Multiplicity results for fractional elliptic equations and system of equations

A thesis

submitted in partial fulfillment of the requirements

of the degree of

Doctor of Philosophy

by

Souptik Chakraborty

ID: 20163477

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

October 8, 2021

Dedicated to

Maa and Baba

Certificate

Certified that the work incorporated in the thesis entitled "*Multiplicity results for fractional elliptic equations and system of equations*", submitted by *Souptik Chakraborty* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

M. Bhakta

Date: October 8, 2021 *Dr. Mousomi Bhakta*

Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Date: October 8, 2021

Souptik Chakraborty

Roll Number: 20163477

Acknowledgements

I would like take this as an opportunity to express my deepest gratitude to my thesis supervisor Dr. Mousomi Bhakta for proposing me the study of these problems, giving me great motivation and inspiration and constant encouragement throughout my entire learning process and research experience. This thesis wouldn't be possible without her constant support throughout my journey. Her scientific insights and passion for Mathematics significantly influence and benefit my own research.

Beside my supervisor, I would also like to thank my Research Advisory Committee members Prof. K Sandeep, Dr. Anup Biswas and Dr. Debdip Ganguly for their insightful comments and encouragements. I have learnt a lot from them. I deeply appreciate their contributions of time and ideas and also their support and encouragements.

I feel myself privileged to be a part of several projects where I got opportunity to work with two outstanding mathematicians Prof. Patrizia Pucci and Prof. Olimpio H. Miyagaki. They were too kind to me, and I am extremely thankful to them for the experience I got during the projects and aspire to work with them in near future as well.

I am also thankful to many mathematicians at Department of Mathematics at IISER Pune. Especially to my Minor thesis supervisor Dr. Krishna Kaipa, Dr. Vivek Mohan Mallick, Dr. Kaneenika Sinha, Dr. Anupam Kumar Singh, Dr. Manish Mishra, Dr. Debargha Banerjee. I owe my understanding of Mathematics to them and highly grateful to them for their contributions. I would like to thank NBHM for providing me research fellowship during my research period at IISER. I would like to acknowledge the support of the administrative staff members for their cooperation, special thanks are due to Mrs. Suvarna Bharadwaj, Mr. Yogesh, Mr. Tushar Kurulkar, Mrs. Sayalee Damle and Mr. Alok Mishra.

I would also like to thank the professors of Mathematics at IIT Bombay for their valuable guidance and support throughout my M.Sc. period. Special thanks to Prof. Neela Nataraj, Prof. Amiya Kumar Pani, Dr. Santanu Dey, Dr. Rekha Santhanam for their constant support and motivation. I am also thankful to the professors of Mathematics Department at RKMV Belur, specially, Dr. Swapan Kumar Ghosh, Dr. Kartick Pal, Dr. Subhankar Roy for teaching me the fundamentals of the subject with very patience, care and love. I am also very grateful to teachers of my school, especially to Mr. Swapan Pal, Mr. Sk. Riasat Ali, Mr. Manotosh Sharma for having faith on me and constantly motivating me and guiding me towards the right direction in my life.

I am thankful to my batch mates, friends at IISER Pune, IIT Bombay and Belur. Many of you might already recognized yourselves and hope you will pardon me if I miss to take your name individually. Special thanks to my PDE group researchers Debangana Di, Prasun, Mitesh and Arghya. It was a pleasant experience to discuss Mathematics with you guys and hope to continue it for long period. Special thanks to my dear friends Basudev, Kartik, Debjit, Arijeet, Projjwal, Riju, Sudipa, Ramya, Debaprasanna, Suraj, Sandipan, Sayan, Anweshi, Amit da for making my Ph.D. journey so memorable and exciting. I am thankful to IIT Bombay batch mates and seniors Anupam, Jyoti, Satavisha, Provanjan Da, Abhijit Da, Tathagata, KT. I am thankful to my RKMV friends Kuntal, Bivas, Aniket, Sukanta, Abhrojyoti, Sirsendu. You guys hold a special position in my heart. I also thank Dr. Santanu Bag for his valuable suggestions in my mathematical journey as well as in my life. Special thank goes to my childhood friends Soumyadeep, Rajat, Deepanjan, Avrajyoti for constantly encourage and support me in every aspect of my life. I owe to you guys.

Last but not the least, a special thanks to my parents and family members for their endless sacrifice, patience, faith and love throughout my whole journey and in all perspective of my life.

Souptik Chakraborty

Contents

Abstract

The major theme of this thesis is the study on multiplicity results for fractional elliptic equations and system of equations. The thesis is mainly divided into three parts. In the first part, existence and multiplicity of positive solutions for perturbed nonlocal scalar field equation with subcritical nonlinearity and nonhomogeneous terms have been studied, and the global compactness result has been proved. The second part deals with Fractional Hardy-Sobolev equation involving critical nonlinearity and nonhomogeneous term. The existence of at least two positive solutions is obtained provided the corresponding nonhomogeneous terms are small enough in the dual space norm. Besides the profile decomposition for the Palais-Smale sequences of the associated energy functional has been accomplished. Third part comprises of the study of nonhomogeneous weakly coupled nonlocal system of equations with critical and subcritical nonlinearities. Firstly, the existence of a positive solution exploiting the local geometry of the associated functional near the origin is achieved. Then proving the global compactness result (which gives the complete description of the associated Palais Smale sequences for the system), the existence of two positive solutions is obtained under some suitable conditions on the nonhomogeneous terms. In addition, considering the corresponding homogeneous system, uniqueness for the ground state solution has been proved.

Notation

The following symbols will be used throughout the thesis.

$\mathbb{R}$: *the set of real numbers.*

$\mathbb{N}$: *the set of natural numbers.*

$\mathbb{R}^N$: $N-$ *fold cartesian product of* $\mathbb{R}$ *with itself.*

B_r : *Ball in* $\mathbb{R}^N$ *of radius* r *centered at origin.*

$B(x, r)$: *Ball in* $\mathbb{R}^N$ *of radius* r *centered at* x.

$C(\mathbb{R}^N)$: *the set of continuous functions on* $\mathbb{R}^N$.

$C_c(\mathbb{R}^N)$: *the set of continuous functions on* $\mathbb{R}^N$ *with compact support.*

$C_0^\infty(\mathbb{R}^N)$: *the space of smooth functions from* $\mathbb{R}^N \to \mathbb{R}$ *with compact support.*

$H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$: *fractional Sobolev space.*

$H^{-s}(\mathbb{R}^N)$: *The dual space of* $H^s(\mathbb{R}^N)$.

$\dot{H}^s(\mathbb{R}^N)$: *Homogeneous fractional Sobolev Space.*

$(\dot{H}^s(\mathbb{R}^N))' :=$ *The dual space of* $\dot{H}^s(\mathbb{R}^N)$

Δ : *the Laplace Operator defined by* $\Delta u = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} u$ *for any function*

$u : \mathbb{R}^N \to \mathbb{R}$, *real valued measurable function.*

$(-\Delta)^s$: *the fractional-Laplacian Operator.*

$⨍_{B(x,R)} u \,\mathrm{d}x$: *average integral of* u *over the ball of radius* R *centered at* x.

$\| \cdot \|_X$: *Norm in the Banach space* X.

S : *Best Sobolev Constant.*

2_s^* : the fractional critical Sobolev exponent $\frac{2N}{N-2s}$.

$\square$: end of a proof.

Chapter 1

Introduction

The main objective of the thesis is to study existence and multiplicity of solutions to various class of nonlocal elliptic equations and system of equations. Over the last few decades fractional Laplace operator drew much attention both in pure and applied mathematics point of view. The fractional Laplacian and these types of operators arise in natural way in many different contexts such as the thin obstacle problem, optimization, mathematical finance, phase transition, anomalous diffusion, crystal dislocation, soft thin films, ultra relativistic limits of quantum mechanics, jump Lévy process in probability theory, minimal surfaces, flame propagation, chemical reactions of liquids, population dynamics etc. To know details about this topics one might refer to [58], [41], [40] and references therein.

In contrast to classical differential operators, such as the Laplacian which is defined for a C^2 function u as, $-\Delta u(x) = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x)$ whose value at any point x, depends on the local behavior of function u in an arbitrarily small neighborhood of x, where as to define $(-\Delta)^s u$ $(s \in (0,1))$, one needs the information about u in the entire space $\mathbb{R}^N$.

The contents of the thesis is mostly divided into three parts. On the

first part we have dealt with existence and multiplicity results for a class of nonlocal scalar field equation. Second part concerns about fractional Hardy-Sobolev type equation and the last part discusses about elliptic system of equations with fractional Laplacian.
Chapter 2 is devoted to the basic tools that are needed to study our problems. The contents of the thesis is mainly corresponds to a paper or preprint as follows:

Part I:

- Bhakta, M.; Chakraborty, S.; Ganguly, D., *Existence and Multiplicity of positive solutions of certain nonlocal scalar field equations*, arXiv: 1910:07919 (To appear in Mathematische Nachrichten (2022)).

Part II:

- Bhakta, M.; Chakraborty, S.; Pucci, P.; *Fractional Hardy-Sobolev equations with nonhomogeneous terms*, Adv. Nonlinear Anal. 10 (2021), no. 1, 1086–1116.

Part III:

- Bhakta, M.; Chakraborty, S.; Pucci, P.; *Nonhomogeneous systems involving critical or subcritical nonlinearities*, Differential Integral Equations 33 (2020), no. 7-8, 323–336.

- Bhakta, M.; Chakraborty, S.; Miyagaki, O. H.; Pucci, P.; *Fractional elliptic systems with critical nonlinearities*, Nonlinearity 34 (2021), no. 11, 7540–7573.

The thesis is organized as follows:

- **Chapter 2** consists of the background materials which are essential to study critical points of the energy functional associated to non-local equation. First we introduce the Fractional Sobolev spaces, Fractional Laplace operator. Then we mention key differences between local and nonlocal operators. Then we define the Lusternik-Schnirelmann Category theory, Morrey Spaces, Fractional Hardy-Sobolev inequality. Contents of this chapter is based on [10, 93, 105].

- **Chapter 3** corresponds to the existence and multiplicity of positive solutions of nonlocal scalar field equation with subcritical nonlinearity and with non-homogeneous term of the type

$$\begin{cases} (-\Delta)^s u + u = a(x)|u|^{p-1}u + f(x) \ \text{ in } \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}$}$$

where $s \in (0,1)$ is fixed parameter, $N > 2s$, $1 < p < 2_s^* - 1 := \frac{N+2s}{N-2s}$, $0 < a \in L^\infty(\mathbb{R}^N)$ and $0 \not\equiv f \in H^{-s}(\mathbb{R}^N)$ is a nonnegative functional i.e., ${}_{H^{-s}}\langle f, u \rangle_{H^s} \geq 0$ whenever $u \geq 0$.
We establish Palais-Smale decomposition of the functional associated with the above equation. Using the decomposition, we prove that $(\mathcal{P})$ admits at least two positive solutions when $a(x) \geq 1$, $a(x) \to 1$ as $|x| \to \infty$ and $\|f\|_{H^{-s}(\mathbb{R}^N)}$ is small enough (but $f \not\equiv 0$). Further, we establish existence of three positive solutions to $(\mathcal{P})$, under the condition that $a(x) \leq 1$ with $a(x) \to 1$ as $|x| \to \infty$ and $\|f\|_{H^{-s}(\mathbb{R}^N)}$ is small enough (but $f \not\equiv 0$). Finally, we prove existence of a positive solution when $f \equiv 0$ under certain asymptotic behavior on the function a. This chapter is based on the paper [24].

- **Chapter 4** deals with existence and multiplicity of positive solutions of the fractional Hardy-Sobolev equations with nonhomogeneous term of the type

$$\begin{cases}(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = K(x)\frac{|u|^{2^*_s(t)-2}u}{|x|^t} + f(x) \quad \text{in} \quad \mathbb{R}^N, \\ \qquad\qquad u \in \dot{H}^s(\mathbb{R}^N), \end{cases} \tag{$E^\gamma_{K,t,f}$}$$

where $N > 2s$, $s \in (0,1)$, $0 \le t < 2s < N$ and $2^*_s(t) := \frac{2(N-t)}{N-2s}$. Clearly, $2 < 2^*_s(t) \le \frac{2N}{N-2s} = 2^*_s$. Here $0 < \gamma < \gamma_{N,s}$, where $\gamma_{N,s}$ is the best Hardy constant in the fractional Hardy inequality

$$\gamma_{N,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2\,\mathrm{d}\xi, \quad \gamma_{N,s} = 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

Here and throughout $\mathscr{F}(u)$ denotes the Fourier transform of u.
In $(E^\gamma_{K,t,f})$, the functions K and f satisfy the properties :

(**K**) $0 < K \in C(\mathbb{R}^N)$, $K(0) = 1 = \lim_{|x|\to\infty} K(x)$.

(**F**) *$f \not\equiv 0$ is a nonnegative functional in the dual space $\dot{H}^s(\mathbb{R}^N)'$ of* $\dot{H}^s(\mathbb{R}^N)$, i.e. whenever u is a nonnegative function in $\dot{H}^s(\mathbb{R}^N)$ then ${}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \ge 0$.

We establish the profile decomposition of the Palais-Smale sequence of the functional associated to $(E^\gamma_{K,t,f})$. Further, if $K \ge 1$ and $\|f\|_{(\dot{H}^s)'}$ is small enough (but $f \not\equiv 0$), we establish existence of at least two positive solutions to the above equation. This chapter is based on the paper [27].

- **Chapter 5** deals with existence, uniqueness/multiplicity of positive solutions to the following nonlocal system of equations

$$\begin{cases}(-\Delta)^s u + \gamma u = \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta + f(x) \;\text{ in } \mathbb{R}^N, \\ (-\Delta)^s v + \gamma v = \dfrac{\beta}{\alpha+\beta}|v|^{\beta-2}v|u|^\alpha + g(x) \;\text{ in } \mathbb{R}^N, \\ u,\, v > 0 \;\text{ in } \mathbb{R}^N, \end{cases} \tag{$\mathcal{S}^\gamma_{\alpha,\beta}$}$$

where $N > 2s$, α, $\beta > 1$, $\alpha + \beta \leq 2^*_s$, $2^*_s := 2N/(N-2s)$, f, g are nontrivial nonnegative functionals in the dual space of $\dot{H}^s(\mathbb{R}^N)$ if $\alpha+\beta = 2^*_s$ and of $H^s(\mathbb{R}^N)$ if $\alpha + \beta < 2^*_s$, while $\gamma = 0$ if $\alpha + \beta = 2^*_s$ and $\gamma = 1$ if $\alpha + \beta < 2^*_s$.

First via a minimization argument exploiting the local geometry of the associated functional near the origin we prove the existence of one positive solution whose energy is negative provided the non-homogeneous terms are small enough in the dual norm. This part is based on the paper [26].

When $\gamma = 0$, $\alpha + \beta = 2^*_s$ and $f = 0 = g$, we show that the ground state solution of $(\mathcal{S}^0_{2^*_s})$ is *unique*. On the other hand, when f and g are nontrivial nonnegative functionals with ker(f)=ker(g), then we establish the existence of at least two different positive solutions of $(\mathcal{S}^0_{2^*_s})$ provided that $\|f\|_{(\dot{H}^s)'}$ and $\|g\|_{(\dot{H}^s)'}$ are small enough. Moreover, we also provide a global compactness result, which gives a complete description of the Palais-Smale sequences of the above system. This part of the chapter is based on the paper [25].

Chapter 2

Nonlocal Framework and Variational Tools

This chapter is devoted to the basic definitions and results regarding fractional Sobolev spaces and fractional nonlocal operators that will be used throughout the thesis. Almost every results presented in this chapter is well known. First three sections of this chapter is written in the spirit of [93], [58]. On the last two sections we briefly discuss about Morrey space and fractional Hardy-Sobolev inequality. Most of the proofs have been omitted.

2.1 Fourier transform of tempered distributions

First we introduce the notion of Fourier transform of a tempered distribution. We consider the Schwartz space consisting of $C^\infty(\mathbb{R}^N)$ functions which, together with all its derivatives of all orders, decrease to zero at infinity faster than any power of $|x|^{-1}$. More precisely, we define

$$\mathscr{S}(\mathbb{R}^N) := \left\{ \phi \in C^\infty(\mathbb{R}^N) \, : \, \forall \alpha, \beta \in \mathbb{N}_0^N, \, \sup_{x \in \mathbb{R}^N} |x^\alpha D^\beta \phi(x)| < \infty \right\},$$

whose topology is generated by the seminorms $\{p_j\}_{j\in\mathbb{N}}$ defined as:

$$p_j(\phi) := \sup_{x\in\mathbb{R}^N} (1+|x|)^j \sum_{|\alpha|\leq j} |D^\alpha \phi(x)|,$$

where $\phi \in \mathscr{S}(\mathbb{R}^N)$.

For any $\phi \in \mathscr{S}(\mathbb{R}^N)$, denoting the space variable $x \in \mathbb{R}^N$ and the frequency variable $\xi \in \mathbb{R}^N$, the Fourier transform of ϕ is defined as

$$\mathscr{F}\phi(\xi) = \hat{\phi}(\xi) := \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} \phi(x)\,\mathrm{d}x. \tag{2.1.1}$$

We note that, for every $\phi \in \mathscr{S}(\mathbb{R}^N)$, we have $\mathscr{F}\phi \in \mathscr{S}(\mathbb{R}^N)$. The inverse Fourier transform is given by

$$\mathscr{F}^{-1}\phi(x) := \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi} \hat{\phi}(\xi)\,\mathrm{d}\xi. \tag{2.1.2}$$

It can be proved that the map $\phi \mapsto \hat{\phi}$ is a continuous linear map of $\mathscr{S}(\mathbb{R}^N)$ into itself with a continuous inverse and hence the Fourier transform is a (topological) isomorphism of $\mathscr{S}(\mathbb{R}^N)$ onto itself.

Now, let $\mathscr{S}'(\mathbb{R}^N)$ be the topological dual of $\mathscr{S}(\mathbb{R}^N)$ and this space is called the space of tempered distributions. If $T \in \mathscr{S}'(\mathbb{R}^N)$, the Fourier transform of T can be defined as the tempered distribution given by

$$\langle \mathscr{F}T, \phi\rangle := \langle T, \mathscr{F}\phi\rangle,$$

for every $\phi \in \mathscr{S}(\mathbb{R}^N)$, where $\langle\cdot,\cdot\rangle$ denotes the usual duality bracket between $\mathscr{S}'(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R}^N)$. We have for

$$\phi \in \mathscr{S}(\mathbb{R}^N), \quad \|\phi\|_{L^2(\mathbb{R}^N)} = \|\hat{\phi}\|_{L^2(\mathbb{R}^N)},$$

which leads us to the extension of the Fourier transform to another class of functions :

Theorem 2.1.1. (Plancherel) *There exists a unique isometry $\mathscr{P} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ which is surjective such that $\mathscr{P}(\phi) = \hat{\phi}$, for every $\phi \in \mathscr{S}(\mathbb{R}^N)$.*

The above formula will be used to establish the equivalence between the fractional spaces $H^s(\mathbb{R}^N)$ and $\hat{H}^s(\mathbb{R}^N)$ (see Proposition 2.4.4).

2.2 Fractional Sobolev spaces

Let Ω be an open, smooth set in $\mathbb{R}^N$ and $p \in [1, +\infty)$. For any $s > 0$, we would define the fractional Sobolev space $W^{s,p}(\Omega)$. If $s \geq 1$ is a positive integer, $W^{s,p}(\Omega)$ denotes the classical Sobolev space equipped with the standard norm

$$\|u\|_{W^{s,p}(\Omega)} := \sum_{0 \leq |\alpha| \leq s} \|D^\alpha u\|_{L^p(\Omega)},$$

for every $u \in W^{s,p}(\Omega)$. We are interested in the cases where $s \notin \mathbb{N}$. Now, for a fixed $s \in (0, 1)$, the Sobolev space $W^{s,p}(\Omega)$ is defined as:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \in L^p(\Omega \times \Omega) \right\} \tag{2.2.1}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u(x)|^p dx + \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} dxdy \right)^{\frac{1}{p}}, \tag{2.2.2}$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} dxdy \right)^{\frac{1}{p}} \tag{2.2.3}$$

is the *Gagliardo seminorm* of u. For $s > 1$, $s \notin \mathbb{N}$, we can write $s = k + \tau$, for $\tau \in (0, 1)$. Then we define,

$$W^{s,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) \, : \, D^\alpha u \in W^{\tau,p}(\Omega) \text{ for any } \alpha \text{ such that } |\alpha| = k \right\}.$$

This space is equipped with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha| = m} \|D^\alpha u\|^p_{W^{\tau,p}(\Omega)} \right)^{\frac{1}{p}},$$

for every $u \in W^{s,p}(\Omega)$. The space $W^{s,p}(\Omega)$ is a well defined Banach space for every $s > 0$.

2.2.1 Embedding results

This subsection deals with the embeddings of fractional Sobolev spaces into Lebesgue spaces. Some basic facts are recalled briefly. For details, see [58, Sections 6 and 7], [93, Section 1.2.1].

Proposition 2.2.1. *Let Ω be an open subset of $\mathbb{R}^N$ and $1 \leq p < \infty$. Then the following assertions hold true:*

(a) If $0 < s \leq s' < 1$, then the embedding

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$

is continuous. Hence, there exists a constant $C_1(N, s, p) \geq 1$ such that

$$\|u\|_{W^{s,p}(\Omega)} \leq C_1(N, s, p)\|u\|_{W^{s',p}(\Omega)},$$

for any $u \in W^{s',p}(\Omega)$.

(b) If $0 < s < 1$ and Ω is of class $C^{0,1}$ (that is, with the Lipschitz boundary) and with bounded boundary $\partial\Omega$, then the embedding

$$W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$

is continuous. Hence, there exists a constant $C_2(N, s, p) \geq 1$ such that

$$\|u\|_{W^{s,p}(\Omega)} \leq C_2(N, s, p)\|u\|_{W^{1,p}(\Omega)},$$

for any $u \in W^{1,p}(\Omega)$.

(c) If $s' \geq s > 1$ and Ω is of class $C^{0,1}$, then the embedding

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$$

is continuous.

Proof. For proofs, see Proposition 2.1, Proposition 2.2 and Corollary 2.3 in [58]. □

Now let us recall some basic properties about continuous (compact) embeddings of the fractional Sobolev spaces $W^{s,p}$ into Lebesgue spaces. We divide our discussion in three different cases : (i) $sp < N$, (ii) $sp = N$, (iii) $sp > N$. Proofs of the following results can be found in [58, Sections 6-8].

Case 1: $sp < N$

Theorem 2.2.2. *Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp < N$. Then there exists a positive constant $C := C(N, p, s)$ such that, for any $u \in W^{s,p}(\mathbb{R}^N)$,*

$$\|u\|^p_{L^{p^*_s}(\mathbb{R}^N)} \leq C \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy,$$

where the exponent

$$p^*_s := \frac{Np}{N - ps}$$

is the so-called fractional critical exponent . Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p^*_s]$. Moreover, the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ is compact for every $q \in [p, p^*_s)$.

Definition 2.2.3 (Extension domain)**.** *For any $s \in (0,1)$ and any $1 \leq p < \infty$, we say that an open set $\Omega \subseteq \mathbb{R}^N$ is an extension domain for $W^{s,p}(\Omega)$ if there exists a positive constant $C \equiv C(n, p, s, \Omega)$ such that for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{s,p}(\Omega)}$.*

In an extension domain Ω, the following embedding result holds:

Theorem 2.2.4. *Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp < N$. Let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C := C(N, p, s, \Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p, p^*_s]$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in

$L^q(\Omega)$ for any $q \in [p, p_s^*]$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*)$.

Case 2: $sp = N$

Theorem 2.2.5. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = N$. Then there exists a positive constant $C := C(N,p,s)$ such that for any $u \in W^{s,p}(\mathbb{R}^N)$,*

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{W^{s,p}(\mathbb{R}^N)},$$

for any $q \in [p,+\infty)$; that is, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p,+\infty)$.

For an extension domain Ω, the following embedding results hold:

Theorem 2.2.6. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = N$. Let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{s,p}(\Omega)$. Then there exists a positive constant $C := C(N,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p,+\infty)$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p,+\infty)$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1,+\infty)$.

Case 3: $sp > N$

We denote by $C^{0,\alpha}(\Omega)$ the space of Hölder continuous functions endowed with the standard norm

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Theorem 2.2.7. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp > N$. Let Ω be a $C^{0,1}$ domain of $\mathbb{R}^N$. Then there exists a positive constant $C := C(N,p,s,\Omega)$ such that for any $u \in W^{s,p}(\Omega)$, we have,*

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

with $\alpha := (sp - N)/p$*; that is, the space* $W^{s,p}(\Omega)$ *is continuously embedded in* $C^{0,\alpha}(\Omega)$.

As a consequence of Theorem 2.2.7, we have the following result.

Corollary 2.2.8. *Let* $s \in (0,1)$ *and* $p \in [1, +\infty)$ *such that* $sp > N$*. Let* Ω *be a* $C^{0,1}$ *bounded domain of* $\mathbb{R}^N$*. Then the embedding*

$$W^{s,p}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$$

is compact for every $\beta < \alpha$*, with* $\alpha := (sp - N)/p$.

2.2.2 The Sobolev space $H^s(\Omega)$

This section is devoted to the case $p = 2$ where we deal its relation with the fractional Laplacian. Let Ω be an open subset of $\mathbb{R}^N$ and denote

$$H^s(\Omega) := W^{s,2}(\Omega),$$

for any $s \in (0,1)$. In this case, we note that the preceding fractional Sobolev space turns out to be a Hilbert space. The inner product on $H^s(\Omega)$ is defined by

$$\langle u, v \rangle_{H^s(\Omega)} := \int_\Omega u(x)v(x)\mathrm{d}x + \iint_{\Omega\times\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \mathrm{d}x\, \mathrm{d}y,$$

for any $u, v \in H^s(\Omega)$ induces the norm given in (2.2.2) when $p = 2$. That is, for every $s \in (0,1)$, we have,

$$H^s(\mathbb{R}^N) := W^{s,2}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_{W^{s,2}(\mathbb{R}^N)} < +\infty\}, \qquad (2.2.4)$$

where $[\cdot]_{W^{s,2}(\mathbb{R}^N)}$ is defined in (2.2.3).

Alternatively, we can also define the space $H^s(\mathbb{R}^N)$ via a Fourier transform, that is, we define

$$\hat{H}^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |y|^{2s})|\mathscr{F}u(y)|^2 dx < +\infty \right\}, \qquad (2.2.5)$$

for any $s > 0$ and

$$\hat{H}^s(\mathbb{R}^N) := \left\{ u \in \mathscr{S}' : \int_{\mathbb{R}^N} (1 + |y|^2)^s |\mathscr{F} u(y)|^2 dx < +\infty \right\},$$

for every $s < 0$.

The equivalence between the space $\hat{H}^s(\mathbb{R}^N)$ defined in (2.2.5) and the one defined by the Gagliardo norm in (2.2.4) is given in Proposition 2.4.4.

Theorem 2.2.2 motivates us to define, the homogeneous fractional Sobolev space is denoted by

$$\dot{H}^s(R^N) := \left\{ u \in L^{2_s^*}(\mathbb{R}^N) \ : \ \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\},$$

which turns out to be a Hilbert space with the inner product

$$\langle u, v \rangle_{\dot{H}^s} = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y)) \, (v(x) - v(y))}{|x - y|^{N+2s}} \, \mathrm{d}x \mathrm{d}y,$$

and corresponding Gagliardo norm is given by

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} := \left(\frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}}.$$

2.3 The Morrey Spaces

The contents of this section can be found in [31, 94].

First we recall the definition of the homogeneous Morrey spaces $L^{r,\gamma}(\mathbb{R}^N)$, introduced by Morrey as a refinement of the usual Lebesgue spaces. A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $L^{r,\gamma}(\mathbb{R}^N)$, with $r \in [1, \infty)$ and $\gamma \in [0, N]$ if and only if

$$\|u\|^r_{L^{r,\gamma}(\mathbb{R}^N)} := \sup_{R>0,\, x\in\mathbb{R}^N} R^\gamma \fint_{B(x,R)} |u|^r \mathrm{d}y < \infty. \qquad (2.3.1)$$

Note that if $\gamma = N$ then $L^{r,N}(\mathbb{R}^N)$ coincides with the usual Lebesgue space $L^r(\mathbb{R}^N)$ for any $r \geq 1$ and similarly $L^{r,0}(\mathbb{R}^N)$ coincides with $L^\infty(\mathbb{R}^N)$. Also we observe that $L^{r,\gamma}(\mathbb{R}^N)$ experiences same translation and dilation invariance as in $L^{2_s^*}(\mathbb{R}^N)$ and therefore of $\dot{H}^s(\mathbb{R}^N)$ if $\frac{\gamma}{r} = \frac{N-2s}{2}$. Let $(u)^{x_0,r}$ be the function defined by

$$(u)^{x_0,r}(\cdot) = r^{-\frac{N-2s}{2}} u\left(\frac{\cdot - x_0}{r}\right).$$

By change of variable formula, one can see that the following equality holds

$$\|(u)^{x_0,r}\|_{L^{r,\frac{N-2s}{2}r}} = \|u\|_{L^{r,\frac{N-2s}{2}r}},$$

for any $r \in [1, 2_s^*]$. We recall a result from [94] which states

Theorem 2.3.1. *[94, Theorem 1] For any $0 < 2s < N$ there exists a constant C depending only on N and s such that, for any $2/2_s^* \le \theta < 1$ and for any $1 \le r < 2_s^*$,*

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \le \|u\|^{\theta}_{\dot{H}^s(\mathbb{R}^N)} \|u\|^{1-\theta}_{L^{r,r(N-2s)/2}} \quad \text{for all } u \in \dot{H}^s(\mathbb{R}^N). \tag{2.3.2}$$

Again, there exists a constant $C = C(N, s)$ such that

$$\|u\|_{L^{r,r(N-2s)/2}(\mathbb{R}^N)} \le C\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \quad \text{for all } u \in L^{2_s^*}(\mathbb{R}^N), \tag{2.3.3}$$

see Theorem 2.3.1 (also see [31, (A.2)]). For further discussion on Morrey spaces, we refer the reader to [94].

Next we define the product space $L^{2,N-2s}(\mathbb{R}^N) \times L^{2,N-2s}(\mathbb{R}^N)$ in the standard way with

$$\|(u, v)\|_{L^p \times L^p} := \left(\|u\|^2_{L^p(\mathbb{R}^N)} + \|v\|^2_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{2}}$$

and

$$\|(u, v)\|_{L^{2,N-2s} \times L^{2,N-2s}} := \left(\|u\|^2_{L^{2,N-2s}} + \|v\|^2_{L^{2,N-2s}}\right)^{\frac{1}{2}}.$$

Therefore, using Sobolev inequality and (2.3.3), it follows that

$$\dot{H}^s \times \dot{H}^s \hookrightarrow L^{2_s^*} \times L^{2_s^*} \hookrightarrow L^{2,N-2s} \times L^{2,N-2s}, \tag{2.3.4}$$

where the embeddings are continuous.

Lemma 2.3.2. *For any $0 < s < N/2$ there exists a constant $C = C(N, s)$ such that, for any $2/2_s^* \leq \theta < 1$ and for any $1 \leq r < 2_s^*$*

$$\|(u,v)\|_{L^{2_s^*}\times L^{2_s^*}} \leq C\|(u,v)\|^{\theta}_{\dot{H}^s\times\dot{H}^s}\|(u,v)\|^{1-\theta}_{L^{2,(N-2s)}\times L^{2,(N-2s)}}$$

for all $(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$.

Proof. Using Theorem 2.3.1,

$$\begin{aligned}
\|(u,v)\|_{L^{2_s^*}\times L^{2_s^*}} &= \left(\|u\|^2_{L^{2_s^*}} + \|v\|^2_{L^{2_s^*}}\right)^{\frac{1}{2}} \\
&\leq \|u\|_{L^{2_s^*}} + \|v\|_{L^{2_s^*}} \\
&\leq C\Big[\|u\|^{\theta}_{\dot{H}^s}\|u\|^{1-\theta}_{L^{2,N-2s}} + \|v\|^{\theta}_{\dot{H}^s}\|v\|^{1-\theta}_{L^{2,N-2s}}\Big] \\
&\leq C\Big[\|(u,v)\|^{\theta}_{\dot{H}^s\times\dot{H}^s}\|u\|^{1-\theta}_{L^{2,N-2s}} + \|(u,v)\|^{\theta}_{\dot{H}^s\times\dot{H}^s}\|v\|^{1-\theta}_{L^{2,N-2s}}\Big] \\
&\leq C\|(u,v)\|^{\theta}_{\dot{H}^s\times\dot{H}^s}\Big[\|u\|^{1-\theta}_{L^{2,N-2s}} + \|v\|^{1-\theta}_{L^{2,N-2s}}\Big] \\
&\leq C\|(u,v)\|^{\theta}_{\dot{H}^s\times\dot{H}^s}\Big[\|(u,v)\|^{1-\theta}_{L^{2,N-2s}\times L^{2,N-2s}} + \|(u,v)\|^{1-\theta}_{L^{2,N-2s}\times L^{2,N-2s}}\Big] \\
&\leq 2C\|(u,v)\|^{\theta}_{\dot{H}^s\times\dot{H}^s}\|(u,v)\|^{1-\theta}_{L^{2,(N-2s)}\times L^{2,(N-2s)}}.
\end{aligned}$$

□

2.4 The fractional Laplacian operator

A very popular non-local operator is given by the fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$. This operator and its generalization appear in many areas of mathematics, like harmonic analysis, probability theory, potential theory, quantum mechanics, statistical physics etc. This section deals with the definition of this operator and its properties. For more complete discussions and comparisons regarding fractional Laplacian see [58], [40], [85].

Let $s \in (0, 1)$ and define the fractional Laplacian operator $(-\Delta)^s : \mathscr{S} \to$

$L^2(\mathbb{R}^N)$ by

$$\begin{aligned}(-\Delta)^s u(x) \; &:= \; C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y \qquad (2.4.1)\\ &:= \; C(N,s)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad x\in\mathbb{R}^N,\end{aligned}$$

where $B(x,\varepsilon)$ is the ball centered at $x\in\mathbb{R}^N$ with radius ε and $C(N,s)$ is the following (positive) normalization constant:

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1-cos(\xi_1)}{|\xi|^{N+2s}} d\xi\right)^{-1} \qquad (2.4.2)$$

with $\xi=(\xi_1,\xi')$, $\xi'\in\mathbb{R}^{N-1}$. The next proposition tells us that the singular integral defined in (2.4.1) can be written as a weighted second-order differential quotient.

Proposition 2.4.1. *Let $s\in(0,1)$. Then for any $u\in\mathscr{S}$,*

$$-(-\Delta)^s u(x) = \frac{1}{2}C(N,s)\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,\mathrm{d}y,\ x\in\mathbb{R}^N. \qquad (2.4.3)$$

For proof, see [93, Proposition 1.10].

Remark 2.4.2. *The above expression in* (2.4.3) *discloses the fact that the fractional Laplacian is a sort of second order difference operator, weighted by a measure supported in $\mathbb{R}^N$ with a polynomial decay, namely*

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^N} \delta_u(x,y)\,\mathrm{d}\mu(y),$$

where $\delta_\mu(x,y) := u(x+y)+u(x-y)-2u(x)$, and $\mathrm{d}\mu(y) := \frac{\mathrm{d}y}{|y|^{N+2s}}$.

Remark 2.4.3. *Let $s\in(0,1/2)$. Notice that for any $u\in\mathscr{S}$ and for a fixed $x\in\mathbb{R}^N$, we have that,*

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y \;&\le\; C\int_{B(x,R)}\frac{|x-y|}{|x-y|^{N+2s}}\,\mathrm{d}y\\ &+\; \|u\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N\setminus B(x,R)}\frac{1}{|x-y|^{N+2s}}\,\mathrm{d}y\\ &\le\; C\left(\int_0^R \frac{1}{\rho^{2s}}\,\mathrm{d}\rho + \int_R^{+\infty}\frac{1}{\rho^{2s+1}}\,\mathrm{d}\rho\right) < +\infty,\end{aligned}$$

where C is a positive constant depending only on the dimension N and the L^∞- norm of the function u. So, in the case $s \in (0, 1/2)$, the integral

$$\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y$$

is not singular near the point x, so one can get rid of the P.V. in (2.4.1).

Proposition 2.4.4. *Let $s \in (0, 1)$ and $C(N, s)$ be the constant defined in* (2.4.2)*. Then, for any $u \in H^s(\mathbb{R}^N)$,*

$$[u]^2_{H^s(\mathbb{R}^N)} = 2C(N, s)^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathscr{F} u(\xi)|^2 d\xi. \tag{2.4.4}$$

Moreover, $H^s(\mathbb{R}^N) = \hat{H}^s(\mathbb{R}^N)$

For proof, see [93, Corollary 1.15].

Now we show that the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$ (see [58, Section 3]).

Proposition 2.4.5. *Let $s \in (0, 1)$. Then, for any $u \in \mathscr{S}(\mathbb{R}^N)$,*

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}(|\xi|^{2s} (\mathscr{F} u)(\xi))(x), x \in \mathbb{R}^N, \tag{2.4.5}$$

where $\mathscr{F}^{-1}$ is the inverse Fourier transform defined in (2.1.2).

For proof, (see [93, Proposition 1.17]).

The following lemma ensures the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space $H^s(\mathbb{R}^N)$ (see [58]).

Proposition 2.4.6. *Let $s \in (0, 1)$ and $C(N, s)$ be the constant defined in (2.4.2). Then, for any $u \in H^s(\mathbb{R}^N)$,*

$$[u]^2_{H^s(\mathbb{R}^N)} = 2C(N, s)^{-1} \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)}. \tag{2.4.6}$$

For proof, see [93, Proposition 1.18].

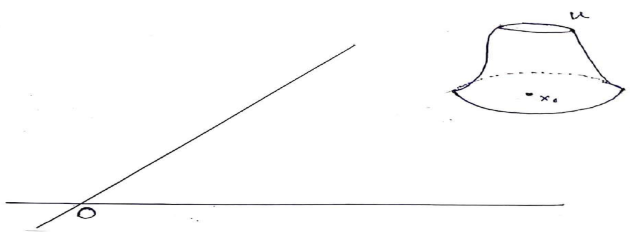
2.4.1 Local vs Nonlocal operator

We list some major differences between the usual Laplacian and the fractional Laplacian. For more details one might see [1], [40] and references therein.

- 1. Let $u \in C_0^\infty(\mathbb{R}^2)$ such that $0 \le u(x) \le 1$ on $\mathbb{R}^2$ and it satisfies

$$u \equiv 1 \text{ on } B(x_0, \tfrac{r}{2}) \text{ and } \operatorname{supp}(u) \subset B(x_0, r),\ 0 \notin \overline{B(x_0, r)}.$$

Consider the figure below



It is easy to see that $(-\Delta)u(0) = 0$. In contrast we notice that,

$$\int_{\mathbb{R}^2} \frac{u(y) - u(0)}{|y|^{2+2s}}\, \mathrm{d}y \ge \int_{B(x_0, r/2)} \frac{1}{|y|^{2+2s}}\, \mathrm{d}y > 0$$

and this gives $(-\Delta)^s u(0) \neq 0$.

The above example conveys a general message that the classical Laplacian maps a $C_0^\infty(\mathbb{R}^N)$ function to again a $C_0^\infty(\mathbb{R}^N)$ function, which is not the case for fractional Laplace operator.

- 2. Classical Laplacian on a function u does not require any integrability assumption on u, whereas in order to define $(-\Delta)^s u$, one must assume,

$$\int_{\mathbb{R}^N} \frac{|u(y)|}{1 + |y|^{N+2s}}\, \mathrm{d}y < +\infty,$$

which can be understood as a local integrability complemented by a growth condition at infinity.

- 3. (All functions are locally s-harmonic up to a small error) It is a very well-known fact that, Harmonic functions are very rigid. For instance, in dimension 1, they reduce to linear functions and in any dimension, they never possess any local extrema. But the situation is completely different for fractional harmonic functions. In fact, we can approximate every function f in $C^k(\overline{B_1})$ by an s-harmonic function in B_1 that vanishes outside a compact set. This was proved in [61].

- 4. (Harnack inequality) The classical Harnack inequality says that for a nonnegative harmonic function on a ball, its oscillation can be controlled on every compactly contained subset of the ball.
  The same does not hold for s-harmonic functions. For Harnack inequality to hold in fractional case one must assume the solution to be nonnegative on whole of $\mathbb{R}^N$, rather than on a given ball. For details one might see [84].

- 5. (Growth from the boundary) Roughly, solution of Laplace equations have "linear (i.e., Lipschitz) growth from the boundary", whereas for s-harmonic function, we only have Hölder growth from the boundary. To understand this, consider $u \in C(\overline{B_1})$ which solves

$$\begin{cases} \Delta u = f & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

  then we can show that

$$|u(x)| \leq C(1-|x|) \sup_{B_1} |f|.$$

  Notice that $(1-|x|)$ represents the distance of the point $x \in B_1$ from the boundary ∂B_1.
  In contrast, the function $u(x) = (x_n)_+^s$ is s-harmonic in the half space $\{x_n > 0\}$. But this function is only Hölder continuous of exponent s near the origin.

- 6. (Regularity up to the boundary) Roughly, solutions of Laplace equations are "smooth up to the boundary", whereas one can expect at most Hölder continuity at the boundary for solutions of fractional Laplace equation. For details regarding boundary regularity for the Dirichlet problem for fractional Laplacian one might see [99].

2.5 Fractional Hardy-Sobolev inequality

We study about fractional Hardy equation and fractional Hardy-Sobolev equation in some detail in Chapter 4. Fractional Hardy inequality states that for $s \in (0,1)$ and $N > 2s$ and for all u in $\dot{H}^s(\mathbb{R}^N)$ the following inequality holds:

$$\gamma_{N,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\, \mathrm{d}x \leq \|u\|^2_{\dot{H}^s(\mathbb{R}^N)},$$

where $\gamma_{N,s}$ is the best Hardy constant, that is

$$\gamma_{N,s} := \inf_{\dot{H}^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\dot{H}^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\, \mathrm{d}x\right)}$$

It has also been shown that $\gamma_{N,s} = 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$. Note that $\gamma_{N,s}$ converges to the best classical Hardy constant $\left(\frac{N-2}{2}\right)^2$, when $s \to 1$. For more details regarding fractional Hardy inequality we refer [73].

Lemma 2.5.1. ([75, Lemma 2.1]) *For $s \in (0,1)$ and $0 \leq t < 2s < N$, there exists positive constants $C_1(N,s)$ and $C_2(N,s)$ such that*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\, \mathrm{d}x\right)^{\frac{2}{2_s^*(t)}} \leq C_1 \|u\|^2_{\dot{H}^s(\mathbb{R}^N)} \quad \forall u \in \dot{H}^s(\mathbb{R}^N), \tag{2.5.1}$$

where $2_s^*(t) = \frac{2(N-t)}{N-2s}$. Moreover, if $\gamma < \gamma_{N,s}$ then

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\, \mathrm{d}x\right)^{\frac{2}{2_s^*(t)}} \leq C_2 \left(\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} - \gamma \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}}\, \mathrm{d}x\right) \quad \forall u \in \dot{H}^s(\mathbb{R}^N). \tag{2.5.2}$$

Proof. Here we briefly sketch the proof. Note that, for $t = 0$, (2.5.1) is fractional Sobolev inequality and for $t = 2s$, (2.5.1) is fractional Hardy inequality. So it is enough to consider the case $0 < t < 2s$, for which $2_s^*(t) > 2$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t} \,\mathrm{d}x &= \int_{\mathbb{R}^N} \frac{|u|^{\frac{t}{s}}}{|x|^t} |u|^{2_s^*(t)-\frac{t}{s}} \,\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} \,\mathrm{d}x\right)^{\frac{t}{2s}} \left(\int_{\mathbb{R}^N} |u|^{2_s^*} \,\mathrm{d}x\right)^{\frac{2s-t}{2s}} \\ &\leq C\|u\|_{\dot{H}^s(\mathbb{R}^N)}^{2_s^*(t)}. \end{aligned}$$

For the first inequality we use Hölder's inequality with exponents $\frac{2s}{t}$ and $\frac{2s}{2s-t}$, whereas for the last inequality we have used fractional Sobolev inequality and fractional Hardy inequality.

The proof of (2.5.2) follows directly from the above and Remark 4.0.2. □

2.6 The Lusternik-Schnirelman Category theory

The following results due to Lusternik and Schnirelman will help us to find critical points of abstract functional on Hilbert Manifolds, in connection with the topological properties of the manifold. For detailed treatment on this topic one might look at [9, 10, 105].

Definition 2.6.1. *For a topological space M, a nonempty subset A of M is said to be contractible in M if the inclusion map $i : A \hookrightarrow M$ is homotopic to a constant $p \in M$, namely there is map $\eta \in C\big([0,\ 1] \times A,\ M\big)$ such that for some $p \in M$*

$$(i)\ \eta(0,\ u) = u \ \text{ for all } u \in A$$

$$(ii)\ \eta(1,\ u) = p \ \text{ for all } u \in A$$

Definition 2.6.2. *The (L-S) category of A with respect to M, denoted by $cat(A, M)$, is the least non-negative integer k such that $A \subset \cup_{i=1}^{k} A_i$, where each A_i $(1 \leq i \leq k)$ is closed and contractible in M. We set $cat(\emptyset, M) = 0$ and $cat(A, M) = +\infty$ if there are no integers with the above property. We write $cat(M)$ to denote $cat(M, M)$.*

Remark 2.6.3. *From the definition we see that for any $A \subseteq M$, $cat(A, M) = cat(\overline{A}, M)$. Moreover, for $A \subset M \subset \tilde{M}$, since contractible closed sets in M are also closed and contractible in $\tilde{M}$, we have, $cat(A, M) \geq cat(A, \tilde{M})$.*

Example 1. *Let S^{N-1} denotes the unit sphere in $\mathbb{R}^N$. Clearly, S^{N-1} is not contractible in itself. But we can consider, two closed hemispheres which covers S^{N-1} and contractible in S^{N-1}. Thus $cat(S^{N-1}) = 2$. But if we consider the closed unit disc in $\mathbb{R}^N$, that is closed and contractible in $\mathbb{R}^N$ and contains S^{N-1} as its boundary. Therefore, $cat(S^{N-1}, \mathbb{R}^N) = 1$.*

Example 2. *We can prove that $cat(\mathbb{T}^2) = 3$, where $\mathbb{T}^2 = S^1 \times S^1$ denotes the two dimensional torus in $\mathbb{R}^3$. In general, $cat(\mathbb{T}^N) = N + 1$, where $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$ denotes the N-dimensional Torus.*

The following fundamental properties of Lusternik-Schnirelman category, can be found in [10] (also see Ambrosetti [9]).

Lemma 2.6.4. *For A, $B \subset M$,*

(i) *if $A \subset B$, then $cat(A, M) \leq cat(B, M)$;*

(ii) $cat(A \cup B, M) \leq cat(A, M) + cat(B, M)$;

(iii) *if A is closed and let $\eta \in C(A, M)$ be a deformation, then*

$$cat(A, M) \leq cat(\overline{\eta(A)}, M).$$

Next we state the following property (see [9], also see [4, Proposition 2.4]) which will play a pivotal role to in the coming chapter.

Proposition 2.6.5. *Suppose M is a Hilbert manifold and $\Psi \in C^1(M,\ \mathbb{R})$. Assume that for $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$,*

(i) $\Psi(x)$ satisfies $(PS)_c$ for $c \le c_0$,

(ii) cat$\Big(\{x \in M\ :\ \Psi(x) \le c_0\}\Big) \ge k$.

Then $\Psi(x)$ has at least k critical points in $\{x \in M\ :\ \Psi(x) \le c_0\}$.

For definition and related discussions on $(PS)_c$ see next chapter, Section 3.2.

Lemma 2.6.6. ([4, Lemma 2.5]) *Let $N \ge 1$ and M be a topological space and S^{N-1} denote the unit sphere in $\mathbb{R}^N$. Suppose the there exists two continuous mapping*

$$F : S^{N-1} \to M, \quad G : M \to S^{N-1},$$

such that $G \circ F$ is homotopic to the identity map $Id : S^{N-1} \to S^{N-1}$, namely there is continuous map $\eta : [0,\ 1] \times S^{N-1} \to S^{N-1}$ such that

$$\eta(0,\ x) = (G \circ F)(x) \ \ \text{for all } x \in S^{N-1}$$
$$\eta(1,\ x) = x \ \ \text{for all } x \in S^{N-1}.$$

Then cat$(M) \ge 2$.

Chapter 3

Existence and Multiplicity of Positive solutions of certain Nonlocal Scalar Field Equations

In this chapter we study the existence and multiplicity of positive solutions to the following fractional elliptic problem in $\mathbb{R}^N$:

$$\begin{cases} (-\Delta)^s u + u = a(x)|u|^{p-1}u + f(x) \;\; \text{in } \mathbb{R}^N, \\ \qquad u > 0 \quad \text{in} \quad \mathbb{R}^N, \\ \qquad u \in H^s(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}$}$$

where $s \in (0,1)$ is fixed parameter, $N > 2s$, $1 < p < 2_s^* - 1 := \frac{N+2s}{N-2s}$, $0 < a \in L^\infty(\mathbb{R}^N)$ and $0 \not\equiv f \in H^{-s}(\mathbb{R}^N)$ is a nonnegative functional i.e., ${}_{H^{-s}}\langle f, u\rangle_{H^s} \geq 0$ whenever $u \geq 0$.

Definition 3.0.1 (**Positive weak solution**). *We say $u \in H^s(\mathbb{R}^N)$ is a positive weak solution of ($\mathcal{P}$) if $u > 0$ in $\mathbb{R}^N$ and for every $\phi \in H^s(\mathbb{R}^N)$, we have*

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} u\phi\,\mathrm{d}x = \int_{\mathbb{R}^N} au^p\phi\,\mathrm{d}x + {}_{H^{-s}}\langle f, \phi\rangle_{H^s},$$

where ${}_{H^{-s}}\langle .,.\rangle_{H^s}$ denotes the duality bracket between f and ϕ.

In recent years, there has been a considerable interest in more general version of nonlinear scalar field equation with fractional diffusion

$$\begin{cases} (-\Delta)^s u + V(x)u = g(x,u) \ \text{ in } \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{3.0.1}$$

see for e.g., the papers ([11, 29, 62, 66, 71, 81, 100]) and the references quoted therein. In the physical context, equations of the type (3.0.1) arise in the study of standing waves for the fractional Schrödinger equations and fractional Klein-Gordon equations. First consider the fractional Schrödinger equation

$$i\frac{\partial\psi}{\partial t} + (-\Delta)^s\psi + (V(x)+\omega)\psi = g(x,\psi),$$

where $\psi = \psi(x,t)$ is a complex valued function defined on $\mathbb{R}^N \times \mathbb{R}$. Suppose we assume

$$g(x, \rho e^{i\theta}) = e^{i\theta} g(x,\rho), \quad \forall\, \rho,\, \theta \in \mathbb{R}, \quad x \in \mathbb{R}^N, \tag{3.0.2}$$

and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $g(x,.)$ is a continuous odd function and $g(x,0) = 0$. Then one can look for standing wave solutions, i.e., $\psi(x,t) = e^{i\omega t}u(x)$, which led us to the following scalar field equation

$$(-\Delta)^s u + V(x)u = g(x,u) \ \text{ in } \mathbb{R}^N. \tag{3.0.3}$$

In context to fractional quantum mechanics, nonlinear fractional Schrödinger equation has been proposed by Laskin in ([86, 87]) in modelling some quantum mechanical phenomenon. In particular it arises in evaluating Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths.

One may also consider fractional nonlinear Klein-Gordon equation:

$$\psi_{tt} + (-\Delta)^s \psi + (V(x) + \omega^2)\psi = g(x, \psi),$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ and g satisfies (3.0.2). Then one can look for standing wave solutions as before and once again this will led us to the equation of type (3.0.3).

Equations of the type (3.0.3) with $s = 1$ arise in various other contexts of physics, for example, the classical approximation in statistical mechanics, constructive field theory, false vacuum in cosmology, nonlinear optics, laser propagation etc (see [13, 52, 70, 78]). They are also known as nonlinear Euclidean scalar field equations (see [21,22]) which has been studied extensively in the last few decades by many Mathematicians. We recall some of the works without any claim of completeness the papers ([15,21,22,57,59,111]) and the references quoted therein. Much of the interest has centered on the existence and multiplicity of solutions under various assumptions on the potential V and the nonlinearity g.

Also in the nonlocal case $s \in (0, 1)$, several existence and multiplicity results have been obtained for (3.0.3) using different variational techniques. Felmer et al. [66] have studied the existence and the symmetry of positive solutions to equation (3.0.3) with $V \equiv 1$ and involving a superlinear nonlinearity $g(x, u)$ satisfying the Ambrosetti-Rabinowitz condition. Frank et al. [71] have proved the uniqueness and nondegeneracy of positive ground state solutions to equation (3.0.3) with $V \equiv 1$ and $g(x, u) = |u|^{p-1}u$ where $p < 2_s^* - 1$. Using minimization on Nehari manifold, Secchi [100] have obtained the existence of ground state solutions to equation (3.0.3) when the nonlinearity is superlinear and subcritical, and the potential V satisfies suitable assumptions as $|x| \to \infty$. Pucci et al. [97] established via Mountain Pass Theorem and Ekeland's variational principle, the existence of multiple solutions for a Kirchhoff fractional Schrödinger equations involving a nonlinearity satisfying the Ambrosetti-Rabinowitz condition, a positive potential V validating suitable assumptions, and in presence of a perturbation term. We refer to [33, 45, 63, 67, 76] for further results related to (3.0.3).

Under the stated assumptions, problem $(\mathcal{P})$ can be considered as a perturbation problem of the following homogeneous equation:

$$\begin{aligned}&(-\Delta)^s w + w = w^p \ \text{ in } \mathbb{R}^N,\\ &w > 0 \ \text{ in } \mathbb{R}^N,\\ &w \in H^s(\mathbb{R}^N).\end{aligned} \tag{3.0.4}$$

In the seminal paper, Frank, Lenzmann and Silvestre in [71] proved that (3.0.4) has a unique (up to a translation) ground state solution. Further, if w is any positive solution of (3.0.4), then w is radially symmetric, strictly decreasing and $w \in H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and satisfies the decay property:

$$\frac{C^{-1}}{1+|x|^{N+2s}} \leq w(x) \leq \frac{C}{1+|x|^{N+2s}}, \tag{3.0.5}$$

with some constant $C > 0$ depending on $N,\ p,\ s$.

Our main question is whether positive solutions can still survive after a perturbation of type $(\mathcal{P})$. This question have been studied by several authors in the local case $s = 1$. The homogeneous case, i.e., $f(x) \equiv 0$ has been studied extensively by Bahri-Li [15], Berestycki-Lions [21] and Ding-Ni [59]. On the other hand for the non homogeneous case, i.e., $f(x) \not\equiv 0$ we refer the works of Bahri-Berestycki [14], Tanaka [106] in the case of bounded domain and Adachi-Tanaka [4], Jeanjean [83] and Zhu [113] in the case of entire $\mathbb{R}^N$ where existence and multiplicity of positive solutions were proved under some assumptions on the potential function and nonhomogeneous term. We also refer the work of Cao-Zhou [42] for the existence of positive solution with more general nonlinearities. In the nonlocal case, there are very few papers where the existence and the multiplicity of solutions for nonhomogeneous problem have been studied, we refer [19,53] for equations studied in bounded domain and [12,31,82] for entire $\mathbb{R}^N$. We also refer [17,18] for closely related problems.

In this article, drawing primary motivation from the above works, we propose to establish the existence and the multiplicity of positive solutions to the equation $(\mathcal{P})$, under the effect of a small perturbation $f \in H^{-s}(\mathbb{R}^N)$, and suitable assumptions on the potential a.

We separate the following two cases:

- $(\mathbf{A_1}):\quad a(x) \in (0,1] \quad \forall\, x \in \mathbb{R}^N, \quad \inf_{x\in\mathbb{R}^N} a(x) > 0,$
  $\mu(\{x : a(x) \neq 1\}) > 0,$ and $a(x) \to 1$ as $|x| \to \infty,$

- $(\mathbf{A_2}):\quad a(x) \geq 1 \quad \forall\, x \in \mathbb{R}^N,\ a \in L^\infty(\mathbb{R}^N), \quad \mu(\{x : a(x) \neq 1\}) > 0,$
  and $\quad a(x) \to 1$ as $|x| \to \infty,$

where $\mu(X)$ denotes the Lebesgue measure of a set X.

3.1 Main Results

Now we state our main theorems

Theorem 3.1.1. *Suppose a satisfies $(\mathbf{A_1})$ and*

$$1 - a(x) \leq \frac{C}{1 + |x|^{\mu(N+2s)}} \quad \forall\, x \in \mathbb{R}^N, \tag{3.1.1}$$

for some $\mu > p + 1 + \frac{N}{N+2s}$. Then there exists $\delta_0 > 0$ such that for any $0 \not\equiv f \in H^{-s}(\mathbb{R}^N)$ with f is a non-negative functional and $\|f\|_{H^{-s}(\mathbb{R}^N)} \leq \delta_0$, problem $(\mathcal{P})$ admits at least three positive solutions.

Theorem 3.1.2. *Let a satisfy $(\mathbf{A_2})$, $0 \not\equiv f \in H^{-s}(\mathbb{R}^N)$ is a nonnegative functional and S_1 be defined as in (3.2.24). Moreover, if*

$$\|f\|_{H^{-s}(\mathbb{R}^N)} < C_p S_1^{\frac{p+1}{2(p-1)}} \quad \textit{where} \quad C_p := (p\|a\|_{L^\infty(\mathbb{R}^N)})^{-\frac{1}{p-1}}\left(\tfrac{p-1}{p}\right),$$

then $(\mathcal{P})$ *admits at least two positive solutions.*

Remark 3.1.3. *For the above two Theorems, it was necessary that* $\|f\|_{H^{-s}(\mathbb{R}^N)}$ *sufficiently small but* $f \not\equiv 0$*. In contrast, our next existence result holds in the case when* $f \equiv 0$*.*

Theorem 3.1.4. *Let* $f \equiv 0$*,* $0 < a \in L^\infty(\mathbb{R}^N)$ *and there exists* $a_0 > 0$ *such that*

$$\lim_{|x|\to\infty} a(x) = a_0 = \inf_{x\in\mathbb{R}^N} a(x).$$

Then, there exists a positive solution to $(\mathcal{P})$ *for every* $1 < p < 2_s^* - 1$*.*

Like in the local case, it is well known that the Sobolev embedding

$$H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \quad \text{for } 2 \le p \le \frac{2N}{N-2s},$$

is not compact. Thus the variational functional associated with $(\mathcal{P})$ fails to satisfy the Palais-Smale (PS) condition. The lack of compactness becomes clear when one looks at the special case (3.0.4). Solutions of (3.0.4) are invariant under translation and therefore, it is not compact. Thus the standard variational technique can not be applied directly. The existence and multiplicity results obtained in the local case were based on the careful analysis of the Palais-Smale level. However one of the major differences in the nonlocal case $s \in (0,1)$ with the local case $s = 1$ is due to the difference in Palais-Smale decomposition theorem.

In the case of $s = 1$, we see that Palais-Smale condition holds for $\bar{I}_{a,f}$ (see Section 3.2 for the definitions) at level c if c can not be decomposed as $c = \bar{I}_{a,f}(\bar{u}) + k\bar{I}_{1,0}(w)$, where $k \ge 1$, $\bar{u}$ is a solution of $(\mathcal{P})$ and w is the unique radial solution of (3.0.4) (with $s = 1$). But in the case of $s \in (0,1)$, uniqueness of positive solution of (3.0.4) is not yet known, only the uniqueness of ground state solution is known ([71]). Therefore, studying the Palais-Smale decomposition theorem (see Proposition 3.2.1), we can not exclude the possibility of breaking down of Palais-Smale condition at the level c for $c \in \Big(\bar{I}_{a,f}(u)+\bar{I}_{1,0}(w^*),\ \bar{I}_{a,f}(u)+2\bar{I}_{1,0}(w^*)\Big)$, where w^* is the unique ground state solution of (3.0.4) and u is any positive solution of $(\mathcal{P})$. Thus one can not argue using Palais-Smale decomposition to obtain positive solutions to $(\mathcal{P})$ whose energy level is strictly greater than $\bar{I}_{a,f}(u)+\bar{I}_{1,0}(w^*)$. For the same reason, arguments of Bahri-Li [15] can not be adopted here to prove Theorem 3.1.4 even if we assume $\lim_{|x|\to\infty} a(x) = 1$.

It is worth mentioning about the novelty of the paper. In the local case $s = 1$, solutions of (3.0.4) has exponential decay, where as for $s \in (0,1)$, solutions of (3.0.4) has polynomial decay of the rate $|x|^{-(N+2s)}$. Thus it is not at all straight forward to guess that the energy estimates would stay in the desired level in the nonlocal case and hence deriving such estimates require a very careful analysis. Due to this fact we are able to prove Theorem 3.1.1 under much weaker growth rate assumption of a at infinity (see (3.1.1)) compared to the local case $s = 1$(see [4]), where it was assumed

$$1 - a(x) \le C \exp\Big(-(2+\delta)|x|\Big) \quad \text{for all}\, x \in \mathbb{R}^N,$$

for some constant $\delta > 0$, $C > 0$.

Now let us briefly explain the methodology to obtain our results. To prove Theorem 3.1.1, we establish existence of first positive solution as a perturbation of 0 (which actually solves the problem for $f \equiv 0$, without the signed condition) via Mountain Pass theorem. We obtain the second and third solutions of $(\mathcal{P})$ using Lusternik-Schnirelman category where the main problem lies in the breaking down of Palais-Smale condition at some level c and we have proved that below the level of breaking down of Palais Smale condition there are two other critical points of the energy functional associated to $(\mathcal{P})$.

To prove Theorem 3.1.2, we first decompose $H^s(\mathbb{R}^N)$ into three components which are homeomorphic to the interior, boundary and the exterior of the unit ball in $H^s(\mathbb{R}^N)$ respectively. Then using assumption (A_2), we prove that the energy functional associated to $(\mathcal{P})$ attains its infimum on one of the components which serves as our first positive solution. The second positive solution is obtained via a careful analysis on the (PS)-sequence associated to the energy functional and we construct a min-max critical level γ, where the (PS) condition holds. That leads to the existence of second positive solution.

In order to prove Theorem 3.1.4, we first establish existence of a positive solution u_k to the following problem:

$$\begin{cases} (-\Delta)^s u + u = a(x)|u|^{p-1}u \ \text{ in } B_k, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_k, \end{cases}$$

where B_k is the ball of radius k centered at 0. Then we show $\|u_k\|_{H^s(\mathbb{R}^N)}$ is uniformly bounded and there exists $0 \leq \bar{u} \in H^s(\mathbb{R}^N)$ such that up to a subsequence $u_k \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$ and $\bar{u}$ is a positive solution of $(\mathcal{P})$. The main difficulty in this proof lies in showing that $\bar{u}$ i.e., the weak limit of the subsequence u_k is a nontrivial element in $H^s(\mathbb{R}^N)$.

This chapter is organised in the following way: In Section 3.2, we prove the Palais-Smale decomposition theorem associated with the functional corresponding to $(\mathcal{P})$. In Section 3.4, we establish existence of three positive solutions of $(\mathcal{P})$, namely Theorem 3.1.1. Section 3.3 deals with existence of two positive solutions of $(\mathcal{P})$ under the assumption (A_2), namely Theorem 3.1.2. In section 3.5, we prove Theorem 3.1.4.

3.2 Palais-Smale characterization

In this section we study the Palais-Smale sequences (in short, PS sequences) of the functional associated to $(\mathcal{P})$.

$$\begin{aligned}\bar{I}_{a,f}(u) &= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x \\ &\qquad -\frac{1}{p+1}\int_{\mathbb{R}^N} a(x)|u|^{p+1}\,\mathrm{d}x - {}_{H^{-s}}\langle f,u\rangle_{H^s} \\ &= \frac{1}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)|u|^{p+1}\,\mathrm{d}x - {}_{H^{-s}}\langle f,u\rangle_{H^s},\end{aligned} \tag{3.2.1}$$

where $0 < a \in L^\infty(\mathbb{R}^N)$, $a(x) \to 1$ as $|x| \to \infty$ and $0 \not\equiv f \in H^{-s}(\mathbb{R}^N)$ is a nonnegative functional i.e., ${}_{H^{-s}}\langle f,u\rangle_{H^s} \geq 0$ whenever $u \geq 0$.

