ON THE DEPTH AND GENERICITY OF REPRESENTATIONS of A p-ADIC GROUP

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by

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To
Maa and Dada

Certificate

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Date: March 1, 2022

Dr. Manish Mishra
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Contents

Abstract

The main theme of the thesis is the study of the depth and genericity of representations of a p-adic group. This thesis is divided into two parts. In the *local Langlands correspondence*(LLC), irreducible representations of the group G(F) of F-points of a reductive group G defined over a non-archimedean local field F are expected to be parametrized by arithmetic objects called *Langlands parameters* in a natural way. One can attach a numerical invariant, namely the '*depth*' to each side of LLC. We will show that for a wildly ramified induced torus, in general the depth is not preserved under LLC for tori. In the second part, we will discuss the principal series component of *Gelfand-Graev representations* of G(F). We describe the component in terms of principal series *Hecke algebra*.

Notation

F : a field

$F^\times$: $F \setminus \{0\}$

$\overline{F}$: an algebraic closure of the field F

F^{sep} : separable closure of the field F in $\overline{F}$

$\mathbb{Z}$: the set of all integers

$\mathbb{Q}$: the set of all rational numbers

$\mathbb{R}$: the set of all real numbers

$\mathbb{R}_{\geq 0}$: the set of all non-negative real numbers

$\mathbb{C}$: the set of all complex numbers

$\mathbb{F}_q$: finite field with q elements

$\cong$: isomorphism

$\mathbf{G}$: a connected reductive algebraic group defined over the field F

$\mathbf{G}(F)$: the group of F- points of the algebraic group $\mathbf{G}$ defined over F

$\mathbb{G}_m$: multiplicative algebraic group

$\mathbb{G}_a$: additive algebraic group

$Z_G(g)$: centralizer of an element g in the group G

$Z(G)$: center of the group G

$(G : H)$: index of the subgroup H in the group G

Aut (V) : group of all automorphisms of V

$GL(V)$ or $GL_n(F)$: general linear group

$SL(V)$ or $SL_n(F)$: special linear group

$GSp_{2n}(F)$: general symplectic group

$[L : F]$: the degree of the finite field extension L/F

Gal (L/F) : the Galois group of a Galois field extension L over F

Card(A) : cardinality of the set A

$\varinjlim$, $\varprojlim$: direct limit and inverse limit respectively

$\square$: end of a proof

1

Introduction

Let F be a non-archimedean local field and W_F be the Weil group of F with respect to a fixed seperable closure F^{sep} of F. For any Galois extension L/F in F^{sep}, we have the upper numbering filtration $\{\mathrm{Gal}(L/F)^u\}_{u\geq 0}$ of the Galois group $\mathrm{Gal}(L/F)$ of the extension L/F (see [Ser, Chap. IV]). If H is any subgroup of the Galois group $\mathrm{Gal}(L/F)$, then in Proposition 10, we will show that the upper numbering filtration subgroups $\{H^u\}_{u\geq 0}$ satisfy the following intersection property:

$$\mathrm{Gal}(L/F)^u \cap H = H^{\psi_{L/L^H}(u)} \text{ for all } u \geq 0 \tag{1.1}$$

where L^H is the fixed field of H in L and φ_{L/L^H} is the classical *Hasse-Herbrand* function of the extension L/L^H, whose inverse is denoted by ψ_{L/L^H}. So, we have an upper numbering filtration subgroups $\{W_F^r\}_{r\geq 0}$ of the Weil group W_F. Parallelly, we have a natural filtration $\{F_r^\times\}_{r\geq 0}$ of the multiplicative group $F^\times$ defined by

$$F_r^\times := \{x \in F^\times \mid \mathrm{val}_F(x-1) \geq r\},$$

where val_F is the normalized valuation of F so that $\mathrm{val}_F(F^\times) = \mathbb{Z}$.

The central result of local class field theory provides us a cononical isomorphism $\tau : F^\times \xrightarrow{\sim} W_F^{ab}$ between the multiplicative group $F^\times$ of F and the abelianization of the Weil group W_F. This gives a bijection

$$\lambda_{\mathbb{G}_m} : \mathrm{Irr}(F^\times) \xrightarrow{\sim} \mathrm{Hom}(W_F, \mathbb{C}^\times) \tag{1.2}$$

between the characters of $F^\times$ and the characters of W_F. Moreover, the isomorphism τ respects the numbering on the filtration subgroups that means $F_r^\times \cong (W_F^r)^{ab}$.

The *local Langlands correspondence* is a family of conjectures that stipulates a vast generalization of the local class field theory isomorphism $\lambda_{\mathbb{G}_m}$ mentioned in Eq. (1.2). Let $\mathbf{G}$ be a connected reductive algebraic group defined over F and $\mathbf{G}(F)$ be the group of F-points of $\mathbf{G}$. Let $\mathbf{G}^\vee(\mathbb{C})$ be the complex dual of $\mathbf{G}(F)$ and W'_F be the Weil-Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$. A *Langlands parameter* for $\mathbf{G}(F)$ is a homomorphism $\phi : W'_F \to \mathbf{G}^\vee(\mathbb{C}) \rtimes W_F$, which are admissible as defined in [Bor1, §8.2]. Let $\mathrm{Irr}(\mathbf{G}(F))$ be the set of isomorphism classes of irreducible smooth complex representations of $\mathbf{G}(F)$ and $\Phi(\mathbf{G}(F))$ be the set of $\mathbf{G}^\vee(\mathbb{C})$-conjugacy classes of Langlands parameters for $\mathbf{G}(F)$. The local Langlands correspondence(LLC) for $\mathbf{G}(F)$ expects that irreducible representations in $\mathrm{Irr}(\mathbf{G}(F))$ can be parametrized by Langlands parameters in $\Phi(\mathbf{G}(F))$ in a natural way. In particular, the correspondence predicts the existance of a finite to one surjection map

$$\lambda_{\mathbf{G}} : \mathrm{Irr}(\mathbf{G}(F)) \to \Phi(\mathbf{G}(F)), \tag{1.3}$$

which satisfies some conditions mentioned in [Bor1, §10].

Here we are assuming that $\mathbf{G}(F)$ admits a local Langlands correspondence $\lambda_{\mathbf{G}}$. One can attached a numerical invariant namely the depth to each side of the LLC $\lambda_{\mathbf{G}}$ in the following way. Let $\mathcal{B}(\mathbf{G}, F)$ be the Bruhat-Tits building of $\mathbf{G}$ defined over F. In [MP3, MP4], Moy and Prasad defined the depth $\mathrm{dep}(\pi)$ of an irreducible smooth complex representation $(\pi, \mathcal{V}_\pi)$ of $\mathbf{G}(F)$ in terms of filtration subgroups $\{\mathbf{G}(F)_{x,r}\}_{r\geq 0}$ of the parahoric subgroup $\mathbf{G}(F)_{x,0}$ for each $x \in \mathcal{B}(\mathbf{G}, F)$ by

$$\mathrm{dep}(\pi) := \inf\{r \in \mathbb{R}_{\geq 0} \mid \exists\ x \in \mathcal{B}(\mathbf{G}, F) \text{ with } \mathcal{V}_\pi^{\mathbf{G}(F)_{x,r+}} \neq 0\},$$

where $\mathbf{G}(F)_{x,r+} = \cup_{s>r} \mathbf{G}(F)_{x,s}$. Also, for each $\phi \in \Phi(\mathbf{G}(F))$, one can define the notion of depth $\mathrm{dep}(\phi)$ of ϕ by the smallest number $\mathrm{dep}(\phi) \geq 0$ such that ϕ is trivial on W_F^r for all $r > \mathrm{dep}(\phi)$.

