exactness just means that the map $\pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0)$ is surjective. An important application of this theorem is to fibration sequences which will be described in a later section.

1.3 Cellular maps and Cellular approximation

In this section we state two important theorems about CW-complexes, Whitehead's theorem [4, Th.4.5] and Cellular approximation [4, Th.4.8]. As CW complexes are built from copies of D^n, which are homeomorphic to I^n, their homotopy groups are easier to study. In particular all n-spheres are CW complexes and the theorem below helps calculate their homotopy groups, but first we make a definition. A map $f : X \to Y$ of CW complexes is called **cellular**, if $f(X_n) \subset Y_n$ for all n, or equivalently it can be constructed as a sequence of compatible maps $f_n : X_n \to Y_n$. Again a map being cellular depends on the cell structure assigned. Now we are ready to state the theorem.

Theorem 1.3.1 (Cellular Approximation)**.** Let $f : X \to Y$ be a continuous function between CW complexes, then f is homotopic to a cellular map. Moreover, if f is already cellular on a sub-complex A of X, then the homotopy can be chosen to be constant on A.

As each S^n has a CW complex structure with one 0-cell and one n-cell, we have the following corollary.

Corollary 1.3.2. $\pi_n(S^m, s_0) = 0$ for all $n < m$.

Definition 1.3.1 (Weak equivalence)**.** A continuous function $f : X \to Y$ is called a weak equivalence if the induced map $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $n \geq 0$ and all $x_0 \in X$. Where at $n = 0$ we mean a bijection of connected components. Then X and Y are said to be weakly equivalent spaces.

Note that the above definition is stronger than all homotopy groups of X and Y being isomorphic, as here we require the isomorphisms to be induced by a map $f : X \to Y$. It is easy to see that every homotopy equivalence is in fact a weak equivalence. The converse is not true in general but is true for CW complexes.

Theorem 1.3.3 (Whitehead's Theorem)**.** If $f : X \to Y$ is a weak equivalence between CW complexes then f is a homotopy equivalence. Moreover, if f is an inclusion of a sub-complex, then X is a deformation retract of Y.

A direct consequence of this result is that if a connected CW complex has all homotopy groups trivial, then it is contractible.

Space of functions

Given topological spaces X and Y we would like to have a suitably nice topology on $\mathrm{Hom}(X, Y)$, the set of all continuous functions from X to Y. Given sets A, B and C, we know there is a bijection of sets between $\mathrm{Hom}(A \times B, C) \leftrightarrow \mathrm{Hom}(A, \mathrm{Hom}(B, C))$. Here the notation $\mathrm{Hom}(A, B)$ denotes the set of all functions from A to B. Our goal is to define a topology on $Hom(X, Y)$ above for all topological spaces such that a similar results holds. That is, for all topological spaces X, Y and Z, there is a homeomorpism $\mathrm{Hom}(X \times Y, Z) \leftrightarrow \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$. Now we try to define such a topological space.

Definition 1.3.2. Let X and Y be topological spaces. We make $\mathrm{Hom}(X, Y)$ a topological space from the following collection: Given $K \subseteq X$ compact and $U \subseteq Y$ open we define $M(K, U)$ to be the subset of $\mathrm{Hom}(X, Y)$ of all continuous functions such that $f(K) \subseteq U$ this is called the *compact open topology* on $\mathrm{Hom}(X, Y)$

A special case of this construction is when X is the n-dimensional cube, this space $\mathrm{Hom}(I^n, X)$ is called the n-dimensional path space of X. We quickly summarise the properties of this construction below, which can be found in [4, A.2].

Theorem 1.3.4. If X is locally compact, then:

(1) the evaluation map $e : \mathrm{Hom}(X, Y) \times X \to Y$, $e(f, x) = f(x)$ is continuous.

(2) A map $f : X \times Z \to Y$ is continuous iff the map $\hat{f} : Z \to \mathrm{Hom}(X, Y)$, $\hat{f}(z)(x) = f(x, z)$ is continuous.

(3) If X is locally compact Hausdorff and Z is Hausdorff, then the map $\mathrm{Hom}(X \times Z, Y) \to \mathrm{Hom}(X, \mathrm{Hom}(Z, Y))$, $f \to \hat{f}$ above is a homeomorphism.

In the case of path spaces we have $\mathrm{Hom}(I^n, X) \cong \mathrm{Hom}(I^{n-1}, \mathrm{Hom(I,X)})$.

1.3.1 Smash products and Loop spaces

As seen in our definition of homotopy groups we are mostly interested in pointed topological spaces and maps that preserve base point. We are therefore interested in $\mathrm{Hom}(X, x_0, Y, y_0)$, the subspace of $\mathrm{Hom}(X, Y)$ consisting of base point preserving maps. These have a relation similar to Theorem 1.3.4. Before we state it we need to make a definition

Definition 1.3.3. Given two pointed space (X, x_0) and (Y, y_0) their *Smash Product* is defined as the quotient $X \wedge Y = X \times Y / X \vee Y$, where $X \vee Y$ is identified homeomorphically with $X \times \{y_0\} \cup \{x_0\} \times Y$.

The smash product gives a way to construct higher dimensional spheres from lower dimensional ones. More precisely,

$$S^n \wedge S^m \cong S^{n+m}$$

where we have suppressed the base point as the spheres are homogeneous.

Theorem 1.3.5. If (X, x_0) is locally compact Hausdorff and (Z, z_0) is Hausdorff, then the map $\mathrm{Hom}(X \wedge Z, Y, y_0) \to \mathrm{Hom}(X, x_0, \mathrm{Hom}(Z, z_0, Y, y_0))$, $f \to \hat{f}$ induced by the map in Theorem 1.3.4, is a homeomorphism.

A special case of this relation arises X and Z are spheres. We fix the notation $\mathrm{Hom}((S^n, s_0), (X, x_0)) = \Omega^n(X, x_0)$, where if $n = 1$, we drop the superscript and call it the *loop space* of (X, x_0). Note that the n^{th} homotopy group of a pointed space is a quotient of its $\Omega^n(X, x_0)$. From here we drop the base point when talking about pointed spaces if it doe not matter in the context. Let $(S^n, s_0) \wedge (X, x_0)$ be denoted by $\sum X$. Then we have, $\mathrm{Hom}_*(\sum X, Y) \cong \mathrm{Hom}_*(X, \Omega Y)$, when X is locally compact

Hausdorff (from here we use the subscript * to denote that we are talking about maps that preserve base points). In particular we have $\mathrm{Hom}_*(S^n, Y) \cong \mathrm{Hom}_*(S^{n-1}, \Omega Y)$. Therefore we have the induced isomorphism on the homotopy groups,

$$\pi_n(\Omega X) \cong \pi_{n+1}(X) \tag{1.4}$$

.From this correspondence it is easy to see that $\Omega^n X \cong \Omega\Omega^{n-1}X$. Hence $\Omega^n X$ are called the iterated loop spaces of X.

1.4 Fibration sequences

Fibration sequences will be an important tool in this thesis. The main reference for this section will be [1, Ch.6]. Before we talk about them we need to make a couple of definitions. A map $p : E \to B$ is said to have **homotopy lifting property** with respect to a space X if given any commutative of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_0"] & E \arrow[d, "p"] \\ X \times I \arrow[r] & B \end{tikzcd}$$

there exists a lift $X \times I \to E$, that makes both the diagrams commute. We we are ready to define fibration sequences.

Definition 1.4.1.

- A map $p : E \to B$ is called a *Serre fibration* if it has the homotopy lifting property with respect to disks D^n for all n.
- For a *Serre fibration* $p : E \to B$ and given $b_0 \in B$, if we define $F = p^{-1}\{b_0\}$, then $F \hookrightarrow E \to B$ is called a fibration sequence.

Theorem 1.4.1. Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration sequence where $F = p^{-1}\{b_0\}$. For all $x_0 \in F$, the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism and hence by Equation 1.3 we have a long exact sequence

$$\ldots \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \ldots\ldots \to \pi_1(B, b_0) \to 0 \tag{1.5}$$

A important class of fibrations are given by path spaces. Given X a topological space and $x_0 \in X$, then $Path(X, x_0) = \mathrm{Hom}_*((I, 0), X, x_0)$ is the space of all paths in X that begin at x_0 with the subspace topology from the path space. Then the evaluation map $Path(X, x_0) \to X$ given by $p \to p(1)$ is a fibration. To see this let us begin with a commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & Path(X, x_0) \\ \downarrow & & \downarrow \\ A \times I & \xrightarrow{H} & X \end{array}$$

Then we can construct the lift $\hat{H} : A \times I \to Path(X, x_0)$ given by

$$\hat{H}(a,t)(s) = \begin{cases} f(a)((t+1)s) & 0 \leq s \leq 1/(t+1) \\ H(a,(1+t)s-1) & 1/(t+1) \leq s \leq 1 \end{cases}$$

This makes the diagram commute. Hence $Path(X, x_0) \to X$ is a fibration with fibre ΩX. We can easily see that $Path(X, x_0)$ is contractible and hence the long exact sequence of homotopy groups (Theorem 1.4.1) give the isomorphisms in Equation 1.4.

Chapter 2

Simplicial sets

In this chapter we define simplicial sets and study some of their important properties. The references for this chapter are [3] and [2]. Simplicial sets provide a way to study and construct CW complexes with purely combinatorial data. We can do this, as giving a CW complex is the same as giving the number of each n-cells and how they are glued together. We first give a rigorous definition through category theory and then look at what the definition entails.

Definition 2.0.2. Let $\mathbf{\Delta}$ denote the category whose objects are sets $[n] = \{0, \dots, n\}$ and a morphism $f : [n] \to [m]$ is an non decreasing function $[n] \to [m]$. That is, $x \leq y \implies f(x) \leq f(y), \forall x, y \in [n]$. It is easy to see that this makes $\mathbf{\Delta}$ a small category with terminal object $[0]$.

Definition 2.0.3 (Category of simplicial sets)**.** A **simplicial set** is a contravariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$. The functor category $[\mathbf{\Delta}^{op}, \mathbf{Set}]$ is called the category of simplicial sets and is denoted by **sSet**. For a simplicial set X,we fix the notation $X_n = X([n])$

As with the case of any functor category, **sSet** has elements $\mathrm{Hom}(-, [n])$, the Hom functors. We denote these by $\mathrm{Hom}(-, [n]) = \Delta^n$. Then by Yoneda embedding, $X_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X)$. Using results about the functor category $[\mathcal{C}^{op}, \mathbf{Set}]$ for a small category $\mathcal{C}$, we get that **sSet** is complete and cocomplete, and every simplicial set is a colimit of Δ^n(as Δ^n are the Hom functors), precisely

$$X \cong \operatorname*{colim}_{\Delta^n \to X} \Delta^n \tag{2.1}$$

Where the colimit notation means taking colimit over the slice category, $\mathbf{\Delta} \downarrow X$.

2.1 Face and degeneracy maps

Every simplicial set is a functor from $\mathbf{\Delta}^{op} \to \mathbf{Set}$, therefore it not only consists of sets X_n, but also maps between them induced by morphisms in $\mathbf{\Delta}$. These are called the *simplicial maps*. Let us look at the morphisms in Δ. We have two special categories of morphisms of Δ. For any $[n]$, let $s_i : [n+1] \to [n]$ be the map $s_i(j) = j$ for $j \leq i$ $s_i(j) = j - i$ for $j > i$. This map repeats i, it sends both i and $i+1$ to i. For any $[n]$ let $d_i : [n] \to [n+1]$ be the map $s_i(j) = j$ for $j < i$ and $s_i(j) = j+1$ for $j \geq i$. This is the map that skips i, meaning i has no pre-image under this map. The simplicial maps induced by d_i and s_j are denoted by d_i^* and s_j^* and are called the face and degeneracy maps respectively. Note that as X is contravariant the induced maps are in the opposite direction, that is, $d_i^* : [n] \to [n+1]$ and $s_j^* : [n+1] \to [n]$. Elements of X_n are called n-simplices of X, (this is consistent with naming of Δ^n as n-simplices due to Yoneda lemma as stated above). An n-simplex x is called degenerate if $x = s_i^*(y)$ for some $y \in X_{n-1}$ and it is called non-degenerate otherwise. The face and degeneracy maps satisfy some relations summarised by [2].

$$\begin{aligned}
d_i^* d_j^* &= d_{j-1}^* d_i^* && i < j \\
d_i^* s_j^* &= s_{j-1}^* d_i^* && i < j \\
d_j^* s_j^* &= d_{j+1}^* s_j^* = id && \\
d_i^* s_j^* &= s_j^* d_{i-1}^* && i > j+1 \\
s_i^* s_j^* &= s_{j+1}^* s_i^* && i \leq j
\end{aligned} \tag{2.2}$$

We can easily show that all other simplicial maps can be obtained from composing the face and degeneracy maps stated above.

2.2 Realisation

As we observed before, for any simplicial set, $X \cong \underset{\Delta^n \to X}{\operatorname{colim}} \Delta^n$. From this we can deduce that to define a functor from $\mathbf{sSet}$ to any category which has all colimits, we only need to define the map on the standard n-simplices and extend it to a colimit preserving functor (this is a property of the category of functors from any small category to sets). We use this to construct a functor $|.| : \mathbf{sSet} \to \mathbf{Top}$ which sends Δ^n to the topological n-simplex $|\Delta^n| = \{(x_0, x_2, ...x_n) | \sum_{i=0}^{n} x_i \leq 1\}$, which is homoemorphic to the n-dimensional disc. Then we can extend it uniquely up to isomorphism to get a functor $|.| : \mathbf{sSet} \to \mathbf{Top}$ called the *realisation functor*. As colimits in the category of topological spaces are constructed by taking disjoint unions and quotients by equivalence relations, we can see that the realisation of a simplicial set is a CW-complex. Note that **sSet** has all limits and colimits and while the realisation functor preserves coproducts, it takes the product of simplicial sets to their product as compactly generated spaces and not the topological product in general. This realisation functor has a left adjoint,

Definition 2.2.1. Let X be a topological space. We define a simplicial set $Sing(X)$ called its singular set as follows. $Sing(X)_n = \{f : |\Delta^n| \to X\}$. The i^th face map is given by the composing with the inclusion $\Delta^{n-1} \hookrightarrow \Delta^n$ of the n^{th} face. The i^{th} degeneracy map is given by collapsing of the i^{th} face (by removing the i^{th} component of each point), giving a simplex of one smaller dimension. As all other simplicial maps are a combination of these we have defined the simplicial set required.