We say that the sequence $u_k \in H^s(\mathbb{R}^N)$ is a PS sequence for $\bar{I}_{a,f}$ at level β if $\bar{I}_{a,f}(u_k) \to \beta$ and $(\bar{I}_{a,f})'(u_k) \to 0$ in $H^{-s}(\mathbb{R}^N)$. It is easy to see that the weak limit of a PS sequence solves $(\mathcal{P})$ (with $f \equiv 0$) except the positivity.

However the main difficulty is that the PS sequence may not converge strongly and hence the weak limit can be zero even if $\beta > 0$. The main purpose of this section is to classify PS sequences for the functional $\bar{I}_{a,f}$. Classification of PS sequences has been done for various problems having lack of compactness, to quote a few, we cite [16, 20, 88]. We establish a classification theorem for the PS sequences of (3.2.1) in the spirit of the above results.

Throughout this section we assume $a(x) \to 1$ as $|x| \to \infty$.

Proposition 3.2.1. *Let $\{u_k\} \subset H^s(\mathbb{R}^N)$ be a PS sequence for $\bar{I}_{a,f}$. Then there exists a subsequence (still denoted by u_k) for which the following hold : there exists an integer $m \geq 0$, sequences x_k^i for $1 \leq i \leq m$, functions $\bar{u}$, w_i for $1 \leq i \leq m$ such that*

$$(-\Delta)^s\bar{u} + \bar{u} = a(x)|\bar{u}|^{p-1}\bar{u} + f \quad \textit{in} \quad \mathbb{R}^N \tag{3.2.2}$$

$$\begin{aligned} (-\Delta)^s w_i + w_i = w_i^p \ in\ \mathbb{R}^N \\ w_i \in H^s(\mathbb{R}^N),\ w_i \not\equiv 0 \end{aligned} \tag{3.2.3}$$

$$\begin{aligned} u_k - \Big(\bar{u} + \sum_{i=1}^m w_i(\cdot - x_k^i)\Big) \to 0 \ as\ k \to \infty \\ \bar{I}_{a,f}(u_k) \to \bar{I}_{a,f}(\bar{u}) + \sum_{i=1}^m \bar{I}_{1,0}(w_i)\ as\ k \to \infty \end{aligned} \tag{3.2.4}$$

$$|x_k^i| \to \infty,\ |x_k^i - x_k^j| \to \infty \ as\ k \to \infty,\ for\ 1 \le i \ne j \le m, \tag{3.2.5}$$

where we agree in the case $m = 0$, the above holds without w_i, x_k^i.

To prove the above proposition, we first need some auxiliary lemmas.

Lemma 3.2.2. *Let $t > 0$ and $2 \le q < 2_s^*$. If $\{w_k\}$ is a bounded sequence in $H^s(\mathbb{R}^N)$ and if*

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,t)} |w_k|^q \mathrm{d}x \longrightarrow 0 \quad as \quad k \to \infty,$$

then $w_k \to 0$ in $L^r(\mathbb{R}^N)$ for all $r \in (2, 2_s^*)$. In addition, if w_k satisfies

$$(-\Delta)^s w_k + w_k - a(x)|w_k|^{p-1} w_k - f \longrightarrow 0 \quad in \quad H^{-s}(\mathbb{R}^N), \tag{3.2.6}$$

then $w_k \to 0$ in $H^s(\mathbb{R}^N)$.

Proof. Choose $\kappa \in (q, 2_s^*)$ arbitrarily. Therefore, using interpolation, we have

$$\|w_k\|_{L^\kappa(B(y,t))} \le \|w_k\|_{L^q(B(y,t))}^{1-\lambda} \|w_k\|_{L^{2_s^*}(B(y,t))}^{\lambda} \le C \|w_k\|_{L^q(B(y,t))}^{1-\lambda} \|w_k\|_{H^s(\mathbb{R}^N)}^{\lambda},$$

where $\frac{1}{\kappa} = \frac{1-\lambda}{q} + \frac{\lambda}{2_s^*}$. Now, covering $\mathbb{R}^N$ by balls of radius t, in such a way that each point of $\mathbb{R}^N$ is contained in at most N_0 balls (for some positive integer N_0), we find

$$\int_{\mathbb{R}^N} |w_k|^\kappa \,\mathrm{d}x \le N_0 C^\kappa \sup_{y \in \mathbb{R}^N} \left(\int_{B(y,t)} |w_k|^q \mathrm{d}x \right)^{(1-\lambda)\frac{\kappa}{q}} \|w_k\|_{H^s(\mathbb{R}^N)}^{\lambda\kappa}.$$

Therefore, the hypothesis of the lemmas implies $w_k \to 0$ in $L^\kappa(\mathbb{R}^N)$ for all $\kappa \in (q, 2_s^*)$. This completes the lemma if $q = 2$, otherwise, if $q > 2$, then again one can argue in similar way by choosing $\kappa \in (2, q)$. In addition, if (3.2.6) is satisfied, then we obtain

$$| {}_{H^{-s}}\langle(-\Delta)^s w_k + w_k - a(x)|w_k|^{p-1}w_k - f,\, w_k\rangle_{H^s}| = o(1)\|w_k\|_{H^s(\mathbb{R}^N)}, \quad (3.2.7)$$

where ${}_{H^{-s}}\langle .,.\rangle_{H^s}$ denotes the duality bracket between $H^{-s}(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$. Since $\{w_k\}$ is bounded in $H^s(\mathbb{R}^N)$, the RHS is $o(1)$. On the other hand, for the LHS we observe that since w_k is bounded in $H^s(\mathbb{R}^N)$ and $w_k \to 0$ in $L^r(\mathbb{R}^N)$, for $r \in (2, 2_s^*)$, we must have $w_k \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ and consequently, ${}_{H^{-s}}\langle f, w_k\rangle_{H^s} = o(1)$. Also, by first part, $w_k \to 0$ in $L^{p+1}(\mathbb{R}^N)$. Hence, (3.2.7) yields $w_k \to 0$ in $H^s(\mathbb{R}^N)$. □

Lemma 3.2.3. *Let ϕ_k weakly converges to ϕ in $H^s(\mathbb{R}^N)$, then we have*

$$a|\phi_k|^{p-1}\phi_k - a|\phi|^{p-1}\phi \longrightarrow 0 \quad in \quad H^{-s}(\mathbb{R}^N).$$

Proof. Defining ψ_k as $\phi_k - \phi$, we see $\psi_k \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. In particular, $\{\psi_k\}$ is bounded in $H^s(\mathbb{R}^N)$. Thus, up to a subsequence, $\psi_k \to 0$ in $L^q_{loc}(\mathbb{R}^N)$ for all $1 < q < 2_s^*$ and $\psi_k \to 0$ a.e.. Consequently, $a|\phi + \psi_k|^{p-1}(\phi + \psi_k) - a|\phi|^{p-1}\phi \to 0$ a.e.. Therefore, using Vitali's convergence theorem, it follows $a|\phi + \psi_k|^{p-1}(\phi + \psi_k) - a|\phi|^{p-1}\phi \to 0$ in $L_{loc}^{\frac{p+1}{p}}(\mathbb{R}^N)$. We also observe that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|a|\phi + \psi_k|^{p-1}(\phi + \psi_k) - a|\phi|^{p-1}\phi\right|^{\frac{p+1}{p}} \leq \varepsilon|\psi_k|^{p+1} + C_\varepsilon|\phi|^{p+1}. \quad (3.2.8)$$

Moreover, since $\psi_k \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ implies ψ_k is uniformly bounded in $L^{p+1}(\mathbb{R}^N)$ and the fact that $|\phi|^{p+1} \in L^1(\mathbb{R}^N)$, it is easy to see from (3.2.8) that given $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0,R)} \left|a|\phi + \psi_k|^{p-1}(\phi + \psi_k) - a|\phi|^{p-1}\phi\right|^{\frac{p+1}{p}} \mathrm{d}x < \varepsilon. \quad (3.2.9)$$

As a result, $a|\phi+\psi_k|^{p-1}(\phi+\psi_k)-a|\phi|^{p-1}\phi \to 0$ in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Since $H^s(\mathbb{R}^N)$ is continuously embedded in $L^{p+1}(\mathbb{R}^N)$, which is the dual space of $L^{\frac{p+1}{p}}(\mathbb{R}^N)$, it follows that $a|\phi+\psi_k|^{p-1}(\phi+\psi_k)-a|\phi|^{p-1}\phi \to 0$ in $H^{-s}(\mathbb{R}^N)$. □

Lemma 3.2.4. *For each $c_0 \geq 0$, there exists $\delta > 0$ such that if $v \in H^s(\mathbb{R}^N)$ solves*

$$(-\Delta)^s v + v = |v|^{p-1}v \text{ in } \mathbb{R}^N, \ v \in H^s(\mathbb{R}^N), \tag{3.2.10}$$

and $\|v\|_{H^s(\mathbb{R}^N)} \leq c_0$, $\|v\|_{L^2(\mathbb{R}^N)} \leq \delta$, then $v \equiv 0$.

Proof. Taking v as a test function, it follows

$$\|v\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |v|^{p+1}\,\mathrm{d}x \leq \|v\|^{\lambda(p+1)}_{L^2(\mathbb{R}^N)}\|v\|^{(1-\lambda)(p+1)}_{L^{2^*_s}(\mathbb{R}^N)} \leq C\delta^{\lambda(p+1)}\|v\|^{(1-\lambda)(p+1)}_{H^s(\mathbb{R}^N)}, \tag{3.2.11}$$

where λ is such that $\frac{1}{p+1} = \frac{\lambda}{2} + \frac{1-\lambda}{2^*_s}$. If $(1-\lambda)(p+1) \geq 2$, i.e., $p \geq 1 + \frac{4s}{N}$, then (3.2.11) implies $v \equiv 0$ as we can choose δ small enough. Now if $p < 1 + \frac{4s}{N}$, then (3.2.11) yields $\|v\|_{H^s(\mathbb{R}^N)} \leq C\delta^{\frac{\lambda(p+1)}{2-(1-\lambda)(p+1)}}$. Therefore, choosing $\delta > 0$ small enough, we can conclude the lemma. □

Proof of Proposition 3.2.1:

Proof. We prove this proposition in the spirit of [16]. We divide the proof into few steps.

Step 1: Using standard arguments it follows that any PS sequence for $\bar{I}_{a,f}$ is bounded in $H^s(\mathbb{R}^N)$. More precisely,

$$\begin{aligned}\lim_{k\to\infty} \bar{I}_{a,f}(u_k) + o(1) + o(1)\|u_k\|_{H^s(\mathbb{R}^N)} &\geq \bar{I}_{a,f}(u_k) - \frac{1}{p+1}(\bar{I}_{a,f})'(u_k)u_k \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_k\|^2_{H^s(\mathbb{R}^N)} \\ &\quad - \left(1 - \frac{1}{p+1}\right) {}_{H^{-s}}\langle f, u_k\rangle_{H^s}.\end{aligned}$$

Hence boundedness follows. Consequently, up to a subsequence $u_k \rightharpoonup u$ in

$H^s(\mathbb{R}^N)$. Moreover, as $(\bar{I}_{a,f})'(u_k)v \to 0$ as $k \to \infty \quad \forall\, v \in H^s(\mathbb{R}^N)$, we have

$$(-\Delta)^s u_k + u_k - a(x)|u_k|^{p-1}u_k - f = \varepsilon_k \xrightarrow{k} 0 \quad \text{in} \quad H^{-s}(\mathbb{R}^N). \tag{3.2.12}$$

Step 2: From (3.2.12) we get by letting $k \to \infty$,

$$\iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} u_k v \,\mathrm{d}x \\ - \int_{\mathbb{R}^N} a(x)\,|u_k|^{p-1} u_k v \,\mathrm{d}x - {}_{H^{-s}}\langle f, v\rangle_{H^s} \to 0,$$

for all $v \in H^s(\mathbb{R}^N)$.

Claim 1: Weak limit u satisfies

$$(-\Delta)^s u + u = a(x)\,|u|^{p-1}u + f \text{ in } \mathbb{R}^N, \quad u \in H^s(\mathbb{R}^N).$$

Indeed, $u_k \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ implies,

$$\iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} u_k v \,\mathrm{d}x \\ \longrightarrow \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} uv \,\mathrm{d}x.$$

Further using Lemma 3.2.3 we conclude

$$\int_{\mathbb{R}^N} a(x)\,|u_k|^{p-1} u_k v \,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^N} a(x)\,|u|^{p-1} uv \,\mathrm{d}x.$$

In view of above the claim follows.

Step 3: In this step we show that $u_k - u$ is a PS sequence for $\bar{I}_{a,0}$ at the level $\lim_{k\to\infty} \bar{I}_{a,f}(u_k) - \bar{I}_{a,f}(u)$ and $u_k - u \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$.

To see this, first we observe that using Brezis-Lieb lemma, we have

$$\iint_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y - \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ = \iint_{\mathbb{R}^{2N}} \frac{|(u_k - u)(x) - (u_k - u)(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y + o(1). \tag{3.2.13}$$

$$\int_{\mathbb{R}^N} |u_k|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |u_k - u|^2 \,\mathrm{d}x + o(1). \tag{3.2.14}$$

$$\int_{\mathbb{R}^N} a(x)|u_k|^{p+1}\,\mathrm{d}x - \int_{\mathbb{R}^N} a(x)|u|^{p+1}\,\mathrm{d}x = \int_{\mathbb{R}^N} a(x)|u_k - u|^{p+1}\,\mathrm{d}x + o(1). \tag{3.2.15}$$

Further as $u_k \rightharpoonup u$ and $f \in H^{-s}(\mathbb{R}^N)$, we also have

$$_{H^{-s}}\langle f, u_k\rangle_{H^s} \longrightarrow {}_{H^{-s}}\langle f, u\rangle_{H^s}. \tag{3.2.16}$$

Using above, it follows that

$$\begin{aligned}
\bar{I}_{a,0}(u_k - u) &= \frac{1}{2}\left(\|u_k\|^2_{H^s(\mathbb{R}^N)} - \|u\|^2_{H^s(\mathbb{R}^N)}\right) \\
&\quad -\frac{1}{p+1}\left(\int_{\mathbb{R}^N} a(x)|u_k|^{p+1} - \int_{\mathbb{R}^N} a(x)|u|^{p+1}\right) + o(1) \\
&\longrightarrow \lim_{k\to\infty} \bar{I}_{a,f}(u_k) + {}_{H^{-s}}\langle f, u\rangle_{H^s} - \bar{I}_{a,0}(u), \quad \text{as } k \to \infty, \\
&= \lim_{k\to\infty} \bar{I}_{a,f}(u_k) - \bar{I}_{a,f}(u).
\end{aligned}$$

Next, note that (3.2.12) and Claim 1 implies

$$(-\Delta)^s(u_k-u)+(u_k-u)-a(x)(|u_k|^{p-1}u_k-|u|^{p-1}u) = \varepsilon_k \to 0 \quad \text{in} \quad H^{-s}(\mathbb{R}^N).$$

Combining this with Lemma 3.2.3, we conclude $I'_{a,0}(u_k-u) \to 0$ in $H^{-s}(\mathbb{R}^N)$. Hence Step 3 follows.

<u>**Step 4:**</u> Using Lemma 3.2.2 we have, either $u_k-u \to 0$ in $H^s(\mathbb{R}^N)$, in that case the proof is over or there exists $\alpha > 0$, such that up to a subsequence

$$Q_k(1) := \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_k - u|^2\,\mathrm{d}x > \alpha > 0.$$

Therefore we can find a sequence $\{y_k\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_k,1)} |u_k - u|^2\,\mathrm{d}x \geq \alpha. \tag{3.2.17}$$

Let us define $\tilde{u}_k(x) := (u_k - u)(y_k + x)$, then using translation invariance of $H^s(\mathbb{R}^N)$, it implies $\tilde{u}_k$ is also bounded in $H^s(\mathbb{R}^N)$ and hence up to a subsequence converges weakly in $H^s(\mathbb{R}^N)$ to $\tilde{u}$. Now we claim that $\tilde{u} \neq 0$. Indeed

Rellich compactness theorem yields $H^s(B(y_k,1)) \hookrightarrow L^2(B(y_k,1))$ compactly embedded and therefore (3.2.17) concludes the claim.

Also it follows from the fact $u_k - u \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ and (3.2.17) that

$$|y_k| \longrightarrow \infty \quad \text{as } k \to \infty.$$

Now define, $v_k := \tilde{u}_k - \tilde{u}$. Note that, $\tilde{u}_k \rightharpoonup \tilde{u}$ implies $v_k \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. Using this and Lemma 3.2.3, in the definition of $\bar{I}'_{1,0}(v_k)$ yields

$$\bar{I}'_{1,0}(v_k) = o(1) \text{ in } H^{-s}(\mathbb{R}^N), \tag{3.2.18}$$

i.e., $(-\Delta)^s v_k + v_k - |v_k|^{p-1}v_k \longrightarrow 0 \quad \text{in } H^{-s}(\mathbb{R}^N)$.

Step 5: In this step we show that

$$(-\Delta)^s\tilde{u} + \tilde{u} = |\tilde{u}|^{p-1}\tilde{u} \quad \text{in } \mathbb{R}^N, \quad \tilde{u} \in H^s(\mathbb{R}^N). \tag{3.2.19}$$

To prove this step, it is enough to show that for arbitrarily chosen $v \in C_0^\infty(\mathbb{R}^N)$, the following holds:

$$\langle \tilde{u}, v\rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\tilde{u}|^{p-1}\tilde{u}v \,\mathrm{d}x. \tag{3.2.20}$$

To show the above, let $v \in C_0^\infty(\mathbb{R}^N)$ be arbitrarily chosen. Since, $\tilde{u}_k \rightharpoonup \tilde{u}$, using Step 3, we estimate the inner product between $\tilde{u}$ and v as follows:

$$\begin{aligned}
\langle \tilde{u}, v\rangle_{H^s(\mathbb{R}^N)} &= \lim_{k\to\infty} \langle \tilde{u}_k, v\rangle_{H^s(\mathbb{R}^N)} \\
= \lim_{k\to\infty} \Bigg[&\iint_{\mathbb{R}^{2N}} \frac{\Big((u_k-u)(x+y_k) - (u_k-u)(y+y_k)\Big)\Big(v(x)-v(y)\Big)}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^N} (u_k-u)(x+y_k)v(x)\,\mathrm{d}x \Bigg] \\
= \lim_{k\to\infty} \Bigg[&\iint_{\mathbb{R}^{2N}} \frac{\Big((u_k-u)(x) - (u_k-u)(y)\Big)\Big(v(x-y_k)-v(y-y_k)\Big)}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^N} (u_k-u)(x)v(x-y_k)\,\mathrm{d}x \Bigg] \\
= \lim_{k\to\infty} &\int_{\mathbb{R}^N} a(x)|(u_k-u)(x)|^{p-1}(u_k-u)(x)v(x-y_k)\,\mathrm{d}x \\
= \lim_{k\to\infty} &\int_{\mathbb{R}^N} a(x+y_k)|\tilde{u}_k(x)|^{p-1}\tilde{u}_k(x)v(x)\,\mathrm{d}x.
\end{aligned} \tag{3.2.21}$$

Claim 2: $\lim_{k\to\infty} \displaystyle\int_{\mathbb{R}^N} a(x+y_k)|\tilde{u}_k(x)|^{p-1}\tilde{u}_k(x)v(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} |\tilde{u}|^{p-1}\tilde{u}v\,\mathrm{d}x.$

To prove the claim, we estimate

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} a(x+y_k)|\tilde{u}_k(x)|^{p-1}\tilde{u}_k(x)v(x)\,\mathrm{d}x - \int_{\mathbb{R}^N} |\tilde{u}|^{p-1}\tilde{u}v\,\mathrm{d}x \right| \\ \leq & \left| \int_{\mathbb{R}^N} a(x+y_k)(|\tilde{u}_k|^{p-1}\tilde{u}_k - |\tilde{u}|^{p-1}\tilde{u})v\,\mathrm{d}x \right| + \left| \int_{\mathbb{R}^N} \Big(a(x+y_k)-1\Big)|\tilde{u}|^{p-1}\tilde{u}v\,\mathrm{d}x \right| \\ = & \; I_k^1 + J_k^1. \end{aligned}$$

Since $|y_k| \to \infty$, $|\tilde{u}|^{p-1}\tilde{u}v \in L^1(\mathbb{R}^N)$, $a \in L^\infty(\mathbb{R}^N)$ and $a(x) \to 1$ as $|x| \to \infty$, using dominated convergence theorem, it follows that

$$\lim_{k\to\infty} J_k^1 = 0. \tag{3.2.22}$$

On the other hand, since v has compact support, using Vitali's convergence theorem

$$\lim_{k\to\infty} I_k^1 \leq \lim_{k\to\infty} \|a\|_{L^\infty(\mathbb{R}^N)} \int_{\operatorname{supp} v} \Big||\tilde{u}_k|^{p-1}\tilde{u}_k - |\tilde{u}|^{p-1}\tilde{u}\Big||v|\mathrm{d}x = 0.$$

Combining the above two estimates, Claim 2 holds. Using Claim 2, we conclude Step 5 from (3.2.21).

Further, by Brezis-Lieb Lemma

$$\begin{aligned} \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \; & - \; \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ & - \; \iint_{\mathbb{R}^{2N}} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \to 0; \end{aligned}$$

$$\int_{\mathbb{R}^N} |\tilde{u}_k|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} |\tilde{u}|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} |v_k|^2\,\mathrm{d}x \to 0.$$

as $k \to \infty$.

In view of the above steps, if $\tilde{u}_k - \tilde{u}$ does not converge to zero in $H^s(\mathbb{R}^N)$, we can repeat the procedure for the Palais-Smale (PS) sequence $\tilde{u}_k - \tilde{u}$ to land in either of the two cases. If it converges to zero then we stop or else we repeat the process. But this process has to stop in finitely many steps and we obtain $\tilde{u}_1,\ \tilde{u}_2,\ \ldots,\ \tilde{u}_n$ denotes the limit solution of (3.2.19) obtained through the procedure, we have

$$\sum_{i=1}^{n}\int_{\mathbb{R}^N}|\tilde{u}_i|^2\mathrm{d}x \le \liminf_{k\to\infty}\int_{\mathbb{R}^N}|u_k-u|^2\,\mathrm{d}x.$$

Thus n can not go to infinity in view of Lemma 3.2.4. □

We end this section with the definition of some functions which will be used throughout the rest of the paper. We define,

$$J(u) := \frac{\|u\|^2_{H^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} a(x)|u(x)|^{p+1}\mathrm{d}x\right)^{\frac{2}{p+1}}},\quad J_\infty(u) := \frac{\|u\|^2_{H^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} |u(x)|^{p+1}\mathrm{d}x\right)^{\frac{2}{p+1}}}, \tag{3.2.23}$$

$$\text{and } S_1 := \inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}} J_\infty(u). \tag{3.2.24}$$

From [71], it is known that S_1 is achieved by unique ground state solution w^* of (3.0.4). Further w^* is radially symmetric positive decreasing smooth function satisfying (3.0.5).

3.3 Proof of Theorem 3.1.1

In this section we prove multiplicity of positive solutions to $(\mathcal{P})$ when a satisfies the assumption $(\mathbf{A_1})$ in the spirit of [4] (also see [15], [5]). Define,

$$I_{a,f}(u) = \frac{1}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}\,\mathrm{d}x - {}_{H^{-s}}\langle f,u\rangle_{H^s}, \tag{3.3.1}$$

where $f \in H^{-s}(\mathbb{R}^N)$ is a nonnegative nontrivial functional. Clearly, if u is a critical points of $I_{a,f}$, then u is solution to

$$\begin{cases} (-\Delta)^s u + u = a(x)u_+^p + f(x) \ \text{ in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N). \end{cases} \tag{3.3.2}$$

Remark 3.3.1. *If u is a weak solution of (3.3.2) and f is a nonnegative functional, then taking $v = u_-$ as a test function in (3.3.2), we obtain*

$$-\|u_-\|^2_{\dot{H}^s(\mathbb{R}^N)} - \iint_{\mathbb{R}^{2N}} \frac{[u_+(y)u_-(x) + u_+(x)u_-(y)]}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y = {}_{H^{-s}}\langle f, u_-\rangle_{H^s} \geq 0.$$

This in turn implies $u_- = 0$, i.e., $u \geq 0$. Therefore, using maximum principle [56, Theorem 1.2], it follows that, $u > 0$ and hence u is a solution to ($\mathcal{P}$).

We set

$$\Sigma := \{u \in H^s(\mathbb{R}^N) : \|u\|_{H^s(\mathbb{R}^N)} = 1\} \quad \text{and} \quad \tilde{\Sigma}_+ := \{u \in \Sigma : u_+ \not\equiv 0\}. \tag{3.3.3}$$

Define a modified functional $J_{a,f} : \tilde{\Sigma}_+ \to \mathbb{R}$ by

$$J_{a,f} := \max_{t>0} I_{a,f}(tv), \tag{3.3.4}$$

where $I_{a,f}$ is defined as in (3.3.1). Set,

$$\underline{a} = \inf_{x\in\mathbb{R}^N} a(x) > 0,$$

$$\bar{a} = \sup_{x\in\mathbb{R}^N} a(x) = 1.$$

From the definition of $J_{a,f}$, a straight forward computation yields

$$J_{a,0}(v) = I_{a,0}\left(\left(\int_{\mathbb{R}^N} a(x)v_+^{p+1}\mathrm{d}x\right)^{-\frac{1}{p-1}} v\right) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\int_{\mathbb{R}^N} a(x)v_+^{p+1}\mathrm{d}x\right)^{-\frac{2}{p-1}}. \tag{3.3.5}$$

Thus,

$$\bar{a}^{-\frac{2}{p-1}} J_{1,0}(v) \leq J_{\bar{a},0}(v) \leq J_{a,0}(v) \leq J_{\underline{a},0}(v) = \underline{a}^{-\frac{2}{p-1}} J_{1,0}(v).$$

Further, as

$$\max_{t\in[0,1]} I_{1,0}(tw^*) = I_{1,0}(w^*),$$

where w^* is the unique (radial) ground state solution of (3.0.4), we obtain

$$\bar{a}^{-\frac{2}{(p-1)}} I_{1,0}(w^*) \leq \inf_{v\in\tilde{\Sigma}_+} J_{a,0}(v) \leq \underline{a}^{-\frac{2}{(p-1)}} I_{1,0}(w^*). \tag{3.3.6}$$

Lemma 3.3.2. *(i) Let $u \in H^s(\mathbb{R}^N)$ and $\varepsilon \in (0,1)$. Then there holds*

$$(1-\varepsilon)I_{\frac{a}{1-\varepsilon},\,0}(u)-\frac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)} \le I_{a,f}(u) \le (1+\varepsilon)I_{\frac{a}{1+\varepsilon},\,0}(u)+\frac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)}. \tag{3.3.7}$$

(ii) For $v \in \tilde{\Sigma}_+$ and $\varepsilon \in (0,1)$, there holds

$$(1-\varepsilon)^{\frac{p+1}{p-1}}J_{a,0}(v)-\tfrac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)} \le J_{a,f}(v) \le (1+\varepsilon)^{\frac{p+1}{p-1}}J_{a,0}(v)+\tfrac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)}. \tag{3.3.8}$$

(iii) In particular, there exists $d_0 > 0$ such that if $\|f\|_{H^{-s}(\mathbb{R}^N)} \le d_0$, then,

$$\inf_{v\in\tilde{\Sigma}_+} J_{a,f}(v) > 0.$$

Proof. Using Young inequality with $\varepsilon > 0$, we can write

$$|\,{}_{H^{-s}}\langle f,u\rangle_{H^s}| \le \|f\|_{H^{-s}(\mathbb{R}^N)}\|u\|_{H^s(\mathbb{R}^N)} \le \frac{\varepsilon}{2}\|u\|^2_{H^s(\mathbb{R}^N)} + \frac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)}.$$

Applying the above inequality in the definition of $I_{a,f}(u)$, we obtain (i). Using (i) in the definition of $J_{a,f}(v)$, we obtain

$$(1-\varepsilon)J_{\frac{a}{1-\varepsilon},0}(v) - \tfrac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)} \le J_{a,f}(v) \le (1+\varepsilon)J_{\frac{a}{1+\varepsilon},0}(v) + \tfrac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)}.$$

Combining this with (3.3.5), we get (ii). Finally, substituting (3.3.6) into (ii) yields (iii). □

Next, for $v \in \tilde{\Sigma}_+$, we study properties of the function $\tilde{g} : [0,\ \infty) \to \mathbb{R}$ defined as

$$\tilde{g}(t) := I_{a,f}(tv). \tag{3.3.9}$$

Lemma 3.3.3. *(i) For every $v \in \tilde{\Sigma}_+$, the function $\tilde{g}$ has at most two critical points in $[0,\ \infty)$.*

(ii) If $\|f\|_{H^{-s}(\mathbb{R}^N)} \le d_0$ (d_0 is chosen as in Lemma 3.3.2), then for any $v \in \tilde{\Sigma}_+$, there exists a unique $t_{a,f}(v) > 0$ such that

$$I_{a,f}\Big(t_{a,f}(v)v\Big) = J_{a,f}(v),$$

where $J_{a,f}$ is defined as in (3.3.4). *Moreover, $t_{a,f}(v) > 0$ satisfies,*

$$t_{a,f}(v) > \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\mathrm{d}x\right)^{-\frac{1}{p-1}} \geq \left(pS_1^{-\frac{(p+1)}{2}}\right)^{-\frac{1}{p-1}}, \tag{3.3.10}$$

and furthermore

$$I''_{a,f}\Big(t_{a,f}(v)v\Big)(v,\ v) < 0. \tag{3.3.11}$$

(iii) If $\tilde{g}$ has a critical point different from $t_{a,f}(v)$, then it lies in $\left[0,\ (1-\frac{1}{p})^{-1}\|f\|_{H^{-s}(\mathbb{R}^N)}\right]$.

This lemma can be proved exactly in the same spirit of [4, Lemma 1.3]. We skip the details. Now we prove the existence of first positive solution in the neighbourhood of 0.

3.3.1 Existence of first solution

The following proposition provides existence of first positive solution.

Proposition 3.3.4. *Let d_0 be as in Lemma 3.3.3. Then there exists $r_1 > 0$ and $d_1 \in (0,\ d_0]$ such that*

(i) $I_{a,f}(u)$ is strictly convex in $B(r_1) = \{u \in H^s(\mathbb{R}^N)\ :\ \|u\|_{H^s(\mathbb{R}^N)} < r_1\}$.

(ii) If $\|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_1$, then

$$\inf_{\|u\|_{H^s(\mathbb{R}^N)}=r_1} I_{a,f}(u) > 0.$$

Moreover, $I_{a,f}$ has a unique critical point $u_{locmin}(a, f; x)$ in $B(r_1)$ and it satisfies,

$$u_{locmin}(a, f; x) \in B(r_1) \quad \text{and } I_{a,f}(u_{locmin}(a, f; x)) = \inf_{u\in B(r_1)} I_{a,f}(u). \tag{3.3.12}$$

i.e., $u_{locmin}(a, f; x)$ is a positive solution to ($\mathcal{P}$) *satisfying* (3.3.12).

Proof. We begin the proof of part (i).

$$I''_{a,f}(u)(h, h) = \|h\|^2_{H^s(\mathbb{R}^N)} - p\int_{\mathbb{R}^N} a(x)u_+^{p-1}h^2\,\mathrm{d}x \tag{3.3.13}$$

Since $a \leq 1$, using Hölder inequality and Sobolev inequality, we estimate the second term on the RHS as follows

$$\begin{aligned}\int_{\mathbb{R}^N} a(x)u_+^{p-1}h^2\,\mathrm{d}x &\leq \left(\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x\right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}^N} |h|^{p+1}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \\ &\leq S_1^{-\frac{p-1}{2}} S_1^{-1} \|u\|_{H^s(\mathbb{R}^N)}^{p-1} \|h\|_{H^s(\mathbb{R}^N)}^2 \\ &= S_1^{-\frac{p+1}{2}} \|u\|_{H^s(\mathbb{R}^N)}^{p-1} \|h\|_{H^s(\mathbb{R}^N)}^2.\end{aligned}$$

Thus substituting the above in (5.2.1) we obtain

$$I''_{a,f}(u)(h,h) \geq \left(1 - pS_1^{-\frac{p+1}{2}} \|u\|_{H^s(\mathbb{R}^N)}^{p-1}\right) \|h\|_{H^s(\mathbb{R}^N)}^2.$$

Therefore, $I''_{a,\,f}(u)$ is positive definite for $u \in B(r_1)$, with $r_1 = p^{-\frac{1}{p-1}} S_1^{\frac{p+1}{2(p-1)}}$ and hence $I_{a,f}(u)$ is strictly convex in $B(r_1)$. This completes the proof of part (i).

(ii) Let $\|u\|_{H^s(\mathbb{R}^N)} = r_1$, then we have

$$\begin{aligned}I_{a,f}(u) &= \tfrac{1}{2}\|u\|_{H^s(\mathbb{R}^N)}^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}\,\mathrm{d}x - {}_{H^{-s}}\langle f, u\rangle_{H^s} \\ &\geq \tfrac{1}{2}r_1^2 - \tfrac{1}{p+1}S_1^{-\frac{p+1}{2}} r_1^{p+1} - r_1\|f\|_{H^{-s}(\mathbb{R}^N)} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}S_1^{-\frac{p+1}{2}} r_1^{p-1}\right) r_1^2 - r_1\|f\|_{H^{-s}(\mathbb{R}^N)}\end{aligned}$$

Since, $r_1^{p-1} = \frac{1}{p}S_1^{\frac{p+1}{2}}$, we obtain

$$I_{a,f}(u) \geq \left(\frac{1}{2} - \frac{1}{p(p+1)}\right) r_1^2 - r_1\|f\|_{H^{-s}(\mathbb{R}^N)}.$$

Thus there exists $d_1 \in (0,\ d_0]$ such that

$$\inf_{\|u\|_{H^s(\mathbb{R}^N)} = r_1} I_{a,f}(u) > 0, \quad \text{for } \ 0 < \|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_1.$$

Since $I_{a,f}(u)$ is strictly convex in $B(r_1)$ and $\inf_{\|u\|_{H^s(\mathbb{R}^N)} = r_1} I_{a,f}(u) > 0 = I_{a,f}(0)$, there exists a unique critical point $u_{locmin}(a, f; x)$ of $I_{a,f}$ in $B(r_1)$ and it satisfies

$$I_{a,f}(u_{locmin}(a, f; x)) = \inf_{\|u\|_{H^s(\mathbb{R}^N)} < r_1} I_{a,f}(u) < I_{a,f}(0) = 0, \tag{3.3.14}$$

where the last inequality is due to strict convexity of $I_{a,f}$ in $B(r_1)$. Combining this with Remark 3.3.1, we conclude the proof of the proposition. □

The next proposition characterises all the critical points of $I_{a,f}$ in terms of the functional $J_{a,f}$.

Proposition 3.3.5. *Let* $d_2 := \min\{d_1,\ (1-\frac{1}{p})r_1\} > 0$*, where* d_1*,* r_1 *be as in Proposition 3.3.4 and suppose that* $0 < \|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_2$*. Then,*

(i) $J_{a,f} \in C^1(\tilde{\Sigma}_+,\ \mathbb{R})$ *and*

$$J'_{a,f}(v)h = t_{a,f}(v) I'_{a,f}\Big(t_{a,f}(v)v\Big)h \tag{3.3.15}$$

for all $h \in T_v\tilde{\Sigma}_+ = \{h \in H^s(\mathbb{R}^N) \mid \langle h,\ v\rangle_{H^s(\mathbb{R}^N)} = 0\}$*.*

(ii) $v \in \tilde{\Sigma}_+$ *is a critical point of* $J_{a,f}(v)$ *iff* $t_{a,f}(v)v \in H^s(\mathbb{R}^N)$ *is a critical point of* $I_{a,f}(u)$*.*

(iii) Moreover, the set of all critical points of $I_{a,f}(u)$ *can be written as*

$$\Big\{t_{a,f}(v)v \mid v \in \tilde{\Sigma}_+,\ J'_{a,f}(v) = 0\Big\} \cup \Big\{u_{locmin}(a, f; x)\Big\} \tag{3.3.16}$$

Proof. (i) Let $\tilde{g}$ be as defined in (3.3.9). Then, from Lemma 3.3.3, we have

$$\tilde{g}'(t_{a,f}(v)) = I'_{a,f}\Big(t_{a,f}(v)v\Big)(v) = 0 \quad \text{and} \quad I''_{a,f}\Big(t_{a,f}(v)v\Big)(v,\ v) < 0.$$

i.e., $\left.\frac{d^2}{dt^2}\right|_{t=t_{a,f}(v)} I_{a,\,f}(tv) < 0$. Therefore, by implicit function theorem (applying implicit function theorem on the function, $\tilde{F} : (0,\infty) \times \tilde{\Sigma}_+ \to \mathbb{R}$, $\tilde{F}(t,v) = I'_{a,f}(tv)(v)$ which is of class C^1), we can see that $t_{a,f}(v) \in C^1(\tilde{\Sigma}_+,\ [0,\ \infty))$. Consequently, $J_{a,f}(v) = I_{a,f}(t_{a,f}(v)v) \in C^1(\tilde{\Sigma}_+,\ \mathbb{R})$. Further, as

$$I'_{a,f}(t_{a,f}(v)v)(v) = 0, \tag{3.3.17}$$

for $h \in T_v\tilde{\Sigma}_+ := \{h \in H^s(\mathbb{R}^N) \mid \langle h,\ v\rangle_{H^s(\mathbb{R}^N)} = 0\}$, we have

$$\begin{aligned} J'_{a,f}(v)h &= I'_{a,f}(t_{a,f}(v)v)\Big(t_{a,f}(v)h + \langle t'_{a,f}(v),\ h\rangle_{H^s(\mathbb{R}^N)}v\Big) \\ &= t_{a,f}(v)I'_{a,f}(t_{a,f}(v)v)h + \langle t'_{a,f}(v),\ h\rangle_{H^s(\mathbb{R}^N)}I'_{a,f}(t_{a,f}(v)v)(v) \\ &= t_{a,f}(v)I'_{a,f}(t_{a,f}(v)v)(h). \end{aligned}$$

Hence (i) follows.

(ii) Applying (i), we have $J'_{a,f}(v) = 0$ if and only if

$$I'_{a,f}(t_{a,f}(v)v)h = 0 \quad \forall\, h \in T_v\tilde{\Sigma}_+. \tag{3.3.18}$$

Since,

$$H^s(\mathbb{R}^N) = \mathrm{Span}\{v\} \oplus T_v\tilde{\Sigma}_+,$$

combining (3.3.17) and (3.3.18), (ii) follows.

(iii) Suppose that $u \in H^s(\mathbb{R}^N)$ is a critical point of $I_{a,f}$. Writting $u = tv$ with $v \in \tilde{\Sigma}_+$ and $t \geq 0$. By lemma 3.3.3, we have either $t = t_{a,f}(v)$ or $t \leq (1-\frac{1}{p})^{-1}\|f\|_{H^{-s}(\mathbb{R}^N)}$.
Thus either $u \in H^s(\mathbb{R}^N)$ corresponds to a critical point of $J_{a,f}$ or, $\|u\|_{H^s(\mathbb{R}^N)} = t\|v\|_{H^s(\mathbb{R}^N)} = t \leq \left(1-\frac{1}{p}\right)^{-1} d_2 \leq r_1$. By Proposition 3.3.4, $I_{a,f}(u)$ has a unique critical point in $B(r_1)$ and it is $u_{locmin}(a,f;x)$. Hence the set of all critical points of $J_{a,f}(v)$ is precisely (3.3.16). □

Next we study the Palais-Smale condition for $J_{a,f}(v)$.

Proposition 3.3.6. *Suppose* $0 < \|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_2$, *where* $d_2 > 0$ *is as found in Proposition 3.3.5. Then,*

(i) $J_{a,f}(v_j) \to \infty$ *whenever* $dist_{H^s(\mathbb{R}^N)}(v_j,\ \partial\tilde{\Sigma}_+) \xrightarrow{j} 0$, *where*

$$dist_{H^s(\mathbb{R}^N)}(v_j,\ \partial\tilde{\Sigma}_+) := \inf\{\|v_j - u\|_{H^s(\mathbb{R}^N)}\ :\ u \in \Sigma,\ u_+ \equiv 0\}.$$

(ii) Suppose that $\{v_j\}_{j=1}^{\infty} \subset \tilde{\Sigma}_+$ *satisfies as* $j \to \infty$

$$J_{a,f}(v_j) \to c, \quad \text{for some } c > 0, \tag{3.3.19}$$

$$\|J'_{a,f}(v_j)\|_{T^*_v\tilde{\Sigma}_+} \equiv \sup\{J'_{a,f}(v_j)h:\ h \in T_{v_j}\tilde{\Sigma}_+,\ \|h\|_{H^s(\mathbb{R}^N)} = 1\} \longrightarrow 0. \tag{3.3.20}$$

Then there exists a subsequence, still we denote by $\{v_j\}$, *a critical point* $u_0(x) \in H^s(\mathbb{R}^N)$ *of* $I_{a,f}(u)$, *an integer* $l \in \mathbb{N} \cup \{0\}$ *and* l *sequences of points*

$\{y_j^{(1)}\}, \ldots, \{y_j^{(l)}\} \subset \mathbb{R}^N$, *critical points* $w_k \in H^s(\mathbb{R}^N)$ $(k = 1, 2, \cdots l)$ *of* (3.0.4) *such that*

1. $|y_j^k| \to \infty$ *as* $j \to \infty$, *for all* $k = 1,\ 2, \ldots,\ l$.

2. $|y_j^{(k)} - y_j^{(k')}| \to \infty$ *as* $j \to \infty$ *for* $k \neq k'$.

3. $\left\| v_j(x) - \frac{u_0(x)+\sum_{k=1}^l w_k(x-y_j^k)}{\|u_0(x)+\sum_{k=1}^l w_k(x-y_j^k)\|_{H^s(\mathbb{R}^N)}} \right\|_{H^s(\mathbb{R}^N)} \to 0$ *as* $j \to \infty$.

4. $J_{a,f}(v_j) \to I_{a,f}(u_0) + \sum_{k=1}^l I_{1,0}(w_k)$ *as* $j \to \infty$.

Proof. (i) Using (3.3.8) and (3.3.5), for any $\varepsilon \in (0,\ 1)$, we have

$$\begin{aligned} J_{a,f}(v_j) &\geq (1-\varepsilon)^{\frac{p+1}{p-1}} J_{a,0}(v_j) - \frac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)} \\ &\geq (1-\varepsilon)^{\frac{p+1}{p-1}} (\tfrac{1}{2} - \tfrac{1}{p+1}) \left(\int_{\mathbb{R}^N} a(x) v_{j+}^{p+1}\, \mathrm{d}x \right)^{-\frac{2}{p-1}} - \frac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)} \end{aligned}$$

Since, $dist(v_j,\ \partial\tilde{\Sigma}_+) \to 0$ implies $(v_j)_+ \to 0$ in $H^s(\mathbb{R}^N)$. Therefore, $(v_j)_+ \to 0$ in $L^{p+1}(\mathbb{R}^N)$. Consequently,

$$\left| \int_{\mathbb{R}^N} a(x)(v_j)_+^{p+1}\, \mathrm{d}x \right| \leq \|a\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} (v_j)_+^{p+1}\, \mathrm{d}x \to 0 \text{ as } j \to \infty.$$

Therefore,

$$J_{a,f}(v_j) \longrightarrow \infty \quad \text{as} \quad dist_{H^s(\mathbb{R}^N)}(v_j,\ \partial\tilde{\Sigma}_+) \longrightarrow 0.$$

Hence (i) follows.

(ii) From (3.3.10) and (3.3.15) we have,

$$\begin{aligned} \left\| I'_{a,f}\Big(t_{a,f}(v_j)v_j\Big) \right\|_{H^{-s}(\mathbb{R}^N)} &= \frac{1}{t_{a,f}(v_j)} \|J'_{a,f}(v_j)\|_{T^*_{v_j}\tilde{\Sigma}_+} \\ &\leq \left(pS_1^{-\frac{p+1}{2}} \right)^{\frac{1}{p-1}} \|J'_{a,f}(v_j)\|_{T^*_{v_j}\tilde{\Sigma}_+} \xrightarrow{j} 0 \end{aligned}$$

We also have, $I_{a,f}(t_{a,f}(v_j)v_j) = J_{a,f}(v_j) \to c$ as $j \to \infty$. Applying Palais-Smale result for $I_{a,f}(u)$ (Proposition 3.2.1), we conclude (ii). $\square$

As a consequence to the above Proposition 3.3.6, we have,

Corollary 3.3.7. *Suppose that* $0 < \|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_2$*, where* $d_2 > 0$ *is as found in Proposition 3.3.5. Then* $J_{a,f}$ *satisfies* $(PS)_c$ *at level*

$$c < I_{a,f}\Big(u_{locmin}(a, f; x)\Big) + I_{1,0}(w^*),$$

where w^* *is the unique ground state solution of* (3.0.4)*.*

Here we say that $J_{a,f}(v)$ satisfies $(PS)_c$ if and only if for any sequence $\{v_j\} \subseteq \tilde{\Sigma}_+$ satisfying (3.3.19) and (3.3.20) has a strongly convergent subsequence in $H^s(\mathbb{R}^N)$.

Proof. By (3.3.14),

$$I_{a,f}(u_{locmin}(a, f; x)) < 0. \tag{3.3.21}$$

On the other hand, from (3.3.16) we see that apart from $u_{locmin}(a, f; x)$, all critical points of $I_{a,f}$ corresponds to a critical point $J_{a,f}$. So, if u_1 is a critical point of $I_{a,f}$, there exists $v_1 \in \tilde{\Sigma}_+$ such that $I_{a,f}(u_1) = J_{a,f}(v_1) > 0$ (here we have used (iii) of Lemma 3.3.2). Hence,

$$I_{a,f}\Big(u_{locmin}(a, f; x)\Big) = \inf\left\{I_{a,f}(u_0) \,\middle|\, u_0 \in H^s(\mathbb{R}^N) \text{ is a critical point of } I_{a,f}\right\}. \tag{3.3.22}$$

Consequently, $I_{a,f}\Big(u_{locmin}(a, f; x)\Big) + I_{1,0}(w^*) \leq I_{a,f}(u_0) + \sum_{i=1}^{l} I_{1,0}(w_i)$, for any critical point u_0 of $I_{a,f}$ and $l \geq 1$, where w^* is the unique positive ground state solution of (3.0.4) and w_i are positive solutions of (3.0.4). From Proposition 3.3.6, we know that if PS sequence for $J_{a,f}$ breaks down at level c, then c must be of the form $I_{a,f}(u_0) + \sum_{i=1}^{l} I_{1,0}(w_i)$, where u_0 is any critical point of $I_{a,f}$ and $l \in \mathbb{N} \cup \{0\}$. Thus, if $l = 0$ and $u_0 = u_{locmin}(a, f; x)$, then applying (3.3.21) to the Proposition 3.3.6(ii)(4), we have $\lim_{j\to\infty} J_{a,f}(v_j) = I_{a,f}(u_{locmin}(a, f; x)) < 0$. On the other hand, from Lemma 3.3.2(iii) we have $\lim_{j\to\infty} J_{a,f}(v_j) > 0$, which gives a contradiction. Therefore, $l = 0$ and $u_0 = u_{locmin}(a, f; x)$ can not happen together. Now, if $l = 0$ and $u_0 \neq$

$u_{locmin}(a, f; x)$, then from Proposition 3.3.6(ii)(3), it follows $v_j \to \frac{u_0}{||u_0||}$ in $H^s(\mathbb{R}^N)$. Hence the Palais-Smale condition at level c is satisfied. Thus the lowest level of breaking down of $(PS)_c$ is $I_{a,f}(u_{locmin}(a, f; x)) + I_{1,0}(w^*)$. Hence the corollary follows. □

3.3.2 Existence of second and third solution

In this subsection, our main aim is to show the existence of second and third positive solution. To this aim we shall use Lusternik-Schnirelman Category theory and a careful analysis of Palais-Smale sequence to prove multiplicity result. We use the following notation.

$$[J_{a,f} \leq c] = \{v \in \tilde{\Sigma}_+ \mid J_{a,f}(v) \leq c\}, \tag{3.3.23}$$

for $c \in \mathbb{R}$. As explained before in order to find the critical points of $J_{a,f}(v)$, we show for a sufficiently small $\varepsilon > 0$,

$$\text{cat}\Bigg([J_{a,f} \leq I_{a,f}(u_{locmin}(a, f; x)) + I_{1,\,0}(w^*) - \varepsilon] \Bigg) \geq 2, \tag{3.3.24}$$

where cat denotes Lusternik-Schnirelman Category.

Now we prove a very delicate energy estimate which plays a pivotal role in the proof of existence of critical points.

Proposition 3.3.8. *Let a be as in Theorem 3.1.1 and f be a nonnegative nontrivial functional in $H^{-s}(\mathbb{R}^N)$ with $\|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_2$, where $d_2 > 0$ is as found in Proposition 3.3.5. Then there exists $R_0 > 0$ such that*

$$I_{a,f}(u_{locmin}(a, f; x) + tw^*(x - y)) < I_{a,f}(u_{locmin}(a, f; x)) + I_{1,0}(w^*), \tag{3.3.25}$$

for all $|y| \geq R_0$ and $t > 0$. Here w^* is the unique ground state solution of (3.0.4).

Proof. It is easy to see that

$$I_{a,f}\Big(u_{locmin}(a, f; x) + tw^*(x - y)\Big) \longrightarrow I_{a,f}(u_{locmin}(a, f; x)) < 0, \text{ as } t \to 0,$$

which follows from the continuity of $I_{a,f}$. It also follows

$$I_{a,f}\Big(u_{locmin}(a,f;x)+tw^*(x-y)\Big)\to-\infty, \quad \text{as} \quad t\to\infty.$$

From these two facts, there exist m, M with $0<m<M$ such that

$$I_{a,f}(u_{locmin}(a,f;x)+tw^*(x-y))<I_{a,f}(u_{locmin}(a,f;x))+I_{1,\,0}(w^*)$$

for all $t\in(0,m)\cup(M,\infty)$.

In view of above it is enough to prove (3.3.25) for all $t\in[m,M]$. We can write

$$\begin{aligned}
&I_{a,f}\Big(u_{locmin}(a,f;x)+tw^*(x-y)\Big)\\
&=\frac{1}{2}\Big\|u_{locmin}(a,f;x)+tw^*(x-y)\Big\|^2_{H^s(\mathbb{R}^N)}\\
&\qquad-\frac{1}{p+1}\int_{\mathbb{R}^N}a(x)\Big(u_{locmin}(a,f;x)+tw^*(x-y)\Big)^{p+1}\,\mathrm{d}x\\
&\qquad\qquad-{}_{H^{-s}}\Big\langle f,\ \Big(u_{locmin}(a,f;x)+tw^*(x-y)\Big)\Big\rangle_{H^s}\\
&=\frac{1}{2}\|u_{locmin}(a,f;x)\|^2_{H^s(\mathbb{R}^N)}+\frac{t^2}{2}\|w^*\|^2_{H^s(\mathbb{R}^N)}+t\langle u_{locmin}(a,f;x),\ w^*(x-y)\rangle_{H^s(\mathbb{R}^N)}\\
&-\frac{1}{p+1}\int_{\mathbb{R}^N}a(x)(u_{locmin}(a,f;x))^{p+1}\,\mathrm{d}x-\frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N}a(x)w^*(x-y)^{p+1}\,\mathrm{d}x\\
&-\frac{1}{p+1}\int_{\mathbb{R}^N}a(x)\Big\{(u_{locmin}(a,f;x)+tw^*(x-y))^{p+1}\\
&\qquad\qquad-(u_{locmin}(a,f;x))^{p+1}-t^{p+1}w^*(x-y)^{p+1}\Big\}\mathrm{d}x\\
&\qquad\qquad-{}_{H^{-s}}\Big\langle f,\ \Big(u_{locmin}(a,f;x)+tw^*(x-y)\Big)\Big\rangle_{H^s}. \qquad (3.3.26)
\end{aligned}$$

Also we have for all $h\in H^s(\mathbb{R}^N)$,

$$\begin{aligned}
0=I'_{a,f}\Big(u_{locmin}(a,f;x)\Big)(h) &= \langle u_{locmin}(a,f;x),\ h\rangle_{H^s(\mathbb{R}^N)}\\
&\quad-\int_{\mathbb{R}^N}a(x)(u_{locmin}(a,f;x))^p h\,\mathrm{d}x-{}_{H^{-s}}\langle f,\ h\rangle_{H^s},
\end{aligned}$$

which in turn implies

$$\langle u_{locmin}(a,f;x),\ h\rangle_{H^s(\mathbb{R}^N)}=\int_{\mathbb{R}^N}a(x)(u_{locmin}(a,f;x))^p h\,\mathrm{d}x+{}_{H^{-s}}\langle f,\ h\rangle_{H^s}.$$

Now by setting $h = tw^*(x-y)$ in above expression, we obtain

$$t\langle u_{locmin}(a,f;x),\ w^*(x-y)\rangle_{H^s(\mathbb{R}^N)} = t\int_{\mathbb{R}^N} a(x)(u_{locmin}(a,f;x))^p w^*(x-y)\,\mathrm{d}x + t\,_{H^{-s}}\langle f,\ w^*(x-y)\rangle_{H^s}.$$

Thus using above and the rearranging the terms in (3.3.26) we have

$$\begin{aligned} I_{a,f}\Big(u_{locmin}(a,f;x) + tw^*(x-y)\Big) &= I_{a,f}\Big(u_{locmin}(a,f;x)\Big) + I_{1,0}(tw^*) \\ &+ \tfrac{t^{p+1}}{p+1}\int_{\mathbb{R}^N}\Big(1-a(x)\Big)w^*(x-y)^{p+1}\,\mathrm{d}x \\ -\tfrac{1}{p+1}\int_{\mathbb{R}^N} a(x)\Big\{\Big(u_{locmin}(a,f;x) &+ tw^*(x-y)\Big)^{p+1} - (u_{locmin}(a,f;x))^{p+1} \\ -t(p+1)(u_{locmin}(a,f;x))^p w^*(x-y) &- t^{p+1}w^*(x-y)^{p+1}\Big\}\,\mathrm{d}x \\ &= I_{a,f}\Big(u_{locmin}(a,f;x)\Big) + I_{1,0}(tw^*) + (I) - (II), \end{aligned} \tag{3.3.27}$$

where

$$(I) := \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N}\Big(1-a(x)\Big)w^*(x-y)^{p+1}\,\mathrm{d}x$$

and

$$\begin{aligned}(II) := \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)\Big\{&\Big(u_{locmin}(a,f;x) + tw^*(x-y)\Big)^{p+1} - (u_{locmin}(a,f;x))^{p+1} \\ &- t(p+1)(u_{locmin}(a,f;x))^p w^*(x-y) - t^{p+1}w^*(x-y)^{p+1}\Big\}\,\mathrm{d}x.\end{aligned}$$

Therefore the proof will be completed if we can show $I < II$. To this aim let us recall a standard fact from calculus. The following inequalities hold true

- $(s+t)^{p+1} - s^{p+1} - t^{p+1} - (p+1)s^p t \geq 0$ for all $(s,\ t) \in [0,\ \infty)\times[0,\ \infty)$.
- For any $r >, 0$ we can find a constant $A(r) > 0$ such that

  $$(s+t)^{p+1} - s^{p+1} - t^{p+1} - (p+1)s^p t \geq A(r)t^2$$

  for all $(s,\ t) \in [r,\ \infty)\times[0,\ \infty)$.

The proof of the above inequalities follows directly using Taylor's theorem on the function $\psi(x) = x^{p+1} - (x-s)^{p+1}$. In particular,

$$\psi(s+t) - \psi(s) = t\psi'(s) + \frac{t^2}{2}\psi''(\xi),$$

where $s \leq \xi \leq s+t$. It's easy to see that $\psi''(\xi) \geq 0$ and thus the 1st inequality follows. For the 2nd inequality, a simple computation yields

$$\psi''(\xi) \geq \begin{cases} p(p+1)r^{p-1} & \text{if} \quad p \geq 2 \\ \dfrac{p(p+1)(p-1)}{2^{2-p}} r^{p-1} & \text{if} \quad 1 < p < 2 \end{cases}$$

Using the above inequality (II) can be estimated as follows : setting $r := \min_{|x|\leq 1} u_{locmin}(a, f; x) > 0$, $A := A(r)$, we have

$$\begin{aligned} (II) &\geq \frac{1}{p+1}\int_{|x|\leq 1} a(x)At^2(w^*)^2(x-y)\,\mathrm{d}x \geq \frac{m^2\,\underline{a}A}{p+1}\int_{|x|\leq 1}(w^*)^2(x-y)\,\mathrm{d}x \\ &\geq \frac{C\,m^2\,\underline{a}A}{p+1}\int_{|x|\leq 1}\frac{\mathrm{d}x}{(1+|x-y|^{N+2s})^2} \geq \frac{C\,m^2\,\underline{a}A}{p+1}|y|^{-2(N+2s)}, \end{aligned} \tag{3.3.28}$$

where in the last inequality we have used the fact that for $|x| \leq 1$, there exists $R > 0$ with $|y| > R$, such that

$$1 + |x-y|^{N+2s} \approx |y|^{(N+2s)}. \tag{3.3.29}$$

On the other hand,

$$\begin{aligned} (I) &= \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N}\Big(1-a(x)\Big)w^*(x-y)^{p+1}\,\mathrm{d}x \\ &\leq \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N}\frac{C}{1+|x|^{\mu(N+2s)}}\left\{\frac{C_2}{1+|x-y|^{N+2s}}\right\}^{p+1}\mathrm{d}x \\ &\leq \frac{CM^{p+1}}{p+1}\int_{\mathbb{R}^N}\frac{\mathrm{d}x}{(1+|x|^{\mu(N+2s)})(1+|x-y|^{N+2s})^{p+1}}. \end{aligned} \tag{3.3.30}$$

Claim: $\displaystyle\int_{\mathbb{R}^N}\frac{\mathrm{d}x}{(1+|x|^{\mu(N+2s)})(1+|x-y|^{N+2s})^{p+1}} \leq \frac{C'}{|y|^{(N+2s)(p+1)}}$ for $|y|$ large enough.

Using the claim, first we complete the proof. Clearly combining the above claim with (3.3.28), it immediately follows that there exists $R_0 > R > 0$ large enough such that

$$(I) < (II) \text{ for } |y| \geq R_0,$$

as $p+1 > 2$. Hence (3.3.25) follows from (3.3.27).

Therefore, we are left to prove the claim.

Note that, to prove the claim, it is enough to show that

$$\int_{\mathbb{R}^N} \frac{|y|^{(p+1)(N+2s)}}{(1+|x|^{\mu(N+2s)})(1+|x-y|^{N+2s})^{p+1}}\,\mathrm{d}x \leq C(N, M, \underline{\mathrm{a}}, p).$$

Therefore we estimate LHS of the above inequality:

$$\begin{aligned}\int_{\mathbb{R}^N} & \frac{|y|^{(p+1)(N+2s)}}{(1+|x|^{\mu(N+2s)})(1+|x-y|^{N+2s})^{p+1}}\,\mathrm{d}x \\ & \leq \underbrace{\int_{\mathbb{R}^N} \frac{|x-y|^{(p+1)(N+2s)}}{(1+|x|^{\mu(N+2s)})(1+|x-y|^{N+2s})^{p+1}}\,\mathrm{d}x}_{:=J_1} \\ & \quad + \underbrace{\int_{\mathbb{R}^N} \frac{|x|^{(p+1)(N+2s)}}{(1+|x|^{\mu(N+2s)})(1+|x-y|^{N+2s})^{p+1}}\,\mathrm{d}x}_{:=J_2}.\end{aligned}$$

Clearly,

$$J_1 \leq \int_{\mathbb{R}^N} \frac{\mathrm{d}x}{1+|x|^{\mu(N+2s)}} = C(N, \mu),$$

since $\mu > \frac{N}{N+2s}$. On the other hand, using (3.3.29), we estimate

$$J_2 \leq \int_{\mathbb{R}^N} \frac{|x|^{(p+1)(N+2s)}}{1+|x|^{\mu(N+2s)}}\,\mathrm{d}x = C(N, \mu, p),$$

since $\mu > (p+1) + \frac{N}{N+2s}$ (by hypothesis of the proposition). Combining the above estimates, claim follows. Hence we conclude the proof of the Proposition. □

We further need several preparatory lemmas and propositions along with the key energy estimate (3.3.25) to prove existence of second and third positive solution. The results below are along the line of [4, 15].

We begin with the properties of the functional $J_{a,0}$ under the condition $(\mathbf{A_1})$.

Lemma 3.3.9. *Let a be as in Theorem 3.1.1 and w^* is unique ground state solution of* (3.0.4). *Then there holds*

(i) $\inf_{v\in\tilde{\Sigma}_+} J_{a,0}(v) = I_{1,0}(w^*)$.

(ii) $\inf_{v\in\tilde{\Sigma}_+} J_{a,0}(v)$ *is not attained.*

(iii) $J_{a,0}(v)$ *satisfies* $(PS)_c$ *for* $c \in (-\infty, I_{1,0}(w^*))$.

Proof. (i) Using (3.3.6), we have $\inf_{v\in\Sigma_+} J_{a,0}(v) \geq I_{1,0}(w^*)$.

Define $w_l(x) = w^*(x + le)$, where e is an unit vector in $\mathbb{R}^N$. Using Lemma 3.3.3, corresponding to $\bar{w}_l = \frac{w_l}{\|w_l\|_{H^s(\mathbb{R}^N)}} \in \tilde{\Sigma}_+$ there exists an unique $t_{a,0}(\bar{w}_l)$ such that

$$J_{a,0}\left(\frac{w_l}{\|w_l\|_{H^s(\mathbb{R}^N)}}\right) = I_{a,0}\left(t_{a,0}(\bar{w}_l)\frac{w_l}{\|w_l\|_{H^s(\mathbb{R}^N)}}\right).$$

Now let us compute

$$I_{a,0}\left(t_{a,0}(\bar{w}_l)\frac{w_l}{\|w_l\|_{H^s(\mathbb{R}^N)}}\right) = \frac{t^2_{a,0}(\bar{w}_l)}{2}\|\bar{w}_l\|^2_{H^s(\mathbb{R}^N)} - \frac{t^{p+1}_{a,0}(\bar{w}_l)}{p+1}\int_{\mathbb{R}^N} a(x)(\bar{w}_l)^{p+1}\,\mathrm{d}x.$$

Moreover from direct computation, we find an explicit form of $t_{a,0}(\bar{w}_l)$ which is given by

$$t_{a,0}(\bar{w}_l) = \left(\int_{\mathbb{R}^N} a(x)\bar{w}_l^{p+1}\mathrm{d}x\right)^{-\frac{1}{p-1}} \xrightarrow{l\to\infty} \left(\frac{\|w^*\|_{H^s(\mathbb{R}^N)}}{\|w^*\|_{L^{p+1}(\mathbb{R}^N)}}\right)^{\frac{p+1}{p-1}},$$

the last limit follows since $a(x) \to 1$ as $|x| \to \infty$. Hence,

$$\begin{aligned} J_{a,0}(\bar{w}_l) \xrightarrow{l\to\infty} & \frac{1}{2}\left\{\frac{\|w^*\|_{H^s(\mathbb{R}^N)}}{\|w^*\|_{L^{p+1}(\mathbb{R}^N)}}\right\}^{\frac{2(p+1)}{(p-1)}} \\ & - \frac{1}{p+1}\left(\left\{\frac{\|w^*\|_{H^s(\mathbb{R}^N)}}{\|w^*\|_{L^{p+1}(\mathbb{R}^N)}}\right\}^{\frac{(p+1)^2}{(p-1)}} \times \frac{\|w^*\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}}{\|w^*\|^{p+1}_{H^s(\mathbb{R}^N)}}\right) \\ = & \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w^*\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = I_{1,0}(w^*). \end{aligned}$$

Hence (i) follows.

