Classification of Two-dimensional Rational Conformal Field Theories

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by

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Certificate

This is to certify that this thesis entitled "Classification of Two-dimensional Rational Conformal Field Theories" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Harsha H R at the Indian Institute of Science Education and Research Pune, under the supervision of Professor Sunil Mukhi during the academic year 2015-2016.

Student
HARSHA H R

Supervisor
PROFESSOR SUNIL MUKHI

To my family and
all my teachers

Declaration

I hereby declare that the matter embodied in the report entitled Classification of Two-dimensional Rational Conformal Field Theories are the results of the investigations carried out by me at the Department of Physics, Indian Institute of Science Education and Research Pune, under the supervision of Professor Sunil Mukhi and the same has not been submitted elsewhere for any other degree.

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Abstract

A complete understanding of the space of Conformal Field Theories is not yet available. However, a particular subclass called Rational Conformal Field Theories (RCFT's) can be classified and understood using modular invariance. In this thesis, we review the formalism of classification of RCFT's using the modular differential equations satisfied by their characters. This classification is based on two parameters n and ℓ. The parameter n is the number of independent characters in the theory and ℓ in an integer, the number of zeroes of the Wronskian of the characters. We study the classification of theories with low number of characters $1-4$. The reconstruction of two-character theories from their characters is also discussed. We discuss the identification of two-character theories with $\ell = 2$ using our generalized coset construction. We use the same method to find new three and four-character theories with $\ell = 0$. For the new three and four character theories the degeneracies of the ground state are also calculated using our generalized coset construction.

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Chapter 1

Conformal Field Theory and Modular Invariance

1.1 Introduction

The study of symmetries of physical systems help us in finding various quantities of interesting properties, without actually analysing the dynamics. Two well-studied symmetries are translational and rotational invariance. Among the symmetries of physical systems, the one that is less studied is the scaling symmetry. Scaling symmetry is not respected in most systems in nature and is a badly broken symmetry (it is a symmetry of the action but not of the macroscopic observables/vacuum of the theory). Most physical systems come with characteristic length or Energy scales.

Statistical systems at criticality are probably the best known physical systems that obey scale invariance. At criticality, such systems show order at all ranges and are characterized by a set of critical exponents. Several statistical systems can have the same set of critical exponents despite having different behaviours away from criticality. All such systems with the same critical exponents fall into the same 'universality class'.

It is an interesting fact that such systems can be described by Conformal Field Theories. Under this identification the critical exponents are mapped to the conformal dimensions of a CFT. This identification is very powerful as we can get the entire spectrum of states from just the critical exponents. Systems that fall in a universality class tend to have the same/same set of partition functions. Thus, a classification of partition functions is equivalent to the classification of various classes of critical behaviours. In CFT, the partition function is built out of functions called characters.(Each character counts the tower of states created by a primary field). Thus, a classification of characters becomes important.

In the past few years the problem of classification of CFT's based on their number of indepenedent characters has been studied extensively. However, a complete understanding of the space of Conformal Field Theories is not yet available. We attempt to gain a better understanding of this scenario for a particular subclass of CFT's in two dimensions.

1.2 Conformal Field Theories

In this section we will briefly review the basics of conformal field theory (CFT) and define many quantities - central charge, primary fields, operator product expansion- that will be used repeatedly in the subsequent sections. We first discuss some general aspects of CFT's in d-dimensions and then specialize to the case of two dimensions. The outline for both discussions is the same- we first find the conformal group and then look at how certain fields transform under conformal transformations [7].

Conformal Field theory in d Euclidean dimensions

A conformal transformation is defined as an invertible map $x \to x'$ which keeps the metric tensor invariant up to a function

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{1.1}$$

$\Lambda(x) = 1$ gives the isometries of the metric. In particular, if the spacetime is flat we get Poincare' symmetry. A general conformal transformation in d-dimensions involves,along with the usual translation and rotation, local dilatation and special conformal transformation (SCT). The finite conformal transformations are:

$$\begin{aligned}(Translation) x'^{\mu} &= x^{\mu} + a^{\mu} \\ (Dilation) x'^{\mu} &= \alpha x^{\mu} \\ (Rotation) x'^{\mu} &= M^{\mu}_{\nu} x^{\mu} \\ (SCT) x'^{\mu} &= \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b.x + b^2 x^2}\end{aligned} \tag{1.2}$$

The generators of these transformation are:

$$\begin{aligned}(Translation)P_\mu &= -i\partial_\mu \\ (Dilation)D_\mu &= -ix^\mu\partial_\mu \\ (Rotation)L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ (SCT)K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\end{aligned}$$

These generators obey the $SO(d+1,1)$ commutation relations. Thus, the conformal group in d Euclidean-dimensions is the group $SO(d+1,1)$.

In principle, one can use the above generators to find out how a spinless field $\phi(x)$ transforms under a finite conformal transformation $x \to x'$.

The field transforms as:

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}}\phi(x) \tag{1.3}$$

Here, $|\frac{\partial x'}{\partial x}|$ is the Jacobian of the transformation. Δ is called the scaling dimension of the field. A field which transforms in this manner is called a 'quasi-primary field'.

Correlation functions

Let us now see the consequence of conformal invariance on the two-point function of two quasi-primary fields ϕ_1 and ϕ_2. Henceforth, we assume that all the fields/operators are time-ordered.

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{Z}\int[d\phi]\phi_1(x_1)\phi_(x_2)exp^{-S[\phi]} \tag{1.4}$$

Here, Z is the partition function and $S[\phi]$ is the action of the theory, with ϕ being the set of all independent fields in the theory including ϕ_1 and ϕ_2. From translational and rotational invariance, we find that:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|) \tag{1.5}$$

Let us now consider the scaling transformation $x \to \lambda x$. Using Eq. (1.3), we find that:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} < \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle \tag{1.6}$$

These two conditions imply that:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{1.7}$$

where C_{12} is a constant. We now demand invariance under special conformal transformation(SCT). For an SCT, we have from Eq. (1.2):

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1-2b.x+b^2x^2)^d} \tag{1.8}$$

This, with Eq. (1.7) implies:

$$\frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}\frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \tag{1.9}$$

This condition is satisfied only when $\Delta_1 = \Delta_2$. Therefore, the two-point function is given by:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \begin{matrix} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & , \Delta_1 = \Delta_2 \\ 0, & \Delta_1 \neq \Delta_2 \end{matrix} \tag{1.10}$$

Conformal Field Theory in two dimensions

It is very interesting to study CFT's in two dimensions as there exist critical systems in two dimensions that are exactly solvable. In two dimensions, the conformal group is easily found by noting that the global conformal transformations (projective transformations) that map the complex plane to itself are:

$$f(z) = \frac{az+b}{cz+d} \qquad , ad-bc=1 \tag{1.11}$$

where a, b, c, d, z are complex numbers. One can associate matrices with such transformations and the set of all matrices of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.12}$$

with $ad - bc = 1$ forms the 6-parameter group $SL(2, C)$. This is the conformal group in two dimensions. It is also known as the 'special conformal group'.

However, the true power of conformal field theories in two dimensions, which renders some systems exactly solvable, are the infinite number of coordinate transformations that are 'locally' conformal -namely holomorphic mappings (with nowhere vanishing derivative) of open subsets of the complex plane onto itself. Since, we are interested in local field theories we need to understand how this local conformal symmetry affects them. Local conformal transformations are generated by the following generators:

$$\ell_n = -z^{n+1}\partial_z \qquad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}_{\bar{z}} \tag{1.13}$$

These generators satisfy an algebra called the **Witt-algebra**.

$$\begin{aligned}
[\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\
[\tilde{\ell}_n, \tilde{\ell}_m] &= (n-m)\tilde{\ell}_{n+m} \\
[\ell_n, \tilde{\ell}_m] &= 0
\end{aligned}$$

Each of the above two infinite-dimensional algebras have a finite subalgebra with generators $\ell_{-1}, \ell_0, \ell_1$. These correspond to the global conformal group.

Let us now see what is the generalization of the quasi-primary field Eq. (1.3) in the two-dimensional case.

Primary fields

In two dimensions the definition of quasi-primary fields also applies to fields with spin. Let us consider a field with spin s and scaling dimension Δ. We define the *holomorphic conformal dimension* h and *antiholomorphic conformal dimension* $\bar{h}$ as:

$$(h, \bar{h}) = \frac{1}{2}\Big(\Delta + s, \Delta - s\Big) \tag{1.14}$$

Under a 'local' conformal transformation $(z, \bar{z}) \to (w(z), \bar{w}(\bar{z}))$, the fields transform as:

$$\phi'(w', \bar{w'}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \tag{1.15}$$

Such a field is called a 'primary field'. It is easy to see that if a field is primary then it is quasi-primary, but the converse is not true. Eq. (1.15) is the generalization of Eq. (1.3). Fields that are not primary are generally called 'secondary' fields. For example, the derivative of a primary field with $h \neq 0$ is a secondary field. In this case the two-point function becomes:

$$\langle\phi_1(z_1, \bar{z_1})\phi_2(z_2, \bar{z_2})\rangle = \begin{matrix} \dfrac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} & , \Delta_1 = \Delta_2 \\ 0, & \Delta_1 \neq \Delta_2 \end{matrix} \tag{1.16}$$

Radial quantization

For a theory in two dimensions one can define a quantization procedure that can help us solve for the energy spectrum. In this picture called the 'radial quantization', we define the time axis to be along the radial direction on the complex plane, with the origin $z = 0$ as the centre. The space axis is orthogonal to the time axis [7]. In this picture, the time ordering of two conformal fields ϕ_1 and ϕ_2 becomes:

$$\begin{aligned} \mathfrak{R}\phi_1(z)\phi_2(w) = \phi_1(z)\phi_2(w), & \qquad |z| > |w| \\ \phi_2(w)\phi_1(z), & \qquad |z| < |w| \end{aligned} \tag{1.17}$$

The correlation function of a product of operators, like the one defined in Eq. (1.17), becomes singular when the two points z and w coincide. Their singular behaviour can be captured using an operator product expansion.

Operator Product Expansion

An operator product expansion is the representation of a product of operators, (at two different positions z and w, respectively) by a sum of single operators that are well-defined when $z \to w$.

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{O_n(w)}{(z-w)^n} \tag{1.18}$$

As a concrete example, one can consider the OPE of the stress-energy tensor and a primary field with conformal dimensions $(h, \bar{h})$ [7]:

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi(w,\bar{w}) \tag{1.19}$$

and the anti-holomorphic counterpart of this equation. The symbol$\sim$ signifies that the equality in 1.19 holds up to regular terms in the limit $w \to z$. The reason for not writing the regular terms is that the OPEs of fields are used in correlation functions and the regular terms do not contribute anything to the integral involved. This point will become clear in the next paragraph.

With the knowledge of OPEs, we now express the commutator of two operators in terms of contour integrals (assuming radial ordering) as follows:

$$[A, B] = \oint_0 dw \oint_w dz a(z)b(w) \tag{1.20}$$

where the integral over z is taken around w and the integral over w is taken around the origin. Here, $a(z)$ and $b(w)$ are holomorphic fields and $A = \oint a(z)dz$ and $B = \oint b(z)dz$. It is understood that the integrand is radially ordered and one uses the OPE of $a(z)$ and $b(w)$ to actually compute the integral. One can clearly see that the regular terms in the OPE, being holomorphic, give no contribution to the contour integral.

Mode expansion and Hilbert Space of a CFT

We now discuss the mode expansion of the fields. The knowledge of this expansion, combined with Eq. (1.20) helps us in finding the algebra of conformal generators for a general two-dimensional Euclidean CFT. Every conformal field $\phi(z, \bar{z})$ with dimensions $(h, \bar{h})$ can be expanded in terms of modes as follows:

$$\phi(z, \bar{z}) = \sum_{m \in Z} \sum_{n \in Z} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n} \tag{1.21}$$

where the modes $\phi_{m,n}$ are given by:

$$\phi_{m,n} = \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} d\bar{z} \bar{z}^{n+\bar{h}-1} \phi(z, \bar{z}) \tag{1.22}$$

Let us now apply this definition to the energy-momentum tensor $T(z, \bar{z})$. We find:

$$T(z) = \sum_{n \in Z} z^{-n-2} L_n \qquad L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \tag{1.23}$$

and the same equation for the anti-holomorphic part. The modes L_n and $\bar{L}_n$ generate local conformal transformations on the Hilbert space. The generators of $SL(2, C)$ on this Hilbert space are L_1, L_0 and L_1. In particular, the operator $L_0 + \bar{L}_0$ generates dilatations (time translations in the radial quantization picture) and is proportional to the Hamiltonian.
The modes $L_n, \bar{L}_n$ satisfy the following algebra called the **Virasoro Algebra**.

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \\
[L_n, \bar{L}_m] &= 0 \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}
\end{aligned} \tag{1.24}$$

The quantity c is called the central charge of the theory.

We now describe some general properties of the Hilbert space of a conformal field theory. Let us consider the vacuum state $|0\rangle$ and demand that $T(z)|0\rangle$ and $T(\bar{z})|0\rangle$ be well-defined in the limit $(z, \bar{z}) \to (0, 0)$. Using, 1.23 see that:

$$\begin{aligned} L_n|0\rangle &= 0 \\ \bar{L}_n|0\rangle &= 0, \qquad\qquad (n \geq 1) \end{aligned} \tag{1.25}$$

One can immediately see that this implies the invariance of the vacuum under global conformal transformations '. Let us now try to understand the action of a primary field on the vacuum $|0\rangle$. To see this, we use the definition of the modes in Eq. (1.22) to get the commutator of a primary field $\phi(z)$ with the modes L_n, $\bar{L}_n$. We have:

$$\begin{aligned} [L_n, \phi(w, \bar{w})] &= h(n+1)w^n\phi(w, \bar{w}) + w^{n+1}\partial\phi(w, \bar{w}); \qquad (n \geq 1) \\ [\bar{L}_n, \phi(w, \bar{w})] &= \bar{h}(n+1)\bar{w}^n\phi(w, \bar{w}) + \bar{w}^{n+1}\bar{\partial}\phi(w, \bar{w}); \qquad (n \geq 1) \end{aligned} \tag{1.26}$$

where $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$

Using Eq. (1.22) and Eq. (1.21),it is easy to see that the action of the primary on the vacuum $|0\rangle$ in the limit $(z, \bar{z}) \to (0, 0)$ reduces to the action of a single mode $\phi_{-h,-\bar{h}}$.

Let us denote this single, asymptotic state as $\phi(0; 0)|0\rangle := |h, \bar{h}\rangle$. This labelling is immediately justified if we look at the action of the operator $L_0 + \bar{L}_0$ on this state. Using Eq. (1.26), we see that:

$$L_0 + \bar{L}_0|h, \bar{h}\rangle = (h + \bar{h})|h, \bar{h}\rangle \tag{1.27}$$

Recalling that the operator $L_0 + \bar{L}_0$ is proportional to the Hamiltonian, we see that the state $|h, \bar{h}\rangle$ is an eigenstate of the Hamiltonian with the eigenvalue $(h + \bar{h})$.

Eq. (1.26) also gives us the following equations:

$$\begin{aligned} L_n|h, \bar{h}\rangle &= 0 \\ \bar{L}_n|h, \bar{h}\rangle &= 0 \qquad if n\rangle 0 \end{aligned} \tag{1.28}$$

Here, we have only considered the holomorphic part of Eq. (1.26). From the Virasoro algebra, it is easy to see that $[L_0, L_{-m}] = mL_{-m}$. From, this it follows that the modes L_n act as ladder operators for the eigenstates of L_0 and the generators $L_m(m > 0)$ raise the conformal dimension of the state by m. The excited states above the state $|h\rangle$ in the Hilbert space look like:

$$L_{k_1} L_{k_2} \cdots L_{k_n} |h\rangle \qquad (1 \leq k_1 \leq k_2 \cdots \leq k_n) \tag{1.29}$$

This state is an eigenstate of the operator L_0 with the eigenvalue

$$h' = k_1 + k_2 \cdots k_n := h + N \tag{1.30}$$

The states Eq. (1.29) are called descendants of the the state $|h\rangle$ and the integer N is called the level of the descendant. The descendant states can be thought of as the states obtained by the action of descendant fields on the vacuum [7]. The subset of the Hilbert space generated by $|h\rangle$ and its descendants is closed under the action of the Virasoro generators and is called a Verma module (This subset forms a representation of the Virasoro algebra).

1.3 Modular Invariance

The main focus of our discussion is to understand the partition function of CFT's. We shall now employ a trick to find the partition function. It is well-known that for a CFT with finite temperature the partition function can be treated as a path integral on the torus. The invariance of the partition function under the choice of lattice vectors used to describe the torus, leads to powerful constraints on the operator content of the theory. The transformation between two bases that describe the same torus are called 'modular transformations'. Let us now study the concept of 'modular invariance' which is central to our program of classifying partition functions.

Modular Invariance

A torus is obtained as follows: One first defines a lattice on the plane with generating vectors ω_1, ω_2. These are also called the periods of the lattice. We identify the points separated by integer multiples of the periods to obtain the torus. Given a point (x, y) on the lattice:

$$(x, y) \sim (x + m\omega_1, y) \sim (x, y + n\omega_2) \tag{1.31}$$

where m and n are integers.

Thus, the whole lattice gets shrunk to a parallelogram. By identifying the opposite sides of this parallelogram, we get a cylinder. Now, we identify the ends of the cylinder to get a torus.

We now consider any CFT on the complex plane. The, partition function of this theory must not depend on the choice of the periods $\omega_{1,2}$. Therefore, we identify an equivalence class of periods for a given lattice. Consider two periods, $\omega'_{1,2}$ and $\omega_{1,2}$, which give rise to the same lattice. The transformation between these two set of periods is of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.32}$$

We see that this matrix belongs to $PSL(2, Z)$. Under such a transformation, also called the modular transformation,we have:

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{1.33}$$

The group of modular transformations is generated by:

$$T : \tau \to \tau + 1 \tag{1.34}$$

$$S : \tau \to -\frac{1}{\tau} \tag{1.35}$$

The action of modular group $PSL(2, Z)$ keeps τ in the upper half-plane if we start with a τ in the upper half-plane. A study of this action reveals that many values of τ are redundant in our search for finding modular invariant partition functions. Thus, we define a region in the upper half plane such that no two values of τ in this region are connected by a modular transformation and any point outside this region can be reached from a unique point inside it, by some modular transformation. Such a region is called the '*fundamental domain*' (F_0) and is conventionally defined as [7]:

$$\begin{gathered} Im(\tau) > 0, \qquad -\frac{1}{2} \leq Re(\tau) \leq 0, \qquad |z| \geq 1 \\ or \\ Im(\tau) > 0, \qquad 0 < Re(\tau) < \frac{1}{2}, \qquad |z| > 1 \end{gathered} \tag{1.36}$$

1.3.1 Basics of modular forms and functions

In this section we discuss some of the basic mathematical concepts needed to understand the subsequent section.

We introduce quantities (modular forms) that transform in a particular way under modular transformations. A modular form of weight $2k$ is a holomorphic function that transforms in the following way under a modular transformation (which can be anything from $PSL_2(Z)$:

$$f'(\tau') = (c\tau + d)^{2k} f(\tau) \tag{1.37}$$

Some best known modular forms are the holomorphic Eisenstein series E_{2k} with $k > 1$:

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{\substack{m,n\in\mathbf{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau + n)^{2k}} \tag{1.38}$$

We define the special Eisenstein series E_2 as:

$$E_2(\tau) = \frac{1}{2\zeta(2)} \sum_{m\neq 0} \sum_{n\in\mathbf{Z}} \frac{1}{(m\tau + n)^2} \tag{1.39}$$

The space of modular forms of weight $2k$ is one for $2k = 4, 6, 8, 10, 14$. From this, the following relations between the Eisenstein series follow:

$$E_4^2 = E_8, \qquad E_4 E_6 = E_{10}, \qquad E_4 E_{10} = E_{14} = E_6 E_8 \tag{1.40}$$

We now define the unique modular invariant called the Klein-j function.

$$j(\tau) = \frac{(E_4)^3}{\Delta}, \quad \Delta = \frac{E_4^3 - E_6^2}{1728} \tag{1.41}$$

Theta function:

$$\theta(\tau, z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z} \tag{1.42}$$

We now describe the Dedekind-eta function:

$$\eta(\tau) = \prod_{n=1}^{\infty} q^{\frac{1}{24}} (1 - q^n) \tag{1.43}$$

We now define the Ramanujan identities satisfied by the Eisenstein series as it will be useful in easily obtaining the recursion relation:

$$\begin{aligned} \tilde{D}E_2 &= -\tfrac{1}{12}E_4 \\ \tilde{D}E_4 &= -\tfrac{1}{3}E_6 \\ \tilde{D}E_6 &= -\tfrac{1}{2}E_4^2 \end{aligned} \tag{1.44}$$

where $\tilde{D}\chi \equiv \tilde{\partial}\chi - \frac{r}{12}E_2\chi$ and r is the weight of χ.