Note: The topological simplices and their face maps were the original motivation for the definition of simplicial sets and the degeneracy maps serve the purpose of treating simplices of lower dimension as degenerate simplices of higher dimension.

We state the adjoint relation here from [2, Th.4.10].

Theorem 2.2.1. The functors $|.|$ and $Sing()$ are adjoints, that is, there is a natural isomorphism,

$$\mathrm{Hom}_{Top}(|X|, Y) \leftrightarrow \mathrm{Hom}_{\mathbf{sSet}}(X, Sing(Y))$$

The study of simplicial sets is deeply rooted in the study of topological spaces. Most of the tools we develop for simplicial sets are done so with the aim of studying topological spaces. With this in mind we define a morphism of simplicial sets $X \to Y$ to be a *weak equivalence* if its realisation $|X| \to |Y|$ is a weak equivalence of topological spaces. Note as realisation of simplicial sets are CW-complexes, such a map is a weak equivalence if and only if it is a homotopy equivalence on the realisations. Therefore for simplicial sets we use the term weak equivalence and homotopy equivalence interchangeably. We try to define paths and homotopy of morphisms of simplicial sets similar to topological spaces. Here Δ^1 plays the role of the unit interval (in fact $|\Delta|$ is the unit interval).

Definition 2.2.2. (1) A *simplicial path* in a simplicial set X is a morphism of simplicial sets $\Delta^1 \to X$. This is nothing but a 1-simplex of X. The end points of this path are defined to be d_0^* and d_1^* of the corresponding 1-simplex.

(2) A *simplicial homotopy* of morphisms of simplicial sets $f, g : X \to Y$ is a map of simplicial sets $F : X \times \Delta^1 \to Y$ such that $F(1, d_0) = f$ and $F(i, d_1)$, where by 1 we mean identity on X and d_0 and d_1 are the maps $\Delta^0 \to \Delta^1$ induced by $d_0, d_1 : [0] \to [1]$ respectively (here as Δ^0 is the terminal object in **sSet** we identify $X \times \Delta^0$ with X).

Note that unlike homotopy in topological spaces, the definition of simplicial homotopy is not symmetric, that is there may not be a morphism, $G : X \times \Delta^1 \to Y$ with $G(1, d_0) = g$ and $G(i, d_1) = f$. It is straightforward to see that a simplicial homotopy of morphisms gives a homotopy of their realisations. Simplicial paths and homotopies are not as well behaved as their topological counterparts. For example, given three 0-simplices a,b and c in X, a path from a to b and a path from b to c does not necessarily give a path from a to c. We now therefore study a class of simplicial sets and morphisms that are more well behaved.

2.3 Kan fibrations

Given the standard n-simplex Δ^n, we define its k^{th} *horn* denoted by Λ_k^n as the simplicial subset of Δ^n generated by all its non degenerate $n-1$-simplices except for the k^{th} face. Where the k^{th} face is $d_k^*(1)$, and 1 is the identity map in $\Delta_n^n = \mathrm{Hom}([n], [n])$. The horns of Δ^n are constructed from gluing n many Δ^{n-1} together along their faces. An important property of the horns is that the realisation of the horn inclusion $|\Lambda_k^n| \hookrightarrow |\Delta^n|$ is a deformation retract (equivalent to the map being a weak equivalence). Any such monomorphism of simplicial sets whose realisation is a deformation retract will be called a *trivial cofibration.* Now we are ready to define Kan fibrations.

Definition 2.3.1. A map of simplicial sets $f : X \to Y$ is called a *Kan fibration* if for every commutative diagram of the type

$$\begin{tikzcd}
\Lambda_k^n \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^n \arrow[r] & Y
\end{tikzcd}$$

There exists a lift (may not be unique) $\Delta^n \to X$ which makes the resultant triangles commute (as in the case of topological spaces,we call this the right lifting property). A simplicial set X is said to be *Kan fibrant*, if the unique map $X \to \Delta^0$ is a Kan fibration.

We state here the important properties of Kan fibrations given in [3, I.10].

Theorem 2.3.1. (1) Kan fibrations have the right lifting property with respect to every trivial cofibration

(2) The realisation of a Kan fibration is a Serre fibration.

(3) Any map of simplicial sets $X \to Y$ can be factored as,

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & X' \arrow[ur] & \end{tikzcd}$$

where $X \to X'$ is a trivial cofibration, $X' \to Y$ is a Kan fibration and this factorisation is unique up to weak equivalence, that is given any other factorisation

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & X'' \arrow[ur] & \end{tikzcd}$$

We would have a homotopy equivalence $X' \to X''$ making the relevant diagrams commute (this can be checked easily using the right lifting property above).

Again mimicking the constructions in topological spaces we call a sequence of maps of simplicial sets $F \to E \to X$ a *fibration sequence* if $E \to X$ is a Kan fibration and $F \to E$ is the pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \Delta^0 \arrow[d] \\ E \arrow[r] & X \end{tikzcd}$$

For some $\Delta^0 \to X$. A sequence $F \to E \to X$ of simplicial sets is called a *homotopy fibration sequence* if for some factorisation $E \to E' \to X$ of $E \to X$ into trivial cofibration and Kan fibration, the map induced from F to the pullback of the diagram $\Delta^0 \to X \leftarrow E''$, is a weak equivalence. Any homotopy fibration sequence will give us a long exact sequence as in section 1.4, when we take the geometric realisation. We will need a slightly more general notion.

Definition 2.3.2. A diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is called *homotopy cartesian* if given a trivial cofibration-Kan fibration factorisation $B \to B' \to D$. The map from A to the pullback

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a weak equivalence.

2.3.1 Bisimplicial sets

Similar to the definition of simplicial sets. Given any category $\mathcal{C}$ we can define the category of *simplicial objects* in $\mathcal{C}$ as the category of functors $\mathbf{\Delta}^{op} \to \mathcal{C}$. In particular we can define the category of simplicial objects in the category of simplicial sets. These can equivalently be viewed as functors $(\mathbf{\Delta}\times\mathbf{\Delta})^{op} \to \mathbf{Set}$. This is called the category of bisimplicial sets, denoted by $\mathbf{s^2Set}$. Any simplicial set can be made into a bisimplicial set in a trivial way by making the functor constant on one component. As morphisms in $\mathbf{\Delta} \times \mathbf{\Delta}$ are ordered pairs of morphisms in $\mathbf{\Delta}$, we have bisimplicial maps generated by (d_i^*, s_j^*),(d_i^*, d_j^*),(s_i^*, s_j^*) and (s_i^*, d_j^*). We have a functor $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$ which sends $[n] \to ([n],[n])$, inducing a functor $d : \mathbf{s^2Set} \to \mathbf{sSet}$ such that, $d(X)_n = X_{n,n}$ for all n. We say a map of bisimplicial sets $X \to Y$ is a weak equivalence if the induced map $d(X) \to d(Y)$ is a weak equivalence. Given a bisimplicial set $X_{\cdot\cdot}$, we can fix one of the components and get a simplicial set $X_{\cdot n}$ for each n. The following lemma [3, IV.1.9] is an important tool from the theory of bisimplicial sets that we will need later.

Lemma 2.3.2 (realisation lemma)**.** Given a morphism of bisimplicial sets $f : X.. \to Y..$, if the induced level wise maps $f_n : X._n \to Y._n$ are weak equivalences for all n then the map of diagonals $d(f) : d(X) \to d(Y)$ is a weak equivalence.

Note that the converse is not true. For example take any simplicial set X and make it a bisimplicial set by taking it to be constant on one component. Then if the corresponding bisimplicial set is $X..$ with $X_{m,n} = X_n$, then $X._n$ is a constant simplicial set and hence given $X \to Y$ of simplicial sets the map $X._n \to Y._n$ is a weak equivalence iff it is a bijection of sets for all n, hence an isomorphism. However we know of weak equivalences which are not isomorphisms.

2.4 Nerve of a Category

In this section we construct a functor that assigns a simplicial set to every small category. That is a functor **Cat** $\to$ **sSet**. The functor will have many important properties, for example it will be a full and faithful functor and hence identifies **Cat** as a subcategory of simplicial sets. In our construction we will use the following observation. Viewing the partially ordered sets $[n] = \{0, 1, \ldots, n\}$ as poset categories, we can identify $\mathbf{\Delta}$ as a full subcategory of **Cat**.

Definition 2.4.1. Let $\mathcal{C}$ be a small category. We define the *Nerve* of $\mathcal{C}$ to be $N(\mathcal{C})_n = \mathrm{Hom}_{\mathbf{Cat}}([n], \mathcal{C})$. Then, the property of Hom functors make $N(\mathcal{C})$ a simplicial set. Explicitly, objects of $N(\mathcal{C})_n$ are sequences $a_0 \to a_1 \to \ldots \to a_n$ of morphisms in $\mathcal{C}$ and the face and degeneracy maps compose maps to decrease the chain length and add identities respectively. Note that $N([n]) = \Delta^n$.

The functorial nature of this construction is easy to see as given a functor $F : \mathcal{C} \to \mathcal{D}$, we get maps $\mathrm{Hom}_{\mathbf{Cat}}([n], \mathcal{C}) \to \mathrm{Hom}_{\mathbf{Cat}}([n], \mathcal{D})$ by composition, which induces a map of simplicial sets $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$.

Theorem 2.4.1. The Nerve functor N is fully faithful.

Proof. Consider $N(\mathcal{C})_0 = \mathrm{Hom}([0], \mathcal{C})$ is nothing but the set of objects in $\mathcal{C}$ and similarly $N(\mathcal{C})_1$ is the set of morphisms in $\mathcal{C}$ with the face maps d_0^* and d_1^* give the source and target of the morphism. Then given a morphism of simplicial sets

$N(\mathcal{C}) \to N(\mathcal{D})$, we construct the functor $\mathcal{C} \to \mathcal{D}$ by defining the map on objects to be the one induced by $N(\mathcal{C})_0 \to N(\mathcal{D})_0$ and the map on morphisms is given by $N(\mathcal{C})_1 \to N(\mathcal{D})_1$. To show that this gives a functor, all we need to show is it preserves composition. But $N(\mathcal{C})_2$ is just the set of ordered pairs of composable morphisms, where if the pair is (f, g),the face maps give the 1-simplices corresponding to f, g and $f \circ g$. Therefore composition is preserved under the map of morphisms we constructed. Hence N is full. To show that N is faithful we see that in the previous construction there was no choice, the functor we constructed was the unique functor that gives this simplicial morphism. Therefore the functor N is faithful. □

We denote the geometric realisation of $N(\mathcal{C})$ by $B\mathcal{C} = |N(\mathcal{C})|$ and call it the *classifying space* of $\mathcal{C}$. Then $B\mathcal{C} \cong B\mathcal{C}^{op}$ as the direction of the arrow does not matter after taking realisation. An important example of the classifying space construction is the classifying space of a group. Given a group G we can consider it as a category with one object $*$ and $\mathrm{Hom}(*, *) = G$. In other words, morphisms are elements of the group and composition is given by the group law. Then its classifying space $|N(\mathcal{C})| = BG$ has the property that, BG is connected, $\pi_1(BG) = G$ and $\pi_n(BG) = 0$ for all $n > 1$.

Remark 2.4.2. Given two small categories $\mathcal{C}$ and $\mathcal{D}$, any natural transformation of functors $F, G : \mathcal{C} \to \mathcal{C}$, induces a homotopy between their realisation. This stems from the fact that any natural transformation of functors $F, G : \mathcal{C} \to \mathcal{C}$, defines a functor $\hat{F} : [1] \times \mathcal{C} \to \mathcal{D}$ and $N([1]) = \Delta^1$. The realisation of any adjoint pair of functors is a homotopy equivalence. In particular, any small category with an initial or final object has a contractible realisation.

Definition 2.4.2 (Comma category)**.** Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an object d of $\mathcal{D}$, the comma category F/d is the category whose objects are ordered pairs (c, f) of an object c in $\mathcal{C}$ and f a morphism $F(c) \xrightarrow{f} d$ and morphisms $(c, f) \to (c', f')$ given by morphisms $c \xrightarrow{g} c'$ such that the diagram

$$\begin{tikzcd} F(c) \arrow[r, "f"] \arrow[d, "F(g)"'] & d \\ F(c') \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes. We also have the dual notion of d/F whose objects are pairs (c, f) with $d \xrightarrow{f} F(c)$ and morphisms given by $c \to c'$ making resultant diagrams commute.

Now we state two important theorems due to Quillen. These can be found in [8, IV.3].

Theorem 2.4.3 (Quillen's Theorem A)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of small categories such that F/d is contractible (by that we mean its classifying space in contractible) for all $d \in$Obj$(\mathcal{D})$. Then the induced map $BF : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.

Theorem 2.4.4 (Quillen's Theorem B)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of small categories such that for every morphism $d \to d'$ in $\mathcal{D}$, the induced map $F/d \to F/d'$ is a homotopy equivalence. Then for each d the sequence $F/d \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ is a homotopy fibration sequence and hence by Theorem 1.4.1, induces a long exact sequence

$$\begin{aligned} \ldots \to \pi_n(B\mathcal{C}) \to \pi_n(B\mathcal{D}) \to \pi_{n-1}(BF/d) \to \ldots \\ \ldots \to \pi_1(B\mathcal{D}) \to \pi_0(BF/d) \to \pi_0(B\mathcal{C}) \to \pi_0(B\mathcal{D}) \to 0 \end{aligned} \tag{2.3}$$

Note that by the observation earlier that $B\mathcal{C} \cong B\mathcal{C}^{op}$, dual of the above theorem, obtained by replacing F/d with d/F, is also true.