(ii) Let us assume on the contrary that there exists $v_0 \in \tilde{\Sigma}_+$ such that $J_{a,0}(v_0) = \inf_{v\in\tilde{\Sigma}_+} J_{a,0}(v) = I_{1,0}(w^*)$. Define, the Nehari manifold $\mathcal{N}$ as

$$\mathcal{N} := \left\{ u \in H^s(\mathbb{R}^N) : \langle (I_{1,0})'(u), u\rangle = 0 \right\}.$$

From a straight forward computation, it is easy to see that there exists $t_{v_0} > 0$ such that $t_{v_0}v_0 \in \mathcal{N}$. Further, observe that for any $v \in \mathcal{N}$, it holds

$$I_{1,0}(v) = \frac{p-1}{2(p+1)}\|v\|^2_{H^s(\mathbb{R}^N)} \geq \frac{p-1}{2(p+1)} S_1^{\frac{p+1}{p-1}},$$

where S_1 is as defined in (3.2.24). therefore, it follows from Remark 3.4.5 that $I_{1,0}(v) \geq I_{1,0}(w^*)$ for all $v \in \mathcal{N}$. Moreover $w \in \mathcal{N}$ and hence

$$\inf_{v\in\mathcal{N}} I_{1,0}(v) = I_{1,0}(w^*).$$

Therefore,

$$\begin{aligned}
I_{1,0}(w^*) &= J_{a,0}(v_0) \\
&:= \max\nolimits_{t>0} I_{a,0}(tv_0) \geq I_{a,0}(t_{v_0}v_0) \\
&= \frac{t_{v_0}^2}{2}\|v_0\|^2_{H^s(\mathbb{R}^N)} - \frac{t_{v_0}^{p+1}}{p+1}\int_{\mathbb{R}^N} a(x)(v_0)_+^{p+1}\,\mathrm{d}x \\
&= \frac{t_{v_0}^2}{2}\|v_0\|^2_{H^s(\mathbb{R}^N)} - \frac{t_{v_0}^{p+1}}{p+1}\int_{\mathbb{R}^N} (v_0)_+^{p+1}\,\mathrm{d}x \\
&\qquad\qquad + \frac{t_{v_0}^{p+1}}{p+1}\int_{\mathbb{R}^N} (1-a(x))(v_0)_+^{p+1}\,\mathrm{d}x \\
&= I_{1,0}(t_{v_0}v_0) + \frac{t_{v_0}^{p+1}}{p+1}\int_{\mathbb{R}^N} (1-a(x))(v_0)_+^{p+1}\,\mathrm{d}x \\
&\geq I_{1,0}(w^*) + \frac{t_{v_0}^{p+1}}{p+1}\int_{\mathbb{R}^N} (1-a(x))(v_0)_+^{p+1}\,\mathrm{d}x.
\end{aligned} \tag{3.3.31}$$

The above inequality and (**A1**) implies

$$\frac{t_{v_0}^{p+1}}{p+1}\int_{\mathbb{R}^N} (1-a(x))(v_0)_+^{p+1}\,\mathrm{d}x = 0. \tag{3.3.32}$$

Therefore

$$(v_0)_+ \equiv 0 \quad \text{in } \{x \in \mathbb{R}^N : a(x) \neq 1\}. \tag{3.3.33}$$

Moreover, substituting (3.3.32) into (3.3.31), we see that inequality in (3.3.31) becomes an equality there. Therefore,

$$\inf_{\mathcal{N}} I_{1,0}(v) = I_{1,0}(w^*) = I_{1,0}(t_{v_0}v_0).$$

Thus $t_{v_0}v_0$ is a constraint critical point of $I_{1,0}$. Therefore using Lagrange multiplier and maximum principle (as before) we conclude that $t_{v_0}v_0 > 0$ which in turn implies $v_0 > 0$ in $\mathbb{R}^N$. This contradicts (3.3.33). Hence (ii) holds.

(iii) From Proposition 3.3.4, we know that $u_{locmin}(a, f; x)$ is the unique critical point of $I_{a,f}$ in $B(r_1)$. Therefore, $u_{locmin}(a, 0; x) = 0$. Consequently, it follows from Corollary 3.3.7 that Palais-Smale condition for $J_{a,0}$ is satisfied at the level $c < I_{1,0}(w^*)$.

This completes the proof. □

The following property of $J_{a,0}(v)$ is important to obtain multiplicity of solutions of ($\mathcal{P}$)

Lemma 3.3.10. ***(Center of mass)***
Let a be as in Theorem 3.1.1. Then there exists a constant $\delta_0 > 0$ such that if $J_{a,0}(v) \le I_{1,0}(w^*) + \delta_0$, then

$$\int_{\mathbb{R}^N} \frac{x}{|x|} |v(x)|^{p+1}\, \mathrm{d}x \neq 0. \tag{3.3.34}$$

Proof. Suppose the conclusion is not true. Then there exists a sequence $\{v_n\} \subset \tilde{\Sigma}_+$ such that

$$J_{a,0}(v_n) \le I_{1,0}(w^*) + \tfrac{1}{n} \text{ and } \int_{\mathbb{R}^N} \frac{x}{|x|} |v_n|^{p+1} \mathrm{d}x \xrightarrow{n\to\infty} 0.$$

Since, by Lemma 3.3.9, we have $\inf_{v\in\tilde{\Sigma}_+} J_{a,0}(v) = I_{1,0}(w^*)$ and the infimum is not attained, applying Ekeland's variational principle, there exists $\tilde{v}_n \subset \tilde{\Sigma}_+$

such that

$$\|v_n - \tilde{v}_n\|_{H^s(\mathbb{R}^N)} \xrightarrow{n\to\infty} 0$$
$$J_{a,0}(\tilde{v}_n) \leq J_{a,0}(v_n) = I_{1,0}(w^*) + \tfrac{1}{n}$$
$$J'_{a,0}(\tilde{v}_n) \xrightarrow{n\to\infty} 0 \text{ in } H^{-s}(\mathbb{R}^N).$$

Therefore, $\{\tilde{v}_n\}$ is a Palais Smale sequence for $J_{a,0}$ at the level $I_{1,0}(w^*)$. Applying Proposition 3.3.6, we get $\{y_n\} \subset \mathbb{R}^N$ such that $|y_n| \xrightarrow{n} \infty$ and

$$\left\|\tilde{v}_n - \frac{w^*(x-y_n)}{\|w^*(x-y_n)\|_{H^s(\mathbb{R}^N)}}\right\|_{H^s(\mathbb{R}^N)} \xrightarrow{n\to\infty} 0.$$

Therefore,

$$\left\|v_n - \frac{w^*(x-y_n)}{\|w^*(x-y_n)\|_{H^s(\mathbb{R}^N)}}\right\|_{H^s(\mathbb{R}^N)} \leq \|v_n - \tilde{v}_n\|_{H^s(\mathbb{R}^N)} + \left\|\tilde{v}_n - \frac{w^*(x-y_n)}{\|w^*(x-y_n)\|_{H^s(\mathbb{R}^N)}}\right\|_{H^s(\mathbb{R}^N)} \xrightarrow{n\to\infty} 0.$$

Therefore the above yields

$$o(\mathbf{1}) = \int_{\mathbb{R}^N} \frac{x}{|x|}|v_n|^{p+1}\mathrm{d}x = \int_{\mathbb{R}^N} \frac{x}{|x|}\left(\frac{w^*(x-y_n)}{\|w^*(x-y_n)\|_{H^s(\mathbb{R}^N)}}\right)^{p+1}\mathrm{d}x + o(\mathbf{1})$$
$$= \frac{1}{\|w\|^{p+1}_{H^s(\mathbb{R}^N)}}\int_{\mathbb{R}^N} \frac{x+y_n}{|x+y_n|}|w^*(x)|^{p+1}\mathrm{d}x \xrightarrow{n\to\infty} e \text{ for some } e \in S^{N-1}.$$

Hence we arrive at a contradiction. □

Lemma 3.3.11. ([4, Lemma 2.5]) *Let $N \geq 1$ and M be a topological space and S^{N-1} denote the unit sphere in $\mathbb{R}^N$. Suppose the there exists two continuous mapping*

$$F : S^{N-1} \to M, \quad G : M \to S^{N-1},$$

such that $G \circ F$ is homotopic to the identity map $Id : S^{N-1} \to S^{N-1}$, namely there is continuous map $\eta : [0,\ 1] \times S^{N-1} \to S^{N-1}$ such that

$$\eta(0,\ x) = (G \circ F)(x) \ \textit{ for all } x \in S^{N-1}$$
$$\eta(1,\ x) = x \ \textit{ for all } x \in S^{N-1}.$$

Then $cat(M) \geq 2$*.*

In view of the above lemma, our next goal will be to construct two mappings:

$$F \ : \ S^{N-1} \to [J_{a,f} \leq I_{a,f}(u_{locmin}(a,f;x)) + I_{1,0}(w^*) - \varepsilon]\,,$$
$$G \ : \ [J_{a,f} \leq I_{a,f}(u_{locmin}(a,f;x)) + I_{1,0}(w^*) - \varepsilon] \to S^{N-1},$$

so that $G \circ F$ is homotopic to the identity.

Proposition 3.3.12. *Let a be as in Theorem 3.1.1 and $d_2 > 0$ and $R_0 > 0$ be as found in Proposition 3.3.5 and Proposition 3.3.8 respectively. Then there exists $d_3 \in (0,\ d_2]$ and $R_1 > R_0$, such that for any $0 < \|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_3$ and for any $|y| \geq R_1$, there exists a unique $t = t(f,y) > 0$ in a neighbourhood of 1 satisfying*

$$u_{locmin}(a,f;x) + tw^*(x-y) \ = \ t_{a,f}\left(\frac{u_{locmin}(a,f;x) + tw^*(x-y)}{\|u_{locmin}(a,f;x) + tw^*(x-y)\|_{H^s(\mathbb{R}^N)}}\right) \times \frac{u_{locmin}(a,f;x) + tw^*(x-y)}{\|u_{locmin}(a,f;x) + tw^*(x-y)\|_{H^s(\mathbb{R}^N)}}.$$

Moreover,

$$\{y \in \mathbb{R}^N \ : \ |y| > R_1\} \to (0,\ \infty); \quad y \mapsto t(f,y)$$

is continuous. Here w^* is the unique ground state solution of (3.0.4)*.*

Proof. Using implicit function theorem, the proof follows exactly in the same spirit of [4, Proposition 2.6]. We skip the details. □

Let us define $F_R : S^{N-1} \to \tilde{\Sigma}_+$ in the following way:

$$F_R(y) = \frac{u_{locmin}(a,f;x) + t(f,Ry)w^*(x-Ry)}{\|u_{locmin}(a,f;x) + t(f,Ry)w^*(x-Ry)\|_{H^s(\mathbb{R}^N)}},$$

for $\|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_3$ and $R \geq R_1$.

In Proposition 3.3.8, we have noticed that for $|y| \geq R_0$, (3.3.25) holds for all $t \geq 0$. For $|y| \geq R_0$, we choose $t = t(f,y)$ such that (3.3.35) holds.

Therefore,

$$\begin{aligned} J_{a,f}\left(\frac{u_{locmin}(a,f;x)+tw^*(x-y)}{\|u_{locmin}(a,f;x)+tw^*(x-y)\|_{H^s(\mathbb{R}^N)}}\right) &= I_{a,f}(u_{locmin}(a,f;x)+tw^*(x-y)) \\ &< I_{a,f}(u_{locmin}(a,f;x))+I_{1,0}(w^*). \end{aligned}$$

Proposition 3.3.13. ([4, Proposition 2.7]) *Let d_3 and R_1 be as found in Proposition 3.3.12. Then, for $0<\|f\|_{H^{-s}(\mathbb{R}^N)}\leq d_3$ and $R\geq R_1$, there exists $\varepsilon_0=\varepsilon_0(R)>0$ such that*

$$F_R(S^{N-1})\subseteq\left[J_{a,f}\leq I_{a,f}(u_{locmin}(a,f;x))+I_{1,0}(w^*)-\varepsilon_0(R)\right],$$

where the notation $[J_{a,f}\leq c]$ is meant in the sense of (3.3.23).

Proof. By construction, we have,

$$F_R(S^{N-1})\subseteq\left[J_{a,f}<I_{a,f}(u_{locmin}(a,f;x))+I_{1,0}(w^*)\right].$$

Since, $F(S^{N-1})$ is compact, the conclusion holds. □

Thus we construct a mapping

$$F_R:S^{N-1}\to\left[J_{a,f}\leq I_{a,f}(u_{locmin}(a,f;x))+I_{1,0}(w^*)-\varepsilon_0(R)\right]$$

Now we will construct G. For the construction of G the following lemma is important.

Lemma 3.3.14. *There exists $d_4\in(0,\ d_3]$ such that if $\|f\|_{H^{-s}(\mathbb{R}^N)}\leq d_4$, then*

$$[J_{a,f}<I_{a,f}(u_{locmin}(a,f;x))+I_{1,0}(w^*)]\subseteq[J_{a,0}<I_{1,0}(w^*)+\delta_0] \tag{3.3.35}$$

where $\delta_0>0$ is given in lemma (3.3.10).

Proof. From (3.3.8), we have for any $\varepsilon\in(0,1)$

$$J_{a,0}(v)\leq(1-\varepsilon)^{-\frac{p+1}{p-1}}\left(J_{a,f}(v)+\frac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)}\right)\text{ for all } v\in\tilde{\Sigma}_+ \tag{3.3.36}$$

From (3.3.21), we also have $I_{a,f}(u_{locmin}(a,f;x))<0$.

Therefore, if $v \in [J_{a,f} < I_{a,f}(u_{locmin}(a,f;x)) + I_{1,0}(w^*)]$ then $J_{a,f}(v) < I_{1,0}(w^*)$. Consequently, from (3.3.36), we have

$$J_{a,0}(v) \leq (1-\varepsilon)^{-\frac{p+1}{p-1}}\left(I_{1,0}(w^*) + \frac{1}{2\varepsilon}\|f\|^2_{H^{-s}(\mathbb{R}^N)}\right)$$

for all $v \in [J_{a,\,f} \leq I_{a,f}(u_{locmin}(a,f;x)) + I_{1,0}(w^*)]$. Since $\varepsilon \in (0,1)$ is arbitrary, we have

$$v \in [J_{a,0} < I_{1,0}(w^*) + \delta_0] \text{ for sufficiently small } \|f\|_{H^{-s}(\mathbb{R}^N)}.$$

Hence the lemma follows. □

Now we can define, $G : [J_{a,f} < I_{a,f}(u_{locmin}(a,f;x)) + I_{1;0}(w^*)] \to S^{N-1}$ by

$$G(v) := \frac{\displaystyle\int_{\mathbb{R}^N} \frac{x}{|x|}|v|^{p+1}\mathrm{d}x}{\left|\displaystyle\int_{\mathbb{R}^N} \frac{x}{|x|}|v|^{p+1}\mathrm{d}x\right|},$$

which is well defined thanks to Lemma 3.3.10 and Lemma 3.3.14. Moreover, we will prove that these constructions F and G serves our purpose.

Proposition 3.3.15. *For a sufficiently large $R \geq R_1$ and for sufficiently small $\|f\|_{H^{-s}(\mathbb{R}^N)} > 0$,*

$$G \circ F_R : S^{N-1} \to S^{N-1}$$

is homotopic to identity.

Proof. This proof follows in the same spirit as in [4, Proposition 2.4]. We skip the details. □

We are now in a position to state our main result in this subsection:

Proposition 3.3.16. *For sufficiently large $R \geq R_1$,*

$$cat\bigg([J_{a,f} < I_{a,f}(u_{locmin}(a,f;x)) + I_{1,\,0}(w^*) - \varepsilon_0(R)]\bigg) \geq 2.$$

Proof. Combining Lemma 3.3.11 and Proposition 3.3.15, this proof follows. □

The above proposition led us to the following multiplicity results.

Theorem 3.3.17. *Let a be as in Theorem 3.1.1. Then there exists $d_5 > 0$ such that if $\|f\|_{H^{-s}(\mathbb{R}^N)} \leq d_5$ and f is nonnegative nontrivial functional in $H^{-s}(\mathbb{R}^N)$, then $J_{a,f}$ has at least two critical points in*

$$[J_{a,f} < I_{a,f}(u_{locmin}(a, f; x)) + I_{1,0}(w^*)] .$$

Proof. From Corollary 3.3.7, we know $(PS)_c$ is satisfied for $J_{a,f}$ when $c \in (-\infty, I_{a,f}(u_{locmin}(a, f; x)) + I_{1,0}(w^*))$. Hence the theorem follows from Proposition 3.3.16 and Proposition 2.6.5. □

Proof of Theorem 3.1.1 concluded:

Proof. We set the first positive solution as $u_1 := u_{locmin}(a, f, x)$ which was found in Proposition 3.3.4. Further, (3.3.21) implies

$$I_{a,f}(u_{locmin}(a, f; x)) < 0.$$

By Theorem 3.3.17, $J_{a,f}$ has at least two critical points v_2, v_3 in

$$[J_{a,f} < I_{a,f}(u_{locmin}(a, f; x)) + I_{1,0}(w^*)] .$$

Using Proposition 3.3.5(iii), $u_2 := t_{a,f}(v_2)v_2$ and $u_3 := t_{a,f}(v_3)v_3$ are the 2nd and 3rd positive solutions of $(\mathcal{P})$. Further, by Lemma 3.3.2(iii), $0 < J_{a,f}(v_i) = I_{a,f}(u_i)$, $i = 2, 3$. Hence

$$0 < I_{a,f}(u_i) < I_{a,f}(u_1) + I_{1,0}(w^*), \quad i = 2, 3.$$

Hence u_1, u_2, u_3 are distinct and $(\mathcal{P})$ has at least 3 distinct solutions. □

3.4 Proof of Theorem 3.1.2

In this section we prove Theorem 3.1.2. To this aim we first establish existence of two positive critical points of $I_{a,f}$ (see (3.3.1)) in the spirit of [83]. Towards that, we partition $H^s(\mathbb{R}^N)$ into three disjoint sets. Let, $g : H^s(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$g(u) := \|u\|^2_{H^s(\mathbb{R}^N)} - p\|a\|_{L^\infty(\mathbb{R}^N)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}.$$

Now, we define

$$U_1 := \{u \in H^s(\mathbb{R}^N) : u = 0 \quad \text{or} \quad g(u) > 0\}, \quad U_2 := \{u \in H^s(\mathbb{R}^N) : g(u) < 0\},$$

$$U := \{u \in H^s(\mathbb{R}^N) \setminus \{0\} : g(u) = 0\}.$$

Remark 3.4.1. *Since $p > 1$, using Sobolev inequality, it is easy to see that $\|u\|_{H^s(\mathbb{R}^N)}$ and $\|u\|_{L^{p+1}(\mathbb{R}^N)}$ are bounded away from 0, for all $u \in U$.*

We define,

$$c_0 := \inf_{U_1} I_{a,f}(u) \quad \text{and} \quad c_1 := \inf_{U} I_{a,f}(u). \tag{3.4.1}$$

Remark 3.4.2. *For any $t > 0$, $g(tu) = t^2\|u\|^2_{H^s(\mathbb{R}^N)} - t^{p+1}p\|a\|_{L^\infty(\mathbb{R}^N)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}$. Moreover $g(0) = 0$ and $t \mapsto g(tu)$ is a strictly concave function, we have for any $u \in H^s(\mathbb{R}^N)$ with $\|u\|_{H^s(\mathbb{R}^N)} = 1$, there exists unique $t = t(u)$ such that $tu \in U$. On the other hand, for any $u \in U$, it holds $g(tu) = (t^2 - t^{p+1})\|u\|^2_{H^s(\mathbb{R}^N)}$. This implies*

$$tu \in U_1 \quad \textit{for all} \quad t \in (0,1) \quad \textit{and} \quad tu \in U_2 \quad \textit{for all} \quad t > 1.$$

Lemma 3.4.3. *Assume C_p is defined as in Theorem 3.1.2. Then there holds,*

$$\tfrac{p-1}{p}\|u\|_{H^s(\mathbb{R}^N)} \geq C_p S_1^{\frac{p+1}{2(p-1)}} \quad \forall \quad u \in U,$$

where S_1 is as defined in (3.2.24).

Proof. $u \in U$ implies, $\|u\|_{L^{p+1}(\mathbb{R}^N)} = \frac{\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2}{p+1}}}{\left(p\|a\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{1}{p+1}}}$. Therefore, combining this with the definition of S_1, we have

$$\|u\|_{H^s(\mathbb{R}^N)} \geq S_1^{\frac{1}{2}} \|u\|_{L^{p+1}(\mathbb{R}^N)} = S_1^{\frac{1}{2}} \frac{\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2}{p+1}}}{\left(p\|a\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{1}{p+1}}} \quad \forall\, u \in U.$$

Therefore, for all $u \in U$, we have

$$\|u\|_{H^s(\mathbb{R}^N)} \geq \frac{S_1^{\frac{p+1}{2(p-1)}}}{\left(p\|a\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{1}{p-1}}} = \frac{p}{p-1} C_p S_1^{\frac{p+1}{2(p-1)}}.$$

Hence the lemma follows. □

Lemma 3.4.4. *Assume C_p is defined as in Theorem 3.1.2 and*

$$\inf_{u \in H^s(\mathbb{R}^N),\, \|u\|_{L^{p+1}(\mathbb{R}^N)}=1} \left\{ C_p \|u\|_{H^s(\mathbb{R}^N)}^{\frac{2p}{p-1}} - {}_{H^{-s}}\langle f, u\rangle_{H^s} \right\} > 0. \tag{3.4.2}$$

Then $c_0 < c_1$, where c_0 and c_1 are defined as in (3.4.1).

Proof. Define,

$$\tilde{J}(u) := \frac{1}{2}\|u\|_{H^s(\mathbb{R}^N)}^2 - \frac{\|a\|_{L^\infty(\mathbb{R}^N)}}{p+1}\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} - {}_{H^{-s}}\langle f, u\rangle_{H^s}, \quad u \in H^s(\mathbb{R}^N). \tag{3.4.3}$$

Step 1: In this step we prove that there exists $\alpha > 0$ such that

$$\frac{d}{dt}\tilde{J}(tu)|_{t=1} \geq \alpha \quad \forall\, u \in U.$$

From the definition of $\tilde{J}$, we have

$$\frac{d}{dt}\tilde{J}(tu)|_{t=1} = \|u\|_{H^s(\mathbb{R}^N)}^2 - \|a\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} - {}_{H^{-s}}\langle f, u\rangle_{H^s}.$$

Therefore, using the definition of U and the value of C_p, we have for $u \in U$

$$\begin{aligned}
\frac{d}{dt}\tilde{J}(tu)|_{t=1} &= \frac{p-1}{p}\|u\|_{H^s(\mathbb{R}^N)}^2 - {}_{H^{-s}}\langle f, u\rangle_{H^s} \\
&= \left(p\|a\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{1}{p-1}} C_p \|u\|_{H^s(\mathbb{R}^N)}^2 - {}_{H^{-s}}\langle f, u\rangle_{H^s} \\
&= \left(\frac{\|u\|_{H^s(\mathbb{R}^N)}^2}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}}\right)^{\frac{1}{p-1}} C_p \|u\|_{H^s(\mathbb{R}^N)}^2 - {}_{H^{-s}}\langle f, u\rangle_{H^s} \\
&= C_p \frac{\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2p}{p-1}}}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{\frac{p+1}{p-1}}} - {}_{H^{-s}}\langle f, u\rangle_{H^s}.
\end{aligned} \tag{3.4.4}$$

Further, (3.4.2) implies there exists $d > 0$ such that

$$\inf_{\substack{u\in H^s(\mathbb{R}^N),\\ \|u\|_{L^{p+1}(\mathbb{R}^N)}=1}} \left\{C_p\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2p}{p-1}} - {}_{H^{-s}}\langle f,u\rangle_{H^s}\right\} \geq d. \tag{3.4.5}$$

Now,

$$\begin{aligned}
(3.4.5) \iff & C_p\frac{\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2p}{(p-1)}}}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{\frac{p+1}{p-1}}} - {}_{H^{-s}}\langle f,u\rangle_{H^s} \geq d, \quad \|u\|_{L^{p+1}(\mathbb{R}^N)} = 1\\
\iff & C_p\frac{\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2p}{(p-1)}}}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{\frac{p+1}{p-1}}} - {}_{H^{-s}}\langle f,u\rangle_{H^s} \geq d\|u\|_{L^{p+1}(\mathbb{R}^N)},\ u\in H^s(\mathbb{R}^N)\setminus\{0\}.
\end{aligned}$$

Hence, plugging back the above estimate into (3.4.4) and using Remark (3.4.1) we complete the proof of Step 1.

Step 2: Let u_n be a minimizing sequence for $I_{a,f}$ on U, i.e., $I_{a,f}(u_n) \to c_1$ and $\|u_n\|_{H^s(\mathbb{R}^N)}^2 = p\|a\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}$. Therefore, for large n

$$c_1+o(1) \geq I_{a,f}(u_n) \geq \tilde{J}(u_n) \geq \left(\frac{1}{2}-\frac{1}{p(p+1)}\right)\|u_n\|_{H^s(\mathbb{R}^N)}^2-\|f\|_{H^{-s}(\mathbb{R}^N)}\|u_n\|_{H^s(\mathbb{R}^N)}.$$

This implies that $\{\tilde{J}(u_n)\}$ is a bounded sequence and $\|u_n\|_{H^s(\mathbb{R}^N)}$ and $\|u_n\|_{L^{p+1}(\mathbb{R}^N)}$ are bounded.

Claim: $c_0 < 0$.

Indeed, to prove the claim, it's enough to show that there exists $v \in U_1$ such that $I_{a,f}(v) < 0$. Note that, thanks to Remark 3.4.2, we can choose $u \in U$ such that ${}_{H^{-s}}\langle f,u\rangle_{H^s} > 0$. Therefore,

$$I_{a,f}(tu) \leq t^2\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}\left[\frac{p\|a\|_{L^\infty(\mathbb{R}^N)}}{2} - \frac{t^{p-1}}{p+1}\right] - t\,{}_{H^{-s}}\langle f,u\rangle_{H^s} < 0.$$

for $t << 1$. Also by Remark 3.4.2, $tu \in U_1$. Hence the claim follows.

Thanks to the above claim, $I_{a,f}(u_n) < 0$ for large n. Consequently,

$$0 > I_{a,f}(u_n) \geq \left(\frac{1}{2} - \frac{1}{p(p+1)}\right)\|u_n\|_{H^s(\mathbb{R}^N)}^2 - {}_{H^{-s}}\langle f,u_n\rangle_{H^s}.$$

This in turn implies ${}_{H^{-s}}\langle f, u_n\rangle_{H^s} > 0$ for all large n (since $p > 1$). Consequently, $\frac{d}{dt}\tilde{J}(tu_n) < 0$ for $t > 0$ small enough. Thus, by Step 1, there exists $t_n \in (0,1)$ such that $\frac{d}{dt}\tilde{J}(t_n u_n) = 0$. Moreover, t_n is unique since,

$$\frac{d^2}{dt^2}\tilde{J}(tu) = \|u\|^2_{H^s(\mathbb{R}^N)} - p\|a\|_{L^\infty(\mathbb{R}^N)} t^{p-1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = (1-t^{p-1})\|u\|^2_{H^s(\mathbb{R}^N)} > 0,$$

for all $u \in U$, for all $t \in [0,\ 1)$.

Step 3: In this step we show that

$$\liminf_{n\to\infty}\{\tilde{J}(u_n) - \tilde{J}(t_n u_n)\} > 0. \tag{3.4.6}$$

We observe that, $\tilde{J}(u_n) - \tilde{J}(t_n u_n) = \displaystyle\int_{t_n}^1 \frac{d}{dt}\{\tilde{J}(tu_n)\}\,\mathrm{d}t$ and that for all $n \in \mathbb{N}$, there is $\xi_n > 0$ such that $t_n \in (0,\ 1-2\xi_n)$ and $\frac{d}{dt}\tilde{J}(tu_n) \geq \alpha$ for $t \in [1-\xi_n,\ 1]$.

To establish (3.4.6), it is enough to show that $\xi_n > 0$ can be chosen independent of $n \in \mathbb{N}$. But this is true since, $\frac{d}{dt}\tilde{J}(tu_n)|_{t=1} \geq \alpha$ and for the boundedness of $\{u_n\}$,

$$\begin{aligned}\left|\frac{d^2}{dt^2}\tilde{J}(tu_n)\right| &= \left|\|u_n\|^2_{H^s(\mathbb{R}^N)} - p\|a\|_{L^\infty(\mathbb{R}^N)} t^{p-1}\|u_n\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}\right| \\ &= \left|(1-t^{p-1})\|u_n\|^2_{H^s(\mathbb{R}^N)}\right| \leq C,\end{aligned}$$

for all $n \geq 1$ and $t \in [0,\ 1]$.

Step 4: From the definition of $I_{a,f}$ and $\tilde{J}$, it immediately follows that $\frac{d}{dt}I_{a,f}(tu) \geq \frac{d}{dt}\tilde{J}(tu)$ for all $u \in H^s(\mathbb{R}^N)$ and for all $t > 0$. Hence,

$$I_{a,f}(u_n) - I_{a,f}(t_n u_n) = \int_{t_n}^1 \frac{d}{dt}(I_{a,f}(tu_n))\ \mathrm{d}t \geq \int_{t_n}^1 \frac{d}{dt}\tilde{J}(tu_n)\ \mathrm{d}t = \tilde{J}(u_n) - \tilde{J}(t_n u_n)$$

Since, $\{u_n\} \in U$ is a minimizing sequence for $I_{a,f}$, and $t_n u_n \in U_1$, we conclude using (3.4.6) that

$$c_0 = \inf_{u\in U_1} I_{a,f}(u) < \inf_{u\in U} I_{a,f}(u) \equiv c_1.$$

□

Next, we introduce the problem at infinity associated to (3.3.2):

$$(-\Delta)^s u + u = u_+^p \quad \text{in} \quad \mathbb{R}^N, \tag{3.4.7}$$

and the corresponding functional $I_{1,0} : H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I_{1,0}(u) = \frac{1}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{p+1}\int_{\mathbb{R}^N} u_+^{p+1}\,\mathrm{d}x.$$

Define,

$$X_1 := \{u \in H^s(\mathbb{R}^N) \setminus \{0\} : (I_{1,0})'(u) = 0\}, \quad S^\infty := \inf_{X_1} I_{1,0}. \tag{3.4.8}$$

Remark 3.4.5. *Clearly* $I_{1,0}(u) = \frac{p-1}{2(p+1)}\|u\|^2_{H^s(\mathbb{R}^N)}$ *on* X_1. *From* (3.2.24), *we also have* $\|u\|^2_{H^s(\mathbb{R}^N)} \geq S_1^{\frac{p+1}{p-1}}$ *on* X_1. *Therefore,* $S^\infty \geq \frac{p-1}{2(p+1)} S_1^{\frac{p+1}{p-1}} > 0$. *Further, it's known from [71] that* S_1 *is achieved by unique positive radial ground state solution* w^* *of* (3.0.4). *Therefore,*

$$I_{1,0}(w^*) = \frac{p-1}{2(p+1)} S_1^{\frac{p+1}{p-1}}.$$

Hence S^∞ *is achieved by* w^*.

Proposition 3.4.6. *Assume* (3.4.2) *holds. Then* $I_{a,f}$ *has a critical point* $u_0 \in U_1$ *with* $I_{a,f}(u_0) = c_0$. *In particular,* u_0 *is a positive weak solution to* $(\mathcal{P})$.

Proof. We decompose the proof into few steps.

Step 1: $c_0 > -\infty$.

Since $I_{a,f}(u) \geq \tilde{J}(u)$, where $\tilde{J}$ is defined as in (3.4.3), in order to prove Step 1, it is enough to show that $\tilde{J}$ is bounded from below. From definition of U_1, it immediately follows that

$$\tilde{J}(u) \geq \left[\tfrac{1}{2} - \tfrac{1}{p(p+1)}\right] \|u\|^2_{H^s(\mathbb{R}^N)} - \|f\|_{H^{-s}(\mathbb{R}^N)}\|u\|_{H^s(\mathbb{R}^N)} \text{ for all } u \in U_1. \tag{3.4.9}$$

As RHS is quadratic function in $\|u\|_{H^s(\mathbb{R}^N)}$, $\tilde{J}$ is bounded from below. Hence Step 1 follows.

Step 2: In this step we show that there exists a bounded PS sequence $\{u_n\} \subset U_1$ for $I_{a,f}$ at level c_0.

Let $\{u_n\} \subset \bar{U}_1$ such that $I_{a,f}(u_n) \to c_0$. Since $I_{a,f}(u) \geq \tilde{J}(u)$ from (3.4.9), it follows that $\{u_n\}$ is a bounded sequence. Since by Lemma 3.4.4, $c_0 < c_1$, without restriction we can assume $u_n \in U_1$. Therefore, by Ekeland's variational principle from $\{u_n\}$, we can extract a PS sequence in U_1 for $I_{a,f}$ at level c_0. We again call it by $\{u_n\}$. That completes the proof of Step 2.

Step 3: In this step we show that there exists $u_0 \in U_1$ such that $u_n \to u_0$ in $H^s(\mathbb{R}^N)$.

Applying Proposition 3.2.1, it follows

$$u_n - u_0 - \sum_{i=1}^{m} w^i(x - x_n^i) \to 0 \ \text{ in } H^s(\mathbb{R}^N) \tag{3.4.10}$$

for some u_0 with $(I_{a,f})'(u_0) = 0$ and some appropriate w^i, $\{x_n^i\}$. To prove Step 3, we need to show that $m = 0$. We argue by method of contradiction. Suppose there is $w^i \neq 0$ ($i \in \{1, 2, \cdots, m\}$) such that $(I_{1,0})'(w^i) = 0$. i.e, $\|w^i\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (w^i_+)^{p+1}\,\mathrm{d}x$. Therefore,

$$\begin{aligned} g(w^i) &= \|w^i\|^2_{H^s(\mathbb{R}^N)} - p\|a\|_{L^\infty(\mathbb{R}^N)}\|w^i\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\ &= \int_{\mathbb{R}^N} (w^i_+)^{p+1}\,\mathrm{d}x - p\|a\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |w^i|^{p+1}\,\mathrm{d}x \\ &\leq \|w^i\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}(1 - p\|a\|_{L^\infty(\mathbb{R}^N)}) < 0. \end{aligned}$$

The last inequality follows from the fact that $p > 1$ and $\|a\|_{L^\infty(\mathbb{R}^N)} \geq 1$. Now from Remark 3.4.5, $I_{1,0}(w^i) \geq S^\infty > 0$ for all $1 \leq i \leq m$. Therefore, $I_{a,f}(u_n) \to I_{a,f}(u_0) + \sum_{i=1}^m I_{1,0}(w_i)$ implies $I_{a,f}(u_0) < c_0$. This in turn implies, $u_0 \notin U_1$. Therefore, $g(u_0) \leq 0$.

Now we evaluate, $g\Big(u_0 + \sum_{i=1}^m w^i(x - x_n^i)\Big)$. Since $u_n \in U_1$, we have $g(u_n) \geq 0$. Therefore, applying uniform continuity of g, we obtain from (3.4.10) that

$$0 \leq \liminf_{n\to\infty} g(u_n) = \liminf_{n\to\infty} g\Big(u_0 + \sum_{i=1}^{m} w^i(x - x_n^i)\Big). \tag{3.4.11}$$

On the other hand, since $|x_n^i| \to \infty$, $|x_n^i - x_n^j| \to \infty$, for $1 \le i \neq j \le m$ the supports of $u_0(\cdot)$ and $w^i(\cdot - x_n^i)$ are going increasingly far away as $n \to \infty$ and we get

$$\lim_{n\to\infty} g\Big(u_0+\sum_{i=1}^{m} w^i(x-x_n^i)\Big) = g(u_0)+\lim_{n\to\infty}\sum_{i=1}^{m} g\left(w^i(x-x_n^i)\right) = g(u_0)+\sum_{i=1}^{m} g(w^i),$$

where the last equality is due to the fact that g is invariant under translation in $\mathbb{R}^N$. Now since $g(u_0) \le 0$ and $g(w^i) < 0$, for $i \le i \neq j \le m$, we get a contradiction to (3.4.11). Hence Step 3 follows.

Step 4: From the previous steps we conclude that $I_{a,f}(u_0) = c_0$ and $(I_{a,f})'(u_0) = 0$. Therefore, u_0 is a weak solution to (3.3.2). Combining this with Remark 3.3.1, we conclude the proof of the proposition. □

Proposition 3.4.7. *Assume* (3.4.2) *holds. Then $I_{a,f}$ has a second critical point $v_0 \neq u_0$. In particular, v_0 is a positive solution to* ($\mathcal{P}$).

Proof. Let u_0 be the critical point obtained in Proposition 3.4.6 and w^* be as in Remark 3.4.5. Set, $w_t(x) := w^*\left(\frac{x}{t}\right)$

Claim 1: $u_0 + w_t \in U_2$ for $t > 0$ large enough.

Indeed, as $p > 1$ and $\|a\|_{L^\infty(\mathbb{R}^N)} \ge 1$,

$$\begin{aligned} g(u_0+w_t) &\le \|u_0\|^2_{H^s(\mathbb{R}^N)} + \|w_t\|^2_{H^s(\mathbb{R}^N)} + 2\langle u_0, w_t\rangle_{H^s} \\ &\qquad -p\left(\|u_0\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \|w_t\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}\right) \\ &\le (1+\varepsilon)\|w_t\|^2_{H^s(\mathbb{R}^N)} + (1+C(\varepsilon))\|u_0\|^2_{H^s(\mathbb{R}^N)} \\ &\qquad -p\left(\|u_0\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \|w_t\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}\right), \end{aligned}$$

where to get the last inequality, we have used Young's inequality with $\varepsilon > 0$. Further, as w^* solves (3.0.4), we have

$$\|w_t\|^2_{H^s(\mathbb{R}^N)} = t^{N-2s}[w^*]^2_{H^s} + t^N\|w^*\|^2_{L^2(\mathbb{R}^N)} \text{ and } \|w_t\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = t^N\|w^*\|^2_{H^s(\mathbb{R}^N)},$$

where $[\cdot]_{H^s}$ denotes the seminorm in $H^s(\mathbb{R}^N)$. Therefore,

$$\begin{aligned} g(u_0+w_t) &\leq (1+C(\varepsilon))\|u_0\|^2_{H^s(\mathbb{R}^N)} - p\|u_0\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\ &\quad +[w^*]^2_{H^s}\Big[(1+\varepsilon)t^{N-2s} - pt^N\Big] + t^N\|w^*\|^2_{L^2(\mathbb{R}^N)}\Big[(1+\varepsilon)-p\Big]. \end{aligned}$$

Choose $\varepsilon > 0$ such that $1+\varepsilon < p$. Therefore, $g(u_0+w_t) < 0$ for t to be large enough. Hence the claim follows.

Claim 2: $I_{a,f}(u_0+w_t) < I_{a,f}(u_0) + I_{1,0}(w_t),\ \forall\, t > 0$.

Indeed, since $u_0,\ w_t > 0$, taking w_t as the test function for (3.3.2) yields

$$\langle u_0,\ w_t\rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} a(x)u_0^p w_t\,\mathrm{d}x + {}_{H^{-s}}\langle f, w_t\rangle_{H^s}.$$

Therefore, using the above expression and the fact that $a \geq 1$, we obtain

$$\begin{aligned} I_{a,f}(u_0+w_t) &= \frac{1}{2}\|u_0\|^2_{H^s(\mathbb{R}^N)} + \frac{1}{2}\|w_t\|^2_{H^s(\mathbb{R}^N)} + \langle u_0,\ w_t\rangle_{H^s(\mathbb{R}^N)} \\ &\quad -\frac{1}{p+1}\int_{\mathbb{R}^N} a(x)(u_0+w_t)^{p+1}\,\mathrm{d}x - {}_{H^{-s}}\langle f, u_0\rangle_{H^s} - {}_{H^{-s}}\langle f, w_t\rangle_{H^s} \\ &= I_{a,f}(u_0) + I_{1,0}(w_t) + \langle u_0,\ w_t\rangle_{H^s(\mathbb{R}^N)} + \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_0^{p+1}\,\mathrm{d}x \\ &\quad +\frac{1}{p+1}\int_{\mathbb{R}^N} w_t^{p+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)(u_0+w_t)^{p+1}\,\mathrm{d}x - {}_{H^{-s}}\langle f, w_t\rangle_{H^s} \\ &\leq I_{a,f}(u_0) + I_{1,0}(w_t) + \\ &\quad \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)\Big[(p+1)u_0^p w_t + u_0^{p+1} + w_t^{p+1} - (u_0+w_t)^{p+1}\Big]\mathrm{d}x \\ &< I_{a,f}(u_0) + I_{1,0}(w_t). \end{aligned}$$

Hence the Claim follows.

Also, by direct computation, it follows

$$I_{1,0}(w_t) = \frac{t^{N-2s}}{2}[w^*]^2_{H^s} + \frac{t^N}{2}\|w^*\|^2_{L^2(\mathbb{R}^N)} - \frac{t^N}{p+1}\|w^*\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \to -\infty \text{ as } t \to \infty, \tag{3.4.12}$$

From (3.4.12), it is also easy to see that

$$\sup_{t>0} I_{1,0}(w_t) = I_{1,0}(w_1) = I_{1,0}(w^*) = S^\infty,$$

where the last equality is due to Remark 3.4.5. Combing this with Claim 2 yields

$$I_{a,f}(u_0 + w_t) < I_{a,f}(u_0) + S^\infty \quad \forall\, t > 0. \tag{3.4.13}$$

Combining (3.4.12) with Claim 2, we have

$$I_{a,f}(u_0 + w_t) < I_{a,f}(u_0) \quad \text{for} \quad t \quad \text{large enough.} \tag{3.4.14}$$

Fix $t_0 > 0$ large enough such that (3.4.14) and Claim 1 are satisfied.

Then we set

$$\gamma := \inf_{i \in \Gamma} \max_{t \in [0,1]} I_{a,f}\Big(i(t)\Big),$$

where

$$\Gamma := \{i \in C\Big([0,1], H^s(\mathbb{R}^N)\Big) : i(0) = u_0, \quad i(1) = u_0 + w_{t_0}\}.$$

As $u_0 \in U_1$ and $u_0 + w_{t_0} \in U_2$, for every $i \in \Gamma$, there exists $t_i \in (0,1)$ such that $i(t_i) \in U$. Therefore,

$$\max_{t \in [0,1]} I_{a,f}(i(t)) \geq I_{a,f}\Big(i(t_i)\Big) \geq \inf_U I_{a,f}(u) = c_1.$$

Thus, $\gamma \geq c_1 > c_0 = I_{a,f}(u_0)$. Here in the last inequality we have used Lemma 3.4.4.

Claim 3: $\gamma < S^\infty$, where S^∞ is as defined in (3.4.8).

It's easy to see that $\lim_{t \to 0} \|w_t\|_{H^s(\mathbb{R}^N)} = 0$. Thus, if we define $\tilde{i}(t) = u_0 + w_{tt_0}$, then $\lim_{t \to 0} \|\tilde{i}(t) - u_0\|_{H^s(\mathbb{R}^N)} = 0$. Consequently, $\tilde{i} \in \Gamma$. Therefore, using (3.4.13), we obtain

$$\gamma \leq \max_{t \in [0,1]} I_{a,f}(\tilde{i}(t)) = \max_{t \in [0,1]} I_{a,f}(u_0 + w_{tt_0}) < I_{a,f}(u_0) + S^\infty.$$

Hence the claim follows.

Hence

$$I_{a,f}(u_0) < \gamma < I_{a,f}(u_0) + S^\infty.$$

Using Ekeland's variational principle, there exists a PS sequence $\{u_n\}$ for $I_{a,f}$ at level γ. Doing a standard computation yields $\{u_n\}$ is bounded sequence. Since, by Remark 3.4.5, we have $S^\infty = I_{1,0}(w^*)$, from Proposition 3.2.1 we can conclude that $u_n \to v_0$, for some $v_0 \in H^s(\mathbb{R}^N)$ such that $(I_{a,f})'(v_0) = 0$ and $I_{a,f}(v_0) = \gamma$. Further, as $I_{a,f}(u_0) < \gamma$, we conclude $v_0 \neq u_0$.

$(I_{a,f})'(v_0) = 0 \implies v_0$ is a weak solution to (3.3.2). Combining this with Remark 3.3.1, we conclude the proof of the proposition. □

Lemma 3.4.8. *If* $\|f\|_{H^{-s}(\mathbb{R}^N)} < C_p S_1^{\frac{p+1}{2(p-1)}}$, *then* (3.4.2) *holds.*

Proof. Using the given hypothesis, we can obtain $\varepsilon > 0$ such that $\|f\|_{H^{-s}(\mathbb{R}^N)} < C_p S_1^{\frac{p+1}{2(p-1)}} - \varepsilon$. Therefore, using Lemma 3.4.3, we have

$$\begin{aligned} {}_{H^{-s}}\langle f, u\rangle_{H^s} &\leq \|f\|_{H^{-s}(\mathbb{R}^N)}\|u\|_{H^s(\mathbb{R}^N)} \\ &< \left[C_p S_1^{\frac{p+1}{2(p-1)}} - \varepsilon\right]\|u\|_{H^s(\mathbb{R}^N)} \\ &\leq \frac{p-1}{p}\|u\|^2_{H^s(\mathbb{R}^N)} - \varepsilon\|u\|_{H^s(\mathbb{R}^N)}, \end{aligned}$$

for all $u \in U$. Therefore,

$$\tfrac{p-1}{p}\|u\|^2_{H^s(\mathbb{R}^N)} - {}_{H^{-s}}\langle f, u\rangle_{H^s} > \varepsilon\|u\|_{H^s(\mathbb{R}^N)} \quad \forall\, u \in U.$$

i.e.,

$$\inf_U \left[\tfrac{p-1}{p}\|u\|^2_{H^s(\mathbb{R}^N)} - {}_{H^{-s}}\langle f, u\rangle_{H^s}\right] \geq \varepsilon \inf_U \|u\|_{H^s(\mathbb{R}^N)}.$$

Since, by Remark 3.4.1, we have $\|u\|_{H^s(\mathbb{R}^N)}$ is bounded away from 0 on U, the above expression implies

$$\inf_U \left[\tfrac{p-1}{p}\|u\|^2_{H^s(\mathbb{R}^N)} - {}_{H^{-s}}\langle f, u\rangle_{H^s}\right] > 0. \tag{3.4.15}$$

On the other hand,

$$(3.4.2) \iff C_p \frac{\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2p}{p-1}}}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{\frac{p+1}{p-1}}} - {}_{H^{-s}}\langle f, u\rangle_{H^s} > 0 \quad \text{for} \quad \|u\|_{L^{p+1}(\mathbb{R}^N)} = 1$$

$$\iff C_p \frac{\|u\|_{H^s(\mathbb{R}^N)}^{\frac{2p}{p-1}}}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{\frac{p+1}{p-1}}} - {}_{H^{-s}}\langle f, u\rangle_{H^s} > 0 \quad \text{for} \quad u \in U$$

$$\iff \tfrac{p-1}{p}\|u\|_{H^s(\mathbb{R}^N)}^2 - {}_{H^{-s}}\langle f, u\rangle_{H^s} > 0 \quad \text{for} \quad u \in U. \tag{3.4.16}$$

Clearly, (3.4.15) insures RHS of (3.4.16) holds. Hence the lemma follows. □

Proof of Theorem 3.1.2 completed:

Proof. Combining Proposition 3.4.6 and Proposition 3.4.7 with Lemma 3.4.8, we conclude the proof of Theorem 3.1.2. □

3.5 Existence Result when $f \equiv 0$

In this section we aim to prove Theorem 3.1.4 in the spirit of [59]. For this using Mountain pass theorem, we first attempt to solve the following problem in the bounded domain with Dirichlet boundary condition:

$$\begin{cases} (-\Delta)^s u + u = a(x)|u|^{p-1}u \ \text{ in } B_k, \\ u > 0 \quad \text{in} \quad B_k, \\ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_k, \end{cases} \tag{$\mathcal{P}_k$}$$

where B_k denotes the ball of radius k, centered at origin, $0 < a \in L^\infty(\mathbb{R}^N)$ satisfies

$$\lim_{|x|\to\infty} a(x) = a_0 = \inf_{x\in\mathbb{R}^N} a(x). \tag{3.5.1}$$

Remark 3.5.1. *Without loss of generality, we can assume $a \not\equiv a_0$, since if $a \equiv a_0$ then $u = a_0^{-\frac{1}{p-1}} w^*$ is a solution of* ($\mathcal{P}$) (with $f \equiv 0$), *where w^* is the* *unique ground state solution of* (3.0.4). *In this case Theorem 3.1.4 follows immediately.*

We fix some notations first. Denote,

$$E_k := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus B_k\}$$

i.e., E_k is the closure of $C_0^\infty(B_k)$ w.r.t. the norm in $H^s(\mathbb{R}^N)$. Therefore,

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq H^s(\mathbb{R}^N)$$

and $\cup_{k\geq 1} E_k$ is dense in $H^s(\mathbb{R}^N)$. We define $I_{a,0}$ as in (3.3.1) (taking $f = 0$ there) and let $I_{a,0}^k$ denote the restriction of $I_{a,0}$ to the subspace E_k. Like before using Remark 3.3.1, we can conclude that any nontrivial critical point of $I_{a,0}^k$ is necessarily a positive solution of $(\mathcal{P}_k)$.

Lemma 3.5.2. *For each $k \geq 1$, Dirichlet problem $(\mathcal{P}_k)$ admits a solution u_k. Moreover, $\{u_k\}$ is uniformly bounded in $H^s(\mathbb{R}^N)$ and so it contains a subsequence that converges weakly to $\bar{u} \geq 0$ in $H^s(\mathbb{R}^N)$.*

Proof. For $u \in E_k$,

$$\begin{aligned} I_{a,0}^k(u) = I_{a,0}(u) &= \frac{1}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}\mathrm{dx} \\ &\geq \frac{1}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{p+1}\|a\|_{L^\infty(\mathbb{R}^N)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\ &\geq \frac{1}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{p+1}\|a\|_{L^\infty(\mathbb{R}^N)} S_1^{-\frac{p+1}{2}}\|u\|^{p+1}_{H^s(\mathbb{R}^N)}, \end{aligned}$$

where S_1 is as defined in (3.2.24). As $\cup_{k\geq 1} E_k$ is dense in $H^s(\mathbb{R}^N)$,

$$I_{a,0}(u) \geq \frac{1}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{p+1}\|a\|_{L^\infty(\mathbb{R}^N)} S_1^{-\frac{p+1}{2}}\|u\|^{p+1}_{H^s(\mathbb{R}^N)} \quad \forall\, u \in H^s(\mathbb{R}^N).$$

Therefore, there exist δ, $\bar{\alpha} > 0$ such that

$$I_{a,0}(u) \geq \bar{\alpha} > 0 \quad \text{on} \quad \|u\|_{H^s(\mathbb{R}^N)} = \delta, \quad u \in H^s(\mathbb{R}^N).$$

Now, choose $u_0 \in E_1$ with $\|u_0\|_{H^s(\mathbb{R}^N)} > \delta$ and $u_0 \geq 0$. As $p > 1$, there exists $t_0 > 0$ such that

$$I_{a,0}(tu_0) \leq \frac{t^2}{2}\|u\|^2_{H^s(\mathbb{R}^N)} - \frac{a_0 t^{p+1}}{p+1}\int_{B_1} u_0^{p+1}\mathrm{d}x < 0 \quad \forall\, t > t_0.$$

If $t_0 \leq 1$, then we define $e := u_0$ otherwise, we define $e := t_0 u_0$. Therefore, $0 \leq e \in E_1$, $\|e\| \geq \delta$ and $I_{a,0}(te) < 0$ for all $t > 1$. We define Γ_k to be the set of all continuous paths in E_k connecting 0 and e and Γ be the set of all continuous paths in $H^s(\mathbb{R}^N)$ connecting 0 and e. Set,

$$\alpha := \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I_{a,0}(u) \tag{3.5.2}$$

and

$$\alpha_k := \inf_{\gamma \in \Gamma_k} \max_{u \in \gamma} I_{a,0}(u). \tag{3.5.3}$$

Observe that, $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma$ and this in turn implies

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha \geq \bar{\alpha} > 0. \tag{3.5.4}$$

Moreover, since $\cup_{k\geq 1} E_k$ is dense in $H^s(\mathbb{R}^N)$, it is easy to check that $\alpha_k \to \alpha$ as $k \to \infty$. Applying Mountain pass lemma, we obtain α_k is a critical point of $I^k_{a,0}$. Let $u_k \in E_k$ be the critical point of $I^k_{a,0}$ corresponding to α_k. Therefore, $I_{a,0}(u_k) = I^k_{a,0}(u_k) = \alpha_k$ and $(I^k_{a,0})'(u_k) = 0$. In particular,

$$\alpha_k = I^k_{a,0}(u_k) - \frac{1}{p+1}(I^k_{a,0})'(u_k)u_k = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_k\|^2_{H^s(\mathbb{R}^N)}.$$

Since $\alpha_k \leq \alpha_1$, for all $k \geq 1$, from the above expression we obtain $\{u_k\}$ is uniformly bounded in $H^s(\mathbb{R}^N)$. Hence there exists $\bar{u}$ in $H^s(\mathbb{R}^N)$ such that, up to a subsequence, $u_k \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$. Moreover, $I_{a,0}(u_k) = \alpha_k \geq \bar{\alpha} > 0$ implies u_k is nontrivial. Therefore, taking $(u_k)_-$ as the test function in

$$\begin{cases} (-\Delta)^s u + u = a(x)u_+^p \ \text{ in } B_k, \\ \qquad\qquad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_k, \end{cases}$$

we obtain $u_k \geq 0$. Therefore, using maximum principle we have $u_k > 0$ in B_k. Hence $\bar{u} \geq 0$. $\square$

Lemma 3.5.3. *Let u_k be a critical point of $I^k_{a,0}$ and $u_k \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$. Then $\bar{u}$ is a critical point of $I_{a,0}$.*

Proof. u_k be a critical point of $I^k_{a,0}$ implies

$$\langle u_k, \psi\rangle_{H^s(\mathbb{R}^N)} - \int_{B_k} a(x)(u_k)^p_+\,\psi\,\mathrm{d}x = 0 \quad \forall\,\psi \in E_k.$$

Let $\phi \in C_0^\infty(\mathbb{R}^N)$. Then $\phi \in E_k$ for large k. Therefore, for large k,

$$\langle u_k, \phi\rangle_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} a(x)(u_k)^p_+\phi\,\mathrm{d}x = 0. \tag{3.5.5}$$

Since $u_k \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$ implies $\langle u_k, \phi\rangle_{H^s(\mathbb{R}^N)} \to \langle \bar{u}, \phi\rangle_{H^s(\mathbb{R}^N)}$ and by Lemma 3.2.3, $\int_{\mathbb{R}^N} a(x)(u_k)^p_+\,\phi\,\mathrm{d}x \to \int_{\mathbb{R}^N} a(x)\bar{u}^p_+\,\phi\,\mathrm{d}x$, as $k \to \infty$. Therefore letting $k \to \infty$ in (3.5.5), yields $I'_{a,0}(\bar{u})(\phi) = 0$. Since $\phi \in C_0^\infty(\mathbb{R}^N)$ is arbitrary, the lemma follows. □

Thanks to Lemma 3.5.2 and Lemma 3.5.3, we are just left to show that $\bar{u} \not\equiv 0$, in order to complete the proof of Theorem 3.1.4.

3.5.1 Comparison argument

For any arbitrarily fixed $R > 0$, define,

$$h_R(x) = \begin{cases} a(x) & \text{if} \quad |x| > R, \\ 0 & \text{if} \quad |x| \le R. \end{cases} \tag{3.5.6}$$

We define the following Nehari manifolds:

$$\mathcal{N} := \{u \in H^s(\mathbb{R}^N) \setminus \{0\} : \|u\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} a(x)u^{p+1}_+\,\mathrm{d}x\}$$

and

$$\mathcal{N}_R := \{u \in H^s(\mathbb{R}^N) \setminus \{0\} : \|u\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} h_R(x)u^{p+1}_+\,\mathrm{d}x\}$$

Set,

$$\alpha^* := \inf_{u\in\mathcal{N}} I_{a,0}(u) \quad \text{and} \quad \beta^*_R := \inf_{u\in\mathcal{N}_R} I_{h_R,0}(u). \tag{3.5.7}$$

Lemma 3.5.4. $\alpha^* < \lim_{R\to\infty} \beta_R^*$.

Proof. From the definition of $I_{a,0}$, we have

$$\begin{aligned}
\alpha^* &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \inf_{\mathcal{N}} \int_{\mathbb{R}^N} a(x) u_+^{p+1} \mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right) \inf_{H^s(\mathbb{R}^N)\setminus\{0\}} \left[\frac{\|u\|_{H^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} a(x) u_+^{p+1} \mathrm{d}x\right)^{\frac{1}{p+1}}}\right]^{\frac{2(p+1)}{p-1}}.
\end{aligned}$$

Similarly,

$$\beta_R^* = \left(\frac{1}{2} - \frac{1}{p+1}\right) \inf_{H^s(\mathbb{R}^N)\setminus\{0\}} \left[\frac{\|u\|_{H^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} h_R(x) u_+^{p+1} \mathrm{d}x\right)^{\frac{1}{p+1}}}\right]^{\frac{2(p+1)}{p-1}}.$$

Therefore, it is enough to prove

$$\inf_{H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{H^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} a(x) u_+^{p+1} \mathrm{d}x\right)^{\frac{1}{p+1}}} < \lim_{R\to\infty} \left(\inf_{H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{H^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} h_R(x) u_+^{p+1} \mathrm{d}x\right)^{\frac{1}{p+1}}}\right).$$

Equivalently, it is enough to show

$$\sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} a(x) u_+^{p+1} \mathrm{d}x > \lim_{R\to\infty} \left(\sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{|x|>R} a(x) u_+^{p+1} \mathrm{d}x\right). \quad (3.5.8)$$

From (3.5.1) we have $\lim_{|x|\to\infty} a(x) = a_0 = \inf_{x\in\mathbb{R}^N} a(x)$. In view of Remark 3.5.1, we first note that, it is enough to consider the case when $\mu(\{x \in \mathbb{R}^N : a(x) \neq a_0\}) > 0$, where $\mu(X)$ denotes the Lebesgue measure of a set X. In this case, we

Claim:

$$\sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} a(x) u_+^{p+1} \mathrm{d}x > M := \sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} a_0 u_+^{p+1} \mathrm{d}x. \quad (3.5.9)$$

To see the claim, first we note that clearly, for each $u \in H^s(\mathbb{R}^N)$ with $\|u\|_{H^s(\mathbb{R}^N)} = 1$, we have

$$\int_{\mathbb{R}^N} a(x) u_+^{p+1} \mathrm{d}x > \int_{\mathbb{R}^N} a_0 u_+^{p+1} \mathrm{d}x.$$

Therefore, the claim will be proved if we show that M is attained. For that, let v_n be a maximizing sequence, i.e.,

$$\|v_n\|_{H^s(\mathbb{R}^N)} = 1, \quad \int_{\mathbb{R}^N} a_0 (v_n)_+^{p+1} \mathrm{d}x \to M.$$

Using symmetric rearrangement technique, without loss of generality, we can assume that v_n is radially symmetric and symmetric decreasing (see [73]). We denote by $H^s_{rad,d}(\mathbb{R}^N)$, the set of all radially symmetric and decreasing functions in $H^s(\mathbb{R}^N)$. Using [28, Lemma 6.1], it is easy to see that

$$H^s_{rad,d}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$$

is compact. Hence by standard argument, it follows that M is attained. Therefore the claim follows.

Thanks to the above above claim, we have

$$\begin{aligned} \sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} a(x) u_+^{p+1} \mathrm{d}x &> \sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} a_0 u_+^{p+1} \mathrm{d}x \\ &> \sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{|x|>R} a_0 u_+^{p+1} \mathrm{d}x \\ &\geq \lim_{R\to\infty} \left(\sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{|x|>R} a_0 u_+^{p+1} \mathrm{d}x \right). \end{aligned} \tag{3.5.10}$$

Using the fact that $\lim_{|x|\to\infty} a(x) \to a_0$, a straight forward computation yields

$$\lim_{R\to\infty} \left(\sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{|x|>R} a_0 u_+^{p+1} \mathrm{d}x \right) = \lim_{R\to\infty} \left(\sup_{\|u\|_{H^s(\mathbb{R}^N)}=1} \int_{|x|>R} a(x) u_+^{p+1} \mathrm{d}x \right).$$

Substituting the above equality into (3.5.10), we obtain (3.5.8). □

Lemma 3.5.5. *$\alpha \leq \alpha^*$, where α and α^* are defined as in* (3.5.2) *and* (3.5.7).

Proof. Let $v \in \mathcal{N}$ be arbitrarily chosen and V denote the 2-dimensional subspace spanned by v and e, where e is as found in the proof of Lemma 3.5.2. Let $V^+ := \{av + be : a \geq 0, b \geq 0\}$. Let S be the circle on V

with radius R large enough such that $I_{a,0} \leq 0$ on $S \cap V^+$ (this follows since $p > 1$ and standard compactness argument on V^+) and v, e lie inside S. Let $l_v := \{tv : t \geq 0\}$ and $l_e := \{te : t \geq 0\}$ intersect S at v_1 and v_2 respectively. We define, $\tilde{\gamma}$ be the path that consists of the segment on l_v with endpoints 0 and v_1, the arc $S \cap V^+$ (connecting v_1 and v_2) and the segment on l_e with endpoints v_2 and e. Therefore, clearly $\tilde{\gamma} \in \Gamma$ and $v \in \tilde{\gamma}$.

Claim: $\max_{u \in \tilde{\gamma}} I_{a,0}(u) = I_{a,0}(v)$.

Indeed, a straight forward computation yields

$$v \in \mathcal{N} \quad \text{implies} \quad \max_{t \geq 0} I_{a,0}(tv) = I_{a,0}(v).$$

Further, from the construction of $\tilde{\gamma}$ it follows $I_{a,0} \leq 0$ on the rest part of $\tilde{\gamma}$ (since $I_{a,0} \leq 0$ on $S \cap V^+$ and $I_{a,0}(te) < 0$ for $t > 1$). Hence the claim follows.

The above claim immediately yields

$$\alpha \leq \max_{u \in \tilde{\gamma}} I_{a,0}(u) = I_{a,0}(v).$$

On the other hand, as $v \in \mathcal{N}$ was arbitrarily chosen, we obtain

$$\alpha \leq \inf_{v \in \mathcal{N}} I_{a,0}(v) = \alpha^*.$$

□

Proof of Theorem 3.1.4

Proof. By Lemma 3.5.3 and Lemma 3.5.2, we know that $\bar{u}$ is a nonnegative critical point of $I_{a,0}$. Therefore, it's enough to show that $\bar{u} \not\equiv 0$ in $H^s(\mathbb{R}^N)$. We prove this by method of contradiction. Suppose $\bar{u} \equiv 0$ in $H^s(\mathbb{R}^N)$. Therefore, using Rellich compactness theorem, $u_k \to 0$ in $L^{p+1}_{loc}(\mathbb{R}^N)$. Hence,

$$0 \leq \varepsilon_k := \int_{B_R} a(x)(u_k)_+^{p+1} \mathrm{d}x \to 0 \quad \text{as} \quad k \to \infty.$$

It is easy to see that for each k, there exists unique $t_{k,R} > 0$ such that $t_{k,R} u_k \in \mathcal{N}_R$, i.e.,

$$t_{k,R}^2 \|u_k\|^2_{H^s(\mathbb{R}^N)} = t_{k,R}^{p+1} \int_{\mathbb{R}^N} h_R(x)(u_k)_+^{p+1} \mathrm{d}x.$$

Claim: $\{t_{k,R}\}_{k=1}^{\infty}$ is a bounded sequence.