Now one fundamental question arises that if ϕ associates to π under LLC, can we expect that the depth will be preserved by LLC, that is

$$\mathrm{dep}(\pi) = \mathrm{dep}(\phi) \text{ if } \pi \in \Pi_\phi \tag{1.4}$$

where $\Pi_\phi \subset \mathrm{Irr}(\mathbf{G}(F))$ is the L-packet of $\phi \in \Phi(\mathbf{G}(F))$ defined by the preimage of ϕ under $\lambda_{\mathbf{G}}$. Though, on each side of LLC the depth is defined in different ways but it has been observed that the depth is preserved in many situations. To be more specific, when $\mathbf{G}$ is GL_n and its inner form, in [ABPS1] Aubert et. all observed that the depth is preserved under the correspondence $\lambda_{\mathbf{G}}$. Also for inner forms of SL_n, they showed in [ABPS1, Theorem 3.8] that the LLC preserves the depth for essencially tame Langlands parameters. In [Gan, §10], Ganapathy showed that LLC for $\mathrm{GSp}_4(F)$ preserves depth. When the residue field characteristic is large enough, M. Oi showed in [Oi2, Oi1] that the depth is preserved under LLC for the unitary groups and for the quasi-split classical groups. When $\mathbf{G}$ is a tamely induced torus $\mathcal{T} = \prod_{i=1}^{k} \mathrm{Res}_{L_i/F}\mathbb{G}_m$, Yu proved the equality (1.4) in [Yu2, Theorem 7.10].

However, for certain classical octahedral representation of $\mathrm{SL}_2(\mathbb{Q}_2)$, Reeder showed that depth will not be preserved under LLC (see [ABPS1, Example 3.5]). Other counter examples for the equality (1.4) have been constructed for inner forms of $\mathrm{SL}_n(F)$ [ABPS1] and in the case of $\mathrm{SL}_2(F)$, when F has characteristic 2 (see [AMPS]).

In this spirit, we will investigate the depth under local Langlands correspondece for wildly ramified induced tori. Now consider the induced F-torus $\mathbf{T} = \mathrm{Res}_{F'/F}\mathbb{G}_{\mathrm{m}}$, where F' is a finite separable extension of F and $\mathrm{Res}_{F'/F}$ denotes the Weil-restriction over F'/F . Let $\lambda_{\mathbf{T}} : \mathrm{Irr}(\mathbf{T}(F)) \to \Phi(\mathbf{T}(F))$ be the local Langlands correspondence for torus $\mathbf{T}$, which is a bijective map in this case. In Theorem 32, we show that $\varphi_{F'/F}(e \cdot \mathrm{dep}(\chi)) = \mathrm{dep}(\lambda_{\mathbf{T}}(\chi))$ for $\chi \in \mathrm{Irr}(\mathbf{T}(F))$, where $\varphi_{F'/F}$ is the Hasse-Herbrand function and e is the ramification index of the extension F'/F. Thus for all postitive depth characters χ of $\mathbf{T}(F)$, we have $\mathrm{dep}(\lambda_{\mathbf{T}}(\chi)) \geq \mathrm{dep}(\chi)$. In particular, when F'/F is a wildly ramified extension, we have $\mathrm{dep}(\lambda_{\mathbf{T}}(\chi)) > \mathrm{dep}(\chi)$ and when F'/F is a tamely ramified extension, $\mathrm{dep}(\lambda_{\mathbf{T}}(\chi)) = \mathrm{dep}(\chi)$. When $\mathbf{T}$ is a tamely induced wildly ramified torus (see Sec. 6.5.1), we show that $\mathbf{T}(F)$ admits characters for which depth is not preserved under LLC. In Section 6.5.2, we compute Hasse-Herbrand function for a certain wildly ramified extension of a cyclotomic field to illustrate the failure of depth preservation.

In the second part of this thesis, we will discuss about the principal series component of Gelfand-Graev representation (in short GGR) of $\mathbf{G}(F)$, where $\mathbf{G}$ is a connected reductive algebraic group defined over a non-archimedean local field F with its F-points group $\mathbf{G}(F)$. Whenever $\mathbf{H}$ is a (Zariski-) closed subgroup of the group $\mathbf{G}$ defined over F, we will denote the group of F-points of $\mathbf{H}$ by $\mathbf{H}(F)$. Fix a minimal F-parabolic subgroup

$\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and whose Levi factor $\mathbf{T}$ contains a maximal F-split torus $\mathbf{S}$ of $\mathbf{G}$. Then, $\mathbf{T}(F)$ acts on the space $\widehat{\mathbf{U}(F)}$ of all smooth characters $\psi : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$ via

$$t \cdot \psi = \psi^t : x \mapsto \psi(txt^{-1}) \text{ for } t \in \mathbf{T}(F) \text{ and } \psi \in \widehat{\mathbf{U}(F)}.$$

A smooth character $\psi : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$ is called non-degenerate (a.k.a. generic) if the stabilizer in $\mathbf{S}(F)$ lies in the center of the group $\mathbf{G}(F)$. We take a generic character ψ of $\mathbf{U}(F)$ and consider the induced representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$ realized by the functions whose support is compact mod $\mathbf{U}(F)$.

Let $\mathcal{R}(\mathbf{G}(F))$ denote the category of all smooth complex representations of $\mathbf{G}(F)$ and $\mathfrak{B}(\mathbf{G}(F))$ be the set of inertial equivalence classes of cuspidal pairs in $\mathbf{G}(F)$. To each class $\mathfrak{s} \in \mathfrak{B}(\mathbf{G}(F))$, Bernstein attached a full subcateory $\mathcal{R}^{\mathfrak{s}}(\mathbf{G}(F))$ of the category $\mathcal{R}(\mathbf{G}(F))$ such that $\mathcal{R}(\mathbf{G}(F))$ is the direct product of these subcategories. Through this decomposition, the induced representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$ decomposed into the direct sum of certain representations $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)_{\mathfrak{s}} \in \mathcal{R}^{\mathfrak{s}}(\mathbf{G}(F))$ for $\mathfrak{s} \in \mathfrak{B}(\mathbf{G}(F))$. Bushnell and Henniart showed in [BH, Theorem 4.2] that the Bernstein component $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)_{\mathfrak{s}}$ is finitely generated over $\mathbf{G}(F)$ for each $\mathfrak{s} \in \mathfrak{B}(\mathbf{G}(F))$. Our purpose is to give a refinement of this finiteness result of Bushnell and Henniart for principal series components. The refinement results can be stated as follows:

(A) Let λ be a smooth character of $\mathbf{T}(F)$. Then the pair $(\mathbf{T}(F), \lambda)$ determines an inertial equivalence class $\mathfrak{s}' := [\mathbf{T}(F), \lambda]_{\mathbf{G}(F)}$ in $\mathfrak{B}(\mathbf{G}(F))$, which gives a Bernstein block $\mathcal{R}^{\mathfrak{s}'}(\mathbf{G}(F))$ in the category $\mathcal{R}(\mathbf{G}(F))$. Bushnell-Kutzko types are known to exist for Bernstein blocks under suitable residue characteristic hypothesis [Fin, KY]. Let (K, ρ) be a $\mathfrak{s}'$-type in $\mathbf{G}(F)$ and $\mathcal{H}(\mathbf{G}(F), \rho)$ is the Hecke algebra associated to the pair (K, ρ). Then the Bernstein component $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)_{\mathfrak{s}'}$ is generated by the ρ-isotypical component $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho}$ of the representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$. In Theorem 40, we will show that the ρ-isotypical component $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho}$ is a cyclic $\mathcal{H}(\mathbf{G}(F), \rho)$-module.

(B) Now assume that $\mathbf{T}$ is split and ψ is a non-degenerate character of $\mathbf{U}(F)$ of generic depth-zero (see §7.5 for definition). If $\lambda \neq 1$, then assume further that the group $\mathbf{G}$ has connected center. In that case, $\mathcal{H}(\mathbf{G}(F), \rho)$ will be an Iwahori-Hecke algebra.

It contains a finite subalgebra $\mathcal{H}_{W_\lambda}$. The algebra $\mathcal{H}_{W_\lambda}$ has a one dimensional representation sgn. In Theorem 43, we will show that as $\mathcal{H}(\mathbf{G}(F),\rho)$-module, the ρ-isotypic component $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^\rho$ is isomorphic to $\mathcal{H}(\mathbf{G}(F),\rho)\otimes_{\mathcal{H}_{W_\lambda}}$ sgn.