2.5 Spectra

Definition 2.5.1. A **Spectrum** is a sequence of pointed simplicial sets (or topological spaces), written as $X = (X_n)_{n \geq 0}$ with simplicial maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$, where S^1 is the pointed simplicial set (resp. topological space) $\Delta_1/\partial\Delta_1$. Note that the smash product of simplicial sets is defined in the same way as the smash product of topological spaces. The maps σ_n induce maps $\pi_i(X_n) \to \pi_{i+1}(X_{n+1})$ by,

$$\pi_i(X_n) \xrightarrow{S^1\wedge} \pi_{i+1}(S^1 \wedge X_n) \xrightarrow{\sigma_n} \pi_{i+1}(X_{n+1})$$

then for every integer n, the n^{th} *stable homotopy group* of X is defined to be,

$$\pi_n(X) := \lim_{\to} \pi_{n+i}(X_i) \tag{2.4}$$

where $\pi_{n+i}(X_i)$ is zero for $n+i \leq 0$. Note that here we have non-zero negative π_n unlike the case of homotopy groups of topological spaces. The morphisms between spectra are defined in the natural way and a morphism between spectra induces a homomorphism between corresponding stable homotopy groups.

Thus spectra form a category with stable homotopy groups giving a functor from the category of spectra to groups. The spectral maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ induce maps $\hat{\sigma_n} : |X_n| \to \Omega|X_{n+1}|$ by the adjoint relation Equation 1.4. A spectrum is called an Ω-*spectrum* if each of these induced maps is a homotopy equivalence. For an Ω-spectrum X we have

$$\pi_k(X) = \pi_{k+i}(X_i) \tag{2.5}$$

for all $k+i > 0$.

Chapter 3

Algebraic K-theory

The main goal of this fifth year project was to study the K-groups of small exact categories. The main references for this chapter are [8] and [7]. Classically, K-groups were invariants assigned to commutative rings based on the structure of the category of finitely generated modules over them. Quillen later gave a definition of K-groups for special subcategories of abelian categories called *exact categories*. This more general definition has the advantage that it is functorial and allows us to construct the K-groups as the homotopy groups of a topological space.

3.1 Definition of K_0

The "Grothendieck group" K_0 of a ring is the starting point of Algebraic K-theory. We construct $K_0(R)$ of a ring R as the group completion of the commutative monoid $\mathbf{P}(R)$ of finitely generated projective R-modules under the direct sum (we identify isomorphic modules so that this is a set and the monoid operation is well defined). Here we give the general construction of the group completion of a commutative monoid.

Definition 3.1.1. Let M be a commutative monoid where the groups operation is given by $'+'$, and let S be a submonoid of M. We define $S^{-1}M = M \times S/\sim$, where $(m, s) \sim (m', s')$ iff $m + s' = m' + s$. Using the fact that S is a submonoid, we can show that this is an equivalence relation and $(m, s) + (m', s') = (m + m', s + s')$ gives a commutative monoid structure on $S^{-1}M$ such that every element of the form $(s, 0)$ has an inverse $(0, s)$. Therefore, we represent the class of an element (m, s) by

$[m] - [s]$. In particular, $M^{-1}M$ is an abelian group and is called the *group completion of* M.

Now we are ready to define $K_0(R)$ of a ring R. Let $\mathbf{P}(R)$ be as above the commutative monoid of finitely generated projective R-modules under direct sum, then $K_0(R) = \mathbf{P}(R)^{-1}\mathbf{P}(R)$. The simplest cases of $K_0(R)$ are for rings where all modules are free or more generally all finitely generated projective modules are free. In these cases $\mathbf{P}(R) \cong \mathbb{N}$, therefore $K_0(R) \cong \mathbb{Z}$. Note that non-isomorphic projective modules may represent the same element in $K_0(R)$, for example if $A \oplus B = A \oplus C$, then by cancellation property of groups $[B] = [C]$. If the ring R is commutative, then we can further define a ring structure on $K_0(R)$ using the tensor product $\otimes$, we define

$$([A] - [B]) \times ([C] - [D]) = [(A \otimes C) \oplus (B \otimes D)] - [(B \otimes C) \oplus (A \otimes D)]$$

which you can check gives a commutative ring structure on $K_0(R)$. In the following section we try to give more generalised version of this construction that makes it more functorial.

K_0 of an symmetric monoidal category

A *symmetric monoidal category* is a category $\mathcal{C}$ along with a functor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a distinguished object e, and natural isomorphisms of functors

$$s\Box t \cong t\Box s \qquad s\Box e \cong e\Box s \cong s \qquad s\Box(t\Box w) \cong (s\Box t)\Box w$$

such that, these natural isomorphisms satisfy some coherence conditions. If the symmetric monoidal category $\mathcal{C}$ is small (or skeletally small), then $\Box$ defines a commutative monoid structure on the set of isomorphism classes of $\mathcal{C}$. Then $K_0^{\Box}(\mathcal{C})$ is defined as the group completion of the monoid structure so induced. An important example of the functor $\Box$ is the product functor (the functor is only unique up to natural isomorphisms) in a category with an initial object and all finite products. Similarly we have a symmetric monoidal structure given by the coproduct in a category with all finite coproducts and a terminal object. The $K_0(R)$ for a ring above is just $K_0^{\oplus}(\mathbf{P}(R))$, where by $\mathbf{P}(R)$ we mean the symmetric monoidal category of finitely generated projective R-modules (which is skeletally small) with $\Box$ being the direct sum $\oplus$.

Note: As most of the constructions of K-groups of categories that follow will rely on there being a set of objects in the category, from now on all categories considered will be small or at the least skeletally small unless explicitly stated otherwise.

K_0 of an exact category

For an abelian category $\mathcal{A}$, we can define $K_0^{\oplus}(\mathcal{A})$ as above with the functor being the direct sum. However we would not only like to identify $[A \oplus B]$ with $[A]+[B]$ but we also want $[A]+[B]=[C]$, for any exact sequence,

$$A \rightarrowtail C \twoheadrightarrow B$$

We formalise this as follows, $K_0(\mathcal{A})$ is defined as the quotient of the free abelian group on the set of isomorphism classes of $\mathcal{A}$ by the subgroup generated by all elements of the form $[A]+[B]-[C]$ for all exact sequences $A \rightarrowtail C \twoheadrightarrow B$. Note that in most interesting cases, such as the category of R-modules, the category is not small and if we take the category of all finitely generated R-modules or finitely generated projective R-modules, they are in general not abelian. However, the above definition only depends on notion of exact sequences in the abelian category. With this in mind we make the following definition:

Definition 3.1.2. An *exact category* is a pair $(\mathcal{C}, \mathcal{E})$ of a category $\mathcal{C}$ with zero objects (denoted by 0) and $\mathcal{E}$, a class of sequences in $\mathcal{C}$ of the form $0 \to A \to C \to B \to 0$, such that:

(i) $\mathcal{C}$ is a full subcategory of some abelian category $\mathcal{A}$

(ii) Each $0 \to A \to C \to B \to 0$ in $\mathcal{E}$ is a short exact sequence in $\mathcal{A}$

(iii) If $0 \to A \to C \to B \to 0$ is a short exact sequence in $\mathcal{A}$, with A,B and C belonging to $\mathcal{C}$, then $0 \to A \to C \to B \to 0$ belongs to $\mathcal{E}$.

(iv) If $0 \to A \to C \to B \to 0$ is a short exact sequence in $\mathcal{A}$, with A, B belonging to $\mathcal{C}$, then C is isomorphic to an object in $\mathcal{C}$.

If a monomorphism $A \rightarrowtail B$ fits in an exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$, then $A \rightarrowtail B$ is called an admissible monomorphism, we define admissible epimorphisms dually. From now on by an exact sequence in an exact category, we mean an element of $\mathcal{E}$ and the arrows $\rightarrowtail$ and $\twoheadrightarrow$ will exclusively denote admissible monomorphisms and admissible epimorphisms respectively. In the last point of the definition, we use isomorphism rather than asking the object itself belong to $\mathcal{C}$ because we will mostly consider small exact subcategories of large abelian categories in this thesis.

Note that (ii) guarantees that $\mathcal{C}$ is closed under direct sum. Therefore we can define $K_0^{\oplus}(\mathcal{C})$ as in the previous section for any exact category. We can also extend the definition of K_0 for an abelian category by making $K_0(\mathcal{C})$ the quotient of the free abelian group on the set of isomorphism classes of objects of $\mathcal{C}$ by the subgroup generated by elements of the form $[A] + [B] - [C]$ for all $0 \to A \to C \to B \to 0$ in $\mathcal{E}$. In general these two definitions may not give the same K_0, but if every sequence in $\mathcal{E}$ splits then these two definitions agree. Therefore for a ring R, the category $\mathbf{P}(R)$ is an essentially small exact category, and $K_0(\mathbf{P}(R)) \cong K_0^{\oplus}(\mathbf{P}(R))$ which we define to be $K_0(R)$. Note that since for any A, B objects in $\mathcal{C}$ we always have the exact sequence $A \rightarrowtail A \oplus B \twoheadrightarrow B$, $[A] + [B] = [A \oplus B]$ in $K_0(\mathcal{C})$. So collecting terms of the form $[A]$ and $-[A]$, every element in $K_0(\mathcal{C})$ is of the form $[A] - [B]$.

3.2 Quillen's Q-construction

The goal of this section is to assign a collections of groups $K_n(\mathcal{C})$ to a small exact category $\mathcal{C}$. This construction is from [8, IV.6]. First, we assign a new category $\mathcal{QC}$ to $\mathcal{C}$. Then we define $K_n(\mathcal{C}) = \pi_{n+1}(B\mathcal{QC})$ where, by BC of a category C we mean its classifying space as defined in section 2.4.

Definition 3.2.1. Let $\mathcal{C}$ be a small exact category. We define $\mathcal{QC}$ to be the category whose objects are the same as the objects in $\mathcal{C}$ and a morphism between two objects A and B in $\mathcal{QC}$ is an equivalence class of diagrams of the form

$$A \twoheadleftarrow C \rightarrowtail B$$

where two such diagrams are equivalent if there is an isomorphism between them which is identity on A and B.

Before we define composition of two morphisms in this category, we need a result about exact categories.

Theorem 3.2.1. Let $A \rightarrowtail B \twoheadrightarrow C$ be an exact sequence in an exact category $\mathcal{C}$ and let $D \twoheadrightarrow B$ be an admissible epimorphism. Then, the *base change sequence* $A' \rightarrowtail D \twoheadrightarrow C$ is an exact sequence in $\mathcal{C}$ where A' is the pullback,

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, two heads] & D \arrow[d, two heads] & \\ A \arrow[r, tail] & B \arrow[r, two heads] & C \end{tikzcd}$$

, $D \twoheadrightarrow C$ is the composite map, and $A' \twoheadrightarrow A$ is an admissible monomorphism.

Proof. First, working in the ambient abelian category, the pullback exists and the pullback of a monomorphism is a monomorphism and the pullback of an epimorphism along a monomorphism in an abelian category is an epimorphism and the resultant sequence is exact as the composition is zero by commutativity of the square and using the property of pullback we can show that $A' \rightarrowtail D$ is the kernel of $D \twoheadrightarrow C$. As $D \twoheadrightarrow B$ is admissible, there is an exact sequence $E \rightarrowtail D \twoheadrightarrow B$. Now by property of pullbacks we have the induced map $E \rightarrowtail A'$ which is the kernel of $A' \twoheadrightarrow A$, then by property (ii), A' is an element of $\mathcal{C}$ and by property (iii), $A' \rightarrowtail D \twoheadrightarrow C$ is an exact sequence in $\mathcal{C}$. $\square$

Invoking duality, we have a similar result for taking pushouts along an admissible monomorphisms. Then we have a simple corollary,

Corollary 3.2.2. In any exact category, composition of two admissible monomorphisms(epimorphisms) is an admissible monomorphism (epimorphism).

Using the above theorem, we can define the composition in $\mathcal{QC}$. Given $A \twoheadleftarrow C \rightarrowtail B$ and $B \twoheadleftarrow C' \rightarrowtail D$, their composition is defined by

$$\begin{tikzcd} & E \arrow[r, tail] \arrow[d, two heads] & C' \arrow[r, tail] \arrow[d, two heads] & D \\ A & C \arrow[l, two heads] \arrow[r, tail] & B & \end{tikzcd}$$

where the square is the pullback and the composite morphisms are admissible by the corollary. It is obvious from this definition that the identity morphism of an object A is $A = A = A$(where the equality denotes the identity morphism of A in $\mathcal{C}$). A thing to note here is, taking equivalence classes of diagrams was important as the pullback above is unique only up to isomorphism.

There are two distinguished classes of morphisms in $\mathcal{QC}$, the first class is the set of diagrams $A \twoheadleftarrow C = C$ which we denote by just $A \twoheadleftarrow C$ and call admissible epics, similarly the second class is the set of diagrams $A = A \rightarrowtail B$, which we denote by $A \rightarrowtail B$ and call admissible monics. Note that $A \twoheadleftarrow C \rightarrowtail B$ is the composition of $A \twoheadleftarrow C$ with $C \rightarrowtail B$. In fact, we can write any morphism as a composition of and admissible epic and monic in a unique way up to isomorphism. In $\mathcal{QC}$, the 0 of $\mathcal{C}$ is no longer the zero object, but for any other object A there are still morphisms $0 \twoheadleftarrow A$ and $A \rightarrowtail 0$, hence its nerve is connected. Now we are ready to define the higher K-groups.

Definition 3.2.2. Let $\mathcal{C}$ be a small exact category, then we define $K_n(\mathcal{C}) = \pi_{n+1}(B\mathcal{QC}) \cong \pi_n(\Omega B\mathcal{QC})$, where $B\mathcal{QC}$ is the geometric realisation of the nerve of $\mathcal{QC}$.

Now we need to check that the above definition of K_0 agrees with our previous definition.

Proposition 3.2.3. For an exact category $\mathcal{C}$, $\pi_1(B\mathcal{QC}) \cong K_0(\mathcal{C})$, where $K_0(\mathcal{C})$ is as defined in the previous section for exact categories. The element of $\pi_1(B\mathcal{QC})$ corresponding to $[A] \in K_0(\mathcal{C})$ is given by,

$$0 \rightarrowtail A \twoheadrightarrow 0$$

where the arrows represent an admissible monic and epic in $\mathcal{QC}$. Hence, their composition gives a loop in $\pi_1(B\mathcal{QC})$.