To prove the claim, first we note that since u_k is critical point of $I^k_{a,0}$, we have

$$\|u_k\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} a(x)(u_k)_+^{p+1}\mathrm{d}x = \varepsilon_k + \int_{|x|>R} a(x)(u_k)_+^{p+1}\mathrm{d}x$$
$$= \varepsilon_k + \int_{|x|>R} h_R(x)(u_k)_+^{p+1}\mathrm{d}x.$$

Further, $p > 1$ implies there exists $\delta > 0$ such that $p+1 > 2+\delta$. Therefore, if $t_{k,R} \geq 1$ then combining the above two expressions, we obtain

$$\varepsilon_k t^2_{k,R} + t^2_{k,R}\int_{|x|>R} h_R(x)(u_k)_+^{p+1}\mathrm{d}x \geq t^{p+1}_{k,R}\int_{|x|>R} h_R(x)(u_k)_+^{p+1}\mathrm{d}x$$
$$\geq t^{2+\delta}_{k,R}\int_{|x|>R} h_R(x)(u_k)_+^{p+1}\mathrm{d}x.$$

Consequently,

$$t^2_{k,R}\varepsilon_k \geq (t^{2+\delta}_{k,R} - t^2_{k,R})\int_{|x|>R} h_R(x)(u_k)_+^{p+1}\mathrm{d}x$$
$$= (t^{2+\delta}_{k,R} - t^2_{k,R})\int_{|x|>R} a(x)(u_k)_+^{p+1}\mathrm{d}x$$
$$= (t^{2+\delta}_{k,R} - t^2_{k,R})(\|u_k\|^2_{H^s(\mathbb{R}^N)} - \varepsilon_k). \qquad (3.5.11)$$

Further, note that $\varepsilon_k \to 0$ and $I^k_{a,0}(u_k) = \alpha_k$ implies

$$\|u_k\|^2_{H^s(\mathbb{R}^N)} = 2\int_{\mathbb{R}^N} a(x)(u_k)_+^{p+1}\mathrm{d}x + 2\alpha_k \geq 2\alpha_k \geq 2\bar{\alpha}$$

(the last inequality follows from (3.5.4)). Therefore from (3.5.11), we have

$$t^2_{k,R}\varepsilon_k \geq \bar{\alpha}(t^{2+\delta}_{k,R} - t^2_{k,R}) \quad \text{for large } k.$$

As a consequence, $t_{k,R} \to 1$ as $k \to \infty$ (for fixed $R > 0$). Hence the claim holds.

Using the above claim, we have $t_{k,R}u_k \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. Further, as u_k is critical point of $I^k_{a,0}$ implies $\|u_k\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} a(x)(u_k)_+^{p+1}\mathrm{d}x$, a straight forward computation yields that $\max_{t\geq 0} I_{a,0}(tu_k) = I_{a,0}(u_k)$. Therefore,

$$\begin{aligned}
\alpha_k = I_{a,0}(u_k) &\geq I_{a,0}(t_{k,R}u_k) \\
&= \frac{t_{k,R}^2}{2}\|u_k\|^2_{H^s(\mathbb{R}^N)} - \frac{t_{k,R}^{p+1}}{p+1}\int_{|x|>R} a(x)(u_k)_+^{p+1}\mathrm{d}x \\
&\qquad -\frac{t_{k,R}^{p+1}}{p+1}\int_{|x|<R} a(x)(u_k)_+^{p+1}\mathrm{d}x \\
&= I_{h_R,0}(t_{k,R}u_k) - \frac{t_{k,R}^{p+1}}{p+1}\int_{|x|<R} a(x)(u_k)_+^{p+1}\mathrm{d}x \\
&\geq \beta_R^* - \frac{t_{k,R}^{p+1}}{p+1}\int_{|x|<R} a(x)(u_k)_+^{p+1}\mathrm{d}x,
\end{aligned}$$

where in the last inequality we have used the fact that $t_{k,R}u_k \in \mathcal{N}_R$. As before, $\frac{t_{k,R}^{p+1}}{p+1}\int_{|x|<R} a(x)(u_k)_+^{p+1}\mathrm{d}x \to 0$ as $k \to \infty$ (keeping $R > 0$ fixed). Thus, taking the limit $k \to \infty$ yields $\alpha \geq \beta_R^*$, where α is as defined in (3.5.2). Consequently, $\alpha \geq \lim_{R\to\infty} \beta_R^*$. Combining this with Lemma 3.5.5, we obtain $\lim_{R\to\infty} \beta_R^* \leq \alpha^*$. This contradicts Lemma 3.5.4. Hence $\bar{u} \neq 0$. Therefore, $\bar{u}$ is a nontrivial nonnegative critical point of $I_{a,0}$. Finally, thanks to maximum principle [56, Theorem 1.2], we get $\bar{u}$ is a positive solution to (3.3.2) (with $f \equiv 0$). Hence, $\bar{u}$ is a positive solution to $(\mathcal{P})$(with $f \equiv 0$). This completes the proof. □

Remark 3.5.6. *It is easy to note that if $a(x) \to 0$ as $|x| \to \infty$ at infinity, once again some "compactness" exists and standard variational arguments leads to the existence of positive solutions in this case.*

Remark 3.5.7. *If $s > \frac{1}{2}$ and $a : \mathbb{R}^N \to [0, \infty)$ is radial function satisfying the growth condition*

$$a(r) \leq C(1 + r^l), \quad r \geq 0,$$

C>0 being a constant and $l < (N-1)(p-1)/2$, then proceeding in the spirit of [59, Lemma 4.8], it follows that (3.5.5) *admits a positive radial solution w_k and $\|w_k\|_{H^s(\mathbb{R}^N)}$ is uniformly bounded above. Therefore, up to a subsequence*

$w_k \rightharpoonup w$ *in* $H^s(\mathbb{R}^N)$ *and* $w \geq 0$. *Next, using the radial lemma [[101], Theorem 7.4(i)], it can be shown in the similar way as in [59, Corollary 4.8] that* $w \not\equiv 0$ *in* $H^s(\mathbb{R}^N)$ *i.e.,* $(\mathcal{P})$ *(with* $f \equiv 0$*) admits a positive radial solution in* $H^s(\mathbb{R}^N)$. *In this case, we do not need to assume any asymptotic behavior of* a *at infinity.*

Conclusion : In this chapter, we have considered nonlocal nonhomogeneous scalar field equation with subcritical nonlinearity multiplied by a positive, bounded coefficient function a whose asymptotic behavior is known. Under the assumption $a \geq 1$ or $a \in (0,1]$, we have obtained multiplicity results under suitable condition on the nonhomogeneous term. While in the homogeneous case we could prove existence of a positive solution under no additional assumption on a except positivity, boundedness and asymptotic behavior. Now, it might be interesting to consider the same question with nonhomogeneous term and weak assumptions on a.

Chapter 4

Fractional Hardy-Sobolev equations with nonhomogeneous terms

The chapter deals with the following fractional Hardy-Sobolev equation with nonhomogeneous term

$$\begin{cases} (-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = K(x)\frac{|u|^{2_s^*(t)-2}u}{|x|^t} + f(x) \quad \text{in} \quad \mathbb{R}^N, \\ u \in \dot{H}^s(\mathbb{R}^N), \end{cases} \qquad (E^{\gamma}_{K,t,f})$$

where $N > 2s$, $s \in (0,1)$, $0 \le t < 2s < N$ and $2_s^*(t) := \frac{2(N-t)}{N-2s}$. Clearly, $2 < 2_s^*(t) \le \frac{2N}{N-2s} = 2_s^*$. Here $0 < \gamma < \gamma_{N,s}$, where $\gamma_{N,s}$ is the best Hardy constant in the fractional Hardy inequality

$$\gamma_{N,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \,\mathrm{d}x \le \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 \,\mathrm{d}\xi, \quad \gamma_{N,s} = 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})},$$

where $\mathcal{F}(u)$ denotes the Fourier transform of u. Moreover,

$$\lim_{s\to 1} \gamma_{N,s} := \left(\frac{N-2}{2}\right)^2,$$

which is exactly the best Hardy constant in the classical case $s = 1$. For the sharp Hardy inequalities in general fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$,

$1 < p < \infty$, as well as for historical comments in the case $p = 2$, we refer the interested reader to [73] and the references therein. While for fractional Hardy-Sobolev-Maz'ya inequality, we mention the recent contribution [90] and for fractional Hardy inequality in Heisenberg group we refer to [6]. In $(E^{\gamma}_{K,t,f})$, the functions K and f satisfy the properties:

(K) $0 < K \in C(\mathbb{R}^N)$, $K(0) = 1 = \lim_{|x|\to\infty} K(x)$.

(F) $f \not\equiv 0$ *is a nonnegative functional in the dual space* $\dot{H}^s(\mathbb{R}^N)'$ *of* $\dot{H}^s(\mathbb{R}^N)$, i.e. whenever u is a nonnegative function in $\dot{H}^s(\mathbb{R}^N)$ then ${}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \geq 0$.

Using the Hardy inequality, it is easy to see that the operator $L_{\gamma,s} := (-\Delta)^s - \frac{\gamma}{|x|^{2s}}$ with $0 \leq \gamma < \gamma_{N,s}$ is a positive operator. The request $\gamma < \gamma_{N,s}$ is fairly natural since we are looking for positive solutions. In this case the Hardy-Sobolev inequality holds for $L_{\gamma,s}$, which states that if $0 \leq t < 2s < N$, then

$$S_{\gamma,t,s} = S_{\gamma,t,s}(\mathbb{R}^N) := \inf_{u\in\dot{H}^s(\mathbb{R}^N)\setminus\{0\}} \frac{\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}\,\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2}{2^*_s(t)}}} \tag{4.0.1}$$

is finite, strictly positive and achieved (see [74, 75]). Observe that thanks to [74], any minimizer for (4.0.1) leads (up to a constant) to a nonnegative variational solution of the

$$(-\Delta)^s u - \gamma\frac{u}{|x|^{2s}} = \frac{|u|^{2^*_s(t)-2}u}{|x|^t}, \quad u \in \dot{H}^s(\mathbb{R}^N). \tag{$E^{\gamma}_{1,t,0}$}$$

If $\gamma = 0 = t$, then $S_{\gamma,t,s}$ reduces to the best Sobolev constant $S_{0,0,s} = S$ which is known to be achieved by $C_{N,s}(1+|x|^2)^{-\frac{N-2s}{2}}$ and any minimizer of S leads (up to a constant) to a nonnegative solution of equation $(E^0_{1,0,0})$ i.e., $(E^{\gamma}_{1,t,0})$ with $\gamma = 0 = t$.

Definition 4.0.1. (Positive weak solution) We say $u \in \dot{H}^s(\mathbb{R}^N)$ is a *positive weak solution of* $(E^{\gamma}_{K,t,f})$ if $u > 0$ in $\mathbb{R}^N$ and for every $\phi \in \dot{H}^s(\mathbb{R}^N)$, we have

$$\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{u\phi}{|x|^{2s}}\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} K(x)\frac{|u|^{2^*_s(t)-2}u\phi}{|x|^t}\mathrm{d}x + {}_{(\dot{H}^s)'}\langle f,\phi\rangle_{\dot{H}^s},$$

where ${}_{(\dot{H}^s)'}\langle .,.\rangle_{H^s}$ denotes the duality bracket between $\dot{H}^s(\mathbb{R}^N)$ and its dual $\dot{H}^s(\mathbb{R}^N)'$.

Remark 4.0.2. For $0 < \gamma < \gamma_{N,s}$,

$$\|u\|_{\gamma} := \left(\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}\,\mathrm{d}x\right)^{\frac{1}{2}}$$

defines a norm in $\dot{H}^s(\mathbb{R}^N)$ which is equivalent to the standard norm in $\dot{H}^s(\mathbb{R}^N)$. In particular,

$$\sqrt{1-\frac{\gamma}{\gamma_{N,s}}}\|u\|_{\dot{H}^s} \le \|u\|_{\gamma} \le \|u\|_{\dot{H}^s}.$$

The corresponding equivalent inner product $\langle\cdot,\cdot\rangle_{\gamma}$ in the fractional homogeneous Hilbert space $\dot{H}^s(\mathbb{R}^N)$ is given by

$$\langle u,v\rangle_{\gamma} := \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{uv}{|x|^{2s}}\,\mathrm{d}x.$$

Finally, for simplicity we endow in what follows the weighted Lebesgue space $L^{2^*_s(t)}(\mathbb{R}^N,|x|^{-t})$ with the norm $\|u\|_{L^{2^*_s(t)}(\mathbb{R}^N,|x|^{-t})} = \left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x\right)^{1/2^*_s(t)}$.

We are going to prove existence and multiplicity of positive solutions of $(E^{\gamma}_{K,t,f})$ in the spirit of [24, 31]. Under the conditions on K and f stated above, equation $(E^{\gamma}_{K,t,f})$ can be regarded as a perturbation problem of the homogeneous equation $(E^{\gamma}_{1,t,0})$. It is known from [75] that when $0 < \gamma < \gamma_{N,s}$ or $\{\gamma = 0$ and $0 < t < 2s\}$, then any nonnegative minimizer for $S_{\gamma,t,s}$ is positive, radially symmetric, radially decreasing, and approaches zero as

$|x| \to \infty$. The main question to be addressed is whether positive solution can survive after a perturbation of type $(E^{\gamma}_{K,t,f})$ or not.

For $\gamma = 0 = t$, this kind of question was recently studied by the first and third author of the current paper in [31]. For Schrödinger operator (without Hardy term), same type of questions were addressed in [24] with subcritical nonlinearity. However for $\gamma \neq 0$ the presence of the Hardy potential requires a new argument to dealt with. One of the key steps to prove the multiplicity result is a careful analysis of the Palais-Smale level. Theorem 4.2.1 studies the profile decomposition of any Palais-Smale sequence possessed by the underlying functional associated to $(E^{\gamma}_{K,t,f})$. We show that concentration takes place along a single profile when $t > 0$, while concentration takes place along two different profiles when $t = 0$. In the local case $s = 1$, $t = 0$ and $f = 0$ Smets deals with the profile decomposition in [103]. In bounded domains and again in the local case $s = 1$, paper [32] treats the case of all $t \geq 0$. However, extension of the latter results in the nonlocal case $s \in (0, 1)$ and in the entire space $\mathbb{R}^N$ is highly nontrivial and requires several delicate estimates and techniques to deal with.

In local case $s = 1$, we refer [65, 103], where authors have studied the local version of $(E^{\gamma}_{K,0,0})$ in $\mathbb{R}^N$. In the nonlocal case, when the domain is a bounded subset of $\mathbb{R}^N$, existence of positive solutions of $(E^{\gamma}_{K,t,f})$ in Ω with $\gamma = 0 = t$ (i.e., without Hardy and Hardy-Sobolev terms) and Dirichlet boundary condition has been proved in [102]. Existence of sign changing solutions of

$$(-\Delta)^s u = |u|^{\frac{4s}{N-2s}} u + \varepsilon f \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $f \geq 0, f \in L^{\infty}(\Omega)$ has been studied in [7] and existence of two positive solutions have been established in [110] when f is a continuous function with compact support in Ω. In the nonlocal case, when the domain is the entire space $\mathbb{R}^N$, but $\gamma = 0$, we refer to [24, 31], where multiplicity of positive solutions have been studied in presence of a nonhomogeneous term.

There is a wide literature regarding problems involving the fractional Hardy potential. Avoiding to disclose the discussion we refer to the following (far from being complete) list of works and references therein [2, 3, 23, 34, 60, 72, 75]. In [60] Dipierro, et al. study the equation $(E^{\gamma}_{1,0,0})$ (i.e., $(E^{\gamma}_{1,t,0})$ with $t = 0$) and prove existence of a ground state solution, qualitative properties of positive solutions and asymptotic behavior of solutions at both 0 and infinity. In [36], the authors studied existence and asymptotic behavior of p-Laplacian problems with Hardy potentials and critical nonlinearities on general open subsets of Heisenberg groups. In [23], the authors deal with the Green function for $L_{\gamma,s}$ $(0 < \gamma < \gamma_{N,s})$ and show when the integral representation of the weak solution is valid.

It is worth noting that solutions of $(E^{\gamma}_{1,t,0})$ do not belong to $L^{\infty}(\mathbb{R}^N)$ as soon as $\gamma > 0$, because of the singularity at zero. In fact solutions blow up at origin (see [60, 75]). For this reason, it seems more difficult to handle $(E^{\gamma}_{K,t,f})$ in the general case using the fine analysis of blow up technique quoted above.

To the best of our knowledge, so far there has been no papers in the literature, where existence and multiplicity of positive solutions of Hardy-Sobolev type equations (with $\gamma \neq 0$ and $t \geq 0$) in $\mathbb{R}^N$, have been established in the nonhomogeneous case $f \neq 0$. Also the profile decomposition in the nonlocal case with the Hardy term is completely new and the proof is very involved, delicate and complicated compared with the local case $s = 1$. The proofs are not at all an easy adoption of the local case or the case $\gamma = 0$. The multiplicity results in this paper is new even in the local case $s = 1$, but we leave the obvious changes, when $s = 1$, to the interested reader.

4.1 Main Results

Below we state the main result.

Theorem 4.1.1. *Assume that* $(\mathbf{F})$ *and* $(\mathbf{K})$ *are satisfied, with* $K \geq 1$ *in* $\mathbb{R}^N$. *If* $\|f\|_{(\dot{H}^s)'} < C_t\sqrt{1-\frac{\gamma}{\gamma_{N,s}}}S_{\gamma,t,s}^{\frac{N-t}{4s-2t}}$, *where*

$$C_t = \left(\frac{4s-2t}{N-2t+2s}\right)\left((2_s^*(t)-1)\|K\|_{L^\infty(\mathbb{R}^N)}\right)^{-\left(\frac{N-2s}{4s-2t}\right)},$$

then

(i) For $t > 0$, *equation* $(E_{K,t,f}^{\gamma})$ *admits two positive solutions;*

(ii) For $t = 0$, *equation* $(E_{K,t,f}^{\gamma})$ *admits a positive solution. In addition, if* $\|K\|_{L^\infty(\mathbb{R}^N)} < \left(\frac{S}{S_{\gamma,0,s}}\right)^{\frac{N}{N-2s}}$ *then* $(E_{K,t,f}^{\gamma})$ *admits two positive solutions.*

Remark 4.1.2. It is worth mentioning that $S > S_{\gamma,0,s}$ for any $\gamma > 0$. To see this, we denote by W the unique positive solution of $(E_{1,0,0}^0)$ and let $W_{\gamma,0}$ be a minimum energy positive solution (ground state solution) of $(E_{1,t,0}^{\gamma})$ with $t = 0$. Then,

$$I_{1,0,0}^{\gamma}(W_{\gamma,0}) \leq I_{1,0,0}^{\gamma}(W) < I_{1,0,0}^{0}(W).$$

A straight forward computation yields that $I_{1,0,0}^{0}(W) = \frac{s}{N}S^{\frac{N}{2s}}$ and $I_{1,0,0}^{\gamma}(W_{\gamma,0}) = \frac{s}{N}S_{\gamma,0,s}^{\frac{N}{2s}}$. Consequently, $S > S_{\gamma,0,s}$ for any $\gamma > 0$. From this observation, it immediately follows that if $K \equiv 1$, then $(E_{1,t,f}^{\gamma})$ admits two positive solutions for all $t \geq 0$ under the given assumption $(\mathbf{F})$ on f.

Note that the Hardy-Sobolev embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*(t)}(\mathbb{R}^N, |x|^{-t})$ for any $0 \leq t < 2s$ is continuous, but not compact. This noncompactness of the embedding even locally in any neighbourhood of zero leads to other additional difficulties, and more importantly, to new phenomenon concerning the possibility of blow up. Thus the variational functional associated to $(E_{K,t,f}^{\gamma})$ does not satisfy the Palais-Smale condition, briefly called (PS) condition. The lack of compactness of the functional associated to $(E_{K,t,f}^{\gamma})$ is due to a concentration phenomenon. We analyze this noncompactness in Theorem 4.2.1, which is one of the most important theorems of the paper. Using this theorem we prove existence and multiplicity of positive solutions to $(E_{K,t,f}^{\gamma})$ in

Theorem 4.1.1. For that first we decompose $\dot{H}^s(\mathbb{R}^N)$ into three components which are homeomorphic to the interior, boundary and the exterior of the unit ball in $\dot{H}^s(\mathbb{R}^N)$ respectively. Then we prove that the energy functional associated to $(E^{\gamma}_{K,t,f})$ attains its infimum on one of the components which serves as our first positive solution. The second positive solution is obtained via a careful analysis on the (PS) sequences associated to the energy functional and we construct a min–max critical level κ_t, where the (PS) condition holds.

This chapter has been organised in the following way. In Section 4.2, we prove the Palais-Smale decomposition theorem associated with the functional corresponding to $(E^{\gamma}_{K,t,f})$ (see Theorem 4.2.1). In Section 4.3, we show existence of two positive solutions of $(E^{\gamma}_{K,t,f})$, namely Theorem 4.1.1. Last section contains some basic estimates which are used in proving the Palais-Smale characterization theorem in Section 4.2.

4.2 Palais-Smale decomposition

In this section we study the Palais-Smale sequences (in short, (PS) sequences) of the functional $\bar{I}^{\gamma}_{K,t,f}$ associated to $(E^{\gamma}_{K,t,f})$

$$\begin{aligned}\bar{I}^{\gamma}_{K,t,f}(u) := \frac{C_{N,s}}{4}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\gamma}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}\,\mathrm{d}x \\ - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{|u|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} \qquad (4.2.1)\\ = \frac{1}{2}\|u\|^2_{\gamma} - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{|u|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s},\end{aligned}$$

where K and f satisfy (**K**) and (**F**) respectively.

We say that the sequence $(u_n)_n \subset \dot{H}^s(\mathbb{R}^N)$ is a (PS) sequence for $\bar{I}_{K,t,f}$ at level β if $\bar{I}_{K,t,f}(u_n) \to \beta$ and $(\bar{I}_{K,t,f})'(u_n) \to 0$ in $(\dot{H}^s)'$. It is easy to see that the weak limit of a (PS) sequence solves $(E^{\gamma}_{K,t,f})$ except the positivity.

However the main difficulty is that the (PS) sequence may not converge strongly and hence the weak limit can be zero even if $\beta > 0$. The main purpose of this section is to classify (PS) sequences of the functional $\bar{I}^{\gamma}_{K,t,f}$. Classification of (PS) sequences has been done for various problems having lack of compactness, to quote a few, we cite [31, 94, 95] in the nonlocal case with $\gamma = 0 = t$, while in the local case [32, 103] with Hardy potentials and in [105] without Hardy potentials. We also refer to [108] for a more abstract approach of the profile decomposition in general Hilbert spaces. We establish a classification theorem for the (PS) sequences of (4.2.1) in the spirit of the above results. In [31, 94], the noncompactness is completely described by the single blow up profile W, which is a solution of

$$(-\Delta)^s W = |W|^{2^*_s - 2} W \quad \text{in} \quad \mathbb{R}^N, \quad W \in \dot{H}^s(\mathbb{R}^N). \qquad (E^0_{1,0,0})$$

In [32, 103] (the local case $s = 1$), the noncompactness are due to concentration occurring through two different profiles. Possibility of two different type of profiles are still present for $(E^{\gamma}_{K,t,f})$ in the case $t = 0$.

Let $t = 0$ and let W be any solution of $(E^0_{1,0,0})$. Then, it can be easily verified that any sequence of the form

$$W^{r_n, y_n}(x) := K(y)^{-\frac{N-2s}{4s}} r_n^{-\frac{N-2s}{4s}} W\Big(\frac{x - y_n}{r_n}\Big), \qquad (4.2.2)$$

is a (PS) sequence for $\bar{I}^{\gamma}_{K,0,0}$ if $y_n \to y \neq 0$ and $r_n \to 0$. If $y = 0$, then W^{r_n, y_n} remains a (PS) sequence for $\bar{I}^{\gamma}_{K,0,0}$ provided that $\frac{|y_n|}{r_n} \to \infty$. Also $W^{r_n, y_n} \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ by [94, Lemma 3].

Further, let $W_{\gamma,t}$ be any solution of $(E^{\gamma}_{1,t,0})$ (where $t \geq 0$). Define a sequence $(W^{R_n,0}_{\gamma,t})_n$ of the form

$$W^{R_n,0}_{\gamma,t}(x) := R_n^{-\frac{N-2s}{4s}} W_{\gamma,t}\Big(\frac{x}{R_n}\Big), \qquad (4.2.3)$$

where $R_n \to 0$. Then $W^{R_n,0}_{\gamma,t} \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ and $(W^{R_n,0}_{\gamma,t})_n$ is a (PS) sequence for $\bar{I}^{\gamma}_{K,t,0}$ for $t \geq 0$.

Theorem 4.2.1. *Let $(u_n)_n$ be a (PS) sequence for $\bar{I}^{\gamma}_{K,t,f}$ at the level β. Then up to a subsequence, still denoted by $(u_n)_n$, the next properties hold.*

If $t = 0$, then there exist $n_1,\ n_2 \in \mathbb{N}$, n_2 sequences $(R^k_n)_n \subset \mathbb{R}^+$ $(1 \leq k \leq n_2)$, n_1 sequences $(r^j_n)_n \subset \mathbb{R}^+$ and $(y^j_n)_n \subset \mathbb{R}^N \setminus \{0\}$ $(1 \leq j \leq n_1)$ and $0 \leq \bar{u} \in \dot{H}^s(\mathbb{R}^N)$ such that

$$
\begin{array}{ll}
(i) & u_n = \bar{u} + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} (W^j)^{r^j_n,\, y^j_n} + \sum_{k=1}^{n_2} (W^k_{\gamma,t})^{R^k_n,0} + o(1) \\
(ii) & (\bar{I}^{\gamma}_{K,t,f})'(\bar{u}) = 0 \\
(iii) & R^k_n \to 0 \ (1 \leq k \leq n_2) \text{ and } r^j_n \to 0 \, (1 \leq j \leq n_1) \\
(iv) & \text{either } \ y^j_n \to y^j \in \mathbb{R}^N \text{or } |y^j| \to \infty \text{ and } \dfrac{r^j_n}{|y^j_n|} \to 0 \ (1 \leq j \leq n_1) \\
(v) & \beta = \bar{I}^{\gamma}_{K,t,f}(\bar{u}) + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{2s}} \bar{I}^0_{1,0,0}(W^j) + \sum_{k=1}^{n_2} \bar{I}^{\gamma}_{1,t,0}(W^k_{\gamma,t}) + o(1) \\
(vi) & \left|\log\left(\dfrac{r^i_n}{r^j_n}\right)\right| + \left|\dfrac{y^i_n - y^j_n}{r^j_n}\right| \underset{n\to\infty}{\longrightarrow} \infty \quad \text{for } i \neq j \\
(vii) & \left|\log\left(\dfrac{R^k_n}{R^l_n}\right)\right| \underset{n\to\infty}{\longrightarrow} \infty \quad \text{for } k \neq l,
\end{array}
$$

where $o(1) \to 0$ in $\dot{H}^s(\mathbb{R}^N)$ as $n \to \infty$, $(W^j)^{r^j_n,\, y^j_n}$ and $(W^k_{\gamma,t})^{R^k_n,0}$ are (PS) sequences of the form (4.2.2) and (4.2.3) respectively, with $W = W^j$ and $W_{\gamma,t} = W^k_{\gamma,t}$.

When $t > 0$, the same conclusions hold, with $W^j = 0$ for all j.

In the case $n_1 = 0,\ n_2 = 0$ the above properties (i)–(vii) are valid without $W,\ W_\gamma,\ R^k_n,\ r^j_n$.

Proof. We prove the theorem in several steps.

<u>**Step 1:**</u> Using standard arguments it follows that there exists $M > 0$ such that

$$\|u_n\|_\gamma < M \quad \text{for all } n \in \mathbb{N}.$$

More precisely, as $n \to \infty$

$$\begin{aligned}\beta + o(1) + o(1)\|u_n\|_\gamma \geq \bar{I}^{\gamma}_{K,t,f}(u_n) &- \frac{1}{2^*_s(t)}\,{}_{(\dot{H}^s)'}\Big\langle (\bar{I}^{\gamma}_{K,t,f})'(u_n), u_n \Big\rangle_{\dot{H}^s} \\ &= \left(\frac{1}{2} - \frac{1}{2^*_s(t)}\right)\|u_n\|^2_\gamma - \left(1 - \frac{1}{2^*_s(t)}\right)\,{}_{(\dot{H}^s)'}\langle f, u_n\rangle_{\dot{H}^s} \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*_s(t)}\right)\|u_n\|^2_\gamma - \left(1 - \frac{1}{2^*_s(t)}\right)\|f\|_{(\dot{H}^s)'}\|u_n\|_{\dot{H}^s}.\end{aligned}$$

As $2^*_s(t) > 2$, from the above estimate it follows that $(u_n)_n$ is bounded in $\dot{H}^s(\mathbb{R}^N)$. Consequently, there exists $\bar{u}$ in $\dot{H}^s(\mathbb{R}^N)$ such that, up to a subsequence, still denoted by $(u_n)_n$, $u_n \rightharpoonup \bar{u}$ in $\dot{H}^s(\mathbb{R}^N)$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^N$. Moreover, as ${}_{(\dot{H}^s)'}\Big\langle (\bar{I}^{\gamma}_{K,t,f})'(u_n), v \Big\rangle_{\dot{H}^s} \to 0$ as $k \to \infty$ for all $v \in \dot{H}^s(\mathbb{R}^N)$, then

$$(-\Delta)^s u_n - \gamma\frac{u_n}{|x|^{2s}} - K(x)|u_n|^{2^*_s(t)-2}u_n - f \longrightarrow 0 \quad \text{in} \quad \dot{H}^s(\mathbb{R}^N)'. \tag{4.2.4}$$

Step 2: From (4.2.4), letting $n \to \infty$, we get

$$\langle u_n, v\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, v\rangle_{\dot{H}^s} \to 0. \tag{4.2.5}$$

As $u_n \rightharpoonup \bar{u}$ in $\dot{H}^s(\mathbb{R}^N)$, it is easy to see that $\langle u_n, v\rangle_\gamma \to \langle \bar{u}, v\rangle_\gamma$ for all $v \in \dot{H}^s(\mathbb{R}^N)$.

Claim 1: $\displaystyle\int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^N} K(x)\frac{|\bar{u}|^{2^*_s(t)-2}\bar{u}v}{|x|^t}\mathrm{d}x$ for all $v \in \dot{H}^s(\mathbb{R}^N)$.

Indeed, $u_n \to \bar{u}$ a.e. in $\mathbb{R}^N$ and

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,\mathrm{d}x &= \int_{B_R} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N\setminus B_R} K(x)\frac{|u_n|^{2^*_s(t)-2}u_n v}{|x|^t}\,\mathrm{d}x.\end{aligned} \tag{4.2.6}$$

On B_R we will show the convergence using Vitali's convergence theorem. For that, given any $\varepsilon > 0$, we choose $\Omega \subset B_R$ such that $\left(\displaystyle\int_\Omega \frac{|v|^{2^*_s(t)}}{|x|^t}\mathrm{d}x\right)^{\frac{1}{2^*_s(t)}} <$

$\dfrac{\varepsilon}{\|K\|_{L^\infty(\mathbb{R}^N)}(MS_{\gamma,t,s}^{-\frac{1}{2}})^{2_s^*(t)-1}}$. Since $\frac{|v|^{2_s^*(t)}}{|x|^t}$ is in $L^1(\mathbb{R}^N)$, the above choice makes sense. Therefore,

$$
\begin{aligned}
&\left|\int_\Omega K(x)\frac{|u_n|^{2_s^*(t)-2}u_n v}{|x|^t}\,\mathrm{d}x\right| \\
\leq\ & \|K\|_{L^\infty(\mathbb{R}^N)}\int_\Omega \frac{|u_n|^{2_s^*(t)-1}|v|}{|x|^t}\,\mathrm{d}x \\
\leq\ & \|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_\Omega \frac{|u_n|^{2_s^*(t)}}{|x|^t}\mathrm{d}x\right)^{\frac{2_s^*(t)-1}{2_s^*(t)}}\left(\int_\Omega \frac{|v|^{2_s^*(t)}}{|x|^t}\mathrm{d}x\right)^{\frac{1}{2_s^*(t)}} \\
\leq\ & \|K\|_{L^\infty(\mathbb{R}^N)} S_{\gamma,t,s}^{-\frac{2^*(t)-1}{2}}\|u_n\|_\gamma^{2_s^*(t)-1}\left(\int_\Omega \frac{|v|^{2_s^*(t)}}{|x|^t}\mathrm{d}x\right)^{\frac{1}{2_s^*(t)}} < \varepsilon
\end{aligned}
$$

Thus $K\frac{|u_n|^{2_s^*(t)-2}u_n v}{|x|^t}$ is uniformly integrable in B_R. Therefore, using Vitali's convergence theorem, we can pass the limit in the 1st integral on RHS of (4.2.6).

To estimate the integral now on B_R^c, we first set $v_n = u_n - \bar{u}$. Then $v_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. It is not difficult to see that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$
\left||v_n+\bar{u}|^{2_s^*(t)-2}(v_n+\bar{u}) - |\bar{u}|^{2_s^*(t)-2}\bar{u}\right| < \varepsilon|v_n|^{2_s^*(t)-1} + C_\varepsilon|\bar{u}|^{2_s^*(t)-1}.
$$

Therefore,

$$
\begin{aligned}
&\left|\int_{B_R^c} K(x)\left\{\frac{|u_n|^{2_s^*(t)-2}u_n}{|x|^t} - \frac{|\bar{u}|^{2_s^*(t)-2}\bar{u}}{|x|^t}\right\}v\,\mathrm{d}x\right| \\
&\leq \|K\|_{L^\infty(\mathbb{R}^N)}\left[\varepsilon\int_{B_R^c}\frac{|v_n|^{2_s^*(t)-1}|v|}{|x|^t}\,\mathrm{d}x + C_\varepsilon\int_{B_R^c}\frac{|\bar{u}|^{2_s^*(t)-1}|v|}{|x|^t}\right] \\
&\leq \|K\|_{L^\infty(\mathbb{R}^N)}\left[\varepsilon\left(\int_{B_R^c}\frac{|v_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2_s^*(t)-1}{2_s^*(t)}}\left(\int_{B_R^c}\frac{|v|^{2_s^*(t)}}{|x|^t}\right)^{\frac{1}{2_s^*(t)}}\right. \\
&\qquad\left. + C_\varepsilon\left(\int_{B_R^c}\frac{|\bar{u}|^{2_s^*(t)}}{|x|^t}\right)^{\frac{2_s^*(t)-1}{2_s^*(t)}}\left(\int_{B_R^c}\frac{|v|^{2_s^*(t)}}{|x|^t}\right)^{\frac{1}{2_s^*(t)}}\right] \\
&\leq C\|K\|_{L^\infty(\mathbb{R}^N)}\left[\varepsilon\|v_n\|_\gamma^{2_s^*(t)-1}\left(\int_{B_R^c}\frac{|v|^{2_s^*(t)}}{|x|^t}\right)^{\frac{1}{2_s^*(t)}} + C_\varepsilon\|\bar{u}\|_\gamma^{2_s^*(t)-1}\left(\int_{B_R^c}\frac{|v|^{2_s^*(t)}}{|x|^t}\right)^{\frac{1}{2_s^*(t)}}\right].
\end{aligned}
$$

Since $(\|v_n\|_\gamma)_n$ is uniformly bounded and $\frac{|v|^{2_s^*(t)}}{|x|^t} \in L^1(\mathbb{R}^N)$, given $\varepsilon > 0$, we can choose $R > 0$ so large that

$$\left| \int_{B_R^c} K(x) \left\{ \frac{|u_n|^{2_s^*(t)-2} u_n}{|x|^t} - \frac{|\bar{u}|^{2_s^*(t)-2} \bar{u}}{|x|^t} \right\} v \, \mathrm{d}x \right| < \varepsilon.$$

This completes the proof of claim 1.

Hence (4.2.5) yields that $\bar{u}$ is a solution of $(E^\gamma_{K,t,f})$.

Step 3: Here we show that $(u_n - \bar{u})_n$ is a (PS) sequence for $\bar{I}^\gamma_{K,t,0}$ at the level $\beta - \bar{I}^\gamma_{K,t,f}(\bar{u})$. To see this, first we observe that as $n \to \infty$,

$$\|u_n - \bar{u}\|_\gamma^2 = \|u_n\|_\gamma^2 - \|\bar{u}\|_\gamma^2 + o(1),$$

and by the Brézis-Lieb lemma as $n \to \infty$

$$\int_{\mathbb{R}^N} K(x) \frac{|u_n - \bar{u}|^{2_s^*(t)}}{|x|^t} \mathrm{d}x = \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{2_s^*(t)}}{|x|^t} \, \mathrm{d}x - \int_{\mathbb{R}^N} K(x) \frac{|\bar{u}|^{2_s^*(t)}}{|x|^t} \, \mathrm{d}x + o(1).$$

Further as $u_n \rightharpoonup u$ and $f \in \dot{H}^s(\mathbb{R}^N)'$, we also have

$${}_{(\dot{H}^s)'}\langle f, u_n \rangle_{\dot{H}^s} \longrightarrow {}_{(\dot{H}^s)'}\langle f, \bar{u} \rangle_{\dot{H}^s}.$$

Therefore, as $n \to \infty$

$$\begin{aligned}
\bar{I}^\gamma_{K,t,0}(u_n - \bar{u}) &= \frac{1}{2}\|u_n - \bar{u}\|_\gamma^2 - \frac{1}{2_s^*(t)} \int_{\mathbb{R}^N} K(x) \frac{|u_n - \bar{u}|^{2_s^*(t)}}{|x|^t} \, \mathrm{d}x \\
&= \frac{1}{2}\|u_n\|_\gamma^2 - \frac{1}{2_s^*(t)} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{2_s^*(t)}}{|x|^t} \, \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u_n \rangle_{\dot{H}^s} \\
&\quad - \left\{ \frac{1}{2}\|\bar{u}\|_\gamma^2 - \frac{1}{2_s^*(t)} \int_{\mathbb{R}^N} K(x) \frac{|\bar{u}|^{2_s^*(t)}}{|x|^t} \, \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, \bar{u} \rangle_{\dot{H}^s} \right\} + o(1) \\
&= \bar{I}^\gamma_{K,t,f}(u_n) - \bar{I}^\gamma_{K,t,f}(\bar{u}) + o(1) \\
&\longrightarrow \beta - \bar{I}^\gamma_{K,t,f}(\bar{u}).
\end{aligned}$$

Further, as ${}_{(\dot H^s)'}\Big\langle(\bar I^{\gamma}_{K,t,f})'(\bar u),v\Big\rangle_{\dot H^s}=0$ for all $v\in \dot H^s(\mathbb{R}^N)$, we obtain

$$
\begin{aligned}
{}_{(\dot H^s)'}\Big\langle(\bar I^{\gamma}_{K,t,0})'(u_n-\bar u),v\Big\rangle_{\dot H^s} &= \langle u_n-\bar u,v\rangle_\gamma-\int_{\mathbb{R}^N}K(x)\frac{|u_n-\bar u|^{2^*_s(t)-2}(u_n-\bar u)v}{|x|^t}\,\mathrm{d}x\\
&= \langle u_n,v\rangle_\gamma-\int_{\mathbb{R}^N}K(x)\frac{|u_n|^{2^*_s(t)-2}u_nv}{|x|^t}\,\mathrm{d}x-{}_{(\dot H^s)'}\langle f,v\rangle_{\dot H^s}\\
&\quad-\bigg(\langle \bar u,v\rangle_\gamma-\int_{\mathbb{R}^N}K(x)\frac{|\bar u|^{2^*_s(t)-2}\bar uv}{|x|^t}\,\mathrm{d}x-{}_{(\dot H^s)'}\langle f,v\rangle_{\dot H^s}\bigg)\\
&\quad+\int_{\mathbb{R}^N}K(x)\bigg\{\frac{|u_n|^{2^*_s(t)-2}u_n}{|x|^t}-\frac{|\bar u|^{2^*_s(t)-2}\bar u}{|x|^t} \qquad (4.2.7)\\
&\qquad-\frac{|u_n-\bar u|^{2^*_s(t)-2}(u_n-\bar u)}{|x|^t}\bigg\}v\mathrm{d}x\\
&= o(1)+\int_{\mathbb{R}^N}K(x)\bigg\{\frac{|u_n|^{2^*_s(t)-2}u_n}{|x|^t}-\frac{|\bar u|^{2^*_s(t)-2}\bar u}{|x|^t}\\
&\qquad-\frac{|u_n-\bar u|^{2^*_s(t)-2}(u_n-\bar u)}{|x|^t}\bigg\}v\mathrm{d}x.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\left|K\left\{|u_n|^{2^*_s(t)-2}u_k-|\bar u|^{2^*_s(t)-2}\bar u-|u_n-\bar u|^{2^*_s-2}(u_n-\bar u)\right\}\right|\\
&\qquad\le C\bigg(|u_n-\bar u|^{2^*_s(t)-2}|\bar u|+|u|^{2^*_s(t)-2}|u_n-\bar u|\bigg).
\end{aligned}
$$

Therefore, following the same method as in the proof of Claim 1 in Step 2, we show that as $n\to\infty$

$$
\int_{\mathbb{R}^N}K(x)\bigg\{\frac{|u_n|^{2^*_s(t)-2}u_n}{|x|^t}-\frac{|\bar u|^{2^*_s(t)-2}\bar u}{|x|^t}-\frac{|u_n-\bar u|^{2^*_s(t)-2}(u_n-\bar u)}{|x|^t}\bigg\}v\mathrm{d}x=o(1) \tag{4.2.8}
$$

for all $v\in \dot H^s(\mathbb{R}^N)$. Plugging this back into (4.2.7), we complete the proof of Step 3.

Step 4: Define $v_n:=u_n-\bar u$. Then $v_n\rightharpoonup 0$ in $\dot H^s(\mathbb{R}^N)$ and by Step 3, $(v_n)_n$ is a (PS) sequence for $\bar I^{\gamma}_{K,t,0}$ at the level $\beta-\bar I^{\gamma}_{K,t,f}(\bar u)$. Thus,

$$
\sup_{n\in\mathbb{N}}\|v_n\|_\gamma\le C \quad\text{and}\quad \langle v_n,\varphi\rangle_\gamma=\int_{\mathbb{R}^N}K(x)\frac{|v_n|^{2^*_s(t)-2}v_n\varphi}{|x|^t}\,\mathrm{d}x+o(1) \tag{4.2.9}
$$

as $n \to \infty$ for all $\varphi \in \dot{H}^s(\mathbb{R}^N)$. Therefore, $\|v_n\|_\gamma^2 = \int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x + o(1)$. Thus, if $\int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \longrightarrow 0$, then we are done when $k = l = 0$ and the (PS) sequence $(u_n)_n$ admits a strongly convergent subsequence.

If not, let $0 < \delta < S_{\gamma,t,s}^{\frac{N-t}{2s-t}} \|K\|_{L^\infty(\mathbb{R}^N)}^{-\frac{N-2s}{2s-t}}$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x > \delta.$$

Up to a subsequence, let $R_n > 0$ be such that

$$\int_{B_{R_n}} K(x)\frac{|v_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x = \delta$$

and R_n being minimal with this property. Define

$$w_n(x) := R_n^{\frac{N-2s}{2}} v_n(R_n x).$$

Therefore, $\|w_n\|_\gamma = \|v_n\|_\gamma$ and

$$\delta = \int_{B_{R_n}} K(x)\frac{|v_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x = \int_{B_1} K(R_n x)\frac{|w_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x. \tag{4.2.10}$$

Therefore, up to a subsequence

$$w_n \rightharpoonup w \text{ in } \dot{H}^s(\mathbb{R}^N) \quad \text{and} \quad w_n \to w \text{ a.e. in } \mathbb{R}^N.$$

Let us now distinguish two cases $w \neq 0$ and $w = 0$.

Step 5: Assume that $w \neq 0$.

Since, $w_n \rightharpoonup w \neq 0$ and $v_n \rightharpoonup 0$, it follows that $R_n \to 0$ as $n \to \infty$. Next, we show that w is a solution of $(E_{1,t,0}^\gamma)$. Indeed, thanks to (4.2.9), for any

$\phi \in C_c^\infty(\mathbb{R}^N)$

$$
\begin{aligned}
\langle w, \phi \rangle_\gamma &= \lim_{n\to\infty} \langle w_n, \phi \rangle_\gamma \\
&= \lim_{n\to\infty} \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(w_n(x)-w_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{w_n \phi}{|x|^{2s}} \,\mathrm{d}x \\
&= \lim_{n\to\infty} \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{R_n^{\frac{N-2s}{2}} (v_n(R_n x)-v_n(R_n y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \\
&\qquad - \gamma \int_{\mathbb{R}^N} \frac{R_n^{\frac{N-2s}{2}} v_n(R_n x)\phi(x)}{|x|^{2s}} \,\mathrm{d}x \qquad (4.2.11) \\
&= \lim_{n\to\infty} \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{R_n^{-\frac{N-2s}{2}} (v_n(x)-v_n(y))(\phi(\frac{x}{R_n})-\phi(\frac{y}{R_n}))}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \\
&\qquad - \gamma \int_{\mathbb{R}^N} \frac{R_n^{-\frac{N-2s}{2}} v_n(x)\phi(\frac{x}{R_n})}{|x|^{2s}} \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} K(x) \frac{|v_n|^{2_s^*(t)-2} v_n}{|x|^t} R_n^{-\frac{N-2s}{2}} \phi\Big(\frac{x}{R_n}\Big) \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} K(R_n x) \frac{|w_n|^{2_s^*(t)-2} w_n}{|x|^t} \phi(x) \,\mathrm{d}x.
\end{aligned}
$$

Clearly $K(R_n x)\frac{|w_n|^{2_s^*(t)-2}w_n}{|x|^t}\phi \to \frac{|w|^{2_s^*(t)-2}w}{|x|^t}\phi$ a.e. in $\mathbb{R}^N$, since $K \in C(\mathbb{R}^N)$, with $K(0)=1$, and $w_n \to w$ a.e. in $\mathbb{R}^N$. Further, arguing as in the proof of Claim 1 in Step 2, we have $K(R_n x)\frac{|w_n|^{2_s^*(t)-2}w_n}{|x|^t}\phi$ is uniformly integrable. Therefore, as ϕ has compact support, using Vitali's convergence theorem we obtain

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} K(R_n x) \frac{|w_n|^{2_s^*(t)-2} w_n}{|x|^t} \phi(x) \,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{|w|^{2s^*(t)-2} w \phi}{|x|^t} \mathrm{d}x. \qquad (4.2.12)
$$

Combining (4.2.12) along with (4.2.11), we conclude that w is a solution of $(E^\gamma_{1,t,0})$.

Define

$$
z_n(x) := v_n(x) - R_n^{-\frac{N-2s}{2}} w(\tfrac{x}{R_n}).
$$

Claim 2: $(z_n)_n$ is a (PS) sequence for $\bar{I}^\gamma_{K,t,0}$ at the level $\beta - \bar{I}^\gamma_{K,t,f}(\bar{u}) - \bar{I}^\gamma_{1,0,0}(w)$.

To prove the claim, set

$$\tilde{z}_n(x) := R_n^{\frac{N-2s}{2}} z_n(R_n x).$$

Then

$$\tilde{z}_n(x) = w_n(x) - w(x) \quad \text{and} \quad \|\tilde{z}_n\|_\gamma = \|w_n - w\|_\gamma = \|z_n\|_\gamma.$$

As $K(0) = 1$ and K is a continuous function, the Brézis-Lieb lemma and a straight forward computation yield as $n \to \infty$

$$\begin{aligned}
\int_{\mathbb{R}^N} K(R_n x)\frac{|w_n(x)|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{|w|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{\left|K^{\frac{1}{2_s^*(t)}}(R_n x) w_n - w\right|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x + o(1) \\
&= \int_{\mathbb{R}^N} K(R_n x)\frac{|w_n - w|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x + o(1).
\end{aligned}$$

Therefore, using the above relations, as $n \to \infty$

$$\begin{aligned}
\bar{I}^\gamma_{K,t,0}(z_n) &= \frac{1}{2}\|z_n\|_\gamma^2 - \frac{1}{2_s^*(t)}\int_{\mathbb{R}^N} K(x)\frac{|z_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \\
&= \frac{1}{2}\|w_n - w\|_\gamma^2 - \frac{1}{2_s^*(t)}\int_{\mathbb{R}^N} K(R_n x)\frac{|w_n - w|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \\
&= \frac{1}{2}\left(\|w_n\|_\gamma^2 - \|w\|_\gamma^2\right) - \frac{1}{2_s^*(t)}\int_{\mathbb{R}^N} K(R_n x)\frac{|w_n(x)|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \\
&\qquad + \frac{1}{2_s^*(t)}\int_{\mathbb{R}^N} \frac{|w|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x + o(1) \\
&= \frac{1}{2}\|v_n\|_\gamma^2 - \frac{1}{2_s^*(t)}\int_{\mathbb{R}^N} K(x)\frac{|v(x)|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \\
&\qquad - \left(\frac{1}{2}\|w\|_\gamma^2 - \frac{1}{2_s^*(t)}\int_{\mathbb{R}^N} \frac{|w|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right) + o(1) \\
&= \bar{I}^\gamma_{K,t,0}(v_n) - \bar{I}^\gamma_{1,0,0}(w) + o(1) \\
&= \beta - \bar{I}^\gamma_{K,t,f}(\bar{u}) - \bar{I}^\gamma_{1,0,0}(w) + o(1).
\end{aligned}$$

Next, let $\phi \in C_0^\infty(\mathbb{R}^N)$ be arbitrary and set $\phi_n(x) := R_n^{\frac{N-2s}{2}} \phi(R_n x)$. This in turn implies that $\|\phi_n\|_\gamma = \|\phi\|_\gamma$ and $\phi_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. Therefore,

$$\begin{aligned}
&{}_{(\dot{H}^s)'}\Big\langle (\bar{I}^\gamma_{K,t,0})'(z_n), \phi \Big\rangle_{\dot{H}^s} \\
&= \langle z_n, \phi\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|z_n|^{2^*_s(t)-2} z_n \phi}{|x|^t}\,\mathrm{d}x \\
&= \langle \tilde{z}_n, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N} K(R_n x)\frac{|\tilde{z}_n|^{2^*_s(t)-2} \tilde{z}_n \phi_n}{|x|^t}\,\mathrm{d}x \\
&= \langle w_n - w, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N} K(R_n x)\frac{|w_n - w|^{2^*_s(t)-2} (w_n - w) \phi_n}{|x|^t}\,\mathrm{d}x \\
&= \langle w_n, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N} K(R_n x)\frac{|w_n|^{2^*_s(t)-2} w_n \phi_n}{|x|^t}\,\mathrm{d}x \\
&\quad - \Big(\langle w, \phi_n\rangle_\gamma - \int_{\mathbb{R}^N} \frac{|w|^{2^*_s(t)-2} w \phi_n}{|x|^t}\,\mathrm{d}x \Big) \qquad (4.2.13)\\
&\quad + \int_{\mathbb{R}^N} (K(R_n x) - 1)\frac{|w|^{2^*_s(t)-2} w \phi_n}{|x|^t}\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} K(R_n x)\Big(\frac{|w_n|^{2^*_s(t)-2} w_n - |w|^{2^*_s(t)-2} w - |w_n - w|^{2^*_s(t)-2}(w_n - w)}{|x|^t} \Big)\phi_n \mathrm{d}x \\
&= \langle v_n, \phi\rangle_\gamma - \int_{\mathbb{R}^N} K(x)\frac{|v_n|^{2^*_s(t)-2} v_n \phi}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\Big\langle \bar{I}^\gamma_{1,0,0}{}'(w), \phi_n \Big\rangle_{\dot{H}^s} + I^1_n + I^2_n \\
&= {}_{(\dot{H}^s)'}\Big\langle (\bar{I}^\gamma_{K,t,0})'(v_n), \phi \Big\rangle_{\dot{H}^s} - 0 + I^1_n + I^2_n = o(1) + I^1_n + I^2_n.
\end{aligned}$$

Now

$$I^1_n := \int_{B_R} \Big(K(R_n x) - 1\Big)\frac{|w|^{2^*_s(t)-2} w \phi_n}{|x|^t}\,\mathrm{d}x + \int_{B^c_R} \Big(K(R_n x) - 1\Big)\frac{|w|^{2^*_s(t)-2} w \phi_n}{|x|^t}\,\mathrm{d}x.$$

Note that as $\frac{|w|^{2^*_s(t)}}{|x|^t} \in L^1(\mathbb{R}^N)$, for $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$\begin{aligned}
&\Big| \int_{B^c_R} \Big(K(R_n x) - 1\Big)\frac{|w|^{2^*_s(t)-2} w \phi_n}{|x|^t}\,\mathrm{d}x \Big| \\
&\le C\Big(\int_{B^c_R} \frac{|w|^{2^*_s(t)}}{|x|^t}\mathrm{d}x \Big)^{\frac{2^*_s(t)-1}{2^*_s(t)}} \Big(\int_{\mathbb{R}^N} \frac{|\phi_n|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \Big)^{\frac{1}{2^*_s(t)}} \\
&\le C\Big(\int_{B^c_R} \frac{|w|^{2^*_s(t)}}{|x|^t}\mathrm{d}x \Big)^{\frac{2^*_s(t)-1}{2^*_s(t)}} \|\phi\|_\gamma < \varepsilon.
\end{aligned}$$

On the other hand, as $K \in C(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} K(x) = 1$ implies that $K \in L^\infty(\mathbb{R}^N)$, applying the Hölder inequality followed by the Hardy-Sobolev inequality, it is easy to see that

$$\Big(K(R_n x)-1\Big)\frac{|w|^{2_s^*(t)-2}w\phi_n}{|x|^t}$$

is uniformly integrable. Therefore, using Vitali's convergence theorem, we get

$$\int_{B_R}\Big(K(R_n x)-1\Big)\frac{|w|^{2_s^*(t)-2}w\phi_n}{|x|^t}\,\mathrm{d}x=o(1).$$

Hence, $I_n^1=o(1)$ as $n\to\infty$.

Next, we aim to show that

$$\begin{aligned} I_n^2 &:= \int_{\mathbb{R}^N}K(R_n x)\Big\{\frac{|w_n|^{2_s^*(t)-2}w_n-|w|^{2_s^*(t)-2}w-|w_n-w|^{2_s^*(t)-2}(w_n-w)}{|x|^t}\Big\}\phi_n\mathrm{d}x\\ &= o(1). \end{aligned}$$

Indeed, this follows as in the proof of (4.2.8), since $\int_{\mathbb{R}^N}\frac{|\phi_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x=\int_{\mathbb{R}^N}\frac{|\phi|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x<\infty$. Hence, from (4.2.13) we conclude the proof of Claim 2.

Step 6: Assume that $w=0$.

Let $\varphi\in C_0^\infty\Big(B_1\Big)$, with $0\le\varphi\le 1$. Set $\psi_n(x):=[\varphi(\frac{x}{R_n})]^2v_n(x)$. Clearly $(\psi_n)_n$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$. Thus,

$$\begin{aligned} o(1) &= {}_{(\dot{H}^s)'}\Big\langle(\bar{I}^\gamma_{K,t,0})'(v_n),\psi_n\Big\rangle_{\dot{H}^s}\\ &= \langle v_n,\psi_n\rangle_\gamma-\int_{\mathbb{R}^N}K(x)\frac{|v_n|^{2_s^*(t)-2}v_n\psi_n}{|x|^t}\mathrm{d}x\\ &= \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(v_n(x)-v_n(y)\Big(\varphi^2(\frac{x}{R_n})v_n(x)-\varphi^2(\frac{y}{R_n})v_n(y)\Big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &\quad-\gamma\int_{\mathbb{R}^N}\frac{v_n^2(x)\varphi^2(\frac{x}{R_n})}{|x|^{2s}}\mathrm{d}x-\int_{\mathbb{R}^N}K(x)\frac{\varphi^2(\frac{x}{R_n})|v_n|^{2_s^*(t)}}{|x|^t}\mathrm{d}x\\ &= \frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{\Big(v_n(R_nx)-v_n(R_ny)\Big)\Big(\varphi^2(x)v_n(R_nx)-\varphi^2(y)v_n(R_ny)\Big)R_n^{N-2s}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &\quad-\gamma\int_{\mathbb{R}^N}\frac{v_n^2(R_nx)\varphi^2(x)R_n^{N-2s}}{|x|^{2s}}\mathrm{d}x-\int_{\mathbb{R}^N}K(x)\frac{|v_n|^{2_s^*(t)-2}\Big(\varphi(\frac{x}{R_n})v_n\Big)^2}{|x|^t}\mathrm{d}x. \end{aligned}$$

Therefore

$$\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{\Big(v_n(R_nx)-v_n(R_ny)\Big)\Big(\varphi^2(x)v_n(R_nx)-\varphi^2(y)v_n(R_ny)\Big)R_n^{N-2s}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$-\gamma\int_{\mathbb{R}^N}\frac{v_n^2(R_nx)\varphi^2(x)R_n^{N-2s}}{|x|^{2s}}\mathrm{d}x=\int_{\mathbb{R}^N}K(x)\frac{|v_n|^{2_s^*(t)-2}\Big(\varphi(\frac{x}{R_n})v_n\Big)^2}{|x|^t}\mathrm{d}x+o(1). \tag{4.2.14}$$

Now,

$$\begin{aligned}
\text{RHS of (4.2.14)} &= \int_{B_1}K(R_nx)\frac{|v_n(R_nx)|^{2_s^*(t)-2}\Big(\varphi(x)v_n(R_nx)\Big)^2R_n^{N-t}}{|x|^t}\,\mathrm{d}x+o(1)\\
&= \int_{B_1}\frac{\Big|K(R_nx)^{\frac{1}{2_s^*(t)-2}}w_n(x)\Big|^{2_s^*(t)-2}\Big(\varphi(x)w_n(x)\Big)^2}{|x|^t}\,\mathrm{d}x+o(1)\\
&\leq \left(\int_{B_1}K(R_nx)^{\frac{2_s^*(t)}{2_s^*(t)-2}}\frac{|w_n(x)|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2_s^*(t)-2}{2_s^*(t)}}\cdot\\
&\quad\times\left(\int_{\mathbb{R}^N}\frac{|\varphi w_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2}{2_s^*(t)}}+o(1) \qquad (4.2.15)\\
&\leq \frac{\|K\|_{L^\infty(\mathbb{R}^N)}^{\frac{2}{2_s^*(t)}}}{S_{\gamma,t,s}}\left(\int_{B_1}K(R_nx)\frac{|w_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{2_s^*(t)-2}{2_s^*(t)}}\|\varphi w_n\|_\gamma^2+o(1)\\
&\leq \frac{\|K\|_{L^\infty(\mathbb{R}^N)}^{\frac{2}{2_s^*(t)}}\delta^{\frac{2s-t}{N-t}}}{S_{\gamma,t,s}}\|\varphi w_n\|_\gamma^2+o(1)\\
&< \|\varphi w_n\|_\gamma^2+o(1)\ \ \text{(By the choice of } \delta \text{ fixed in Step 4).}
\end{aligned}$$

Claim 3: As $n\to\infty$

$$\text{LHS of (4.2.14)} = \|\varphi w_n\|_\gamma^2+o(1). \tag{4.2.16}$$

Indeed,

LHS of (4.2.14)

$$=\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{\Big(v_n(R_nx)-v_n(R_ny)\Big)\Big(\varphi^2(x)v_n(R_nx)-\varphi^2(y)v_n(R_ny)\Big)R_n^{N-2s}}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y$$
$$-\gamma\int_{\mathbb{R}^N}\frac{|\varphi w_n|^2}{|x|^{2s}}\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\Big(w_n(x) - w_n(y)\Big)\Big(\varphi^2(x)w_n(x) - \varphi^2(y)w_n(y)\Big)}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \\
&\qquad - \gamma \int_{\mathbb{R}^N} \frac{|\varphi w_n|^2}{|x|^{2s}}\mathrm{d}x \\
&= \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|\varphi(x)w_n(x) - \varphi(y)w_n(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{|\varphi w_n|^2}{|x|^{2s}} \,\mathrm{d}x \\
&\qquad - \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&= \|\varphi w_n\|_\gamma^2 - \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \qquad (4.2.17)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\iint_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y &= \int_{x\in B_1}\int_{y\in B_1} + \int_{x\in B_1}\int_{y\in B_1^c} + \int_{x\in B_1^c}\int_{y\in B_1} \\
&=: \mathcal{J}_n^1 + \mathcal{J}_n^2 + \mathcal{J}_n^3.
\end{aligned}
$$

Of course, $\mathcal{J}_n^2 = \mathcal{J}_n^3$, as the integral is symmetric with respect to x and y.

$$
\begin{aligned}
\mathcal{J}_n^1 &= \int_{x\in B_1}\int_{y\in B_1} \frac{(\varphi(x) - \varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \\
&\le C \int_{x\in B_1}\int_{y\in B_1} \frac{|w_n(x)||w_n(y)|}{|x-y|^{N+2s-2}} \,\mathrm{d}x\mathrm{d}y \\
&\le C \bigg(\int_{x\in B_1}\int_{y\in B_1} \frac{|w_n(x)|^2}{|x-y|^{N+2s-2}} \,\mathrm{d}x\mathrm{d}y \bigg)^{\frac{1}{2}} \cdot \\
&\qquad \times \bigg(\int_{x\in B_1}\int_{y\in B_1} \frac{|w_n(y)|^2}{|x-y|^{N+2s-2}}\mathrm{d}x\mathrm{d}y \bigg)^{\frac{1}{2}} \qquad (4.2.18) \\
&\le C \int_{x\in B_1}\int_{y\in B_1} \frac{|w_n(x)|^2}{|x-y|^{N+2s-2}} \,\mathrm{d}x\mathrm{d}y \\
&\le C \int_{x\in B_1} \bigg(\int_{|z|<2} \frac{1}{|z|^{N+2s-2}} \,\mathrm{d}z \bigg) |w_n(x)|^2 \mathrm{d}x \\
&\le C\|w_n\|_{L^2(B_1)}^2 = o(1) \ \Big(\text{as } w = 0 \text{ implies } w_n \to 0 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^N)\Big).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathcal{J}_n^2 &= \int_{x\in B_1}\int_{y\in B_1^c} \frac{(\varphi(x)-\varphi(y))^2 w_n(x)w_n(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\leq \int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\leq 1\}} + \int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\geq 1\}} \qquad (4.2.19)\\
&=: \mathcal{J}_n^{21} + \mathcal{J}_n^{22},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{J}_n^{21} \leq C&\Big(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\leq 1\}} \frac{|w_n(x)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x\Big)^{\frac{1}{2}}\cdot \\
&\times\Big(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\leq 1\}} \frac{|w_n(y)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x\Big)^{\frac{1}{2}} \\
=:&\, C J_n^1\cdot J_n^2.
\end{aligned}
$$

Now,

$$
|J_n^1|^2 \leq \int_{x\in B_1}\Big(\int_{|z|<1}\frac{1}{|z|^{N+2s-2}}\,\mathrm{d}z\Big)|w_n(x)|^2\mathrm{d}x \leq C\|w_n\|^2_{L^2(B_1)} = o(1),
$$

and

$$
\begin{aligned}
|J_n^2|^2 &= \int_{x\in B_1}\int_{y\in B_1^c}\frac{\mathbf{1}_{\{|x-y|<1\}}(x,y)|w_n(y)|^2}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x \\
&\leq \int_{y\in B_1^c}\Big(\int_{x\in B_1}\frac{\mathbf{1}_{\{|x-y|<1\}}(x,y)}{|x-y|^{N+2s-2}}\,\mathrm{d}x\Big)|w_n(y)|^2\,\mathrm{d}y \\
&\leq \int_{y\in B_2}\Big(\int_{x\in B_1}\frac{\mathbf{1}_{\{|x-y|<1\}}(x,y)}{|x-y|^{N+2s-2}}\,\mathrm{d}x\Big)|w_n(y)|^2\,\mathrm{d}y \\
&\leq C\|w_n\|^2_{L^2(B_2)} \leq C'.
\end{aligned}
$$

Therefore, $\mathcal{J}_n^{21} = o(1)$ as $n\to\infty$. Moreover,

$$
\begin{aligned}
\mathcal{J}_n^{22} &= \int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\geq 1\}}\frac{|w_n(x)||w_n(y)||\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x \\
&\leq C\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\geq 1\}}\frac{|w_n(x)||w_n(y)|}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\le C\Big(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\ge 1\}}\frac{|w_n(x)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\Big)^{\frac12}\cdot\\
&\qquad\times\Big(\int_{x\in B_1}\int_{y\in B_1^c\cap\{|x-y|\ge 1\}}\frac{|w_n(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x\Big)^{\frac12}\\
&\le C\Big(\int_{x\in B_1}\Big(\int_{|z|\ge 1}\frac{1}{|z|^{N+2s}}\,\mathrm{d}z\Big)|w_n(x)|^2\mathrm{d}x\Big)^{\frac12}\cdot\\
&\qquad\times\Big(\int_{x\in B_1}\int_{|z|\ge 1}\frac{|w_n(x+z)|^2}{|x+z|^{2s}}\frac{|x+z|^{2s}}{|z|^{N+2s}}\,\mathrm{d}z\mathrm{d}x\Big)^{\frac12}\\
&\le C'\|w_n\|_{L^2(B_1)}\Big[\int_{x\in B_1}\Big(\int_{\mathbb{R}^N}\frac{|w_n(x+z)|^2}{|x+z|^{2s}}\mathrm{d}z\Big)\mathrm{d}x\Big]^{\frac12},
\end{aligned}
$$

since $|z|\ge 1$ and $|x|<1$ implies $\frac{|x+z|^{2s}}{|z|^{N+2s}}\le C$. Therefore, using the Hardy inequality, we obtain from the last of the above estimate that as $n\to\infty$

$$
\mathcal{I}_n^{22}\le C''\|w_n\|_{L^2(B_1)}\|w_n\|^2_{\dot{H}^s(\mathbb{R}^N)}=o(1).
$$

Putting the above estimates together, we obtain from (4.2.19) that $\mathcal{I}_n^2=o(1)$ as $n\to\infty$. This, along with (4.2.18), concludes the proof of Claim 3.