For positive depth character λ of $\mathbf{T}(F)$, Theorem 43 assumes that characteristic of F is 0 and the residue characteristic of F is not too small. Then, Theorems 40 and Theorem 43 generalize the main result of Chan and Savin in [CS] who treat the case $\lambda = 1$ for $\mathbf{T}$ split, i.e., unramified principal series blocks of split groups. Our proofs benefit from the ideas in [CS]. However they are quite different. The existence of a generator in $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^\rho$ is concluded by specializing quite general results in [BH, BK]. For Theorem 43, instead of computing the effect of intertwiners on the generator as in [CS], we make a reduction to depth-zero case i.e., we will show that if the result holds for depth-zero characters λ, then it holds for positive depth characters also under the above conditions. Then the depth-zero situation can be reduced to a finite group analogue of the question. There it holds by a result in [Ree, §7.2] of Reeder .

Structure of the thesis:

In Part I, we first recall some preliminaries such as local fields (in §2.1), the Weil group (in §2.3), ramification groups (in §2.4), structure of root subgroups (in §3.4), Bruhat-Tits buildings of $\mathbf{G}(F)$ (in §3.5), Moy-Prasad filtration of $\mathbf{G}(F)$ (in §3.6), Bernstein decomposition (in §4.1), Hecke algebra (in §4.2), and the notion of types (in §4.3).

In Part II, we will start with the investigation of the depth under induction and Shapiro's isomorphism. In section 5.1 and 5.2 , we will give depth-comparison results for induction and Shapiro's isomorphism respectively using the Hasse-Herbrand function. Then, in Chapter 6, we will review the statement of the local Langlands correspondence for tori and there, we investigate the depth under LLC for induced tori by proving Theorem 32. Moreover, in section 6.6, we will mention a recent work [AP] of Aubert and Plymen, who have generalized our Theorem 32 by giving a depth comparison result under the enhanced local Langlands correspondence for the Weil restricted groups.

In Part III, we will study about Gelfand-Graev representations of a p-adic group $\mathbf{G}(F)$. In section 7.3, we show that the ρ-isotypical component of a Gelfand-Graev representation of $\mathbf{G}(F)$ is a cyclic Hecke algebra module (in Theorem 40). In section 7.4, we will briefly discuss about the principal series Hecke algebra as constructed by

Roche in [Roc]. In section 7.5, we will conclude the thesis by proving Theorem 43 about principal series component of Gelfand-Graev representation.

Part I

Preliminaries

2

Basic Theory

2.1 Local Fields

Let F be a *non-archimedean local field* with respect to a (non-trivial) *discrete valuation* $\mathrm{val}_F : F \to \mathbb{R} \cup \{\infty\}$ defined on it. The discrete valuation val_F determines a topology on the field F such that F is complete with respect to that topology. For example, *the completion* $\mathbb{Q}_p$ *of the set* $\mathbb{Q}$ of rational numbers with respect to the p-adic valuation, *the field* $\mathbb{F}_p((t))$ *of formal power series* over the finite field $\mathbb{F}_p$ and any finite extension of $\mathbb{Q}_p$ or, $\mathbb{F}_p((t))$ are non-archimedean local fields, where p is a prime number.

We write $\mathcal{O}_F := \{x \in F \mid \mathrm{val}_F(x) \geq 0\}$ for the valuation ring (a.k.a. *ring of integers*) of F with it's unique maximal ideal $\mathcal{P}_F := \{x \in F \mid \mathrm{val}_F(x) > 0\}$ and ϖ_F for a uniformizer. Let k_F be the residue field $\mathcal{O}_F/\mathcal{P}_F$ of F. Since F is a local field, k_F is a finite field, say $\mathbb{F}_q$ with cardinality q equal to p^r for some natural number r, where the characteristic of k_F is the prime number p.

2.2 Unramified and Ramified extensions:

Fix an algebraic closure $\overline{F}$ of the local field F. Throughout this thesis, we will assume every field extension of F to be contained in $\overline{F}$. Consider a degree n field extension E/F, where $F \subset E$ are local fields. Then the valuation val_F of F can be extended uniquely to a valuation val_E of E. Let $\mathcal{O}_E, \mathcal{P}_E$ and k_E respectively denote the ring of integers, the unique maximal ideal and the residue field of E. Then we have the inclusions $\mathrm{val}_F(F^\times) \subset \mathrm{val}_E(E^\times)$ and $k_F \subset k_E$. The *ramification index* of the finite extension E/F is defined by the index $e_{E/F} := \big(\mathrm{val}_E(E^\times) : \mathrm{val}_F(F^\times)\big)$ and the *inertia degree* of the extension E/F is defined by the degree $f_{E/F} := [k_E : k_F]$ of the residue field extension k_E/k_F.

Definition 1 (Unramified Extension)**.** *A finite extension E/F is called unramified extension if the corresponding residue field extension k_E/k_F is separable and the degrees of the field extensions E/F and k_E/k_F are equal, i.e. $[E : F] = [k_E : k_F]$.*

Let L/F is an algebraic extension and as before we denote the corresponding invariant as $\mathcal{O}_L, \mathcal{P}_L$ and k_L respectively. Then the extension L/F is said to be unramified extension if it is the union of finite unramified subextensions. It is an well known fact that the composite of any two unramified extensions of F is again an unramified extension. Consider the maximal unramified subextension L^{ur}/F of L/F, which is the composition of all unramified subextensions in L/F. Then the residue field of L^{ur} is the separable closure k_F^{sep} of k_F in the residue field extension k_L/k_F of L/F

Definition 2 (Ramified Extension)**.** *(i) An algebraic extension L/F is said to be purely ramified if $L^{ur} = F$.*

(ii) An algebraic extension L/F is said to be tamely ramified if the residue fields' extension k_L/k_F is separable and the degree of every finite subextension of L/L^{ur} is prime to p.

(iii) The composite L_t/F of all tamely ramified subextensions in an algebraic extension L/F is called the maximal tamely ramified subextension of L/F.

(iv) An extension L/F is said to be wildly ramified if it is not a tamely ramified extension i.e., $L_t \neq L$.

Example 3. Consider $F = \mathbb{Q}_p$ and $L = \mathbb{Q}_p(\zeta_n)$, where ζ_n is the primitive n^{th} root of unity. If n is prime to p, then the extension L/F is an unramified extension of degree d such that d is the smallest natural number with $p^d \equiv 1 \bmod n$. If $n = p^m$ for some natural number m, then the extension L/F is a purely ramified extension with degree

$(p-1)p^{m-1}$. Suppose $n = n'p^m$ for some natural numbers n', m with $(n', p) = 1$. Then, the maximal unramified subextension L^{ur} of L/F is $\mathbb{Q}_p(\zeta_{n'})$ and the maximal tamely ramified sub-extension L_t of L/F is $L^{ur}(\zeta_p)$.

2.3 The Weil Group

Consider the local field F with its residue field k_F, where we assume k_F to be the finite field $\mathbb{F}_q$. Let F^{sep} and k_F^{sep} be fixed separable closures of F and k_F respectively. If F^{ur} denotes the unique maximal unramified extension of F contained in F^{sep}, then from [Fro, §7, Cor.2] the Galois group $\text{Gal}(F^{\text{ur}}/F)$ is topologically isomorphic with the Galois group $\text{Gal}(k_F^{\text{sep}}/k_F)$. Since the only finite extensions of $\mathbb{F}_q$ are $\mathbb{F}_{q^n}$ for various natural number $n \in \mathbb{Z}_{>0}$ and $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \simeq \mathbb{Z}/n\mathbb{Z}$ with the canonical generator being the automorphism defined by $x \mapsto x^q$, then by definition

$$\text{Gal}(k_F^{\text{sep}}/k_F) = \varprojlim_{l/k_F \text{ finite}} \text{Gal}(l/k_F) \simeq \varprojlim_{n} \mathbb{Z}/n\mathbb{Z} =: \widehat{\mathbb{Z}}$$

with the topological generator being the (Frobenius) automorphism $\text{Frob}_{k_F} : x \mapsto x^q$ i.e., Frob_{k_F} generates the dense subgroup $\mathbb{Z}$ of $\widehat{\mathbb{Z}}$. Therefore, $\text{Gal}(F^{\text{ur}}/F)$ is topologically isomorphic with the pro-finite group $\widehat{\mathbb{Z}}$. Let $I_F := \text{Gal}(F^{\text{sep}}/F^{\text{ur}})$ be the inertia group of F. Then we have the following exact sequence of topological groups:

$$1 \to I_F = \text{Gal}(F^{\text{sep}}/F^{\text{ur}}) \to \text{Gal}(F^{\text{sep}}/F) \to \text{Gal}(F^{\text{ur}}/F) \simeq \widehat{\mathbb{Z}} \to 1.$$