Proof. The proof here will closely follow the proof in [8, Prop.IV.6.2]. Let T denote the maximal tree whose elements are $0 \rightarrowtail A$ for each A in $\mathcal{C}$, by an application of van-Kampen theorem and cellular approximation (Theorem 1.3.1), we can see that $\pi_1(B\mathcal{QC})$ is generated by $[f]$ where each $[f]$ is a morphism in $\mathcal{QC}$, such that $[0 \rightarrowtail A] = 1$ for all A, and $[f \circ g] = [f] * [g]$ generate all the relations in $\pi_1(B\mathcal{QC})$. Now for any admissible monic $B \rightarrowtail C$, we have $0 \rightarrowtail B \rightarrowtail C = 0 \rightarrowtail C$, where the composition on the right is the composition in $\mathcal{QC}$. Therefore, $[B \rightarrowtail C] = 1$, giving

$$[A \twoheadleftarrow B \rightarrowtail C] = [A \twoheadleftarrow B][B \rightarrowtail C] = [A \twoheadleftarrow B]$$

, but $[0 \twoheadleftarrow B] = [A \twoheadleftarrow B][0 \twoheadleftarrow A]$ by composition, giving $[A \twoheadleftarrow B] = [0 \twoheadleftarrow B][0 \twoheadleftarrow A]^{-1}$. Hence $\pi_1(B\mathcal{QC})$ is generated by elements of the form $[0 \twoheadleftarrow A]$. As stated previously $[0 \twoheadleftarrow A]$ is the loop $0 \rightarrowtail A \twoheadrightarrow 0$ and under our identification goes to $[A]$ in $K_0(\mathcal{C})$.

Now for any exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{C}$, the composition of $C \twoheadleftarrow B$ and $0 \rightarrowtail C$ in $\mathcal{C}$ is the morphism $0 \twoheadleftarrow A \rightarrowtail B$, giving

$$[0 \twoheadleftarrow B] = [C \twoheadleftarrow B][0 \twoheadleftarrow C] = [0 \twoheadleftarrow A][0 \twoheadleftarrow C]$$

Where the last equality arises from $[C \twoheadleftarrow B][0 \rightarrowtail C] = [A \rightarrowtail B][0 \twoheadleftarrow A]$ and cancelling the terms equal to identity. Applying this to the split exact sequences $A \rightarrowtail A \oplus B \twoheadrightarrow B$ and $B \rightarrowtail A \oplus B \twoheadrightarrow A$, we get

$$[0 \twoheadleftarrow A][0 \twoheadleftarrow B] = [0 \twoheadleftarrow A \oplus B] = [0 \twoheadleftarrow B][0 \twoheadleftarrow A]$$

Hence the group is abelian and the group homomorphism $\pi_1(B\mathcal{QC}) \to K_0(\mathcal{C})$ sending $[0 \twoheadleftarrow A] \to [A]$ is well defined and surjective. If we show all the composition relations $[f] * [g] = [f \circ g]$ arise from these then the above morphism will be an isomorphism. Now the composition of $A \twoheadleftarrow C \rightarrowtail B$ and $B \twoheadleftarrow C' \rightarrowtail D$ is $A \twoheadleftarrow F \rightarrowtail D$ where F is the pullback. But the relation $[B \twoheadleftarrow C' \rightarrowtail D][A \twoheadleftarrow C \rightarrowtail B] = [A \twoheadleftarrow F \rightarrowtail D]$ can be simplified to $[0 \twoheadleftarrow C'][0 \twoheadleftarrow B]^{-1}[0 \twoheadleftarrow C][0 \twoheadleftarrow A]^{-1} = [0 \twoheadleftarrow F][0 \twoheadleftarrow A]^{-1}$, we can rewrite this as,

$$[0 \twoheadleftarrow C'][0 \twoheadleftarrow C] = [0 \twoheadleftarrow F][0 \twoheadleftarrow B]$$

Now by construction of F we have two exact sequences $F \rightarrowtail C' \twoheadrightarrow G$ and $C \rightarrowtail B \twoheadleftarrow G$. It is easy to see that the relation we obtained in $\pi_1(B\mathcal{QC})$ from the composition can be obtained from multiplying the relations obtained from the above two exact sequences and cancelling $[0 \twoheadleftarrow G]$ on both sides. Thus no new relation is obtained and hence $\pi_1(B\mathcal{QC}) \cong K_0(\mathcal{C})$. $\square$

Properties of Quillen's Q-construction:

The $\mathcal{Q}$-construction in functorial in the following sense,

Theorem 3.2.4. Let **ExCat** be the category of small exact categories where the morphisms are *exact functors*, that is, functors which take exact sequences to exact sequences. Then $\mathcal{Q}$ is a functor from **ExCat** to **Cat**, the category of small categories.

Proof. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an exact functor between two exact categories, then by definition $\mathcal{F}$ takes admissible monics to admissible monics and admissible epics to admissible epics. Hence it takes diagrams of the form $A \twoheadleftarrow C \rightarrowtail B$ to diagrams of the same form. To show $\mathcal{QF}$ is a functor all we have to do is show that $\mathcal{F}$ preserves pullbacks of diagrams of the form

$$\begin{tikzcd} & B \arrow[d, two heads] \\ A \arrow[r, tail] & C \end{tikzcd}$$

But if $A \rightarrowtail B \twoheadrightarrow D$ is exact then the pullback F of the above diagram is the kernel of the composite $C \twoheadrightarrow B \twoheadrightarrow D$. (Note then the map $F \to A$ is uniquely determined by kernel property). As exact functors preserve kernels (and cokernels) of admissible epics (monics), $\mathcal{QF}$ preserves such pullbacks. The functorial composition rule $\mathcal{Q}(\mathcal{F} \circ \mathcal{G}) = \mathcal{QF} \circ \mathcal{QG}$ also follows easily. Hence $\mathcal{Q}$ is functorial and by functoriality of the nerve construction, so are the K-groups. $\square$

If $\mathcal{C}$ and $\mathcal{D}$ are small exact categories, the product category $\mathcal{C} \times \mathcal{D}$ is naturally an exact category, where the exact sequences are component wise exact sequences. Then it is east to see $\mathcal{Q}(\mathcal{C} \times \mathcal{D}) = \mathcal{QC} \times \mathcal{QD}$ and hence, $K_n(\mathcal{C} \times \mathcal{D}) = K_n(\mathcal{C}) \oplus K_n(\mathcal{D})$.

For any exact category $\mathcal{C}$, its opposite category $\mathcal{C}^{op}$ is an exact category with admissible monics and epics interchanged. Then $\mathcal{QC}^{op} \cong \mathcal{QC}$ by sending the diagram $A \twoheadleftarrow C \rightarrowtail B$ to its pushout, and using the fact that in an abelian category, a diagram of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] & B \arrow[d, two heads] \\ C \arrow[r, tail] & D \end{tikzcd}$$

is a pushout iff it is a pullback. The $\mathcal{Q}$ construction is useful in many cases, but the difficulty of calculating higher homotopy groups for topological spaces restricts its usefulness especially in explicit calculations. One way to get around this problem is to try and define K-groups as stable homotopy groups of spectra (section 2.5), rather than homotopy groups of topological spaces. In the next section we will look at Waldhausen's S-construction which does exactly that and see its usefulness by deriving several results from it.

3.3 Waldhausen's S construction

Waldhuasen's S construction gives a way to define the K-groups of a small Waldhausen category as the stable homotopy groups of a spectrum of simplicial sets, in the case of exact categories this gives the same K-groups as Quillen's Q-construction. The treatment here will closely follow Wladhausen's original paper [7], but we will restrict ourselves to exact categories.

3.3.1 S. of an exact category

In this section let $\mathcal{C}$ be a small exact category with a distinguished zero object 0. For the poset $[n] = \{0, 1, ...n\}$, viewed as a category, we define $Ar[n]$ to be the category whose objects are morphisms of $[n]$ and morphisms are commutative squares. Simply put, $Ar[n]$ is a poset category with objects (i, j),where $i \leq j$ and a morphism $(i, j) \to (k, l)$ exits if and only if $i \leq k$ and $j \leq l$. Then the category of functors from $Ar[n]$ to $\mathcal{C}$, i.e., $[Ar[n], \mathcal{C}]$ has a natural exact category structure where admissible monics (rep. epics), are given objectwise. Note that this is also a small category.

Definition 3.3.1. $\mathcal{S}_n(\mathcal{C})$ is the full subcategory of $[Ar[n], \mathcal{C}]$ of functors $A : Ar[n] \to (\mathcal{C})$ with the following properties,

1. for all j, $A_{j,j} = 0$ the distinguished zero object,
2. for all $i \leq j \leq k$, $A_{i,j} \to A_{i,k}$ is an admissible monic,
3. the commutative diagram,

$$\begin{tikzcd} A_{i,j} \arrow[r] \arrow[d] & A_{i,k} \arrow[d] \\ A_{j,j} \arrow[r] & A_{j,k} \end{tikzcd}$$

   is a pushout. That is, $A_{i,j} \rightarrowtail A_{i,k} \twoheadrightarrow A_{j,k}$ is an exact sequence.

Proposition 3.3.1. The above definition is equivalent to defining objects of $\mathcal{S}_n$ to be sequences of monomorphisms $0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail ...... \rightarrowtail A_n$ with a choice of cokernals $A_i \rightarrowtail A_j \twoheadrightarrow A_j/A_i$ for all $i \leq j$, with $A_i/A_i = 0$ for all i. Then morphisms are morphisms between such sequences which naturally induce a morphism between corresponding cokernels by universal property.

Proof. An object A in $\mathcal{S}_n(\mathcal{C})$ gives rise to a sequence as above in a natural way by letting $A_i = A_{0,i}$ and $A_j/A_i = A_{i,j}$ for all $i \leq j$. For the converse, we have to show the reverse of the construction above gives an object in $\mathcal{S}_n(\mathcal{C})$. Let $0 = A_0 \rightarrowtail A_1 \rightarrowtail .... \rightarrowtail A_n$ be a sequence as above, we define $A_{i,j} = A_j/A_i$ and $A_{0,i} = A_i$. Then by definition, $A_{0,i} \rightarrowtail A_{0,j}$ is a monomorphism for all $i \leq j$. For $i > 0$ and $i \leq j \leq k$ we have the commutative diagram,

$$\begin{tikzcd}
A_{0,i} \arrow[r, equal] \arrow[d, tail] & A_{0,i} \arrow[d, tail] & \\
A_{0,j} \arrow[r, tail] \arrow[d, two heads] & A_{0,k} \arrow[r, two heads] \arrow[d, two heads] & A_{j,k} \arrow[d, equal] \\
A_{i,j} \arrow[r] & A_{i,k} \arrow[r] & A_{j,k}
\end{tikzcd}$$

where the maps $A_{i,j} \to A_{i,k}$ and $A_{i,k} \to A_{j,k}$ are induced by the universal properties of $A_{i,j} = A_j/A_i$ and $A_{i,k} = A_k/A_i$ respectively. All we need to show is that $A_{i,j} \to A_{i,k}$ is monic, $A_{i,k} \to A_{j,k}$ is epic, and the sequence is exact. As we can embed any exact category in an abelian category, a mono epi sequence is exact if and only if their composition is zero, and we see that the composition is zero by pre-composing with the epimorphism $A_{0,i} \to A_{i,j}$ and using the fact that the diagrams commute. Next, the map $A_{i,k} \to A_{j,k}$ is epic as the composition $A_{0,k} \to A_{i,k} \to A_{j,k}$ is epic, and $A_{i,j} \to A_{i,k}$ is monic because the lower left square is a pushout, and $A_{0,j} \to A_{0,k}$ is monic. $\square$

Now, each $\theta : [n] \to [m]$, a morphism between $[n]$ and $[m]$ as objects of $mathbf\Delta$, induces $Ar\theta : Ar[n] \to Ar[m]$, a functor which is just θ acting on each component. Therefore we have $\theta^* : [Ar[m], \mathcal{C}] \to [Ar[n], \mathcal{C}]$, and this restricts to a functor from $\theta^* : \mathcal{S}_m(\mathcal{C}) \to \mathcal{S}_n(\mathcal{C})$. Hence, $\mathcal{S}.(\mathcal{C})$ is a *Simplicial exact category.*

Simplicial maps

As stated above $\mathcal{S}.(\mathcal{C})$ is a simplicial exact category. So, what do the simplicial maps look like?

Let us look at the face and degeneracy maps. Similar to the nerve construction, the i^{th} degeneracy takes $(0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \ldots \rightarrowtail A_n, A_j/A_k)$ to $(0 = A_0 \rightarrowtail$

$A_1 \rightarrowtail A_2 \rightarrowtail \ldots \rightarrowtail A_i \to A_i \rightarrowtail A_j \rightarrowtail \ldots \rightarrowtail A_n, A_j/A_k)$ that is, insertion of an identity morphism at the i^{th} position and setting the corresponding quotient to be the distinguished zero object. The similar statement is true for face maps except for $i = 0$ in which case the degeneracy map is given by $(0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \ldots \rightarrowtail A_n, A_j/A_k) \to (0 = A_1/A_1 \rightarrowtail A_2/A_1 \rightarrowtail A_3/A_1 \rightarrowtail \ldots \rightarrowtail A_n/A_1, A_j/A_k = (A_j/A_1)/(A_k/A_1))$.

We have defined a small exact category $\mathcal{S}_n\mathcal{C}$ for each n. Of particular interest is $\mathcal{S}_2\mathcal{C}$ whose objects are nothing but exact sequences $A \rightarrowtail B \twoheadrightarrow C$. We will denote $\mathcal{S}_2\mathcal{C}$ by $E(\mathcal{C})$ as this particular category will be of importance later.