Combining Claim 3 with (4.2.15) yields

$$
\|\varphi w_n\|_\gamma=o(1)\text{ as }n\to\infty. \tag{4.2.20}
$$

Substituting this into (4.2.16) and comparing with (4.2.14) yields as $n\to\infty$

$$
\int_{\mathbb{R}^N}K(R_nx)\frac{\varphi^2(x)|w_n(x)|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x=o(1).
$$

Therefore,

$$
\int_{B_r}K(R_nx)\frac{|w_n|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x=o(1),\quad\text{for any}\quad 0<r<1. \tag{4.2.21}
$$

But this contradicts (4.2.10) when $t>0$. Therefore, $w=0$ cannot happen in the case $t>0$, i.e.,

$$
t>0\implies w\ne 0.
$$

Consequently, from now onwards, we restrict ourselves to the case $t=0$ and $w=0$.

Step 7: Let $t = 0$ and $w = 0$. First we consider the tight case, $(v_n)_n \subseteq \dot{H}^s_0(B_R)$, for some fixed ball of radius $R > 0$ (where $\dot{H}^s_0(B_R)$ is the closure of $C_0^\infty(B_R)$ with respect to the $\dot{H}^s(\mathbb{R}^N)$ norm). The remaining case will be obtained by a splitting argument together with a Kelvin transform.

Therefore, in view of (4.2.10) and (4.2.21), using the concentration-compactness principle in the tight case [89], it follows that in the sense of measure,

$$K(R_n x)|w_n|^{2_s^*}\mathrm{d}x\Big|_{\{|x|\le 1\}} \overset{*}{\rightharpoonup} \sum_j C_{x_j}\delta_{x_j}, \tag{4.2.22}$$

where $x_j \in \mathbb{R}^N$ satisfies $|x_j| = 1$. Let $\bar{C} := \max_j C_{x_j}$ and define

$$Q_n(r) := \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} K(R_n x)|w_n|^{2_s^*}\,\mathrm{d}x. \tag{4.2.23}$$

Clearly, $Q_n(r) > C/2$ for each $r > 0$ large enough. Moreover, (4.2.22) gives

$$\liminf_{n\to\infty} Q_n(r) \ge \frac{C}{2}.$$

Hence, there exist sequences $(s_n)_n \subset \mathbb{R}^+$ and $(q_n)_n \subset \mathbb{R}^N$ such that $s_n \to 0$ and $|q_n| > 1/2$ and

$$\frac{C}{2} = \sup_{q\in\mathbb{R}^N} \int_{B_{s_n}(q)} K(R_n x) w_n^{2_s^*}\mathrm{d}x = \int_{B_{s_n}(q_n)} K(R_n x) w_n^{2_s^*}\,\mathrm{d}x. \tag{4.2.24}$$

Define $\theta_n(x) := s_n^{\frac{N-2s}{2}} w_n(s_n x + q_n)$. Thus $\|\theta_n\|_\gamma = \|w_n\|_\gamma$ for any $n \in \mathbb{N}$. Consequently, up to a subsequence, there exists $\theta \in \dot{H}^s(\mathbb{R}^N)$ such that $\theta_n \rightharpoonup \theta$ in $\dot{H}^s(\mathbb{R}^N)$ and $\theta_n \to \theta$ a.e. in $\mathbb{R}^N$.

First note that $\theta \neq 0$. Otherwise, choosing $\varphi \in C_0^\infty\big(B_1(x)\big)$, with $0 \le \varphi \le 1$, for an arbitrary but fixed $x \in \mathbb{R}^N$, and proceeding exactly as in obtaining (4.2.20), we are able to show that $\theta_n \to 0$ in $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$. On the other hand, from (4.2.24) it follows that

$$\int_{B_1} K(s_n R_n x + q_n)\theta_n^{2_s^*}\,\mathrm{d}x = \frac{C}{2} > 0.$$

which leads to a contradiction. Thus, $\theta \neq 0$. Recall that

$$\theta_n(x) = s_n^{\frac{N-2s}{2}} w_n(s_n x + q_n) = (s_n R_n)^{(\frac{N-2s}{2})} v_n(s_n R_n x + R_n q_n).$$

Define $r_n = s_n R_n = o(1)$ and $y_n = R_n q_n$. Hence, $\frac{r_n}{|y_n|} < 2s_n = o(1)$ and, up to a subsequence, $y_n \to y$ in $\mathbb{R}^N$. From Lemma 4.4.1, we deduce that

$$\theta = K(y)^{-\frac{N-2s}{4s}} W^{\tau,a} \quad \text{for some } \tau > 0,\ a \in \mathbb{R}^N,$$

where W is a solution of $(E^0_{1,0,0})$ and that $n \mapsto \tilde{v}_n(x) := v_n(x) - K(y)^{\frac{4s}{N-2s}} W^{r_n\tau, y_n + r_n a}(x)$ is a (PS) sequence for $\bar{I}^{\gamma}_{K,t,0}$ at level $\beta - \bar{I}^{\gamma}_{K,t,f}(\bar{u}) - K(y)^{-\frac{N-2s}{2s}} \bar{I}^{(0)}_{1,0,0}(W)$, where W is a solution of $(E^0_{1,0,0})$.

In summary, in both cases $t > 0$ and $t = 0$, starting from a (PS) sequence $(v_n)_n$ of $\bar{I}^{\gamma}_{K,t,0}$ we have found another (PS) sequence $(\tilde{v}_n)_n$ of $\bar{I}^{\gamma}_{K,t,0}$ at a strictly lower level, with a fixed minimum amount of decrease. Since $\sup_n \|v_n\|_\gamma \leq C < \infty$, the process should stop after finitely many steps.

Step 8: When $t = 0$ we only dealt with the case $(v_n)_n \subset \dot{H}^s_0(B_R)$ for some fixed $R > 0$. Now we are going to relax the assumption $(v_n)_n \subset \dot{H}^s_0(B_R)$.

Let us define

$$\tilde{f}(k) := \liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} K(x)|v_n|^{2^*_s}\, \mathrm{d}x.$$

We claim that $\tilde{f}(k) = 0$ for all but finitely many k's.

Indeed, if $\tilde{f}(k) > 0$ for some k, then $\liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} K(x)|v_n|^{2^*_s}\, \mathrm{d}x > 0$. Therefore,

$$\liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} |v_n|^{2^*_s}\, \mathrm{d}x > 0. \tag{4.2.25}$$

By Step 6, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ as $n \to \infty$

$$\begin{aligned} \|K\|_{L^\infty}^{\frac{2}{2^*_s}} \left(\int_{\mathrm{supp}(\varphi)} K(R_n x)|w_n|^{2^*_s} \mathrm{d}x \right)^{\frac{2^*_s - 2}{2^*_s}} \left(\int_{\mathbb{R}^N} |\varphi w_n|^{2^*_s} \mathrm{d}x \right)^{\frac{2}{2^*_s}} &\geq \|\varphi w_n\|_\gamma^2 + o(1) \\ &\geq S_{\gamma,0,s} \left(\int_{\mathbb{R}^N} |\varphi w_n|^{2^*_s} \right)^{\frac{2}{2^*_s}} \\ &\quad + o(1). \end{aligned} \tag{4.2.26}$$

Fix any $\varepsilon > 0$ and choose $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in $B_{k+1} \setminus B_k$ and $\mathrm{supp}(\varphi) \subseteq B_{k+1+\varepsilon} \setminus B_{k-\varepsilon}$ and $0 \leq \varphi \leq 1$. Define, $\varphi_n(x) = \varphi(R_n x)$. Then

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\varphi_n w_n|^{2_s^*} \mathrm{d}x = \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\varphi v_n|^{2_s^*} \mathrm{d}x > \liminf_{n\to\infty} \int_{B_{k+1}\setminus B_k} |v_n|^{2_s^*} \mathrm{d}x > 0.$$

Now (4.2.26), with $\varphi = \varphi_n$, yields as $n \to \infty$

$$\int_{B_{k+1+\varepsilon}\setminus B_{k-\varepsilon}} K(x)|v_n|^{2_s^*}\ \mathrm{d}x \geq \|K\|_{L^\infty(\mathbb{R}^N)}^{-\frac{N-2s}{2s}} S_{\gamma,0,s}^{\frac{N}{2s}} + o(1).$$

Combining the above, as $\varepsilon > 0$ is arbitrary, we obtain $\tilde{f}(k) \geq \|K\|_{L^\infty(\mathbb{R}^N)}^{-\frac{N-2s}{2s}} S_{\gamma,0,s}^{\frac{N}{2s}}$. Therefore, since $(v_n)_n$ is bounded in $L^{2_s^*}(\mathbb{R}^N)$, it follows that $\tilde{f}(k) = 0$ for all but finitely many k's and this completes the proof of the claim.

Now given such a k for which $\tilde{f}(k) = 0$, we take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi \equiv 1$ on B_k and $\chi \equiv 0$ on B_{k+1}^c and $0 \leq \chi \leq 1$. We shall show that both $(\chi v_n)_n$ and $\Big((1-\chi)v_n\Big)_n$ are (PS) sequences for $I_{K,0,0}^\gamma$. Indeed for $h \in C_0^\infty(\mathbb{R}^N)$ as $n \to \infty$

$$\begin{aligned}
&\langle \chi v_n, h\rangle_\gamma \\
&= \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)v_n(x) - \chi(y)v_n(y)\big)\big(h(x)-h(y)\big)}{|x-y|^{N+2s}}\ \mathrm{d}x\mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{\chi v_n h}{|x|^{2s}} \mathrm{d}x \\
&= \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\big(v_n(x) - v_n(y)\big)\big(\chi(x)h(x) - \chi(y)h(y)\big)}{|x-y|^{N+2s}}\ \mathrm{d}x\mathrm{d}y - \gamma \int_{\mathbb{R}^N} \frac{v_n(\chi h)}{|x|^{2s}} \mathrm{d}x \\
&\quad + \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(x)v_n(y)}{|x-y|^{N+2s}}\ \mathrm{d}x\mathrm{d}y \\
&\quad - \frac{C_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(y)v_n(x)}{|x-y|^{N+2s}}\ \mathrm{d}x\mathrm{d}y \qquad (4.2.27) \\
&= \langle v_n, \chi h\rangle_\gamma + C_{N,s} \iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(x)v_n(y)}{|x-y|^{N+2s}}\ \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^N} K(x)|v_n|^{2_s^*-2} v_n(\chi h)\ \mathrm{d}x + C_{N,s}\mathbb{I}_n + o(\|h\|),
\end{aligned}$$

where $\mathbb{I}_n := \displaystyle\iint_{\mathbb{R}^{2N}} \frac{\big(\chi(x)-\chi(y)\big)h(x)v_n(y)}{|x-y|^{N+2s}}\ \mathrm{d}x\mathrm{d}y.$

Claim 4: $\mathbb{I}_n = o(\|h\|_\gamma)$ as $n \to \infty$.

Indeed,

$$\mathbb{I}_n \le \left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 h^2(x)}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

Now,

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y &= \int_{y\in B_{k+1}} \int_{x\in B_{k+1}} + \int_{y\in B_{k+1}} \int_{x\in B_{k+1}^c} + \int_{y\in B_{k+1}^c} \int_{x\in B_{k+1}} \\ &=: \mathbb{I}_n^1 + \mathbb{I}_n^2 + \mathbb{I}_n^3.\end{aligned}$$

Since $v \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ implies $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we see that as $n \to \infty$

$$\begin{aligned}\mathbb{I}_n^1 &= \int_{y\in B_{k+1}} \int_{x\in B_{k+1}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\le C \int_{y\in B_{k+1}} \int_{x\in B_{k+1}} \frac{v_n^2(y)}{|x-y|^{N+2s-2}} \, \mathrm{d}x\mathrm{d}y \\ &\le C \int_{y\in B_{k+1}} \left(\int_{x\in B_{k+1}\cap\{|x-y|<1\}} \frac{\mathrm{d}x}{|x-y|^{N+2s-2}} + \int_{x\in B_{k+1}\cap\{|x-y|\ge 1\}} \mathrm{d}x \right) v_n^2(y)\mathrm{d}y \qquad (4.2.28) \\ &\le C' \int_{y\in B_{k+1}} v_n^2(y)\mathrm{d}y \\ &= o(1);\end{aligned}$$

$$\begin{aligned}\mathbb{I}_n^2 &= \int_{y\in B_{k+1}} \int_{x\in B_{k+1}^c} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \\ &\le C \int_{y\in B_{k+1}} \left(\int_{x\in B_{k+1}^c\cap\{|x-y|\le 1\}} \frac{\mathrm{d}x}{|x-y|^{N+2s-2}} + \int_{x\in B_{k+1}^c\cap\{|x-y|\ge 1\}} \frac{\mathrm{d}x}{|x-y|^{N+2s}} \right) v_n^2(y)\mathrm{d}y \qquad (4.2.29) \\ &\le C'' \int_{y\in B_{k+1}} v_n^2(y)\mathrm{d}y = o(1);\end{aligned}$$

$$
\begin{aligned}
\mathbb{I}_n^3 &= \int_{y\in B_{k+1}^c}\int_{x\in B_{k+1}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x \\
&= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\geq 1\}} + \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\leq 1\}} \\
&=: \mathbb{I}_n^{31} + \mathbb{I}_n^{32}.
\end{aligned}
$$

For estimating $\mathbb{I}_n^{31}$, we choose $\varepsilon > 0$ arbitrary and $R >> k+1$ so that

$$
\begin{aligned}
\mathbb{I}_n^{31} &= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\geq 1\}} \frac{\chi^2(x) v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x \\
&\leq \int_{x\in B_{k+1}}\left(\int_{y:|x-y|\geq 1} \frac{v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\right)\mathrm{d}x \\
&\leq \int_{x\in B_{k+1}}\left(\int_{B_R} v_n^2(y)\mathrm{d}y + \int_{B_R^c\cap\{|x-y|\geq 1\}} \frac{v_n^2(y)}{|y|^{N+2s}}\frac{|y|^{N+2s}}{|x-y|^{N+2s}}\mathrm{d}y\right)dx
\end{aligned}
\tag{4.2.30}
$$

$$
\begin{aligned}
&\leq \int_{x\in B_{k+1}}\left(o(1) + C\int_{B_R^c}\frac{v_n^2(y)}{|y|^{N+2s}}\mathrm{d}y\right)dx \\
&\leq C'''\left(o(1) + C\Big(\int_{B_R^c}|v_n|^{2_s^*}\mathrm{d}y\Big)^{\frac{2}{2_s^*}}\Big(\int_{B_R^c}\frac{\mathrm{d}y}{|y|^{(N+2s)N/2s}}\Big)^{\frac{2s}{N}}\right) < \varepsilon \text{ for } R >> k+1,
\end{aligned}
$$

since $(v_n)_n$ is uniformly bounded in $L^{2_s^*}(\mathbb{R}^N)$ and $|y|^{(N+2s)N/2s} \in L^1(\{|y| > 1\})$. Moreover,

$$
\begin{aligned}
\mathbb{I}_n^{32} &= \int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c\cap\{|x-y|\leq 1\}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x \\
&\leq C\int_{x\in B_{k+1}}\int_{y\in B_{k+1}^c} \frac{\mathbf{1}_{|x-y|\leq 1}(x,y) v_n^2(y)}{|x-y|^{N+2s-2}}\,\mathrm{d}y\mathrm{d}x \\
&= C\int_{y\in B_{k+1}^c}\left(\int_{x\in B_{k+1}}\frac{\mathbf{1}_{|x-y|\leq 1}(x,y)}{|x-y|^{N+2s-2}}\mathrm{d}x\right)v_n^2(y)\mathrm{d}y \\
&= C\int_{y\in B_{k+2}}\left(\int_{x\in B_{k+1}}\frac{\mathbf{1}_{|x-y|\leq 1}(x,y)}{|x-y|^{N+2s-2}}\mathrm{d}x\right)v_n^2(y)\mathrm{d}y \\
&\leq C''''\int_{y\in B_{k+2}} v_n^2(y)\mathrm{d}y = o(1).
\end{aligned}
\tag{4.2.31}
$$

Combining (4.2.28)–(4.2.31), we obtain

$$\left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 v_n^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} = o(1)$$

as $n \to \infty$. Similarly, it follows that

$$\left(\iint_{\mathbb{R}^{2N}} \frac{|\chi(x)-\chi(y)|^2 h^2(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \le \|h\|_{\dot{H}^s(\mathbb{R}^N)}.$$

Hence Claim 4 is proved.

Therefore, using (4.2.27) and the fact that $\tilde{f}(k) = 0$, we obtain as $n \to \infty$

$$\begin{aligned}
{}_{(\dot{H}^s)'}\langle (\bar{I}^{\gamma}_{K,t,0})'(\chi v_n), h\rangle_{\dot{H}^s} &= \langle \chi v_n, h\rangle_\gamma - \int_{\mathbb{R}^N} K(x)|\chi v_n|^{2^*_s-2}(\chi v_n)h\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} K(x)\{\chi - \chi^{2^*_s-1}\}|v_n|^{2^*_s-2}v_n h\,\mathrm{d}x + o(\|h\|) \\
&\le C\|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_{B_{k+1}\setminus B_k} |v_n|^{2^*_s}\mathrm{d}x\right)^{\frac{2^*_s-1}{2^*_s}} \|h\|_\gamma + o(\|h\|) \\
&= o(\|h\|).
\end{aligned}$$

This is the required inequality.

Now, as $n \to \infty$

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x)|v_n|^{2^*_s}\mathrm{d}x &= \int_{\mathbb{R}^N} K(x)|\chi v_n + (1-\chi)v_n|^{2^*_s}\mathrm{d}x \\
&= \int_{\mathbb{R}^N} K(x)|\chi v_n|^{2^*_s}\mathrm{d}x + \int_{\mathbb{R}^N} K(x)|(1-\chi)v_n|^{2^*_s}\mathrm{d}x + o(1).
\end{aligned} \tag{4.2.32}$$

The last line in (4.2.32) follows from the fact that $\mathrm{supp}(\chi) \subseteq B_{k+1}$ and $\mathrm{supp}(1-\chi) \subset \mathbb{R}^N \setminus B_k$ and all the remaining terms in the expansion of $|\chi v_n + (1-\chi)v_n|^{2^*_s}$ involves product of some powers of χv_n and $(1-\chi)v_n$ whose support lies in $B_{k+1} \setminus B_k$, but in the definition of χ we have chosen the same k for which $\tilde{f}(k) = 0$.

We know that $(v_n)_n$ is a (PS) sequence of $\bar{I}^{\gamma}_{K,0,0}$ at the level $\beta - \bar{I}^{\gamma}_{K,t,f}(\bar{u})$. Hence, from (4.2.32) the level of the (PS) sequence $(v_n)_n$ of $\bar{I}^{\gamma}_{K,0,0}$ is integrally split between the two new (PS) sequences $(\chi v_n)_n$ and $\Big((1-\chi)v_n\Big)_n$.

Let $\mathcal{K}$ denote the Kelvin transform in $\dot{H}^s(\mathbb{R}^N)$ given by,

$$\mathcal{K}u(x) := \frac{1}{|x|^{N-2s}}u(|x|^{-2}x).$$

Therefore, it is known that (see [99]),

$$(-\Delta)^s\mathcal{K}u(x) = \frac{1}{|x|^{N+2s}}(-\Delta)^s u(|x|^{-2}x).$$

Claim 5: $\|\mathcal{K}(u)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u\|_{\dot{H}^s(\mathbb{R}^N)}$.

To prove the claim, first assume that $u \in C_0^\infty(\mathbb{R}^N)$. Thus

$$|(-\Delta)^s\mathcal{K}u(x)| \leq \frac{C}{1+|x|^{N+2s}}. \tag{4.2.33}$$

Therefore,

$$\begin{aligned}
\|\mathcal{K}(u)\|^2_{\dot{H}^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\mathcal{K}u(x)|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} (-\Delta)^s\mathcal{K}(u(x))\mathcal{K}u(x) \,\mathrm{d}x \ \ (\text{Using (4.2.33) and } \mathcal{K}(u) \in \dot{H}^s(\mathbb{R}^N)) \\
&= \int_{\mathbb{R}^N} \frac{1}{|x|^{N+2s}}(-\Delta)^s u(|x|^{-2}x)\frac{1}{|x|^{N-2s}}u(|x|^{-2}x)\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \Big((-\Delta)^s u(x)\Big)u(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u(x)|^2\mathrm{d}x \ \ \Big(\text{as } u \in C_0^\infty(\mathbb{R}^N)\Big) \\
&= \|u\|^2_{\dot{H}^s(\mathbb{R}^N)}
\end{aligned}$$

Next for any $u \in \dot{H}^s(\mathbb{R}^N)$, let $(u_n)_n \in C_0^\infty(\mathbb{R}^N)$ be such that $u_n \to u$ in $\dot{H}^s(\mathbb{R}^N)$. Then

$$\|\mathcal{K}(u_n)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u_n\|_{\dot{H}^s(\mathbb{R}^N)} \to \|u\|_{\dot{H}^s(\mathbb{R}^N)}. \tag{4.2.34}$$

Thus,

$$\|\mathcal{K}(u_n) - \mathcal{K}(u_m)\|_{\dot{H}^s(\mathbb{R}^N)} = \|\mathcal{K}(u_n-u_m)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u_n-u_m\|_{\dot{H}^s(\mathbb{R}^N)} \underset{n,m\to\infty}{\longrightarrow} 0.$$

Hence, $(\mathcal{K}(u_n))_n$ is a Cauchy sequence in $\dot{H}^s(\mathbb{R}^N)$, so there exists $v \in \dot{H}^s(\mathbb{R}^N)$ such that $\mathcal{K}(u_n) \to v$. Now, as $u_n \to u$ a.e. in $\mathbb{R}^N$ so $\mathcal{K}(u_n) \to \mathcal{K}(u)$ a.e.

in $\mathbb{R}^N$. Consequently, $v = \mathcal{K}(u)$. Therefore, passing the limit in (4.2.34), we have $\|\mathcal{K}(u)\|_{\dot{H}^s(\mathbb{R}^N)} = \|u\|_{\dot{H}^s(\mathbb{R}^N)}$ for all $u \in \dot{H}^s(\mathbb{R}^N)$.

Using Claim 5 along with standard change of variable, it is easy to see that

$$\bar{I}^{\gamma}_{K,0,0}\Big(\mathcal{K}(u)\Big) = \frac{1}{2}\|u\|^2_{\dot{H}^s} - \frac{\gamma}{2}\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\mathrm{d}x - \frac{1}{2^*_s}\int_{\mathbb{R}^N} \mathcal{K}\Big(|x|^{-2}x\Big)|u|^{2^*_s}(x)\mathrm{d}x,$$

that is, $\bar{I}^{\gamma}_{K,0,0} \circ \mathcal{K}$ has the same expression as $\bar{I}^{\gamma}_{K,0,0}$ except that $K(x)$ has to be replaced by $K(|x|^{-2}x)$. Hence, Steps 5 and 7 can be applied to $(\mathcal{K}\Big((1-\chi)v_n\Big))_n$, since this sequence is now a (PS) sequence for $\bar{I}^{\gamma}_{K,0,0} \circ \mathcal{K}$ in $\dot{H}^s_0(B_{\frac{1}{k}})$. Using again either Step 5 or Step 7, we obtain the characterization of $(\mathcal{K}\Big((1-\chi)v_n\Big))_n$ and from that we deduce the characterization of $((1-\chi)v_n)_n$; the only point which needs to be taken care of $\mathcal{K}\Big(W(\frac{x-y^j_n}{r^j_n})\Big)$. This is the concern in Lemma 4.4.2.

Finally (vi) and (vii) follow as in [94, Theorem 4]. Thus the proof is completed. □

We mention the proofs of Lemma 4.4.1 and Lemma 4.4.2 in the end of this chapter, namely in Section 4.4.

4.3 Proof of the main Theorem 4.1.1

In this section we assume without further mentioning that all the assumptions of Theorem 4.1.1 are satisfied. We first establish existence of two positive critical points for the functional

$$I^{\gamma}_{K,t,f}(u) = \frac{1}{2}\|u\|^2_{\gamma} - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{u_+^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s}.$$

Clearly, if u is a critical point of $I^{\gamma}_{K,t,f}$, then u solves

$$\begin{cases} (-\Delta)^s u - \gamma\dfrac{u}{|x|^{2s}} = K(x)\dfrac{u_+^{2^*_s(t)-1}}{|x|^t} + f(x) \quad \text{in } \mathbb{R}^N, \\ u \in \dot{H}^s(\mathbb{R}^N). \end{cases} \tag{4.3.1}$$

Remark 4.3.1. If u is a weak solution of (4.3.1) and f is a nonnegative functional in $\dot{H}^s(\mathbb{R}^N)'$, then taking $v = u_-$ as a test function in (4.3.1), we obtain

$$-\|u_-\|_\gamma^2 - \iint_{\mathbb{R}^{2N}} \frac{|u_+(y)u_-(x) + u_+(x)u_-(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y = {}_{(\dot{H}^s)'}\langle f, u_-\rangle_{\dot{H}^s} \geq 0,$$

which in turn implies that $u_- \equiv 0$, i.e., $u \geq 0$. Therefore, the maximum principle [56, Theorem 1.2] yields that u is a positive solution to (4.3.1). Hence u is a solution to $(E^\gamma_{K,t,f})$.

To establish the existence of two critical points for $I^\gamma_{K,t,f}$, we first need to prove some auxiliary results. Towards that, we partition $\dot{H}^s(\mathbb{R}^N)$ into three disjoint sets. Let $\psi_t : \dot{H}^s(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\psi_t(u) := \|u\|_\gamma^2 - \Big(2_s^*(t) - 1\Big)\|K\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x$$

and set

$$\Sigma_1^t := \Big\{u \in \dot{H}^s(\mathbb{R}^N) \ : \ u = 0 \text{ or } \ \psi_t(u) > 0\Big\}, \quad \Sigma_2^t := \Big\{u \in \dot{H}^s(\mathbb{R}^N) \ : \ \psi_t(u) < 0\Big\},$$
$$\Sigma^t := \Big\{u \in \dot{H}^s(\mathbb{R}^N) \ : \ \psi_t(u) = 0\Big\}.$$

Remark 4.3.2. If $u \in \Sigma^t$, then

$$\|u\|_\gamma^2 = \Big(2_s^*(t) - 1\Big)\|K\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \leq \Big(2_s^*(t) - 1\Big)\|K\|_{L^\infty(\mathbb{R}^N)} S_{\gamma,t,s}^{-\frac{2_s^*(t)}{2}} \|u\|_\gamma^{2_s^*(t)}.$$

Therefore, $\|u\|_\gamma$ and $\|u\|_{L^{2_s^*(t)}(\mathbb{R}^N, |x|^{-t})}$ are bounded away from 0 for all $u \in \Sigma^t$.

Set

$$c_0^t := \inf_{\Sigma_1^t} I^\gamma_{K,t,f}(u), \quad c_1^t := \inf_{\Sigma^t} I^\gamma_{K,t,f}(u), \quad t \geq 0. \tag{4.3.2}$$

Remark 4.3.3. For any $\lambda > 0$ and $u \in \dot{H}^s(\mathbb{R}^N)$

$$\psi_t(\lambda u) = \lambda^2\|u\|_\gamma^2 - \lambda^{2_s^*(t)}\Big(2_s^*(t) - 1\Big)\|K\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x.$$

Moreover, $\psi_t(0) = 0$ and $\lambda \mapsto \psi_t(\lambda u)$ is a strictly concave function. Thus for any $u \in \dot{H}^s(\mathbb{R}^N)$ with $\|u\|_\gamma = 1$, there exists a unique $\lambda = \lambda(u)$ such that $\lambda u \in \Sigma^t$. Moreover, as $\psi_t(\lambda u) = \left(\lambda^2 - \lambda^{2_s^*(t)}\right)\|u\|_\gamma^2$ for all $u \in \Sigma^t$, then $\lambda u \in \Sigma_1^t$ for all $\lambda \in (0,1)$ and $\lambda u \in \Sigma_2^t$ for all $\lambda > 1$.

Lemma 4.3.4. *Assume that C_t is defined as in Theorem* 4.1.1. *Then*

$$\frac{4s-2t}{N-2t+2s}\|u\|_\gamma \geq C_t S_{\gamma,t,s}^{\frac{N-t}{4s-2t}} \quad \text{for all } \; u \in \Sigma^t, \; t \geq 0.$$

Proof. Fix $u \in \Sigma^t$. Then

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{1}{2_s^*(t)}} = \frac{\|u\|_\gamma^{\frac{2}{2_s^*(t)}}}{\left(\left(2_s^*(t)-1\right)\|K\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{1}{2_s^*(t)}}}.$$

Combining this with the definition of $S_{\gamma,t,s}$ yields

$$\|u\|_\gamma \geq S_{\gamma,t,s}^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{1}{2_s^*(t)}} = S_{\gamma,t,s}^{\frac{1}{2}}\frac{\|u\|_\gamma^{\frac{2}{2_s^*(t)}}}{\left(\left(2_s^*(t)-1\right)\|K\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{1}{2_s^*(t)}}}$$

for all $u \in \Sigma^t$. From here using the definition of C_t, we conclude the proof of the lemma. □

Lemma 4.3.5. *Assume that $t \geq 0$, C_t is given as in Theorem* 4.1.1 *and c_0^t, c_1^t are defined as in* (4.3.2). *Further if*

$$\inf_{\substack{u\in \dot{H}^s(\mathbb{R}^N)\\ \|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})}=1}} \left\{C_t\|u\|_\gamma^{\frac{N-2t+2s}{2s-t}} - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s}\right\} > 0, \tag{4.3.3}$$

then $c_0^t < c_1^t$.

Proof. Define

$$\tilde{J}_t(u) := \frac{1}{2}\|u\|_\gamma^2 - \frac{\|K\|_{L^\infty}}{2_s^*(t)}\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s}. \tag{4.3.4}$$

Step I: In this step we prove that there exists $\beta_t > 0$ such that

$$\frac{d}{dp}\tilde{J}_t(pu)\bigg|_{p=1} \geq \beta_t \quad \text{for all } \; u \in \Sigma^t.$$

Indeed, using the definition of Σ^t and the value of C_t, we have for $u \in \Sigma^t$

$$\begin{aligned}
\frac{d}{dp}\tilde{J}_t(pu)\Big|_{p=1} &= \|u\|_\gamma^2 - \|K\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \\
&= \left(1 - \frac{1}{2_s^*(t)-1}\right)\|u\|_\gamma^2 - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \qquad (4.3.5)\\
&= \frac{4s-2t}{N-2t+2s}\|u\|_\gamma^2 - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \\
&= C_t \frac{\|u\|_\gamma^{\frac{N+2s-2t}{2s-t}}}{\|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})}^{\frac{N-t}{2s-t}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s}.
\end{aligned}$$

Furthermore, (4.3.3) implies that there exists $d > 0$ such that

$$\inf_{\substack{u\in \dot{H}^s(\mathbb{R}^N) \\ \|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})}=1}} \left\{C_t\|u\|_\gamma^{\frac{N-2t+2s}{2s-t}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s}\right\} \geq d. \qquad (4.3.6)$$

Observe that,

$$\begin{aligned}
(4.3.6) \quad &\iff \quad C_t \frac{\|u\|_\gamma^{\frac{N+2s-2t}{2s-t}}}{\|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})}^{\frac{N-t}{2s-t}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \geq d, \ \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x = 1, \\
&\iff \quad C_t \frac{\|u\|_\gamma^{\frac{N+2s-2t}{2s-t}}}{\|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})}^{\frac{N-t}{2s-t}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} \geq d\|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})}, \\
&\qquad\qquad u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}.
\end{aligned}$$

Hence, plugging back the above estimate into (4.3.5) and using Remark 4.3.2, we complete the proof of Step I.

Step II: Let $(u_n^t)_n$ be a minimizing sequence for $I_{K,t,f}^\gamma$ on Σ^t, that is,

$$I_{K,t,f}^\gamma(u_n^t) \to c_1^t \quad \text{and} \quad \|u_n^t\|_\gamma^2 = \|K\|_{L^\infty(\mathbb{R}^N)}\Big(2_s^*(t)-1\Big)\int_{\mathbb{R}^N} \frac{|u_n^t|^{2_s^*(t)}}{|x|^t}\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
c_1^t + o(1) = I_{K,t,f}^\gamma(u_n) &\geq \tilde{J}_t(u_n^t) \\
&\geq \left(\frac{1}{2} - \frac{1}{2_s^*(t)\Big(2_s^*(t)-1\Big)}\right)\|u_n^t\|_\gamma^2 - \|f\|_{(\dot{H}^s)'}\|u_n\|_\gamma.
\end{aligned}$$

This implies that $(\tilde{J}_t(u_n^t))_n$ is bounded and $(\|u_n^t\|_\gamma)_n$, $(\|u_n^t\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})})_n$ are bounded.

Claim: $c_0^t < 0$ for all $t \geq 0$.

To prove this claim, it is enough to show that there exists $v^t \in \Sigma_1^t$ such that $I_{K,t,f}^\gamma(v^t) < 0$. Note that, thanks to Remark 4.3.3, we can choose $u^t \in \Sigma^t$ such that ${}_{(\dot{H}^s)'}\langle f, u^t\rangle_{\dot{H}^s} > 0$.

Therefore,

$$\begin{aligned} I_{K,t,f}^\gamma(pu^t) &= p^2\left[\frac{\Big(2_s^*(t)-1\Big)\|K\|_{L^\infty(\mathbb{R}^N)}}{2} - \frac{p^{2_s^*(t)-2}}{2_s^*(t)}\right]\int_{\mathbb{R}^N}\frac{|u^t|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x \\ &\quad -p\,{}_{(\dot{H}^s)'}\langle f, u^t\rangle_{\dot{H}^s} < 0 \end{aligned}$$

for $p << 1$. Moreover, $pu^t \in \Sigma_1^t$ by Remark 4.3.3. Hence the claim follows.

Thanks to the above claim, $I_{K,t,f}^\gamma(u_n^t) < 0$ for large n. Consequently,

$$0 > I_{K,t,f}^\gamma(u_n^t) \geq \left(\frac{1}{2} - \frac{1}{2_s^*(t)\Big(2_s^*(t)-1\Big)}\right)\|u_n^t\|_\gamma^2 - {}_{(\dot{H}^s)'}\langle f, u_n^t\rangle_{\dot{H}^s}$$

for large n. This in turn implies that ${}_{(\dot{H}^s)'}\langle f, u_n^t\rangle_{\dot{H}^s} > 0$ for n large enough. Hence, $\frac{d}{dp}\tilde{J}_t(pu_n^t) < 0$ for $p > 0$ small enough. Thus, by Step I there exists $p_n^t \in (0,1)$ such that $\frac{d}{dp}\tilde{J}_t(p_n^t u_n^t) = 0$.

Moreover, it is easy to check that for all $u^t \in \Sigma^t$, the map $p \mapsto \frac{d}{dp}\tilde{J}_t(pu^t)$ is strictly increasing in $[0,1)$ and therefore, we can conclude that p_n^t is unique.

Step III: In this step we show that

$$\liminf_{n\to\infty}\left\{\tilde{J}_t(u_n^t) - \tilde{J}_t(p_n^t u_n^t)\right\} > 0. \tag{4.3.7}$$

We observe that $\tilde{J}_t(u_n^t) - \tilde{J}_t(p_n^t u_n^t) = \displaystyle\int_{p_n^t}^1 \frac{d}{dp}\tilde{J}_t(pu_n)\mathrm{d}p$ and that for all $n \in \mathbb{N}$ there exists $\xi_n^t > 0$ such that $p_n^t \in (0, 1-2\xi_n^t)$ and $\frac{d}{dp}\tilde{J}_t(pu_n^t) \geq \frac{\beta_t}{2}$ for $p \in [1-\xi_n^t, 1]$.

To establish (4.3.7), it is enough to show that $\xi_n^t > 0$ can be chosen independently of $n \in \mathbb{N}$. This is possible, since $\frac{d}{dp}\tilde{J}_t(pu_n^t)\Big|_{p=1} \geq \beta_t$, and $(u_n^t)_n$

is bounded, so that for all n and $p \in [0,1]$

$$\left|\frac{d^2}{dp^2}\tilde{J}_t(pu_n^t)\right| = \left|\|u_n^t\|_\gamma^2 - \Big(2_s^*(t)-1\Big)\|K\|_{L^\infty} p^{2_s^*(t)-2}\int_{\mathbb{R}^N}\frac{|u_n^t|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right|$$
$$= \left|\Big(1-p^{2_s^*(t)-2}\Big)\|u_n^t\|_\gamma^2\right| \le C.$$

Step IV: From the definition of $I^\gamma_{K,t,f}$ and $\tilde{J}_t$, it immediately follows that $\frac{d}{dp}I^\gamma_{K,t,f}(pu) \ge \frac{d}{dp}\tilde{J}_t(pu)$ for all $u \in \dot{H}^s(\mathbb{R}^N)$ and for all $p > 0$. Hence,

$$\begin{aligned} I^\gamma_{K,t,f}(u_n^t) - I^\gamma_{K,t,f}(p_n^t u_n^t) = \int_{p_n^t}^1 \frac{d}{dp} I^\gamma_{K,t,f}(pu_n^t)\,\mathrm{d}p \ &\ge \int_{p_n^t}^1 \frac{d}{dp}\tilde{J}_t(pu_n^t)\,\mathrm{d}p \\ &= \tilde{J}_t(u_n^t) - \tilde{J}_t(p_n^t u_n^t). \end{aligned}$$

Since $(u_n^t)_n \subset \Sigma^t$ is a minimizing sequence for $I^\gamma_{K,t,f}$ on Σ^t and $p_n^t u_n^t \in \Sigma_1^t$, then by (4.3.7)

$$c_0^t = \inf_{\Sigma_1^t} I^\gamma_{K,,t,f}(u) < \inf_{\Sigma^t} I^\gamma_{K,t,f}(u) = c_1^t.$$

□

Proposition 4.3.6. *Assume that $t \ge 0$ and (4.3.3) holds. Then $I^\gamma_{K,t,f}$ has a critical point $u_t \in \Sigma_1^t$ with $I^\gamma_{K,t,f}(u_t) = c_0^t$. In particular, u_t is a positive solution to $(E^\gamma_{K,t,f})$.*

Proof. We divide the proof in few steps.

Step 1: In this step we show that $c_0^t > -\infty$.

From the definition of $\tilde{J}_t$ in (4.3.4), we have $I^\gamma_{K,t,f}(u) > \tilde{J}_t(u)$. Therefore, in order to prove Step 1, it is enough to show that $\tilde{J}_t$ is bounded from below. From the definition of Σ_1^t,

$$\tilde{J}_t(u) \ge \left(\frac{1}{2} - \frac{1}{2_s^*(t)\Big(2_s^*(t)-1\Big)}\right)\|u\|_\gamma^2 - \|f\|_{(\dot{H}^s)'}\|u\|_\gamma \quad \text{for all } u \in \Sigma_1^t. \tag{4.3.8}$$

As the RHS is a quadratic function in $\|u\|_\gamma$, then $\tilde{J}_t$ is bounded from below and thus so is $I^\gamma_{K,t,f}$.

Step 2: In this step we show that there exists a bounded nonnegative (PS) sequence $(u_n^t)_n \subset \Sigma_1^t$ for $I^\gamma_{K,t,f}$ at the level c_0^t. Let $(u_n^t)_n \subset \bar{\Sigma}_1^t$ such that

$I^{\gamma}_{K,t,f}(u^t_n) \to c^t_0$. Since Lemma 4.3.5 implies $c^t_0 < c^t_1$, without any restriction we can assume that $(u_n)_n \subset \Sigma^t_1$. Further, using Ekeland's variational principle, $(u^t_n)_n$ admits a (PS) subsequence, still called $(u^t_n)_n$, in Σ^t_1 for $I^{\gamma}_{K,t,f}$ at the level c^t_0. Moreover, as $I^{\gamma}_{K,t,f}(u) \geq \tilde{J}_t(u)$, from (4.3.8) it follows that $(u^t_n)_n$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$. Therefore, up to a subsequence, $u^t_n \rightharpoonup u_t$ in $\dot{H}^s(\mathbb{R}^N)$ and $u^t_n \to u_t$ a.e. in $\mathbb{R}^N$. In particular, $(u^t_n)_+ \to (u_t)_+$ and $(u^t_n)_- \to (u_t)_-$ a.e. in $\mathbb{R}^N$. Moreover, the fact that f is a nonnegative functional gives as $n \to \infty$

$$\begin{aligned}
o(1) &= {}_{(\dot{H}^s)'}\Big\langle (I^{\gamma}_{K,t,f})'(u^t_n), (u^t_n)_- \Big\rangle_{\dot{H}^s} \\
&= \langle u^t_n, (u^t_n)_- \rangle_{\gamma} - \int_{\mathbb{R}^N} \frac{K(x)(u^t_n)_+^{2^*_s(t)-1}(u^t_n)_-}{|x|^t}\, \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, (u^t_n)_- \rangle_{\dot{H}^s} \\
&\leq -\|(u^t_n)_-\|^2_{\gamma} - \iint_{\mathbb{R}^{2N}} \frac{(u^t_n)_-(x)(u^t_n)_+(y) + (u^t_n)_+(x)(u^t_n)_-(y)}{|x-y|^{N+2s}}\, \mathrm{d}x\mathrm{d}y \\
&\leq -\|(u^t_n)_-\|^2_{\gamma}.
\end{aligned}$$

This implies that $(u_n)_- \to 0$ in $\dot{H}^s(\mathbb{R}^N)$ and so $(u_n)_- \to 0$ in a.e. in $\mathbb{R}^N$, which in turn yields that $(u_0)_- \equiv 0$, that is, $u_0 \geq 0$ a.e. in $\mathbb{R}^N$. Consequently, without loss of generality, we can assume $(u^t_n)_n$ is a nonnegative (PS) sequence. This completes the proof of Step 2.

Step 3: In this step we show that $u^t_n \to u_t$ in $\dot{H}^s(\mathbb{R}^N)$.

Applying Theorem 4.2.1, we get as $n \to \infty$

$$u^t_n = u_t + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r^j_n,\, y^j_n} + \sum_{k=1}^{n_2} (W^k_{\gamma,t})^{R^k_n,0} + o(1), \quad \text{if } t = 0, \quad (4.3.9)$$

and

$$u^t_n = u_t + \sum_{k=1}^{n_2} (W^k_{\gamma,t})^{R^k_n,0} + o(1), \quad \text{if } t > 0, \quad (4.3.10)$$

where $(I^{\gamma}_{K,t,f})'(u_t) = 0$, W is the unique positive solution of $(E^0_{1,0,0})$, where $W^k_{\gamma,t}$, $k = 1, 2, \cdots n_2$ are positive ground state solutions of $(E^{\gamma}_{1,t,0})$ ($\{u^t_n\}$ is a (PS) sequence for $I^{\gamma}_{K,t,f}$ implies $W^k_{\gamma,t}$ is a solution of (4.3.1), with $f \equiv 0$, and therefore by Remark (4.3.1), $W^k_{\gamma,t}$ is a nonnegative solution of $(E^{\gamma}_{1,t,0})$).

Moreover, $(y_n^j)_n$, $(r_n^j)_n$ and $(R_n^k)_n$ are some appropriate sequences with $R_n^k \to 0$ for each $k = 1, \cdots n_2$, $r_n^j \to 0$, $\frac{r_n^j}{y_n^j} \to 0$ and either $y_n^j \to y^j$ or $|y_n^j| \to \infty$ and for all $j = 1, \cdots n_1$, are appropriate sequences. To prove Step 3, we need to show that $n_1 = 0 = n_2$. We prove this by the method of contradiction.

Suppose $t = 0$. The case $t > 0$ is comparatively easier and the proof of that case will easily follow from arguments that we present in the case of $t = 0$. Also for $t > 0$, one can argue as in [24, Proposition 3.1].

Thus, let us assume that $t = 0$ and $u_n^t \not\to u_t$ in $\dot{H}^s(\mathbb{R}^N)$. For simplicity of notations, we denote u_n^0 by u_n.

Then either $n_1 \neq 0$ or $n_2 \neq 0$ or both n_1, $n_2 \neq 0$ in (5.5.9). Here we prove the last case that is when n_1 and n_2 both are non zero. If one of them is zero, that case is again comparatively easier and argument in that case will follow from this case. First we observe that

$$\begin{aligned}
&\psi_0\Big(K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big) \\
&= K(y^j)^{-\frac{N-2s}{2s}} \|W\|_\gamma^2 - (2_s^* - 1)\|K\|_{L^\infty(\mathbb{R}^N)} K(y^j)^{-\frac{N}{2s}} \|W\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} \\
&= K(y^j)^{-\frac{N}{2s}} \Big(K(y^j) - (2_s^* - 1)\|K\|_{L^\infty(\mathbb{R}^N)}\Big) \|W\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} \\
&\qquad - \gamma K(y^j)^{-\frac{N-2s}{2s}} \int_{\mathbb{R}^N} \frac{|W|^2}{|x|^{2s}} \mathrm{d}x < 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\psi_0\Big((W_{\gamma,0}^k)^{R_n^k, 0}\Big) = \psi_0(W_{\gamma,0}^k) &= \|W_{\gamma,0}^k\|_\gamma^2 - (2_s^* - 1)\|K\|_{L^\infty(\mathbb{R}^N)} \|W_{\gamma,0}^k\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} \\
&= \Big(1 - (2_s^* - 1)\|K\|_{L^\infty(\mathbb{R}^N)}\Big) \|W_{\gamma,0}^k\|_\gamma^2 < 0.
\end{aligned}$$

Theorem 4.2.1 gives

$$o(1) + c_0^0 = I_{K,0,f}^\gamma(u_n) \to I_{K,0,f}^\gamma(u_0) + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{2s}} I_{1,0,0}^0(W) + \sum_{k=1}^{n_2} I_{1,0,0}^\gamma(W_{\gamma,0}^k).$$

As $K > 0$, $I_{1,0,0}^0(W) = \frac{s}{N}\|W\|_{\dot{H}^s}^2 > 0$ and $I_{1,0,0}^\gamma(W_{\gamma,0}) = \frac{s}{N}\|W_{\gamma,0}\|_\gamma^2 > 0$, from the above expression we obtain $I_{K,0,f}^\gamma(u_0) < c_0^0$. This in turn yields $u_0 \notin \Sigma_1^0$

and

$$\psi_0(u_0) \le 0. \tag{4.3.11}$$

Next, we evaluate $\psi_0\Big(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j,\, y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\Big)$. Since $u_n \in \Sigma_1^0$, we have $\psi_0(u_n) \ge 0$. Therefore, the uniform continuity of ψ_0 and (5.5.9) imply

$$0 \le \liminf_{n\to\infty} \psi_0(u_n) = \liminf_{n\to\infty} \psi_0\Big(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\Big). \tag{4.3.12}$$

Since u_0, W, $W_{\gamma,0}^k \ge 0$ for all $k = 1, \cdots, n_1$,

$$\begin{aligned}
&\psi_0\Big(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\Big) \\
&\le \|u_0\|_\gamma^2 + \sum_{k=1}^{n_2} \|W_{\gamma,0}^k\|_\gamma^2 + \sum_{j=1}^{n_1} \|K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\|_\gamma^2 + 2\Big\langle u_0, \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}\Big\rangle_\gamma \\
&\quad + 2\Big\langle u_0, \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big\rangle_\gamma + 2\Big\langle \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0}, \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big\rangle_\gamma \\
&\quad + \sum_{i,k=1,\, i\ne k}^{n_2} \Big\langle (W_{\gamma,0}^i)^{R_n^i,0}, (W_{\gamma,0}^k)^{R_n^k,0}\Big\rangle_\gamma \\
&\quad + \sum_{l,j=1,\, l\ne j}^{n_1} \Big\langle K(y^l)^{-\frac{N-2s}{4s}} W^{r_n^l, y_n^l}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big\rangle_\gamma \\
&\quad - (2_s^* - 1)\|K\|_{L^\infty(\mathbb{R}^N)}\Big(\|u_0\|_{2_s^*}^{2_s^*} + \sum_{j=1}^{n_1} \|K(y^j)^{-\frac{N-2s}{4s}}\big(W^{r_n^j, y_n^j}\big)\|_{2_s^*}^{2_s^*} + \sum_{k=1}^{n_2} \|W_{\gamma,0}^k\|_{2_s^*}^{2_s^*}\Big) \\
&\le \psi_0(u_0) + \sum_{k=1}^{n_2} \psi_0(W_{\gamma,0}^k) + \sum_{j=1}^{n_1} \psi_0\Big(K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big) + \text{the above inner products.}
\end{aligned} \tag{4.3.13}$$

We now prove that all the five inner products in the RHS of (4.3.13) approaches 0 as $n \to \infty$. As $r_n^j \to 0$ and $\frac{|y_n^j|}{|r_n^j|} \to \infty$, it follows that $W^{r_n^j, y_n^j} \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ (see [94, Lemma 3]) and $W^{r_n^j, y_n^j} \to 0$ a.e. in $\mathbb{R}^N$. Choosing $R > 0$

large enough as $n \to \infty$

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u_0 W^{r_n^j, y_n^j}}{|x|^{2s}} \mathrm{d}x &\leq \int_{B_R} \frac{u_0 W^{r_n^j, y_n^j}}{|x|^{2s}} \mathrm{d}x + \int_{|x|>R} \frac{u_0 W^{r_n^j, y_n^j}}{|x|^{2s}} \mathrm{d}x \\ &\leq \int_{B_R} \frac{u_0 W^{r_n^j, y_n^j}}{|x|^{2s}} \mathrm{d}x + \left(\int_{|x|>R} \frac{|u_0|^2}{|x|^{2s}} \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{|x|>R} \frac{|W|^2}{|x + \frac{y_n^j}{r_n^j}|^{2s}} \mathrm{d}x\right)^{\frac{1}{2}} \\ &= o(1),\end{aligned}$$

where in the 1st integral we have passed the limit using Vitali's convergence theorem via the Hölder inequality, while in the 2nd integral simply using the Hardy inequality. Therefore, as $n \to \infty$

$$\begin{aligned}\left\langle u_0,\, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j} \right\rangle_\gamma &= K(y^j)^{-\frac{N-2s}{4s}} \left[\left\langle u_0,\, W^{r_n^j, y_n^j} \right\rangle_{\dot{H}^s} - \gamma \int_{\mathbb{R}^N} \frac{u_0 W^{r_n^j, y_n^j}}{|x|^{2s}} \mathrm{d}x \right] \\ &= o(1). \end{aligned} \tag{4.3.14}$$

Since $R_n^k \to 0$ as $n \to \infty$, similarly we also have

$$\left\langle u_0, \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k, 0} \right\rangle_\gamma = o(1). \tag{4.3.15}$$

Now,

$$\begin{aligned}&\left\langle K(y^l)^{-\frac{N-2s}{4s}} \left(W^{r_n^l, y_n^l}\right), K(y^j)^{-\frac{N-2s}{4s}} \left(W^{r_n^j, y_n^j}\right) \right\rangle_\gamma \\ &= K(y^l)^{-\frac{N-2s}{4s}} K(y^j)^{-\frac{N-2s}{4s}} (r_n^l)^{-\frac{N-2s}{2}} (r_n^j)^{-\frac{N-2s}{2}} \\ &\quad \times \Bigg[\iint_{\mathbb{R}^{2N}} \frac{\Big(W(\frac{x-y_n^l}{r_n^l}) - W(\frac{y-y_n^l}{r_n^l})\Big)\Big(W(\frac{x-y_n^j}{r_n^j}) - W(\frac{y-y_n^j}{r_n^j})\Big)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\qquad - \gamma \int_{\mathbb{R}^N} \frac{W(\frac{x-y_n^l}{r_n^l}) W(\frac{x-y_n^j}{r_n^j})}{|x|^{2s}} \mathrm{d}x \Bigg] \\ &= K(y^l)^{-\frac{N-2s}{4s}} K(y^j)^{-\frac{N-2s}{4s}} (r_n^l)^{\frac{N-2s}{2}} (r_n^j)^{-\frac{N-2s}{2}} \\ &\quad \times \Bigg[\iint_{\mathbb{R}^{2N}} \frac{\Big(W(x) - W(y)\Big)\Big(W(\frac{r_n^l x + y_n^l - y_n^j}{r_n^j}) - W(\frac{r_n^l y + y_n^l - y_n^j}{r_n^j})\Big)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\qquad - \gamma \int_{\mathbb{R}^N} \frac{W(x) W(\frac{r_n^l y + y_n^l - y_n^j}{r_n^j})}{|x + \frac{y_n^l}{r_n^l}|^{2s}} \mathrm{d}x \Bigg] \\ &= K(y^l)^{-\frac{N-2s}{4s}} K(y^j)^{-\frac{N-2s}{4s}} \left[\left\langle W, W_n \right\rangle_{\dot{H}^s(\mathbb{R}^N)} - \gamma \int_{\mathbb{R}^N} \frac{W W_n}{|x + \frac{y_n^l}{r_n^l}|^{2s}} \mathrm{d}x \right],\end{aligned}$$

where $W_n := (\frac{r_n^l}{r_n^j})^{\frac{N-2s}{2}} W\Big(\frac{r_n^l}{r_n^j}x + \frac{y_n^l - y_n^j}{r_n^j}\Big)$. Theorem 4.2.1 (vi) yields

$$\Big|\log\Big(\frac{r_n^l}{r_n^j}\Big)\Big| + \Big|\frac{y_n^l - y_n^j}{r_n^j}\Big| \longrightarrow \infty.$$

Thus $W_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ (see [94, Lemma 3]). Hence, as $n \to \infty$

$$\Big\langle K(y^l)^{-\frac{N-2s}{4s}}\Big(W^{r_n^l, y_n^l}\Big), K(y^j)^{-\frac{N-2s}{4s}}\Big(W^{r_n^j, y_n^j}\Big)\Big\rangle_\gamma = o(1). \tag{4.3.16}$$

Similarly,

$$\Big\langle (W_{\gamma,0}^i)^{R_n^i,0}, (W_{\gamma,0}^k)^{R_n^k,0}\Big\rangle_\gamma = o(1) \tag{4.3.17}$$

as $|\log \frac{R_n^j}{R_n^k}| \to \infty$.

Finally, we estimate $\Big\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big\rangle_\gamma$. First we note that $|\log \frac{R_n^j}{R_n^k}| \to \infty$ implies that either $\frac{R_n^j}{R_n^k} \to 0$ or $\frac{R_n^j}{R_n^k} \to \infty$. Suppose $\frac{R_n^j}{R_n^k} \to 0$. Then

$$\begin{aligned}
&\Big\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big\rangle_\gamma \\
&= K(y^j)^{-\frac{N-2s}{4s}} (R_n^k)^{\frac{N-2s}{2}} (r_n^j)^{-\frac{N-2s}{2}} \cdot \\
&\times \Bigg[\iint_{\mathbb{R}^{2N}} \frac{\Big(W_{\gamma,0}^k(x) - W_{\gamma,0}^k(y)\Big)\Big(W(\frac{R_n^k x - y_n^j}{r_n^j}) - W(\frac{R_n^k y - y_n^j}{r_n^j})\Big)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\qquad - \gamma \int_{\mathbb{R}^N} \frac{W_{\gamma,0}^k(x) W(\frac{R_n^k x - y_n^j}{r_n^j})}{|x|^{2s}} \mathrm{d}x\Bigg] \\
&= K(y^j)^{-\frac{N-2s}{4s}} \Bigg[\Big\langle W_{\gamma,0}^k, W^n\Big\rangle_{\dot{H}^s(\mathbb{R}^N)} - \gamma \int_{\mathbb{R}^N} \frac{W_{\gamma,0}^k W^n}{|x|^{2s}} \mathrm{d}x\Bigg],
\end{aligned}$$

where $W^n := (\frac{r_n^j}{R_n^k})^{-\frac{N-2s}{2}} W\left(\frac{x - \frac{y_n^j}{R_n^k}}{r_n^j/R_n^k}\right)$. The proof of Theorem 4.2.1 gives $\frac{r_n^j}{R_n^k} = \frac{s_n^j R_n^j}{R_n^k}$ for any j and k. As $s_n^j \to 0$ and $\frac{R_n^j}{R_n^k} \to 0$, we have $\frac{r_n^j}{R_n^k} \to 0$. Moreover, $\frac{|y_n^j|}{r_n^j} \to \infty$ implies that $\frac{|y_n^j/R_n^k|}{r_n^j/R_n^k} \to \infty$. Thus $|\log \frac{r_n^j}{R_n^k}| + |y_n^j/R_n^k| \to \infty$. Consequently by [94, Lemma 3], $W^n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. Hence, an argument similar to (4.3.14) yields

$$\Big\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j}\Big\rangle_\gamma = o(1). \tag{4.3.18}$$

On the other hand, if $\frac{R_n^j}{R_n^k} \to \infty$ then $\frac{R_n^k}{R_n^j} \to 0$. Then similarly, we also show that

$$\Big\langle (W_{\gamma,0}^k)^{R_n^k,0}, K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j} \Big\rangle_{\dot{H}^s(\mathbb{R}^N)} = K(y^j)^{-\frac{N-2s}{4s}} \langle W_\gamma^n, W \rangle,$$

where $W_\gamma^n(x) = \left(\frac{R_n^k}{R_n^j}\right)^{-\frac{N-2s}{2s}} W_{\gamma,0}^k \left(\frac{x - \frac{y_n^j}{r_n^j}}{R_n^k / r_n^j}\right)$. Since $\frac{R_n^k}{R_n^j} \to 0$ and $\frac{|y_n^j|}{r_n^j} \to \infty$, again applying [94, Lemma 3], we get $W_\gamma^n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$. Hence, in any case (4.3.18) holds.

Combining (4.3.14)–(4.3.18) along with (4.3.13), we have

$$\psi_0\Bigg(u_0 + \sum_{j=1}^{n_1} K(y^j)^{-\frac{N-2s}{4s}} W^{r_n^j, y_n^j} + \sum_{k=1}^{n_2} (W_{\gamma,0}^k)^{R_n^k,0} \Bigg) < 0.$$

This contradicts (4.3.12). Therefore, $n_1 = 0$ and $n_2 = 0$ in (5.5.9). Hence, $u_n \to u_0$ in $\dot{H}^s(\mathbb{R}^N)$. Consequently, $\psi_0(u_n) \to \psi_0(u_0)$, which in turn implies that $u_0 \in \bar{\Sigma}_1^t$. But, since $c_0^0 < c_1^0$, we conclude$u_0 \in \Sigma_1^t$. Hence Step 3 follows. □

Proposition 4.3.7. *Assume that $t \geq 0$ and (4.3.3) holds. Then $I_{K,t,f}^\gamma$ has a second critical point $v_t \neq u_t$. In particular, v_t solves $(E_{K,t,f}^\gamma)$.*

Proof. Let $t \geq 0$ and let u_t be the critical point of $I_{K,t,f}^\gamma$ obtained in Proposition 4.3.6. Let $W_{\gamma,t}$ be a positive radial ground state solution of $(E_{1,t,0}^\gamma)$. Set $w_{\gamma,t}^\tau(x) := W_{\gamma,t}(\frac{x}{\tau})$. Let $\bar{x}_0 \in \mathbb{R}^N$ be such that $K(\bar{x}_0) = \|K\|_{L^\infty(\mathbb{R}^N)}$.

<u>**Claim 1:**</u> $u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w_{\gamma,t}^\tau \in \Sigma_2^t$ for $\tau > 0$ large enough.

Indeed, as $\|K\|_{L^\infty(\mathbb{R}^N)} \geq 1$, $0 \leq t < 2s$ and u_t, $w_{\gamma,t}^\tau > 0$, using Cauchy's inequality, with $\varepsilon > 0$, we have

$$\begin{aligned}
&\psi_t\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}\Big)\\
&= \|u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}\|^2_\gamma\\
&- \Big(2^*_s(t)-1\Big)K(\bar{x}_0)\int_{\mathbb{R}^N}\frac{|u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x\\
&\le \|u_t\|^2_\gamma + K(\bar{x}_0)^{-\frac{N-2s}{2s}}\|w^\tau_{\gamma,t}\|^2_\gamma + 2K(\bar{x}_0)^{-\frac{N-2s}{4s}}\langle u_t, w^\tau_{\gamma,t}\rangle_\gamma\\
&- \Big(2^*_s(t)-1\Big)\Big\{\int_{\mathbb{R}^N}\frac{|u_t|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x + K(\bar{x}_0)^{-\frac{N-t}{2s}}\int_{\mathbb{R}^N}\frac{|w^\tau_{\gamma,t}|^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x\Big\}\\
&\le \|u_t\|^2_\gamma + K(\bar{x}_0)^{-\frac{N-2s}{2s}}\|w^\tau_{\gamma,t}\|^2_\gamma + 2K(\bar{x}_0)^{-\frac{N-2s}{4s}}\Big(\frac{\varepsilon}{2}\|w^\tau_{\gamma,t}\|^2_\gamma + \frac{1}{2\varepsilon}\|u_t\|^2_\gamma\Big)\\
&\quad - \Big(2^*_s(t)-1\Big)\Big\{\int_{\mathbb{R}^N}\frac{|u_t|^{2^*_s(t)}}{|x|^t}\mathrm{d}x + K(\bar{x}_0)^{-\frac{N-t}{2s}}\tau^{N-t}\int_{\mathbb{R}^N}\frac{|W_{\gamma,t}|^{2^*_s(t)}}{|x|^t}\mathrm{d}x\Big\}
\end{aligned}$$

$$\begin{aligned}
&= \Big(1+\frac{1}{\varepsilon}\Big)\|u_t\|^2_{\dot{H}^s(\mathbb{R}^N)} - (2^*_s(t)-1)\|u_t\|^{2^*_s(t)}_{L^{2^*_s(t)}(\mathbb{R}^N,|x|^{-t})}\\
&\quad + \|W_{\gamma,t}\|^2_\gamma\Big[(1+\varepsilon)\tau^{N-2s} - \Big(2^*_s(t)-1\Big)K(\bar{x}_0)^{-\frac{N-t}{2s}}\tau^{N-t}\Big]\\
&< 0 \quad \text{for } \tau > 0 \text{ large enough.}
\end{aligned}$$

Therefore, $u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t} \in \Sigma^t_2$ for $\tau > 0$ large enough. Hence, Claim 1 follows.

Claim 2: $I^\gamma_{K,t,f}\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}\Big) < I^\gamma_{K,t,f}(u_t) + I^\gamma_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}\Big)$ for all $\tau > 0$.