Using the Ramanujan identities we see that the covariant derivative of the Klein's j-function is given by:

$$\tilde{D}j = -\frac{E_6}{E_4}j \tag{1.45}$$

Rimeann- Roch theorem/ Valence relation: If χ is a non-zero modular function of weight k, then the total number of zeroes of χ in the moduli space is given by:

$$\nu_\infty(\chi) + \frac{1}{3}\nu_\rho(\chi) + \frac{1}{2}\nu_i(\chi) + \nu_\tau(\chi) = \frac{k(k-1)}{12} \tag{1.46}$$

where ν_x is the number of zeroes of χ at the point x.

1.3.2 Transformation from the torus onto the plane

One can map functions on the torus to that on the complex plane. (This transformation maps six copies of the fundamental region in the Teichmuller space onto the complex plane with ramification points $0, 1$ and ∞). To achieve this, we use the modular invariant lambda function defined as:

$$\lambda = \frac{\theta_2^4}{\theta_3^4} \tag{1.47}$$

where θ_2 and θ_4 are theta functions defined in Eq. (1.42). The transformation

$$\tau \to \lambda \tag{1.48}$$

maps functions on the torus to those on the complex plane [7].

1.4 Rational Conformal field theories

The characters of 2d RCFT are a set of p, holomorphic functions $\chi_i(\tau)$ on the moduli space of the torus, such that the CFT partition function can be written as a bilinear:

$$Z(\tau,\bar{\tau}) = \sum_{i,j=0}^{p-1} M_{ij}\chi_i(\tau)\bar{\chi}_j(\bar{\tau}) \tag{1.49}$$

The partition function is required to be modular invariant:

$$Z(\gamma\tau,\gamma\bar{\tau}) = Z(\tau,\bar{\tau}), \quad \gamma \in PSL(2,Z) \tag{1.50}$$

This is ensured if the characters transform as vector-valued modular forms:

$$\chi_i(\gamma\tau) = \sum_k V_{ik}(\gamma)\chi_k(\tau) \tag{1.51}$$

RCFT's have rational values of c and the conformal dimension h. The best examples of RCFT's are minimal models and WZW models.

1.4.1 The standard coset construction

Construction of new CFT's using cosets is well-known (see Ref.[7]). Typically, one takes the coset of an affine theory by an other affine theory, to obtain an affine theory. An affine theory is one in which all the integrable primaries contribute to the partition function.

Let $\mathcal{G}$ and $\mathcal{P}$ be two CFT's with the algebras $\mathfrak{g}$ and $\mathfrak{p}$, respectively, with the latter being a subalgebra of the former. Let k be the level of $\mathfrak{g}$. Then, the level of $\mathfrak{p}$ is given by $x_e k$ where x_e is the embedding index of $\mathfrak{p}$ in $\mathfrak{g}$ [7]. We want to describe the coset $\mathcal{C} = \mathcal{G}/\mathcal{P}$.

Let $L_m^{\mathcal{G}}$ and $L_m^{\mathcal{P}}$ be the Virasoro modes of $\mathcal{G}$ and $\mathcal{P}$, respectively, obtained from the Sugawara construction and $J_n^a, \tilde{J}_n^{a'}$ be the corresponding generators of the respective algebras. Using the property that the generators of $\mathfrak{p}$ are linear combinations of those of $\mathfrak{g}$ one can show [7] that the Virasoro modes of the coset $L_m^{\mathcal{C}} = L_m^{\mathcal{G}} - L_m^{\mathcal{P}}$ satisfy the Virasoro algebra:

$$[L_m^{\mathcal{C}}, L_n^{\mathcal{C}}] = (m-n)L_{m+n}^{\mathcal{C}} + (c_{\mathcal{G}} - c_{\mathcal{P}})\frac{(m^3-m)}{12}\delta_{m+n,0} \tag{1.52}$$

where $c_{\mathcal{G}}$ and $c_{\mathcal{P}}$ are the central charges of theories $\mathcal{G}$ and $\mathcal{P}$, respectively. The central charge of the coset theory is given by:

$$c_{\mathcal{C}} = \frac{kdim\mathfrak{g}}{k+g} - \frac{x_e kdim\mathfrak{p}}{x_e k + p} \tag{1.53}$$

Here, g, p are the dual coxeter numbers and $(dim\mathfrak{g}, dim\mathfrak{p})$ are the dimensions of the algebras $(\mathfrak{g}, \mathfrak{p})$.

The well-known unitary minimal models can be described as cosets. The coset

$$\frac{su(2)_k \oplus su(2)_1}{su(2)_{k+1}} \tag{1.54}$$

describes the minimal model $\mathcal{M}$(k,k+1)with the central charge [7].

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)} \tag{1.55}$$

With this brief introduction we turn our attention to the discussion of modular differential equations and classification of RCFT's.

Chapter 2

Modular differential equations and the Wronskian

2.1 Modular differential equations and the Wronskian

Since characters are modular covariant Eq. (1.51) the differential equation satisfied by them must be modular invariant i.e, they must be the independent solutions to a degree-p modular invariant differential equation. The most general such linear, holomorphic, modular invariant differential equation of degree-p is [1]:

$$\left(D^p + \sum_{k=0}^{p-1} \phi_k(\tau) D^k\right) \chi = 0 \tag{2.1}$$

where D is a covariant derivative defined as:

$$D \equiv \frac{\partial}{\partial \tau} - \frac{i\pi r}{6} E_2(\tau) \tag{2.2}$$

where r is the modular weight of the function on which it acts, and $E_2(\tau)$ is the Eisenstein E_2 series defined as:

$$E_2(\tau) = \frac{1}{2\zeta(2)} \sum_{\substack{m,n \in \mathbb{Z} \\ m \neq 0, n \neq 0}} \frac{1}{(m\tau + n)^2} \tag{2.3}$$

The covariant derivative defined in Eq. (2.2) is similar to the covariant derivative encountered in general theory of relativity. Such a covariant derivative with a connection term is required for modular invariance. For the differential equation to be invariant under a modular transformation, each term in the differential equation must transform with the same weight. When an ordinary partial derivative acts on a modular function the result is not necessarily another modular function. Instead, there is an additional inhomogeneous term and this is compensated by the connection term to ensure the right homogeneous transformation of each term. Note that D increases the modular weight by 2, so an expression like $D^n\chi$ stands for:

$$\left(\partial_\tau - \frac{i\pi(n-1)}{3}\right)\left(\partial_\tau - \frac{i\pi(n-2)}{3}\right)\cdots\left(\partial_\tau - \frac{i\pi}{3}\right)\partial_\tau\chi \tag{2.4}$$

At this point, we notice that modular invariance of the differential equation forces that the coefficient functions $\phi_k(\tau)$ must be modular covariant with a modular weight $2(p-k)$. The classification of RCFT's is based on the order of the differential equation and the number of singular points of the coefficient function $\phi_k(\tau)$. This number can be recast as the number of zeroes of a certain Wronskian. To see this we express following the analysis of Ref.[1] the coefficient functions in terms of the independent solutions to the differential equation i.,e the characters, as follows:

$$\phi_k(\tau) = (-1)^{n-k}\frac{W_k}{W} \tag{2.5}$$

where W_k are the Wronskian determinants:

$$W_k \equiv \begin{pmatrix} \chi_0 & \cdots & \chi_{p-1} \\ D\chi_0 & \cdots & D\chi_{p-1} \\ \vdots & & \vdots \\ D^{k-1}\chi_0 & \cdots & D^{k-1}\chi_{p-1} \\ D^{k+1}\chi_0 & \cdots & D^{k+1}\chi_{p-1} \\ \vdots & & \vdots \\ D^p\chi_0 & \vdots & D^p\chi_{p-1} \end{pmatrix} \tag{2.6}$$

and $W = W_p$ and from now on, we shall refer to W as "the Wronskian'.

The characters $\chi_0, \chi_2, \cdots \chi_{p-1}$ count the number of states above various primaries and can't be singular at any point in the moduli space. It follows that the numerator in Eq. (2.5) can't be singular and the only points at which $\phi_k(\tau)$ are singular are those at which the denominator is zero. Thus, we classify the differential equations by the number of zeroes of W in moduli space. As the torus moduli space has two orbifold points $\tau = e^{\frac{i\pi}{3}}$ of order $\frac{1}{3}$ and i of order $\frac{1}{2}$ one can combine the degrees and the numbers of zeroes of W can be expressed in the form $\frac{\ell}{6}$ with $\ell = 0, 2, 3, 4, \cdots$. Following [1] we note that the number of zeroes of the Wronskian can be expressed in terms of the central charge and the primary dimensions. To achieve this we take the $\tau \to i\infty$ limit and see that a character χ_i with corresponding to a primary of dimension h_i behaves in the following way:

$$\chi_i \sim exp\Big[2\pi i\Big(h_i - \frac{c}{24}\Big)\Big] \tag{2.7}$$

Consequently the Wronskian behaves as:

$$W \sim exp\Big[2\pi i\Big(\sum_{i=0}^{p-1} h_i - \frac{pc}{24}\Big)\Big] \tag{2.8}$$

The Wronskian involves $\frac{1}{2}p(p-1)$ derivatives and as each derivative increases the weight by two the Wronskian behaves like a modular form (because it has no poles in the interior of moduli space) of weight $p(p-1)$. From the above expression we see that W has a $-\sum_{i=0}^{i=p-1} h_i + \frac{pc}{24}$ order pole at $\tau = i\infty$. Thus, from the Riemann-Roch theorem Eq. (1.46) we find that the sum of total number of zeroes and poles of the Wronskian is given by:

$$\frac{\ell}{6} = -\sum_{i=0}^{i=p-1} h_i + \frac{pc}{24} + \frac{1}{2}p(p-1) \tag{2.9}$$

Since the characters are non-singular W has no poles in the interior of moduli space. It follows that $\ell = 0$ or any integer ≥ 2 [1]. In terms of the exponents $\alpha_i = h_i - \frac{c}{24}$:

$$\sum_{i=0}^{p-1} \alpha_i = \frac{p(p-1)}{12} - \frac{\ell}{6} \tag{2.10}$$

Now, the strategy is to fix p and ℓ. This will fix the coefficients(ϕ_k of the differential equation. In general ϕ_k are meromorphic modular functions. We know that the space of meromorphic modular functions is spanned by rational functions of the form $E_4^a E_6^b$ for integers a and b. For a fixed value of p and ℓ, a and b can be determined and thus ϕ_k can be completely determined. This leaves only a finite number of real parameters in the differential equation.

Once we have a differential equation for fixed values of p and ℓ we solve it by power-series method. To do this, we expand the character as the following power series:

$$\chi_i(\tau) = \sum_{n=0}^{\infty} a_n^{(i)} q^{\alpha_i + n} \tag{2.11}$$

with $q = e^{2\pi i \tau}$.

Similarly, we expand the relevant Eisenstein series E_k as:

$$E_k(\tau) = \sum_{n=0}^{\infty} E_{k,n} q^{\alpha_i + n} \tag{2.12}$$

The power series expansion in the Eq. (2.11) must have the property that the ratio of any coefficient to the leading one should be a non-negative rational number. This must be so as the coefficients in the character count the number of states at each level in a CFT. In particular, for the identity character ,corresponding to the exponent $\alpha_0 = -\frac{c}{24}$, these ratios are non-negative integers as the identity is assumed to be non-degenerate and has only one state at the lowest level. These conditions are powerful enough to constrain the allowed values of exponents α_i and sometimes gives a finite set [1].

Once the allowed exponents are known one can reconstruct the entire CFT from it using the prescription in [2]. In a nutshell the procedure can be described as follows: After finding the allowed values of α_i we try to understand how the characters χ_i transform under the transformation $S : \tau \to -\frac{1}{\tau}$. The matrix corresponding to this transformation determines the partition function of the theory. This is equivalent to knowing the number of times a character appears in the partition function. Once this is done, we use the fact that the matrix S_{ij} corresponding to the transformation S diagonalizes the fusion rules of the primary fields [7]. This, with the physical condition that the fusion rule matrix for the identity character is the identity matrix, gives us the fusion rules. Lastly, we find the spectrum-generating algebra of the CFT's by systematically figuring out the number of spin-n holomorphic currents from the powers series expansion of the characters. A detailed analysis of the reconstruction program can be found in [2].

In what follows we shall try to classify RCFT's with low number of characters. We shall, however, illustrate the reconstruction procedure only for two-character theories with $\ell = 0$.

2.2 Tensor-product theories

A search for RCFT's based on characters poses an interesting complication. It can so happen that we 'rediscover' tensor-products of well-known theories as solutions of the differential equation. It is important to understand how tensor products work so that we know if there are any product theories that are solutions to a particular differential equation. In this section we briefly discuss the effect of tensoring on the number of characters and ℓ. We mainly follow the analysis of Ref.[4].

Let us try to understand the behaviour of the parameter ℓ under tensoring. For simplicity, let us consider a pair of distinct pair of CFT's with p and p' characters respectively. Let the characters, exponents and parameters of these two theories be $(\chi_i, \chi'_{i'})$, $(\alpha_i, \alpha'_{i'})$ and (ℓ, ℓ') respectively. If we assume that the conformal dimensions of distinct primaries are different we find that the tensor product is a CFT with pp' characters $\tilde{\chi}_{(ii')} = \chi_i \chi'_{i'}$. And the exponents of this theory are $\tilde{\alpha}_{(ii')} = \alpha_i + \alpha'_{i'}$. Let $\tilde{\ell}$ be the parameter for this theory. Using the relation Eq. (2.10), we find:

$$\tilde{\ell} = \frac{1}{2}\, pp'(p-1)(p'-1) + p'\ell + p\ell' \tag{2.13}$$

Let us see what are the consequences of the above formula for theories with less number of characters. Let us consider the tensor product of a one character theory ($p = 1$) with fixed ℓ and a theory with p' characters and fixed ℓ'. We find that $\tilde{\ell} = p'\ell + \ell'$. Since, $\ell > 0$ for a one-character theory, we see that the the tensor product has an $\tilde{\ell}$ greater than that of both the theories. In particular if $\ell' = 0$ we find that $\tilde{\ell} = p'\ell$. Thus, the tensor product theory acquires a non-zero value of ℓ. For particular values of p', we can get more information from this equation. If $p' = 2$, we can conclude that there are no two-character theories with $\ell = 2, 3, 6$ that are tensor products of $\ell' = 0$, two-character theories and a one-character theory (This is because $\ell \geq 2$ for one-character theories and there is no one character theory with $\ell = 3$). It is also clear that if p' is even then there is no way to get an odd value for $\tilde{\ell}$.

Let us consider the case of $p, p' > 1$ and $\ell = \ell' = 0$. We find:

$$\tilde{\ell} = \frac{1}{2} pp'(p-1)(p'-1) \tag{2.14}$$

This clearly implies that if we study $\ell = 0$ theories for arbitrary number of characters we will never encounter tensor products of theories with lower number of characters.

Now, let us turn our attention to tensor products of identical theories. As an example let us tensor a p (all independent and distinct) character theory with itself. The product theory has p^2 primaries. But, exactly $\binom{p}{2}$ of them are degenerate and the number of independent characters is $\frac{p(p+1)}{2}$. In this case, using the valence formula we get the following expression:

$$\tilde{\ell} = \frac{(p+1)p(p-1)(p-2)}{8} + (p+1)\ell \tag{2.15}$$

It is interesting to note that if the theory we start with has more than p distinct primaries, of which some are degenerate such that we only have p distinct characters, then the product theory will have higher degeneracies. However, the number of distinct characters is still given by the above formula.

For one character theories this formula implies that the product theory is always a one-character theory with a $\tilde{\ell}$ that is twice the value of ℓ for the original theory. This can be easily cross-checked using Eq. (3.3). In particular the n^{th} tensor product gives:

$$\tilde{\ell} = 2^n \ell \tag{2.16}$$

Now, we consider the tensor product of a pair of two-character, $\ell = 0$ theories. From the above discussion it is clear that we get a three-character theory with $\ell = 0$. Using all the facts we have discussed above we see that the n^{th} tensor product of a two-character CFT with a fixed value of ℓ gives a theory with $n+1$ characters and:

$$\tilde{\ell} = \frac{n(n+1)\ell}{2} \tag{2.17}$$

This proves the statement that the set of CFT's with p characters and $\ell = 0$ is always non-empty $\forall p$. Also, it is now obvious that during study of p-character theories with fixed ℓ we will always encounter powers of $k(< p)$-character, $\ell = 0$ theories.

Let us finally consider the general case of n^{th} product of a p-character theory. The resulting theory will have p^n primaries, but many of them are degenerate and the number of the characters is :

$$\tilde{p} = \binom{p+n-1}{p-1} \tag{2.18}$$

Now, we need to find $\tilde{\ell}$ for the product theory and this involves a summation over the exponents $\tilde{\alpha}_i$. To do this, let us consider ordered partitions of n_i of the integer n into p numbers n_i. Therefore $\sum_{i=1}^{i=p} n_i$ where the sample space of n_i is $0, 1, \cdots n$. We now consider n_1 and calculate the quantity $q = \sum_{(n_i)} n_1$, where the sum is over all possible partitions. Since this sum considers all possible values that n_1 can take in all possible ordered partitions of n, it is clear that this sum must be the same $\forall n_i$. The quantity q counts the number of times a particular α_i occurs in the product theory. Thus, we have:

$$\sum_j \tilde{\alpha}_j = q \sum_i \alpha_i \tag{2.19}$$

where i runs over the characters of the original theory and j, over the characters of the composite theory. One can check that:

$$q = \binom{p+n-1}{p} \tag{2.20}$$

Thus, the formula for $\tilde{\ell}$ of the product theory is given by:

$$\tilde{\ell} = \frac{\tilde{p}(\tilde{p}-1)}{2} - q\,\frac{p(p-1)}{2} + q\ell \tag{2.21}$$

where $\tilde{p}$ and q are defined by Eqs.(2.2) and (2.20) respectively.

It is easy to check that this reproduces the formula in Eq. (2.15) for $n = 2$ and arbitrary p, as well as Eq. (2.17) for $p = 2$ and arbitrary n. Moreover, one can verify that the first two terms in the above equation always add up to a positive number whenever $p > 2$. This is a proof of our statement that when we tensor theories with $p > 2$, the value of ℓ always increases.

Chapter 3

One character theories

The most general one-character,homogeneous differential equation with arbitrary ℓ is of the form:

$$[D + \mu\phi_0(\tau)]\,\chi = 0 \tag{3.1}$$

where μ is a real parameter. Here, ϕ, ϕ_0 are modular functions of weight two and are determined by the value of ℓ. Since D acts on the character, which has weight zero, we have $D = \partial$ and the differential equation is:

$$(\partial_\tau + \mu\phi_0)\chi + = 0 \tag{3.2}$$

Using the valence formula 2.10 it is easy to see that the value of ℓ fixes the value of the central charge.

$$\alpha = -\frac{c}{24} = -\frac{\ell}{6}. \tag{3.3}$$

This relation tells us that classifying one-character theories based onℓ is equivalent to classifying them based on central charge. This property is unique to one-character theories.

The problem of classifying all possible theories with a single character ($p = 1$) can be addressed without taking recourse to the modular differential equation. In this case, the single character, by itself, is modular transformation invariant up to phase and the partition function, which is defined as the modulus-squared of the character $\chi_0(\tau)$, is modular invariant. Moreover, the character should have no singularities in τ space except at $\tau = i\infty$ and have a q-expansion with only non-negative integer coefficients. The well- known modular function of weight zero - Klein j-function defined in Eq. (1.41) satisfies all these properties. The function j has a q-expansion given by:

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots \tag{3.4}$$

Comparing this with the character formula $\chi_0 = \sum_{n=0}^{\infty} a_n q^{-\frac{c}{24}}$, we find that j corresponds to a CFT with $c = 24$ and $\ell = 6$. Various combinations of the j-function are also allowed characters. One can consider a fractional power of j,$j^{1/3}$. It turns out that this function also has non-negative, integer coefficients in the q-expansion. One can also consider polynomials in j with appropriately tuned coefficients. In fact, the single character can be built entirely out of the j-function and the most general expression for the character is given by:

$$\chi(\tau) = j^{\delta}(j - 1728)^{\beta} P(j) \tag{3.5}$$

where $\delta = 0, \frac{1}{3}, \frac{2}{3}$, $\beta = 0, \frac{1}{2}$ and $P(j)$ is a polynomial in j whose coefficients must be adjusted so that the q-expansion has non-negative coefficients. Other functions of j like $j^{\frac{1}{6}}$ are not allowed as they do not satisfy non-negativity of the coefficients. Moreover, the factor of $(j - 1728)^{\frac{1}{2}}$ by itself does not satisfy non-negativity, so this case should be excluded [1].