3.3.2 $S^n(\mathcal{C})$ of an exact category $\mathcal{C}$

Now as each $\mathcal{S}_n(\mathcal{C})$ is an exact category, we can apply $\mathcal{S}.$ again on each $\mathcal{S}_n(\mathcal{C})$ and get a bisimplicial exact category. Recursively we define $\mathcal{S}^n(\mathcal{C})$ as,

$$\mathcal{S}^0(\mathcal{C}) = \mathcal{C}$$

$$\mathcal{S}^n(\mathcal{C}) = \mathcal{S}.\mathcal{S}^{n-1}(\mathcal{C})$$

Therefore, for each $n \geq 0$ we get an *n-simplicial exact category*. Hence we will use the notation $\mathcal{S}^n(\mathcal{C})[-,-,...-]$, to represent the fact that for every n-tuple $(a_1, a_2, ..., a_n)$ with $a_i \in \mathbb{N} \cup \{0\}$ we have an exact category $\mathcal{S}^n(\mathcal{C})[a_1, a_2, ...a_n]$.

Functoriality:

Note that the given constructions are *functorial*, in the sense that given an *exact* functor between two exact categories $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, we get an exact functor $\mathcal{S}_n(\mathcal{F}) : \mathcal{S}_n(\mathcal{C}) \to \mathcal{S}_n(\mathcal{D})$, for each n, which is a restriction of the functor,

$$[Ar[n], \mathcal{C}] \to [Ar[n], \mathcal{D}]$$

$$\mathcal{G} \to \mathcal{F} \circ \mathcal{G}$$

and further induces a morphism of simplicial exact categories from $\mathcal{S}.(\mathcal{C}) \to \mathcal{S}.(\mathcal{D})$. Inductively, $\mathcal{S}.$ is a functor from the category of small n-simplicial categories to the category of small $n+1$-simplicial categories.

3.3.3 Construction of K-theory

Let $\mathcal{S}^n(\mathcal{C})$ be as above, define $i\mathcal{S}^n(\mathcal{C})$ to be the n-simplicial subcategory of isomorphisms of $\mathcal{S}^n(\mathcal{C})$. That is, for all $(a_1, a_2, ..., a_n)$, $i\mathcal{S}^n(\mathcal{C})[a_1, a_2, ..., a_n]$ is the subcategory of $\mathcal{S}^n(\mathcal{C})[a_1, a_2, ..., a_n]$, of all isomorphisms. Therefore its *nerve*, $N(i\mathcal{S}^n(\mathcal{C}))$ is an $n+1$-simplicial set. Now we are ready to construct the desired spectrum. For $n \geq 0$ define

$$K(\mathcal{C})_n = |N(i\mathcal{S}^n(\mathcal{C}))|,$$

where $|.|$ is the diagonal functor to the category of simplicial sets. Now to construct the spectrum $K(\mathcal{C})$, we need to define the maps

$$\sigma_n : S^1 \wedge K(\mathcal{C})_n \to K(\mathcal{C})_{n+1}, \forall n \in \mathbb{N}$$

we do this by first constructing a map $\sigma_n : \Delta_1 \times K(\mathcal{C})_n \to K(\mathcal{C})_{n+1}$, and showing that restricted to $\partial\Delta_1 \times K(\mathcal{C})_n \cup \Delta_1 \times \{0\}$, the map is constant ($0 \in K_0(\mathcal{C}) = \{0\}$). First we observe that, $\mathcal{S}_1(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$, and inductively, $\mathcal{S}_1(\mathcal{S}^n(\mathcal{C})) \xrightarrow{\sim} \mathcal{S}^n(\mathcal{C})$. We then have an induced map, $|N(i\mathcal{S}_1(\mathcal{S}^n(\mathcal{C}))| \cong |N(i\mathcal{S}^n(\mathcal{C}))|$. So we can construct maps,

$$\sigma_n : \Delta_1 \times |N(i\mathcal{S}_1\mathcal{S}^n(\mathcal{C}))| \to |N(i\mathcal{S}^{n+1}(\mathcal{C}))|$$

By, $(\theta, x) \to \theta^*(x)$ where $\theta \in \mathrm{Hom}([m], [1])$ and $x \in i\mathcal{S}^n(\mathcal{C})[m, m, ..., 1, m]$. Let $\theta \in \mathrm{Hom}([m], [1])$. Then, either $\theta(i) = 0 \ \forall i \in [m]$ or $\exists j \leq m$ such that, $\theta(i) = 0$ for all $i < j$ and $\theta(i) = 1$ for all $i \geq j$. We will therefore denote by v_j the morphism whose first non-zero is at its j^{th} position. Then we define the map to be,

$$(v_j, 0 = c_0 \rightarrowtail c_1 \rightarrowtail ... \rightarrowtail c_m) \to \begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
c_0 & \rightarrowtail & c_1 & \rightarrowtail & ... & \rightarrowtail & c_m \\
\| & & \| & & \| & & \| \\
c_0 & \rightarrowtail & c_1 & \rightarrowtail & ... & \rightarrowtail & c_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
c_0 & \rightarrowtail & c_1 & \rightarrowtail & ... & \rightarrowtail & c_m
\end{array}$$

Where the first non-zero sequence occurs at the j^{th} position, and it is the constant 0 sequence when θ maps all values to 0. Note here if the sequence was 0 to begin with, it would go to the diagram with all zeroes, and if $\theta \in \partial\Delta_1 = \mathrm{Hom}([0],[1])$, then still we would get a diagram of zeroes. Hence we have constructed the maps,

$$\sigma_n : S^1 \wedge K(\mathcal{C})_n \to K(\mathcal{C})_{n+1}, \forall n \in \mathbb{N}$$

Therefore $K(\mathcal{C}) = (K(\mathcal{C}_n))_{n\geq 0}$ is the desired spectrum and the K-groups are,

$$K_n(\mathcal{C}) = \pi_n(K(\mathcal{C}))$$

We can again check that the construction of the K-theory is functorial as we did above for $\mathcal{S}.$.

Chapter 4

The Additivity Theorem

In the last chapter, we have given two different constructions of K-groups of an exact category. We would like to show that these two definitions are in fact equivalent. To do this we first prove an important result about the S-constructions called the additivity theorem. The version we prove is the one given by Waldhausen [7, 1.4]. This theorem is the main goal of this thesis and is a very important tool in proving several important properties of the K-groups which we will prove in the next chapter. There are several equivalent formulations of the additivity theorem, but before we state them here we need a few more definitions. We had defined $E(\mathcal{C})$ earlier for an exact category $\mathcal{C}$. We now generalise this and define $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ for an exact category $\mathcal{C}$ and exact subcategories $\mathcal{A}$ and $\mathcal{B}$ as the full subcategory of $E(\mathcal{C})$ of exact sequences of the form,

$$A \rightarrowtail C \twoheadrightarrow B, \qquad A \in \mathcal{A},\ B \in \mathcal{B}$$

Let $\mathcal{C}$, $\mathcal{D}$ be exact categories. Then an *exact sequence* of exact functors $\mathcal{C} \to \mathcal{D}$ is a sequence of natural transformations $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$, where $\mathcal{F}$,$\mathcal{F}'$,$\mathcal{F}'' : \mathcal{C} \to \mathcal{D}$ are exact functors such that, $\mathcal{F}'(C) \to \mathcal{F}(C) \to \mathcal{F}''(C)$ is an exact sequence for all $C \in \mathcal{C}$.

Theorem 4.0.2 (Equivalent statements of Additivity theorem)**.** The following statements are equivalent:

(1) The following projection is a homotopy equivalence for all small exact categories $\mathcal{C}$,

$$i\mathcal{S}.(E(\mathcal{C})) \longrightarrow i\mathcal{S}.(\mathcal{C}) \times i\mathcal{S}.(\mathcal{C})$$

$$A \rightarrowtail C \twoheadrightarrow B \longrightarrow (A, B) \tag{4.1}$$

(2) The following projection is a homotopy equivalence for all small exact categories $\mathcal{C}$,

$$i\mathcal{S}.(E(\mathcal{A}, \mathcal{C}, \mathcal{B})) \longrightarrow i\mathcal{S}.(\mathcal{A}) \times i\mathcal{S}.(\mathcal{B})$$

$$A \rightarrowtail C \twoheadrightarrow B \longrightarrow (A, B)$$

(3) The following two maps are homotopic for all small exact categories $\mathcal{C}$,

$$i\mathcal{S}.E(\mathcal{C}) \longrightarrow i\mathcal{S}.\mathcal{C}$$

$$A \rightarrowtail C \twoheadrightarrow B \longrightarrow C, \;\; resp. A \oplus B$$

(4) Given an exact sequence of exact functors $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$, then there is homotopy ,

$$|i\mathcal{S}.\mathcal{F}| \simeq |i\mathcal{S}.(\mathcal{F}' \oplus \mathcal{F}'')|$$

Here all homotopies are taken to mean homotopies after taking the diagonal of the nerve to obtain a simplicial set.

Where $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is as defined in the previous section.

Proof. (2) is a special case of (1) when $\mathcal{A} = \mathcal{B} = \mathcal{C}$, and (3) is a special case of (4) when the three functors are the projections. So we have (1) $\implies$ (2) and (3) $\implies$ (4). Therefore it is enough to show (2) $\implies$ (3) $\implies$ (4) and (4) $\implies$ (1).

(3) $\implies$ (4). Lets denote the projections $i\mathcal{S}.E(\mathcal{C}) \to i\mathcal{S}.(\mathcal{C})$ by p_i, $i = 1, 2, 3$. Then (3) can be rephrased as $|i\mathcal{S}.(p_2)| \simeq |i\mathcal{S}.(p_1 \oplus p_3)|$. Now given an exact sequence of exact functors $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$, $\mathcal{D} \to \mathcal{C}$, we get an exact functor $G : \mathcal{D} \to E(\mathcal{C})$ sending $A \to \mathcal{F}'(A) \rightarrowtail \mathcal{F}(A) \twoheadrightarrow \mathcal{F}''(A)$. Then $\mathcal{F} = p_2 G$, $\mathcal{F}' = p_1 G$ and $\mathcal{F} = p_3 G$. Hence (3) $\implies$ (4) as composition preserves homotopies.

(2) $\implies$ (3). We have a map $i\mathcal{S}.(\mathcal{C}) \times i\mathcal{S}.(\mathcal{C}) \longrightarrow i\mathcal{S}.(E(\mathcal{C}))$, given by $(A, B) \longrightarrow A \rightarrowtail A \oplus B \twoheadrightarrow B$, (where you make a choice of direct sum). Composing this with p_2 and $p_1 \oplus p_3$ gives us the same maps up to isomorphism and hence are homotopic. So to show that the maps themselves are homotopic, it is enough to show that the above map is a homotopy equivalence. But the map is a section of the projection

map of (2) which by hypothesis is a homotopy equivalence, hence is also a homotopy equivalence. Therefore (2) $\implies$ (3).

(4) $\implies$ (1). Restricting the map from previous argument, we get that $p : i\mathcal{S}.(E(\mathcal{A},\mathcal{C},\mathcal{B})) \to i\mathcal{S}.\mathcal{A} \times i\mathcal{S}.\mathcal{B}$ sending $A \rightarrowtail C \twoheadrightarrow B \longrightarrow (A,B)$ has a section $(A,B) \longrightarrow A \rightarrowtail A \oplus B \twoheadrightarrow B$, lets denote it by i. therefore to show p is a homotopy equivalence we just need to show ip is homotopic to the identity. For this we will use (4) on the exact sequence of exact functors,

$$\begin{tikzcd} & (A = A \to 0) \arrow[d, tail] \\ A \rightarrowtail C \twoheadrightarrow B \arrow[r] & (A \rightarrowtail C \twoheadrightarrow B) \\ & (0 \to B = B) \arrow[u, tail] \end{tikzcd}$$

Applying (4) to this sequence gives us the desired homotopy and hence all the statements are equivalent. $\square$

4.1 Proving Additivity Theorem

In this section we will try and prove the additivity theorem in steps. The proof given here is identical to that in [7, 1.4]. We will first define a simplicial set that contains the essential information from $i\mathcal{S}.(\mathcal{C})$.

If $\mathcal{C}$ is a small exact category, we define the $s_n\mathcal{C} = Ob(\mathcal{S}_n(\mathcal{C}))$ the set of objects of $\mathcal{S}_n(\mathcal{C})$. Then $s.\mathcal{C}$ is a simplicial set.

Lemma 4.1.1. An exact functor of exact categories $f : \mathcal{C} \to \mathcal{D}$ induces a map of simplicial sets $s.f : s.\mathcal{C} \to s.\mathcal{D}$. A natural transformation of functors f and f' induces a homotopy of maps $s.f$ and $s.f'$.

Proof. The fact that we have an induced map is clear as it follows from the result for $\mathcal{S}.(\mathcal{C})$. To show there is a homotopy, we will give an explicit simplicial homotopy between the maps, that is a map $s.\mathcal{C} \times \Delta^1 \to s.\mathcal{D}$ which restricts to the given maps when restricted via the face maps of Δ^1. To do this we give an alternate and equivalent definition of simplicial homotopy. Let X be a simplicial set, then X is a functor

$X : \mathbf{\Delta}^{op} \to \mathbf{Set}$. Let X^* denote the composite functor

$$(\mathbf{\Delta}/[1])^{op} \longrightarrow \mathbf{\Delta}^{op} \xrightarrow{X} \mathbf{Set}$$

$$([n] \to [1]) \longrightarrow [n] \longrightarrow X[n]$$

Where $\mathbf{\Delta}/[1]$ is the slice category of objects over $[1]$. Then we have maps $\mathbf{\Delta}^{op} \cong (\mathbf{\Delta}/[0])^{op} \rightrightarrows (\mathbf{\Delta}/[1])^{op}$ induced by the face maps $d_0, d_1 : [1] \to [0]$ let's denote these by $\hat{d}_0$,$\hat{d}_1$. A *simplicial homotopy of maps* $f, g : X \to Y$ is then equivalent to a natural transformation $F : X^* \to Y^*$ such that $F\hat{d}_0 = f$ and $F\hat{d}_1 = g$. The equivalence of these definitions can be easily obtained the identification of X^* with $X \times \Delta^1$ by $(x, \theta) \in X_n \to x \in X^*_\theta$. Note that simplicial homotopy is not symmetric, that is we might not be able to find a homotopy with $\hat{d}_0$ and $\hat{d}_1$ interchanged. But we can always find such a map after realisation.