Indeed, as $u_t,\ w^\tau_{\gamma,t} > 0$ taking $K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}$ as the test function in $(E^\gamma_{K,t,f})$, we get

$$\begin{aligned}
\langle u_t, K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}\rangle_\gamma &= K(\bar{x}_0)^{-\frac{N-2s}{4s}}\int_{\mathbb{R}^N} K(x)\frac{u_t^{2^*_s(t)-1} w^\tau_{\gamma,t}}{|x|^t}\mathrm{d}x\\
&\quad + {}_{(\dot{H}^s)'}\langle f, K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^\tau_{\gamma,t}\rangle_{\dot{H}^s}. \qquad (4.3.19)
\end{aligned}$$

Therefore, using the above equality together with the fact that $K \geq 1$ yields

$$
\begin{aligned}
& I^{\gamma}_{K,t,f}\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) \\
&= \frac{1}{2}\|u_t\|^2_{\gamma} + \frac{1}{2}K(\bar{x}_0)^{-\frac{N-2s}{2s}}\|w^{\tau}_{\gamma,t}\|^2_{\gamma} + K(\bar{x}_0)^{-\frac{N-2s}{4s}}\langle u_t, w^{\tau}_{\gamma,t}\rangle_{\gamma} \\
&\quad - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u_t\rangle_{\dot{H}^s} \\
&\quad - K(\bar{x}_0)^{-\frac{N-2s}{4s}}\ {}_{(\dot{H}^s)'}\langle f, w^{\tau}_{\gamma,t}\rangle_{\dot{H}^s} \\
&= I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) + K(\bar{x}_0)^{-\frac{N-2s}{4s}}\langle u_t, w^{\tau}_{\gamma,t}\rangle_{\gamma} \\
&\quad + \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{u_t^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x + \frac{K(\bar{x}_0)^{-\frac{N-t}{2s}}}{2^*_s(t)}\int_{\mathbb{R}^N}\frac{(w^{\tau}_{\gamma,t})^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \\
&\quad - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x - K(\bar{x}_0)^{-\frac{N-2s}{4s}}\ {}_{(\dot{H}^s)'}\langle f, w^{\tau}_{\gamma,t}\rangle_{\dot{H}^s}
\end{aligned}
$$

$$
\begin{aligned}
&\leq I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) \\
&\quad + K(\bar{x}_0)^{-\frac{N-2s}{4s}}\int_{\mathbb{R}^N} K(x)\frac{u_t^{2^*_s(t)-1} w^{\tau}_{\gamma,t}}{|x|^t}\,\mathrm{d}x + \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{u_t^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \\
&\quad + \frac{K(\bar{x}_0)^{-\frac{N-t}{2s}}}{2^*_s(t)}\int_{\mathbb{R}^N}\frac{(w^{\tau}_{\gamma,t})^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x - \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\frac{\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)^{2^*_s(t)}}{|x|^t}\,\mathrm{d}x \\
&\leq I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) \\
&\quad + \frac{1}{2^*_s(t)}\int_{\mathbb{R}^N} K(x)\Bigg[2^*_s(t)K(\bar{x}_0)^{-\frac{N-2s}{4s}}\frac{u_t^{2^*_s(t)-1} w^{\tau}_{\gamma,t}}{|x|^t} \\
&\quad + \frac{u_t^{2^*_s(t)}}{|x|^t} + K(\bar{x}_0)^{-\frac{N-t}{2s}}\frac{(w^{\tau}_{\gamma,t})^{2^*_s(t)}}{|x|^t} - \frac{\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)^{2^*_s(t)}}{|x|^t}\Bigg]\,\mathrm{d}x \\
&< I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big).
\end{aligned}
$$

Hence the claim follows. As

$$
\|w^{\tau}_{\gamma,t}\|^2_{\gamma} = \tau^{N-2s}\|W_{\gamma,t}\|^2_{\gamma}, \quad \|w^{\tau}_{\gamma,t}\|^{2^*_s(t)}_{L^{2^*_s(t)}(\mathbb{R}^N,|x|^{-t})} = \tau^{N}\|W_{\gamma,t}\|^2_{\gamma},
$$

and $0 \leq t < 2s < N$, it is easy to see using the definition of $I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big)$ that

$$\lim_{\tau\to\infty} I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) = -\infty \tag{4.3.20}$$

Consequently, a straight forward computation yields that

$$\sup_{\tau>0} I^{\gamma}_{1,t,0}\Big(K(x_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) = I^{\gamma}_{1,t,0}\Big(K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau_{\max}}_{\gamma,t}\Big), \quad \text{where } \tau_{\max} = K(\bar{x}_0)^{\frac{1}{2s}}.$$

Therefore, substituting the value of $\tau_{\max}$ in the definition of $I^{\gamma}_{1,t,0}$, it is not difficult to check that

$$\sup_{\tau>0} I^{\gamma}_{1,t,0}\Big(K(x_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) = I^{\gamma}_{1,t,0}(W_{\gamma,t}).$$

Combining the above relation with Claim 2 and (5.5.3), we obtain

$$I^{\gamma}_{K,t,f}\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t}) \quad \text{for all } \tau > 0, \tag{4.3.21}$$

$$I^{\gamma}_{K,t,f}\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau}_{\gamma,t}\Big) < I^{\gamma}_{K,t,f}(u_t) \quad \text{for } \tau \text{ large enough.} \tag{4.3.22}$$

Now, fix $\tau_0 > 0$ large enough such that Claim 1 and (4.3.22) are satisfied. Set

$$\kappa_t := \inf_{\theta\in\Theta_t} \max_{r\in[0,1]} I^{\gamma}_{K,t,f}\Big(\theta(r)\Big),$$

where

$$\Theta_t := \Big\{\theta \in C\Big([0,1], \dot{H}^s(\mathbb{R}^N)\Big) \;:\; \theta(0) = u_t,\;\; \theta(1) = u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau_0}_{\gamma,t}\Big\}.$$

As $u_t \in \Sigma^t_1$, $u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{\tau_0}_{\gamma,t} \in \Sigma^t_2$ for every $\theta \in \Theta_t$, there exists $r_\theta \in (0,1)$ such that $\theta(r_\theta) \in \Sigma^t$. Thus

$$\max_{r\in[0,1]} I^{\gamma}_{K,t,f}(\theta(r)) \geq I^{\gamma}_{K,t,f}(\theta(r_\theta)) \geq \inf_{u\in\Sigma^t} I^{\gamma}_{K,t,f}(u) = c^t_1.$$

Hence,

$$\kappa_t \geq c^t_1 > c^t_0 = I^{\gamma}_{K,t,f}(u_t).$$

Here in the last inequality we have used Lemma 4.3.5.

Claim 3: $\kappa_t < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t})$.

Note that $\lim_{\tau\to 0}\|w^{\tau}_{\gamma,t}\|_{\gamma} = 0$, thus if we define $\bar{\theta}(r) := u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{r\tau_0}_{\gamma,t}$, then $\bar{\theta} \in \Theta_t$ and $\lim_{r\to 0}\|\bar{\theta}(r) - u_t\|_{\gamma} = 0$. Therefore by (4.3.21),

$$\begin{aligned}\kappa_t \le \max_{r\in[0,1]} I^{\gamma}_{K,t,f}\Big(\bar{\theta}(r)\Big) &= \max_{r\in[0,1]} I^{\gamma}_{K,t,f}\Big(u_t + K(\bar{x}_0)^{-\frac{N-2s}{4s}} w^{r\tau_0}_{\gamma,t}\Big)\\ &< I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t}),\end{aligned}$$

that is,

$$I^{\gamma}_{K,t,f}(u_t) < \kappa_t < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t}) \quad \text{for all } t \ge 0. \tag{4.3.23}$$

Using Ekeland's variational principle, there exists a (PS) sequence $(v^t_n)_n$ of $I^{\gamma}_{K,t,f}$ at level κ_t for all $t \ge 0$. Since any (PS) for $I^{\gamma}_{K,t,f}$ is bounded and $\kappa_t < I^{\gamma}_{K,t,f}(u_t) + I^{\gamma}_{1,t,0}(W_{\gamma,t})$, using Theorem 4.2.1, in the case of $t > 0$, there exists $v_t \in \dot{H}^s(\mathbb{R}^N)$ such that $v^t_n \to v_t$ in $\dot{H}^s(\mathbb{R}^N)$, with $I^{\gamma}_{K,t,f}(v_t) = \kappa_t$ and $(I^{\gamma}_{K,t,f})'(v_t) = 0$. Moreover, $I^{\gamma}_{K,t,f}(u_t) < \kappa_t$ implies that $u_t \neq v_t$. Hence we have proved the proposition for $t > 0$.

Next let us assume that $t = 0$ so that we are in case (ii) of Theorem 4.1.1 and so

$$K(\bar{x}_0) = \|K\|_{L^{\infty}(\mathbb{R}^N)} < \left(\frac{S}{S_{\gamma,0,s}}\right)^{\frac{N}{N-2s}} \tag{4.3.24}$$

holds by assumption. Let W denote the unique positive solution of $(E^0_{1,0,0})$. As $W_{\gamma,0}$ is a minimum energy positive solution (ground state solution) of $(E^{\gamma}_{1,t,0})$, with $t = 0$, it follows that

$$I^{\gamma}_{1,0,0}(W_{\gamma,0}) \le I^{\gamma}_{1,0,0}(W) < I^0_{1,0,0}(W),$$

where the last inequality is due to the fact that $W > 0$ and so $\int_{\mathbb{R}^N} \frac{|W|^2}{|x|^{2s}}\mathrm{d}x > 0$. Since S and $S_{\gamma,0,s}$ are achieved by W and $W_{\gamma,0}$ respectively, it is easy to see that $\|W\|^2_{\dot{H}^s(\mathbb{R}^N)} = S^{\frac{N}{2s}}$ and $\|W_{\gamma,0}\|^2_{\gamma} = S^{\frac{N}{2s}}_{\gamma,0,s}$. On the other hand, as

$I^0_{1,0,0}(W) = \frac{s}{N}\|W\|^2_{\dot{H}^s}$ and $I^\gamma_{1,0,0}(W_{\gamma,0}) = \frac{s}{N}\|W_{\gamma,0}\|^2_\gamma$, we obtain $\frac{I^\gamma_{1,0,0}(W_{\gamma,0})}{I^0_{1,0,0}(W)} = \left(\frac{S_{\gamma,0,s}}{S}\right)^{\frac{N}{2s}}$. This together with (4.3.24) yields

$$I^\gamma_{1,0,0}(W_{\gamma,0}) < K(\bar{x}_0)^{-\frac{N-2s}{2s}} I^0_{1,0,0}(W) \leq K(x)^{-\frac{N-2s}{2s}} I^0_{1,0,0}(W) \quad \text{for all } x \in \mathbb{R}^N.$$

Combining the above inequality with (4.3.23) yields

$$I^\gamma_{K,0,f}(u_0) < \kappa_0 < \min\left\{I^\gamma_{K,0,f}(u_0)+K(x)^{-\frac{N-2s}{2s}} I^0_{1,0,0}(W),\ I^\gamma_{K,0,f}(u_0)+I^\gamma_{1,0,0}(W_{\gamma,0})\right\}.$$

Hence, again using Theorem 4.2.1 (as in the case $t > 0$), we can conclude that the (PS) sequence $(v^0_n)_n$ converges strongly to some $v_0 \in \dot{H}^s(\mathbb{R}^N)$, with $I^\gamma_{K,0,f}(v_0) = \kappa_0$ and $(I^\gamma_{K,0,f})'(v_0) = 0$. As before, $I^\gamma_{K,0,f}(u_0) < \kappa_0$ implies that $u_0 \neq v_0$. Hence we have completed the proof for all $t \geq 0$. □

Lemma 4.3.8. *If* $\|f\|_{(\dot{H}^s)'} < C_t\sqrt{1-\frac{\gamma}{\gamma_{N,s}}} S_{\gamma,t,s}^{\frac{N-t}{4s-2t}}$, *then* (4.3.3) *holds.*

Proof. By the given assumption, there exists $\varepsilon > 0$ such that

$$\|f\|_{(\dot{H}^s)'} < C_t\sqrt{1-\frac{\gamma}{\gamma_{N,s}}} S_{\gamma,t,s}^{\frac{N-t}{4s-2t}} - \varepsilon.$$

Combining this with Lemma 4.3.4, for all $u^t \in \Sigma^t$, it holds

$$\begin{aligned}
{}_{(\dot{H}^s)'}\langle f, u^t\rangle_{\dot{H}^s} &\leq \|f\|_{(\dot{H}^s)'}\|u^t\|_{\dot{H}^s(\mathbb{R}^N)} \\
&\leq \left(1-\frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}} \|f\|_{(\dot{H}^s)'}\|u^t\|_\gamma \\
&< C_t S_{\gamma,t,s}^{\frac{N-t}{4s-2t}}\|u^t\|_\gamma - \varepsilon\left(1-\frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}}\|u^t\|_\gamma \\
&\leq \frac{4s-2t}{N-2t+2s}\|u^t\|^2_\gamma - \varepsilon\left(1-\frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}}\|u^t\|_\gamma.
\end{aligned}$$

Hence,

$$\inf_{u\in\Sigma^t}\left[\frac{4s-2t}{N-2t+2s}\|u\|^2_\gamma - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s}\right] \geq \varepsilon\left(1-\frac{\gamma}{\gamma_{N,s}}\right)^{-\frac{1}{2}} \inf_{u\in\Sigma^t}\|u\|_\gamma.$$

Since $\|u\|_\gamma$ is bounded away from 0 on Σ^t by Remark 4.3.2, the above expression implies that

$$\inf_{u\in\Sigma^t}\left[\frac{4s-2t}{N-2t+2s}\|u\|^2_\gamma - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s}\right] > 0. \tag{4.3.25}$$

On the other hand,

$$(4.3.3) \iff C_t \frac{\|u\|_\gamma^{\frac{N-2t+2s}{2s-t}}}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{N-2s}{4s-2t}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} > 0, \text{ for } \|u\|_{L^{2_s^*(t)}(\mathbb{R}^N,|x|^{-t})} = 1$$

$$\iff C_t \frac{\|u\|_\gamma^{\frac{N-2t+2s}{2s-t}}}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(t)}}{|x|^t}\,\mathrm{d}x\right)^{\frac{N-2s}{4s-2t}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} > 0 \quad \text{for } u \in \Sigma^t \tag{4.3.26}$$

$$\iff \frac{4s-2t}{N-2t+2s}\|u\|_\gamma^2 - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} > 0 \quad \text{for } u \in \Sigma^t.$$

Clearly, (5.5.16) ensures the RHS of (4.3.26) holds. The lemma now follows.

□

Proof of Theorem 4.1.1 *completed.* Combining Propositions 4.3.6 and 4.3.7 with Lemma 4.3.8, we conclude the proof of Theorem 4.1.1. □

4.4 Necessary Lemmas to complete the proof of Proposition (4.2.1):

Lemma 4.4.1. *Let $(v_n)_n \subseteq \dot{H}^s(\mathbb{R}^N)$ be a (PS) sequence for $\bar{I}^\gamma_{K,0,0}$ at the level d. Assume that, there exist sequences $(y_n)_n \to y \in \mathbb{R}^N$, $r_n \to 0 \in \mathbb{R}^+ \cup \{0\}$ such that $w_n(x) = r_n^{\frac{N-2s}{2}} v_n(r_n x + y_n)$ converges weakly in $\dot{H}^s(\mathbb{R}^N)$ and a.e. to some $w \in \dot{H}^s(\mathbb{R}^N)$. If $\frac{|y_n|}{r_n} \to \infty$, then $K(y)^{\frac{N-2s}{4s}} w$ solves $(E^0_{1,0,0})$. Moreover,*

$$z_n := v_n - r_n^{-\frac{N-2s}{2}} w(\frac{x-y_n}{r_n})$$

is a (PS) sequence for $\bar{I}^\gamma_{K,0,0}$ at the level $d - K(y)^{-\frac{N-2s}{2s}} \bar{I}^0_{1,0,0}(K(y)^{\frac{N-2s}{4s}} w)$.

Proof. Let $(v_n)_n \subseteq \dot{H}^s(\mathbb{R}^N)$ be a (PS) sequence for $\bar{I}^\gamma_{K,0,0}$ at the level d and

ϕ be an arbitrary $C_c^\infty(\mathbb{R}^N)$ function. Put $\phi_n(x) := r_n^{-\frac{N-2s}{2}}\phi(\frac{x-y_n}{r_n})$. Thus,

$$\begin{aligned}
\langle w, \phi\rangle_{\dot{H}^s} &= \lim_{n\to\infty}\langle w_n, \phi\rangle_{\dot{H}^s} \\
&= \lim_{n\to\infty}\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(w_n(x)-w_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&= \lim_{n\to\infty}\frac{C_{N,s}}{2}\iint_{\mathbb{R}^{2N}}\frac{(v_n(x)-v_n(y))(\phi_n(x)-\phi_n(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \qquad (4.4.1)\\
&= \lim_{n\to\infty}\gamma\int_{\mathbb{R}^N}\frac{v_n\phi_n}{|x|^{2s}}\,\mathrm{d}x + \int_{\mathbb{R}^N}K(x)|v_n|^{2_s^*-2}v_n\phi_n\,\mathrm{d}x \\
&= \lim_{n\to\infty}\left[\gamma\int_{\mathbb{R}^N}\frac{w_n\phi}{|x+r_n^{-1}y_n|^{2s}}\,\mathrm{d}x + \int_{\mathbb{R}^N}K(r_nx+y_n)|w_n|^{2_s^*-2}w_n\phi\,\mathrm{d}x\right].
\end{aligned}$$

Since $r_n^{-1}|y_n| \to \infty$, for each fixed ϕ we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{w_n\phi}{|x+r_n^{-1}y_n|^{2s}}\,\mathrm{d}x = 0.$$

Therefore, taking the limit as $n\to\infty$ in (4.4.1), we obtain $(-\Delta)^s w = K(y)|w|^{2_s^*-2}w$, or equivalently $K(y)^{\frac{N-2s}{4s}}w$ solves $(E^0_{1,0,0})$. Moreover,

$$\int_{\mathbb{R}^N}\frac{w_n w}{|x-r_n^{-1}y_n|^{2s}}\mathrm{d}x = \int_{\mathbb{R}^N}\frac{|w|^2}{|x-r_n^{-1}y_n|^{2s}}\mathrm{d}x = o(1).$$

Therefore, proceeding as in Claim 2 of Step 5 in the proof of Theorem 4.2.1, we obtain as $n\to\infty$

$$\bar{I}^\gamma_{K,0,0}(z_n) = \bar{I}^\gamma_{K,0,0}(v_n) - K(y)^{-\frac{N-2s}{2s}}\bar{I}^0_{1,0,0}(K(y)^{\frac{N-2s}{4s}}w) + o(1).$$

To prove that ${}_{(\dot{H}^s)'}\left\langle \bar{I}^\gamma_{K,0,0}(z_n), \varphi\right\rangle_{\dot{H}^s} = o(\|\varphi\|)$, we proceed as in the proof of Claim 2 of Step 5 in Theorem 4.2.1, the only additional estimate we need to check is

$$\int_{\mathbb{R}^N}\frac{w\varphi_n}{|x-r_n^{-1}y_n|^{2s}}\mathrm{d}x = o(\|\varphi_n\|),$$

where $\varphi_n = r_n^{\frac{N-2s}{2s}}\varphi(r_nx+y_n)$, $\|\varphi_n\| = \|\varphi\|$. This estimate follows from the Cauchy-Schwartz and the Hölder inequalities. □

Lemma 4.4.2. *Let $\mathcal{K}$ denote the Kelvin transform in $\mathbb{R}^N$. If $(r_n)_n \subset \mathbb{R}^+\cup\{0\}$ and $(y_n)_n \subset \mathbb{R}^N$ are sequences such that $\frac{|y_n|}{r_n} \to \infty$ and W is a positive* *solution of* $(E^0_{1,0,0})$, *then in the sense of* $\dot{H}^s(\mathbb{R}^N)$*-norm as* $n \to \infty$

$$r_n^{-\frac{N-2s}{2}}\mathcal{K}\left(W(\tfrac{x-y_n}{r_n})\right) = \left(r_n|y_n|^{-2}\right)^{-\frac{N-2s}{2}} W\left(\frac{x-\frac{y_n}{|y_n|^2}}{r_n|y_n|^{-2}}\right) + o(1). \tag{4.4.2}$$

Proof. Let the assumptions and notation of the statement hold. Let W be a positive solution of $(E^0_{1,0,0})$. Then $W(x) = C_{N,s}(1+|x|^2)^{-\frac{N-2s}{2}}$ thanks to [46]. The $\dot{H}^s(\mathbb{R}^N)$ norm is invariant under the scaling so that

$$v \mapsto \tilde{v}(x) := \left(\frac{r_n}{|y_n|^2}\right)^{\frac{N-2s}{2}} v\left(\frac{r_n}{|y_n|^2}x + \frac{y_n}{|y_n|^2}\right), \tag{4.4.3}$$

we can apply it to each side of (4.4.2) to check the convergence. The RHS of (4.4.2) becomes $W + o(1)$. The LHS of (4.4.2), after some algebraic computation, is transformed into

$$W^n(x) := C_{N,s}\left(1 + \frac{r_n}{|y_n|}\langle x, y_n\rangle + \left(1 + r_n^2|y_n|^{-2}\right)|x|^2)\right)^{-\frac{N-2s}{2}}.$$

As $\frac{r_n}{|y_n|} \to 0$, clearly $W^n \to W$ in $\dot{H}^s(\mathbb{R}^N)$. Hence the proof is complete. □

Conclusion: In this chapter we consider, nonlocal Hardy-Sobolev equation with nonhomogeneous term in $\mathbb{R}^N$. Here the nonlinearity is multiplied by a positive continuous coefficient function $K \geq 1$, whose asymptotic behavior is known. First, we prove the profile decomposition of the PS sequences for the associated energy functional. Then, using that we prove multiplicity result for the equation under certain restrictions on the nonhomogeneous term. It might be interesting to know whether we can prove multiplicity results under weaker assumptions on the coefficient function K, say what happens when $K \in (0, 1]$ or K is only positive and asymptotically converges to 1?

Chapter 5

Fractional Elliptic Systems with Critical or Subcritical nonlinearities

In this chapter we study existence, uniqueness and multiplicity of positive solutions to the following fractional nonhomogeneous elliptic system in $\mathbb{R}^N$

$$\begin{cases} (-\Delta)^s u + \gamma u = \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + f(x) \ \text{ in } \mathbb{R}^N, \\ (-\Delta)^s v + \gamma v = \dfrac{\beta}{\alpha+\beta}|v|^{\beta-2}v|u|^{\alpha} + g(x) \ \text{ in } \mathbb{R}^N, \\ u,\, v > 0 \ \text{ in } \ \mathbb{R}^N, \end{cases} \tag{$\mathcal{S}^{\gamma}_{\alpha,\beta}$}$$

where $N > 2s,\ \alpha,\ \beta > 1,\ \alpha+\beta \leq 2^*_s,\ 2^*_s := 2N/(N-2s),\ f,\ g$ are non-trivial nonnegative functionals in the dual space of $\dot{H}^s(\mathbb{R}^N)$ if $\alpha+\beta = 2^*_s$ and of $H^s(\mathbb{R}^N)$ if $\alpha+\beta < 2^*_s$, while $\gamma = 0$ if $\alpha+\beta = 2^*_s$ and $\gamma = 1$ if $\alpha+\beta < 2^*_s$.

In the vectorial case, the natural solution space for $(\mathcal{S}^{\gamma}_{\alpha,\beta})$ is the Hilbert space $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, equipped with the inner product

$$\Big\langle (u,v),(\phi,\psi) \Big\rangle_{\dot{H}^s \times \dot{H}^s} := \langle u, \phi\rangle_{\dot{H}^s} + \langle v, \psi\rangle_{\dot{H}^s},$$

and the norm

$$\|(u,v)\|_{\dot{H}^s\times\dot{H}^s} := \left(\|u\|^2_{\dot{H}^s} + \|v\|^2_{\dot{H}^s}\right)^{\frac{1}{2}},$$

when $\alpha+\beta = 2^*_s$, while is $H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$ equipped with the inner product

$$\Big\langle (u,v),(\phi,\psi)\Big\rangle_{H^s\times H^s} := \langle u,\phi\rangle_{\dot{H}^s} + \langle v,\psi\rangle_{\dot{H}^s} + \langle u,\phi\rangle_{L^2} + \langle v,\psi\rangle_{L^2},$$

and the norm

$$\|(u,v)\|_{H^s\times H^s} := \left(\|u\|^2_{H^s} + \|v\|^2_{H^s}\right)^{\frac{1}{2}},$$

if $\alpha+\beta < 2^*_s$.

In general, given any two Banach spaces X and Y, the product space $X\times Y$ is endowed with the following product norm

$$\|(x,y)\|_{X\times Y} := \left(\|x\|^2_X + \|y\|^2_Y\right)^{\frac{1}{2}}.$$

For instance, $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$ $(p>1)$ is equipped with the product norm

$$\|(u,v)\|_{L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)} := \left(\|u\|^2_{L^p(\mathbb{R}^N)} + \|v\|^2_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{2}}.$$

When $\alpha+\beta = 2^*_s$, we consider the corresponding weakly coupled Fractional elliptic system

$$\begin{cases} (-\Delta)^s u = \dfrac{\alpha}{2^*_s}|u|^{\alpha-2}u|v|^{\beta} + f(x) \ \text{ in } \mathbb{R}^N, \\ (-\Delta)^s v = \dfrac{\beta}{2^*_s}|v|^{\beta-2}v|u|^{\alpha} + g(x) \ \text{ in } \mathbb{R}^N, \\ u,\, v > 0 \ \text{ in } \ \mathbb{R}^N. \end{cases} \tag{$\mathcal{S}^0_{2^*_s}$}$$

It is well-known that $u\in\dot{H}^s(\mathbb{R}^N)$ implies $u\in L^p_{\text{loc}}(\mathbb{R}^N)$ for any $p\in[2,2^*_s]$.

Definition 5.0.1 (**Positive Weak Solution).** When $\alpha+\beta = 2^*_s$, we say $(u,v)\in\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ is a *positive weak solution* of $(\mathcal{S}^0_{2^*_s})$ if $u,\, v>0$ in $\mathbb{R}^N$ and

$$\Big\langle (u,v),(\phi,\psi)\Big\rangle_{\dot{H}^s\times\dot{H}^s} = \frac{\alpha}{2^*_s}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^{\beta}\phi\,\mathrm{d}x + \frac{\beta}{2^*_s}\int_{\mathbb{R}^N}|v|^{\beta-2}v|u|^{\alpha}\psi\,\mathrm{d}x$$
$$+ {}_{(\dot{H}^s)'}\langle f,\phi\rangle_{\dot{H}^s} + {}_{(\dot{H}^s)'}\langle g,\psi\rangle_{\dot{H}^s}$$

holds for every $(\phi,\psi)\in\dot H^s(\mathbb{R}^N)\times\dot H^s(\mathbb{R}^N)$, while if $\alpha+\beta<2_s^*$ a couple $(u,v)\in H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$ is said to be a *positive weak solution* of $(\mathcal{S}^\gamma_{\alpha,\beta})$ if $u,\,v>0$ in $\mathbb{R}^N$ and

$$\Big\langle (u,v),(\phi,\psi)\Big\rangle_{H^s\times H^s}=\frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^\beta\phi\,\mathrm{d}x+\frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N}|v|^{\beta-2}v|u|^\alpha\psi\,\mathrm{d}x$$
$$+\,{}_{H^{-s}}\langle f,\phi\rangle_{H^s}+{}_{H^{-s}}\langle g,\psi\rangle_{H^s}$$

holds for every $(\phi,\psi)\in H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$.

Define

$$S=\inf_{u\in\dot H^s(\mathbb{R}^N)\setminus\{0\}}\frac{\|u\|^2_{\dot H^s}}{\|u\|^2_{2_s^*}},\qquad S_{\alpha+\beta}=\inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}}\frac{\|u\|^2_{\dot H^s}}{\|u\|^2_{\alpha+\beta}},$$

and

$$S_{(\alpha,\beta)}=\begin{cases}\displaystyle\inf_{(u,v)\in\dot H^s(\mathbb{R}^N)\times\dot H^s(\mathbb{R}^N)\setminus\{(0,0)\}}\frac{\|u\|^2_{\dot H^s}+\|v\|^2_{\dot H^s}}{\left(\displaystyle\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x\right)^{2/2_s^*}}, & \text{if } \alpha+\beta=2_s^*\\[3ex] \displaystyle\inf_{(u,v)\in H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)\setminus\{(0,0)\}}\frac{\|u\|^2_{\dot H^s}+\|v\|^2_{\dot H^s}}{\left(\displaystyle\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x\right)^{2/(\alpha+\beta)}}, & \text{if } \alpha+\beta<2_s^*.\end{cases}$$

In the celebrated paper [46] Chen, Li and Ou prove that when $\alpha+\beta=2_s^*$ the Sobolev constant $S_{\alpha+\beta}=S$ is achieved by w, where w is the unique positive solution (up to translations and dilations) of

$$(-\Delta)^s w=w^{2_s^*-1}\ \text{ in }\mathbb{R}^N,\quad w\in\dot H^s(\mathbb{R}^N).\tag{5.0.1}$$

Indeed, any positive solution of the above equation is radially symmetric, with respect to some point $x_0\in\mathbb{R}^N$, strictly decreasing in $r=|x-x_0|$, of class $C^\infty(\mathbb{R}^N)$ and so of the explicit parametric form

$$w(x)=c_{N,s}\left(\frac{\lambda}{\lambda^2+|x-x_0|^2}\right)^{\frac{N-2s}{2}},$$

for some $\lambda > 0$. On the other hand, when $2 < \alpha + \beta < 2_s^*$, Frank, Lenzmann and Silvestre in their celebrated paper [71] prove that $S_{\alpha+\beta}$ is achieved by unique (up to a translation) positive ground state solution w of

$$(-\Delta)^s w + w = w^{\alpha+\beta-1} \text{ in } \mathbb{R}^N, \quad w \in H^s(\mathbb{R}^N).$$

Furthermore, w is radially symmetric, symmetric decreasing $C^\infty(\mathbb{R}^N)$ function which satisfies the following decay property in $\mathbb{R}^N$

$$\frac{C^{-1}}{1 + |x|^{N+2s}} \leq w(x) \leq \frac{C}{1 + |x|^{N+2s}},$$

with some constant $C > 0$ depending on $N, \ \alpha + \beta, \ s$.

Next, we recall a result from [64] ([8] in the local case) which states the relation between $S_{\alpha,\beta}$ and $S_{\alpha+\beta}$.

Lemma 5.0.2. *[64, Lemma 5.1] In all cases $\alpha > 1$, $\beta > 1$, with $\alpha+\beta \leq 2_s^*$, it results*

$$S_{\alpha,\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}} \right] S_{\alpha+\beta}.$$

Moreover, if w achieves $S_{\alpha+\beta}$ then (Bw, Cw) achieves $S_{\alpha,\beta}$ for all positive constants B and C such that $B/C = \sqrt{\alpha/\beta}$.

The scalar version of $(\mathcal{S}^0_{2_s^*})$ has been considered by Bhakta and Pucci in [31], where they prove existence of at least two positive solutions. This class of problems in the scalar and local cases, involving Sobolev critical exponents was treated in the pioneering paper [39]. Then existence was extended in [107] to multiplicity results. These kind of problems were studied in several directions. Let us mention [43, 44, 91, 104, 109] for more general perturbations and [50] for existence of sign changing solutions. Versions for systems were extended, for instance, in [37, 79, 80, 112] and in the references therein.

Elliptic systems arise in biological applications (e.g. population dynamics) or physical applications (e.g. models of a nuclear reactor) and have drawn a lot of attentions (see [8, 51, 92, 98] and references therein). For systems in bounded domains with nonhomogeneous terms we refer to [35]. Problems involving the fractional Laplace operator appear in several areas such as phase transitions, flames propagation, chemical reaction in liquids, population dynamics, finance, etc., see for e.g. [41, 58].

In the nonlocal case, there are not so many papers, in which weakly coupled systems of equations have been studied. We refer to [30,47,55,64,77,80], where Dirichlet systems of equations in bounded domains have been treated. For the nonlocal systems of equations in the entire space $\mathbb{R}^N$, we cite [26, 68, 69] and the references therein. In the very recent work [26], we have proved existence of one solution to $(\mathcal{S}^{\gamma}_{\alpha,\beta})$ when f and g are nontrivial but $\|f\|_{(\dot{H}^s)'}$ and $\|g\|_{(\dot{H}^s)'}$ are small enough. To the best of our knowledge, so far there have been no papers in the literature, where uniqueness/multiplicity of positive solutions have been established for $(\mathcal{S}^0_{2^*_s})$, with the fractional Laplacian and the critical exponents in $\mathbb{R}^N$. The main results in the chapter are new even in the local case $s = 1$.

5.1 Main Results

Our first result concerns about a general existence result for our Fractional elliptic system with both critical and subcritical nonlinearity.

Theorem 5.1.1. *(i) If $\alpha + \beta = 2^*_s$, and f, g are nontrivial nonnegative functionals in the dual space $\dot{H}^s(\mathbb{R}^N)'$ of $\dot{H}^s(\mathbb{R}^N)$ such that* $\ker(f)= \ker(g)$, *then system $(\mathcal{S}^0_{2^*_s})$ admits a nontrivial solution $(\bar{u}, \bar{v})$ such that $\bar{u} > 0$ and $\bar{v} > 0$, provided that $0 < \max\{\|f\|_{(\dot{H}^s)'}, \|g\|_{(\dot{H}^s)'}\} \le d$ for some $d > 0$ sufficiently small.*

(ii) If $\alpha + \beta < 2^*_s$, and f, g are nontrivial nonnegative functionals in the dual space $H^{-s}(\mathbb{R}^N)$ of $H^s(\mathbb{R}^N)$ such that $\ker(f)$=$\ker(g)$, *then* $(\mathcal{S}^{\gamma}_{\alpha,\beta})$

admits a nontrivial solution $(\bar{u}, \bar{v})$ *such that* $\bar{u} > 0$ *and* $\bar{v} > 0$*, provided that* $0 < \max\{\|f\|_{H^{-s}}, \|g\|_{H^{-s}}\} \leq d$ *for some* $d > 0$ *sufficiently small.*

Furthermore, in both the cases (i) *and* (ii) *if* $f \equiv g$*, then the solution* $(\bar{u}, \bar{v})$ *has the property that* $\bar{u} \not\equiv \bar{v}$*, whenever* $\alpha \neq \beta$*. Finally, if* $\alpha = \beta$ *but* $f \not\equiv g$*, then* $\bar{u} \not\equiv \bar{v}$*.*

This is the main result proved in [26].

Definition 5.1.2 (Ground state solution of $(\mathcal{S}^0_{2^*_s})$)**.** *We say that a pair* $(u, v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ *is a ground state solution or least energy solution for* $(\mathcal{S}^0_{2^*_s})$*, with* $f = 0 = g$*, if* (u, v) *is a minimizer of* $S_{\alpha,\beta}$*.*

Lemma 5.0.2 poses a natural question: are all the ground state solutions of $(\mathcal{S}^0_{2^*_s})$, with $f = 0 = g$, of the form (Bw, Cw), where w is the unique positive solution of (5.0.1)?

We answer this question affirmatively in our first main theorem which is stated as below.

Theorem 5.1.3 (Uniqueness of ground state for homogeneous system)**.** *Let* (u_0, v_0) *be a minimizer of* $S_{\alpha,\beta}$*. Then there exist* $\tau, B > 0$ *such that*

$$(u_0, v_0) = (Bw, Cw), \text{ with } C = B\tau, \quad \tau = \sqrt{\frac{\beta}{\alpha}},$$

where w *is the unique positive solution of* (5.0.1)*.*

The above result partially extends the uniqueness theorem due to Chen, Li and Ou [46] from the scalar case (5.0.1) to the system $(\mathcal{S}^0_{2^*_s})$ with $f = 0 = g$. Theorem 5.1.3 proves the uniqueness of ground state solution of the system $(\mathcal{S}^0_{2^*_s})$ when $f = 0 = g$ and also generalizes [64, Lemma 5.1], where as in [46] uniqueness has been established among all positive solutions of (5.0.1).

Our next main result is the multiplicity of solutions for the nonhomogeneous system $(\mathcal{S}^0_{2^*_s})$.

Theorem 5.1.4 (Multiplicity for nonhomogeneous system)**.** *Assume that f, g are nontrivial nonnegative functionals in the dual space of $\dot{H}^s(\mathbb{R}^N)$ with $ker(f) = ker(g)$ and*

$$\max\{\|f\|_{(\dot{H}^s)'}, \|g\|_{(\dot{H}^s)'}\} < C_0 S_{\alpha,\beta}^{\frac{N}{4s}}, \quad \text{where} \quad C_0 := \left(\frac{4s}{N+2s}\right)(2^*_s - 1)^{-\frac{N-2s}{4s}},$$

then $(\mathcal{S}^0_{2^*_s})$ admits at least two positive solutions.

Furthermore, if $f \equiv g$, then the solution (u, v) of $(\mathcal{S}^0_{2^*_s})$ has the property that $u \not\equiv v$, whenever $\alpha \neq \beta$. Finally, if $\alpha = \beta$ but $f \not\equiv g$, then $u \not\equiv v$.

Theorem 5.1.3 and Theorem 5.1.4 are the main results of [25]. Theorem 5.1.4 complements the mentioned work [26] on $(\mathcal{S}^0_{2^*_s})$.

The proof of the uniqueness Theorem 5.1.3 is inspired by some arguments made in [49] and [96] (also see [48]). The main difference is that in our case the nontrivial solution (u, v) has both components nontrivial, that is $u \neq 0$ and $v \neq 0$, and in the proof it was necessary to deal with a non symmetric system.

To prove the multiplicity Theorem 5.1.4, the main difficulty is the lack of compactness of the Sobolev space $\dot{H}^s(\mathbb{R}^N)$ into the Lebesgue space $L^{2^*_s}(\mathbb{R}^N)$. For this reason the functional associated to system $(\mathcal{S}^0_{2^*_s})$ may fail to satisfy the Palais-Smale condition at some critical levels. To overcome this, it is necessary to look for a nice energy range where the (PS) condition holds in order to use variational arguments. Classification of (PS) sequences associated with a scalar equation (local/nonlocal) has been done in many papers, to quote a few, we cite [31, 54, 88, 94, 95, 105]. To the best of our knowledge, the (PS) decomposition associated to systems of equations has not been studied much. We quote the recent work [96], where in the local case the (PS) decomposition was done for systems of equations in bounded domains.

Again to the best of our knowledge, in both the local and nonlocal cases, Proposition 5.4.1 (see, Section (5.4)) is the first result where the (PS) decomposition has been established for system of equations in the whole space $\mathbb{R}^N$. Next, to prove multiplicity of solutions, we decompose the space $\dot{H}^s(\mathbb{R}^N)$ into three disjoint components. The first solution is constructed using a minimization argument in one of the components. Another solution is obtained by combining the Ekeland's variational principle with a careful analysis of the critical levels by using the homogeneous unique solution with some estimates in a slightly larger Morrey space.

The chapter has been organized as follows. In Section (5.2), we prove existence of one positive solution of the system $(\mathcal{S}^{\gamma}_{\alpha,\beta})$, namely Theorem 5.1.1. In Section (5.3), we prove the uniqueness for the ground state solution of the homogeneous system, namely Theorem 5.1.3. Section (5.4) deals with the Palais-Smale decomposition associated with the functional of $(\mathcal{S}^{0}_{2^*_s})$. In Section (5.5), we prove Theorem 5.1.4.

Remark 5.1.5. Adapting the arguments in the proof of Theorem 5.1.3 and Theorem 5.1.4, the results of uniqueness and multiplicity can be obtained for the following system of equations:

a)
$$\begin{cases} (-\Delta)^s u + u = \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + f(x) \ \text{ in } \mathbb{R}^N, \\ (-\Delta)^s v + v = \dfrac{\beta}{\alpha+\beta}|v|^{\beta-2}v|u|^{\alpha} + g(x) \ \text{ in } \mathbb{R}^N, \\ u,\, v > 0 \ \text{ in } \ \mathbb{R}^N, \end{cases} \tag{5.1.1}$$

where $N > 2s$, $\alpha,\ \beta > 1$ and $\alpha+\beta < 2^*_s$, and $f,\ g$ are nonnegative functionals in the dual space of $H^s(\mathbb{R}^N)$ (see Theorem 5.1.1, for existence of solutions). It is known that the scalar equation

$$(-\Delta)^s u + u = |u|^{\alpha+\beta-2}u \quad \text{in } \mathbb{R}^N \tag{5.1.2}$$

has a unique ground state solution (see [71]). If ω denotes the unique ground state solution of (5.1.2), then it can be shown that $(r\omega, t\omega)$ is a ground state solution of (5.1.1) when $f = 0 = g$ and $r/t = \sqrt{\alpha/\beta}$. Next, following an argument similar to Theorem 5.1.3, with obvious modifications, it can be shown that any ground state solution of (5.1.1) with $f = 0 = g$ is of the form $(r\omega, t\omega)$ where $r/t = \sqrt{\alpha/\beta}$.

b)

$$\begin{cases} (-\Delta)^s u = \dfrac{\alpha}{2_s^*} a(x)|u|^{\alpha-2}u|v|^{\beta} + f(x) \ \text{ in } \mathbb{R}^N, \\ (-\Delta)^s v = \dfrac{\beta}{2_s^*} b(x)|v|^{\beta-2}v|u|^{\alpha} + g(x) \ \text{ in } \mathbb{R}^N, \\ u,\, v > 0 \ \text{ in } \mathbb{R}^N, \end{cases} \tag{5.1.3}$$

where α, β, f, g are as in $(\mathcal{S}^0_{2_s^*})$ and the potentials a, b are continuous functions in $\mathbb{R}^N$ with $a, b \geq 1$ and $a(x)$, $b(x) \to 1$ as $|x| \to \infty$. See for instance [31] in the scalar case.

c) One can also try to adopt the methodology of this paper in order to study the system of equations involving the Hardy operator i.e., if $(-\Delta)^s$ is replaced by the Hardy operator $(-\Delta)^s - \frac{\gamma}{|x|^{2s}}$, where $\gamma \in (0, \gamma_{N,s})$ and $\gamma_{N,s}$ is the best Hardy constant in the fractional Hardy inequality. The multiplicity question in the scalar case was already solved for this problem in the recent paper [27].

Remark 5.1.6. Theorem 5.1.3 proves uniqueness of ground state solutions of $(\mathcal{S}^0_{2_s^*})$ with $f = 0 = g$. Therefore, it is interesting to ask if any positive solution of $(\mathcal{S}^0_{2_s^*})$ with $f = 0 = g$ is of the form (rw, tw), where $r/t = \sqrt{\alpha/\beta}$ and w is the *unique* positive solution of (5.0.1).

5.2 Proof of Theorem 5.1.1

Before proving the main Theorem 5.1.1 let us present some auxiliary results.

Lemma 5.2.1. *There exists a positive constant $C = C(\alpha, \beta, s, N)$ such that when $\alpha + \beta = 2_s^*$*

$$\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \mathrm{d}x \right)^{1/2_s^*} \leq C \|(u,v)\|_{\dot{H}^s \times \dot{H}^s}$$

for all $(u, v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, while if $\alpha + \beta < 2_s^*$

$$\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \mathrm{d}x \right)^{1/(\alpha+\beta)} \leq C \|(u,v)\|_{H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)}$$

for all $(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$.

Proof. It easily follows from the definition of $S_{\alpha+\beta}$ and the inequality

$$|t|^\alpha |\tau|^\beta \leq |t|^{\alpha+\beta} + |\tau|^{\alpha+\beta}$$

for all $(t, \tau) \in \mathbb{R}^2$. □

Finally we prove a short useful result

Lemma 5.2.2. *In all cases $\alpha > 1$, $\beta > 1$, with $\alpha + \beta \leq 2_s^*$,*

$$S_{(\alpha,\beta)} > S_{\alpha+\beta}$$

holds true.

Proof. If $\alpha > \beta$, then using Lemma 5.0.2,

$$\frac{S_{(\alpha,\beta)}}{S_{\alpha+\beta}} = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}} = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \frac{\alpha+\beta}{\alpha} > 1.$$

Similarly, if $\alpha < \beta$ then

$$\frac{S_{(\alpha,\beta)}}{S} = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}} = \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} = \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left[1 + \left(\frac{\beta}{\alpha}\right)^{-1}\right]$$
$$= \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \frac{\alpha+\beta}{\beta} > 1.$$

Further, $S_{(\alpha,\beta)} > 2S_{\alpha+\beta}$ for $\alpha = \beta$. □

We are finally in a position to prove the main result and we simply say that a couple (u,v) is positive if both components are positive.

Proof of Theorem 5.1.4 – *Part* (i). Let $\alpha+\beta=2_s^*$. We note that system $(\mathcal{S}^0_{2_s^*})$ is variational and the underlying functional is

$$I_{f,g}(u,v):=\frac{1}{2}\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s}-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x-{}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s}-{}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s},$$

which is well defined in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ and of class $C^1\Big(\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)\Big)$. Moreover, if (u,v) is a solution of $(\mathcal{S}^0_{2_s^*})$, then (u,v) is a positive critical point of $I_{f,g}$ and vice versa.

Let us now introduce the auxiliary functional

$$J_{f,g}(u,v):=\frac{1}{2}\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s}-\frac{1}{2_s^*}\int_{\mathbb{R}^N}u_+^\alpha v_+^\beta\,\mathrm{d}x-{}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s}-{}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s},$$

which is well defined in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ and of class $C^1\Big(\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)\Big)$, with second derivative. Indeed, for all (u,v), $(\phi,\psi)\in\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$

$$\begin{aligned}J''_{f,g}(u,v)\Big((\phi,\psi),(\phi,\psi)\Big)&=\|(\phi,\psi)\|^2_{\dot{H}^s\times\dot{H}^s}-\frac{\alpha(\alpha-1)}{2_s^*}\int_{\mathbb{R}^N}u_+^{\alpha-2}v_+^\beta\phi^2\,\mathrm{d}x\\&-\frac{\beta(\beta-1)}{2_s^*}\int_{\mathbb{R}^N}u_+^\alpha v_+^{\beta-2}\psi^2\,\mathrm{d}x-\frac{2\alpha\beta}{2_s^*}\int_{\mathbb{R}^N}u_+^{\alpha-1}v_+^{\beta-1}\phi\psi\,\mathrm{d}x.\end{aligned}\tag{5.2.1}$$

Using Hölder's and Sobolev's inequalities, we estimate the second term on the RHS as follows

$$\begin{aligned}\int_{\mathbb{R}^N}u_+^{\alpha-2}v_+^\beta\phi^2\mathrm{d}x&\le\left(\int_{\mathbb{R}^N}|\phi|^{2_s^*}\mathrm{d}x\right)^{\frac{2}{2_s^*}}\left(\int_{\mathbb{R}^N}|u|^{2_s^*}\mathrm{d}x\right)^{\frac{\alpha-2}{2_s^*}}\left(\int_{\mathbb{R}^N}|v|^{2_s^*}\mathrm{d}x\right)^{\frac{\beta}{2_s^*}}\\&\le S^{-1-\frac{\alpha-2}{2}-\frac{\beta}{2}}\|u\|^{\alpha-2}_{\dot{H}^s}\|v\|^\beta_{\dot{H}^s}\|\phi\|^2_{\dot{H}^s}\\&\le S^{-\frac{2_s^*}{2}}\|(u,v)\|^{2_s^*-2}_{\dot{H}^s\times\dot{H}^s}\|(\phi,\psi)\|^2_{\dot{H}^s\times\dot{H}^s}.\end{aligned}$$

In the inequality we have used the fact that $\|u\|_{\dot{H}^s}\le\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}$ and $\alpha+\beta=2_s^*$. Similarly,

$$\int_{\mathbb{R}^N}u_+^\alpha v_+^{\beta-2}\psi^2\,\mathrm{d}x\le S^{-\frac{2_s^*}{2}}\|(u,v)\|^{2_s^*-2}_{\dot{H}^s\times\dot{H}^s}\|(\phi,\psi)\|^2_{\dot{H}^s\times\dot{H}^s}.$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}^N} u_+^{\alpha-1} v_+^{\beta-1}\phi\psi \,\mathrm{d}x &\le \left(\int_{\mathbb{R}^N}|\phi|^{2_s^*}\,\mathrm{d}x\right)^{\frac{1}{2_s^*}}\left(\int_{\mathbb{R}^N}|\psi|^{2_s^*}\,\mathrm{d}x\right)^{\frac{1}{2_s^*}}\\ &\quad\left(\int_{\mathbb{R}^N}|u|^{2_s^*}\,\mathrm{d}x\right)^{\frac{\alpha-1}{2_s^*}}\left(\int_{\mathbb{R}^N}|v|^{2_s^*}\,\mathrm{d}x\right)^{\frac{\beta-1}{2_s^*}}\\ &\le S^{-\frac12-\frac12-\frac{\alpha-1}{2}-\frac{\beta-1}{2}}\|\phi\|_{\dot H^s}\|\psi\|_{\dot H^s}\|u\|_{\dot H^s}^{\alpha-1}\|v\|_{\dot H^s}^{\beta-1}\\ &\le \frac{S^{-\frac{2_s^*}{2}}}{2}\|(\phi,\psi)\|^2_{\dot H^s\times\dot H^s}\|(u,v)\|^{2_s^*-2}_{\dot H^s\times\dot H^s}.\end{aligned}$$

Thus, substituting the above three estimates in (5.2.1), we obtain

$$\begin{aligned}J''_{f,g}(u,v)\Big((\phi,\psi),(\phi,\psi)\Big) \ge \left(1-\frac{S^{-\frac{2_s^*}{2}}}{2_s^*}\|(u,v)\|^{2_s^*-2}_{\dot H^s\times\dot H^s}\Big[\alpha(\alpha-1)+\beta(\beta-1)+\alpha\beta\Big]\right)\cdot\\ \times\|(\phi,\psi)\|^2_{\dot H^s\times\dot H^s}.\end{aligned}$$

Therefore, $J''_{f,g}(u,v)$ is positive definite for (u,v) in the ball centered at 0 and of radius r in $\dot H^s(\mathbb{R}^N)\times\dot H^s(\mathbb{R}^N)$, where

$$r=\left(\frac{2_s^*}{\alpha^2+\beta^2+\alpha\beta-2_s^*}\right)^{\frac{1}{2_s^*-2}} S^{\frac{N}{4s}}.$$

Hence $J_{f,g}$ is strictly convex in B_r. For $(u,v)\in\dot H^s(\mathbb{R}^N)\times\dot H^s(\mathbb{R}^N)$, with $\|(u,v)\|_{\dot H^s\times\dot H^s}=r$,

$$\begin{aligned}J_{f,g}(u,v) &= \frac12\|(u,v\|^2_{\dot H^s\times\dot H^s}-\frac{1}{2_s^*}\int_{\mathbb{R}^N}u_+^{\alpha}v_+^{\beta}\mathrm{d}x - {}_{(\dot H^s)'}\langle f,u\rangle_{\dot H^s} - {}_{(\dot H^s)'}\langle g,v\rangle_{\dot H^s}\\ &\ge\left(\frac12-\frac{1}{2_s^*}S_{(\alpha,\beta)}^{-\frac{2_s^*}{2}}r^{2_s^*-2}\right)r^2-(\|f\|_{(\dot H^s)'}\|u\|_{\dot H^s}+\|g\|_{(\dot H^s)'}\|v\|_{\dot H^s})\\ &\ge\left(\frac12-\frac{1}{2_s^*}S_{(\alpha,\beta)}^{-\frac{2_s^*}{2}}r^{2_s^*-2}\right)r^2-\Big(\|f\|_{(\dot H^s)'}+\|g\|_{(\dot H^s)'}\Big)r.\end{aligned}$$

As $r^{2_s^*-2}=\frac{2_s^*}{\alpha^2+\beta^2+\alpha\beta-2_s^*}S^{\frac{2_s^*}{2}}$, we obtain

$$J_{f,g}(u,v)\ge\left[\frac12-\frac{1}{\alpha^2+\beta^2+\alpha\beta-2_s^*}\left(\frac{S}{S_{(\alpha,\beta)}}\right)^{\frac{2_s^*}{2}}\right]r^2-r(\|f\|_{(\dot H^s)'}+\|g\|_{(\dot H^s)'}). \tag{5.2.2}$$

We claim that

$$(\alpha^2+\beta^2+\alpha\beta-2_s^*)\left(\frac{S_{(\alpha,\beta)}}{S}\right)^{2_s^*/2} > 2. \tag{5.2.3}$$

By Lemma 5.2.2 and $\alpha+\beta=2_s^*$, we have

$$\begin{aligned}(\alpha^2+\beta^2+\alpha\beta-2_s^*)\left(\frac{S_{(\alpha,\beta)}}{S}\right)^{\frac{2_s^*}{2}} &> (\alpha^2+\beta^2+\alpha\beta-2_s^*)\frac{S_{(\alpha,\beta)}}{S}\\ &= \Big[2_s^*(2_s^*-1)-\alpha\beta\Big]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}}\frac{2_s^*}{\alpha}.\end{aligned}$$

Since $2_s^*>2$, to prove (5.2.3) it is enough to show that

$$\Big[2_s^*(2_s^*-1)-\alpha\beta\Big]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}}\frac{1}{\alpha}>1.$$

Now,

$$\begin{aligned}\Big[2_s^*(2_s^*-1)-\alpha\beta\Big]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}}\frac{1}{\alpha}>1 &\Longleftrightarrow 2_s^*(2_s^*-1)-\alpha\beta>\beta^{\frac{\beta}{2_s^*}}\alpha^{\frac{2_s^*-\beta}{2_s^*}}\\ &\Longleftrightarrow 2_s^*(2_s^*-1)>\alpha\beta\Bigg[1+\frac{1}{\alpha^{\frac{\beta}{2_s^*}}\beta^{\frac{2_s^*-\beta}{2_s^*}}}\Bigg].\end{aligned}$$

Since, α, $\beta>1$ and $\alpha+\beta=2_s^*$, we have

$$\alpha\beta\Bigg[1+\frac{1}{\alpha^{\frac{\beta}{2_s^*}}\beta^{\frac{2_s^*-\beta}{2_s^*}}}\Bigg]<2\alpha\beta\le\frac{(\alpha+\beta)^2}{2}=\frac{(2_s^*)^2}{2}<2_s^*(2_s^*-1).$$

Hence the claim (5.2.3) follows.

Now, by (5.2.2) and (5.2.3) there exists a number $d>0$ such that

$$\inf_{\|(u,v)\|_{\dot H^s\times\dot H^s}=r} J_{f,g}(u,v)>0,\ \text{ provided that } 0<\max\{\|f\|_{(\dot H^s)'},\|g\|_{(\dot H^s)'}\}\le d.$$

Furthermore, for $(u,v)\in\dot H^s(\mathbb{R}^N)\times\dot H^s(\mathbb{R}^N)$, with $u>0$ and $v\ge0$,

$$J_{f,g}(tu,tv)\begin{cases}<0 \text{ for } t>0 \text{ small enough}\\ >0 \text{ for } t<0 \text{ small enough,}\end{cases} \tag{5.2.4}$$

since f and g are nontrivial. Combining this along with the fact that $J_{f,g}$ is strictly convex in B_r and

$$\inf_{\|(u,v)\|_{\dot H^s\times\dot H^s}=r} J_{f,g}(u,v)>0=J_{f,g}(0,0),$$

we conclude that there exists a unique critical point $(\bar{u},\bar{v})$ of $J_{f,g}$ in B_r such that

$$J_{f,g}(\bar{u},\bar{v}) = \inf_{\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}<r} J_{f,g}(u,v) < J_{f,g}(0,0) = 0.$$

Therefore, $(\bar{u},\bar{v})$ is a nontrivial solution of

$$\begin{cases} (-\Delta)^s u = \dfrac{\alpha}{2_s^*} u_+^{\alpha-1} v_+^{\beta} + f(x) \ \text{ in } \mathbb{R}^N, \\ (-\Delta)^s v = \dfrac{\beta}{2_s^*} v_+^{\beta-1} u_+^{\alpha} + g(x) \ \text{ in } \mathbb{R}^N, \\ u,\, v \in \dot{H}^s(\mathbb{R}^N). \end{cases} \tag{5.2.5}$$

Since, f and g are nonnegative functionals, then taking $(\phi,\psi) = (\bar{u}_-,\bar{v}_-)$ as a test function in (5.2.5), we obtain

$$\begin{aligned} -\|\bar{u}_-\|_{\dot{H}^s}^2 &- \iint_{\mathbb{R}^{2N}} \frac{[\bar{u}_+(y)\bar{u}_-(x) + \bar{u}_+(x)\bar{u}_-(y)]}{|x-y|^{N+2s}} \mathrm{d}x\,\mathrm{d}y - \|\bar{v}_-\|_{\dot{H}^s}^2 \\ &- \iint_{\mathbb{R}^{2N}} \frac{[\bar{v}_+(y)\bar{v}_-(x) + \bar{v}_+(x)\bar{v}_-(y)]}{|x-y|^{N+2s}} \mathrm{d}x\,\mathrm{d}y \\ &= {}_{H^{-s}}\langle f,\bar{u}_-\rangle_{H^s} + {}_{H^{-s}}\langle g,\bar{v}_-\rangle_{H^s} \geq 0. \end{aligned}$$

This in turn implies $\bar{u}_- = 0$ and $\bar{v}_- = 0$, i.e., $\bar{u} \geq 0$ and $\bar{v} \geq 0$. Therefore, $(\bar{u},\bar{v})$ is nontrivial nonnegative solution of $(\mathcal{S}^0_{2_s^*})$.

Next we assert that $(\bar{u},\bar{v}) \neq (0,0)$ implies $\bar{u} \neq 0$ and $\bar{v} \neq 0$. Suppose not, that is assume for instance that $\bar{u} \neq 0$ but $\bar{v} = 0$. Then taking the test function $(\phi,\psi) = (\bar{u},0)$ we get

$$\|\bar{u}\|_{\dot{H}^s}^2 = {}_{(\dot{H}^s)'}\langle f,\bar{u}\rangle_{\dot{H}^s}.$$

Next, choose as test function $(\phi,\psi) = (0,\bar{u})$, so that

$${}_{(\dot{H}^s)'}\langle g,\bar{u}\rangle_{\dot{H}^s} = 0.$$

Hence, $\|\bar{u}\|_{\dot{H}^s} = 0$, since $\ker(f) = \ker(g)$ by assumption. This contradicts the fact that $(\bar{u},\bar{v}) \neq (0,0)$. Similarly, we can show that if $\bar{u} = 0$ then $\bar{v} = 0$ too. Hence the assertion follows.

Let us claim that $\bar{u} > 0$ and $\bar{v} > 0$ in $\mathbb{R}^N$. To prove the claim, first we note that taking the test function $(\phi, \psi) = (\phi, 0)$, where $\phi \in \dot{H}^s(\mathbb{R}^N)$ with $\phi \geq 0$, we obtain

$$\langle \bar{u}, \phi \rangle_{\dot{H}^s} = \frac{\alpha}{2_s^*} \int_{\mathbb{R}^N} \bar{u}^{\alpha-1} \bar{v}^{\beta} \phi \, \mathrm{d}x + {}_{(\dot{H}^s)'}\langle f, \phi \rangle_{\dot{H}^s} \geq 0,$$

as f is a nonnegative functional and $\bar{u}$, $\bar{v} \geq 0$. This implies $\bar{u}$ is a weak supersolution to

$$(-\Delta)^s u = 0.$$

Therefore, applying the maximum principle [56, Theorem 1.2 (ii)], with $c \equiv 0$ and $p = 2$ there, it follows that $\bar{u} > 0$ in $\mathbb{R}^N$. Similarly, taking the test function $(\phi, \psi) = (0, \psi)$, with $\psi \in \dot{H}^s(\mathbb{R}^N)$ and $\psi \geq 0$, yields $\bar{v} > 0$ in $\mathbb{R}^N$. This proves the claim.

The final assertion will be shown below by the method of contradiction. Therefore, let us suppose $\bar{u} \equiv \bar{v}$ and divide the proof in the two cases covered by the theorem.

First, we assume $f \equiv g$ but $\alpha \neq \beta$. Then, taking the test function $(\phi, \psi) = (\bar{u}, -\bar{u})$ yields

$$\frac{1}{2_s^*}(\alpha - \beta) \int_{\mathbb{R}^N} \bar{u}^{\alpha+\beta} dx = 0.$$

This is impossible since $\bar{u}$ is positive in $\mathbb{R}^N$.

In the remaining case, we assume $\alpha = \beta$ but $f \not\equiv g$ and $\ker(f) = \ker(g)$. Then taking the test function $(\phi, \psi) = (\phi, -\phi)$, where $\phi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$${}_{(\dot{H}^s)'}\langle f - g, \phi \rangle_{\dot{H}^s} = 0.$$

This in turn implies $f \equiv g$ as $\phi \in C_0^\infty(\mathbb{R}^N)$ is arbitrary. This contradiction completes the proof of Part (i).

Part (ii). The proof follows along the same lines as in Part (i), therefore we just mention only the differences. It is easy to see that the associated functional corresponding to $(\mathcal{S}^{\gamma}_{\alpha,\beta})$ is now

$$\tilde{I}_{f,g}(u,v) := \frac{1}{2}\|(u,v)\|^2_{H^s\times H^s} - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\,\mathrm{d}x - {}_{H^{-s}}\langle f,u\rangle_{H^s} - {}_{H^{-s}}\langle g,v\rangle_{H^s}.$$

Let us introduce the auxiliary functional as

$$\tilde{J}_{f,g}(u,v) := \frac{1}{2}\|(u,v)\|^2_{H^s\times H^s} - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N} u_+^{\alpha}v_+^{\beta}\,\mathrm{d}x - {}_{H^{-s}}\langle f,u\rangle_{H^s} - {}_{H^{-s}}\langle g,v\rangle_{H^s},$$

which is well defined in $H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$ and of class $C^1\Big(H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)\big)$, with second derivative. Arguing as before, we obtain for all (u,v), $(\phi,\psi)\in H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$

$$\begin{aligned}
&\tilde{J}''_{f,g}(u,v)\Big((\phi,\psi),(\phi,\psi)\Big)\\
&= \|(\phi,\psi)\|^2_{H^s\times H^s} - \frac{\alpha(\alpha-1)}{\alpha+\beta}\int_{\mathbb{R}^N} u_+^{\alpha-2}v_+^{\beta}\phi^2\mathrm{d}x\\
&\quad - \frac{\beta(\beta-1)}{\alpha+\beta}\int_{\mathbb{R}^N} u_+^{\alpha}v_+^{\beta-2}\psi^2\mathrm{d}x - \frac{2\alpha\beta}{\alpha+\beta}\int_{\mathbb{R}^N} u_+^{\alpha-1}v_+^{\beta-1}\phi\psi\mathrm{d}x.\\
&\geq \left(1 - \frac{S_{\alpha+\beta}^{-\frac{\alpha+\beta}{2}}}{\alpha+\beta}\|(u,v)\|^{\alpha+\beta-2}_{H^s\times H^s}\Big[\alpha(\alpha-1)+\beta(\beta-1)+\alpha\beta\Big]\right)\times\|(\phi,\psi)\|^2_{H^s\times H^s}.
\end{aligned}$$

Therefore, $\tilde{J}''_{f,g}(u,v)$ is positive definite for (u,v) in the ball centered at 0 and of radius r in $H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$, where

$$r = \left(\frac{\alpha+\beta}{\alpha^2+\beta^2+\alpha\beta-(\alpha+\beta)}\right)^{\frac{1}{\alpha+\beta-2}} S_{\alpha+\beta}^{\frac{\alpha+\beta}{2(\alpha+\beta-2)}}.$$

Hence $\tilde{J}_{f,g}$ is strictly convex in B_r. Furthermore, for all $(u,v)\in H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$, with $\|(u,v)\|_{H^s\times H^s}=r$,

$$\tilde{J}_{f,g}(u,v) \geq \left[\frac{1}{2} - \frac{1}{\alpha^2+\beta^2+\alpha\beta-(\alpha+\beta)}\left(\frac{S_{\alpha+\beta}}{S_{(\alpha,\beta)}}\right)^{\frac{\alpha+\beta}{2}}\right]r^2 - r\left(\|f\|_{H^{-s}}+\|g\|_{H^{-s}}\right). \tag{5.2.6}$$

Since $S_{(\alpha,\beta)} > S_{\alpha+\beta}$ by Lemma 5.2.2, we have

$$\begin{aligned}
&\Big(\alpha^2+\beta^2+\alpha\beta-(\alpha+\beta)\Big)\left(\frac{S_{(\alpha,\beta)}}{S_{\alpha+\beta}}\right)^{(\alpha+\beta)/2}\\
&\geq \Big(\alpha^2+\beta^2+\alpha\beta-(\alpha+\beta)\Big)\frac{S_{(\alpha,\beta)}}{S_{\alpha+\beta}}\\
&= \Big[(\alpha+\beta)(\alpha+\beta-1)-\alpha\beta\Big]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\frac{\alpha+\beta}{\alpha}.
\end{aligned}$$

Therefore, to prove

$$\Big[\alpha^2+\beta^2+\alpha\beta-(\alpha+\beta)\Big]\left(\frac{S_{(\alpha,\beta)}}{S_{\alpha+\beta}}\right)^{(\alpha+\beta)/2}>2,$$

it is enough to show that

$$\Big[(\alpha+\beta)(\alpha+\beta-1)-\alpha\beta\Big]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\frac{1}{\alpha}>1,$$

since $\alpha+\beta>2$. Actually, the above expression is equivalent to

$$(\alpha+\beta)(\alpha+\beta-1)>\alpha\beta\left[1+\frac{1}{\alpha^{\frac{\beta}{\alpha+\beta}}\beta^{\frac{\alpha}{\alpha+\beta}}}\right].$$

As α, $\beta>1$, a straight forward computation yields

$$\alpha\beta\left[1+\frac{1}{\alpha^{\frac{\beta}{\alpha+\beta}}\beta^{\frac{\alpha}{\alpha+\beta}}}\right]<2\alpha\beta\leq\frac{(\alpha+\beta)^2}{2}<(\alpha+\beta)(\alpha+\beta-1).$$

Therefore, (5.2.6) implies the existence of a number $d>0$ such that

$$\inf_{\|(u,v)\|_{H^s\times H^s}=r}\tilde{J}_{f,g}(u,v)>0,\quad \text{provided that } 0<\max\{\|f\|_{H^{-s}},\|g\|_{H^{-s}}\}\leq d.$$

From here on, proceeding as in the proof of Part (i), with obvious changes, we get the assertion. □

5.3 Uniqueness for the homogeneous system

First we need an auxiliary lemma which will be used to prove Theorem 5.1.3. Consider the following system with a parameter $\mu>0$

$$\begin{cases}(-\Delta)^s u=\mu|u|^{2_s^*-2}u+\dfrac{\alpha}{2_s^*}|u|^{\alpha-2}u|v|^{\beta} \text{ in } \mathbb{R}^N,\\ (-\Delta)^s v=\dfrac{\beta}{2_s^*}|v|^{\beta-2}v|u|^{\alpha} \text{ in } \mathbb{R}^N,\\ u,\, v>0 \text{ in } \mathbb{R}^N.\end{cases}\tag{5.3.1}$$

Associated to (5.3.1), we define

$$S_{\mu,\alpha,\beta}:=\inf_{(u,v)\in\dot{H}^s\times\dot{H}^s\setminus\{(0,0)\}}\frac{\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s}}{\left(\mu\displaystyle\int_{\mathbb{R}^N}|u|^{2_s^*}\mathrm{d}x+\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\mathrm{d}x\right)^{2/2_s^*}}.\tag{5.3.2}$$

Lemma 5.3.1. *(i) Let* $h(\tau) := \dfrac{1+\tau^2}{(\mu+\tau^\beta)^{2/2_s^*}}$, $\tau > 0$. *Then there exists* $\mu_0 > 0$ *such that for* $\mu \in (0, \mu_0)$,

$$S_{\mu,\alpha,\beta} = h(\tau_0)S, \quad \textit{where} \quad h(\tau_0) = \min_{\tau>0} h(\tau).$$

Furthermore, $\tau_0 = \tau_0(\mu, \alpha, \beta, N, s) > 0$.