The above expression gives valuable information about the central charge and via Eq. (3.3), also about the parameter ℓ. One can easily see that multiplying two functions of j always increases the central charge. It follows from Eq. (3.3) that this also increases the value of ℓ.

Eq. (3.5) is a statement only about the characters of candidate one-character CFT's. It does not imply that each such function corresponds to a genuine CFT . For example, consider polynomials of the form $j_N = j + \mathcal{N}$ corresponding to central charge $c = 24$ and $\ell = 6$. Using Eq. (1.45) it is easy to see that this satisfies the inhomogeneous equation:

$$\tilde{D} j_{\mathcal{N}} + \frac{E_6}{E_4}(j) = 0 \tag{3.6}$$

This inhomogeneous equation can be recast to get a homogeneous first order equation. We have:

$$\Big(\tilde{D} + \frac{E_6}{E_4}\frac{j}{j_{\mathcal{N}}}\Big) j_{\mathcal{N}} = 0 \tag{3.7}$$

There are infinitely many candidate characters of this form with $-744 \leq \mathcal{N} < \infty$. However, it has been shown that there only 71 CFT's with $c = 24$ ([11]). Among these theories, $j - 744$ has been successfully identified as the character of the Monster CFT.

Another example of a class of functions that have been identified as the characters of genuine CFT's are monomials of the type $j^{\frac{n}{3}}$. They correspond to the character of the tensor-product $\otimes^n (E_8)_{k=1}$. From Eq. (3.3), it is easy to see that this function has $\ell = 2n$ and central charge $c = 8n$. Using Eq. (1.45) it is easy to see that the differential equation satisfied by this class of characters is:

$$\tilde{D}j^{\frac{n}{3}} + \frac{n}{3}\Big(\frac{E_6}{E_4}\Big)j^{\frac{n}{3}} = 0 \tag{3.8}$$

A particular example of this class is the E_8 theory at level 1, with $n = 1$. This theory has $\ell = 2$, the lowest possible ℓ for a one-character theory.

There exists a proposal that polynomials in j with certain coefficients equal to zero correspond to “extremal” CFT’s[6]. However, some doubt has been cast on the existence of CFT’s corresponding to these characters [16, 17]. A list of all possible functions that correspond to genuine one-character conformal field theories is not yet available.

With this brief discussion we turn our attention to the more complex case of two-character CFT’s.

Chapter 4

Two character theories

We now consider the case of two-character CFT's. This is also the first case where modular covariance is non-trivial. As we shall see this is qualitatively very different from the one-character case. The most general second-order differential equation is of the form:

$$\left(\tilde{D}^2 + \phi_1(\tau)\tilde{D} + \phi_0(\tau)\right)\chi = 0 \tag{4.1}$$

where $\tilde{D}$ is the covariant derivative defined as

$$\tilde{D} = \frac{1}{2i\pi}\left(\partial_\tau - \frac{i\pi r}{6}E_2(\tau)\right) \tag{4.2}$$

where r is the weight of the function on which it acts.

The coefficients functions $\phi_i(\tau)$ can be determined by fixing the value of ℓ. In this section only the lowest allowed values of ℓ, $\ell = 0, 2, 3, 4, 5$, are discussed. It turns out that the analysis of theories with $\ell \geq 4$ is harder because of an additional parameter in the differential equation. This complicates the standard Diophantine analysis which works for the previous cases.

Now, let us understand what the coefficient functions are in different cases. Firstly, we note that the differential equation is modular invariant and each term carries the same weight. We also know that the action of $\tilde{D}$ on a function increases its modular weight by 2. Therefore, the first term has weight 4. This forces ϕ_1 to have modular weight 2 and ϕ_0 to be a weight-4 modular function. For the $\ell = 0$ case they are both non-singular. Since there is no modular form of weight two, the coefficient $\phi_1 = 0$. The other coefficient is the $\phi_0 \sim E_4(\tau)$, where the Eisenstein E_4 series is the unique modular form of weight four.

The 2^{nd} order equation with $\ell = 0$, therefore, is:

$$(\tilde{D}^2 + \mu E_4)\chi = 0 \tag{4.3}$$

where μ is an arbitrary real parameter.

Before we proceed further with the analysis, we need to understand the zeros/poles of some modular forms in the moduli space. We are concerned mainly with the behaviour of E_4 and E_6 at three points: a) Any point in the interior of moduli space b) The point $\rho = e^{\frac{i\pi}{3}}$ c) The point i. We use the Riemann-Roch formula Eq. (1.46) to conclude that the Eisenstein E_4 series has a double zero at ρ and no zero at i. Similarly, E_6 has a zero of degree- $\frac{1}{2}$ at i and is non-vanishing at ρ. Since both of them are modular forms, by definition, they don't have poles in the interior of the moduli space. Equipped with this information we can write down the non-singular differential equations for $\ell = 2, 3, 4, 5$

We now consider $\ell = 2$. In this case both ϕ_0 and ϕ_1 can have a pole of maximum degree $\frac{1}{3}$ (the fractional degree implies that the pole must occur at ρ. This counts as a $\frac{1}{6}$-order pole). This is equivalent to demanding that the denominator have a degree- $\frac{1}{3}$ zero. Therefore, we find $\phi_1 \sim \frac{E_6}{E_4}$. On the other hand, one cannot construct a weight-4 expression from E_4, E_6 without increasing the degree of zeros in the denominator beyond $\frac{1}{3}$. It is easy to see this. Consider $\frac{E_6^a}{E_4^b}$ for some integers a and b. To get a weight-4 object we need $6a - 4b = 4$. The integer solutions to this equation are $(a, b) = (2, 2)$. But, the maximum value of b is one. Therefore, we are forced to have $\phi_0 \sim E_4$. The $\ell = 2$ equation is therefore:

$$\left(\tilde{D}^2 + \mu_1 \frac{E_6}{E_4}\tilde{D} + \mu_2 E_4\right)\chi = 0 \tag{4.4}$$

This might give the impression that we have two parameters. But, once we start analysing the differential equation the indicial equation, together with the valence relation Eq. (2.10), fixes one of the parameters and we will have to scan over only one real parameter.

For $\ell = 3$, the maximum degree of the pole allowed is $\frac{1}{2}$. Thus, the only possibility is a zero at the point $\tau = i$ and this implies that we can only have E_6 in the denominator. We conclude that $\phi_1 \sim \frac{E_4^2}{E_6}$ and $\phi_0 \sim E_4$ (There is no weight-4 modular function with only E_6 in the denominator). Thus, the $\ell = 3$ equation is:

$$\left(\tilde{D}^2 + \mu_1 \frac{E_8}{E_6}\tilde{D} + \mu_2 E_4\right)\chi = 0 \tag{4.5}$$

As in the previous cases, it turns out that μ_1 is determined by the valence relation

Eq. (2.10).

For $\ell = 4$, we can allow ϕ_k with at most E_4^2 in the denominator. We argue as before and get $\phi_1 \sim \frac{E_6}{E_4}$ and $\phi_0 \sim \frac{E_6^2}{E_4^2}, E_4$. The $\ell = 4$ equation ,therefore, is:

$$\left(\tilde{D}^2 + \mu_1 \frac{E_6}{E_4} \tilde{D} + \nu E_4 + \mu \frac{E_6^2}{E_4^2}\right) \chi = 0 \tag{4.6}$$

We observe that the equation is qualitatively different from the previous ones and now has three parameters. One of them gets fixed by the valence formula as always and we are left with a space of two free parameters to scan.

Finally, we consider the $\ell = 5$ case. Here we can allow E_4, E_6 and $E_4 E_6$ in the denominator, however there is no expression of weights 2 or 4 that can have $E_4 E_6$ in the denominator. Thus, the most general equation for this case is given by:

$$\left(\tilde{D}^2 + \left(\mu_1 \frac{E_6}{E_4} + \nu \frac{E_4^2}{E_6}\right) \tilde{D} + \mu E_4\right) \chi = 0 \tag{4.7}$$

$\ell = 6$ onwards we can admit a full zero in the interior of moduli space. Thus, we have an additional parameter at $\ell = 6$. In fact, for large ℓ the number of parameters grows as $\sim \ell^2$. In the discussion that follows we restrict ourselves to the case of $\ell \leq 4$.

4.1 $\ell = 0$ theories

In this section we consider the non-singular two-character differential equation.The detailed procedure of how to find potential CFT's for a second order differential equation will be discussed with this example (Ref [1, 4, 2]). Other cases follow similarly.

The differential equation for this case , in terms of ordinary derivatives, is given by:

$$\left(\tilde{\partial}^2 - \frac{1}{6} E_2 \tilde{\partial} + \mu E_4\right) \chi = 0 \tag{4.8}$$

We solve the differential expansion by power-series method. Accordingly, we plug in the series expansion $\chi = \sum_{n=0}^{\infty} a_n q^{\alpha+n}$ into the above differential equation. We expand the Eisenstein series in terms of its Fourier components as $E_a(\tau) = \sum_{k=0}^{\infty} E_{a,k}\, q^k$. Substituting this leads to the following recursion relation:

$$(n+\alpha)^2 a_n - \frac{1}{6} \sum_{k=0}^{n} (n-k+\alpha) E_{2,k} a_{n-k} + \mu \sum_{k=0}^{n} E_{4,k} a_{n-k} = 0 \tag{4.9}$$

The $n = 0$ equation or the "indicial equation" is given by:

$$\alpha^2 - \tfrac{1}{6}\alpha + \mu = 0 \tag{4.10}$$

Let us denote the roots of the above quadratic equation as α_0, α_1 where α_0 is the exponent corresponding to the identity character and α_1 corresponds to the non-trivial primary, then:

$$\begin{aligned} \alpha_0 + \alpha_1 &= \frac{1}{6} \\ \mu = \alpha_0\alpha_1 &= \alpha_0\left(\frac{1}{6} - \alpha_0\right) \end{aligned} \tag{4.11}$$

Using the $n = 1$ equation we get:

$$m_1 \equiv \frac{a_1}{a_0} = -\frac{24\alpha - 360\mu}{5 + 12\alpha} \tag{4.12}$$

The above expression is satisfied by both the exponents α_0 and α_1 and let us denote the corresponding quantities as $m_1^{(0)}$ and $m_1^{(1)}$, respectively. The former is the degeneracy of the first excited state in the identity character (We assume that the vacuum of the theory is non-degenerate) and the latter is the ratio of the degeneracy of the first excited to that of the ground state of the other character. It is important to note that The degeneracy of the ground state of the non-trivial primary cannot be determined by the homogeneous differential equation. This is a general feature of the differential equation method and will be dealt with, in detail, in section 4.3.4 .

Continuing with our analysis we substitute for μ in Eq. (4.12) to obtain:

$$m_1^{(i)} = \frac{24\alpha_i(60\alpha_i - 11)}{5 + 12\alpha_i} \tag{4.13}$$

for $i = 0, 1$.

One can derive interesting constraints on candidate theories just by demanding that the $m_1^{(i)}$ be non-negative. Let us consider $m_1^{(0)}$ and re-express the function as a function of central charge using $\alpha_0 = -\frac{c}{24}$ to get:

$$m_1^{(0)} = \frac{c(5c + 22)}{10 - c} \tag{4.14}$$

This means that the allowed values of central charge have an upper bound $c < 10$. For $m_1^{(0)}$ to be non-negative both the numerator and denominator must have the same sign. For positive values of $c < 10$ this is obviously true. If we consider $c < 0$ the denominator is always positive and the numerator can only be non-negative for $c \geq -\frac{22}{5}$. Thus, the central charge is bounded $-\frac{22}{5} \leq c < 10$. The numerator vanishes when $c = -\frac{22}{5}$. This, along with the fact that minimal models have $a_1^{(0)} = 0$, (for an explanation see section **??**) tells us that the only minimal model that can be a two-character, $\ell = 0$ theory must have $c = -\frac{22}{5}$.

Manipulating the above equation and dropping the superscript we get:

$$(5c)^2 + 5c(m_1 + 22) = 50m_1 \tag{4.15}$$

from which it immediately follows that $5c$ is an integer.

Now let us focus on finding all possible candidate characters that satisfy this equation. We notice from Eq. (4.15) that $5c$ will be rational only if the discriminant:

$$\sqrt{m_1^2 + 244m_1 + 484} \tag{4.16}$$

is rational. Since m_1 has to be a non-negative integer, this can only happen if the term inside the bracket is the square of an integer. Imposing this we have:

$$m_1^2 + 244m_1 + 484 = N^2 \tag{4.17}$$

for some integer N. This is a Diophantine equation and the standard trick to solve this is to shift N by an integer amount so that the first two terms on the left-hand-side are absorbed. Thus, we define:

$$N = \tilde{N} + m_1 + 122 \tag{4.18}$$

Inserting this into the above equation and solving for m_1, we find:

$$m_1 = -\frac{\tilde{N}}{2} - 122 - \frac{7200}{\tilde{N}} \tag{4.19}$$

From the above equation one can infer that $\tilde{N}$ is a negative even integer with $0 < N < 120$, and it must divide 7200. The equation also has interesting symmetry . If one replaces the $\tilde{N}$ by $\frac{14400}{\tilde{N}}$ the first and third term go to each other and the second term is invariant i.e, we get the same m_1. We consider all the even factors of 7200 and choose only one of the pair of values which give the same m_1 to obtain the following 22 values of $\tilde{N}$:

$$\tilde{N} = -\{2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 50, 60, 72, 80, 90, 96, 100\} \tag{4.20}$$

Now we compute the central charge for the above candidates using Eq. (4.15). Once the central charge is known the two exponents are given by $\alpha_0 = -\frac{c}{24}$ and $\alpha_1 = \frac{1}{6} - \alpha_0$. Using the recursion relation 4.9 we find $a_n^{(i)}$ ($i = 0, 1$) for large n and in the process we reject all the candidates that do not give non-negative integral a_n^0 and non-negative rational a_n^1 for any n. We are left with the following ten values at $n = 5000$.

$$\tilde{N} = -\{96, 90, 80, 72, 60, 48, 40, 30, 24, 20\} \tag{4.21}$$

The central charges, non-trivial primary and m_1 values for the above theories is listed in the table 4.1.3. These results first were reported and analysed in Ref.[1]. In what follows we shall summarize this analysis. We first find the characters by explicitly solving the differential equation. We then find the fusion rules and finally recover the chiral algebra and identify the CFT's.

4.1.1 Transformation from the torus onto the plane

In the previous sections we discussed the possible $\ell = 0$ theories by checking the integrality of m_1^0 to very high levels. But, we did not focus on getting a closed-form expression for the character. This can be achieved by 're-casting' the differential equation in terms of a different variable and will be the primary focus of this section. This procedure was first discussed in [2] for two and three character theories $\ell = 0$ theories. Using a similar transformation the characters of two-character $\ell = 0$ theories have also been found in Ref.[3]. Here, we content ourselves with a discussion of two-character $\ell = 0$ theories. The arbitrary ℓ case will be discussed in the subsequent sections. The case of three-character $\ell = 0$ theories is similar and the reader is referred to Ref.[2] for a detailed analysis.

As discussed in 1.3.2, we consider the two-character $\ell = 0$ equation in the variable $\lambda = \frac{\theta_2^4}{\theta_3^4}$ where θ_3 and θ_4 are Jacobi's elliptic theta functions described in **??**. Under this transformation the differential equation Eq. (**??**) becomes:

$$\frac{\partial^n \chi}{\partial \lambda^n} + \sum_{k=0}^{k=n} \psi_k(\lambda) \frac{\partial^k \chi}{\partial \lambda^k} = 0 \tag{4.22}$$

where ψ_k are rational functions of the variable λ. These coefficients can be fixed by looking at the modular transformation properties of λ. To check this we need only look at the two transformations that generate the full modular group ($PSL_2(Z)$)-S and T (for more details see Eq. (1.34), Eq. (1.35)). The transformations $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$

correspond, respectively, to the following transformations on λ:

$$\lambda \to \frac{\lambda}{\lambda - 1} \tag{4.23}$$

$$\lambda \to 1 - \lambda \tag{4.24}$$

By noting that under these transformations the differential equation has to remain invariant one can fix ψ_k. We now focus our attention on two-character $\ell = 0$ theories. After fixing the coefficients Eq. (4.9) becomes:

$$\lambda^2(1-\lambda)^2\frac{\partial^2\chi}{\partial\lambda^2} - \frac{2}{3}\lambda(1-\lambda)(2\lambda-1)\frac{\partial\chi}{\partial\lambda} + \mu(\lambda(1-\lambda)-1)\chi = 0 \tag{4.25}$$

It is easy to check that the above differential equation is invariant under the transformations Eq. (4.23) [2].

Our goal was to solve for the characters and by recasting the differential equation in terms of the variable λ the solutions are easy to obtain. We note that the 4.25 is a hypergeometric equation and the solutions to this equation are:

$$\chi_0 = \Big(\frac{1}{16}\lambda(1-\lambda)\Big)^{\frac{(1-x)}{6}} F\Big(\frac{1}{2} - \frac{x}{6}, \frac{1}{2} - \frac{x}{2}, 1 - \frac{1}{3}x; \lambda\Big) \tag{4.26}$$

$$\chi_1 = N\Big(\frac{1}{16}\lambda(1+\lambda)\Big)^{\frac{(1+x)}{6}} F\Big(\frac{1}{2} + \frac{x}{6}, \frac{1}{2} - \frac{x}{2}, 1 + \frac{1}{3}x; \lambda\Big) \tag{4.27}$$

(4.28)

where N is a normalization constant and

$$x = 1 + \frac{1}{2}c \tag{4.29}$$

4.1.2 Monodromy of the characters and Fusion Rules

Let us focus on finding out the monodromy properties of the solutions of Eq. (4.25). This question can be posed as follows:We know that Eq. (4.22) is modular invariant and therefore, one may choose the solutions to be eigenstates of the transformation T Eq. (1.34). Then, the solutions are of the form $\sum_{n=0}^{n=\infty} a_n q^{n+\alpha}$ with some rational α. Let us assume that we have normalized the solution so that coefficients are positive integers. In this case, the transformation S in Eq. (1.35) (also called the monodromy matrix) corresponds to an $n \times n$ matrix acting on the column vector of n characters. We want to find out what this matrix S is.

The monodromy matrix plays a crucial role in reconstructing any CFT. It counts the number of primary fields corresponding to a given character. For example, let us consider a theory with n characters $\chi_0, \chi_1 \cdots \chi_n$. In this case, if the $n \times n$ matrix S leaves the diagonal matrix $diag(1, Y_1, Y_2 \cdots Y_n)$ invariant, then the modular invariant partition function corresponding to this case is:

$$Z(\tau) = |\chi_0|^2 + \sum_{i=1}^{n-1} Y_i |\chi_i|^2 \tag{4.30}$$

where Y_i are integers that correspond to the number of primary fields associated with χ_i. If S does not leave a matrix of the above form invariant, then the corresponding differential equation does not have conformally invariant field theories as solutions.

Let us now consider the case where $Y_i = 1 \forall i$. In this case, each character has only one primary associated with it. For this case, S has to be a symmetric, unitary matrix. Using this fact, we can compute the fusion rules aN_{ijk} for fusing the primary fields ϕ_i, ϕ_j and ϕ_k. We need the following formula due to Verlinde [7],

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{kn}}{S_{0n}} \tag{4.31}$$

where the sum is over all the primary fields and 0 corresponds to the identity field. The fusion rules N_{ijk} correspond to consistent CFT's only when $N_{ijk} \geq 0$. When the Y_i s are all not one, needs to find an $1 + \sum_{i=1}^{n-1} \times 1 + \sum_{i=1}^{n-1}$ matrix $\widehat{S}$ which diagonalizes the fusion rules and gives $N_{ijk} \geq 0$. the matrix S imposes constraints on $\widehat{S}$. Other constraints are obtained by demanding that $\widehat{S}$ be symmetric and unitary.