We will now construct the desired maps. A natural transformation of two functors $f, g : \mathcal{C} \to \mathcal{D}$ is equivalent to a functor $F : \mathcal{C} \times [1] \to \mathcal{D}$, where $[1]$ is the poset $[1] = \{0 < 1\}$ taken to be a category. We then obtain the required simplicial homotopy by

$$(a : [n] \to [1]) \longrightarrow ((A : Ar[n] \to \mathcal{C}) \longrightarrow (A' : Ar[n] \to \mathcal{D})$$

where A' is defined as the composition

$$Ar[n] \xrightarrow{(A,a_*)} C \times Ar[1] \xrightarrow{id \times p} \mathcal{C} \times [1] \xrightarrow{F} \mathcal{D}$$

$p : Ar[1] \to [1]$ is a map of posets given by $(0,0) \to 0$, $(1,1) \to 1$ and $(0,1) \to 1$. We can see that this works by composing this map with $\hat{d}_0$ and $\hat{d}_1$ which from the definition of p and the fact that F restricted to 0 and 1 in $[1]$ give f and g respectively, gives us that the composition map above is the desired homotopy. □

This lemma gives us an important corollary immediately.

Corollary 4.1.2. (1) An exact equivalence of exact categories $\mathcal{C} \to \mathcal{D}$ gives a homotopy equivalence of simplicial sets $s.\mathcal{C} \to s.\mathcal{D}$.

(2) Let $\mathcal{C}$ be a small exact category. Then there is a homotopy equivalence $s.\mathcal{C} \to N(i\mathcal{S}.(\mathcal{C})$, where $s.\mathcal{C}$ is viewed as bisimplicial set which is constant in the second component.

Proof. (1) is clear from the lemma. For (2) the map is very straightforward, we send

$$(0 \rightarrowtail a_1 \rightarrowtail a_2 \ldots \rightarrowtail a_n) \to \begin{array}{ccccccccc}
0 & \rightarrowtail & a_1 & \rightarrowtail & a_2 & \rightarrowtail & \ldots & \rightarrowtail & a_n \\
\| & & \| & & \| & & \| & & \| \\
0 & \rightarrowtail & a_1 & \rightarrowtail & a_2 & \rightarrowtail & \ldots & \rightarrowtail & a_n \\
\| & & \| & & \| & & \| & & \| \\
0 & \rightarrowtail & \vdots & \rightarrowtail & \vdots & \rightarrowtail & \vdots & \rightarrowtail & \vdots \\
\| & & \| & & \| & & \| & & \| \\
0 & \rightarrowtail & a_1 & \rightarrowtail & a_2 & \rightarrowtail & \ldots & \rightarrowtail & a_n \\
\| & & \| & & \| & & \| & & \| \\
0 & \rightarrowtail & a_1 & \rightarrowtail & a_2 & \rightarrowtail & \ldots & \rightarrowtail & a_n
\end{array}$$

Where on $s_{n,m}\mathcal{C}$ we send it to m-copies of the equality. To show that this is a homotopy equivalence, we observe that $N(i\mathcal{S}_n(\mathcal{C}))_m$ is $s_n\mathcal{C}([m], i)$, where $\mathcal{C}([m], i)$ is the subcategory of the functor category $[[m], \mathcal{C}]$ where all the morphisms go to isomorphisms. Note that there is an equivalence of categories $\mathcal{C} \to \mathcal{C}([m], i)$ sending each object to the identity sequence of m elements and the map above is just the induced map $s.\mathcal{C} \to s.\mathcal{C}([m], i)$ at each $[m]$ and hence by the lemma above is a homotopy equivalence. Therefore by the realisation lemma (Theorem 2.3.2) $s.\mathcal{C} \to N(i\mathcal{S}.(\mathcal{C})$ is a homotopy equivalence. $\square$

Recall that one of the equivalent statements of the additivity theorem was that $i\mathcal{S}.E(\mathcal{C}) \longrightarrow i\mathcal{S}.\mathcal{C} \times i\mathcal{S}.(\mathcal{C})$ is a homotopy equivalence. As this map was induced by an exact functor $E(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$, it also induces a map $s.E(\mathcal{C}) \to s.\mathcal{C} \times s.\mathcal{C}$. Therefore by the above corollary, the additivity theorem is equivalent to showing this map is a homotopy equivalence. Before we prove this lemma we need to prove a version of Quillen's theorem A and B(Theorem 2.4.3 and Theorem 2.4.4).

Let Δ^n be the standard n-simplex as always. Let $f : X \to Y$ be a map of simplicial sets and let $y \in Y_n$. We define the simplicial set $f/(n, y)$ to be the pullback

$$\begin{array}{ccc}
f/(n,y) & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\Delta^n & \xrightarrow{y} & Y
\end{array}$$

We then have the following lemmas

Lemma 4.1.3 (lemma A)**.** If $f/(n,y)$ is contractible for every (n,y) then f is a homotopy equivalence.

Lemma 4.1.4 (lemma B)**.** If for every $a : [m] \to [n]$, and every $y \in Y_n$, the induced map $f/(m, a^*y) \to f/(n,y)$ is a homotopy equivalence then for every (n,y) the pullback diagram above is a homotopy cartesian (2.3.2).

To prove this using Quillen's Theorem A and B we need to translate this into a statement about categories. We define the functor $simp : \mathbf{sSet} \to \mathbf{Cat}$, which takes a simplicial set Y to a small category $simp(Y)$, whose objects are ordered pairs (n,y) for every $y \in Y_n$, and where a morphism $(n', y') \to (n,y)$ is a morphism $a : [n'] \to [n]$ such that $a^*(y) = y'$. It can be easily checked that this construction is functorial.

Lemma 4.1.5.

1. $simp(f/(n,y))$ is naturally isomorphic to the fibre over (n,y) of the induced map of categories $simp(f)$.
2. For all simplicial sets X, $Nsimp(X)$ is homotopy equivalent to X.

Proof. For (1) we see that m-simplices of $f/(n,y)$ correspond to pairs (x,θ) where $x \in X_m$ and $\theta \in \mathrm{Hom}([m],[n])$ such that $\theta^*(y) = f(x)$, so objects of $simp(f/(n,y))$ will be of the form $(m,(x,\theta))$, but elements of $simp(f)/(n,y)$ will be of the same form if we unravel the definition of $\mathcal{F}/d$ for a functor. This will give a natural isomorphism between the corresponding categories.

For (2) we will first prove the statement for Δ^n. An m-simplex in $Nsimp(\Delta^n)$ is a sequence of maps in $\mathbf{\Delta}$ (after applying Yoneda lemma),

$$[n_0] \xrightarrow{a_0} [n_1] \xrightarrow{a_1} [n_2] \xrightarrow{a_2} \ldots \to [n_m] \xrightarrow{a_m} [n]$$

We send each of these to a map $[m] \to [n]$ given by

$$b(i) = a_m a_{m-1} \ldots a_i(n_i)$$

Now we know that Δ^n is always contractible and $simp(\Delta^n)$ has a terminal object (n, Id) and is therefore contractible (section 2.4). Therefore the above map gives us a homotopy equivalence. To extend this map to all simplicial sets we note that $simp$ preserves colimit, as colimits of simplicial sets are defined obejctwise. Therefore using $X \cong \underset{\Delta^n \to X}{\operatorname{colim}} \Delta^n$ to get a map $Nsimp(X) \to X$ for every simplicial set. The map preserves colimits we can use the homotopy equivalence for the standard simplices and the *Gluing lemma*[3, II.9.8] for pushouts if we can construct each simplicial set as a sequence of pushouts of disjoint unions of Δ^n, which we will do now. Let $X_{\leq n} = \underset{\substack{\Delta^k \to X \\ k \leq n}}{\operatorname{colim}} \Delta^k$. Its realisation is then the n-skeleton of X. Hence if for each n $Nsimp(X_{\leq n})$ is homotopy equivalent to $X_{\leq n}$ under the same map, then $Nsimp(X)$ is homotopy equivalent to X. We prove this statement by induction. For $n = 0$ this is true as we only have a discrete set of point. Assuming the homotopy equivalence for all $k < n$ we prove the statement for n. This we obtain as $X_{\leq n}$ is the pushout

$$\begin{tikzcd}
\coprod_{k<n} \Delta^k \arrow[r] \arrow[d] & \coprod\limits_{\Delta^n \to X} \Delta^n \arrow[d] \\
X_{\leq n-1} \arrow[r] & X_{\leq n}
\end{tikzcd}$$

Where we have on diagram of the form

$$\begin{tikzcd}
\Delta^k \arrow[r] \arrow[d] & \Delta^n \\
X_{\leq n-1} &
\end{tikzcd}$$

For each map $\Delta^k \to X$ for $k < n$ such that it factors as $\Delta^k \to \Delta^n \to X$. Therefore by induction hypothesis and gluing lemma we have a homotopy equivalence for all simplicial sets. $\square$

Using the above lemma, the proof of lemma A and B follow directly from theorem A and B.

Now we state the most tricky part of proving the additivity theorem.

Sublemma: The map $f : s.E(\mathcal{C}) \to s.\mathcal{C}$, $(A \rightarrowtail C \twoheadrightarrow B) \to A$ satisfies the hypothesis of lemma B above.

Proof of sublemma. In this context, the assertion is that for all $y \in s_n\mathcal{C}$ and $w : [m] \to [n]$ in $\mathbf{\Delta}$, the map induced by pullback property, $w_* : f/(m, w^*y) \to f/(n, y)$ is a homotopy equivalence. For every such w we have commutative diagrams of the form

$$
\begin{tikzcd}
{[m]} \arrow[rr, "w"] & & {[n]} \\
& {[0]} \arrow[ul, "u"] \arrow[ur, "v"'] &
\end{tikzcd}
$$

Therefore it is enough to prove the assertion for the case $[0] \to [n]$, as in the above diagram, u^* and v^* being homotopy equivalences guarantees that w^* is a homotopy equivalence.

Now the statement we need to prove is the following: let A' be an n-simplex of $s.\mathcal{C}$, and 0 the unique 0-simplex os $s.\mathcal{C}$. Let $v_i : [0] \to [n]$ for $i \leq n$ denote the map that takes 0 to i. The for every i the induced map

$$v_{i*} : f/(0,0) \to f/(n, A')$$

is a homotopy equivalence. An m-simplex of $s.E(\mathcal{C})$ can be identified with an object of $E(\mathcal{S}_m(\mathcal{C}))$. That is, an exact sequence in the exact category $\mathcal{S}_m(\mathcal{C})$. Similarly an m-simplex in $f/(n, A')$ is then a pair consisting of an exact sequence $A \rightarrowtail C \twoheadrightarrow B$ in $\mathcal{S}_m(\mathcal{C})$ along with a map $u : [m] \to [n]$ such that the composition,

$$Ar[m] \xrightarrow{u_*} Ar[n] \xrightarrow{A'} \mathcal{C}$$

is equal to A. If we look at $f/(0,0)$ then by above identification, its m-simplices are exact sequences in $\mathcal{S}_m(\mathcal{C})$ such that the first term is 0, hence by exactness the second map is an isomorphism. $f/(0,0)$ is then $s.\mathcal{C}'$ where $\mathcal{C}'$ is the subcategory of $E(\mathcal{C})$ consisting of sequences of the form $0 \rightarrowtail B \xrightarrow{\sim} C$. But this category is equivalent to $\mathcal{C}$ and we have a homotopy equivalence, $s.\mathcal{C} \xrightarrow{j_*} f/(0,0)$. Where j_* acts by $A \to (0 \rightarrowtail A = A)$. The composition of the quotient projection $(A \rightarrowtail C \twoheadrightarrow B) \to B$ with the pullback map $f/(n, A') \to s.E(\mathcal{C})$ gives us a map $p : f/(n, A') \to s.\mathcal{C}$. This map is the left inverse of the composition,

$$s.\mathcal{C} \xrightarrow{j_*} f/(0,0) \xrightarrow{v_{i*}} f/(n, A')$$

and since j_* is already a homotopy equivalence, to show v_{i*} is a homotopy equivalence for all i it is enough to show that p is a homotopy equivalence.

To show that p is homotopy equivalence, it is enough to show that $v_{n*}j_*p$:

$f/(n, A') \to f/(n, A')$ is homotopic to the identity map. We have the pullback map $f/(n, A') \to \Delta^n$, we will construct the required homotopy by lifting the simplicial homotopy that contracts Δ^n to its last vertex. This is a homotopy given by a map of the composed functor

$$(\mathbf{\Delta}/[1])^{op} \longrightarrow \mathbf{\Delta}^{op} \longrightarrow \mathbf{Set}$$

$$([m] \to [1]) \longrightarrow [m] \longrightarrow Hom([m], [n])$$

to itself namely,

$$(v : [m] \to [1]) \longrightarrow ((u : [m] \to [n]) \to (u'' : [m] \to [n])$$

where u'' is defined by the composition,

$$[m] \xrightarrow{(u,v)} [n] \times [1] \xrightarrow{w} [n]$$

with $w(j, 0) = j$, $w(j, 1) = n$. A lifting of this homotopy to $f/(n, A')$ will be a map taking,

$$(v : [m] \to [1]) \to ((A \rightarrowtail C \twoheadrightarrow B, u : [m] \to [1]) \longrightarrow (A'' \rightarrowtail C'' \twoheadrightarrow B'', u'' : [m] \to [n])$$

where u'' is as obtained above and A'' must be equal to the composition

$$Ar[m] \xrightarrow{u''_*} Ar[n] \xrightarrow{A'} C$$

and thus is fixed. We wish that this is possible and will give us the remaining conditions. For this let us observe that for all $j \in [m]$ we have

$$u(j) \leq u''(j)$$

by our above definition. Viewing $[m]$ and $[n]$ as poset categories, this says that there exists a natural transformation of functors $u \to u''$ (which is necessarily unique). This induces a natural transformation

$$(u_* : Ar[m] \to Ar[n]) to (u''_* : Ar[m] \to Ar[n])$$

giving a map of composed functors,

$$Ar[m] \to Ar[n] \to \mathcal{C}$$

which is map from A to A'' in $\mathcal{S}_m\mathcal{C}$. This map is forced to be unique as it is induced from a natural transformation between poset categories, which if existing must be unique. Now we define our define the required sequence $A'' \rightarrowtail C'' \twoheadrightarrow B''$ from $A \rightarrowtail C \twoheadrightarrow B$ by taking pushout along $A \to A''$ to obtain C'' and by property of pushouts we can take $B'' = B$

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & C \arrow[r, two heads] \arrow[d] & B \arrow[d, "||"] \\
A'' \arrow[r, tail] & C'' \arrow[r, two heads] & B''
\end{tikzcd}$$

Here the pushouts are only unique up to isomorphisms. So we make a choice but we make choices in $\mathcal{C}$ and these choices will fix all choices of pushouts in $\mathcal{S}_m(\mathcal{C})$ as they are defined object wise. The only special choices we make are, when $A \to A''$ is identity, we choose $C \to C''$ to be identity and if $A'' = 0$ we choose $C'' \to B''$ to be identity. These conditions make sure that the homotopy starts at identity and the image of $v_{n*}j_*$ is fixed under the homotopy. As we have made the construction of $(A'' \rightarrowtail C'' \twoheadrightarrow B'')$ object wise, we need to verify that this construction is compatible with the structure maps of $\mathbf{\Delta}/[1]$. To see first composing $A \to A''$ with a map induced by some $t : [m'] \to [m]$ gives us is a natural transformation between $At_* \to A''t_*$ but this must be the natural transformation we constructed between them as it is unique. For the pushouts note that we made choices in $\mathcal{C}$ and since an element in $\mathcal{S}_m(\mathcal{C})$ is a diagram over $\mathcal{C}$ on which the structure acts by omission or/and repetition, we will then obtain the required compatibility. Hence this is a homotopy and by above choices it is a homotopy between the desired maps. Hence the sublemma is proved. □

Using the sublemma, we now prove the lemma we require.