Let W_F be the inverse image of $\langle \text{Frob}_{k_F} \rangle$ (i.e., equal to $\mathbb{Z}$) in $\text{Gal}(F^{\text{sep}}/F)$. Now, the topology in W_F will be that for which the inertia subgroup I_F has the pro-finite topology induced from the natural topology of $\text{Gal}(F^{\text{sep}}/F)$. Then the topological group W_F is called the *Weil group* of F relative to the separable closure F^{sep}.

2.4 Ramification Groups

Consider F to be a complete field under a non-archimedean valuation val_F. Let $\mathcal{O}_F$ denotes the corresponding valuation ring of F with the unique maximal ideal $\mathcal{P}_F$ and the corresponding residue field k_F, which is equal to $\mathcal{O}_F/\mathcal{P}_F$. Let L be a Galois extension (may be infinite) of the field F. Consider the Galois group $G := \mathrm{Gal}(L/F)$ of the extension L/F. In this section, we will mainly study about some filtration subgroups of the group $G = \mathrm{Gal}(L/F)$. In particular, we study some chain of subgroups of G with some nice properties.

2.4.1 Ramification Groups in Lower Numbering

Let F be a complete field and L be a finite Galois extension of F. Then L is a again a complete field under extended non-archimedean valuation val_L. Write $\mathcal{O}_L$, $\mathcal{P}_L$ for the ring of integers of L and the unique maximal ideal of $\mathcal{O}_L$ respectively, with the residue field k_L. Further assume that the residue field extension k_L/k_F is separable. If $e_{L/F}$ and $f_{L/F}$ are the ramification index and inertia degree of the field extension L/F respectively, then we have the identity $[L:F] = e_{L/F} \cdot f_{L/F}$. Naturally, the Galois group $\mathrm{Gal}(L/F)$ acts on the valuation ring $\mathcal{O}_L$. Then, we have the following

Lemma 4. *[Ser, Chap. IV, Lemma 1] Let $\sigma \in G := \mathrm{Gal}(L/F)$ and i be an integer ≥ -1. Then σ operates trivially on the quotient ring $\mathcal{O}_L/\mathcal{P}_L^{i+1}$ if and only if $\mathrm{val}_L(\sigma(x)-x) \geq i+1$ for all $x \in \mathcal{O}_L$.*

Proof. Lemma follows from the fact that for any $[a] = a + \mathcal{P}_L^{i+1} \in \mathcal{O}_L/\mathcal{P}_L^{i+1}$ with $a \in \mathcal{O}_L$,

$$\begin{aligned} \sigma \cdot [a] = [a] &\text{ if and only if } \sigma(a) + \mathcal{P}_L^{i+1} = a + \mathcal{P}_L^{i+1} \\ &\text{ if and only if } \sigma(a) - a \in \mathcal{P}_L^{i+1} \\ &\text{ if and only if } \mathrm{val}_L(\sigma(a) - a) \geq i+1. \end{aligned}$$

□

For integer $i \geq -1$, define G_i to be the set of all $\sigma \in G = \mathrm{Gal}(L/F)$ such that σ operates trivially on $\mathcal{O}_L/\mathcal{P}_L^{i+1}$. Then $G_{-1} = G$. The groups G_i are called *ramification groups*. They form a decreasing filtration

$$G = G_{-1} \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$$

of normal subgroups of G. Conventionally, G_0 and G_1 are called the inertia and wild inertia subgroup of G respectively with the quotient G_0/G_1 is called the tame quotient.

Extend the definition of G_u for all real numbers $u \geq -1$ by setting

$$G_u = G_i, \text{ where } i \text{ is the least integer } \geq u. \tag{2.1}$$

This numbering of ramification groups is called *lower numbering*. Lower numbering behaves nice with respect to intersections i.e.,

Proposition 5. *[Ser, Chap. IV, Prop. 2] If H is a subgroup of $G = \mathrm{Gal}(L/F)$, then $G_u \cap H = H_u$ for $u \in \mathbb{R}_{\geq -1}$*

Proof. Let L^H be the subextension of L fixed by H. Then $H = \mathrm{Gal}(L/L^H) \subset G$. Therefore, for each integer $i \geq -1$,

$$\begin{aligned} \sigma \in H_i &\iff \sigma \in H \text{ and } \sigma \text{ operates trivially on } \mathcal{O}_L/\mathcal{P}_L^{i+1} \\ &\iff \{\sigma \in G \mid \sigma \text{ operates trivially on } \mathcal{O}_L/\mathcal{P}_L^{i+1}\} \cap H \\ &\iff G_i \cap H. \end{aligned}$$

Now the Proposition follows from Eq.(2.1). □

2.4.2 Hasse-Herbrand Function and Upper Numbering

In this section, we will mainly describe how the lower numbering of the ramification groups behaves under quotients and will define another filtration of the Galois group to extend the filtration for infinite Galois extensions.

Hasse-Herbrand function: As before, consider the finite Galois group $G = \mathrm{Gal}(L/F)$ and it's lower numbering ramification subgroups G_u for all real numbers $u \geq -1$. For $t > 0$, $(G_0 : G_t)$ denotes the index of the subgroup G_t in G_0 and for $t \in [-1, 0]$, $(G_0 : G_t)$ is $(G_t : G_0)^{-1}$. Therefore, $(G_0 : G_t) = (G_t : G_0)^{-1} = (G_0 : G_0)^{-1} = 1$ for $-1 < t \leq 0$. Now define $\varphi_{L/F} : [-1, \infty) \to \mathbb{R}$ to be the map

$$r \mapsto \int_0^r \frac{1}{(G_0 : G_t)} dt.$$

Explicitly, one can write the function $\varphi_{L/F}$ as:

$$\varphi_{L/F}(r) = \begin{cases} r, & \text{if } -1 \leq r \leq 0 \\ \frac{g_1}{g_0} r, & \text{if } 0 \leq r \leq 1 \\ \frac{1}{g_0}[g_1 + g_2 + \cdots + g_m + (r-m)g_{m+1}], & \text{if } 0 < m \leq r \leq m+1, m \in \mathbb{Z}, \end{cases} \tag{2.2}$$

here $g_i = \text{Card}(G_i)$ denotes the cardinality of G_i for integers $i \geq 0$. The function $\varphi_{L/F}$ is called the *Hasse-Herbrand* function. It has the basic properties [Ser, Chap. IV, §3]:

(a) $\varphi_{L/F}$ is continuous, piecewise linear, increasing and concave.

(b) $\varphi_{L/F}(0) = 0$.

(c) $\varphi_{L/F}$ is a homeomorphism of $[-1, \infty)$ onto itself.

(d) If $\varphi'_{L/F}$ be the left derivative of $\varphi_{L/F}$ then for $r \in \mathbb{R}_{\geq -1}$

$$\varphi'_{L/F}(r) = \frac{1}{(G_0 : G_r)}. \tag{2.3}$$

(e) If $\varphi'_{L/F}$ be the right derivative of $\varphi_{L/F}$ then

$$\varphi'_{L/F}(r) = \begin{cases} \frac{1}{(G_0:G_r)} & \text{if } r \text{ is not an integer}, \\ \frac{1}{(G_0:G_{r+1})} & \text{if } r \text{ is an integer}. \end{cases} \tag{2.4}$$

If an extension L/F is not Galois, define $\varphi_{L/F} = \varphi_{E'/F} \circ \varphi^{-1}_{L/E'}$, where E' is any Galois extension of F contained in L. Convensionaly, the inverse $\varphi^{-1}_{L/F}$ is denoted $\psi_{L/F}$. Then the function $\psi_{L/F}$ has also the following properties [Ser, Chap. IV, §3]:

(a) $\psi_{L/F}$ is continuous, piecewise linear, increasing and convex.