We now recall that 4.26 are eigenfunctions of the transformation T and consequently have non-negative integer coefficients in the q-expansion. We check, using the fact that for small q,$\lambda \sim 16 q^{\frac{1}{2}}$, that $a_0 = 1$ for χ_0 and $a_1 = N$ for χ_1. We observe that under $S : \lambda \to 1 - \lambda$ the characters 4.26 transform as follows:

$$\begin{pmatrix} \chi_0(1-\lambda) \\ \chi_1(1-\lambda) \end{pmatrix} = \begin{pmatrix} \dfrac{\Gamma(1-\frac{x}{3})\Gamma(\frac{x}{3})}{\Gamma(\frac{1}{2}-\frac{x}{6})\Gamma(\frac{1}{2}+\frac{x}{6})} & \dfrac{(16)^{x/3}}{N} \dfrac{\Gamma(1-\frac{x}{3})\Gamma(-\frac{x}{3})}{\Gamma(\frac{1}{2}-\frac{x}{6})\Gamma(\frac{1}{2}-\frac{x}{2})} \\ \dfrac{N}{(16)^{x/3}} \dfrac{\Gamma(1+\frac{x}{3})\Gamma(\frac{x}{3})}{\Gamma(\frac{1}{2}+\frac{x}{6})\Gamma(\frac{1}{2}+\frac{x}{2})} & \dfrac{\Gamma(1+\frac{x}{3})\Gamma(-\frac{x}{3})}{\Gamma(\frac{1}{2}+\frac{x}{6})\Gamma(\frac{1}{2}-\frac{x}{6})} \end{pmatrix} \times \begin{pmatrix} \chi_0(\lambda) \\ \chi_1(\lambda) \end{pmatrix} \tag{4.32}$$

$$\equiv S \begin{pmatrix} \chi_0(\lambda) \\ \chi_1(\lambda) \end{pmatrix} \tag{4.33}$$

We now take a small digression and try to address the question of identification of candidate theories.

4.1.3 Chiral Algebra for $\ell = 0$ theories

In this section we discuss the identification of the two-character, $\ell = 0$ candidate theories (Ref. [1]). With the knowledge of only m_1 (degeneracy of the first level above the identity character) and c (central charge), one can identify the theories. It turns out that most of the candidates can be described by simple Kac-Moody algebras.

Lie Algebra	dim(G)	$\mathfrak{g}$	c at $k=1$	c at $k=2$
A_n	$n(n+2)$	$n+1$	n	$\frac{2n(n+2)}{n+3}$
B_n	$n(2n+1)$	$2n-1$	$n+\frac{1}{2}$	$2n$
C_n	$n(2n+1)$	$n+1$	$\frac{n(2n+1)}{n+2}$	$\frac{2n(2n+1)}{n+3}$
D_n	$n(2n-1)$	$2n-2$	n	$2n-1$
E_6	78	12	6	$\frac{78}{7}$
E_7	133	18	7	$\frac{103}{10}$
E_8	248	30	8	$\frac{31}{2}$
F_4	52	9	$\frac{26}{5}$	$\frac{104}{11}$
G_2	14	4	$\frac{14}{5}$	$\frac{14}{3}$

Table 4.1: Dimensions, dual Coxeter numbers and central charges at levels $k = 1, 2$.

Let us summarize the arguments used in the identification of the candidate CFT's. A first look at the table reveals that the m_1 values correspond to the dimensions of many well-known simple Lie algebras. The identifications in the table has been made by following the procedure outlined in [2]. The central idea behind the procedure is that the states at the first level above the identity character are generated by spin-1 currents. All such states are of the form $J^a_{-1}|0\rangle$ where $|0\rangle$ is the vacuum state and thus, m_1 counts the total number of Kac-Moody currents in the theory. To help the identification a list of simple Lie Algebras and their properties have been given in 4.1.

We assume that the chiral algebras are semisimple(direct sum of simple algebras). We use the fact that each component of the direct sum satisfies [1]:

$$\frac{\mathfrak{g}^{(i)}}{k_i} = \frac{m_1}{c} - 1 \tag{4.34}$$

where $\mathfrak{g}^{(i)}$ is the dual Coxeter number and k_i is the level of the i^{th} component in the direct sum. Now, we shall analyse the candidate theories case by case. We first discuss

No.	c	m_1	h	Identification
1	$\frac{2}{5}$	1	$\frac{1}{5}$	$c = -\frac{22}{5}$ minimal model $(c \to c - 24h)$
2	1	3	$\frac{1}{4}$	$k = 1$ SU(2) WZW
3	2	8	$\frac{1}{3}$	$k = 1$ SU(3) WZW
4	$\frac{14}{5}$	14	$\frac{2}{5}$	$k = 1$ G_2 WZW
5	4	28	$\frac{1}{2}$	$k = 1$ SO(8) WZW
6	$\frac{26}{5}$	52	$\frac{3}{5}$	$k = 1$ F_4 WZW
7	6	78	$\frac{2}{3}$	$k = 1$ E_6 WZW
8	7	133	$\frac{3}{4}$	$k = 1$ E_7 WZW
9	$\frac{38}{5}$	190	$\frac{4}{5}$	?
10	8	248	$\frac{5}{6}$	$\supset k = 1$ E_8 WZW

Table 4.2: Chiral algebras for the $\ell = 0$ theories. Here m_1 is the degeneracy of the first excited above the identity character, c is the central charge and h is the conformal dimension of the non-trivial primary.

the entries $2-9$ of 4.2 which can be identified with simple Kac-Moody algebras. The entries 1 and 10 are rather special and are discussed at the end of the section.

Case 2: The central charge is 1 and the $m_1 = 3$. We have $\frac{m_1}{c} - 1 = 2$. Thus $\mathfrak{g} = 2k$. The possible candidates are A_1, C_1 and D_2. However, the dimension is 3. Only, $A_1 = SU(2)$ at level $k = 1$ fits the bill (also C_1). Therefore the theory is identified as the $SU(2)$ Wess-Zumino-Witten model (WZW) at level 1.

Case 3: For this case the central charge $c = 2$ and $\frac{m_1}{c} - 1 = 3$. For small values of central charge like this it is fairly obvious that the chiral algebra is simple and their is no direct sum, as that would lead to a sum of central charges greater than 2 or not give the right dimension . By staring at the table 4.1 we see that A_2 and B_2 both at level 1 have dual coxeter number 3. The dimensions of B_2 are 10. The dimension of A_2 on the other hand is 8 and this is same as m_1 for this case. Thus, we identify the theory to be $A_2 = SU(3)$ at level 1.

There is a small subtlety about the number of characters and primary fields in this theory and is worth mentioning. This theory has three primary fields corresponding to $1, 3$ and $\bar{3}$ representations, but the characters corresponding to 3 and $\bar{3}$ are complex conjugates of each other. Thus, there are only two independent characters. It is important to note that the classification scheme is based on number of 'independent' characters

(p) and this number need not agree with the number of primary fields in the theory.

Case 4: In this case we have $\frac{m_1}{c} - 1 = 4$ and the dual coxeter number is $4k$ and $m_1 = 14$. The only algebra with $\mathfrak{g} \geq 4$ and dimension ≤ 14 is G_2. Thus, this is the G_2 WZW model at level 1.

Case 5: The table 4.9 tells us that $\frac{m_1}{c} - 1 = 6$ and $m_1 = 28$. By referring to the table 4.1. The only candidate satisfying both the conditions is $D_4 = SO(8)$. Thus, this is the $SO(8)$ WZW model at level 1, with central charge $c = 4$.

Analogous to the case of $SU(3)$ WZW model at level 1, the $SO(8)$ theory also has different number of fields and characters. It has four primary fields each corresponding to the representations $1, 8_v, 8_s$ and $8_s'$. But, there are only two independent characters as the corresponding to $8_v, 8_s$ and $8_s'$ representations are the same because of triality of $SO(8)$. .

Case 6: For this case we find that the dual coxeter number is $\mathfrak{g} = 9k$ and $m_1 = 52$. Only F_4 matches this requirement. Therefore, this theory is identified as the F_4 WZW model.

Case 7: We find that the values of central charge are and m_1 are 6 and 78 respectively. As before, we calculate $\frac{m_1}{c} - 1$ and it is equal to 12. Thus, the dual coxeter number is of the form $12k$. A look at the table 4.1 reveals that this is the E_6 WZW model.

Case 8: In this case the central charge is $c = 7$ and $m_1 = 133$. One can already see that this matches the corresponding values for E_7. We check the dual coxeter number and it is equal to $\frac{m_1}{c} - 1 = 18$.

Case 9: For this case we find that c, m_1 and $\mathfrak{g}$ are $\frac{38}{5}, 190$ and 24. We are unable to find any simple Lie algebra that corresponds to this case. However, as it will be explained (Ref [2]) in the next section this is not a consistent CFT as the fusion rules turn out to be negative.

Case 1:The first entry in the table 4.2 has a subtlety and merits more discussion. The entry corresponds to the value $m_1 = 1$. If one naively substitutes this in Eq. (4.14) one finds that $c = \frac{2}{5}$. However, there is no known theory with this value of central charge. Moreover, by looking at the value of the conformal dimension of the other primary -

$-\frac{1}{5}$, one suspects that this theory must be identified with the $c = -\frac{22}{5}$ minimal model. On closer investigation following the discussion in section 2.1 one finds that $c = \frac{2}{5}$ was obtained by assuming that the identity character is the dominant one in the $\tau \to i\infty$ limit and this gives $\chi_0\ e^{-2\pi i \frac{c}{24}\tau}$. However, in this case it is clear that the dominant character is actually the one over the other primary and it behaves as $e^{-2\pi i(h-\frac{c}{24})\tau}$ with $h = -\frac{1}{5}$ in the same limit. Thus, one needs to swap $\alpha_0 = -\frac{c}{24}$ and $\alpha_1 = h - \frac{c}{24}$. After doing this, one finds that $c = -\frac{22}{5}$ and $m_1 = 1$. It is extremely important to note that this cannot be done in general, as the other character can have fractional values of $m_n = \frac{a_n}{a_0}$. Minimal models, however, are special and the degeneracy (the number of states at the lowest level) is always one ($a_0 = 1$)for them. Therefore, each character will have integer values of m_n and swapping the exponents makes sense. An important check consistent with this identification is the fact that minimal models have no states at the first excited level above the identity character, since the Virasoro generator L_{-1} gives rise to a null state (and there are no spin-1 currents in a minimal model). With $\alpha_0 = \frac{11}{60}$ we indeed find that $m_1^0 = 0$.

We now look at the value of $\frac{\mathfrak{g}}{k} = \frac{m_1}{c} - 1$ to find $\frac{3}{2}$. By staring at table 4.1 we indeed find that no combination of the Lie algebras gives $g = \frac{3}{2}$.

Case 10 : The 10^{th} entry corresponding to $m_1 = 248, c = 8$ has one character which is spurious i.e, the denominator of m_n^1 grows with n and no finite degeneracy(a_0) can be assigned to the character. The other (dominant) character corresponds to $j^{\frac{1}{3}}$ which is the character of the E_8 WZW model. It is indeed true that this character arises as the solution to a one-character differential equation. Unlike all other cases where the pair of characters are modular covariant here the 'identity' character is modular invariant. This example raises the question of theories with low number of characters appearing as solutions to a higher order differential equation. We shall not dwell on this question. The usual analysis yields $\frac{m_1}{c} - 1 = 30$ and this does not correspond to any simple algebra. Thus, we conclude that this example is actually a one-character theory.

We have identified the chiral algebras for almost all the candidates with $\ell = 0$. It is interesting to note that most of them are simple Kac-Moody algebras. We shall now get back to discussing the fusion rules for each of these theories.

4.1.4 Monodromy and fusion rules for two-character, $\ell = 0$ theories

Case 1 Candidate theories corresponding to $c = 1$, $N = 2$ and $c = 7$, $N = 56$ respectively. For these theories, $x = \frac{3}{2}$ and $x = \frac{9}{2}$ respectively. We use the following identities to simplify the matrix S.

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) \tag{4.35}$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)} \tag{4.36}$$

In this case, using Eq. (4.32) the transformation matrix for the character gives:

$$\begin{pmatrix} \chi_0(1-\lambda) \\ \chi_1(1-\lambda) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \chi_0(\lambda) \\ \chi_1(\lambda) \end{pmatrix}$$

The matrix is unitary and the corresponding partition function is given by:

$$Z = |\chi_0|^2 + |\chi_1|^2 \tag{4.37}$$

From the matrix S in this case, we read off the fusion rules:

$$N_{000} = N_{011} = 1, \qquad N_{001} = N_{111} = 0 \tag{4.38}$$

These are nothing but the fusion rules of $SU(2)$ and E_7 WZW models at level 1. Case 2 $c = \frac{14}{5}, N = 7$ and $c = \frac{26}{5}, N = 26$. For these theories, $x = \frac{12}{5}$ and $x = \frac{18}{5}$ respectively. And the matrix S is:

$$S = \frac{1}{2} \begin{pmatrix} \frac{1}{\sin(2\pi/5)} & \frac{1}{\sin(\pi/5)} \\ \frac{1}{\sin(\pi/5)} & -\frac{1}{\sin(2\pi/5)} \end{pmatrix}$$

We see that S is unitary. Therefore, there is one primary per character and the diagonal partition function is of the form Eq. (4.37). The fusion rules are :

$$N_{000} = N_{011} = N_{111} = 1, \qquad N_{001} = 0 \tag{4.39}$$

These correspond to the fusion rules of G_2 and F_4 WZW models at level 1. Case 3 $c = \frac{2}{5}, N = 1$ and $c = \frac{38}{5}, N = 57$. For these theories we have $x = \frac{6}{5}$ and $x = \frac{24}{5}$, respectively. Using Eq. (4.32) we get the following S -matrix:

$$S = \frac{1}{2} \begin{pmatrix} \frac{1}{\sin(\pi/5)} & \frac{1}{\sin(2\pi/5)} \\ \frac{1}{\sin(2\pi/5)} & -\frac{1}{\sin(\pi/5)} \end{pmatrix}$$

The matrix is unitary, so there is only one primary field per character:

$$N_{000} = N_{011} = 1, \qquad N_{111} = -1, \qquad N_{001} = 1 \tag{4.40}$$

One of the fusion rules turn out to be negative and this cannot be a consistent CFT. But, we can try to reinterpret the characters to get a consistent CFT. If we go back and question our assumptions we realize that we had assumed that the dominant character when $q \to 0$ is the identity character. But, this result was derived based on the assumption that such a theory has $c > 0$ and conformal dimension $h > 0$. This assumption breaks down for a non-unitary theory. Thus, we reverse the roles of the two characters and find the S matrix to find:

$$S = \frac{1}{2}\begin{pmatrix} -\frac{1}{\sin(\pi/5)} & \frac{1}{\sin(2\pi/5)} \\ \frac{1}{\sin(2\pi/5)} & \frac{1}{\sin(\pi/5)} \end{pmatrix}$$

and the fusion rules are:

$$N_{000} = N_{011} = N_{111} = 1, \qquad N_{001} = 0 \tag{4.41}$$

Thus, we obtain sensible fusion rules. But, the exchange of characters is possible only for the theory with $N = 1$ and after the reversal $c = -\frac{22}{5}$ and the conformal dimension of the non-trivial primary is $h = -\frac{1}{5}$. However, for the second theory after the exchange the identity becomes 57-fold degenerate. This is not permissible as the identity is assumed to be non-degenerate. Thus, this case does not describe a consistent CFT. Case 4 4) $c = 2, N = 3$ and $c = 6, N = 27$ For these theories $x = 2$ and $x = 4$, respectively. From Eq. (4.32) , we have the following S matrix:

$$S = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

It is clear that the matrix is not unitary. We find that it leaves the the matrix diag(1,2). Therefore, the partition function is:

$$Z = |\chi_0|^2 + 2|\chi_1|^2 \tag{4.42}$$

This means that there are two primary fields corresponding to the character χ_1. Let us denote the three characters of the theory by $\chi_0, \hat{\chi_1}, \hat{\chi_2}$ with $\hat{\chi_1}, = \hat{\chi_2} = \chi_1$

$$\begin{pmatrix} \chi_0(1-\lambda) \\ \hat{\chi}_1(1-\lambda) \\ \hat{\chi}_2(1-\lambda) \end{pmatrix} \equiv \widehat{S} \begin{pmatrix} \chi_0(\lambda) \\ \chi_1(\lambda) \\ \chi_2(\lambda) \end{pmatrix}$$

$\widehat{S}$ is not uniquely fixed by S. But, we get the following constraints:

$$\begin{gathered} \widehat{S}_{00} = \widehat{S}_{10} = \widehat{S}_{20} = \frac{1}{\sqrt{3}}, \qquad \widehat{S}_{01} + \widehat{S}_{02} = \frac{2}{\sqrt{3}} \\ \widehat{S}_{11} + \widehat{S}_{12} = \widehat{S}_{21} + \widehat{S}_{22} = -\frac{1}{\sqrt{3}} \end{gathered} \tag{4.43}$$

Now, by demanding that $\widehat{S}$ be unitary and symmetric we get:

$$\widehat{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sqrt{3}a & -(\sqrt{3}a+1) \\ 1 & -(\sqrt{3}a+1) & \sqrt{3}a \end{pmatrix}$$

The demand for unitarity gives:

$$\frac{1}{3} + |a|^2 + |a + \frac{1}{\sqrt{3}}|^2 = 1 \tag{4.44}$$

This implies:

$$a = -\frac{1}{2\sqrt{3}} \pm \frac{1}{2}, -\frac{1}{\sqrt{3}} e^{i\frac{\pi}{3}} \tag{4.45}$$

The equation for fusion rule coefficient now becomes,

$$N_{ijk} = \sum_n \frac{\widehat{S_{in}}\widehat{S_{jn}}\widehat{S_{kn}}}{\widehat{S_{0n}}} \tag{4.46}$$

The requirement that $N_{011} \geq 0$ gives:

$$Im\Big[\frac{1}{3} + a^2 + \Big(\frac{1}{\sqrt{3}} + a\Big)^2\Big] = 0 \tag{4.47}$$

By checking the other fusion coefficients we find that the first of the two values in Eq. (4.45) gives $N_{112} = \frac{1}{2}$ and this is not allowed. The other value of a gives the following set of consistent fusion rules:

$$N_{000} = N_{012} = N_{111} = N_{222} = 1,$$
$$N_{000} = N_{002} = N_{011} = N_{022} = N_{122} = N_{112} = 0$$

These are the fusion rules of the $SU(3)$ and E_6 WZW model at level 1. The primary fields corresponding to a single character in each case are $(3, \bar{3}$ and $27, \bar{27}$, respectively and they correspond to the indices $(1, 2)$ in the fusion rules. The final form of $\widehat{S}$ is :

$$\widehat{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -e^{\pm i\pi/3} & -e^{\mp i\pi/3} \\ 1 & -e^{\mp i\pi/3} & -e^{\pm i\pi/3} \end{pmatrix}$$

Case 5 $c = 4, N = 8$ For this theory we find $x = 3$. Naively substituting this into the character transformation formula, we get divergences. To regularize this, we take $x = 3 + \epsilon$ and finally take $\epsilon \to 0$. We get the following S matrix:

$$S = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

This non-unitary matrix leaves the matrix diag(1,3) invariant. Thus, the partition function is :

$$Z = |\chi_0|^2 + 3|\chi_1|^2 \tag{4.48}$$

There are 3 primary fields associated with the identity character and we must find a 4×4 matrix $\widehat{S}$ which diagonalizes the fusion rules. Following the same steps as in the previous case, we find:

$$\widehat{S} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

The fusion rules are:

$$N_{000} = N_{011} = N_{022} = N_{033} = N_{123} = 1 \tag{4.49}$$

with all other N_{ijk} being zero. These are exactly the fusion rules of the $SO(8)$ WZW model at level $k = 1$ and has three degenerate primary fields $(8_V, 8_S, 8_{S'}$. Case 6 $c = 8$ As we have explained in the previous section, this theory has one spurious character. Therefore, there is no finite N which χ_1 has an integer q-expansion. However, for any finite value of

N, the S matrix has the form

$$\widehat{S} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}$$

This leaves the partition function $Z = |\chi_0|^2$ invariant. Thus, there in no primary corresponding to the character χ_1. We conclude that this is a one-character theory and can be identified with E_8 WZW model at level one.

After a detailed discussion of two-character $\ell = 0$ theories which were first reported in [2, 1] we turn our attention to $\ell > 0$ theories.