Proof of lemma. We can apply lemma B to obtain a homotopy cartesian square for each simplex (n, y) of $s.\mathcal{C}$. For the 0-simplex we then have a fibration up to homotopy $f/(0,0) \to s.E(\mathcal{C}) \to s.\mathcal{C}$. From previous observation the fibre is homotopic to $s.\mathcal{C}$, then we get the sequence

$$s.\mathcal{C} \to s.E(\mathcal{C}) \to s.\mathcal{C}$$

which sends $B \to 0 \rightarrowtail B = B$ and $A \rightarrowtail C \twoheadrightarrow B$ to A is a fibration up to homotopy. There is a map to this sequence from the product fibration sequence which is identity at the ends and is $s.\mathcal{C} \times s.\mathcal{C} \to s.E(\mathcal{C})$ sending $(A, B) \to (A \rightarrowtail A \oplus B \twoheadrightarrow B)$ at the middle. Hence this map must be a homotopy equivalence. Then, this is a section of the desired map and hence it is also a homotopy equivalence as desired. □

4.2 Applications of additivity theorem

We will here see some applications of additivity theorem. One of the main results we will prove is that the K-theory spectrum we constructed is an Ω spectrum (section 2.5) beyond the first term. This will give us two important results, namely a version of additivity theorem for the K-groups and the equivalence between the two definition of K-theory. First we need some tools.

Definition 4.2.1 (Path object)**.** The shift functor is a functor $shift : \mathbf{\Delta} \to \mathbf{\Delta}$ that takes $[n]$ to $[n+1]$ and takes a morphism f to f', where $f'(0) = 0$ and $f(i+1) = f(i)$ for all $0 < i \leq n$. We also then have a natural transformation $Id \implies shift$ which is at each $[n]$ the morphism that takes i to $i + 1$. Then for a simplicial object $X : \mathbf{\Delta} \to \mathcal{C}$ in a category $\mathcal{C}$ the *associated path object* PX is the composite simplicial object $X \circ shift$.

Lemma 4.2.1. PX is simplicially homotopy equivalent to the constant simplicial object $[n] \to X_0$

Proof. We have the composite map $PX \to X_0 \to PX$ induced by

$$[n] \to ([n + 1] \to [0] \to [n + 1])$$

, where the last map is given by $0 \to 0$ for all n. So it is enough to show that the identity on PX is homotopic to this composite map. The homotopy is given by the natural transformation

$$(a : [n] \to [1]) \to (\phi_a^* : X_{n+1} \to X_{n+1})$$

where ϕ_a^* is the simplicial map induced by the morphism $\phi_a : [n+1] \to [n+1]$ where

$$\phi_a(i) = \begin{cases} 0 & i = 0 \\ i & a(i-1) = 1 \\ 0 & a(i-1) = 0 \end{cases}$$

□

Now $PX_0 = X_1$, so we have an inclusion of the constant simplicial object X_1 into PX. We also have a projection $PX \to X$ which is obtained by applying the d_0 of X to obtain $PX_n = X_{n-1} \to X_n$ from the properties of simplicial objects, we can check that this is a morphism of simplicial sets. Now the composite sequence $X_1 \to PX \to X$ takes every 1-simplex to its first vertex and hence contained in the constant simplicial object X_0. If we apply this to the simplicial category $i\mathcal{S}.\mathcal{C}$ we obtain a sequence $i\mathcal{S}_1\mathcal{C} \to P(i\mathcal{S}.) \to i\mathcal{S}.\mathcal{C}$ and we know $iS_1(C)$ is isomorphic to $i\mathcal{C}$ the category of all isomorphisms in $\mathcal{C}$. Then we get a sequence

$$i\mathcal{C} \to P(i\mathcal{S}.(\mathcal{C}) \to i\mathcal{S}.(\mathcal{C})$$

. As the composite is always contained in $i\mathcal{S}_0(\mathcal{C})$ which has a unique 0-simplex, the realisation of the composite is constant and $|P(i\mathcal{S}.(\mathcal{C}))|$ is contractible by above lemma, so we obtain a map well defined upto homotopy,

$$|i\mathcal{C}| \to \Omega|i\mathcal{S}.\mathcal{C}|$$

But we already have a natural choice of such a map which is adjoint (subsection 1.3.1) to the map we used in our definition of $K(\mathcal{C})$(subsection 3.3.3).

In this construction, we can replace $\mathcal{C}$ with $\mathcal{S}.(\mathcal{C})$ to get another sequence,

$$i\mathcal{S}.(\mathcal{C}) \to P(i\mathcal{S}.\mathcal{S}.(\mathcal{C})) \to i\mathcal{S}.\mathcal{S}.(\mathcal{C})$$

Proposition 4.2.2. The above sequence is a fibration up to homotopy. Therefore as $|P(i\mathcal{S}.\mathcal{S}.(\mathcal{C}))|$ is contractible as before, we have a homotopy equivalence $|i\mathcal{S}.\mathcal{C}| \to \Omega|i\mathcal{S}.\mathcal{S}.\mathcal{C}|$.

We will prove this proposition as a special case of the next proposition.

Definition 4.2.2. Let $f : \mathcal{A} \to \mathcal{B}$ be an exact functor of exact categories. Then $\mathcal{S}.(f : \mathcal{A} \to \mathcal{B})$ is the pullback

$$\begin{tikzcd} \mathcal{S}.(f : \mathcal{A} \to \mathcal{B}) \arrow[r] \arrow[d] & P\mathcal{S}.\mathcal{B} \arrow[d] \\ \mathcal{S}.\mathcal{A} \arrow[r] & \mathcal{S}.\mathcal{B} \end{tikzcd}$$

Where the map $P\mathcal{S}.\mathcal{B} \to \mathcal{S}.\mathcal{B}$ is the map $PX \to X$ as before. Using the sequence

$$\mathcal{B} \to P(i\mathcal{S}.(\mathcal{B}) \to i\mathcal{S}.(\mathcal{B})$$

as above whose composition is constant along with the trivial map $\mathcal{B} \to \mathcal{S}.\mathcal{A}$ induces a map $\mathcal{B} \to \mathcal{S}.(f : \mathcal{A} \to \mathcal{B})$ giving a sequence

$$\mathcal{B} \to \mathcal{S}.(f : \mathcal{A} \to \mathcal{B}) \to \mathcal{S}.\mathcal{A}$$

whose composition is trivial.

Proposition 4.2.3. The sequence

$$i\mathcal{S}.\mathcal{B} \to i\mathcal{S}.\mathcal{S}.(f : \mathcal{A} \to \mathcal{B}) \to i\mathcal{S}.\mathcal{S}.\mathcal{A}$$

is a fibration up to homotopy.

Proof. We will use the fibration criterion ([6, lemma 5.13]), which says that it is enough to show that for all n, the sequence

$$i\mathcal{S}.\mathcal{B} \to i\mathcal{S}.\mathcal{S}_n(f : \mathcal{A} \to \mathcal{B}) \to i\mathcal{S}.\mathcal{S}_n\mathcal{A}$$

is a fibration up to homotopy. Using additivity theorem we will show that this sequence is, up to homotopy, the trivial fibration sequence associated to the product $i\mathcal{S}.\mathcal{B} \times i\mathcal{S}.\mathcal{S}_n\mathcal{A}$.

Objects in $\mathcal{S}_n(f : \mathcal{A} \to \mathcal{B})$ are pairs of sequences $0 \rightarrowtail A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \ldots \rightarrowtail A_{0,n}$ and $0 \rightarrowtail B_{0,1} \rightarrowtail B_{0,2} \rightarrowtail \ldots \rightarrowtail B_{0,n+1}$ with $f(A_{0,1}) \rightarrowtail f(A_{0,2}) \rightarrowtail \ldots \rightarrowtail f(A_{0,n}) \cong B_{0,2}/B_{0,1} \rightarrowtail B_{0,3}/B_{0,1} \ldots \rightarrowtail B_{0,n+1}/B_{0,1}$ plus some choice of quotients. As dropping the choices of quotients will give us an equivalent category we will do so. Let $\mathcal{C}'$ denote the subcategory of objects where all the maps $B_{0,1} \rightarrowtail B_{0,2} \rightarrowtail$

$\ldots \rightarrowtail B_{0,n+1}$ are identity and all $A_{0,i}$ are the distinguished 0. Then objects in $\mathcal{C}'$ are uniquely determined by $B_{0,1}$ and is hence $\mathcal{C}'$ is isomorphic to $\mathcal{B}$. Let $\mathcal{C}''$ denote another subcategory where $B_{0,1} = 0$ then $\mathcal{C}''$ is isomorphic to $\mathcal{S}_n\mathcal{A}$ as the sequence $f(A_{0,1}) \rightarrowtail f(A_{0,2}) \rightarrowtail \ldots \rightarrowtail f(A_{0,n})$ then has unique lift. We then have functors $j' : \mathcal{S}_n(f : \mathcal{A} \to \mathcal{B}) \to \mathcal{C}'$, given by

$$\begin{aligned}(0 \rightarrowtail A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \ldots \rightarrowtail A_{0,n}, 0 \rightarrowtail B_{0,1} \rightarrowtail B_{0,2} \rightarrowtail \ldots \rightarrowtail B_{0,n+1}) \longrightarrow \\ (0 \rightarrowtail A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \ldots \rightarrowtail A_{0,n}, 0 \rightarrowtail 0 \rightarrowtail f(A_{0,1}) \rightarrowtail f(A_{0,2}) \rightarrowtail \ldots \rightarrowtail f(A_{0,n}))\end{aligned} \tag{4.2}$$

and $j'' : \mathcal{S}_n(f : \mathcal{A} \to \mathcal{B}) \to \mathcal{B}$ given by

$$\begin{aligned}(0 \rightarrowtail A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \ldots \rightarrowtail A_{0,n}, 0 \rightarrowtail B_{0,1} \rightarrowtail B_{0,2} \rightarrowtail \ldots \rightarrowtail B_{0,n+1}) \longrightarrow \\ (0 \rightarrowtail 0 \rightarrowtail \ldots \rightarrowtail 0, 0 \rightarrowtail B_{0,1} = B_{0,1} = \ldots = B_{0,1})\end{aligned} \tag{4.3}$$

This gives an exact sequence of exact functors

$$j' \rightarrowtail id \twoheadrightarrow j''$$

then by additivity theorem we have the identity on $i\mathcal{S}.\mathcal{S}_n(f : \mathcal{A} \to \mathcal{B})$ is homotopic to the sum of $i\mathcal{S}.j'$ and $i\mathcal{S}.j''$ whose images are isomorphic to $i\mathcal{S}.\mathcal{S}_n\mathcal{A}$ and $i\mathcal{S}.\mathcal{B}$ respectively. Using the exactness of the sequence of functor we can construct a functor,

$$i\mathcal{S}.\mathcal{B} \times i\mathcal{S}.\mathcal{S}_n\mathcal{A} \to i\mathcal{S}.\mathcal{S}_n(f : \mathcal{A} \to \mathcal{B})$$

which is a retraction upto homotopy by the homotopy above. As j' and j'' are constant on $\mathcal{C}'$ and $\mathcal{C}''$, left composing the above map with the sum of $i\mathcal{S}.j'$ and $i\mathcal{S}.j''$ (identifying the isomorphic categories) gives identity on $i\mathcal{S}.\mathcal{B} \times i\mathcal{S}.\mathcal{S}_n\mathcal{A}$. Therefore the given map is a homotopy equivalence. It is not hard to see that this extends to give a homotopy equivalence of the product fibration sequence and the desired sequence. Hence $i\mathcal{S}.\mathcal{B} \to i\mathcal{S}.\mathcal{S}.(f : \mathcal{A} \to \mathcal{B}) \to i\mathcal{S}.\mathcal{S}.\mathcal{A}$ is a fibration sequence up to homotopy. □

Now if we choose f to be the identity functor $\mathcal{C} \to \mathcal{C}$, then we get the result desired.

Corollary 4.2.4. The K-theory spectrum is an Ω-spectrum beyond the first term.