(b) $\psi_{L/F}(0) = 0$.

(c) $\psi_{L/F}$ is a homeomorphism of $[-1, \infty)$ onto itself.

(d) If z is an integer, the $\psi_{L/F}(z)$ is also an integer.

Using the Hasse-Herbrand function, we have the following quotient relation of lower numbering ramification groups:

Theorem 6 (Herbrand)**.** *Let H be a normal subgroup of the Galois group $G = \mathrm{Gal}(L/F)$ and L^H be the corresponding subfield of L fixed by H i.e., $H = \mathrm{Gal}(L/L^H)$. Then we have the following lower numbering relation under quotient:*

$$G_r H/H = (G/H)_s, \text{ for } s = \varphi_{L/L^H}(r).$$

Here, we omit the proof of this standard theorem. Interested readers may look [Neu, Theorem 10.7] for the proof. The functions $\varphi_{L/F}$ and $\psi_{L/F}$ satisfy the following Chain relations or transitivity formulas:

Proposition 7. *[Ser, Chap. IV, Prop. 15] Suppose H be a normal subgroup of the Galois group $G = \mathrm{Gal}(L/F)$ with the corresponding subfield L^H of L fixed by H. Then,*

$$\varphi_{L/F} = \varphi_{L^H/F} \circ \varphi_{L/L^H} \text{ and } \psi_{L/F} = \psi_{L/L^H} \circ \psi_{L^H/F}.$$

Ramification groups in upper numbering: The lower numbering of the ramification groups G_s has many nice properties, including the compatible with subgroup intersections but it does not behave well while taking quotients (which follows from Herbrand's Theorem 6), so it is very natural to look for another numbering or filtrations of the Galois group that behaves well under quotients. And as a result, one will able to define that filtrations for infinite Galois extensions.

Let L/F be a finite Galois extension and the corresponding finite Galois group $G = \mathrm{Gal}(L/F)$ has the lower numbering filtration $\{G_s\}_{s\in\mathbb{R}_{\geq -1}}$. Then using the Herbrand function $\varphi_{L/F}$, one can define the following filtration of G:

Definition 8. *Define an upper numbering on ramification groups by setting*

- $G^v = G_u$ *if* $v = \varphi_{L/F}(u)$ *for* $u \in [-1, \infty)$ *or,*
- $G^v = G_{\psi_{L/F}(v)}$ *for* $v \in [-1, \infty)$.

Upper numbering filtration determines the ramification groups of a quotient group and it is given by the following result

Proposition 9. *If H is a normal subgroup of G, then for any $v \in [-1, \infty)$,*

$$(G/H)^v = G^v H/H.$$

Proof. Let L^H be the subfield of L fixed by the normal subgroup H. So $G/H = \mathrm{Gal}(L^H/F)$. Fix an element $v \in [-1, \infty)$. Then,

$$\begin{aligned} G^v H/H &= G_{\psi_{L/F}(v)} H/H \\ &= (G/H)_{\varphi_{L/L^H} \circ \psi_{L/F}(v)} \quad \text{(Using Herbrand Theorem 6)} \\ &= (G/H)_{\psi_{L^H/F}(v)} \quad \text{(Using Proposition 7)} \\ &= (G/H)^v. \end{aligned}$$

□

So upper ramification groups behave well under quotients and therefore, one can define the upper numbering ramification groups for infinite extensions. For an infinite Galois extension Ω of F, define the ramification groups on $G = \mathrm{Gal}(\Omega/F)$ by:

$$G^v = \varprojlim_{E/F \text{be finite, } E \subset \Omega} \mathrm{Gal}(E/F)^v.$$

for $v \in [-1, \infty)$. As a consequence, we can say that the upper numbering ramification groups are more natural than the lower numbering ramification groups.

Now let L/F be any Galois extension (may be infinite) of local fields and E be a finite extension of F contained in L. Write $G = \mathrm{Gal}(L/F)$ and $H = \mathrm{Gal}(L/E)$.

Proposition 10 (Lemma 1, [MP1])**.** *For all $r \geq 0$, $G^r \cap H = H^{\psi_{E/F}(r)}$.*

Proof. Let E' be a finite Galois extension of F in L containing E. Write $I := \mathrm{Gal}(L/E')$.

Then

$$\begin{aligned}
(G/I)^r \cap (H/I) &= (G/I)_{\psi_{E'/F}(r)} \cap (H/I) \\
&= (H/I)_{\psi_{E'/F}(r)} \\
&= (H/I)^{\varphi_{E'/E}(\psi_{E'/F}(r))} \\
&= (H/I)^{\varphi_{E'/E}\psi_{E'/E}\psi_{E/F}(r)} \\
&= (H/I)^{\psi_{E/F}(r)}.
\end{aligned}$$

The proposition now follows by taking inverse limit over E'. $\square$

3

Buildings and filtrations

Let F be a complete field under a non-trivial discrete valuation val_F and the field is strictly Henseline. Let $\mathcal{O}_F$ be the corresponding valuation ring of F with the unique maximal ideal $\mathcal{P}_F$ and the corresponding residue class field $k_F = \mathcal{O}_F/\mathcal{P}_F$. Let $\mathbf{G}$ be a connected reductive algebraic group defined over F and $\mathbf{G}(F)$ be the F-points group of $\mathbf{G}$. This chapter contains a brief review of the construction of Bruhat-Tits building on $\mathbf{G}(F)$ and Moy-Prasad filtration of parahoric subgroups of $\mathbf{G}(F)$ associated to each point of the building. For a detailed and comprehensive discussion, interested readers can refer to the book [Lan1] by Landvogt.

3.1 Root datum

By a *root datum* we mean a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, where X and $X^\vee$ are two finitely generated torsion free $\mathbb{Z}$-modules with a perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbb{Z}$. Here, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ respectively with a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha \in \Phi$, define the endomorphisms $s_\alpha : X \to X$ and $s_{\alpha^\vee} : X^\vee \to X^\vee$ respectively by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \, \alpha \text{ and } s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \, \alpha^\vee,$$

for $x \in X$ and $y \in X^\vee$. Then the above data is subjected to the following conditions:

(R1) We have $\langle \alpha, \alpha^\vee \rangle = 2$ for each $\alpha \in \Phi$, (R2) We have $s_\alpha(\Phi) \subset \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$ for each $\alpha \in \Phi$.

The elements of the finite set Φ (resp. $\Phi^\vee$) are called roots (resp. coroots) of the root datum Ψ. It is clear that if $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is a root datum, so is $\Psi^\vee = (X^\vee, \Phi^\vee, X, \Phi)$ and $\Psi^\vee$ is called the dual of Ψ.

Let $\mathbf{Q} := \mathbf{Q}(\Phi)$ be the submodule of X generated by Φ and $V := \mathbf{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the corresponding vector space over $\mathbb{Q}$. Similarly we have $\mathbf{Q}^\vee := \mathbf{Q}(\Phi^\vee)$ and $V^\vee := \mathbf{Q}^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. One can identify $V^\vee$ with the dual space of the vector-space V. If $\Phi \neq \emptyset$, then Φ (identify with $\Phi \otimes 1 \subset V$) is a *root system* on V in the sense of [Bou2]. Let $W = W(\Phi)$ be the finite group generated by reflection maps s_α for $\alpha \in \Phi$. Then W is called the spherical *Weyl group* of Φ and W can also be identified with the group of automorphisms of $X^\vee$ generated by the $s_{\alpha^\vee}$ for $\alpha^\vee \in \Phi^\vee$.