4.2 $\ell > 0$ theories

Now, let us consider the case of non-singular second order differential equations. Not much was known about this case despite intense investigation over the years. Though a few candidates were listed in[3] for $2 < \ell < 12$, no conclusive identification of CFT's has been made. In the following sections we discuss our recent work on two-character theories with $\ell = 2, 3$ and 4, that sheds more light on the subject [4, 5].

4.3 $\ell = 2$ theories

This is the first non-singular case we analyse and we work with the equation:

$$\left(\tilde{D}^2 + \mu_1 \frac{E_6}{E_4}\tilde{D} + \mu_2 E_4\right)\chi = 0 \tag{4.50}$$

In terms of ordinary derivatives we get:

$$\left(\tilde{\partial}^2 - \tfrac{1}{6}E_2\tilde{\partial} + \mu_1 \frac{E_6}{E_4}\tilde{\partial} + \mu_2 E_4\right)\chi = 0 \tag{4.51}$$

We know from the discussion in section 2.2 that we don't expect tensor products as solutions to the $\ell = 2$, two-character differential equation. With this intuition, we proceed and use Ramanujan identities 1.44 to rewrite the recursion relations with each term linear in the Eisenstein series. Accordingly, we multiply the above differential equation by E_4 and use the second Ramanujan identity in Eq. (1.44) to get:

$$\left(E_4\tilde{\partial}^2 - \tfrac{1}{2}\tilde{\partial}E_4\tilde{\partial} + \left(\mu_1 - \tfrac{1}{6}\right)E_6\tilde{\partial} + \mu_2 E_8\right)\chi = 0 \tag{4.52}$$

Inserting the mode expansion for the character and the Eisenstein series , we get:

$$\sum_{k=0}^{n}\Big[(n+\alpha-k)^2E_{4,k}-\tfrac{1}{2}k(n+\alpha-k)E_{4,k}+\tilde{\mu}_1(n+\alpha-k)E_{6,k}+\mu_2E_{8,k}\Big]a_{n-k}=0 \tag{4.53}$$

where $\tilde{\mu}_1 = \mu_1 - \frac{1}{6}$.

The $n = 0$ case gives the indicial equation:

$$\alpha^2 + \tilde{\mu}_1\alpha + \mu_2 = 0 \tag{4.54}$$

As promised, we now determine $\tilde{\mu}_1$ using the identity Eq. (2.10) with $p = 2, \ell = 2$:

$$\alpha_0 + \alpha_1 = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6} \tag{4.55}$$

Form the sum of the roots of the indicial equation we get $\alpha_0 + \alpha_1 = -\tilde{\mu}_1$. This fixes the value of $\tilde{\mu}_1$to be $\frac{1}{6}$. From the product of the roots we obtain:

$$\mu_2 = -\alpha^2 - \tfrac{1}{6}\alpha \tag{4.56}$$

We consider the $n = 1$ equation to find:

$$m_1^{(i)} = \frac{a_1^{(i)}}{a_0^{(i)}} = \frac{24\alpha_i(71 + 60\alpha_i)}{7 + 12\alpha_i} \tag{4.57}$$

We work with $i = 0$ and drop the superscript on m_1 as before. Assuming that $\alpha_0 = -\frac{c}{24}$ corresponds to the dominant character we find:

$$m_1 = \frac{c(5c - 142)}{(14 - c)} \tag{4.58}$$

By demanding non-negativity of m_1 we find that $14 < c \leq \frac{142}{5}$. Following the same procedure as in the previous case we conclude that if at all there is a minimal model solution ($m_1^0 = 0$) to this differential equation it must have $\alpha_0 = -\frac{71}{60}$. It turns out that this case is ruled out later. Now, we rewrite the formula for m_1 in terms of the central charge to obtain :

$$(5c)^2 + 5c(m_1 - 142) = 70m_1 \tag{4.59}$$

This shows that even for $\ell = 2$, $5c$ is an integer. We carry out the analysis done in the previous case to get:

$$m_1^2 - 4m_1 + 20164 = N^2 \tag{4.60}$$

where N is an integer. Now we make the following shift:

$$N = \tilde{N} + m_1 - 2 \tag{4.61}$$

and solve for m_1 to find:

$$m_1 = -\frac{\tilde{N}}{2} + 2 + \frac{10080}{\tilde{N}} \tag{4.62}$$

As done in the previous case, we conclude that $\tilde{N}$ is an even integer. It should also divide 10080. Here, we find that the formula is invariant under $\tilde{N} \to -\frac{20160}{\tilde{N}}$. For $m_1 \geq 0$ we must have $|\tilde{N}| \leq 144$ or negative otherwise. We can neglect the negative values of $\tilde{N}$ because of the symmetry. This restricts $\tilde{N}$ to the following 31 values:

$$\begin{aligned}\tilde{N} = \{&2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48, 56, 60, 70,\\ &72, 80, 84, 90, 96, 112, 120, 126, 140, 144\}\end{aligned} \tag{4.63}$$

For each of these cases we use Eq. (4.62) to compute the values of m_1. These are nothing but m_1^0 values for these cases (We had suppressed the superscript earlier for notational simplicity). In the next step we compute α_0 from Eq. (4.57). The values of can be immediately found using $\alpha_1 = -\frac{1}{6} - \alpha_0$ and consequently, we obtain m_1^1 by a second use of Eq. (4.57). We find that all the $m_1^{(1)}$ are non-negative.

Applying the same method as in the $\ell = 0$ case we compute $m_2^{(0)}$ from Eq. (4.53). At this stage the values $\tilde{N} = \{2, 4, 6, 10, 14, 18, 28, 32\}$ are ruled out due to fractional values and $\tilde{N} = \{126, 140, 144\}$ due to negative values of $m_2^{(0)}$. Also,$\tilde{N} = 120$ gives rise to a divergence in the value of $m_2^{(0)}$. Thus, we rule out all these candidates. Continuing with our analysis, we find that the cases $\tilde{N} = \{8, 36, 56, 70\}$ give fractional values of $m_3^{(0)}$ and $\tilde{N} = \{12, 16, 42, 84, 112\}$ give fractional values of m_4^0 . We eliminate all these cases and are left with the following 10 values [4]:

$$\tilde{N} = \{20, 24, 30, 40, 48, 60, 72, 80, 90, 96\} \tag{4.64}$$

The next step is to consider $m_n^{(1)}$ for these cases. We examine these cases till $n = 20$ and find that all of them give positive fractions for m_n^1. However, we find two very different kind of behaviours in the denominators of these fractions. They are generically fractional and display two markedly different kinds of behaviours. For$\tilde{N} = 20$ the denominator of m_n^1 keeps growing with n (akin to the 'spurious 'character' explained in the $\ell = 0$ case) and thus, we eliminate this theory. (It is interesting to note that in many of the the already ruled out cases,the non-trivial primaries lead to such 'spurious' characters).

However, for the other cases the denominator of $m_n^{(1)}$ is relatively small and contains only two or three prime factors and no new factors appear after $n = 10$. In this sense the denominator "stabilizes". We further check these cases till $n = 5000$ and find that the denominators remain stable. This strongly suggests that these cases correspond to CFT. In these cases one may take the largest denominator encountered to be the lower bound on the ground state degeneracy. This means that all the excited states till m^i_{5000} will definitely have positive integer values for the degeneracies.

We list our findings for the 9 candidate theories in Table 4.3 [4, 3]. This list first appeared in [3]. However, in the of [3] the integrality was checked only up to very low levels $n \sim 3$. Moreover, there was no construction of the candidates.

No.	$\tilde{N}$	c	h	m_1	Apparent Degen
1	96	$\frac{118}{5}$	$\frac{9}{5}$	59	32509
2	48	$\frac{94}{5}$	$\frac{7}{5}$	188	4794
3	24	$\frac{82}{5}$	$\frac{6}{5}$	410	902
4	30	17	$\frac{5}{4}$	323	51
5	40	18	$\frac{4}{3}$	234	1
6	60	20	$\frac{3}{2}$	140	5
7	72	$\frac{106}{5}$	$\frac{8}{5}$	106	15847
8	80	22	$\frac{5}{3}$	88	22
9	90	23	$\frac{7}{4}$	69	253

Table 4.3: Potentially consistent CFT's with $\ell = 2$.

We note that central charges for these candidate lie in the range $16 < c < 24$. An other interesting observation is that the table has an evident correspondence with the table for $\ell = 0$ theories. The sum of central charges of theories 9 to 1 in the table 4.3 and theories 1 to 9 in the table 4.1.3, in that order, add to 24. For example, the $\ell = 2$ theory 9 has central charge 23 and the $\ell = 0$ theory 1 has central charge 1 and their sum is 24. Similarly, when we traverse the tables in opposite order we see that the corresponding conformal dimensions add up to two. This strongly suggests that there is some relation between $\ell = 0$ and $\ell = 2$ two-character theory. Recall that from the discussion in section 2.2 we have ruled out the possibility that these candidates can be tensor products. Thus, these candidates, if they exist, must be independent, new CFT's. The relation between these pairs of theories will become clear when we discuss the generalised coset construction4.3.2 and we will see why one has to compare the theory number x in the table 4.1.3 with the the theory $9 - x$ in 4.3.

We now try to identify the potential chiral algebra for these theories.

4.3.1 Potential Chiral Algebras for $\ell = 2$ theories

In this section we discuss the potential chiral algebras for the 9 $\ell = 2$ theories. Following our approach in the previous case and the discussion in **??**, we assume that the chiral algebra of these theories is the sum of a number of simple Kac-Moody algebras: $G = \oplus_i G_i$. We start with the Sugawara relation:

$$c_i = \frac{k_i \dim G_i}{k_i + (\mathfrak{g})_i} \tag{4.65}$$

where k_i is the level of the ith algebra, $\dim G_i$ is the dimension of the associated finite-dimensional Lie algebra and $(\mathfrak{g})_i$ is the dual Coxeter number of this algebra.

Now, let us try to guess a combination of Kac-Moody algebras that underlie a given candidate CFT using the known values of the central charge c and the integer m_1, the degeneracy of the first excited state above the identity character. We follow the procedure outlined in the case of $\ell = 0$ theories.

As we have assumed the chiral algebra to be semisimple it follows that:

$$\begin{aligned} c &= \sum_i \frac{k_i \dim G_i}{k_i + (\mathfrak{g})_i} \\ m_1 &= \sum_i \dim G_i \end{aligned} \tag{4.66}$$

We follow the procedure outlined in the $\ell = 0$ case and note that each constituent algebra separately satisfies (generalizing the arguments of [2]):

$$\frac{(\mathfrak{g})_i}{k_i} = \frac{m_1}{c} - 1 \tag{4.67}$$

Thus, we only need to look for factors that all have the same value of $\frac{\mathfrak{g}}{k}$. Let us apply this case-by-case to the theories listed in Table 4.3. For simplicity, we assume that $k_i = 1$. We first list all possible algebras whose $\mathfrak{g}_{\mathfrak{i}} = \frac{m_1}{c} - 1$ for any candidate in the table 4.3. Then, we only keep those theories whose dimensions, with possible positive integral coefficients, add up to the value of m_1 for that candidate theory. These two steps ensure that the central charges of add up to the central charge of the candidate $\ell - 2$ theory. We list our results in 4.4.

The goal of this exercise is to see if there are any current algebras that can potentially arise in our $\ell = 2$ theories. Despite the fact that we restricted our search to level-1 semisimple Lie algebras we find at least one solution for all the cases except theories 1 and 9. But, these theories can be ruled out by the following argument: These theories correspond to $\ell = 0$ theories with central charge $c = \frac{38}{5}$ and $\frac{2}{5}$ respectively which have negative fusion rules. Now, it has been shown in Ref [3] that the $\ell = 0$ and $\ell = 2$ theories have the same fusion rules. The $\ell = 0$ theory with central charge $\frac{2}{5}$ was interpreted as a non-unitary minimal model by exchanging the role of the identity and the non-trivial character,as the ground state degeneracy of both was one. We cannot repeat this exercise with theory 9 of table 4.3 as both the characters have degeneracies of ground state greater than one (at least 902 and 32509 respectively). For that matter, we cannot swap the characters of the theory with $c = \frac{38}{5}$ as the other character has a 57-fold degeneracy of the ground state. Thus, we rule out this theory [2] and consequently rule out theory 1 in 4.3.

The candidates in 4.3 are not WZW models of the corresponding chiral algebras, yet the non-trivial primary we have obtained must be a subset of the primaries of those algebras. Moreover, the characters of these theories must arise as combinations of the characters corresponding to WZW models of the chiral algebras. Thus, we need to check if the non-trivial primaries listed in the table 4.3 can be obtained by the level-1 primaries of the corresponding chiral algebras. For a detailed discussion on this we refer the reader to Ref. [4]

No.	c	$\mathfrak{g} = \frac{m_1}{c} - 1$	Algebras with given $\mathfrak{g}$	Combinations with right c, m_1
1	$\frac{82}{5}$	24	A_{23}, C_{23}, D_{13}	None
2	17	18	$A_{17}, C_{17}, D_{10}, E_7$	$A_{17}, D_{10} \oplus E_7$
3	18	12	A_{11}, C_{11}, D_7, E_6	$A_{11} \oplus D_7, (E_6)^3$
4	$\frac{94}{5}$	9	A_8, B_5, C_8, F_4	$C_8 \oplus F_4$
5	20	6	A_5, C_5, D_4	$(A_5)^4, (D_4)^5$
6	$\frac{106}{5}$	4	A_3, C_3, G_2	$A_3 \oplus C_3 \oplus (G_2)^5, A_3 \oplus (C_3)^3 \oplus (G_2)^2$
7	22	3	A_2, B_2	$(A_2)^{11}, (A_2)^6 \oplus (B_2)^4, A_2 \oplus (B_2)^8$
8	23	2	A_1	$(A_1)^{23}$
9	$\frac{118}{5}$	$\frac{5}{2}$	None	None

Table 4.4: Potential level-1 chiral algebras for the $\ell = 2$ theories.

Apparently, this is the best we can do using differential equations. However, the striking similarity between the central charges and the conformal dimensions of the

$\ell = 0$ and $\ell = 2$ theories is intriguing. This merits closer investigation, which we carry out in the next section.

4.3.2 The Generalised coset construction

It turns out that the $\ell = 2$ theories arise as cosets of certain $c = 24$, meromorphic, one-character theories. This proves the claim that the $\ell = 2$ candidates are indeed consistent conformal field theories. However, to understand this one needs to generalize the standard coset construction of section1.4.1. We now take a small digression and understand the generalized coset construction[5] and then go on to show how the $\ell = 2$ theories arise as cosets.

In the section 1.2 we described the usual coset construction where a coset of an affine theory is taken by another affine theory to get new CFT's [8]. Here, we shall generalize this construction [5] to describe the coset of a non-affine theory by an affine theory. We use our coset construction to identify the two-character $\ell = 2$ CFT's. Such generalizations have been considered before in [9, 10]. Since, this construction is not standard we describe it in detail.

Consider a meromorphic conformal field theory $\mathcal{H}$. As an example, one could think of a self-dual theory defined on a lattice which has an affine symmetry algebra, but a chiral algebra not necessarily generated just by the currents. Let us consider $\mathcal{D}$, an affine sub-theory of $\mathcal{H}$, associated with a semi-simple Lie algebra $\mathfrak{h}$ at some level k (Note that k is a positive integer). Then, we describe the following coset:

$$\mathcal{C} = \mathcal{H}/\mathcal{D} \ , \tag{4.68}$$

It is important to understand the chiral algebra of $\mathcal{C}$. It contains all the chiral fields in $\mathcal{H}$ that have a trivial OPE(Operator Product Expansion) with the fields of $\mathcal{D}$.

One needs to show that this generalized coset construction leads to a consistent CFT and the proof is very similar to the one for the standard coset construction in 1.4.1. We need to show that the coset has a stress-energy tensor with the central charge $c^{\mathcal{C}} = c^{\mathcal{H}} - c^{\mathcal{D}}$. Since the denominator is an affine theory, its stress energy tensor is given by the Sugawara construction **??** and this involves currents J^a only from $\mathfrak{h}$. On the other hand, we know that the denominator is also a sub-theory of $\mathcal{H}$. Therefore, we have:

$$[L_n^{\mathcal{H}}, J_m^a] = -mJ_{n+m}^a \ , \qquad \text{and} \qquad [L_n^{\mathcal{D}}, J_m^a] = -mJ_{n+m}^a \ , \tag{4.69}$$

Here $L_n^{\mathcal{H}}$ and $L_n^{\mathcal{D}}$ are the Virasoro modes of the numerator and denominator theory, respectively. As a consequence, we have:

$$[L_n^{\mathcal{C}}, J_m^a] = 0 \ , \qquad \text{where} \qquad L_n^{\mathcal{C}} = L_n^{\mathcal{H}} - L_n^{\mathcal{D}} \ . \tag{4.70}$$

The modes $L_n^{\mathcal{C}}$ commute with the currents J_m^a and are thus a part of the chiral algebra of the coset theory. Moreover, we know that $L_m^{\mathcal{D}}$, the Virasoro generators of the denominator theory ($\mathcal{D}$ are bilinears in the currents J_n^a. Using this we can show that:

$$[L_n^{\mathcal{C}}, L_m^{\mathcal{D}}] = 0 \ . \tag{4.71}$$

Till now, we have discussed the tools required to show that the coset modes $L_n^{\mathcal{C}}$ form a Virasoro algebra for the coset theory with a central charge that is the difference of the central charges of the numerator and denominator theory. We now find the Virasoro algebra of the coset theory :

$$\begin{aligned}
[L_m^{\mathcal{C}}, L_n^{\mathcal{C}}] &= [L_m^{\mathcal{C}}, L_n^{\mathcal{H}}] - [L_m^{\mathcal{C}}, L_n^{\mathcal{D}}] \\
&= [L_m^{\mathcal{H}} - L_m^{\mathcal{D}}, L_n^{\mathcal{H}}] \\
&= (m-n)L_{m+n}^{\mathcal{H}} + c^{\mathcal{H}} m(m^2-1)\delta_{m,-n} - [L_m^{\mathcal{D}}, L_n^{\mathcal{C}}] - [L_m^{\mathcal{D}}, L_n^{\mathcal{D}}] \\
&= (m-n)L_{m+n}^{\mathcal{C}} + (c^{\mathcal{H}} - c^{\mathcal{D}}) m(m^2-1)\delta_{m,-n} \ ,
\end{aligned} \tag{4.72}$$

(4.71),

We now specialize to the case where $\mathcal{H}$ is a self-dual theory (i.e, it has only the vacuum representation). This, as we shall see, will help us in describing the two-character $\ell = 0$ theories as cosets. If we demand modular invariance, we see that such self-dual theories exist for $c = 24N$, where N is an integer (This can be easily seen from the character). We only work with $c = 24$. For this case, 71 theories are believed to exist[11] . 70 of these have been constructed [12, 13]. Only one of these, namely the E_8^3 theory is an affine theory.

For central charge $c = 24$ such meromorphic, self-dual theories have the single character given by:

$$\chi_0^{\mathcal{H}}(\tau) = J(\tau) + \mathcal{N} \ , \qquad \text{with} \qquad J(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \cdots \ , \tag{4.73}$$

where $q = e^{2i\pi\tau}$ and $j(\tau)$ is the Klein j-function. $\mathcal{N}$ is the dimension of the Lie algebra $\}$ whose affine Kac-Moody algebra is contained in $\mathcal{H}$. Since $\mathcal{D}$ is a subtheory of $\mathcal{H}$, the numerator can be decomposed in terms of the irreducible representations of the denominator algebra. This implies that for characters, we have the following identity:

$$\chi_0^{\mathcal{H}}(\tau) = \chi_0^{\mathcal{D}}(\tau) \cdot \chi_0^{\mathcal{C}}(\tau) + \sum_{i=1}^{p-1} d_i \, \chi_i^{\mathcal{D}}(\tau) \cdot \chi_i^{\mathcal{C}}(\tau) \; , \tag{4.74}$$

where $\chi_0^{\mathcal{D}}$ and $\chi_0^{\mathcal{C}}$ are the identity characters of $\mathcal{D}$ and $\mathcal{C}$, respectively and $\chi_i^{\mathcal{D}}$ for $i = 1, \ldots, p-1$ are the remaining $p-1$ irreducible characters of $\mathcal{D}$.