Proof. As shown earlier, the map $|i\mathcal{C}| \to \Omega|i\mathcal{S}.\mathcal{C}|$ was the same as the map in the spectrum. Repeatedly applying $i\mathcal{S}.$ we get maps $|i\mathcal{S}.^n(\mathcal{C})| \to \Omega|i\mathcal{S}.^{n-1}(\mathcal{C})|$ and from the above theorem all but the first if these is a homotopy equivalence. Therefore the K-theory spectrum is an Ω-spectrum beyond the first term and $\pi_k(K(\mathcal{C})) \cong \pi_k(i\Omega\mathcal{S}.\mathcal{C})$ for all $k \geq 0$. □

Also note that since each of $K(\mathcal{C})_n = |i\mathcal{S}.^n(\mathcal{C})|$ is connected, we can show that the negative homotopy groups are all 0(Equation 2.5) and as we have shown $d(i\mathcal{S}.\mathcal{C})$ is homotopy equivalent to $s.\mathcal{C}$ and therefore $\pi_k(K(\mathcal{C})) \cong \pi_k(\Omega(s.\mathcal{C}))$.

Given two exact functors between exact categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ we have the following commutative diagram,

$$\begin{tikzcd}
i\mathcal{S}.\mathcal{B} \arrow[r] \arrow[d] & i\mathcal{S}.\mathcal{S}.(f : \mathcal{A} \to \mathcal{B}) \arrow[r] \arrow[d] & i\mathcal{S}.\mathcal{S}.\mathcal{A} \arrow[d, equal] \\
i\mathcal{S}.\mathcal{B} \arrow[r] & i\mathcal{S}.\mathcal{S}.(f : \mathcal{A} \to \mathcal{C}) \arrow[r] & i\mathcal{S}.\mathcal{S}.\mathcal{A}
\end{tikzcd}$$

Since the rows are fibrations up to homotopy, we can prove a result, similar to that about pullbacks, that then the first square,

$$\begin{tikzcd}
i\mathcal{S}.\mathcal{B} \arrow[r] \arrow[d] & i\mathcal{S}.\mathcal{S}.(f : \mathcal{A} \to \mathcal{B}) \arrow[d] \\
i\mathcal{S}.\mathcal{B} \arrow[r] & i\mathcal{S}.\mathcal{S}.(f : \mathcal{A} \to \mathcal{C})
\end{tikzcd} \tag{4.4}$$

is homotopy cartesian.

4.3 Equivalence of definitions of K-theory

Here we prove that the K-groups defined by Quillen's Q-construction and Waldhausen's S-construction are isomorphic for any small exact category $\mathcal{C}$. For that it is enough to show that $\mathcal{QC}$ viewed as a constant simplicial category is homotopy equivalent to $i\mathcal{S}.\mathcal{C}$. To do this we replace both these with equivalent categories and then show that they are equivalent. The proof is sketched in [7].

4.3.1 Segal-Quillen subdivision

Consider the functor $T : \mathbf{\Delta} \to \mathbf{\Delta}$ that sends $[n] \to [2n+1] = [0, 1, \ldots, n, 0', \ldots, n']$ we have written $[2n+1]$ this way as it makes describing the morphisms easier. Given an morphism f in $\mathbf{\Delta}$, we define $T(f)(i) = f(i)$ and $T(f)(i') = f(i)'$. Then composition with T gives a functor $[\mathbf{\Delta}^{op}, \mathcal{D}] \to [\mathbf{\Delta} op, \mathcal{D}]$ for any category $\mathcal{D}$. We denote this functor by $X \to X^e$. Here $X^e_n = X_{2n+1}$.

Suppose $\mathcal{C}$ is a small category, define $\mathcal{C}^e$ to be the category whose objects are morphisms of $\mathcal{C}$ and whose morphisms from $A \xrightarrow{i} B$ to $A' \xrightarrow{i'} B'$ is a pair of morphisms $A' \to A$ and $B \to B'$ such that the composite $A' \to A \xrightarrow{i} B \to B'$ is i'. It can be seen that this construction is functorial. Note that objects in $\mathcal{C}^e$ are 1-simplices in $N(\mathcal{C})$ and morphisms in $\mathcal{C}^e$ are 3-simplices in $N(\mathcal{C})$. Therefore we can show that $N(\mathcal{C}^e) = N(\mathcal{C})^e$.

Proposition 4.3.1. For any category $\mathcal{C}$, the source and target functors $\mathcal{C}^{op} \leftarrow \mathcal{C}^e \to \mathcal{C}$ are homotopy equivalences.

Proof. We prove the target functor is a homotopy equivalence, the other case is similar. Let $F : \mathcal{C}^e \to \mathcal{C}$ be the target functor. Then for any C in $\mathcal{C}$, objects in F/C are pairs $(A \to B, B \to C)$ and morphisms $(A \to B, B \to C) \to (A' \to B', B' \to C)$ are

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & C \\ & B' \arrow[ur] & \end{tikzcd}$$

such that $A' \to A \to B \to B'$ is the original $A' \to B'$ for some $A' \to A$, this has the obvious terminal object $(C \xrightarrow{id} C, C \xrightarrow{id} C)$ and it therefore contractible (section 2.4). Hence by Quillen' Theorem A, the target map is a homotopy equivalence. □

By realisation lemma we can extend this to a statement about simplicial categories. Therefore $i\mathcal{S}.\mathcal{C}$ is homotopy equivalent to $i\mathcal{S}.\mathcal{C}^e$. For convenience we will denote $i\mathcal{S}.\mathcal{C}^e$ by $i\mathcal{S}^e.\mathcal{C}$.

4.3.2 Swallowing Lemma

In the previous subsection, we constructed a simplicial category which is homotopy equivalent to $i\mathcal{S}.\mathcal{C}$, now we will construct a simplicial category which is homotopy equivalent to $\mathcal{QC}$. Let $\mathcal{C}$ be any category and let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of $\mathcal{C}$, with $Ob(\mathcal{A}) = Ob(\mathcal{B}) = Ob(\mathcal{C})$. Then we define a simplicial category $\mathcal{A}.\mathcal{B}(\mathcal{C})$, where for any $n \geq 0$, $\mathcal{A}_n\mathcal{B}(\mathcal{C})$ is the category whose objects are sequences

$$A. := A_0 \to A_1 \to \ldots \to A_n$$

where each morphism $A_i \to A_i + 1$ is in $\mathcal{A}$ and whose morphisms from $A. \to A.'$ are compatible maps $A_i \to A_i'$ each contained in $\mathcal{B}$. The simplicial functor defined by $\theta : [m] \to [n]$ is $\theta* : \mathcal{A}_n\mathcal{B}(\mathcal{C}) \to \mathcal{A}_m\mathcal{B}(\mathcal{C})$ such that $\theta * (A.)_i = A_(\theta(i))$. We define

$$\mathcal{A}.\mathcal{B}.(\mathcal{C}) = dN(\mathcal{A}.\mathcal{B}(\mathcal{C}))$$

Note that $\mathcal{A}.\mathcal{B}.(\mathcal{C}) = \mathcal{B}.\mathcal{A}.(\mathcal{C})$, and $N(\mathcal{C}) = Id.\mathcal{C}.(\mathcal{C})$, where Id is the subcategory of identity maps.

Lemma 4.3.2 (swallowing lemma)**.** Let $\mathcal{C}$ be any small category and $\mathcal{A}$ be a subcategory with the same set of objects. The natural map

$$N(\mathcal{C}) = Id.\mathcal{C}.(\mathcal{C}) \to \mathcal{A}.\mathcal{C}.(\mathcal{C})$$

is a weak equivalence.

Proof. We have $N(\mathcal{C}) = Id.\mathcal{C}.(\mathcal{C}) \to \mathcal{A}.\mathcal{C}.(\mathcal{C})$ induced by the inclusion $Id \rightarrowtail \mathcal{A}$. We show this is a weak equivalence by showing that for each n, $Id_n\mathcal{C}(\mathcal{C}) \to \mathcal{A}_n\mathcal{C}(\mathcal{C})$ induces a homotopy equivalence on the nerves. For this we construct $F_n : \mathcal{A}_n\mathcal{C}(\mathcal{C}) \to Id_n\mathcal{C}(\mathcal{C})$ which sends

$$A_0 \to A_1 \to \ldots \to A_n \longrightarrow A_0 \to A_0 \to \ldots \to A_0$$

, where each $A_0 \to A_0$ is identity. If $i_n : Id_n\mathcal{C}(\mathcal{C}) \to \mathcal{A}_n\mathcal{C}(\mathcal{C})$ is the map induced by the inclusion, we see that $F_n i_n$ is identity and there is a natural transformation from $i_n F_n$ to the identity on $\mathcal{A}_n\mathcal{C}(\mathcal{C})$. Hence this is a homotopy equivalence. □

4.3.3 Equivalence of Q construction and S construction

Let $i\mathcal{Q}.\mathcal{C}$ denote $\mathcal{QC}.i\mathcal{C}(\mathcal{C})$, where $i\mathcal{C}$ is the category of all isomorphisms in $i\mathcal{C}$. From the swallowing lemma $i\mathcal{Q}.\mathcal{C}$ is homotopy equivalent to $N(\mathcal{QC})$. Therefore to prove the definitions are equivalent, we show $i\mathcal{Q}.\mathcal{C}$ is homotopy equivalent to $i\mathcal{S}^e.\mathcal{C}$.

Theorem 4.3.3. $iQ.\mathcal{C}$ is homotopy equivalent to $i\mathcal{S}^e.\mathcal{C}$.

Proof. We prove the assertion by showing the category $iQ_n\mathcal{C}$ is equivalent to $i\mathcal{S}^e_n\mathcal{C}$. An object in $i\mathcal{S}^e_n\mathcal{C}$ is a sequence

$$0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \ldots \rightarrowtail A_n \rightarrowtail A_{0'} \rightarrowtail A_{1'} \rightarrowtail \ldots \rightarrowtail A_{n'}$$

with choice of quotients. From this we diagrams

$$\begin{tikzcd} A_{i/(i+1)'} \arrow[r, tail] \arrow[d, two heads] & A_{(i+1)/(i+1)'} \\ A_{i/i'} & \end{tikzcd}$$

This gives a sequence $A_{0/0'} \to A_{1/1'} \to \ldots \to A_{n/n'}$ in $Q\mathcal{C}$. For the other direction, given a sequence $A_{0/0'} \to A_{1/1'} \to \ldots \to A_{n/n'}$ in $Q\mathcal{C}$, taking successive pushouts of the diagrams given by $A_{i/i'} \to A_{(i+1)/(i+1)'}$ will give us all the required quotients up to isomorphism. Hence there is an equivalence of categories between $iQ_n\mathcal{C}$ and $i\mathcal{S}^e_n\mathcal{C}$. □

This shows that both constructions give the same K-groups.

4.4 Fundamental theorems of Algebraic K-theory

In this section we state some fundamental theorems of algebraic K-theory. Their proof follow quiet straightforwardly from the results stated above and are given elaborately in [5]. These theorems relate the K-theory of an exact category $\mathcal{A}$ to the K-theory of some special subcategories.

Theorem 4.4.1 (Cofinality Theorem)**.** Let $\mathcal{A}$ be a small exact category and let $\mathcal{B}$ be a full exact subcategory which is *cofinal* in $\mathcal{A}$. That is, $\mathcal{B}$ is closed under extensions and for each A in $\mathcal{A}$, there exists a A' in $\mathcal{A}$, such that, $A \oplus A'$ is in $\mathcal{B}$. We define $G = K(\mathcal{A})/K(\mathcal{B})$. Then there is a homotopy fibration sequence

$$i\mathcal{S}.\mathcal{B} \to i\mathcal{S}.\mathcal{A} \to BG$$

In particular $K_n(\mathcal{A}) \cong K_n(\mathcal{B})$ for all $n \geq 1$.

An important example of a cofinal subcategory is the category of finitely generated free modules as a subcategory of the category finitely generated projective modules over a ring R.

Theorem 4.4.2 (Resolution Theorem)**.** Let $\mathcal{A}$ be a small exact category and $\mathcal{B}$ be an full exact subcategory that is closed under extensions, exact sequences and cokernels and for every object A in $\mathcal{A}$ there is an exact sequence

$$A \rightarrowtail B \twoheadrightarrow B'$$

where B and B' are objects of $\mathcal{B}$. That is every object of $\mathcal{A}$ is a kernel of some morphism in $\mathcal{B}$. Then the morphism

$$i\mathcal{S}.\mathcal{B} \to i\mathcal{S}.\mathcal{A}$$

is a homotopy equivalence.

Note that since a category $\mathcal{C}$ and its dual $\mathcal{C}^{op}$ give the same K-spectrum, the dual of the above statement is also true.

Theorem 4.4.3 (Devissage Theorem)**.** Let $\mathcal{A}$ be a small abelian category and $\mathcal{B}$ a full abelian subcategory which is closed under direct sum, subobjects,and quotient objects. If every A object in $\mathcal{A}$ has a finite filtration,

$$0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \ldots \rightarrowtail A_n = A$$

where each quotient A_i/A_{i-1} is in $\mathcal{B}$ and so is A_1. Then the morphism

$$i\mathcal{S}.\mathcal{B} \to i\mathcal{S}.\mathcal{A}$$

is a homotopy equivalence.

An abelian subcategory of an abelian category is called a *Serre subcategory* if it is closed under extensions, subobjects and quotients. Given a small abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{B}$, the quotient category $\mathcal{A}/\mathcal{B}$ is defined as the category obtained by inverting all morphisms in $\mathcal{A}$ whose kernel and cokernel are in $\mathcal{B}$ (as all categories considered are small we can do this and obtain a category). Then we can show that $\mathcal{A}/\mathcal{B}$ is an abelian category and $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ is an exact functor [8, II.A].

The localisation theorem is stated in [5] is for the particular case of localisation of rings at multiplicatively closed sets, but the proof for general Serre subcategories goes through similarly.

Theorem 4.4.4 (Localisation Theorem)**.** Let $\mathcal{A}$ be a small abelian category and $\mathcal{B}$ be a Serre subcategory. Then

$$i\mathcal{S}.\mathcal{B} \to i\mathcal{S}.\mathcal{A} \to i\mathcal{S}.\mathcal{A}/\mathcal{B}$$

is a homotopy fibration sequence. In particular we have a long exact sequence

$$\ldots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \to K_n(\mathcal{B}) \to K_n(\mathcal{A}/\mathcal{B}) \to \ldots \to K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B})$$

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