3.2 Root datum of a reductive group:

Let $\mathbf{G}$ be a connected reductive algebraic group defined over a non-archimedean local field F. Consider $\mathbf{G}(F)$ to be the group of F-points of $\mathbf{G}$. Naturally, $\mathbf{G}(F)$ has the structure of a Hausdorff, locally compact and totally disconnected topological group.

Let $\mathbf{S}$ be a maximal F-split torus (i.e., F-split torus and maximal for these properties) of $\mathbf{G}$. Let $\mathcal{N} = N_{\mathbf{G}}(\mathbf{S})$ denotes the normalizer and $\mathcal{Z} = Z_{\mathbf{G}}(\mathbf{S})$ denotes the centralizer of $\mathbf{S}$ in $\mathbf{G}$. Both $\mathcal{N}$ and $\mathcal{Z}$ are F-subgroups of $\mathbf{G}$. Denote $X^*(S) = \mathrm{Hom}_F(\mathbf{S}, \mathbf{G}_m)$ for the group of rational characters of $\mathbf{S}$ defined over F and $X_*(S) = \mathrm{Hom}_F(\mathbf{G}_m, \mathbf{S})$ for the group of co-characters of $\mathbf{S}$ defined over F. Both $X_*(S)$ and $X^*(S)$ are free abelian groups of finite rank. Also, there exists the following perfect pairing of these groups:

$$\langle , \rangle : X_*(S) \times X^*(S) \to \mathbb{Z},$$

where $\langle \lambda, \chi \rangle$ is defined by the integer such that $(\chi \circ \lambda)(t) = t^{\langle \lambda, \chi \rangle}$ for all $t \in F^\times$ with $\lambda \in X_*(S), \chi \in X^*(S)$.

Fix $V_0 = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ for the $\mathbb{R}$-vector space corresponding to $X_*(S)$. One can identify its dual space $V_0^\vee$ with $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ and get a canonical pairing $\langle , \rangle : V_0 \times V_0^\vee \to \mathbb{R}$, which extends the above pairing $\langle , \rangle$.

As in [Spr, §3.5], let $\Phi_F = \Phi(\mathbf{G}, \mathbf{S}, F) \subset X^*(\mathbf{S})$ be the relative root system of $\mathbf{G}$ with respect to the maximal F-split torus $\mathbf{S}$. This is a root system in the above sense, lying in a subspace of $X^*(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ spanned by Φ_F. As usual, we will denote $\Phi_F^\vee \subset X_*(S)$ for the dual root system of Φ_F. One can associate to the tuple $(\mathbf{G}, \mathbf{S}, F)$ a root datum $\Psi_F = \Psi(\mathbf{G}, \mathbf{S}, F) = (X^*(S), \Phi_F, X_*(S), \Phi_F^\vee)$.

A root $\alpha \in \Phi_F$ is called *divisible* if $\frac{1}{2}\alpha \in \Phi_F$. For any arbitrary subset $\Phi \subset \Phi_F$, we will denote its non-divisible elements by $\Phi^{\text{red}} = \{\alpha \in \Phi \mid \frac{1}{2}\alpha \notin \Phi\}$. A subset $\Phi \subset \Phi_F$ is called *closed* if for any $\alpha, \beta \in \Phi$, we have $\{n\alpha + m\beta \mid m, n \in \mathbb{Z}_{>0}\} \cap \Phi_F \subset \Phi$.

From [Bor2], we obtain that there exists a unique closed and connected unipotent F-subgroup $\mathbf{U}_\alpha$ of $\mathbf{G}$ for each $\alpha \in \Phi_F$. Each $\mathbf{U}_\alpha$ is normalized by $\mathcal{Z} = Z_{\mathbf{G}}(\mathbf{S})$. The subgroup $\mathbf{U}_\alpha$ is called root subgroup of $\mathbf{G}$ associated to the root $\alpha \in \Phi_F$.

One can define an equivalence relation $\sim$ on V_0 as follows: for $v_1, v_2 \in V_0$, $v_1 \sim v_2$ if and only if $\alpha(v_1)$ and $\alpha(v_2)$ have the same sign or $\alpha(v_1) = \alpha(v_1) = 0$ for all $\alpha \in \Phi_F$. The equivalence classes are called faces in V_0 with respect to Φ_F. One can identify the Weyl group $W = W(\Phi_F)$ of Φ_F with the quotient group $W(\mathbf{G}, \mathbf{S}, F) := \mathcal{N}(F)/\mathcal{Z}(F)$, which operates on V_0 and $V_0^\vee$ in a natural way. The field F being discrete valuation field, we will have an affine action of $\mathcal{N}(F)$ on V_0 lifting the action of $\mathcal{N}(F)/\mathcal{Z}(F)$ with translation action of $\mathcal{Z}(F)$.

3.3 The Apartments

In this section we will associate to the tuple $(\mathbf{G}, \mathbf{S}, F)$ an affine space $A_0(\mathbf{G}, \mathbf{S}, F)$ under some vector subspace V on which the group $\mathcal{N}(F)$ operates. Let $\mathcal{Z}_c$ be the maximal central F-torus of $\mathbf{G}$ with its maximal F-split F-subtorus $\mathcal{Z}_{c,s}$.

Let $X_F^*(\mathcal{Z})$ be the group of F-rational characters of $\mathcal{Z}$. Then $X_F^*(\mathcal{Z})$ can be identified with a finite index subgroup of $X^*(\mathbf{S})$. Then we have the following standard group homomorphism from $\mathcal{Z}(F)$ to $V_0 = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$:

Proposition 11 (Lemma 1.1, [Lan1])**.** *There exists a unique group homomorphism*

$$\nu_0 : \mathcal{Z}(F) \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{Q} \subset V_0$$

characterized by $\langle \nu_0(z), \chi \rangle = -\mathrm{val}_F(\chi(z))$ *for all* $z \in \mathcal{Z}(F)$ *and* $\chi \in X_F^*(\mathcal{Z})$.

We will denote $\mathcal{Z}_b(F)$ for the subgroup $\ker(\nu_0)$, the kernel of the group homomorphism

ν_0. In the following Proposition, we will recall some properties of $\mathcal{Z}_b(F)$ described in [Lan1, Prop. 1.2. and Prop. 1.3.].

Proposition 12. *If one consider the F-analytic topology on the group $\mathcal{Z}(F)$, that induces a topology on $\mathcal{Z}_b(F)$ also. Then,*

(i) $\mathcal{Z}_b(F)$ is the maximal compact open subgroup of $\mathcal{Z}(F)$.

(ii) $\mathcal{Z}_b(F)$ is a normal subgroup of $\mathcal{N}(F)$ (follows from the normality of $\mathcal{Z}(F)$ in $\mathcal{N}(F)$).

(iii) We have the following short exact sequence of groups:

$$0 \to \mathcal{Z}(F)/\mathcal{Z}_b(F) \to \mathcal{N}(F)/\mathcal{Z}_b(F) \to \mathcal{N}(F)/\mathcal{Z}(F) \to 1.$$

(iv) $\mathcal{Z}(F)/\mathcal{Z}_b(F)$ is a free abelian group containing $X_*(\mathbf{S})$ and its rank is $\dim(V_0)$.

Now we will consider the subspace V' of V_0 defined by

$$V' := \{v \in V_0 \mid \alpha(v) = 0 \text{ for all } \alpha \in \Phi_F = \Phi(\mathbf{G}, \mathbf{S}, F)\},$$

which can also be identified with $X_*(\mathcal{Z}_{c,s}) \otimes_{\mathbb{Z}} \mathbb{R}$ i.e., the group of co-characters of the central maximal split torus $\mathcal{Z}_{c,s}$ of $\mathbf{G}$ tensored with $\mathbb{R}$. Then, V' is the zero space when $\mathbf{G}$ is semi-simple. In general, V' is not zero but when $\mathbf{G}$ is reductive, then the group $W \simeq \mathcal{N}(F)/\mathcal{Z}(F)$ acts trivially on V'.