Using $c^{\mathcal{D}} + c^{\mathcal{C}} = c^{\mathcal{H}} = 24$ we can see that the q^{-1} term of 4.73 is produced by the first term of (4.74).Moreover, since all other terms on the left-hand-side have only non-negative integer powers of q , we are forced to conclude that the conformal dimensions of the non-trivial primaries of the coset (h_i^C) and that of the denominator (h_i^D) add pair-wise to give a natural number n_i i.e, $h_i^C + h_i^D = n_i$.

For the case where the Lie algebra $\mathfrak{h}$ of $\mathcal{D}$ is a direct summand of the Lie algebra $\mathfrak{g}$ of $\mathcal{H}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, we find that the coset algebra will contain the affine algebra based on $\mathfrak{k}$. In this case the q^0 term of (4.74) will also arise from the first summand on the right-hand-side; in that case we have that $h_i^{\mathcal{D}} + h_i^{\mathcal{C}} = n_i \geq 2$. If $\mathfrak{h}$, on the other hand, is not a direct summand of $\mathfrak{g}$, then $h_i^{\mathcal{D}} + h_i^{\mathcal{C}} = n_i = 1$ for at least one $i \in \{1, \ldots, p-1\}$.

Now, we try to see how the parameter ℓ behaves after taking the coset. Both the theories $\mathcal{D}$ and $\mathcal{C}$ have p distinct characters. Thus, using Eq. (2.10), the fact that central charges add up to $24N$ and the conformal dimensions add pairwise to some fixed integer n_i, we find, for the coset theory:

$$\ell^{\mathcal{C}} = p^2 + (6N-1)\,p - 6\sum_{i=1}^{p-1} n_i - \ell \; . \tag{4.75}$$

For our case, we have $N = 1$ and $\forall n_i = 2$. Then the above gives:

$$\ell^{\mathcal{C}} = (p-3)(p-4) - \ell \; . \tag{4.76}$$

Thus, if $p = 2$, $\ell^{\mathcal{C}} = 2 - \ell$. In particular, if the denominator theory has $\ell = 0$ the coset has $\ell^{\mathcal{C}} = 2$. We will argue that our candidate theories can be described as such cosets . We also find that for $p = 3, 4$ the only solutions are $\ell = \ell^{\mathcal{C}} = 0$. We will use this facts in the subsequent sections to find new three and four-character CFT's.

4.3.3 $\ell = 2$ theories as cosets

In this section, we describe the relation between two-character $\ell = 0$ and $\ell = 2$ theories first noted in [4]. All the relevant features have been listed in table 4.5.

We know that the $\ell = 0$ theories are affine theories with simple Kac-Moody algebras as their chiral algebras. We consider the cosets of self-dual Schellekens theories (at $c = 24$) with affine two-character, $\ell = 0$ theories $\mathcal{D}$, with the current algebra of the denominator being a direct summand of the current algebra of the numerator. From the discussion at the end of section 4.3.2 we know that this condition implies that $n_1 \geq 2$. Then, by demanding that $ell, ell^{\mathcal{C}} \geq 0$ in Eq. (4.75), we find that $n_1 = 2$ and $\ell^{\mathcal{C}} = 2 - \ell$. Thus, this construction map an $\ell = 0$ affine two-character theory (whose Lie algebra appears as a direct summand in one of the Schellekens self-dual theories) to a two-character theory with $\ell = 2$.

	$\ell = 0$				$\tilde{\ell} = 2$				
No.	c	h	m_1	Algebra	$\tilde{c}$	$\tilde{h}$	$\tilde{m}_1$	$m_1 + \tilde{m}_1$	Schellekens No.
1	1	$\frac{1}{4}$	3	$\mathfrak{a}_1$	23	$\frac{7}{4}$	69	72	$15 - 21$
2	2	$\frac{1}{3}$	8	$\mathfrak{a}_2$	22	$\frac{5}{3}$	88	96	$24, 26 - 28$
3	$\frac{14}{5}$	$\frac{2}{5}$	14	$\mathfrak{g}_2$	$\frac{106}{5}$	$\frac{8}{5}$	106	120	$32, 34$
4	4	$\frac{1}{2}$	28	$\mathfrak{d}_4$	20	$\frac{3}{2}$	140	168	$42, 43$
5	$\frac{26}{5}$	$\frac{3}{5}$	52	$\mathfrak{f}_4$	$\frac{94}{5}$	$\frac{7}{5}$	188	240	$52, 53$
6	6	$\frac{2}{3}$	78	$\mathfrak{e}_6$	18	$\frac{4}{3}$	234	312	$58, 59$
7	7	$\frac{3}{4}$	133	$\mathfrak{e}_7$	17	$\frac{5}{4}$	323	456	$64, 65$

Table 4.5: Characters with $\ell = 0$ and $\ell = 2$. Here $c, \tilde{c}$ are the central charges, $h, \tilde{h}$ the conformal dimensions of the primary and $m_1, \tilde{m}_1$ the degeneracy of the first excited state in the identity character. All the Lie algebras of the $\ell = 0$ theories are at level 1.

Table 4.5 lists the Schellekens theories that contain the Lie algebra of one of the $\ell = 0$ theories as a direct summand. For example, the first entry means that the theories $15 - 21$ in Schellekens list have $\mathfrak{a}_1$ as their direct summand. It is easy to calculate the central charge of the coset theory using $24 - c^{\mathcal{D}} = c^{\mathcal{C}}$. The non-trivial conformal dimension of the coset primary can be found using $\tilde{h} = 2 - h$. The entries in this table agree precisely with the observations of [3, 4].

The chiral algebra of the coset theory contains the affine algebra that is left by deleting the affine algebra of the denominator from the direct sum of affine algebras in the numerator. However, the full chiral algebra of the resulting coset is not described just by these currents and hence, the cosets are non-affine theories.. Entry 1 clearly shows that there are different $c = 24$, meromorphic, one-character theories that have the Lie algebra of a given $\ell = 0$ theory in the direct summand, but the rest of the algebra is different. This means that there are different $\ell = 2$ coset theories with the same pair of characters.

From the coset construction we have the following identity for characters:

$$J(\tau) + \mathcal{N} = \chi_0(\tau)\tilde{\chi}_0(\tau) + \chi_1(\tau)\tilde{\chi}_1(\tau)\ , \tag{4.77}$$

where $\mathcal{N} = m_1 + \tilde{m}_1$ (since $n_1 = 2$). The characters of the theories listed in the table are the standard Gauss-hypergeometric functions and we have the following formulae from[3]

$$\begin{aligned}
\chi_0 &= j^{\frac{c}{24}}\ {}_2F_1\Big(-\tfrac{1}{2}\big(h-\tfrac{1}{6}\big), -\tfrac{1}{2}\big(h-\tfrac{5}{6}\big); 1-h; \tfrac{1728}{j}\Big)\\
\chi_1 &= \sqrt{m}\ j^{\frac{c}{24}-h}\ {}_2F_1\Big(\tfrac{1}{2}\big(h+\tfrac{1}{6}\big), \tfrac{1}{2}\big(h+\tfrac{5}{6}\big); 1+h; \tfrac{1728}{j}\Big)\\
\tilde{\chi}_0 &= j^{\frac{\tilde{c}}{24}}\ {}_2F_1\Big(-\tfrac{1}{2}\big(\tilde{h}+\tfrac{1}{6}\big), -\tfrac{1}{2}\big(\tilde{h}-\tfrac{7}{6}\big); 1-\tilde{h}; \tfrac{1728}{j}\Big)\\
\tilde{\chi}_1 &= \sqrt{\tilde{m}}\ j^{\frac{\tilde{c}}{24}-\tilde{h}}\ {}_2F_1\Big(\tfrac{1}{2}\big(\tilde{h}-\tfrac{1}{6}\big), \tfrac{1}{2}\big(\tilde{h}+\tfrac{7}{6}\big); 1+\tilde{h}; \tfrac{1728}{j}\Big)\ ,
\end{aligned} \tag{4.78}$$

where

$$\begin{aligned}
\sqrt{m} &= (1728)^h \left(\frac{\sin\frac{\pi}{2}\left(\frac{1}{6}-h\right)\sin\frac{\pi}{2}\left(\frac{5}{6}-h\right)}{\sin\frac{\pi}{2}\left(\frac{1}{6}+h\right)\sin\frac{\pi}{2}\left(\frac{5}{6}+h\right)}\right)^{\frac{1}{2}} \frac{\Gamma(1-h)\Gamma\Big(\frac{1}{2}\left(\frac{11}{6}+h\right)\Big)\Gamma\Big(\frac{1}{2}\left(\frac{7}{6}+h\right)\Big)}{\Gamma(1+h)\Gamma\Big(\frac{1}{2}\left(\frac{11}{6}-h\right)\Big)\Gamma\Big(\frac{1}{2}\left(\frac{7}{6}-h\right)\Big)}\\
\sqrt{\tilde{m}} &= (1728)^{\tilde{h}} \left(\frac{\sin\frac{\pi}{2}\left(\frac{1}{6}+\tilde{h}\right)\sin\frac{\pi}{2}\left(\frac{7}{6}-\tilde{h}\right)}{\sin\frac{\pi}{2}\left(\frac{1}{6}-\tilde{h}\right)\sin\frac{\pi}{2}\left(\frac{7}{6}+\tilde{h}\right)}\right)^{\frac{1}{2}} \frac{\Gamma(1-\tilde{h})\Gamma\Big(\frac{1}{2}\left(\frac{13}{6}+\tilde{h}\right)\Big)\Gamma\Big(\frac{1}{2}\left(\frac{5}{6}+\tilde{h}\right)\Big)}{\Gamma(1+\tilde{h})\Gamma\Big(\frac{1}{2}\left(\frac{13}{6}-\tilde{h}\right)\Big)\Gamma\Big(\frac{1}{2}\left(\frac{5}{6}-\tilde{h}\right)\Big)}\ .
\end{aligned} \tag{4.79}$$

$\sqrt{m}$ and $\sqrt{\tilde{m}}$ correspond to the degeneracies of the ground state of the other character, including a factor to account for the case where multiple primaries correspond to the same character. There are only two cases when such multiplicities can occur: 1) Two primaries correspond to characters that are complex conjugates of each other. In this case, $\sqrt{m}$ and $\sqrt{\tilde{m}}$ have a factor of $\sqrt{2}$. 2) The triality of $\mathfrak{d}_4$ leads to three primaries giving the same character. In this case, $\sqrt{m}$ and $\sqrt{\tilde{m}}$ have a factor of $\sqrt{3}$.

Barring these cases $\sqrt{m}$ and $\sqrt{\tilde{m}}$ are integers. Moreover, the product $\sqrt{m}\sqrt{\tilde{m}}$ is ‘ always’ an integer. For the specific values of h corresponding to the known $\ell = 0$ CFTs $\sqrt{m}$ and $\sqrt{\tilde{m}}$ can easily be calculated. We have verified that the relation Eq. (4.77) works out in each case. In fact, Eq. (4.77) is true not just for the special values for h that correspond to CFT's, it is true for arbitrary h. This is so because of an identity of gauss hypergeometric functions:

$$
\begin{aligned}
&{}_2F_1(r, r+\tfrac{1}{3}; 2r+\tfrac{5}{6}; x)\, {}_2F_1(-r-1, -r-\tfrac{1}{3}; -2r-\tfrac{5}{6}; x) \\
&\quad + Mx^2\, {}_2F_1(-r+\tfrac{1}{6}, -r+\tfrac{1}{2}; -2r+\tfrac{7}{6}; x)\, {}_2F_1(r+\tfrac{5}{6}, r+\tfrac{3}{2}; 2r+\tfrac{17}{6}; x) = 1 - \frac{2(3r+1)}{(12r+5)}x\ ,
\end{aligned} \tag{4.80}
$$

where

$$
M = \frac{216(2r+1)(r+1)(3r+1)r}{(12r+11)(12r+5)^2(12r-1)} = (1728)^{-2}\sqrt{m\tilde{m}}\ . \tag{4.81}
$$

This, in turn, is a special case of the following hypergeometric identity:

$$
\begin{aligned}
&{}_2F_1(a, b; c; x)\, {}_2F_1(-a-1, -b, -c; x) \\
&\quad + \frac{ab(a+1)(b-c)x^2}{c^2(1-c^2)}\, {}_2F_1(a-c+1, b-c+1; 2-c; x)\, {}_2F_1(-a+c, 1-b+c; 2+c; x) \\
&\quad + \frac{b}{c}x = 1\ .
\end{aligned} \tag{4.82}
$$

This relation can be proven just by an analysis of the poles on both sides of the equation. Firstly, we note that the Guass hypergeometric function ${}_2F_1(a, b; c; x)$ is meromorphic in the parameter c, with simple poles at all non-positive integers. At the poles the behaviour is given by:

$$
\lim_{c \to -n} \frac{{}_2F_1(a, b; c; x)}{\Gamma(c)} = \frac{(a)_{n+1}(b)_{n+1}x^{n+1}}{(n+1)!}\, {}_2F_1(a+n+1, b+n+1; n+2; x)\ . \tag{4.83}
$$

This implies that each of the first two terms in eq. (4.82) has a single pole for all $c \in \mathbb{Z}$, $c \neq 0$. We use the identity above and see that the poles from these two terms cancel. It only remains to check the $c \to 0$ limit. We see that both these terms have both double and simple poles and the third term has a simple pole. Again it turns out that the residues cancel. Sice all the poles cancel, we conclude that the left-hand-side is a constant in c. We now choose a particular value to fix this constant. By substituting $c = b$, we find that this constant is equal to 1, independent of a, b and x.

Eq. (4.80) can be obtained from Eq. (4.82) from the following substitution $a = r$, $b = r + \frac{1}{3}$, $c = 2r + \frac{5}{6}$. From this, one can obtain the character identities eq. (4.77) by making the following substitution:

$$
x = \frac{1728}{j}\ , \quad r = -\frac{1}{2}(h - \tfrac{1}{6})\ . \tag{4.84}
$$

In terms of r, the constant term (Schellekens number) is given by:

$$\mathcal{N} = 744 - 1728\frac{2(3r+1)}{(12r+5)} \; ; \tag{4.85}$$

This expression will be an integer only for special values of r that correspond to CFT's.

Thus, we have shown that $\ell = 2$, two-character theories arise as cosets of some Schellekens theories by two-character $\ell = 0$ theories. This completes the identification of $\ell = 2$ theories.

In the next section we use eq. (4.77) to obtain the degeneracies of the ground state for the non-trivial primary.

4.3.4 Degeneracies

Although in the above discussion we used the degeneracy of primaries calculated in Ref.[3], we can actually derive these degeneracies directly from the coset construction.

The relation 4.77 furnishes an infinite set of constraints between the coefficients of various powers of q on both sides of the equation and in general, we can find the degeneracies (of the non-trivial primaries) for any n-character coset theory if we know the degeneracies of the corresponding denominator theory. For the case of two-character theories, we know m as the $\ell = 0$ theories are well-known simple algebras. We need to find $\tilde{m}$ - the degeneracy of the non-trivial primary of the coset theory. To find $\tilde{m}_1$ we only need one constraint which is obtained by equating the q^{th} order term on both sides of 4.77.We have:

$$\tilde{m} = (196884 - m_2^0 - \tilde{m}_2^0 - m_1^0\tilde{m}_1^0)/(m) \tag{4.86}$$

where m_1^0, m_2^0 are the degeneracies of the first and second excited states, respectively, of the identity character of the denominator theory. $\tilde{m}_1^0$ and $\tilde{m}_2^0$ are the corresponding values for the coset theory. In the table 4.6, we summarize the results for two-character $\ell = 2$ theories and see that they exactly agree with Table 2 of [3]. Moreover, they divide the 'apparent degeneracies' listed in Eq. (4.3).

	$\ell = 0$				$\tilde{\ell} = 2$				
No.	c	$\sqrt{m}$	m_1	Algebra	$\tilde{c}$	$\sqrt{\tilde{m}}$	$\tilde{m}_1$	$m_1 + \tilde{m}_1$	Schellekens No.
1	1	2	3	$\mathfrak{a}_1$	23	32384	69	72	$15 - 21$
2	2	$3\sqrt{2}$	8	$\mathfrak{a}_2$	22	$16038\sqrt{2}$	88	96	$24, 26 - 28$
3	$\frac{14}{5}$	7	14	$\mathfrak{g}_2$	$\frac{106}{5}$	15847	106	120	$32, 34$
4	4	$8\sqrt{3}$	28	$\mathfrak{d}_4$	20	$5120\sqrt{3}$	140	168	$42, 43$
5	$\frac{26}{5}$	26	52	$\mathfrak{f}_4$	$\frac{94}{5}$	4794	188	240	$52, 53$
6	6	$27\sqrt{2}$	78	$\mathfrak{e}_6$	18	$2187\sqrt{2}$	234	312	$58, 59$
7	7	56	133	$\mathfrak{e}_7$	17	1632	323	456	$64, 65$

Table 4.6: Two-character theories with $\ell = 0$ and $\ell = 2$. Here $c, \tilde{c}$ are the central charges,$h, \tilde{h}$ are the conformal dimensions of the primaries, $\sqrt{m}, \sqrt{\tilde{m}}$ the degeneracies of the primaries and $m_1, \tilde{m}_1$ the degeneracy of the first excited state in the identity character.

4.4 $\ell = 3$ theories

As discussed in Section 4 , the differential equation in this case is:

$$\left(\tilde{D}^2 + \mu_1 \frac{E_8}{E_6}\tilde{D} + \mu_2 E_4\right)\chi = 0 \tag{4.87}$$

Following the procedure used for $\ell = 0, 2$ we express the above equation in terms of ordinary derivatives as:

$$\left(\tilde{\partial}^2 - \frac{1}{6}E_2\tilde{\partial} + \mu_1 \frac{E_8}{E_6}\tilde{\partial} + \mu_2 E_4\right)\chi = 0 \tag{4.88}$$

In order to make the equation linear in E_k $(k = 2, 4, 6)$ we multiply the equation by E_6 and use the Ramanujan identity Eq. (1.44) to eliminate $E_2 E_6$ from the equation. Finally, we substitute the mode expansion for the character and the Eisenstein series to get:

$$\sum_{k=0}^{n}\Big[(n+\alpha-k)^2 E_{6,k} - \tfrac{1}{3}k(n+\alpha-k)E_{6,k} + \tilde{\mu}_1(n+\alpha-k)E_{8,k} + \mu_2 E_{10,k}\Big]a_{n-k} = 0 \tag{4.89}$$

where $\tilde{\mu}_1 = \mu_1 - \frac{1}{6}$.

The indicial $(n = 0)$ equation gives:

$$\alpha^2 + \tilde{\mu}_1\alpha + \mu_2 = 0 \tag{4.90}$$

From this equation it follows that:

$$\alpha_0 + \alpha_1 = -\tilde{\mu}_1$$
$$\alpha_0\alpha_1 = \mu_2$$

By using the valence identity Eq. (2.10) with $p = 2, \ell = 3$ we get the sum the two α s to be:

$$\alpha_0 + \alpha_1 = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3} \tag{4.91}$$

This and the indicial equation together imply that $\tilde{\mu}_1 = \frac{1}{3}$.

Expressing m_1 as a function of α^i we find:

$$m_1^{(i)} = \frac{24\alpha^{(i)}(15\alpha^{(i)} - 26)}{2 + 3\alpha^{(i)}} \tag{4.92}$$

At this point, it is important to reiterate that one can know a lot about the theories just by manipulating the expression of m_1. Equipped with the knowledge of previous sections and of tensor products we can now ask whether a particular one or two-character theory can 're-appear'here. As an example we can ask whether the $E_8, k = 1$ character with $\alpha = -\frac{1}{3}$ can re-appear. To check this we set $m_1 = 248$ (which is the m_1 value of $E_8, k = 1$ character) and solve for α. We get $\alpha_0 = -\frac{1}{3}, \frac{62}{15}$. The corresponding values of α_1 are 0 and $-\frac{67}{15}$. Since, α_1 also solves the same equation it could so happen that $\alpha_1 = -\frac{1}{3}, \frac{62}{15}$. This shows that the $E_8, k = 1$ character will re-appear twice corresponding to central charges $c = 0, 8$ with the tower corresponding to the other character absent (This is the because m_n^i is proportional to $\alpha_i \; \forall \; n$). The case of $(\frac{62}{15}, -\frac{67}{15})$ (unordered pair) gives negative m_n^i for $i = 0, 1$ at higher n. This shows that a lot of interesting questions can be answered even before going through the Diophantine analysis. We now re-express m_1 in terms of the central charge and try to derive more general properties. Using $\alpha_0 = -\frac{c}{24}$, we have (We have dropped the superscript to simplify the notation):

$$m_1 = \frac{c(208 + 5c)}{16 - c} \tag{4.93}$$

We can easily conclude that $c < 16$. After a couple of manipulations we find:

$$(5c)^2 + (m_1 + 208)(5c) = 80m_1 \tag{4.94}$$

This shows that $5c$ has to be an integer.