(ii) For any $r > 0$, $(rw, r\tau_0 w)$ *achieves* $S_{\mu,\alpha,\beta}$, *where* w *is the unique positive solution of* (5.0.1).

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $S_{\mu,\alpha,\beta}$. Choose $\tau_n > 0$ such that $\|v_n\|_{L^{2_s^*}(\mathbb{R}^N)} = \tau_n \|u_n\|_{L^{2_s^*}(\mathbb{R}^N)}$. Now set, $z_n = \dfrac{v_n}{\tau_n}$. Therefore, $\|u_n\|_{L^{2_s^*}(\mathbb{R}^N)} = \|z_n\|_{L^{2_s^*}(\mathbb{R}^N)}$ and applying Young's inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta \mathrm{d}x &\leq \frac{\alpha}{2_s^*}\int_{\mathbb{R}^N} |u_n|^{2_s^*}\mathrm{d}x + \frac{\beta}{2_s^*}\int_{\mathbb{R}^N} |z_n|^{2_s^*}\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |u_n|^{2_s^*}\mathrm{d}x = \int_{\mathbb{R}^N} |z_n|^{2_s^*}\mathrm{d}x.\end{aligned}$$

Hence,

$$\begin{aligned}S_{\mu,\alpha,\beta} + o(1) &= \frac{\|u_n\|_{\dot{H}^s}^2 + \|v_n\|_{\dot{H}^s}^2}{\left(\mu \int_{\mathbb{R}^N} |u_n|^{2_s^*}\mathrm{d}x + \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \mathrm{d}x\right)^{2/2_s^*}} \\ &= \frac{\|u_n\|_{\dot{H}^s}^2}{\left(\mu \int_{\mathbb{R}^N} |u_n|^{2_s^*}\mathrm{d}x + \tau_n^\beta \int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta \mathrm{d}x\right)^{2/2_s^*}} \\ &\quad + \frac{\tau_n^2 \|u_n\|_{\dot{H}^s}^2}{\left(\mu \int_{\mathbb{R}^N} |z_n|^{2_s^*}\mathrm{d}x + \tau_n^\beta \int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta \mathrm{d}x\right)^{2/2_s^*}} \\ &\geq \frac{1}{(\mu+\tau_n^\beta)^{2/2_s^*}} \frac{\|u_n\|_{\dot{H}^s}^2}{\left(\int_{\mathbb{R}^N} |u_n|^{2_s^*}\mathrm{d}x\right)^{2/2_s^*}} \\ &\qquad + \frac{\tau_n^2}{(\mu+\tau_n^\beta)^{2/2_s^*}} \frac{\|z_n\|_{\dot{H}^s}^2}{\left(\int_{\mathbb{R}^N} |z_n|^{2_s^*}\mathrm{d}x\right)^{2/2_s^*}} \\ &\geq \frac{1+\tau_n^2}{(\mu+\tau_n^\beta)^{2/2_s^*}} S \geq \min_{\tau>0} h(\tau) S.\end{aligned}$$

Note that h is a C^1 function with $h(\tau) > 0$ for all $\tau \geq 0$, $h(\tau) \to \infty$ as $\tau \to \infty$ and $h(\tau) \to \mu^{-\frac{2}{2_s^*}}$ as $\tau \to 0$. Therefore, there exists $\tau_0 \geq 0$ such that $\min_{\tau > 0} h(\tau) = h(\tau_0)$. Next, we claim that $\tau_0 > 0$, if we choose $\mu > 0$ small enough. To prove the claim, first we note that $h(0) = \mu^{-\frac{2}{2_s^*}}$ and $h(1) = 2(1+\mu)^{-2/2_s^*}$. Therefore, we can choose $\mu_0 > 0$ small enough such that for $\mu \in (0, \mu_0)$, $h(0) > h(1)$. Thus, h can not attain global minimum at 0, if $\mu \in (0, \mu_0)$. Hence $\tau_0 > 0$.

Consequently, $S_{\mu,\alpha,\beta} + o(1) \geq h(\tau_0)S$, and as $o(1) \to 0$ as $n \to \infty$, we get $S_{\mu,\alpha,\beta} \geq h(\tau_0)S$. On the other hand, choosing $(u, v) = (w, \tau_0 w)$, we easily see that $S_{\mu,\alpha,\beta} \leq h(\tau_0)S$. Hence $S_{\mu,\alpha,\beta} = h(\tau_0)S$.

Since τ_0 is the minimum point for h, clearly $h'(\tau_0) = 0$. Thus τ_0 satisfies

$$\tau\Big(\mu 2_s^* + \alpha\tau^\beta - \beta\tau^{\beta-2}\Big) = 0.$$

But $\tau_0 > 0$, and so τ_0 satisfies $\mu 2_s^* + \alpha\tau^\beta - \beta\tau^{\beta-2} = 0$. This proves (i).

(ii) Note that for $(u, v) = (rw, r\tau_0 w)$, an easy computation yields

$$\frac{\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s}}{\left(\mu\displaystyle\int_{\mathbb{R}^N}|u|^{2_s^*}\mathrm{d}x + \int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x\right)^{2/2_s^*}} = h(\tau_0)S.$$

Hence using (i), we conclude that $S_{\mu,\alpha,\beta}$ is achieved by $(rw, r\tau_0 w)$. $\square$

Proof of Theorem 5.1.3

Proof. Suppose that (u_0, v_0) and w achieves $S_{\alpha,\beta}$ and S respectively. We are going to prove that there are $r, t > 0$ such that

$$(u_0, v_0) = (rw, tw).$$

Claim.

a)

$$\int_{\mathbb{R}^N}|u_0|^\alpha|v_0|^\beta\,\mathrm{d}x = r^\alpha t^\beta\int_{\mathbb{R}^N}w^{2_s^*}\,\mathrm{d}x, \qquad \text{whenever} \quad \frac{r}{t} = \sqrt{\frac{\alpha}{\beta}}.$$

b) There exists $r > 0$ such that

$$\int_{\mathbb{R}^N} |u_0|^{2_s^*}\,\mathrm{d}x = r^{2_s^*} \int_{\mathbb{R}^N} w^{2_s^*}\,\mathrm{d}x.$$

Assuming the Claim for a while, first we complete the proof.

Indeed, fix r as found in claim b) and set $t = r\sqrt{\beta/\alpha}$. Therefore, by Lemma 5.0.2, (rw, tw) achieves $S_{\alpha,\beta}$. Consequently, (rw, tw) solves $(\mathcal{S}^0_{2_s^*})$ with $f = 0 = g$ and

$$\frac{\alpha}{2_s^*} r^{\alpha-2} t^{\beta} = 1 = \frac{\beta}{2_s^*} r^{\alpha} t^{\beta-2}. \tag{5.3.3}$$

Now define $(u_1, v_1) = (\frac{u_0}{r}, \frac{v_0}{t})$. Then, by Claim a) we have

$$\|u_1\|^2_{\dot{H}^s} = \frac{1}{r^2}\|u_0\|^2_{\dot{H}^s} = \frac{\alpha}{2_s^* r^2}\int_{\mathbb{R}^N} |u_0|^{\alpha}|v_0|^{\beta}\,\mathrm{d}x = \frac{\alpha r^{\alpha} t^{\beta}}{2_s^* r^2}\int_{\mathbb{R}^N} |w|^{2_s^*}\,\mathrm{d}x = \|w\|^2_{\dot{H}^s},$$

where for the last equality we have used (5.3.3). Similarly, it follows that

$$\|v_1\|^2_{\dot{H}^s} = \|w\|^2_{\dot{H}^s}.$$

Therefore

$$\|u_1\|^2_{\dot{H}^s} = \|w\|^2_{\dot{H}^s} = \|v_1\|^2_{\dot{H}^s}. \tag{5.3.4}$$

Further, using Claim b) in the definition of u_1 yields

$$\int_{\mathbb{R}^N} |u_1|^{2_s^*}\,\mathrm{d}x = \int_{\mathbb{R}^N} |w|^{2_s^*}\,\mathrm{d}x. \tag{5.3.5}$$

Combining (5.3.4) and (5.3.5), by the uniqueness result in the scalar case, see [48], we conclude that

$$u_1 = w, \text{ that is } u_0 = rw.$$

Now we prove that $v_1 = w$. Indeed, by Claim a)

$$\begin{aligned}\int_{\mathbb{R}^N} |w|^{2_s^*}\,\mathrm{d}x = \int_{\mathbb{R}^N} |u_1|^{\alpha}|v_1|^{\beta}\mathrm{d}x &\leq \left(\int_{\mathbb{R}^N} |u_1|^{2_s^*}\mathrm{d}x\right)^{\alpha/2_s^*}\left(\int_{\mathbb{R}^N} |v_1|^{2_s^*}\mathrm{d}x\right)^{\beta/2_s^*}\\ &= \left(\int_{\mathbb{R}^N} |w|^{2_s^*}\mathrm{d}x\right)^{\alpha/2_s^*}\left(\int_{\mathbb{R}^N} |v_1|^{2_s^*}\mathrm{d}x\right)^{\beta/2_s^*}.\end{aligned}$$

Consequently, $\|w\|_{L^{2_s^*}} \leq \|v_1\|_{L^{2_s^*}}$. Combining this with (5.3.4) and the fact that w achieves S, we obtain

$$S^{-1/2}\|v_1\|_{\dot{H}^s} = S^{-1/2}\|w\|_{\dot{H}^s} = \|w\|_{L^{2_s^*}} \leq \|v_1\|_{L^{2_s^*}} \leq S^{-1/2}\|v_1\|_{\dot{H}^s}.$$

Hence the inequality becomes equality in the above expression, i.e., v_1 achieves S. Again by the uniqueness result in the scalar case, we conclude that

$$v_1 = w, \text{ that is } v_0 = tw.$$

This proves Theorem 5.1.3. Now we are going to prove the Claim. First, we prove **Claim a)**.

Consider the following problem with a parameter $\mu > 0$

$$\begin{cases} (-\Delta)^s u = \dfrac{\mu\alpha}{2_s^*}|u|^{\alpha-2}u|v|^{\beta} \text{ in } \mathbb{R}^N, \\ (-\Delta)^s v = \dfrac{\mu\beta}{2_s^*}|v|^{\beta-2}v|u|^{\alpha} \text{ in } \mathbb{R}^N, \\ u,\, v > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{S}_\mu$}$$

Associated to $(\mathcal{S}_\mu)$, define the following min-max problem

$$B(\mu) := \inf_{(u,v)\in \dot{H}^s\times\dot{H}^s\setminus\{(0,0)\}} \max_{t>0} E_\mu(tu,tv),$$

where

$$E_\mu(u,v) := \frac{1}{2}\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s} - \frac{\mu}{2_s^*}\int_{\mathbb{R}^N} |u|^{\alpha}|v|^{\beta}\,\mathrm{d}x.$$

Note that there exists $t_\mu > 0$ such that

$$\max_{t>0} E_\mu(tu_0,tv_0) = E_\mu(t_\mu u_0, t_\mu v_0),$$

where t_μ satisfies

$$H(\mu,t_\mu) = 0 \quad \text{and} \quad H(\mu,t) := C - \mu D t^{2_s^*-2},$$

with

$$C = \|(u_0,v_0)\|^2_{\dot{H}^s\times\dot{H}^s} \text{ and } D = \int_{\mathbb{R}^N} |u_0|^{\alpha}|v_0|^{\beta}\,\mathrm{d}x.$$

Since (u_0, v_0) is a least energy solution of $(\mathcal{S}^0_{2^*_s})$ with $f = 0 = g$,

$$H(1,1) = 0,\ \frac{\partial}{\partial t}H(1,1) < 0 \text{ and } H(\mu, t_u) = 0.$$

By the implicit function theorem t_μ is a C^1 function near of $\mu = 1$, and

$$t'_\mu\ |_{\mu=1} = -\frac{\frac{\partial}{\partial\mu}H}{\frac{\partial}{\partial t}H}\bigg|_{\mu=1=t} = -\frac{1}{(2^*_s - 2)}.$$

By the Taylor formula around $\mu = 1$, we have

$$t_\mu(\mu) = 1 + t'_\mu(1)(\mu - 1) + O(|\mu - 1|^2)$$

Consequently,

$$t^2_\mu(\mu) = 1 + 2t'_\mu(1)(\mu - 1) + O(|\mu - 1|^2).$$

Further, as $H(\mu, t_\mu) = C - \mu D t_\mu^{2^*_s - 2} = 0$ and $C = D$, we have $t_\mu^{2^*_s-2} = \mu^{-1}$. Therefore,

$$\begin{aligned} B(\mu) \leq E_\mu(t_\mu u_0, t_\mu v_0) &= t^2_\mu\left(\frac{1}{2} - \frac{\mu t_\mu^{2^*_s-2}}{2^*_s}\right)\|(u_0, v_0)\|^2_{\dot{H}^s\times\dot{H}^s} \\ &= t^2_\mu \frac{s}{N}\|(u_0, v_0)\|^2_{\dot{H}^s\times\dot{H}^s} = t^2_\mu B(1) \\ &= B(1) - \frac{2}{(2^*_s - 2)}B(1)(\mu - 1) + O(|\mu - 1|^2). \end{aligned} \tag{5.3.6}$$

From the definition of $B(1)$, a direct computation yields

$$\begin{aligned} B(1) &= \inf_{(u,v)\in\dot{H}^s\times\dot{H}^s\setminus\{(0,0)\}} E_1(\tilde{t}u, \tilde{t}v), \quad \text{where} \quad \tilde{t} = \left(\frac{\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s}}{\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta \mathrm{d}x}\right)^{1/(2^*_s-2)} \\ &= \inf_{(u,v)\in\dot{H}^s\times\dot{H}^s\setminus\{(0,0)\}} \frac{s}{N}\left(\frac{\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s}}{\left(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta \mathrm{d}x\right)^{2/2^*_s}}\right)^{2^*_s/(2^*_s-2)} \\ &= \frac{s}{N}S_{\alpha,\beta}^{\frac{2^*_s}{2^*_s-2}} = \frac{s}{N}\left(\frac{\|(u_0,v_0)\|^2_{\dot{H}^s\times\dot{H}^s}}{\left(\int_{\mathbb{R}^N}|u_0|^\alpha|v_0|^\beta \mathrm{d}x\right)^{2/2^*_s}}\right)^{2^*_s/(2^*_s-2)} = \frac{sD}{N}. \end{aligned} \tag{5.3.7}$$

Substituting the above value of $B(1)$ in (5.3.6) yields

$$B(\mu) \leq B(1) - \frac{D}{2^*_s}(\mu - 1) + O(|\mu - 1|^2).$$

Thus

$$\frac{B(\mu)-B(1)}{\mu-1}\begin{cases}\le -\frac{D}{2_s^*}+O(|\mu-1|) & \text{if} \quad \mu>1\\ \ge -\frac{D}{2_s^*}+O(|\mu-1|) & \text{if} \quad \mu<1.\end{cases} \tag{5.3.8}$$

The first inequality in (5.3.8) implies $B'(1)\le -\dfrac{D}{2_s^*}$ and the second inequality in (5.3.8) implies $B'(1)\ge -\dfrac{D}{2_s^*}$. Hence,

$$B'(1)=-\frac{D}{2_s^*}=-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u_0|^\alpha|v_0|^\beta\,\mathrm{d}x. \tag{5.3.9}$$

On the other hand, proceeding as in (5.3.7), we derive that

$$\begin{aligned} B(\mu) &= \frac{s}{N}\frac{1}{\mu^{\frac{2}{2_s^*-2}}}\inf_{(u,v)\in\dot{H}^s\times\dot{H}^s\setminus\{(0,0)\}}\left(\frac{\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s}}{\left(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x\right)^{\frac{2}{2_s^*}}}\right)^{2_s^*/(2_s^*-2)}\\ &= \frac{s}{N}\frac{1}{\mu^{\frac{2}{2_s^*-2}}}S_{\alpha,\beta}^{\frac{2_s^*}{2_s^*-2}}. \end{aligned}$$

Since, (rw,tw) (for any $r,t>0$ with $r/t=\sqrt{\alpha/\beta}$) is also a ground state solution of $(\mathcal{S}^0_{2_s^*})$ with $f=0=g$, from the above expression of $B(\mu)$, we obtain

$$B(\mu)=\frac{s}{N}\frac{1}{\mu^{\frac{2}{2_s^*-2}}}r^\alpha t^\beta\int_{\mathbb{R}^N}|w|^{2_s^*}\mathrm{d}x \quad\implies\quad B'(1)=-\frac{r^\alpha t^\beta}{2_s^*}\int_{\mathbb{R}^N}|w|^{2_s^*}\,\mathrm{d}x.$$

Comparing this with (5.3.9), we conclude that

$$\int_{\mathbb{R}^N}|u_0|^\alpha|v_0|^\beta\,\mathrm{d}x=r^\alpha t^\beta\int_{\mathbb{R}^N}|w|^{2_s^*}\,\mathrm{d}x,$$

where $r,t>0$ are arbitrary with $r/t=\sqrt{\alpha/\beta}$. This proves Claim a).

Let us turn to the proof of **Claim b)**. Let μ_0 be as in Lemma 5.3.1. Consider the system (5.3.1) with $\mu\in(0,\mu_0)$ and define the following min-max problem

$$\tilde{B}(\mu):=\inf_{(u,v)\in\dot{H}^s\times\dot{H}^s\setminus\{(0,0)\}}\max_{t>0}\tilde{E}_\mu(tu,tv),$$

where

$$\tilde{E}_\mu(u,v) := \frac{1}{2}\|(u,v)\|^2_{\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)} - \frac{\mu}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}\,\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x.$$

Note that there exists $t_\mu > 0$ such that

$$\max_{t>0}\tilde{E}_\mu(tu_0,tv_0) = \tilde{E}_\mu(t_\mu u_0, t_\mu v_0),$$

where t_μ satisfies

$$\tilde{H}(\mu,t_\mu) = 0 \text{ and } \tilde{H}(\mu,t) = C - (\mu G + D)t^{2_s^*-2}$$

with

$$C = \|(u_0,v_0)\|^2_{\dot{H}^s\times\dot{H}^s} \quad G = \int_{\mathbb{R}^N}|u_0|^{2_s^*}\,\mathrm{d}x \quad \text{and} \quad D = \int_{\mathbb{R}^N}|u_0|^\alpha|v_0|^\beta\,\mathrm{d}x.$$

Since (u_0,v_0) is a ground state solution of $(\mathcal{S}^0_{2_s^*})$ with $f = 0 = g$,

$$\tilde{H}(0,1) = C - D = 0, \quad \frac{\partial}{\partial t}\tilde{H}(0,1) = -(2_s^*-2)D \quad \text{and} \quad \frac{\partial}{\partial\mu}\tilde{H}(0,1) = -G,$$

evaluated at $t = 1$ and $\mu = 0$. By the implicit function theorem t_μ is a C^1 function near of $\mu = 0$, and

$$t'_\mu\,|_{\mu=0} = -\frac{\frac{\partial}{\partial\mu}\tilde{H}}{\frac{\partial}{\partial t}\tilde{H}}\,|_{\mu=0,t=1} = -\frac{G}{(2_s^*-2)D}.$$

The Taylor formula around $\mu = 0$ and $t_\mu = 1$ yields

$$t_\mu(\mu) = 1 + \mu t'_\mu(0) + O(|\mu|^2),$$

consequently,

$$t_\mu^2(\mu) = 1 + 2\mu t'_\mu(0) + O(|\mu|^2).$$

Now $\tilde{B}(0) = B(1)$, where $B(.)$ is as defined in the proof of Claim a). Therefore, $\tilde{B}(0) = \dfrac{sD}{N}$.

Since $\tilde{H}(\mu,t_\mu) = C - (\mu G + D)t^{2_s^*-2} = 0$, and $C = D$ using an argument as before, it follows that

$$\begin{aligned}
\tilde{B}(\mu) \le \tilde{E}_\mu(t_\mu u_0, t_\mu v_0) &= \frac{t_\mu^2}{2}C - \frac{t_\mu^{2_s^*}}{2_s^*}(\mu G + D) \\
&= t_\mu^2 \frac{sD}{N} = t_\mu^2 \tilde{B}(0) \\
&= \tilde{B}(0) - \frac{2G}{(2_s^* - 2)D}\mu \tilde{B}(0) + O(|\mu|^2) \\
&= \tilde{B}(0) - \frac{1}{2_s^*} G\mu + O(|\mu|^2).
\end{aligned}$$

Then

$$\tilde{B}'(0) = \lim_{\mu \to 0} \frac{\tilde{B}(\mu) - \tilde{B}(0)}{\mu} = -\frac{G}{2_s^*} = -\frac{1}{2_s^*}\int_{\mathbb{R}^N} |u_0|^{2_s^*}\, \mathrm{d}x. \tag{5.3.10}$$

On the other hand, from the definition of $\tilde{B}(\mu)$, a straight forward computation yields

$$\begin{aligned}
\tilde{B}(\mu) = & \inf_{(u,v)\in \dot{H}^s \times \dot{H}^s \setminus \{(0,0)\}} \tilde{E}_\mu(\tilde{t}u, \tilde{t}v), \\
& \qquad \text{where } \tilde{t} = \left(\frac{\|(u,v)\|^2_{\dot{H}^s \times \dot{H}^s}}{\mu \int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x + \int_{\mathbb{R}^N} |u|^\alpha |u|^\beta \mathrm{d}x} \right)^{1/(2_s^*-2)}, \\
= & \frac{s}{N} \inf_{(u,v)\in \dot{H}^s \times \dot{H}^s \setminus \{(0,0)\}} \left[\frac{\|(u,v)\|^2}{\left(\mu \int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x + \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \mathrm{d}x\right)^{2/2_s^*}} \right]^{2_s^*/(2_s^*-2)}.
\end{aligned}$$

Since by Lemma 5.3.1, $S_{\mu,\alpha,\beta}$ is achieved by $(rw, \tau_0 rw)$, an easy computation yields

$$\tilde{B}(\mu) = \frac{s}{N}\left(\frac{1+\tau_0^2}{(\mu + \tau_0^\beta)^{2/2_s^*}} \right)^{2_s^*/(2_s^*-2)} \int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x.$$

As a consequence,

$$\tilde{B}'(0) = -\frac{1}{2_s^*}\left(\frac{1+\tau_0^2}{\tau_0^\beta} \right)^{2_s^*/(2_s^*-2)} \int_{\mathbb{R}^N} |w|^{2_s^*}\, \mathrm{d}x.$$

Now set

$$\tilde{r} = \left(\frac{1+\tau_0^2}{\tau_0^\beta} \right)^{1/(2_s^*-2)}.$$

Therefore, $\tilde{B}'(0) = -\frac{\tilde{r}^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N} |w|^{2_s^*}\,\mathrm{d}x$. Comparing this with (5.3.10) yields

$$\int_{\mathbb{R}^N} |u_0|^{2_s^*}\,\mathrm{d}x = \tilde{r}^{2_s^*}\int_{\mathbb{R}^N} |w|^{2_s^*}\,\mathrm{d}x.$$

This proves Claim b). Thus, we conclude the proof of Theorem 5.1.3. □

5.4 The Palais-Smale decomposition

In this section we study the Palais-Smale sequences (in short, (PS) sequences) of the functional associated to $(\mathcal{S}^0_{2_s^*})$, namely,

$$I_{f,g}(u,v) := \frac{1}{2}\|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s} - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s}. \tag{5.4.1}$$

We say that the sequence $\{(u_n,v_n)\}\subset \dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ is a (PS) sequence for $I_{f,g}$ at level β if $I_{f,g}(u_n,v_n)\to\beta$ and $I'_{f,g}(u_n,v_n)\to 0$ in $\Big(\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)\Big)'$. It is easy to see that the weak limit of a (PS) sequence of $I_{f,g}$ solves $(\mathcal{S}^0_{2_s^*})$ except the positivity.

However the main difficulty is that the (PS) sequence may not converge strongly and hence the weak limit can be zero even if $\beta>0$. The main purpose of this section is to classify (PS) sequences for the functional $I_{f,g}$.

Proposition 5.4.1. *Let $\{(u_n,v_n)\}\subset\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ be a (PS) sequence for $I_{f,g}$ at a level γ. Then there exists a subsequence (still denoted by $\{(u_n,v_n)\}$) for which the following hold :*
there exist an integer $k\geq 0$, sequences $\{x_n^i\}_n\subset\mathbb{R}^N$, $r_n^i>0$ for $1\leq i\leq k$, pair of functions (u,v), $(\tilde{u}_i,\tilde{v}_i)\in\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ for $1\leq i\leq k$ such that (u,v) satisfies $(\mathcal{S}^0_{2_s^*})$ without the signed restrictions and

$$\begin{cases}(-\Delta)^s\tilde{u}_i = \frac{\alpha}{2_s^*}|\tilde{u}_i|^{\alpha-2}\tilde{u}_i|\tilde{v}_i|^\beta \quad in\ \mathbb{R}^N,\\ (-\Delta)^s\tilde{v}_i = \frac{\beta}{2_s^*}|\tilde{v}_i|^{\beta-2}\tilde{v}_i|\tilde{u}_i|^\alpha \quad in\ \mathbb{R}^N,\end{cases} \tag{5.4.2}$$

$$\begin{aligned} (u_n, v_n) &= (u, v) + \sum_{i=1}^{k} (\tilde{u}_i, \tilde{v}_i)^{r_n^i, x_n^i} + o(1), \\ \textit{where } (\tilde{u}_i, \tilde{v}_i)^{r,y} &:= r^{-\frac{N-2s}{2}} \Big(\tilde{u}_i(\tfrac{x-y}{r}), \tilde{v}_i(\tfrac{x-y}{r})\Big) \\ \textit{and } \ o(1) &\to 0 \quad \textit{in } \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N), \\ \gamma &= I_{f,g}(u, v) + \sum_{i=1}^{k} I_{0,0}(\tilde{u}_i, \tilde{v}_i) + o(1), \end{aligned} \tag{5.4.3}$$

$$\begin{aligned} r_n^i \to 0 \textit{ and either } x_n^i \to x^i \in \mathbb{R}^N \textit{ or } |x_n^i| \to \infty, \quad 1 \le i \le k, \\ \left|\log\left(\frac{r_n^i}{r_n^j}\right)\right| + \left|\frac{x_n^i - x_n^j}{r_n^i}\right| \longrightarrow \infty \quad \textit{for } i \neq j,\ 1 \le i, j \le k, \end{aligned} \tag{5.4.4}$$

where in the case $k = 0$ the above expressions hold without $(\tilde{u}_i, \tilde{v}_i)$, x_n^i and r_n^i.

Remark 5.4.2. From Proposition 5.4.1, we see that if $\{(u_n, v_n)\}$ is any nonnegative (PS) sequence for $I_{f,g}$ at level γ, then $\{(u_n, v_n)\}$ satisfies the (PS) condition if γ can not be decomposed as $\gamma = I_{f,g}(u, v) + \sum_{i=1}^{k} I_{0,0}(\tilde{u}_i, \tilde{v}_i)$, where $k \ge 1$ and $(\tilde{u}_i, \tilde{v}_i)$ is a solution of (5.4.2).

Before starting the proof of this proposition, we prove some lemmas which will be used in proving Proposition 5.4.1.

Lemma 5.4.3. [38, Theorem 2] *Let $j : \mathbb{C} \to \mathbb{C}$ be a continuous function with $j(0) = 0$ and satisfy the following hypothesis that for every $\varepsilon > 0$, there exists two continuous functions φ_ε and ψ_ε such that*

$$|j(\tilde{a} + \tilde{b}) - j(\tilde{a})| \le \varepsilon \varphi_\varepsilon(\tilde{a}) + \psi_\varepsilon(\tilde{b}) \quad \forall\, \tilde{a}, \tilde{b} \in \mathbb{C}.$$

Further, let $f_n = f + g_n$ be a sequence of measurable functions from $\mathbb{R}^N$ to $\mathbb{C}$ such that

(i) $g_n \to 0$ a.e.

(ii) $j(f) \in L^1(\mathbb{R}^N)$.

(iii) $\int_{\mathbb{R}^N} \varphi_\varepsilon\big(g_n(x)\big)\mathrm{d}x \le C < \infty$, for some constant C, independent of ε and n.

(iv) $\int_{\mathbb{R}^N} \psi_\varepsilon(f(x))\mathrm{d}x < \infty$ *for all* $\varepsilon > 0$.

Then

$$\int_{\mathbb{R}^N} |j(f+g_n) - j(f) - j(g_n)|\mathrm{d}x \longrightarrow 0, \quad as \quad n \to \infty.$$

Lemma 5.4.4. *Let* $\alpha, \beta > 1$. *Then for every* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$\Big||x+a|^\alpha |y+b|^\beta - |x|^\alpha |y|^\beta\Big| \le \varepsilon(|x|^{\alpha+\beta} + |y|^{\alpha+\beta}) + C_\varepsilon(|a|^{\alpha+\beta} + |b|^{\alpha+\beta})$$

holds for all x, y, a, $b \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$ be arbitrary. Then there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
|x+a|^\alpha |y+b|^\beta - |x|^\alpha |y|^\beta &= |y+b|^\beta(|x+a|^\alpha - |x|^\alpha) + |x|^\alpha(|y+b|^\beta - |y|^\beta) \\
&\le 2^{\beta-1}(|y|^\beta + |b|^\beta)\left(\frac{\varepsilon/2}{2^\beta - 1}|x|^\alpha + C_\varepsilon |a|^\alpha\right) \\
&\quad + |x|^\alpha\left(\frac{\varepsilon}{2}|y|^\beta + C_\varepsilon |b|^\beta\right) \\
&\le \varepsilon\left(|x|^\alpha |y|^\beta + \frac{1}{2}|b|^\beta |x|^\alpha\right) \\
&\quad + C_\varepsilon(|x|^\alpha |b|^\beta + |y|^\beta |a|^\alpha + |a|^\alpha |b|^\beta) \\
&\le \varepsilon(|x|^{\alpha+\beta} + |y|^{\alpha+\beta}) + C'_\varepsilon(|a|^{\alpha+\beta} + |b|^{\alpha+\beta}),
\end{aligned}$$

where in the last inequality we have used Young's inequality with different ε. This completes the proof. □

Lemma 5.4.5. *If* $u_n \rightharpoonup u$ *and* $v_n \rightharpoonup v$ *in* $\dot{H}^s(\mathbb{R}^N)$. *Then*

$$\int_{\mathbb{R}^N} \left(|u_n|^\alpha |v_n|^\beta - |u|^\alpha |v|^\beta - |u_n - u|^\alpha |v_n - v|^\beta\right)\mathrm{d}x = o(1).$$

Proof. Define $j : \mathbb{R}^2 \to \mathbb{R}$ defined by $j(x,y) = |x|^\alpha |y|^\beta$. Then j satisfies the hypothesis of Lemma 5.4.3. Next considering

$$f_n := (u_n, v_n), \quad f = (u,v), \quad g_n = (u_n - u, v_n - v),$$

we see that all the hypothesis of Lemma 5.4.3 are satisfied. Hence the lemma follows from Lemma 5.4.3. □

Lemma 5.4.6. *Let $\{(u_n, v_n)\}$ weakly converge to (u, v) in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ and pointwise a.e. in $\mathbb{R}^N \times \mathbb{R}^N$, then*

$$\int_{\mathbb{R}^N} |u_n|^{\alpha-2} u_n |v_n|^\beta \phi \, \mathrm{d}x \longrightarrow \int_{\mathbb{R}^N} |u|^{\alpha-2} u |v|^\beta \phi \, \mathrm{d}x \quad \text{as } n \to \infty, \tag{5.4.5}$$

$$\int_{\mathbb{R}^N} |u_n|^{\alpha} |v_n|^{\beta-2} v_n \psi \, \mathrm{d}x \longrightarrow \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta-2} v \psi \, \mathrm{d}x \quad \text{as } n \to \infty, \tag{5.4.6}$$

for all $(\phi, \psi) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$.

Proof. Set

$$M := \max\left\{ \|u_n\|^{\alpha-1}_{L^{2^*_s}(\mathbb{R}^N)}, \; \|v_n\|^{\beta}_{L^{2^*_s}(\mathbb{R}^N)} \; \|u\|^{\alpha-1}_{L^{2^*_s}(\mathbb{R}^N)} \; \|v\|^{\beta}_{L^{2^*_s}(\mathbb{R}^N)} \right\}.$$

Using the Sobolev inequality we see that M is well-defined. Let $\phi \in \dot{H}^s(\mathbb{R}^N)$ and $\varepsilon > 0$ be arbitrary. Then, there exists $R = R(\varepsilon) > 0$ such that $\left(\int_{B(0,R)^c} |\phi|^{2^*_s} \mathrm{d}x \right)^{\frac{1}{2^*_s}} < \frac{\varepsilon}{2M^2}$. Note that,

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|u_n|^{\alpha-2} u_n |v_n|^\beta \right. & - \left. |u|^{\alpha-2} u |v|^\beta \right) \phi \, \mathrm{d}x \\ & = \left(\int_{B(0,R)} + \int_{B(0,R)^c} \right) \left(|u_n|^{\alpha-2} u_n |v_n|^\beta - |u|^{\alpha-2} u |v|^\beta \right) \phi \end{aligned}$$

and using Hölder inequality

$$\begin{aligned} & \int_{B(0,R)^c} \left(|u_n|^{\alpha-2} u_n |v_n|^\beta - |u|^{\alpha-2} u |v|^\beta \right) \phi \, \mathrm{d}x \\ \leq & \left(\int_{\mathbb{R}^N} |u_n|^{2^*_s} \mathrm{d}x \right)^{(\alpha-1)/2^*_s} \left(\int_{\mathbb{R}^N} |v_n|^{2^*_s} \mathrm{d}x \right)^{\beta/2^*_s} \left(\int_{B(0,R)^c} |\phi|^{2^*_s} \mathrm{d}x \right)^{1/2^*_s} \\ & + \left(\int_{\mathbb{R}^N} |u|^{2^*_s} \mathrm{d}x \right)^{(\alpha-1)/2^*_s} \left(\int_{\mathbb{R}^N} |v|^{2^*_s} \mathrm{d}x \right)^{\beta/2^*_s} \left(\int_{B(0,R)^c} |\phi|^{2^*_s} \mathrm{d}x \right)^{1/2^*_s} \\ < & \ \varepsilon. \end{aligned}$$

On the other hand, using Hölder inequality as above, it is also easily checked that $\left(|u_n|^{\alpha-2} u_n |v_n|^\beta - |u|^{\alpha-2} u |v|^\beta \right) \phi$ is equi-integrable in $B(0, R)$. Therefore, applying Vitaly's convergence theorem it follows that

$$\lim_{n \to \infty} \int_{B(0,R)} \left(|u_n|^{\alpha-2} u_n |v_n|^\beta - |u|^{\alpha-2} u |v|^\beta \right) \phi \, \mathrm{d}x = 0.$$

Hence the lemma follows. □

Proof of Proposition 5.4.1:

Proof. We divide the proof into few steps.

Step 1: Using standard arguments it follows that (PS) sequences for $I_{f,g}$ are bounded in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$. More precisely, as $n\to\infty$

$$\begin{aligned}
&\gamma+o(1)+o(1)\|(u_n,v_n)\|_{\dot{H}^s\times\dot{H}^s}\\
\geq\ & I_{f,g}(u_n,v_n)\ -\ \frac{1}{2_s^*}\,{}_{(\dot{H}^s\times\dot{H}^s)'}\langle I'_{f,g}(u_n,v_n),(u_n,v_n)\rangle_{\dot{H}^s\times\dot{H}^s}\\
\geq\ & \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|(u_n,v_n)\|^2_{\dot{H}^s\times\dot{H}^s}\\
&\qquad-\left(1-\frac{1}{2_s^*}\right)\left(\|f\|_{(\dot{H}^s)'}\|u_n\|_{\dot{H}^s}+\|g\|_{(\dot{H}^s)'}\|v_n\|_{\dot{H}^s}\right)\\
\geq\ & \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|(u_n,v_n)\|^2_{\dot{H}^s\times\dot{H}^s}\\
&\qquad-\left(1-\frac{1}{2_s^*}\right)(\|f\|_{(\dot{H}^s)'}+\|g\|_{(\dot{H}^s)'})\|(u_n,v_n)\|_{\dot{H}^s\times\dot{H}^s}.
\end{aligned}$$

This immediately implies that $\{(u_n,v_n)\}$ is bounded in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$. Consequently, up to a subsequence, $(u_n,v_n)\rightharpoonup(u,v)$ in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$. Further, ${}_{(\dot{H}^s\times\dot{H}^s)'}\langle I'_{f,g}(u_n,v_n),(\phi,\psi)\rangle_{\dot{H}^s\times\dot{H}^s}\to 0$ implies

$$\begin{aligned}
&\Big\langle(u_n,v_n),(\phi,\psi)\Big\rangle_{\dot{H}^s\times\dot{H}^s}-\frac{\alpha}{2_s^*}\int_{\mathbb{R}^N}|u_n|^{\alpha-2}u_n|v_n|^{\beta}\phi\,\mathrm{d}x\\
&\qquad-\frac{\beta}{2_s^*}\int_{\mathbb{R}^N}|v_n|^{\beta-2}v_n|u_n|^{\alpha}\psi\,\mathrm{d}x-{}_{(\dot{H}^s)'}\langle f,\phi\rangle_{\dot{H}^s}-{}_{(\dot{H}^s)'}\langle g,\psi\rangle_{\dot{H}^s}=o(1).
\end{aligned}$$

Passing to the limit using Lemma 5.4.6, we see that (u,v) satisfies $(\mathcal{S}^0_{2_s^*})$ without signed restrictions.

Step 2: In this step we show that $\{(u_n-u,v_n-v)\}$ is a (PS) sequence for $I_{0,0}$ at the level $\gamma-I_{f,g}(u,v)$

To see this, first we observe that as $n\to\infty$

$$\|u_n-u\|_{\dot{H}^s}=\|u_n\|^2_{\dot{H}^s}-\|u\|^2_{\dot{H}^s}+o(1),\quad \|v_n-v\|^2_{\dot{H}^s}=\|v_n\|^2_{\dot{H}^s}-\|v\|^2_{\dot{H}^s}+o(1).$$

Using this along with the fact that $(u_n, v_n) \rightharpoonup (u, v)$, $f,\ g \in (\dot{H}^s(\mathbb{R}^N))'$ and Lemma 5.4.5 yields

$$
\begin{aligned}
& I_{0,0}(u_n - u, v_n - v) \\
= \;& \frac{1}{2}\|(u_n - u, v_n - v)\|^2_{\dot{H}^s \times \dot{H}^s} - \frac{1}{2^*_s}\int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta \mathrm{d}x \\
= \;& \frac{1}{2}(\|u_n\|^2_{\dot{H}^s} - \|u\|^2_{\dot{H}^s}) + \frac{1}{2}(\|v_n\|^2_{\dot{H}^s} - \|v\|^2_{\dot{H}^s}) - \frac{1}{2^*_s}\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \mathrm{d}x \\
& + \frac{1}{2^*_s}\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u_n\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v_n\rangle_{\dot{H}^s} \\
& + {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} + {}_{(\dot{H}^s)'}\langle g, v\rangle_{\dot{H}^s} \\
& + \frac{1}{2^*_s}\int_{\mathbb{R}^N} \left\{|u_n|^\alpha |v_n|^\beta - |u|^\alpha |v|^\beta - |u_n - u|^\alpha |v_n - v|^\beta\right\} \mathrm{d}x + o(1) \\
= \;& I_{f,g}(u_n, v_n) - I_{f,g}(u, v) + o(1).
\end{aligned}
$$

Next, as $(u_n - u, v_n - v) \rightharpoonup (0, 0)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, applying Lemma 5.4.6, we obtain

$$
\begin{aligned}
& {}_{(\dot{H}^s \times \dot{H}^s)'}\langle I'_{0,0}(u_n - u, v_n - v),\ (\phi, \psi)\rangle_{\dot{H}^s \times \dot{H}^s} \\
= \;& \langle (u_n - u, v_n - v), (\phi, \psi)\rangle_{\dot{H}^s \times \dot{H}^s} - \frac{\alpha}{2^*_s}\int_{\mathbb{R}^N} |u_n - u|^{\alpha-2}(u_n - u)|v_n - v|^\beta \phi \,\mathrm{d}x \\
& \quad - \frac{\beta}{2^*_s}\int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^{\beta-2}(v_n - v)\psi \,\mathrm{d}x \\
= \;& o(1). \qquad (5.4.7)
\end{aligned}
$$

This completes Step 2.

Step 3: Rescaling of $\{(u_n, v_n)\}_n$ in the nontrivial case.

If $(u_n, v_n) \to (u, v)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, then the theorem is proved with $k = 0$. Therefore, we assume $(u_n, v_n) \not\to (u, v)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$. Set,

$$\tilde{u}_n := u_n - u, \quad \tilde{v}_n := v_n - v.$$

Therefore, we are in the case where $(\tilde{u}_n, \tilde{v}_n) \not\longrightarrow (0, 0)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$. Since, by Step 2, $\{(\tilde{u}_n, \tilde{v}_n)\}$ is a bounded (PS) sequence for $I_{0,0}$, we have

$I'_{0,0}(\tilde{u}_n,\tilde{v}_n)(\tilde{u}_n,\tilde{v}_n) = o(1)$. Therefore, up to a subsequence

$$\begin{aligned} 0 < \|(\tilde{u}_n,\tilde{v}_n)\|^2_{\dot{H}^s\times\dot{H}^s} &= \int_{\mathbb{R}^N} |\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta \mathrm{d}x \\ &\le \int_{\mathbb{R}^N} |\tilde{u}_n|^{2_s^*}\mathrm{d}x + \int_{\mathbb{R}^N} |\tilde{v}_n|^{2_s^*}\mathrm{d}x \le \|(\tilde{u}_n,\tilde{v}_n)\|^{2_s^*}_{L^{2_s^*}\times L^{2_s^*}}. \end{aligned}$$

Thus $(\tilde{u}_n,\tilde{v}_n) \not\longrightarrow (0,0)$ in $L^{2_s^*}(\mathbb{R}^N)\times L^{2_s^*}(\mathbb{R}^N)$. Consequently,

$$\inf_n \|(\tilde{u}_n,\tilde{v}_n)\|_{L^{2_s^*}\times L^{2_s^*}} \ge \delta \quad \text{for some} \quad \delta > 0.$$

Hence, applying Lemma 2.3.2,

$$\delta \le C\|(\tilde{u}_n,\tilde{v}_n)\|^\theta_{\dot{H}^s\times\dot{H}^s}\|(\tilde{u}_n,\tilde{v}_n)\|^{1-\theta}_{L^{2,(N-2s)}\times L^{2,(N-2s)}} \le C'\|(\tilde{u}_n,\tilde{v}_n)\|^{1-\theta}_{L^{2,(N-2s)}\times L^{2,(N-2s)}},$$

that is,

$$\|(\tilde{u}_n,\tilde{v}_n)\|_{L^{2,(N-2s)}\times L^{2,(N-2s)}} \ge C_1 \quad \text{for some} \quad C_1 > 0.$$

Comparing the above inequality with (2.3.4) yields existence of some $\bar{C} > 0$ such that

$$\bar{C} \le \|(\tilde{u}_n,\tilde{v}_n)\|^2_{L^{2,(N-2s)}\times L^{2,(N-2s)}} \le \bar{C}^{-1},$$

that is

$$\bar{C} \le \sup_{x\in\mathbb{R}^N,\, R>0} R^{N-2s} \fint_{B(x,R)} \Big(|\tilde{u}_n|^2 + |\tilde{v}_n|^2\Big)\mathrm{d}y \le \bar{C}^{-1}.$$

As a result, for every $n\in\mathbb{N}$, there exists $x_n\in\mathbb{R}^N$ and $r_n > 0$ such that

$$r_n^{N-2s} \fint_{B(x_n,r_n)} \Big(|\tilde{u}_n|^2 + |\tilde{v}_n|^2\Big)\mathrm{d}y \ge \|(\tilde{u}_n,\tilde{v}_n)\|_{L^{2,(N-2s)}\times L^{2,(N-2s)}} - \frac{\bar{C}}{2n} \ge \frac{\bar{C}}{2} > 0. \tag{5.4.8}$$

Now define

$$\tilde{\tilde{u}}_n := r_n^{\frac{N-2s}{2}}\tilde{u}_n(r_nx + x_n), \quad \tilde{\tilde{v}}_n := r_n^{\frac{N-2s}{2}}\tilde{v}_n(r_nx + x_n).$$

In view of the scaling invariance of the $\dot{H}^s(\mathbb{R}^N)$ norm and $L^{2_s^*}(\mathbb{R}^N)$ norm, $\{(\tilde{\tilde{u}}_n,\tilde{\tilde{v}}_n)\}$ is a bounded sequence in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ and up to a subsequence

$$(\tilde{\tilde{u}}_n,\tilde{\tilde{v}}_n) \rightharpoonup (\tilde{u},\tilde{v}) \quad \text{in } \dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N) \text{ and } (\tilde{\tilde{u}}_n,\tilde{\tilde{v}}_n) \longrightarrow (\tilde{u},\tilde{v}) \text{ in } L^2_{loc}(\mathbb{R}^N)\times L^2_{loc}(\mathbb{R}^N).$$

Therefore, using change of variable, we observe from (5.4.8) that

$$\begin{aligned} 0 &< r_n^{-2s}\int_{B(x_n,r_n)} \Big(|\tilde{u}_n|^2+|\tilde{v}_n|^2\Big)\mathrm{d}y \\ &= \int_{B(0,1)} \Big(|\tilde{\tilde{u}}_n(z)|^2+|\tilde{\tilde{v}}_n(z)|^2\Big)\mathrm{d}z \longrightarrow \int_{B(0,1)} (|\tilde{u}|^2+|\tilde{v}|^2)\mathrm{d}x. \end{aligned}$$

Hence $(\tilde{u},\tilde{v}) \neq (0,0)$. Clearly, up to a subsequence, either $x_n \to x_0 \in \mathbb{R}^N$ or $|x_n| \to \infty$. Further, as $(\tilde{\tilde{u}}_n,\tilde{\tilde{v}}_n) \rightharpoonup (\tilde{u},\tilde{v}) \neq (0,0)$ in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ and $(\tilde{u}_n,\tilde{v}_n) \rightharpoonup (0,0)$ in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$, we infer that $r_n \to 0$.

Step 4: In this step we prove that $(\tilde{u},\tilde{v})$ solves

$$\begin{cases} (-\Delta)^s\tilde{u} = \dfrac{\alpha}{2_s^*}|\tilde{u}|^{\alpha-2}\tilde{u}|\tilde{v}|^{\beta} \ \text{ in } \mathbb{R}^N, \\ (-\Delta)^s\tilde{v} = \dfrac{\beta}{2_s^*}|\tilde{u}|^{\alpha}|\tilde{v}|^{\beta-2}\tilde{v} \ \text{ in } \mathbb{R}^N. \end{cases} \tag{5.4.9}$$

To this aim, it is enough to show that for arbitrary $(\varphi,\psi) \in C_c^\infty(\mathbb{R}^N)\times C_c^\infty(\mathbb{R}^N)$ it holds

$$\langle \tilde{u},\varphi\rangle_{\dot{H}^s} + \langle \tilde{v},\psi\rangle_{\dot{H}^s} = \frac{\alpha}{2_s^*}\int_{\mathbb{R}^N} |\tilde{u}|^{\alpha-2}\tilde{u}|\tilde{v}|^{\beta}\varphi\,\mathrm{d}x + \frac{\beta}{2_s^*}\int_{\mathbb{R}^N} |\tilde{u}|^{\alpha}|\tilde{v}|^{\beta-2}\tilde{v}\psi\,\mathrm{d}x.$$

Let $\varphi,\psi \in C_c^\infty(\mathbb{R}^N)$ be arbitrary. As $(\tilde{\tilde{u}}_n,\tilde{\tilde{v}}_n) \rightharpoonup (\tilde{u},\tilde{v})$ in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$, using change of variables and Step 2, that is $\{(\tilde{u}_n,\tilde{v}_n)\}$ is a (PS) sequence for $I_{0,0}$, we deduce

$$\begin{aligned} &\langle \tilde{u},\,\varphi\rangle_{\dot{H}^s} + \langle \tilde{v},\,\psi\rangle_{\dot{H}^s} \\ &= \lim_{n\to\infty}\Big(\langle \tilde{\tilde{u}}_n,\varphi\rangle_{\dot{H}^s} + \langle \tilde{\tilde{v}}_n,\,\psi\rangle_{\dot{H}^s}\Big) \\ &= \lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{r_n^{\frac{N-2s}{2}}\Big(\tilde{u}_n(r_nx+x_n)-\tilde{u}_n(r_ny+x_n)\Big)\Big(\varphi(x)-\varphi(y)\Big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &\quad + \lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{r_n^{\frac{N-2s}{2}}\Big(\tilde{v}_n(r_nx+x_n)-\tilde{v}_n(r_ny+x_n)\Big)\Big(\psi(x)-\psi(y)\Big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &= \lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{r_n^{-\frac{N-2s}{2}}\Big(\tilde{u}_n(x)-\tilde{u}_n(y)\Big)\Big(\varphi\big(\frac{x-x_n}{r_n}\big)-\varphi\big(\frac{y-x_n}{r_n}\big)\Big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$+\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{r_n^{-\frac{N-2s}{2}}\Big(\tilde{v}_n(x)-\tilde{v}_n(y)\Big)\Big(\psi\Big(\frac{x-x_n}{r_n}\Big)-\psi\Big(\frac{y-x_n}{r_n}\Big)\Big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$=\lim_{n\to\infty}\left[\frac{\alpha}{2_s^*}\int_{\mathbb{R}^N}|\tilde{u_n}|^{\alpha-2}\tilde{u_n}|\tilde{v_n}|^{\beta}\tilde{\varphi}_n\mathrm{d}x+\frac{\beta}{2_s^*}\int_{\mathbb{R}^N}|\tilde{u_n}|^{\alpha}|\tilde{v_n}|^{\beta-2}\tilde{v_n}\tilde{\psi}_n\mathrm{d}x\right],\tag{5.4.10}$$

where

$$\tilde{\varphi}_n(x):=r_n^{-\frac{N-2s}{2}}\varphi\Big(\frac{x-x_n}{r_n}\Big)\quad\text{and}\quad\tilde{\psi}_n(x):=r_n^{-\frac{N-2s}{2}}\psi\Big(\frac{x-x_n}{r_n}\Big).$$

Again applying change of variable to (5.4.10) yields us

$$\begin{aligned}\text{RHS of (5.4.10)} &= \lim_{n\to\infty}\left[\frac{\alpha}{2_s^*}\int_{\mathbb{R}^N}|\tilde{\tilde{u}}_n|^{\alpha-2}\tilde{\tilde{u}}_n|\tilde{\tilde{v}}_n|^{\beta}\varphi\,\mathrm{d}x+\frac{\beta}{2_s^*}\int_{\mathbb{R}^N}|\tilde{\tilde{u}}_n|^{\alpha}|\tilde{\tilde{v}}_n|^{\beta-2}\tilde{\tilde{v}}_n\psi\,\mathrm{d}x\right]\\ &= \frac{\alpha}{2_s^*}\int_{\mathbb{R}^N}|\tilde{u}|^{\alpha-2}\tilde{u}|\tilde{v}|^{\beta}\varphi\,\mathrm{d}x+\frac{\beta}{2_s^*}\int_{\mathbb{R}^N}|\tilde{u}|^{\alpha}|\tilde{v}|^{\beta-2}\tilde{v}\psi\,\mathrm{d}x,\end{aligned}$$

where the last equality is obtained by Lemma 5.4.6. This completes Step 4.

Now define,

$$w_n(x):=\tilde{u}_n(x)-r_n^{-\frac{N-2s}{2}}\tilde{u}\left(\frac{x-x_n}{r_n}\right)\quad\text{and}\quad z_n(x):=\tilde{v}_n(x)-r_n^{-\frac{N-2s}{2}}\tilde{v}\left(\frac{x-x_n}{r_n}\right).\tag{5.4.11}$$

Step 5: In this step we show that $\{(w_n,z_n)\}$ is a (PS) sequence for $I_{0,0}$ at the level $\gamma-I_{f,g}(u,v)-I_{0,0}(\tilde{u},\tilde{v})$.

To prove that, first we set

$$\tilde{w}_n:=r_n^{\frac{N-2s}{2}}w_n(r_nx+x_n),\quad\text{and}\quad\tilde{z}_n:=r_n^{\frac{N-2s}{2}}z_n(r_nx+x_n).\tag{5.4.12}$$

Combining (5.4.11) and (5.4.12) yields

$$\tilde{w}_n=\tilde{\tilde{u}}_n-\tilde{u},\quad\tilde{z}_n=\tilde{\tilde{v}}_n-\tilde{v},$$

and from the scaling invariance in the norm of $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ gives

$$\|(w_n,z_n)\|_{\dot{H}^s\times\dot{H}^s}=\|(\tilde{w}_n,\tilde{z}_n)\|_{\dot{H}^s\times\dot{H}^s}=\|(\tilde{\tilde{u}}_n-\tilde{u},\tilde{\tilde{v}}_n-\tilde{v})\|_{\dot{H}^s\times\dot{H}^s}.$$

A straight forward computation using the above equality, change of variables and Lemma 5.4.5 yields

$$\begin{aligned}
I_{0,0}(w_n, z_n) &= \frac{1}{2}\|\tilde{\tilde{u}}_n - \tilde{u}\|^2_{\dot{H}^s} + \frac{1}{2}\|\tilde{\tilde{v}}_n - \tilde{v}\|^2_{\dot{H}^s} - \frac{1}{2^*_s}\int_{\mathbb{R}^N} |w_n|^\alpha |z_n|^\beta \mathrm{d}x \\
&= \frac{1}{2}\Big(\|\tilde{\tilde{u}}_n\|^2_{\dot{H}^s} - \|\tilde{u}\|^2_{\dot{H}^s} + \|\tilde{\tilde{v}}_n\|^2_{\dot{H}^s} - \|\tilde{v}\|^2_{\dot{H}^s} + o(1)\Big) \\
&\quad - \frac{1}{2^*_s}\int_{\mathbb{R}^N} |\tilde{\tilde{u}}_n - \tilde{u}|^\alpha |\tilde{\tilde{v}}_n - \tilde{v}|^\beta \mathrm{d}x \\
&= \frac{1}{2}\|(\tilde{\tilde{u}}_n, \tilde{\tilde{v}}_n)\|^2_{\dot{H}^s\times\dot{H}^s} - \frac{1}{2}\|(\tilde{u}, \tilde{v})\|^2_{\dot{H}^s\times\dot{H}^s} \\
&\quad - \frac{1}{2^*_s}\left[\int_{\mathbb{R}^N} |\tilde{\tilde{u}}_n|^\alpha |\tilde{\tilde{v}}_n|^\beta \mathrm{d}x - \int_{\mathbb{R}^N} |\tilde{u}|^\alpha |\tilde{v}|^\beta \mathrm{d}x\right] + o(1) \\
&= I_{0,0}(\tilde{\tilde{u}}_n, \tilde{\tilde{v}}_n) - I_{0,0}(\tilde{u}, \tilde{v}) + o(1) \\
&= I_{0,0}(\tilde{u}_n, \tilde{v}_n) - I_{0,0}(\tilde{u}, \tilde{v}) + o(1) \\
&= \gamma - I_{f,g}(u, v) - I_{0,0}(\tilde{u}, \tilde{v}) + o(1),
\end{aligned}$$

where in the last equality we have used Step 2. Now, to complete the proof of Step 5, we left to show that $\langle I'_{0,0}(w_n, z_n)(\varphi, \psi)\rangle = 0$ for all $(\varphi, \psi) \in C^\infty_c(\mathbb{R}^N) \times C^\infty_c(\mathbb{R}^N)$. Let $(\varphi, \psi) \in C^\infty_c(\mathbb{R}^N) \times C^\infty_c(\mathbb{R}^N)$ be arbitrary and set

$$\varphi_n := r_n^{\frac{N-2s}{2}}\varphi(r_n x + x_n), \quad \psi_n := r_n^{\frac{N-2s}{2}}\psi(r_n x + x_n).$$

Thus $\varphi_n \rightharpoonup 0$ and $\psi_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$ as $r_n \to 0$. Observe that applying change of variables,

$$\begin{aligned}
\Big\langle (w_n, z_n), (\varphi, \psi)\Big\rangle_{\dot{H}^s\times\dot{H}^s} &= \langle w_n, \varphi\rangle_{\dot{H}^s} + \langle z_n, \psi\rangle_{\dot{H}^s} \\
&= \langle \tilde{w}_n, \varphi_n\rangle_{\dot{H}^s} + \langle \tilde{z}_n, \psi_n\rangle_{\dot{H}^s} \\
&= \langle \tilde{\tilde{u}}_n - \tilde{u}, \varphi_n\rangle_{\dot{H}^s} + \langle \tilde{\tilde{v}}_n - \tilde{v}, \psi_n\rangle_{\dot{H}^s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle I'_{0,0}(w_n, z_n)(\varphi, \psi)\rangle &= \langle \tilde{\tilde{u}}_n - \tilde{u}, \varphi_n\rangle_{\dot{H}^s} + \langle \tilde{\tilde{v}}_n - \tilde{v}, \psi_n\rangle_{\dot{H}^s} \\
&\quad - \frac{\alpha}{2^*_s}\int_{\mathbb{R}^N} |\tilde{\tilde{u}}_n - \tilde{u}|^{\alpha-2}(\tilde{\tilde{u}}_n - \tilde{u})|\tilde{\tilde{v}}_n - \tilde{v}|^\beta \varphi_n \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\beta}{2_s^*}\int_{\mathbb{R}^N} |\tilde{\tilde{u}}_n - \tilde{u}|^{\alpha}|\tilde{\tilde{v}}_n - \tilde{v}|^{\beta-2}(\tilde{\tilde{v}}_n - \tilde{u})\psi_n \mathrm{d}x \\
&= o(1),
\end{aligned}
$$

where the last equality follows by change of variable and an argument similar to Step 2. This concludes Step 5.

Now, starting from a (PS) sequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ for $I_{0,0}$ we have extracted another (PS) sequence $\{(w_n, z_n)\}$ at a level which is strictly lower than the previous one, with a fixed minimum amount of decrease (as it is easy to check that $I_{0,0}(\tilde{u}, \tilde{v}) \geq \frac{s}{N} S_{\alpha,\beta}^{\frac{N}{2s}}$). On the other hand, as $\sup_n \|(\tilde{u}_n, \tilde{v}_n)\|_{\dot{H}^s \times \dot{H}^s} \leq C$ (finite), this process should terminate after finitely many steps and the last (PS) sequence strongly converges to 0. Further, $\left|\log\left(\frac{r_n^i}{r_n^j}\right)\right| + \left|\frac{x_n^i - x_n^j}{r_n^i}\right| \longrightarrow \infty$ for $i \neq j$, $1 \leq i, j \leq m$ (see [95, Theorem 1.2]). This completes the proof.

□

5.5 Multiplicity in the nonhomogeneous case

In this section we aim to prove Theorem 5.1.4. For that first we would like to establish existence of two positive critical points of the functional

$$
J_{f,g}(u,v) = \frac{1}{2}\|(u,v)\|^2_{\dot{H}^s \times \dot{H}^s} - \frac{1}{2_s^*}\int_{\mathbb{R}^N} u_+^{\alpha} v_+^{\beta}\, \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v\rangle_{\dot{H}^s} \tag{5.5.1}
$$

where f, g are nontrivial nonneagtive functionals on $(\dot{H}^s(\mathbb{R}^N))'$ with $\ker(f) = \ker(g)$.

Remark 5.5.1. If $(u, v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ is a nontrivial critical point of $J_{f,g}$ then (u, v) solves

$$
\begin{cases}
(-\Delta)^s u = \dfrac{\alpha}{2_s^*} u_+^{\alpha-1} v_+^{\beta} + f(x) & \text{in } \mathbb{R}^N, \\
(-\Delta)^s v = \dfrac{\beta}{2_s^*} u_+^{\alpha} v_+^{\beta-1} + g(x) & \text{in } \mathbb{R}^N.
\end{cases} \tag{5.5.2}
$$

Note that taking $(\phi,\psi)=(u_-,v_-)$ as a test function in (5.5.2), we obtain

$$\begin{aligned}-\|(u_-,v_-)\|^2_{\dot{H}^s\times\dot{H}^s} &- \iint_{\mathbb{R}^{2N}} \frac{[u_+(y)u_-(x)+u_+(x)u_-(y)]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &- \iint_{\mathbb{R}^{2N}} \frac{[v_+(y)v_-(x)+v_+(x)v_-(y)]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &= {}_{(\dot{H}^s)'}\langle f,u_-\rangle_{H^s} + {}_{(\dot{H}^s)'}\langle g,v_-\rangle_{H^s} \geq 0.\end{aligned}$$

which in turn implies $u_-=0$ and $v_-=0$. Therefore $u,v\geq 0$ and (u,v) is a solution of $(\mathcal{S}^0_{2^*_s})$ without strict positivity condition.