Define the $\mathbb{R}$-vector space V by the following quotient space:

$$V = V(\mathbf{G}, \mathbf{S}, F) = V_0/V'$$

which can be identified with $X_*(\mathbf{S}/\mathcal{Z}_{c,s}) \otimes_{\mathbb{Z}} \mathbb{R}$. Consider the map $\nu_0 : \mathcal{Z}(F) \to V$, which is the composition $(\mathrm{pr} \circ \nu_0)$ of the map $\nu_0 : \mathcal{Z}(F) \to V_0$, and the natural projection map $\mathrm{pr} : V_0 \twoheadrightarrow V = V_0/V'$. Therefore, we have the homomorphism $\nu_0 : \mathcal{Z}(F)/\mathcal{Z}_b(F) \to V$.

The conjugation action of W on $\mathbf{S}$ induces a linear action on V_0, which induces a canonical group homomorphism

$$j : W \to GL(V_0)$$

and the image of this group homomorphism j acts trivially on V'. Therefore, we have the group homomorphism $j' : W \to GL(V)$ induces from j.

Now let us consider any affine space A_0 under the vector space V. If $\mathrm{Aff}(A_0)$ denotes the set of affine bijections $A_0 \to A_0$, we have $\mathrm{Aff}(A_0) \simeq V \rtimes GL(V)$.Then the group homomorphisms $\nu_0 : \mathcal{Z}(F)/\mathcal{Z}_b(F) \to V$ and $j' : W \to GL(V)$ assure the action of both the groups $\mathcal{Z}(F)/\mathcal{Z}_b(F)$ and $\mathcal{N}(F)/\mathcal{Z}(F)$ on the affine space A_0. Now, one can construct a group homomorphism

$$\nu : \mathcal{N}(F) \to \mathrm{Aff}(A_0) \simeq V \rtimes GL(V),$$

extending the homomorphism $\nu_0 : \mathcal{Z}(F)/\mathcal{Z}_b(F) \to V$. In the following Proposition, we will have the clear statement regarding this.

Proposition 13. *[Lan1, Proposition 1.8] Up to unique isomorphism, there exists a canonical affine space $A_0 = A_0(\mathbf{G}, \mathbf{S}, F)$ under $V = V(\mathbf{G}, \mathbf{S}, F)$ together with a group homomorphism $\nu : \mathcal{N}(F) \to \mathrm{Aff}(A_0)$ extending the map $\nu_0 : \mathcal{Z}(F) \to V$.*

Now, we can define the so called **apartment** of $\mathbf{G}$ associated to $\mathbf{S}$ in the following way:

Definition 14 (The standard apartment:)**.** *The affine space(as mentioned in Proposition 13) $A_0 = A_0(\mathbf{G}, \mathbf{S}, F)$ together with a group homomorphism $\nu : \mathcal{N}(F) \to \mathrm{Aff}(A_0)$ is called the standard apartment of $\mathbf{G}$ with respect to the maximal F-split torus $\mathbf{S}$.*

3.4 Structure of the root subgroups.

In this section, we recall the structure of the root groups as described in the book [Lan1] or, [BT2, section 4]. As before, let $\mathbf{G}$ be a connected reductive algebraic group defined over the local field F. Moreover, for this section we will assume further that $\mathbf{G}$ is F-quasi-split. Let $\mathbf{S}$ be a maximal F-split torus of $\mathbf{G}$ and $\mathcal{Z} = Z_{\mathbf{G}}(\mathbf{S})$ denotes the centralizer of $\mathbf{S}$ in $\mathbf{G}$. Then, $\mathcal{Z}$ is a F-subgroups of $\mathbf{G}$ and in particular, $\mathcal{Z}$ is a maximal torus of $\mathbf{G}$. Suppose $\widetilde{F}$ be the splitting field of $\mathcal{Z}$, where $\widetilde{F}/F$ is a Galois extension with the corresponding Galois group $\mathrm{Gal}(\widetilde{F}/F)$. Consider the root system $\Phi_{\widetilde{F}} = \Phi(\mathbf{G}, \mathcal{Z}, \widetilde{F})$ of $\mathbf{G}$ with respect to the maximal $\widetilde{F}$-split torus $\mathcal{Z}$. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing

$\mathcal{Z}$, which will ensure a system of positive roots $\Phi^+_{\widetilde{F}}$ in $\Phi_{\widetilde{F}}$ together with the associated simple root system $\Delta_{\widetilde{F}}$. Let Φ_F be the restricted root system of $\mathbf{G}$ with respect to $\mathbf{S}$ i.e., Φ_F contains the restrictions of roots in $\Phi_{\widetilde{F}}$ from $\mathcal{Z}$ to $\mathbf{S}$. Let Δ_F be a basis of Φ_F consisting of restriction of roots in $\Delta_{\widetilde{F}}$ to $\mathbf{S}$. Since there exists an F-Borel subgroup, one can assume that $\Delta_{\widetilde{F}}$ is $\mathrm{Gal}(\widetilde{F}/F)$-invariant. Then, each fibre of the (restriction) map $\alpha \mapsto \alpha|_{\mathbf{S}}$ is a single Galois orbit in $\Delta_{\widetilde{F}}$.

Through out this section, we will label the roots in $\Phi_{\widetilde{F}}$ (resp. in Φ_F) by Greek letters :$\alpha, \beta, ...$ (resp. by Latin letters:$a, b, ...$). Let $\Phi^a_{\widetilde{F}}$ be the pre-image of $a \in \Phi_F$ in $\Phi_{\widetilde{F}}$. We denote by $\widetilde{\mathbf{U}}_\alpha$ (resp. $\mathbf{U}_a$) the root subgroup of $\mathbf{G}_{\widetilde{F}} = \mathbf{G} \times_F \widetilde{F}$ (resp. $\mathbf{G}$) corresponding to the root $\alpha \in \Phi_{\widetilde{F}}$ (resp. $a \in \Phi_F$).

3.4.1 Chevalley-Steinberg system

The Galois group $\mathrm{Gal}(\widetilde{F}/F)$ acts on the set $\{\widetilde{\mathbf{U}}_\alpha \mid \alpha \in \Phi_{\widetilde{F}}\}$ as follows: for $\alpha \in \Phi_{\widetilde{F}}$ and $\sigma \in \mathrm{Gal}(\widetilde{F}/F)$, we have $\sigma(\widetilde{\mathbf{U}}_\alpha) = \widetilde{\mathbf{U}}_{\sigma(\alpha)}$. Let $\widetilde{F}_\alpha$ be the fixed subfield of $\widetilde{F}$ of the stabilizer subgroup $\mathrm{Stab}_{\mathrm{Gal}(\widetilde{F}/F)}(\alpha)$ of α in $\mathrm{Gal}(\widetilde{F}/F)$. Then $\widetilde{\mathbf{U}}_\alpha$ is defined over $\widetilde{F}_\alpha$.

Consider a system $(\widetilde{x}_\alpha)_{\alpha \in \Phi_{\widetilde{F}}}$ of $\widetilde{F}$-group isomorphisms $\widetilde{x}_\alpha : \mathbf{G}_a \to \widetilde{\mathbf{U}}_\alpha$. Two $\widetilde{F}$-group isomorphisms $\widetilde{x}_\alpha : \mathbf{G}_a \to \widetilde{\mathbf{U}}_\alpha$ and $\widetilde{x}_{-\alpha} : \mathbf{G}_a \to \widetilde{\mathbf{U}}_{-\alpha}$ are called *associated* if there exists a $\widetilde{F}$-group monomorphism $\epsilon_\alpha : SL_2 \to \mathbf{G}$ such that for all $u \in \widetilde{F} = \mathbf{G}_a(\widetilde{F})$, we have

$$\widetilde{x}_\alpha(u) = \epsilon_\alpha \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad and \quad \widetilde{x}_{-\alpha}(u) = \epsilon_\alpha \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}.$$

Let $m_\alpha := \widetilde{x}_\alpha(1)\widetilde{x}_{-\alpha}(1)\widetilde{x}_\alpha(1)$.