We follow the steps outlined in the previous subsections and solve the Diophantine equation to obtain:

$$m_1 = -\frac{\tilde{N}}{2} - \frac{46080}{\tilde{N}} - 368 \tag{4.95}$$

This equation tells us that $\tilde{N}$ has to be a negative even integer which is a factor 46080.For values of $160 < \tilde{N} < 576$ m_1 is negative and the set of values of m_1 for values of $\tilde{N} > 576$ and $\tilde{N} < 160$ are identical. This is because of the invarince of m_1 under $\tilde{N} \to \frac{92160}{\tilde{N}}$. Thus, we can restrict $\tilde{N}$ to the range $N \leq 160$. The allowed values of $\tilde{N}$ are:

$$\tilde{N} = -\{2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 60, 64, 72, 80, 90, 96, 120, 128, 144, 160\} \tag{4.96}$$

We find the relevant α values and subject these candidates to the same tests as in the cases of $\ell = 0, 2$. At the first level, ruling out the theories for which $m_1^{(1)}$ is negative, we are left with $\tilde{N} = -\{80, 90, 96, 120, 128, 144, 160\}$. Next we check the first few $m_n^{(0)}$ and eliminate theories for which the value is fractional (or negative). The cases $\tilde{N} = -128, -144$ and $\tilde{N} = -90, -96$ get eliminated at level 2 and 3 respectively. This leaves us with three candidates- $\tilde{N} = -80, 120, 160$. These correspond to central charges $c = 0, 4, 8$ respectively. We immediately notice that the character $j(\tau)^{\frac{1}{3}}$ of the $E_8, k = 1$ theory has indeed re-appeared twice!. Once as the non-identity character in the $c = 0$ case and once as the identity character of $c = 8$ theory. As discussed before, in both cases the tower corrresponding to the other character is absent. The remaining case can be analyzed right away. The $c = 4$ case gives $h = 0$ and hence $\alpha_0 = \alpha_1 = -\frac{1}{6}$and the two characters have identical towers and are essentially the same. This looks unphysical. On the other hand, one cannot treat this as the case of a single character appearing twice as there is no single-character CFT with $c = 4$.

Thus, we conclude that there are no new two-character CFT's with $\ell = 3$.

4.5 $\ell = 4$ theories

The case of $\ell = 4$ is qualitatively different from the previous cases. In the previous cases we noted that the indicial equation would fix the unknown parameter and there would be no free parameters $m_1^i(\alpha_i)$ onwards. Here, however we find that there are more free parameters and that the indicial equation has lesser roots than the number of free parameters. We also find that the Diophantine analysis for this is not straightforward.

We use the differential equation Eq. (4.6) . We express it in terms of ordinary derivatives and multiply by E_4^2 (Recall that $E_4^2 = E_8$) to get:

$$\left(E_8\tilde{\partial}^2 - \tfrac{1}{6}E_2E_4^2\tilde{\partial} + \mu_1 E_{10}\tilde{\partial} + \nu E_4^3 + \mu\Delta\right)\chi = 0 \tag{4.97}$$

Here we have made use of the fact that the space of modular forms of weight 12 is spanned by E_4^3 and E_6^2, and have chosen the linear combinations Δ (defined in Eq. (1.41)) as this has no constant term in the q-expansion.

We use the Ramanujan identity Eq. (1.44) to linearise the differential equation to get:

$$\left(E_8\tilde{\partial}^2 - \tfrac{1}{4}\tilde{\partial}E_8\tilde{\partial} + (\mu_1 - \tfrac{1}{6})E_{10}\tilde{\partial} + \nu E_4^3 + \mu\Delta\right)\chi = 0 \tag{4.98}$$

The above differential equation is linear in the Eisenstein series and consequently the recursion relations are much easier to handle.

We substitute the the q-expansion of χ and the Eisenstein series to obtain:

$$\sum_{k=0}^{n}\left[(n+\alpha-k)^2E_{8,k} + \left((\mu_1 - \tfrac{1}{6})E_{10,k} - \tfrac{1}{4}kE_{8,k}\right)(n+\alpha-k) + \left(\nu(E_4^3)_{,k} + \mu\Delta_{,k}\right)\right]a_{n-k} = 0 \tag{4.99}$$

The indicial equation we get is:

$$\alpha^2 + (\mu_1 - \tfrac{1}{6})\alpha + \nu = 0 \tag{4.100}$$

Now, we use the valence formula Eq. (2.10) with $\ell = 4$ to fix the parameter μ_1. We have $\mu_1 = \frac{2}{3}$.

From the indicial equation we have:

$$\begin{aligned}\alpha_0 + \alpha_1 &= -\frac{1}{2}\\ \alpha_0\alpha_1 &= \nu\end{aligned} \tag{4.101}$$

The $n = 1$ equation gives:

$$m_1 = \frac{24\alpha(20\alpha+51)}{4\alpha+3} - \frac{2\mu}{4\alpha+3} \tag{4.102}$$

We notice that there is an arbitrary parameter μ left and will need an other constraint, namely m_2 , to fix it. In this sense, the analysis of $\ell \geq 4$ two-character theories is similar to that of three-character $\ell = 0$ theories discussed in the next section.

We also note that μ is a rational number and scanning the space of two rational numbers (α, μ) will lead to infinite set of possibilities. This is so because if we find a pair (α, μ) for which m_1 is an integer, then the pair $(\alpha, \mu - n\frac{4\alpha+3}{2})$ gives the integer $m_1 + n$, for any integer n. Thus, the Diophantine analysis which worked for $\ell = 0, 2, 3$ will not work in this case. Let us, nevertheless, analyse the indicial equation. By demanding the descriminant (as described in the previous sections) to be zero we find:

$$93636 + 240\mu - 252m_1 + m_1^2 \tag{4.103}$$

Since μ is a rational number we cannot demand the above expression to be the square of an integer in general. However, if we arbitrarily set $\mu = 0$ the familiar Diophantine analysis applies and we get:

$$m_1 = \frac{38880}{\tilde{N}} + 126 - \frac{\tilde{N}}{2} \tag{4.104}$$

This shows that $\tilde{N}$ must be an even integer that divides 38880. As before, we find a symetry here. Under $\tilde{N} \to -\frac{77760}{\tilde{N}}$ m_1 is invariant. Thus, we restrict the possible values of $\tilde{N}$ to the following values:

$$\begin{aligned}\tilde{N} = \{&2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 54, 60, 72, 80, 90,\\ &96, 108, 120, 144, 160, 162, 180, 216, 240, 270, 288, 324, 360, 432\}\end{aligned} \tag{4.105}$$

It turns out that all of them are ruled out by level 10 (see Ref.[4]). Thus, we see that for the case of $\mu = 0$ there are no $\ell = 4$ theories.

One can do slightly better than such an arbitrary choice of μ by following the reasoning that has been repeatedly stressed in the previous subsections and ask if CFT's with certain properties can occur as solutions to the $\ell = 4$ equation. It turns out that there are interesting answers to such questions. Let us consider the tensor product of a two-character and a one-character theory and use Eq. (2.13) with $(p, p') = (1.2)$ to get:

$$\tilde{\ell} = 2\ell + \ell' \tag{4.106}$$

Since $\ell = 0$ or $\ell \geq 2$, the only solution to the above equation for $\tilde{\ell} = 4$ is $(\ell, \ell') = (2, 0)$. Thus, we can choose the $j^{\frac{1}{3}}$ character which has $\ell = 2$ (corresponding to E_8 at level 1) and tensor it with any two-character theory with $\ell = 0$. For any such two-character theory with the identity exponent α_0, central charge c and the degeneracy of the excited state above identity m_1, this tensoring gives $\alpha_0 \to \alpha_0 - \frac{1}{3}$, $c \to c + 8$ and $m_1 \to m_1 + 248$, where 248 is the dimension of E_8 . Accordingly, it turns out that for $\mu = -384$ Eq. (4.102) reduces to Eq. (4.13) on making the transformation $m_1 \to m_1 + 248$ and $\alpha_0 \to \alpha_0 - \frac{1}{3}$.

As promised we now return to the case of general μ and show that we can use m_2 to fix μ. The recursion relation for $n = 2$ gives:

$$m_2 = \frac{2(264\mu + \mu^2 + 142938\alpha - 216\mu\alpha + 284976\alpha^2 - 480\mu\alpha^2 + 298080\alpha^3 + 57600\alpha^4)}{(3+4\alpha)(5+4\alpha)} \tag{4.107}$$

Eliminating μ between the above equation and Eq. (4.102) we get:

$$m_2 = \frac{406008\alpha + 246240\alpha^2 - 528m_1 - 2016\alpha m_1 + 3m_1^2 + 4\alpha m_1^2}{2(5+4\alpha)} \tag{4.108}$$

This equation has two integer parameters m_1 and m_2 and is quadratic in α unlike the previous one. Solving this equation for m_2 and demanding th discriminant to be an integer we get:

$$(-406008 + 2016m_1 - 4m_1^2 + 8m_2)^2 + 984960(528m_1 - 3m_1^2 + 10m_2) = N^2 \tag{4.109}$$

Thus, using m_2 we do get a Diophantine equation (with two parameters m_1, m_2).

Chapter 5

Three character theories

5.1 $\ell = 0$ theories

The case of three character theories is qualitatively different from the cases of two-character theories that have been discussed in the earlier sections. The formalism of classification extends directly to this case. However, here, m_1 is a function of two α s and the Diophantine analysis becomes complicated . However, one can conclude a couple of general properties of three character theories.

Unlike the case of two-character theories where the number of theories is finite for $\ell = 0, 2, 3$ there are infinitely many three-character, $\ell = 0$ theories. These were first reported in [2]. In this section we revise the results and analysis of [2] and describe our recent work ([5]) that has led to the discovery of new three-character theories using coset construction. We restrict ourselves to a discussion of $\ell = 0$ theories.

For theories with three independent characters and $\ell = 0$ the modular invariant differential equation takes the form:

$$\left(D_\tau^3 + \pi^2\mu_1 E_4 D_\tau + i\pi^3\mu_2 E_6\right)\chi(\tau) = 0 \tag{5.1}$$

In terms of ordinary derivatives the above equation can be written as:

$$\left(\partial_\tau^3 - \frac{i\pi}{3}(\partial_\tau E_2)\partial_\tau - i\pi E_2\partial_\tau^2 - \frac{2\pi^2}{9}E_2^2\partial_\tau + \mu_1\pi^2 E_4\partial_\tau + i\mu_2\pi^3 E_6\right)\chi = 0 \tag{5.2}$$

Here we have made use of the fact that there is no modular form of weight two. After substituting for E_2^2 using the Ramanujan identity:

$$\frac{1}{2i\pi}[\partial_\tau E_2] = \frac{E_2^2 - E_4}{12} \tag{5.3}$$

we find:

$$\left(\partial_\tau^3 + i\pi(\partial_\tau E_2)\partial_\tau - i\pi * E_2\partial_\tau^2 - \frac{2\pi^2}{9}E_4\partial_\tau + \mu_1\pi^2 E_4\partial_\tau + i\mu_2\pi^3 E_6\right)\chi = 0 \tag{5.4}$$

By substituting the mode expansions we get the recursion relation:

$$\begin{aligned} &-8(n+\alpha)^3 a_n - 4\sum_{k=0}^{k=n} E_{2,k}a_{n-k}k(n-k+\alpha) + 4\sum_{k=0}^{k=n}(n-k+\alpha)^2 a_{n-k}E_{2,k} \\ &-\frac{4}{9}\sum_{k=0}^{k=n} E_{4,k}(n-k+\alpha)a_{n-k} + 2\mu_1\sum_{k=0}^{k=n}(n-k+\alpha)E_{4,k}a_{n-k} + \mu_2\sum_{k=0}^{k=n} E_{6,k}a_{n-k} = 0 \end{aligned} \tag{5.5}$$

For $n = 0$ and $n = 1$ we get the following polynomial equations in α:

$$-8\alpha^3 + 4\alpha^2 + -\frac{4}{9}\alpha + 2\mu_1\alpha + \mu_2 = 0 \tag{5.6}$$

and

$$a_1[-24\alpha^2 - 16\alpha - \frac{40}{9} + 2\mu_1] + a_0[-4E_{2,1}\alpha + 4\alpha^2 E_{2,1} - \frac{4}{9}E_{4,1}\alpha + \mu_2 E_{6,1} + 2\mu_1 E_{4,1}\alpha] = 0 \tag{5.7}$$

From these equations we immediately see that:

$$2\mu_1 = \frac{4}{9} - 8(\alpha_0\alpha_1 + \alpha_1\alpha_2 + \alpha_0\alpha_2), \qquad \mu_2 = 8\alpha_0\alpha_1\alpha_2 \tag{5.8}$$

Using these two equations and substituting for the Fourier coefficients of the Eisenstien series (see appendix) we get,

$$m_1^{(i)} = \frac{24\alpha_i(20\alpha_i^2 + (62\alpha_j - 11)\alpha_i + 62\alpha_j^2 - 31\alpha_j + 1)}{(\alpha_i - \alpha_j + 1)(4\alpha_i + 2\alpha_j + 1)}, j \neq i \tag{5.9}$$

It is a remarkable fact that the denominator has factorised into a product of two linear factors. In fact, is a general feature of the degeneracies even at higher levels. To see this, we note that at each level the extra factor in the denominator comes from the coefficient of a_n in the recursion relation 5.5. Let us denote the coefficient of a_n by A_n and the denominator of m_n as D_n (We have dropped superscripts for notational simplification). Thus,

$$D_n = A_n A_{n-1} \cdots A_1 \tag{5.10}$$

We substitute $k = 0$ in Eq. (5.5) to get the coefficient of a_n and after simplification we find:

$$A_n = -4n(n + \alpha_i - \alpha_j)(4\alpha_i + 2\alpha_j + 2n - 1) \tag{5.11}$$

Let us try to find more properties of the expression m_1 in 5.9. Suppose we denote the right-hand-side by $F(\alpha_0, \alpha_1)$. Then this function has the property:

$$F(\alpha_0, \alpha_1) = F\left(\alpha_0, \frac{1}{2} - \alpha_0 - \alpha_1\right) = F(\alpha_0, \alpha_2) \tag{5.12}$$

reflecting the fact that μ_1, μ_2 are symmetric functions of the α_i. The same formula determines the degeneracies above the non-identity characters. All we have to do is insert $\alpha = \alpha_i$, $i \neq 0$ instead of $\alpha = \alpha_0$. The results can be summarised as:

$$m_1^{(i)} = \frac{24\alpha_i\left(20\alpha_i^2 + (62\alpha_j - 11)\alpha_i + 62\alpha_j^2 - 31\alpha_j + 1\right)}{(\alpha_i - \alpha_j + 1)(4\alpha_i + 2\alpha_j + 1)}, \quad j \neq i \tag{5.13}$$

Note that after fixing i, the above holds for *both* the values $j \neq i$. The property of Eq. (5.12) ensures that we get the same answer with either choice.

As a check of the above equations, we consider the SO(N) WZW models at level $k = 1$. These are all 3-character theories with $c = \frac{N}{2}$ [2]. (In the subsequent discussion we shall prove that this family is a solution to Eq. (5.2)). We have:

$$\alpha_0 = -\frac{N}{48}, \quad \alpha_1 = -\frac{N}{48} + \frac{1}{2}, \quad \alpha_2 = \frac{N}{24} \tag{5.14}$$

Inserting these values into Eq. (5.13), we find:

$$m_1^{(0)} = \frac{N(N-1)}{2}, \quad m_1^{(1)} = \frac{N^2 - 3N + 8}{6}, \quad m_1^{(2)} = N \tag{5.15}$$

The numerator of Eq. (5.9) does not factorize in general. Then, one might naively think that a given set of exponents that correspond to a valid CFT's may render one of the factors in the denominator zero and cause m_1 to blow up. However, for such values of the exponents α , the numerator factorizes and cancels the factor from the denominator so that there is no pole. In fact, this happens for the $SO(N)$ models discussed above.

After a discussion of the various properties of m_1 let us shift our focus back on finding candidate CFT's. The form of 5.9 makes it obvious that the Diophantine analysis is not straightforward any more, as one has to scan a space of two parameters. We obtain m_2 by substituting $n = 2$ in Eq. (5.5). After some manipulation we obtain (We have dropped the superscripts on m_n):

$$
\begin{aligned}
N^4 &+ (m_1^2 + 93m_1 - 2m_2 + 955)N^3 - (2380m_1^2 - 28770m_1 - 7700m_2 - 167160)N^2 \\
&+ (1372000m_1^2 - 9800m_1m_2 + 10760400m_1 - 7330400m_2)N \\
&- (32980000m_1^2 - 13720000m_1m_2) = 0
\end{aligned}
$$

where $N = -1680\alpha_0$ is rational. If we substitute $N = \frac{p}{q}$ with p and q co-prime, we see that the first term cannot be cancelled by any other term except when $q = 1$. Therefore, N is an integer[2]. One can proceed and try to derive a Diophantine equation using Eq. (5.9) similar to what was done for $\ell = 4$ theories in the two-character case.

5.2 New three-character theories from Coset construction

In this section we show that the generalized coset construction described in 4.3.2 can be used to construct new three-character, $\ell = 0$ CFT's.Recall from Eq. (4.76) that the coset of a Schellekens theory by an affine three-character $\ell = 0$ theory gives another three-character theory with $\ell = 0$ (not necessarily affine). We consider affine $\ell = 0$, three-character theories $\mathcal{D}$ whose chiral algebra is a direct summand of any of the 71 self-dual $c = 24$ theories $\mathcal{H}$ in [11]. We shall see the new theories we find will be solutions to Eq. (5.2) and the character will satisfy the integrality conditions.

Let us illustrate this with an example and take $\mathcal{D} = \mathfrak{a}_3$ theory at level 1. This is a 3-character theory and solves an equation of the type Eq. (5.2). It has central charge $c = 3$, and conformal dimensions $(h_1, h_2) = (\frac{3}{8}, \frac{1}{2})$. This theory is contained, as a direct summand, in the theory number 30 in Schellekens' list ([11]). The latter, self-dual theory is $\mathfrak{a}_3^{\oplus 8}$. Since this is the current algebra of the theory, the corresponding Schellekens number, $\mathcal{N} = 120$. (The dimension of $\mathfrak{a}_3$ is 15).

Following the discussion in 4.3.2, we know that the coset theory will have the current algebra $\mathfrak{a}_3^{\oplus 7}$ of dimension 105, central charge $\tilde{c} = 24 - 3 = 21$ and $n_i = 2$. Thus, the conformal dimensions of the coset are $(\tilde{h}_1, \tilde{h}_2) = (\frac{13}{8}, \frac{3}{2})$. This, along with the fact that the coset has $\ell^{\mathcal{C}} = 0$ is then sufficient to fix the differential equation satisfied by the characters of the coset theory completely.and as a consequence allows one to calculate the characters (as solutions of the modular differential equation). Consequently, we can calculate the degeneracies to arbitrary levels for each characters. This is indeed true and the characters satisfy the required consistency conditions to very high orders

n 5000. In particular, the first level above the identity character gives the dimension of the current algebra, 105.

The remaining cases follow exactly like the example discussed and our results for the three-character theories has been summarised in table 5.1. All the coset theories are non-affine theories (not just generated by the affine currents), except the theories in lines 12 and 13 of the table. In these cases two affine theories, namely $\mathfrak{e}_{8,2}$ and $\mathfrak{b}_{8,1}$, arise as cosets of each other. Except for these two cases all other examples seem to be new.