Next, we assert that $(u,v)\neq(0,0)$ implies $u\neq 0$ and $v\neq 0$. Suppose not, that is assume for instance that $u\neq 0$ but $v=0$. Then taking $(\phi,\psi)=(u,0)$ as test function we get $\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} = {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s}$. Further choosing $(\phi,\psi)=(0,u)$ as test function, we have ${}_{(\dot{H}^s)'}\langle g,u\rangle_{\dot{H}^s}=0$. These together with the hypothesis that ker(f)=ker(g) implies $\|u\|_{\dot{H}^s}=0$. This contradicts $(u,v)\neq(0,0)$. Similarly we can show that if $u=0$ then $v=0$ too. Hence our assertion follows. Next, we claim that $u>0$, and $v>0$ in $\mathbb{R}^N$. Taking $(\phi,0)$ as test function where $\phi\geq 0$ in $\dot{H}^s(\mathbb{R}^N)$ we get,

$$\langle u,\phi\rangle_{\dot{H}^s} = \frac{\alpha}{2^*_s}\int_{\mathbb{R}^N} u^{\alpha-1}v^{\beta}\phi\,\mathrm{d}x + {}_{(\dot{H}^s)'}\langle f,\phi\rangle_{\dot{H}^s} \geq 0.$$

This implies $0\leq u\in\dot{H}^s(\mathbb{R}^N)$ is a weak supersolution to $(-\Delta)^s u=0$. Therefore applying maximum principle [56, Theorem 1.2(2)], with $c=0$ and $p=2$ there, we obtain $u>0$ in $\mathbb{R}^N$. Similarly we can show that $v>0$ in $\mathbb{R}^N$. Hence, if (u,v) is a critical point of $J_{f,g}$ then (u,v) is a solution of $(\mathcal{S}^0_{2^*_s})$.

To prove, existence of two critical points for $J_{f,g}$, first we partition the space $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ into three disjoint sets via the function $\Psi:\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$\Psi(u,v) := \|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s} - (2^*_s-1)\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\,\mathrm{d}x.$$

Set

$$\Omega_1 := \{(u,v)\in\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N) : (u,v)=(0,0) \text{ or } \Psi(u,v)>0\},$$

$$\Omega_2 := \{(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) : \Psi(u,v) < 0\},$$

$$\Omega := \{(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N) \setminus \{(0,0)\} : \Psi(u,v) = 0\}.$$

Put

$$c_0 := \inf_{\Omega_1} J_{f,g}(u,v), \quad c_1 := \inf_{\Omega} J_{f,g}(u,v). \tag{5.5.3}$$

Remark 5.5.2. Note that for all $\lambda > 0$ and $(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$

$$\Psi(\lambda u, \lambda v) = \lambda^2 \|(u,v)\|^2_{\dot{H}^s \times \dot{H}^s} - \lambda^{2^*_s}(2^*_s - 1) \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x.$$

Moreover, $\Psi(0,0) = 0$ and $\lambda \mapsto \Psi(\lambda u, \lambda v)$ is a strictly concave function in $\mathbb{R}^+$. Thus for any $(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ with $\|(u,v)\|_{\dot{H}^s \times \dot{H}^s} = 1$, there exists a unique λ (λ depends on (u,v)) such that $(\lambda u, \lambda v) \in \Omega$. Moreover as

$$\Psi(\lambda u, \lambda v) = (\lambda^2 - \lambda^{2^*_s}) \|(u,v)\|^2_{\dot{H}^s \times \dot{H}^s} \text{ for all } (u,v) \in \Omega,$$

$(\lambda u, \lambda v) \in \Omega_1$ for all $\lambda \in (0,1)$ and $(\lambda u, \lambda v) \in \Omega_2$ for all $\lambda > 1$.

Lemma 5.5.3. *Assume C_0 is defined as in Theorem 5.1.4 and c_0 and c_1 are defined as in* (5.5.3)*. Further, if*

$$\inf_{\substack{(u,v) \in \dot{H}^s \times \dot{H}^s, \\ \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x = 1}} \left\{ C_0 \|(u,v)\|^{\frac{N+2s}{2s}}_{\dot{H}^s \times \dot{H}^s} - {}_{(\dot{H}^s)'}\langle f, u \rangle_{H^s} - {}_{(\dot{H}^s)'}\langle g, v \rangle_{H^s} \right\} > 0, \tag{5.5.4}$$

then $c_0 < c_1$.

Proof. Step 1: First we assert that, there exists $\delta > 0$ such that

$$\frac{d}{dt} I_{f,g}(tu, tv) \Big|_{t=1} \geq \delta \quad \forall \ (u,v) \in \Omega.$$

Doing a straight forward computation, it is easy to see that for any $(u,v) \in \Omega$

$$\begin{aligned} \frac{d}{dt} \tilde{I}_{f,g}(tu, tv) \Big|_{t=1} &= \frac{4s}{N+2s} \|(u,v)\|^2_{\dot{H}^s \times \dot{H}^s} - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v \rangle_{\dot{H}^s} \\ &= C_0 \frac{\|(u,v)\|^{\frac{N+2s}{2s}}_{\dot{H}^s \times \dot{H}^s}}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x \right)^{\frac{N-2s}{4s}}} - {}_{(\dot{H}^s)'}\langle f, u \rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v \rangle_{\dot{H}^s} \end{aligned} \tag{5.5.5}$$

Further, (5.5.4) implies there exists $d > 0$ such that

$$\inf_{\substack{(u,v)\in \dot{H}^s\times\dot{H}^s,\\ \int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x=1}} \left\{ C_0\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}^{\frac{N+2s}{2s}} - {}_{(\dot{H}^s)'}\langle f,u\rangle_{H^s} - {}_{(\dot{H}^s)'}\langle g,v\rangle_{H^s} \right\} \geq d. \quad (5.5.6)$$

Now,

$$\begin{aligned}
(5.5.6) \iff\ & C_0 \frac{\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}^{\frac{N+2s}{2s}}}{\left(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x\right)^{\frac{N-2s}{4s}}} - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s} \geq d,\\
& \qquad\qquad\qquad\qquad\qquad \text{with } \int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x = 1.\\
\iff\ & C_0 \frac{\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}^{\frac{N+2s}{2s}}}{\left(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x\right)^{\frac{N-2s}{4s}}} - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s}\\
& \qquad\qquad \geq d\left(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x\right)^{1/2_s^*},\\
& \qquad \text{for all } (u,v)\in \dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)\setminus\{(0,0)\}.
\end{aligned}$$

Observe that $\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,\mathrm{d}x$ is bounded away from 0 for all $(u,v)\in\Omega$. Therefore, plugging back the above estimate into (5.5.5) proves Step 1.

Step 2: Let $\{(u_n,v_n)\}$ be a minimizing sequence for $J_{f,g}$ on Ω, i.e., $J_{f,g}(u_n,v_n)\to c_1$ and $\|(u_n,v_n)\|^2_{\dot{H}^s\times\dot{H}^s} = (2_s^*-1)\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x$. Therefore, for large n

$$\begin{aligned}
c_1+o(1) \geq J_{f,g}(u_n,v_n) \geq I_{f,g}(u_n,v_n) \geq & \left(\frac{1}{2}-\frac{1}{2_s^*(2_s^*-1)}\right)\|(u_n,v_n)\|^2_{\dot{H}^s\times\dot{H}^s}\\
& - \left(\|f\|_{(\dot{H}^s)'}+\|g\|_{(\dot{H})'}\right)\|(u_n,v_n)\|_{\dot{H}^s\times\dot{H}^s}.
\end{aligned}$$

This implies that $\{I_{f,g}(u_n,v_n)\}$ is a bounded sequence and $\{\|(u_n,v_n)\|_{\dot{H}^s\times\dot{H}^s}\}$ and $\left\{\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x\right\}$ are bounded.

Claim: $c_0 < 0$.

Observe that to prove the claim, it is sufficient to show that there exists $(u,v)\in\Omega_1$ such that $J_{f,g}(u,v) < 0$. Using Remark 5.5.2, we can choose

$(u,v) \in \Omega$ such that ${}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} + {}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s} > 0$. Therefore,

$$\begin{aligned} J_{f,g}(tu,tv) &= t^2\left[\frac{2_s^*-1}{2} - \frac{t^{2_s^*-2}}{2_s^*}\right]\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \,\mathrm{d}x \\ &\quad -t\,{}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} - t\,{}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s} < 0, \end{aligned}$$

for $t << 1$. Moreover, $(tu,tv) \in \Omega_1$ by Remark 5.5.2. Hence the claim follows.

Due to the above claim, $J_{f,g}(u_n,v_n) < 0$ for large n. Consequently, for large n

$$0 > J_{f,g}(u_n,v_n) \geq \left(\frac{1}{2} - \frac{1}{2_s^*(2_s^*-1)}\right)\|(u_n,v_n)\|^2_{\dot{H}^s\times\dot{H}^s} - {}_{(\dot{H}^s)'}\langle f,u_n\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g,v_n\rangle_{\dot{H}^s}.$$

This in turn implies ${}_{(\dot{H}^s)'}\langle f,u_n\rangle_{\dot{H}^s} + {}_{(\dot{H}^s)'}\langle g,v_n\rangle_{\dot{H}^s} > 0$ for all large n. Consequently, $\frac{d}{dt}I_{f,g}(tu_n,tv_n) < 0$ for $t > 0$ small enough. Thus, by Step 1, there exists $t_n \in (0,1)$ such that $\frac{d}{dt}I_{f,g}(t_nu_n,t_nv_n) = 0$. Since for all $(u,v) \in \Omega$, the function $\frac{d}{dt}I_{f,g}(tu,tv)$ is strictly increasing in $t \in [0,1)$, we can conclude that t_n is unique.

Step 3: In this step we show that

$$\liminf_{n\to\infty}\{I_{f,g}(u_n,v_n) - I_{f,g}(t_nu_n,t_nv_n)\} > 0. \tag{5.5.7}$$

Observe that $I_{f,g}(u_n,v_n) - I_{f,g}(t_nu_n,t_nv_n) = \int_{t_n}^1 \frac{d}{dt}\{I_{f,g}(tu_n,tv_n)\}\,\mathrm{d}t$ and that for all $n \in \mathbb{N}$ there is $\xi_n > 0$ such that $t_n \in (0,\ 1-2\xi_n)$ and $\frac{d}{dt}I_{f,g}(tu_n) \geq \delta/2$ for $t \in [1-\xi_n,\ 1]$.

To establish (5.5.7), it is enough to show that $\xi_n > 0$ can be chosen independent of $n \in \mathbb{N}$. This is possible as $\frac{d}{dt}I_{f,g}(tu_n,tv_n)|_{t=1} \geq \frac{\delta}{2}$ for $t \in [1-\xi_n, 1]$ and $\{(u_n,v_n)\}$ is bounded in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, so that for all $n \in \mathbb{N}$ and $t \in [0,1]$

$$\begin{aligned} \left|\frac{d^2}{dt^2}I_{f,g}(tu_n,tv_n)\right| &= \left|\|(u_n,v_n)\|^2_{\dot{H}^s\times\dot{H}^s} - (2_s^*-1)t^{2_s^*-2}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,\mathrm{d}x\right| \\ &= |1-t^{2_s^*-2}|\,\|(u_n,v_n)\|^2_{\dot{H}^s\times\dot{H}^s} \leq C, \end{aligned}$$

for all $n \geq 1$ and $t \in [0,\ 1]$.

Step 4: From the definition of $J_{f,g}$ and $I_{f,g}$, it immediately follows that $\frac{d}{dt}J_{f,g}(tu,tv) \geq \frac{d}{dt}I_{f,g}(tu,tv)$ for all $(u,v) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ and for all $t > 0$. Hence,

$$\begin{aligned} J_{f,g}(u_n,v_n) - J_{f,g}(t_nu_n,t_nv_n) &= \int_{t_n}^1 \frac{d}{dt}(J_{f,g}(tu_n,tv_n))\ \mathrm{d}t \\ &\geq \int_{t_n}^1 \frac{d}{dt}I_{f,g}(tu_n,tv_n)\ \mathrm{d}t \\ &= I_{f,g}(u_n,v_n) - I_{f,g}(t_nu_n,t_nv_n). \end{aligned}$$

Since $\{(u_n,v_n)\} \subset \Omega$ is a minimizing sequence for $J_{f,g}$ on Ω, and $(t_nu_n,t_nv_n) \in \Omega_1$, we conclude using (5.5.7) that

$$c_0 = \inf_{(u,v)\in\Omega_1} J_{f,g}(u,v) < \inf_{(u,v)\in\Omega} J_{f,g}(u,v) = c_1.$$

□

Proposition 5.5.4. *Assume that* (5.5.4) *holds. Then $J_{f,g}$ has a critical point $(u_0,v_0) \in \Omega_1$, with $J_{f,g}(u_0,v_0) = c_0$. In particular, (u_0,v_0) is a positive weak solution to* $(\mathcal{S}^0_{2^*_s})$.

Proof. We decompose the proof into few steps.

Step 1: $c_0 > -\infty$.

Since $J_{f,g}(u,v) \geq I_{f,g}(u,v)$, it is enough to show that $I_{f,g}$ is bounded from below. From the definition of Ω_1, it immediately follows that for all $(u,v) \in \Omega_1$,

$$I_{f,g}(u,v) \geq \left[\frac{1}{2} - \frac{1}{2^*_s(2^*_s-1)}\right] \|(u,v)\|^2_{\dot{H}^s\times\dot{H}^s} - (\|f\|_{(\dot{H}^s)'} + \|g\|_{(\dot{H})'})\|(u_n,v_n)\|_{\dot{H}^s\times\dot{H}^s}. \tag{5.5.8}$$

As RHS is quadratic function in $\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}$, $I_{f,g}$ is bounded from below. Hence Step 1 follows.

Step 2: In this step we show that there exists a bounded nonnegative (PS) sequence $\{(u_n,v_n)\} \subset \Omega_1$ for $J_{f,g}$ at level c_0.

Let $\{(u_n, v_n)\} \subset \bar{\Omega}_1$ such that $J_{f,g}(u_n, v_n) \to c_0$. Since Lemma 5.5.3 implies that $c_0 < c_1$, without restriction we can assume $\{(u_n, v_n)\} \subset \Omega_1$. Further, using Ekeland's variational principle from $\{(u_n, v_n)\}$, we can extract a (PS) sequence in Ω_1 for $J_{f,g}$ at level c_0. We again call it by $\{(u_n, v_n)\}$. Moreover, as $J_{f,g}(u, v) \geq I_{f,g}(u, v)$, from (5.5.8) it follows that $\{(u_n, v_n)\}$ is a bounded sequence. Therefore, up to a subsequence $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ and $(u_n, v_n) \to (u_0, v_0)$ a.e. in $\mathbb{R}^N$. In particular, $(u_n)_+ \to (u_0)_+$, $(v_n)_+ \to (v_0)_+$ and $(u_n)_- \to (u_0)_-$, $(v_n)_- \to (v_0)_-$ a.e. in $\mathbb{R}^N$. Moreover, as f, g are nonnegative functionals, a straight forward computation yields

$$\begin{aligned} o(1) &= {}_{\dot{H}^s)'}\langle J'_{f,g}(u_n, v_n), ((u_n)_-, (v_n)_-)\rangle_{\dot{H}^s \times \dot{H}^s} \\ &= \langle (u_n, v_n), ((u_n)_-, (v_n)_-)\rangle_{\dot{H}^s \times \dot{H}^s} - {}_{(\dot{H}^s)'}\langle f, (u_n)_-\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, (v_n)_-\rangle_{\dot{H}^s} \\ &\leq -\| ((u_n)_-, (v_n)_-) \|^2_{\dot{H}^s \times \dot{H}^s}. \end{aligned}$$

Therefore, $\Big((u_n)_-, (v_n)_-\Big) \longrightarrow (0, 0)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, which in turn implies up to a subsequence $(u_n)_- \to 0$ and $(v_n)_- \to 0$ a.e. in $\mathbb{R}^N$ and thus $(u_0)_- = 0$ and $(v_0)_- = 0$ a.e. in $\mathbb{R}^N$. Consequently, without loss of generality, we can assume that $\{(u_n, v_n)\}$ is a nonnegative sequence. This completes the proof of Step 2.

Step 3: In this step we show that $(u_n, v_n) \to (u_0, v_0)$ in $\dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ and $(u_0, v_0) \in \Omega_1$.

Applying Proposition 5.4.1, we get

$$(u_n, v_n) - \left((u_0, v_0) + \sum_{j=1}^{m} (\tilde{u}_j, \tilde{v}_j)^{r_n^j, x_n^j} \right) \longrightarrow (0, 0) \ \text{ in } \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N). \tag{5.5.9}$$

with $J'_{f,g}(u_0, v_0) = 0$, $(\tilde{u}_j, \tilde{v}_j)$ is a nonnegative solution of (5.4.2) ($\{(u_n, v_n)\}$ is a (PS) sequence for $J_{f,g}$ implies $(\tilde{u}_j, \tilde{v}_j)$ is a solution of (5.5.2), with $f \equiv g \equiv 0$, and therefore by Remark (5.5.1), $(\tilde{u}_j, \tilde{v}_j)$ is a nonnegative solution of (5.4.2)), and $\{x_n^j\} \subset \mathbb{R}^N$, $\{r_n^j\} \subset \mathbb{R}^+$ are some appropriate sequences such that $r_n^j \overset{n\to\infty}{\longrightarrow} 0$ and either $x_n^j \overset{n\to\infty}{\longrightarrow} x^j$ or $|x_n^j| \overset{n\to\infty}{\longrightarrow} \infty$. To prove Step 3, we need to show that $m = 0$. Arguing by contradiction, suppose that $j \neq 0$ in (5.5.9). Therefore,

$$\begin{aligned}\Psi\left((\tilde{u}_j, \tilde{v}_j)^{r_n^j, x_n^j}\right) &= \| (\tilde{u}_j, \tilde{v}_j) \|^2_{\dot{H}^s \times \dot{H}^s} - (2_s^* - 1) \int_{\mathbb{R}^N} |\tilde{u}_j|^\alpha |\tilde{v}_j|^\beta \, \mathrm{d}x \\ &= (2 - 2_s^*) \| (\tilde{u}_j, \tilde{v}_j) \|^2_{\dot{H}^s \times \dot{H}^s} < 0. \qquad (5.5.10)\end{aligned}$$

From Proposition 5.4.1, we also have

$$c_0 = \lim_{n\to\infty} J_{f,g}(u_n, v_n) = J_{f,g}(u_0, v_0) + \sum_{j=1}^m J_{0,0}(\tilde{u}_j, \tilde{v}_j).$$

Since $(\tilde{u}_j, \tilde{v}_j)$ is a solution to (5.4.2), it is easy to see that $J_{0,0}(\tilde{u}_j, \tilde{v}_j) = \frac{s}{N} \|(\tilde{u}_j, \tilde{v}_j)\|^2_{\dot{H}^s \times \dot{H}^s}$ and $S_{(\alpha,\beta)} \leq \|(\tilde{u}_j, \tilde{v}_j)\|^{\frac{4s}{N}}_{\dot{H}^s \times \dot{H}^s}$, which in turn implies $J_{0,0}(\tilde{u}_j, \tilde{v}_j) = \frac{s}{N} S^{\frac{N}{2s}}_{(\alpha,\beta)}$. Consequently, $J_{f,g}(u_0, v_0) < c_0$. Therefore, $(u_0, v_0) \notin \Omega_1$ and

$$\Psi(u_0, v_0) \leq 0. \qquad (5.5.11)$$

Next, we evaluate $\Psi\left((u_0, v_0) + \sum_{j=1}^m (\tilde{u}_j, \tilde{v}_j)^{r_n^j, x_n^j}\right)$. We observe that $(u_n, v_n) \in \Omega_1$ implies $\Psi(u_n, v_n) \geq 0$. Combining this with the uniform continuity of Ψ and (5.5.9) yields

$$0 \leq \liminf_{n\to\infty} \Psi(u_n, v_n) = \liminf_{n\to\infty} \Psi\left((u_0, v_0) + \sum_{j=1}^m (\tilde{u}_j, \tilde{v}_j)^{r_n^j, x_n^j}\right). \qquad (5.5.12)$$

Note that from Step 2, we already have u_0, $v_0 \geq 0$. We also have $(\tilde{u}_j, \tilde{v}_j)$ is nonnegative for all j (see the paragraph after (5.5.9)) (since $\{(u_n, v_n)\}$ is a nonnegative sequence). Therefore, as $\alpha, \beta > 1$,

$$\begin{aligned}
&\Psi\bigg((u_0,v_0)+\sum_{j=1}^{m}(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j}\bigg)\\
&=\|(u_0,v_0)\|^2_{\dot H^s\times\dot H^s}+\bigg\|\sum_{j=1}^{m}(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j}\bigg\|^2_{\dot H^s\times\dot H^s}+2\bigg\langle(u_0,v_0),\sum_{j=1}^{m}(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j}\bigg\rangle_{\dot H^s\times\dot H^s}\\
&\qquad-(2_s^*-1)\int_{\mathbb R^N}\bigg|u_0+\sum_{j=1}^{m}\tilde u_j^{r_n^j,x_n^j}\bigg|^{\alpha}\bigg|v_0+\sum_{j=1}^{m}\tilde u_j^{r_n^j,x_n^j}\bigg|^{\beta}\mathrm dx\\
&\le\|(u_0,v_0)\|^2_{\dot H^s\times\dot H^s}+\sum_{j=1}^{m}\bigg\|(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j}\bigg\|^2_{\dot H^s\times\dot H^s}\\
&\quad+2\sum_{j=1}^{m}\sum_{i=1}^{m}\bigg\langle(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j},(\tilde u_i,\tilde v_i)^{r_n^i,x_n^i}\bigg\rangle_{\dot H^s\times\dot H^s}\\
&\quad+2\bigg\langle(u_0,v_0),\sum_{j=1}^{m}(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j}\bigg\rangle_{\dot H^s\times\dot H^s}\\
&\quad-(2_s^*-1)\bigg(\int_{\mathbb R^N}|u_0|^{\alpha}|v_0|^{\beta}\mathrm dx+\sum_{j=1}^{m}\int_{\mathbb R^N}|\tilde u_j^{r_n^j,x_n^j}|^{\alpha}|\tilde v_j^{r_n^j,x_n^j}|^{\beta}\mathrm dx\bigg)\\
&=\Psi(u_0,v_0)+\sum_{j=1}^{m}\Psi\Big((\tilde u_j,\tilde v_j)^{r_n^j,x_n^j}\Big)+\text{the above inner products.}
\end{aligned}\tag{5.5.13}$$

We now prove that all the inner products in the RHS of (5.5.13) approaches 0 as $n\to\infty$. As $r_n^j\overset{n\to\infty}{\longrightarrow}0$, it follows that $u_j^{r_n^j,x_n^j}\rightharpoonup 0$ and $v_j^{r_n^j,x_n^j}\rightharpoonup 0$ in $\dot H^s(\mathbb R^N)$ as $n\to\infty$ (see [94, Lemma 3]). Therefore, $\langle u_0,u_j^{r_n^j,x_n^j}\rangle_{\dot H^s}\overset{n\to\infty}{\longrightarrow}0$ and $\langle v_0,v_j^{r_n^j,x_n^j}\rangle_{\dot H^s}\overset{n\to\infty}{\longrightarrow}0$ for all $j=1,\cdots,m$. Hence,

$$2\bigg\langle(u_0,v_0),\sum_{j=1}^{m}(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j}\bigg\rangle_{\dot H^s\times\dot H^s}=o(1)\quad\text{as } n\to\infty.$$

Next,

$$\begin{aligned}
&\bigg\langle(\tilde u_j,\tilde v_j)^{r_n^j,x_n^j},(\tilde u_i,\tilde v_i)^{r_n^i,x_n^i}\bigg\rangle_{\dot H^s\times\dot H^s}\\
=\;&(r_n^i)^{\frac{N-2s}{2}}(r_n^j)^{-\frac{N-2s}{2}}\iint_{\mathbb R^{2N}}\frac{\Big(\tilde u_j(\frac{x-x_n^j}{r_n^j})-\tilde u_j(\frac{y-x_n^j}{r_n^j})\Big)\Big(\tilde u_i(\frac{x-x_n^i}{r_n^i})-\tilde u_i(\frac{x-x_n^i}{r_n^i})\Big)}{|x-y|^{N+2s}}\mathrm dx\mathrm dy\\
&+(r_n^i)^{\frac{N-2s}{2}}(r_n^j)^{-\frac{N-2s}{2}}\iint_{\mathbb R^{2N}}\frac{\Big(\tilde v_j(\frac{x-x_n^j}{r_n^j})-\tilde v_j(\frac{y-x_n^j}{r_n^j})\Big)\Big(\tilde v_i(\frac{x-x_n^i}{r_n^i})-\tilde v_i(\frac{x-x_n^i}{r_n^i})\Big)}{|x-y|^{N+2s}}\mathrm dx\mathrm dy\\
=\;&(r_n^i)^{\frac{N-2s}{2}}(r_n^j)^{-\frac{N-2s}{2}}\iint_{\mathbb R^{2N}}\frac{\Big(\tilde u_i(x)-\tilde u_i(y)\Big)\Big(\tilde u_j(\frac{r_n^ix+x_n^i-x_n^j}{r_n^j})-\tilde u_j(\frac{r_n^iy+x_n^i-x_n^j}{r_n^j})\Big)}{|x-y|^{N+2s}}\mathrm dx\mathrm dy
\end{aligned}$$

$$+(r_n^i)^{\frac{N-2s}{2}}(r_n^j)^{-\frac{N-2s}{2}}\iint_{\mathbb{R}^{2N}}\frac{\Big(\tilde{v}_i(x)-\tilde{v}_i(y)\Big)\Big(\tilde{v}_j(\frac{r_n^i x+x_n^i-x_n^j}{r_n^j})-\tilde{v}_j(\frac{r_n^i y+x_n^i-x_n^j}{r_n^j})\Big)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$= \langle \tilde{u}_i, \tilde{u}_j^n\rangle_{\dot{H}^s} + \langle \tilde{v}_i, \tilde{v}_j^n\rangle_{\dot{H}^s},$$

where $\tilde{u}_j^n(x) := \Big(\frac{r_n^i}{r_n^j}\Big)^{\frac{N-2s}{2}}\tilde{u}_j\Big(\frac{r_n^i}{r_n^j}x+\frac{x_n^i-x_n^j}{r_n^j}\Big)$, and $\tilde{v}_j^n(x) := \Big(\frac{r_n^i}{r_n^j}\Big)^{\frac{N-2s}{2}}\tilde{v}_j\Big(\frac{r_n^i}{r_n^j}x+\frac{x_n^i-x_n^j}{r_n^j}\Big)$. Further, we observe that using the following

$$\left|\log\left(\frac{r_n^i}{r_n^j}\right)\right|+\left|\frac{x_n^i-x_n^j}{r_n^i}\right|\longrightarrow\infty$$

from Proposition 5.4.1, it is easy to see that $\tilde{u}_i^n \rightharpoonup 0$ and $\tilde{v}_i^n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$ as $n\to\infty$ for each fixed i and j (see [94, Lemma 3]). Hence

$$\Big\langle (\tilde{u}_j,\tilde{v}_j)^{r_n^j,x_n^j}, (\tilde{u}_i,\tilde{v}_i)^{r_n^i,x_n^i}\Big\rangle_{\dot{H}^s\times\dot{H}^s} = o(1).$$

Substituting this back into (5.5.13) and using (5.5.10) and (5.5.11) gives a contradiction to (5.5.12). Therefore, $m=0$ in (5.5.9). Hence, $(u_n,v_n)\to(u_0,v_0)$ in $\dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)$. Consequently, $\Psi(u_n,v_n)\to\Psi(u_0,v_0)$, which in turn implies $(u_0,v_0)\in\bar{\Omega}_1$. But, since $c_0<c_1$, we conclude $(u_0,v_0)\in\Omega_1$. Thus Step 3 follows.

Step 4: From the previous steps we see that $J_{f,g}(u_0,v_0)=c_0$ and $J'_{f,g}(u_0,v_0)=0$. Therefore, (u_0,v_0) is a weak solution to (5.5.2). Combining this with Remark 5.5.1, we end the proof of the proposition. □

Proposition 5.5.5. *Assume that* (5.5.4) *holds. Then,* $J_{f,g}$ *has a second critical point* $(u_1,v_1)\neq(u_0,v_0)$, *where* (u_0,v_0) *is the positive solution to* $(\mathcal{S}^0_{2_s^*})$ *obtained in Proposition 5.5.4. In particular,* (u_1,v_1) *is a second positive solution to* $(\mathcal{S}^0_{2_s^*})$.

Proof. Let (u_0,v_0) be the critical point obtained in Proposition 5.5.4 and (Bw,Cw) (with $C=B\sqrt{\frac{\beta}{\alpha}}$) be a positive ground state solution of (5.4.2) described as in Theorem 5.1.3. A standard computation yields that $I_{0,0}(Bw,Cw)=\frac{s}{N}S_{\alpha,\beta}^{\frac{N}{2s}}$. For $t>0$, define

$$w_t(x) := w\Big(\frac{x}{t}\Big),\quad \tilde{u}_t(x) := Bw_t(x),\quad \tilde{v}_t(x) := Cw_t(x).$$

Claim 1: $(u_0 + \tilde{u}_t, v_0 + \tilde{v}_t) \in \Omega_2$ for $t > 0$ large enough.

Indeed, as (u_0, v_0) and $(\tilde{u}_t, \tilde{v}_t)$ are positive and $\alpha, \beta > 1$, using Young's inequality with $\varepsilon > 0$, we have

$$\begin{aligned}
& \Psi(u_0 + \tilde{u}_t, v_0 + \tilde{v}_t) \\
= \;& \|(u_0 + \tilde{u}_t)\|^2_{\dot{H}^s} + \|(v_0 + \tilde{v}_t)\|^2_{\dot{H}^s} - (2^*_s - 1)\int_{\mathbb{R}^N} |u_0 + \tilde{u}_t|^\alpha |v_0 + \tilde{v}_t|^\beta \mathrm{d}x \\
\leq \;& \|(u_0, v_0)\|^2_{\dot{H}^s \times \dot{H}^s} + \|(\tilde{u}_t, \tilde{v}_t)\|^2_{\dot{H}^s \times \dot{H}^s} + 2\langle u_0, u_t\rangle_{\dot{H}^s} + 2\langle v_0, v_t\rangle_{\dot{H}^s} \\
& \quad -(2^*_s - 1)\left(\int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \, \mathrm{d}x + \int_{\mathbb{R}^N} |\tilde{u}_t|^\alpha |\tilde{v}_t|^\beta \, \mathrm{d}x\right) \\
\leq \;& (1+\varepsilon)\|(\tilde{u}_t, \tilde{v}_t)\|^2_{\dot{H}^s \times \dot{H}^s} + (1 + C_\varepsilon)\|(u_0, v_0)\|^2_{\dot{H}^s \times \dot{H}^s} \\
& -(2^*_s - 1)\int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \mathrm{d}x - (2^*_s - 1)t^N B^\alpha C^\beta \int_{\mathbb{R}^N} |w|^{2^*_s} \mathrm{d}x \\
\leq \;& \left((1+\varepsilon)(B^2 + C^2)t^{N-2s} - (2^*_s - 1)t^N B^\alpha C^\beta\right)\|w\|^2_{\dot{H}^s} \\
& +(1 + C_\varepsilon)\|(u_0, v_0)\|^2_{\dot{H}^s \times \dot{H}^s} - (2^*_s - 1)\int_{\mathbb{R}^N} |u_0|^\alpha |v_0|^\beta \mathrm{d}x \\
< \;& 0 \quad \text{for } t > 0 \text{ large enough.}
\end{aligned}$$

Hence the claim follows.

Claim 2: $J_{f,g}\Big(u_0 + \tilde{u}_t, v_+\tilde{v}_t\Big) < J_{f,g}(u_0, v_0) + J_{0,0}(\tilde{u}_t, \tilde{v}_t) \ \forall \ t > 0.$

Indeed, since u_0, v_0, w_t, $B > 0$, taking $(\tilde{u}_t, \tilde{v}_t)$ as the test function for (5.5.2) yields

$$\Big\langle (u_0, v_0), \ (\tilde{u}_t, \tilde{v}_t)\Big\rangle_{\dot{H}^s \times \dot{H}^s} = \frac{\alpha}{2^*_s}\int_{\mathbb{R}^N} u_0^{\alpha-1} v_0^\beta \tilde{u}_t \, \mathrm{d}x + \frac{\beta}{2^*_s}\int_{\mathbb{R}^N} u_0^\alpha v_0^{\beta-1} \tilde{v}_t \, \mathrm{d}x + {}_{(\dot{H}^s)'}\langle f, \tilde{u}_t\rangle_{\dot{H}^s} + {}_{(\dot{H}^s)'}\langle g, \tilde{v}_t\rangle_{\dot{H}^s}.$$

Consequently, using the above expression, we obtain

$$\begin{aligned}
& J_{f,g}\Big(u_0 + \tilde{u}_t, v_0 + \tilde{v}_t\Big) \\
&= \frac{1}{2}\|(u_0, v_0)\|^2_{\dot{H}^s \times \dot{H}^s} + \frac{1}{2}\|(\tilde{u}_t, \tilde{v}_t)\|^2_{\dot{H}^s \times \dot{H}^s} + \Big\langle (u_0, v_0), \ (\tilde{u}_t, \tilde{v}_t)\Big\rangle_{\dot{H}^s \times \dot{H}^s} \\
&\quad - \frac{1}{2^*_s}\int_{\mathbb{R}^N} (u_0 + \tilde{u}_t)^\alpha (v_0 + \tilde{v}_t)^\beta \, \mathrm{d}x - {}_{(\dot{H}^s)'}\langle f, u_0 + \tilde{u}_t\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v_0 + \tilde{v}_t\rangle_{\dot{H}^s}
\end{aligned}$$

$$
\begin{aligned}
&= J_{f,g}(u_0,v_0) + J_{0,0}(\tilde{u}_t,\tilde{v}_t) + \frac{1}{2_s^*}\int_{\mathbb{R}^N} u_0^\alpha v_0^\beta \,\mathrm{d}x + \frac{1}{2_s^*}\int_{\mathbb{R}^N} |\tilde{u}_t|^\alpha |\tilde{v}_t|^\beta \mathrm{d}x \\
&\quad + \frac{\alpha}{2_s^*}\int_{\mathbb{R}^N} u_0^{\alpha-1} v_0^\beta \tilde{u}_t \,\mathrm{d}x + \frac{\beta}{2_s^*}\int_{\mathbb{R}^N} u_0^\alpha v_0^{\beta-1}\tilde{v}_t \,\mathrm{d}x \\
&\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^N} (u_0+\tilde{u}_t)^\alpha (v_0+\tilde{v}_t)^\beta \,\mathrm{d}x \\
&\leq J_{f,g}(u_0,v_0) + J_{0,0}(\tilde{u}_t,\tilde{v}_t) + \frac{1}{2_s^*}\int_{\mathbb{R}^N} \Big[u_0^\alpha v_0^\beta + \tilde{u}_t^\alpha \tilde{v}_t^\beta \\
&\qquad\qquad + \alpha u_0^{\alpha-1} v_0^\beta \tilde{u}_t + \beta u_0^\alpha v_0^{\beta-1}\tilde{v}_t - (u_0+\tilde{u}_t)^\alpha (v_0+\tilde{v}_t)^\beta \Big] \mathrm{d}x \\
&< J_{f,g}(u_0,v_0) + J_{0,0}(\tilde{u}_t,\tilde{v}_t).
\end{aligned}
$$

Hence the Claim follows.

Using the definition of $\tilde{u}_t$ and $\tilde{v}_t$, it immediately follows

$$
\lim_{t\to\infty} J_{0,0}\Big(\tilde{u}_t,\tilde{v}_t\Big) = -\infty, \tag{5.5.14}
$$

and

$$
\sup_{t>0} J_{0,0}\Big(\tilde{u}_t,\tilde{v}_t\Big) = J_{0,0}(\tilde{u}_{t'},\tilde{v}_{t'}), \quad \text{where} \quad t' = \left(\frac{B^2+C^2}{B^\alpha C^\beta}\right)^{\frac{1}{2s}}.
$$

Therefore, doing a straight forward computation and using Lemma 5.0.2, we get that

$$
\sup_{t>0} J_{0,0}\Big(\tilde{u}_t,\tilde{v}_t\Big) = \frac{s}{N}\frac{(B^2+C^2)^{\frac{N}{2s}}}{(B^\alpha C^\beta)^{\frac{N-2s}{2s}}} S^{\frac{N}{2s}} = \frac{s}{N} S_{\alpha,\beta}^{\frac{N}{2s}}.
$$

Combining this with Claim 2 and (5.5.14) yields

$$
\begin{aligned}
& J_{f,g}(u_0+\tilde{u}_t, v_0+\tilde{v}_t) < J_{f,g}(u_0,v_0) + \frac{s}{N} S_{\alpha,\beta}^{\frac{N}{2s}} \quad \forall\, t>0 \\
\text{and} \qquad & J_{f,g}(u_0+\tilde{u}_t, v_0+\tilde{v}_t) < J_{f,g}(u_0,v_0) \quad \text{for } t \text{ large enough.}
\end{aligned} \tag{5.5.15}
$$

Fix $t_0 > 0$ large enough such that (5.5.15) and Claim 1 are satisfied.

Next, we set

$$
\eta := \inf_{\gamma\in\Gamma} \max_{r\in[0,1]} J_{f,g}\Big(\gamma(r)\Big),
$$

where

$$
\Gamma := \Big\{ \gamma \in C\Big([0,1],\, \dot{H}^s(\mathbb{R}^N)\times\dot{H}^s(\mathbb{R}^N)\Big) : \gamma(0) = (u_0,v_0),\ \gamma(1) = (u_0+\tilde{u}_{t_0}, v_0+\tilde{v}_{t_0}) \Big\}.
$$

As $(u_0, v_0) \in \Omega_1$ and $(u_0 + \tilde{u}_{t_0}, v_0 + \tilde{v}_{t_0}) \in \Omega_2$, for every $\gamma \in \Gamma$, there exists $r_\gamma \in (0,1)$ such that $\gamma(r_\gamma) \in \Omega$. Therefore,

$$\max_{r\in[0,1]} J_{f,g}(\gamma(r)) \geq J_{f,g}\Big(\gamma(r_\gamma)\Big) \geq \inf_{\Omega} J_{f,g}(u,v) = c_1.$$

Thus, $\eta \geq c_1 > c_0 = J_{f,g}(u_0, v_0)$. Here in the last inequality we have used Lemma 5.5.3.

Claim 3: $J_{f,g}(u_0,v_0) < \eta < J_{f,g}(u_0,v_0) + \frac{s}{N} S_{\alpha,\beta}^{\frac{N}{2s}}$.

Since $\lim_{t\to 0} \|w_t\|_{\dot{H}^s(\mathbb{R}^N)} = 0$, we also have $\lim_{t\to 0} \|(\tilde{u}_t, \tilde{v}_t)\|_{\dot{H}^s\times\dot{H}^s} = 0$. Thus, if we define $\tilde{\gamma}(r) := (u_0, v_0) + (\tilde{u}_{rt_0}, \tilde{u}_{rt_0})$, then $\lim_{r\to 0} \|\tilde{\gamma}(r) - (u_0,v_0)\|_{\dot{H}^s\times\dot{H}^s} = 0$. Consequently, $\tilde{\gamma} \in \Gamma$. Therefore, using (5.5.15), we obtain

$$\eta \leq \max_{r\in[0,1]} J_{f,g}(\tilde{\gamma}(r)) = \max_{r\in[0,1]} J_{f,g}\left(u_0 + \tilde{u}_{rt_0}, v_0 + \tilde{v}_{rt_0}\right) < J_{a,f}(u_0,v_0) + \frac{s}{N} S_{\alpha,\beta}^{\frac{N}{2s}}.$$

Hence Claim 3 follows.

Using Ekeland's variational principle, there exists a (PS) sequence $\{(u_n, v_n)\}$ for $J_{f,g}$ at level η. Arguing as before we see that $\{(u_n,v_n)\}$ is a bounded sequence. Further, since Claim 3 holds, from Proposition 5.4.1 we conclude that $(u_n, v_n) \to (u_1, v_1)$, for some $(u_1,v_1) \in \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ such that $J'_{f,g}(u_1,v_1) = 0$ and $J_{f,g}(u_1,v_1) = \eta$. On the other hand, as $J_{f,g}(u_0,v_0) < \eta$, we conclude $(u_0,v_0) \neq (u_1,v_1)$.

$J'_{f,g}(u_1,v_1) = 0 \Longrightarrow (u_1,v_1)$ is a weak solution to (5.5.2). Combining this with Remark 5.5.1, we complete the proof of the proposition. □

Lemma 5.5.6. *Let C_0 be as defined in Theorem 5.1.4. If* $\max\{\|f\|_{(\dot{H}^s)'}, \|g\|_{(\dot{H}^s)'}\} < C_0 S_{\alpha,\beta}^{\frac{N}{4s}}$*, then* (5.5.4) *holds.*

Proof. **Assertion 1**:

$$\frac{4s}{N+2s} \|(u,v)\|_{\dot{H}^s\times\dot{H}^s} \geq C_0 S_{\alpha,\beta}^{\frac{N}{4s}} \quad \forall\ (u,v) \in \Omega.$$

To see this, we fix $(u,v) \in \Omega$. Therefore, using the definition of $S_{\alpha,\beta}$ we have

$$\begin{aligned} \|(u,v)\|_{\dot{H}^s\times\dot{H}^s} &\geq S_{\alpha,\beta}^{1/2}\left(\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\,\mathrm{d}x\right)^{1/2_s^*} \\ \implies \|(u,v)\|_{\dot{H}^s\times\dot{H}^s} &\geq S_{\alpha,\beta}^{1/2}\frac{\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}^{2/2_s^*}}{(2_s^*-1)^{1/2_s^*}}. \end{aligned}$$

From here, using the definition of C_0, the assertion follows.

Note that by the given hypothesis, there exists $\varepsilon > 0$ such that

$$\|f\|_{(\dot{H}^s)'} + \|g\|_{(\dot{H}^s)'} < C_0 S_{\alpha,\beta}^{\frac{N}{4s}} - \varepsilon.$$

Combining this with the above Assertion 1, for all $(u,v) \in \Omega$, it holds

$$\begin{aligned} {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} + {}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s} &\leq \left(\|f\|_{(\dot{H}^s)'} + \|g\|_{(\dot{H}^s)'}\right)\|(u,v)\|_{\dot{H}^s\times\dot{H}^s} \\ &< \left(C_0 S_{\alpha,\beta}^{\frac{N}{4s}} - \varepsilon\right)\|(u,v)\|_{\dot{H}^s\times\dot{H}^s} \\ &\leq \frac{4s}{N+2s}\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}^2 - \varepsilon\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}. \end{aligned}$$

Consequently,

$$\inf_{(u,v)\in\Omega}\left[\frac{4s}{N+2s}\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}^2 - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s}\right] > \varepsilon \inf_{(u,v)\in\Omega}\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}.$$

Since $\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}$ is bounded away from 0 on Ω, the above expression implies that

$$\inf_{(u,v)\in\Omega}\left[\frac{4s}{N+2s}\|(u,v)\|_{\dot{H}^s\times\dot{H}^s}^2 - {}_{(\dot{H}^s)'}\langle f,u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g,v\rangle_{\dot{H}^s}\right] > 0. \quad (5.5.16)$$

On the other hand,

$$
\begin{aligned}
(5.5.4) \iff & C_0 \frac{\|(u,v)\|_{\dot{H}^s \times \dot{H}^s}^{\frac{N+2s}{2s}}}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x\right)^{\frac{N-2s}{4s}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v\rangle_{\dot{H}^s} > 0 \\
& \qquad\qquad\qquad \text{for } \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x = 1 \\
\iff & C_0 \frac{\|(u,v)\|_{\dot{H}^s \times \dot{H}^s}^{\frac{N+2s}{2s}}}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, \mathrm{d}x\right)^{\frac{N-2s}{4s}}} - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v\rangle_{\dot{H}^s} > 0 \\
& \qquad\qquad\qquad \text{for } (u,v) \in \Omega \\
\iff & \frac{4s}{N+2s} \|(u,v)\|_{\dot{H}^s \times \dot{H}^s}^2 - {}_{(\dot{H}^s)'}\langle f, u\rangle_{\dot{H}^s} - {}_{(\dot{H}^s)'}\langle g, v\rangle_{\dot{H}^s} > 0 \\
& \qquad\qquad\qquad \text{for } (u,v) \in \Omega. \qquad (5.5.17)
\end{aligned}
$$

Clearly, (5.5.16) ensures that the RHS of (5.5.17) holds. The lemma now follows. □

End of Proof of Theorem 5.1.4 Combining Propositions 5.5.4 and 5.5.5 with Lemmas 5.5.6 and 5.5.3, we complete the proof of Theorem 5.1.4.

Conclusion : In this chapter we consider nonlocal weakly coupled elliptic system of equations with both critical and subcritical nonlinearity and with nonhomogeneous term in $\mathbb{R}^N$. First, we prove existence of one positive solution for the system as a perturbation of 0. Then we consider the corresponding homogeneous system with critical nonlinearity and prove uniqueness for ground state solutions. Then characterizing the PS sequences for the associated energy functional, we prove multiplicity result for the nonhomogeneous critical system under certain assumption on the nonhomogeneous terms.

We could able to prove uniqueness for ground state solutions for the corresponding critical system. It will be an interesting question if we can prove uniqueness/multiplicity of solutions for the homogeneous system with critical nonlinearity.

Bibliography

[1] N. Abatangelo and E. Valdinoci, *Getting acquainted with the fractional Laplacian*, in Contemporary research in elliptic PDEs and related topics, Vol. 33 of *Springer INdAM Ser.*, 1–105, Springer, Cham (2019).

[2] B. Abdellaoui, A. Attar, A. Dieb, and I. Peral, *Attainability of the fractional Hardy constant with nonlocal mixed boundary conditions: applications*, Discrete Contin. Dyn. Syst. **38** (2018), no. 12, 5963–5991.

[3] B. Abdellaoui, M. Medina, I. Peral, and A. Primo, *The effect of the Hardy potential in some Calderón-Zygmund properties for the fractional Laplacian*, J. Differential Equations **260** (2016), no. 11, 8160–8206.

[4] S. Adachi and K. Tanaka, *Four positive solutions for the semilinear elliptic equation:* $-\Delta u + u = a(x)u^p + f(x)$ *in* $\mathbf{R}^N$, Calc. Var. Partial Differential Equations **11** (2000), no. 1, 63–95.

[5] ———, *Existence of positive solutions for a class of nonhomogeneous elliptic equations in* $\mathbf{R}^N$, Nonlinear Anal. **48** (2002), no. 5, Ser. A: Theory Methods, 685–705.

[6] Adimurthi and A. Mallick, *A Hardy type inequality on fractional order Sobolev spaces on the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18** (2018), no. 3, 917–949.

[7] S. Alarcón and J. Tan, *Sign-changing solutions for some nonhomogeneous nonlocal critical elliptic problems*, Discrete Contin. Dyn. Syst. **39** (2019), no. 10, 5825–5846.

[8] C. O. Alves, D. C. de Morais Filho, and M. A. S. Souto, *On systems of elliptic equations involving subcritical or critical Sobolev exponents*, Nonlinear Anal. **42** (2000), no. 5, Ser. A: Theory Methods, 771–787.

[9] A. Ambrosetti, *Critical points and nonlinear variational problems*, Mém. Soc. Math. France (N.S.) (1992), no. 49, 139.

[10] A. Ambrosetti and A. Malchiodi, Nonlinear analysis and semilinear elliptic problems, Vol. 104 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge (2007), ISBN 978-0-521-86320-9; 0-521-86320-1.

[11] V. Ambrosio and G. M. Figueiredo, *Ground state solutions for a fractional Schrödinger equation with critical growth*, Asymptot. Anal. **105** (2017), no. 3-4, 159–191.

[12] V. Ambrosio and H. Hajaiej, *Multiple solutions for a class of nonhomogeneous fractional Schrödinger equations in $\mathbb{R}^N$*, J. Dynam. Differential Equations **30** (2018), no. 3, 1119–1143.

[13] D. L. T. Anderson and G. H. Derrick, *Stability of Time-Dependent Particlelike Solutions in Nonlinear Field Theories. I*, Journal of Mathematical Physics **11** (1970), no. 4, 1336–1346.

[14] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc. **267** (1981), no. 1, 1–32.

[15] A. Bahri and Y. Y. Li, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbf{R}^N$, Rev. Mat. Iberoamericana **6** (1990), no. 1-2, 1–15.

[16] A. Bahri and P.-L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), no. 3, 365–413.

[17] A. Bahrouni, H. Ounaies, and V. D. Rădulescu, *Bound state solutions of sublinear Schrödinger equations with lack of compactness*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 2, 1191–1210.

[18] A. Bahrouni, V. D. Rădulescu, and P. Winkert, *A critical point theorem for perturbed functionals and low perturbations of differential and nonlocal systems*, Adv. Nonlinear Stud. **20** (2020), no. 3, 663–674.

[19] R. Bartolo, P. L. De Nápoli, and A. Salvatore, *Infinitely many solutions for non-local problems with broken symmetry*, Adv. Nonlinear Anal. **7** (2018), no. 3, 353–364.

[20] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains*, Arch. Rational Mech. Anal. **99** (1987), no. 4, 283–300.

[21] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345.

[22] ———, *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 347–375.

[23] M. Bhakta, A. Biswas, D. Ganguly, and L. Montoro, *Integral representation of solutions using Green function for fractional Hardy equations*, J. Differential Equations **269** (2020), no. 7, 5573–5594.

[24] M. Bhakta, S. Chakraborty, and D. Ganguly, *Existence and Multiplicity of positive solutions of certain nonlocal scalar field equations* (to appear in Mathematische Nachrichten (2022)).

[25] M. Bhakta, S. Chakraborty, O. H. Miyagaki, and P. Pucci, *Fractional elliptic systems with critical nonlinearities*, Nonlinearity **34** (2021), no. 11, 7540–7573.

[26] M. Bhakta, S. Chakraborty, and P. Pucci, *Nonhomogeneous systems involving critical or subcritical nonlinearities*, Differential Integral Equations **33** (2020), no. 7-8, 323–336.

[27] ———, *Fractional Hardy-Sobolev equations with nonhomogeneous terms*, Adv. Nonlinear Anal. **10** (2021), no. 1, 1086–1116.

[28] M. Bhakta and D. Mukherjee, *Semilinear nonlocal elliptic equations with critical and supercritical exponents*, Commun. Pure Appl. Anal. **16** (2017), no. 5, 1741–1766.

[29] ———, *Nonlocal scalar field equations: qualitative properties, asymptotic profiles and local uniqueness of solutions*, J. Differential Equations **266** (2019), no. 11, 6985–7037.

[30] M. Bhakta and P.-T. Nguyen, *On the existence and multiplicity of solutions to fractional Lane-Emden elliptic systems involving measures*, Adv. Nonlinear Anal. **9** (2020), no. 1, 1480–1503.

[31] M. Bhakta and P. Pucci, *On multiplicity of positive solutions for nonlocal equations with critical nonlinearity*, Nonlinear Anal. **197** (2020) 111853, 22.

[32] M. Bhakta and K. Sandeep, *Hardy-Sobolev-Maz'ya type equations in bounded domains*, J. Differential Equations **247** (2009), no. 1, 119–139.

[33] G. M. Bisci and V. D. Rădulescu, *Ground state solutions of scalar field fractional Schrödinger equations*, Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2985–3008.

[34] K. Bogdan, T. Grzywny, T. Jakubowski, and D. Pilarczyk, *Fractional Laplacian with Hardy potential*, Comm. Partial Differential Equations **44** (2019), no. 1, 20–50.

[35] D. Bonheure and M. Ramos, *Multiple critical points of perturbed symmetric strongly indefinite functionals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 2, 675–688.

[36] S. Bordoni, R. Filippucci, and P. Pucci, *Existence problems on Heisenberg groups involving Hardy and critical terms*, J. Geom. Anal. **30** (2020), no. 2, 1887–1917.

[37] M. Bouchekif and Y. Nasri, *On a nonhomogeneous elliptic system with changing sign data*, Nonlinear Anal. **65** (2006), no. 7, 1476–1487.

[38] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.

[39] H. Brézis and L. Nirenberg, *A minimization problem with critical exponent and nonzero data*, Symmetry in nature, Symp. in Honour of L. A. Radicati di Brozolo, Pisa/Italy 1989, 129-140 (1989). (1989).

[40] C. Bucur and E. Valdinoci, Nonlocal diffusion and applications, Vol. 20 of *Lecture Notes of the Unione Matematica Italiana*, Springer, [Cham];

Unione Matematica Italiana, Bologna (2016), ISBN 978-3-319-28738-6; 978-3-319-28739-3.

[41] L. Caffarelli, *Non-local diffusions, drifts and games*, in Nonlinear partial differential equations, Vol. 7 of *Abel Symp.*, 37–52, Springer, Heidelberg (2012).

[42] D.-M. Cao and H.-S. Zhou, *Multiple positive solutions of nonhomogeneous semilinear elliptic equations in* $\mathbf{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), no. 2, 443–463.

[43] ———, *On the existence of multiple solutions of nonhomogeneous elliptic equations involving critical Sobolev exponents*, Z. Angew. Math. Phys. **47** (1996), no. 1, 89–96.

[44] R. Castro and M. Zuluaga, *Existence results for a class of nonhomogeneous elliptic equations with critical Sobolev exponent*, Note Mat. **13** (1993), no. 2, 269–276.

[45] X. Chang and Z.-Q. Wang, *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity*, Nonlinearity **26** (2013), no. 2, 479–494.

[46] W. Chen, C. Li, and B. Ou, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math. **59** (2006), no. 3, 330–343.

[47] W. Chen and M. Squassina, *Critical nonlocal systems with concave-convex powers*, Adv. Nonlinear Stud. **16** (2016), no. 4, 821–842.

[48] Z. Chen and W. Zou, *Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent*, Arch. Ration. Mech. Anal. **205** (2012), no. 2, 515–551.

[49] ———, *Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent: higher dimensional case*, Calc. Var. Partial Differential Equations **52** (2015), no. 1-2, 423–467.

[50] M. Clapp, M. del Pino, and M. Musso, *Multiple solutions for a non-homogeneous elliptic equation at the critical exponent*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 1, 69–87.

[51] P. Clément, J. Fleckinger, E. Mitidieri, and F. de Thélin, *Existence of positive solutions for a nonvariational quasilinear elliptic system*, J. Differential Equations **166** (2000), no. 2, 455–477.

[52] S. Coleman, *Fate of the false vacuum: Semiclassical theory*, Phys. Rev. D **15** (1977) 2929–2936.

[53] E. Colorado, A. de Pablo, and U. Sánchez, *Perturbations of a critical fractional equation*, Pacific J. Math. **271** (2014), no. 1, 65–85.

[54] J. N. Correia and G. M. Figueiredo, *Existence of positive solution of the equation* $(-\Delta)^s u + a(x)u = |u|^{2_s^*-2}u$, Calc. Var. Partial Differential Equations **58** (2019), no. 2, Paper No. 63, 39.

[55] D. G. Costa, O. H. Miyagaki, M. Squassina, and J. Yang, *Asymptotics of ground states for fractional Hénon systems*, in Contributions to nonlinear elliptic equations and systems, Vol. 86 of *Progr. Nonlinear Differential Equations Appl.*, 133–161, Birkhäuser/Springer, Cham (2015).

[56] L. M. Del Pezzo and A. Quaas, *A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian*, J. Differential Equations **263** (2017), no. 1, 765–778.

[57] M. del Pino and P. L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations **4** (1996), no. 2, 121–137.

[58] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[59] W. Y. Ding and W.-M. Ni, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Rational Mech. Anal. **91** (1986), no. 4, 283–308.

[60] S. Dipierro, L. Montoro, I. Peral, and B. Sciunzi, *Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential*, Calc. Var. Partial Differential Equations **55** (2016), no. 4, Art. 99, 29.

[61] S. Dipierro, O. Savin, and E. Valdinoci, *All functions are locally s-harmonic up to a small error*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 4, 957–966.

[62] J. a. M. do Ó, O. H. Miyagaki, and M. Squassina, *Ground states of nonlocal scalar field equations with Trudinger-Moser critical nonlinearity*, Topol. Methods Nonlinear Anal. **48** (2016), no. 2, 477–492.

[63] M. M. Fall, F. Mahmoudi, and E. Valdinoci, *Ground states and concentration phenomena for the fractional Schrödinger equation*, Nonlinearity **28** (2015), no. 6, 1937–1961.

[64] L. F. O. Faria, O. H. Miyagaki, F. R. Pereira, M. Squassina, and C. Zhang, *The Brezis-Nirenberg problem for nonlocal systems*, Adv. Nonlinear Anal. **5** (2016), no. 1, 85–103.

[65] V. Felli and A. Pistoia, *Existence of blowing-up solutions for a nonlinear elliptic equation with Hardy potential and critical growth*, Comm. Partial Differential Equations **31** (2006), no. 1-3, 21–56.

[66] P. Felmer, A. Quaas, and J. Tan, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 6, 1237–1262.

[67] A. Fiscella and P. Pucci, *p-fractional Kirchhoff equations involving critical nonlinearities*, Nonlinear Anal. Real World Appl. **35** (2017) 350–378.

[68] A. Fiscella, P. Pucci, and S. Saldi, *Existence of entire solutions for Schrödinger-Hardy systems involving two fractional operators*, Nonlinear Anal. **158** (2017) 109–131.

[69] A. Fiscella, P. Pucci, and B. Zhang, *p-fractional Hardy-Schrödinger-Kirchhoff systems with critical nonlinearities*, Adv. Nonlinear Anal. **8** (2019), no. 1, 1111–1131.

[70] P. H. Frampton, *Consequences of vacuum instability in quantum field theory*, Phys. Rev. D **15** (1977) 2922–2928.

[71] R. L. Frank, E. Lenzmann, and L. Silvestre, *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math. **69** (2016), no. 9, 1671–1726.

[72] R. L. Frank, E. H. Lieb, and R. Seiringer, *Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. **21** (2008), no. 4, 925–950.

[73] R. L. Frank and R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities*, J. Funct. Anal. **255** (2008), no. 12, 3407–3430.

[74] N. Ghoussoub, F. Robert, S. Shakerian, and M. Zhao, *Mass and asymptotics associated to fractional Hardy-Schrödinger operators in critical regimes*, Comm. Partial Differential Equations **43** (2018), no. 6, 859–892.

[75] N. Ghoussoub and S. Shakerian, *Borderline variational problems involving fractional Laplacians and critical singularities*, Adv. Nonlinear Stud. **15** (2015), no. 3, 527–555.

[76] J. Giacomoni, P. K. Mishra, and K. Sreenadh, *Critical growth problems for $\frac{1}{2}$-Laplacian in* $\mathbb{R}$, Differ. Equ. Appl. **8** (2016), no. 3, 295–317.

[77] J. Giacomoni, T. Mukherjee, and K. Sreenadh, *Doubly nonlocal system with Hardy-Littlewood-Sobolev critical nonlinearity*, J. Math. Anal. Appl. **467** (2018), no. 1, 638–672.

[78] B. Gidas, *Euclidean Yang-Mills and related equations*, in Bifurcation phenomena in mathematical physics and related topics (Proc. NATO Advanced Study Inst., Cargèse, 1979), Vol. 54 of *NATO Adv. Study Inst. Ser. C: Math. Phys. Sci.*, 243–267, Reidel, Dordrecht-Boston, Mass. (1980).

[79] P. Han, *Multiple positive solutions of nonhomogeneous elliptic systems involving critical Sobolev exponents*, Nonlinear Anal. **64** (2006), no. 4, 869–886.

[80] X. He, M. Squassina, and W. Zou, *The Nehari manifold for fractional systems involving critical nonlinearities*, Commun. Pure Appl. Anal. **15** (2016), no. 4, 1285–1308.

[81] A. Iannizzotto, K. Perera, and M. Squassina, *Ground states for scalar field equations with anisotropic nonlocal nonlinearities*, Discrete Contin. Dyn. Syst. **35** (2015), no. 12, 5963–5976.

[82] T. Isernia, *Positive solution for nonhomogeneous sublinear fractional equations in* $\mathbb{R}^N$, Complex Var. Elliptic Equ. **63** (2018), no. 5, 689–714.

[83] L. Jeanjean, *Two positive solutions for a class of nonhomogeneous elliptic equations*, Differential Integral Equations **10** (1997), no. 4, 609–624.

[84] M. Kassmann, *A new formulation of Harnack's inequality for nonlocal operators*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 11-12, 637–640.

[85] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal. **20** (2017), no. 1, 7–51.

[86] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A **268** (2000), no. 4-6, 298–305.

[87] ———, *Fractional Schrödinger equation*, Phys. Rev. E (3) **66** (2002), no. 5, 056108, 7.

[88] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 223–283.

[89] ———, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana **1** (1985), no. 2, 45–121.

[90] A. Mallick, *Extremals for fractional order Hardy-Sobolev-Maz'ya inequality*, Calc. Var. Partial Differential Equations **58** (2019), no. 2, Paper No. 45, 37.

[91] J. a. Marcos do Ó, E. Medeiros, and U. Severo, *On a quasilinear nonhomogeneous elliptic equation with critical growth in* $\mathbb{R}^N$, J. Differential Equations **246** (2009), no. 4, 1363–1386.

[92] E. Mitidieri, *Nonexistence of positive solutions of semilinear elliptic systems in* $\mathbf{R}^N$, Differential Integral Equations **9** (1996), no. 3, 465–479.

[93] G. Molica Bisci, V. D. Radulescu, and R. Servadei, Variational methods for nonlocal fractional problems, Vol. 162 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge (2016), ISBN 978-1-107-11194-3. With a foreword by Jean Mawhin.

[94] G. Palatucci and A. Pisante, *Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 799–829.

[95] ———, *A global compactness type result for Palais-Smale sequences in fractional Sobolev spaces*, Nonlinear Anal. **117** (2015) 1–7.

[96] S. Peng, Y.-f. Peng, and Z.-Q. Wang, *On elliptic systems with Sobolev critical growth*, Calc. Var. Partial Differential Equations **55** (2016), no. 6, Art. 142, 30.

[97] P. Pucci, M. Xiang, and B. Zhang, *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2785–2806.

[98] W. Reichel and H. Zou, *Non-existence results for semilinear cooperative elliptic systems via moving spheres*, J. Differential Equations **161** (2000), no. 1, 219–243.

[99] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. (9) **101** (2014), no. 3, 275–302.

[100] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, J. Math. Phys. **54** (2013), no. 3, 031501, 17.

[101] ———, *On fractional Schrödinger equations in* $\mathbb{R}^N$ *without the Ambrosetti-Rabinowitz condition*, Topol. Methods Nonlinear Anal. **47** (2016), no. 1, 19–41.

[102] X. Shang, J. Zhang, and Y. Yang, *Positive solutions of nonhomogeneous fractional Laplacian problem with critical exponent*, Commun. Pure Appl. Anal. **13** (2014), no. 2, 567–584.

[103] D. Smets, *Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2909–2938.

[104] M. Squassina, *Two solutions for inhomogeneous nonlinear elliptic equations at critical growth*, NoDEA Nonlinear Differential Equations Appl. **11** (2004), no. 1, 53–71.

[105] M. Struwe, Variational methods, Vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, Springer-Verlag, Berlin, fourth edition (2008), ISBN 978-3-540-74012-4. Applications to nonlinear partial differential equations and Hamiltonian systems.

[106] K. Tanaka, *Periodic solutions for singular Hamiltonian systems and closed geodesics on non-compact Riemannian manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), no. 1, 1–33.

[107] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), no. 3, 281–304.

[108] K. Tintarev and K.-H. Fieseler, Concentration compactness, Imperial College Press, London (2007), ISBN 978-1-86094-667-7; 1-86094-667-4. Functional-analytic grounds and applications.

[109] Y. Wan and J. Yang, *Multiple solutions for inhomogeneous critical semilinear elliptic problems*, Nonlinear Anal. **68** (2008), no. 9, 2569–2593.

[110] F. Wang and Y. Zhang, *Existence of multiple positive solutions for nonhomogeneous fractional Laplace problems with critical growth*, Bound. Value Probl. (2019) Paper No. 169, 21.

[111] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Comm. Math. Phys. **153** (1993), no. 2, 229–244.

[112] G. Zhang and S. Liu, *Solution of a class of nonhomogeneous elliptic system involving critical Sobolev exponent*, Ann. Differential Equations **21** (2005), no. 1, 85–92.

[113] X. P. Zhu, *A perturbation result on positive entire solutions of a semilinear elliptic equation*, J. Differential Equations **92** (1991), no. 2, 163–178.

Index