	$\mathcal{D}$					$\mathcal{C}$					
No.	c	h_1	h_2	m_1	Algebra	$\tilde{c}$	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{m}_1$	$m_1+\tilde{m}_1$	Schellekens No.
1	$\frac{3}{2}$	$\frac{3}{16}$	$\frac{1}{2}$	3	$\mathfrak{a}_{1,2}$	$\frac{45}{2}$	$\frac{29}{16}$	$\frac{3}{2}$	45	48	$5,7,8,10$
2	$\frac{5}{2}$	$\frac{5}{16}$	$\frac{1}{2}$	10	$\mathfrak{c}_{2,1}$	$\frac{43}{2}$	$\frac{27}{16}$	$\frac{3}{2}$	86	96	$25,26,28$
3	3	$\frac{3}{8}$	$\frac{1}{2}$	15	$\mathfrak{a}_{3,1}$	21	$\frac{13}{8}$	$\frac{3}{2}$	105	120	$30,31,33-35$
4	$\frac{7}{2}$	$\frac{7}{16}$	$\frac{1}{2}$	21	$\mathfrak{b}_{3,1}$	$\frac{41}{2}$	$\frac{25}{16}$	$\frac{3}{2}$	123	144	$39,40$
5	4	$\frac{2}{5}$	$\frac{3}{5}$	24	$\mathfrak{a}_{4,1}$	20	$\frac{8}{5}$	$\frac{7}{5}$	120	144	$37,40$
6	$\frac{9}{2}$	$\frac{9}{16}$	$\frac{1}{2}$	36	$\mathfrak{b}_{4,1}$	$\frac{39}{2}$	$\frac{23}{16}$	$\frac{3}{2}$	156	192	$47,48$
7	5	$\frac{5}{8}$	$\frac{1}{2}$	45	$\mathfrak{d}_{5,1}$	19	$\frac{11}{8}$	$\frac{3}{2}$	171	216	49
8	$\frac{11}{2}$	$\frac{11}{16}$	$\frac{1}{2}$	55	$\mathfrak{b}_{5,1}$	$\frac{37}{2}$	$\frac{21}{16}$	$\frac{3}{2}$	185	240	53
9	6	$\frac{3}{4}$	$\frac{1}{2}$	66	$\mathfrak{d}_{6,1}$	18	$\frac{5}{4}$	$\frac{3}{2}$	198	264	$54,55$
10	$\frac{13}{2}$	$\frac{13}{16}$	$\frac{1}{2}$	78	$\mathfrak{b}_{6,1}$	$\frac{35}{2}$	$\frac{19}{16}$	$\frac{3}{2}$	210	288	56
11	7	$\frac{7}{8}$	$\frac{1}{2}$	91	$\mathfrak{d}_{7,1}$	17	$\frac{9}{8}$	$\frac{3}{2}$	221	312	59
12	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	$\mathfrak{b}_{8,1}$	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	384	62
13	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	$\mathfrak{e}_{8,2}$	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	384	62
14	9	$\frac{9}{8}$	$\frac{1}{2}$	153	$\mathfrak{d}_{9,1}$	15	$\frac{7}{8}$	$\frac{3}{2}$	255	408	63
15	10	$\frac{5}{4}$	$\frac{1}{2}$	190	$\mathfrak{d}_{10,1}$	14	$\frac{3}{4}$	$\frac{3}{2}$	266	456	64

Table 5.1: Three-character theories with $\ell = 0$. Here $c, \tilde{c}$ are the central charges, $h_1, h_2, \tilde{h}_1, \tilde{h}_2$ the conformal dimensions of the primaries and $m_1, \tilde{m}_1$ the degeneracy of the first excited state in the identity character.

5.3 Degeneracies for three character theories

We now calculate the degeneracies of the ground states for the non-trivial characters of the new coset theories listed in 5.1. In this section, whenever we refer to just degeneracies, we mean the degeneracies of the ground state of the non-identity characters.

We follow the steps outlined for the two-character, $\ell = 2$ case in section 4.3.4. The only difference in the case of three-character theories is that we need two constraints to find out the degeneracies of the two non-trivial primaries. We equate the coefficients of q^0 and q^1 in Eq. (4.77) to get:

$$D_1\tilde{D}_1 + D_2\tilde{D2} + m_2^0 + m_1^0\tilde{m}_1^0 + \tilde{m}_2^0 = 196884 \quad (5.16)$$

$$m_3^{(0)}+D_1\tilde{D}_1m_1^{(1)}+D_2\tilde{D}_2m_1^{(2)}+m_2^{(0)}\tilde{m}_1^{(0)}+m_1^{(0)}\tilde{m}_2^{(0)}+\tilde{m}_3^{(0)}+D_1\tilde{D}_1\tilde{m_1^{(1)}}+D_2\tilde{D}_2\tilde{m_1^{(2)}} = 21493760 \quad (5.17)$$

where m_n^i are the degeneracies at level n for the i^{th} character. D_1, D_2 are the degeneracies of the non-trivial characters of the denominator theory and $\tilde{D}_1, \tilde{D}_2$ are the corresponding values for the coset theory.

We solve the above equations for $\tilde{D}_1, \tilde{D}_2$ to get:

$$\begin{aligned}\tilde{D}_1 =&(21493760 - m_3^{(0)} - 196884m_1^{(2)} + m_2^{(0)}m_1^{(2)} - m_2^{(0)}\tilde{m}_1^{(0)} \\ &+ m_1^{(0)}m_1(2)\tilde{m}_1^{(0)} - m_1^{(0)}\tilde{m}_2 + m_1^{(2)}\tilde{m}_2^{(0)} - \tilde{m}_3^{(0)} - 196884\tilde{m}_1^{(2)} + \\ &m_2^{(0)}\tilde{m}_1^{(2)} + m_1^{(0)}\tilde{m}_1^{(0)}\tilde{m}_1^{(2)} + \tilde{m}_2^{(0)}\tilde{m}_1^{(2)})/D_1(m_1^{(1)} - m_1^{(2)} + \tilde{m}_1^{(1)} - \tilde{m}_1^{(2)})\end{aligned}$$

$$\tilde{D}_2 = \frac{196884 - D_1\tilde{D}_1 - m_2^{(0)} - m_1^{(0)}\tilde{m}_1^{(0)} - \tilde{m}_2^{(0)}}{D_2} \quad (5.18)$$

Using the above two equations we find the degeneracies for the new coset theories with 3-character theories. The degeneracies for the affine theories D_1, D_2 are well-known (for example see [14]). Our results are summarised in table 5.2

	$\mathcal{D}$				$\mathcal{C}$				
No.	c	D_1	D_2	Algebra	$\tilde{c}$	$\tilde{D}_1$	$\tilde{D}_2$	$m_1 + \tilde{m}_1$	Schellekens No.
1	$\frac{3}{2}$	2	3	$\mathfrak{a}_{1,2}$	$\frac{45}{2}$	46,080	4785	48	$5, 7, 8, 10$
2	$\frac{5}{2}$	4	5	$\mathfrak{c}_{2,1}$	$\frac{43}{2}$	22,016	5031	96	$25, 26, 28$
3	3	4	6	$\mathfrak{a}_{3,1}$	21	21,504	5096	120	$30, 31, 33-35$
4	$\frac{7}{2}$	8	7	$\mathfrak{b}_{3,1}$	$\frac{41}{2}$	$10,496$	5125	144	$39, 40$
5	4	5	10	$\mathfrak{a}_{4,1}$	20	16,250	5000	144	$37, 40$
6	$\frac{9}{2}$	16	9	$\mathfrak{b}_{4,1}$	$\frac{39}{2}$	4992	5083	192	$47, 48$
7	5	16	10	$\mathfrak{d}_{5,1}$	19	4864	5016	216	49
8	$\frac{11}{2}$	32	11	$\mathfrak{b}_{5,1}$	$\frac{37}{2}$	2368	4921	240	53
9	6	32	12	$\mathfrak{d}_{6,1}$	18	2304	4800	264	$54, 55$
10	$\frac{13}{2}$	64	13	$\mathfrak{b}_{6,1}$	$\frac{35}{2}$	1120	4655	288	56
11	7	64	14	$\mathfrak{d}_{7,1}$	17	1088	4488	312	59
12	$\frac{17}{2}$	256	17	$\mathfrak{b}_{8,1}$	$\frac{31}{2}$	248	3875	384	62
13	$\frac{31}{2}$	248	3875	$\mathfrak{e}_{8,2}$	$\frac{17}{2}$	256	17	384	62
14	9	256	18	$\mathfrak{d}_{9,1}$	15	240	3640	408	63
15	10	1024	20	$\mathfrak{d}_{10,1}$	14	56	3136	456	64

Table 5.2: Three-character theories with $\ell = 0$. Here $c, \tilde{c}$ are the central charges, $D_1, D_2, \tilde{D}_1, \tilde{D}_2$ the degeneracies of the primaries and $m_1, \tilde{m}_1$ the degeneracy of the first excited state in the identity character.

Chapter 6

Four Character theories

For theories with four independent characters and $\ell = 0$ the modular invariant differential equation takes the form:

$$\left(D_\tau^4 + \mu_1\pi^2 E_4 D_\tau^2 + i\mu_2\pi^3 E_6 D_\tau + \mu_3\pi^4 E_8\right)\chi = 0 \tag{6.1}$$

Now we express D_τ^4 in terms of ordinary derivatives:

$$\begin{aligned} D_\tau^4 &= (\partial_\tau - i\pi E_2)\left(\partial_\tau^3 - \frac{i\pi}{3}(\partial_\tau E_2)\partial_\tau - i\pi E_2\partial_\tau^2 - \frac{2\pi^2}{9}E_2^2\partial_\tau\right) \\ &= \partial_\tau^4 - (2i\pi)E_2\partial_\tau^3 - \frac{4i\pi}{3}(\partial_\tau E_2)\partial_\tau^2 - \frac{11\pi^2}{9}E_2^2\partial_\tau^2 - \frac{7\pi^2}{9}E_2\partial_\tau E_2\partial_\tau - \frac{i\pi}{3}\partial_\tau^2 E_2\partial_\tau + \frac{i2\pi^3}{9}E_2^3\partial_\tau \end{aligned} \tag{6.2}$$

Inserting this and lower powers of D_τ into the equation, we find:

$$\begin{aligned} &\left(\partial_\tau^4 - (2i\pi)E_2\partial_\tau^3 - \frac{4i\pi}{3}(\partial_\tau E_2)\partial_\tau^2 - \frac{11\pi^2}{9}E_2^2\partial_\tau^2 - \frac{7\pi^2}{9}E_2\partial_\tau E_2\partial_\tau - \frac{i\pi}{3}\partial_\tau^2 E_2\partial_\tau\right. \\ &\left.+\frac{i2\pi^3}{9}E_2^3\partial_\tau + \mu_1 E_4\pi^2\left[\partial_\tau^2 - \frac{i\pi}{3}E_2\partial_\tau\right] + i\mu_2\pi^3 E_6\partial_\tau + \mu_3\pi^4 E_8\right)\chi = 0 \end{aligned} \tag{6.3}$$

Using Ramanujan identities this can be converted into an equation that is linear in all the Eisenstein series. The identities are:

$$\begin{aligned} \frac{1}{2i\pi}[\partial_\tau E_2] &= \frac{E_2^2 - E_4}{12} \\ \frac{1}{2i\pi}[\partial_\tau E_4] &= \frac{E_2 E_4 - E_6}{3} \end{aligned} \tag{6.4}$$

and the resulting linearised equation is:

$$\left(\partial_\tau^4 - (2i\pi)E_2\partial_\tau^3 + 6i\pi(\partial_\tau E_2)\partial_\tau^2 - \frac{11\pi^2}{9}E_4\partial_\tau^2 + \frac{11\pi^2}{18}\partial_\tau E_4\partial_\tau - 2i\pi\partial_\tau^2 E_2\partial_\tau + \frac{i2\pi^3}{9}E_6\partial_\tau \right. \\ \left. +\mu_1\left[\pi^2 E_4\partial_\tau^2 - \frac{i\pi^3}{3}E_6\partial_\tau - \frac{\pi^2}{2}\partial_\tau E_4\partial_\tau\right] + i\mu_2\pi^3 E_6\partial_\tau + \mu_3\pi^4 E_8\right)\chi = 0 \tag{6.5}$$

Inserting the power series expansions for $E_2, E_4, E_6\ \chi$, we get:

$$16(n+\alpha)^4 a_n - 16\sum_{k=0}^{n} E_{2,k}(n-k+\alpha)\Big((n-k+\alpha)^2 - 3k(n-k+\alpha) + k^2\Big)a_{n-k} \\ - 4\tilde{\mu}_1\sum_{k=0}^{n} E_{4,k}(n-k+\alpha)\Big(n - \tfrac{3}{2}k + \alpha\Big)a_{n-k} - 2\tilde{\mu}_2\sum_{k=0}^{n} E_{6,k}(n-k+\alpha)a_{n-k} \\ + \mu_3\sum_{k=0}^{n} E_{8,k}a_{n-k} = 0 \tag{6.6}$$

Specialising to $n = 0$, we get the indicial equation:

$$16\alpha^4 - 16\alpha^3 + \frac{44}{9}\alpha^2 - \frac{4}{9}\alpha - 4\alpha^2\mu_1 + \frac{2}{3}\mu_1\alpha - 2\mu_2\alpha + \mu_3 = 0 \tag{6.7}$$

From this, it follows that $\sum_{i=0}^{3}\alpha_i = 1$ and:

$$\tilde{\mu}_1 = -4\sum_{i<j=0}^{3}\alpha_i\alpha_j, \quad \tilde{\mu}_2 = 8\sum_{i<j<k=0}^{3}\alpha_i\alpha_j\alpha_k, \quad \mu_3 = 16\,\alpha_0\alpha_1\alpha_2\alpha_3 \tag{6.8}$$

We shall not go into the detailed analysis of four-character theories from the viewpoint of the modular invariant differential equation.However, wee note that the four-character case is qualitatively similar to that of three characters. Even here we can see that the denominator factorizes at each level and we can figure out the factor that first appears at the n^{th} level by substituting $k = 0$ in Eq. (6.6). We find:

$$A_n = 16n(n+\alpha_i-\alpha_j)(n+\alpha_i-\alpha_k)(-1+n+2\alpha_i+\alpha_j+\alpha_k) \tag{6.9}$$

In the next section we discuss how new $\ell = 0$, four-character theories can be obtained using the generalized coset construction of section 4.3.2.

6.1 New four-character theories from coset construction

We follow the procedure outlined in the case of three-character theories. We consider affine $\ell = 0$, four-character theories $\mathcal{D}$ whose chiral algebra is a direct summand of any of the 71 self-dual $c = 24$ theories $\mathcal{H}$ in [11]. We find theories that are solutions to 6.5 and the characters satisfy the integrality conditions.

	$\mathcal{D}$						$\mathcal{C}$						
No.	c	h_1	h_2	h_3	m_1	Algebra	$\tilde{c}$	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{h}_3$	$\tilde{m}_1$	$m_1 + \tilde{m}_1$	Schellekens No.
1	$\frac{16}{5}$	$\frac{4}{15}$	$\frac{2}{3}$	$\frac{3}{5}$	8	$\mathfrak{a}_{2,2}$	$\frac{104}{5}$	$\frac{26}{15}$	$\frac{4}{3}$	$\frac{7}{5}$	52	60	$13, 14$
2	$\frac{14}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{7}{9}$	14	$\mathfrak{g}_{2,2}$	$\frac{58}{3}$	$\frac{5}{3}$	$\frac{4}{3}$	$\frac{11}{9}$	58	72	21
3	$\frac{21}{5}$	$\frac{7}{20}$	$\frac{3}{5}$	$\frac{3}{4}$	21	$\mathfrak{c}_{3,1}$	$\frac{99}{5}$	$\frac{33}{20}$	$\frac{7}{5}$	$\frac{5}{4}$	99	120	33
4	5	$\frac{5}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	35	$\mathfrak{a}_{5,1}$	19	$\frac{19}{12}$	$\frac{4}{3}$	$\frac{5}{4}$	133	168	$43 - 45$

Table 6.1: Four-character theories with $\ell = 0$. Here $c, \tilde{c}$ are the central charges, $h_1, h_2, h_3, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ the conformal dimensions of the primaries and $m_1, \tilde{m}_1$ the degeneracy of the first excited state in the identity character.

6.1.1 Ground state degeneracies from coset construction

The degeneracies of the ground stated of non-trivial primaries for the four-character coset theories follows from Eq. (4.77) analogous to the three-character case, except now we need 3 constraints. These are obtained by equating the coefficients of q, q^2, q^3 order terms, respectively, on both sides of Eq. (4.73). For the degeneracies of the denominator theory we refer to [14]. Our results are tabulated in table 6.2.

	$\mathcal{D}$					$\mathcal{C}$					
No.	c	D_1	D_2	D_3	Algebra	$\tilde{c}$	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$	$m_1 + \tilde{m}_1$	Schellekens No.
1	$\frac{16}{5}$	3	6	8	$\mathfrak{a}_{2,2}$	$\frac{104}{5}$	38,556	2106	2652	60	$13, 14$
2	$\frac{14}{3}$	7	14	27	$\mathfrak{g}_{2,2}$	$\frac{58}{3}$	3190	8294	1044	72	21
3	$\frac{21}{5}$	6	$14\sqrt{2}$	21	$\mathfrak{c}_{3,1}$	$\frac{99}{5}$	14,280	$1683\sqrt{2}$	528	120	33
4	5	6	15	20	$\mathfrak{a}_{5,1}$	19	12,960	3078	912	168	$43 - 45$

Table 6.2: Four-character theories with $\ell = 0$. Here $c, \tilde{c}$ are the central charges, $D_1, D_2, D_3, \tilde{D}_1, \tilde{D}_2, \tilde{D}_3$ the degeneracies of the primaries and $m_1, \tilde{m}_1$ the degeneracy of the first excited state in the identity character.

We won't explore higher character theories. The case of five-character theories is similar to the discussion in this section. With this section, we conclude our discussion of classification of RCFT's theories with low number of characters.

Chapter 7

Summary and Conclusion

In this thesis, we have discussed our finding of new RCFT's with $2, 3$ and 4 characters. Our search for RCFT's was based on the modular differential equation satisfied by the characters of RCFT's. We found 7 new, two-character theories with $\ell = 2$. These theories had striking resemblance with the two-character $\ell = 0$ theories in terms of central charges and conformal dimensions of the primaries. Using our novel generalized coset construction, which involves taking the coset of a non-affine theory by an affine theory, we could identify these theories as the cosets of some $c = 24$, one-character theories by the well-known $\ell = 0$ theories with two-characters.

We used the generalized coset construction to find 15 new, non-affine 3-character theories and 4 new four-character theories. The coset implies the identity Eq. (4.77) at the level of characters, using which we were successful in finding the ground state degeneracies of the new theories. It is important to note that this is a non-trivial result as the differential equation approach cannot give these degeneracies. Our coset construction can also act as a tool to construct many non-affine theories.

In addition to finding new RCFT's we have reviewed the method of modular differential equations and improved the understanding of the topic by discussing the behaviour of the parameter ℓ under tensoring.

We shall now summarize the modular differential equation method of classifying CFT's. This classification is based on two parameters- n -number of independent characters and ℓ the number of poles of the Wronskian of characters. We fix n and ℓ and this, in turn, fixes the differential equation up to some constants. Then, we substitute the mode expansion for the character and the Eisenstein series to get a recursion relation. The indicial equation fixes one of the parameters. By demanding that m_1^0 is an integer we get a Diophantine equation for two-character theories till $\ell = 3$. For $\ell \geq 4$ or for higher character theories one needs more constraints to get a Diophantine equation. We analyse the Diophantine equation to find the set of candidate theories. In cases where this list is finite, we check m_n^i for all candidates and rule out cases that give fractional or negative value of m_n^0. For the other characters we only demand the m_n^i to be non-negative rational numbers. We reject cases where the denominator does not 'stabilize' at the 10^{th} level.

For candidates that pass the integrality tests, we try to find the characters by mapping the corresponding differential equation onto the plane. The characters help us in finding the fusion rules using Verlinde's formula. Finally, we try to identify the potential chiral algebras for these theories by assuming that the chiral algebra is semisimple.

We discussed the classification of one-character theories without using the differential equation. Then, we studied the well-known [1, 2] two-character theories with $\ell = 0$ zero. We then studied the case of $\ell = 2, 3, 4$ theories. We find that there is no two-character theory with $\ell = 3$. In the case of $\ell = 4$, two-character theories an additional parameter renders the usual Diophantine analysis difficult.

In the case of three and four-character theories with $\ell = 0$, we note that there are infinitely many solutions.Moreover, we find new 3 and 4-character CFT's from the generalized coset construction. The chiral algebra and degeneracies of the ground states immediately follow.

It is interesting to see how the extra parameter in the two-character $\ell \geq 4$ case can be tackled. In the case of three-character theories, though the general solution is very difficult to obtain, classification of specific subclasses of CFT's can be attempted. For example, one can try to answer the question -" What are all the CFT's with no Kac-Moody algebra ($m_1^0 = 0$)?

We have considered the cosets of $c = 24$ one-character CFT's. It is interesting to see whether this can be generalized to $c = 24k$ with $k \geq 1$.

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