Definition 15. *A system $(\widetilde{x}_\alpha)_{\alpha \in \Phi_{\widetilde{F}}}$ of $\widetilde{F}$-group isomorphisms $\widetilde{x}_\alpha : \mathbf{G}_a \to \widetilde{\mathbf{U}}_\alpha$ is called an $\widetilde{F}$-Chevalley system of $\mathbf{G}$ with respect to $\mathcal{Z}$, if it satisfies the following properties for all $\alpha \in \Phi_{\widetilde{F}}$.*

(i) $\widetilde{x}_\alpha$, $\widetilde{x}_{-\alpha}$ are associated and,

(ii) for any $\alpha, \beta \in \Phi_{\widetilde{F}}$, there exists an $\epsilon \in \{1, -1\}$ such that for all $u \in \mathbf{G}_a(\widetilde{F})$, we have $\widetilde{x}_{s_\alpha(\beta)}(u) = m_\alpha \widetilde{x}_\beta(\epsilon u) m_\alpha^{-1}$.

In order to parameterize the root groups $\{\mathbf{U}_a\}_{a \in \Phi_F}$ of $\mathbf{G}$ over F, we have to introduce a *Chevalley-Steinberg system* $(\widetilde{x}_\alpha)_{\alpha\in\Phi_{\widetilde{F}}}$, which is a generalization of Chevalley system for non-split groups.

Definition 16. *An $\widetilde{F}$-Chevalley system $(\widetilde{x}_\alpha)_{\alpha\in\Phi_{\widetilde{F}}}$ of $\mathbf{G}$ is said to be a Chevalley-Steinberg system of $\mathbf{G}$, if the following properties hold for all root $\alpha \in \Phi_{\widetilde{F}}$.*

(i) The isomorphism $\widetilde{x}_\alpha$ is defined over $\widetilde{F}_\alpha$,

(ii) if the restriction $a \in \Phi_F$ of α to $\mathbf{S}$ is not a divisible root, then $\widetilde{x}_{\sigma(\alpha)} = \sigma \circ \widetilde{x}_\alpha \circ \sigma^{-1}$ for all $\sigma \in \mathrm{Gal}(\widetilde{F}/F)$ and

(iii) if the restriction $a \in \Phi_F$ of α to $\mathbf{S}$ is a divisible root, then there exist $\beta, \beta' \in \Phi_{\widetilde{F}}$ restricting to $\frac{a}{2}$ such that $\widetilde{F}_\beta = \widetilde{F}_{\beta'}$ is a quadratic extension of $\widetilde{F}_\alpha$, and $\widetilde{x}_{\sigma(\alpha)} = \sigma \circ \widetilde{x}_\alpha \circ \sigma^{-1} \circ \epsilon$ for all $\sigma \in \mathrm{Gal}(\widetilde{F}/\widetilde{F}_\alpha)$, where $\epsilon \in \{1, -1\}$ with $\epsilon = 1$ if and only if σ induces the identity automorphism on $\widetilde{F}_\beta$.

According to [Lan1, Proposition 4.4], a Chevalley-Steinberg system of $\mathbf{G}$ always exists. Even if the group $\mathbf{G}$ is non-split, a Chevalley-Steinberg system of $\mathbf{G}$ will allow us to define a valuation of root subgroups of $\mathbf{G}$ and that will be discussed in the next section.

3.4.2 Valuation of root groups

Consider the root subgroup $\mathbf{U}_a$ of $\mathbf{G}$ corresponding to the root $a \in \Phi_F$. In order to describe the root subgroup $\mathbf{U}_a$, one needs to consider the universal semisimple covering $\pi : G^a \to \langle \mathbf{U}_a, \mathbf{U}_{-a}\rangle$ and the unipotent subgroups of G^a. The map π induces an isomorphism between $\mathbf{U}_a$ and a root subgroup U^a_+ of G^a. Now, two distinguished cases may arise:

Case(1) : The root $a \in \Phi_F$ such that $\frac{a}{2} \notin \Phi_F$, and $2a \notin \Phi_F$.
Let $\alpha \in \Phi^a_{\widetilde{F}}$ such that $\alpha|_{\mathbf{S}} = a$. There exists an $\widetilde{F}$-isomorphism between $G^a \times_F \widetilde{F}$ and a product of SL_2 indexed by $\Delta^a_{\widetilde{F}}$ together with an action of $\mathrm{Gal}(\widetilde{F}/F)$ on $G^a \times_F \widetilde{F}$ by permuting the components. Then $\mathrm{Res}_{\widetilde{F}_\alpha/F} SL_2$ (the Weil restriction of SL_2 over $\widetilde{F}_\alpha/F$) is isomorphic to G^a. Similarly, the inclusion $\widetilde{\mathbf{U}}_\alpha \subset \mathbf{U}_a \times_F \widetilde{F}$ induces a canonical F-isomorphism $\mathrm{Res}_{\widetilde{F}_\alpha/F}\widetilde{\mathbf{U}}_\alpha \simeq \mathbf{U}_a$. From the Chevalley-Steinberg system $(\widetilde{x}_\alpha)_{\alpha\in\Phi_{\widetilde{F}}}$ of $\mathbf{G}$, the $\widetilde{F}_\alpha$-group isomorphisms $\widetilde{x}_{\pm\alpha} : \mathbf{G}_a \to \widetilde{\mathbf{U}}_{\pm\alpha}$ induces F-isomorphisms

$$x_{\pm a} = \mathrm{Res}_{\widetilde{F}_\alpha/F}(\widetilde{x}_{\pm\alpha}) : \mathrm{Res}_{\widetilde{F}_\alpha/F}(\mathbf{G}_a) \to \mathbf{U}_{\pm a},$$

which are called parametrizations of $\mathbf{U}_a$. These isomorphism induce group isomorphism

$$x_{\pm a} : \mathrm{Res}_{\widetilde{F}_\alpha/F}(\mathbf{G}_a)(F) = \widetilde{F}_\alpha \to \mathbf{U}_{\pm a}(F),$$

This isomorphism allows us to define the valuation $\varphi_a : \mathbf{U}_a(F) \to \mathbb{R} \cup \{\infty\}$ of root group $\mathbf{U}_a(F)$ as follows: $\varphi_a(x_a(u)) = \mathrm{val}_F(u)$ for $u \in F$.

Set $\Gamma_a = \Gamma'_a := \varphi_a(\mathbf{U}_a(F) \backslash \{1\}) \subset \mathbb{R}$ and for each $l \in \mathbb{R}$, define $U_{a,l} := \varphi_a^{-1}([l, \infty))$ with $U_{a,\infty} = \{1\}, U_{a,-\infty} = \mathbf{U}_a(F)$ and $U_{a,l+} = \cup_{k>l} U_{a,k}$. Then, $(U_{a,l})_{l \in \mathbb{R}}$ defines a filtration of $\mathbf{U}_a(F)$ and according to [Lan1, Lemma 4.9], this filtration is independent of the choice of $\alpha \in \Phi^a_{\widetilde{F}}$.

Case(2) : The root $a \in \Phi_F$ such that $\frac{a}{2} \in \Phi_F$ or $2a \in \Phi_F$.
We assume that $2a \in \Phi_F$. Then there exists an $\widetilde{F}$-isomorphism between $G^a \times_F \widetilde{F}$ and a product of SL_3 indexed by the family of pairs (α, α'), where $\alpha, \alpha' \in \Delta^a_{\widetilde{F}}$ with $\alpha + \alpha' \in \Phi^a_{\widetilde{F}}$. Again the Galois group $\mathrm{Gal}(\widetilde{F}/F)$ acts on $G^a \times_F \widetilde{F}$ by permuting the components. Note that $\widetilde{F}_\alpha = \widetilde{F}_{\alpha'}$ is a quadratic separable extension of $\widetilde{F}_{\alpha+\alpha'}$. For the non-trivial automorphism $\sigma \in \mathrm{Gal}(\widetilde{F}_\alpha/\widetilde{F}_{\alpha+\alpha'})$, there exists a Hermitean form $\mathfrak{h}$ on $\widetilde{F}^3_\alpha$, defined by

$$\mathfrak{h}(t_{-1}, t_0, t_1) = \sigma(t_{-1})t_1 + \sigma(t_0)t_1 + \sigma(t_1)t_{-1} \text{ for } t_{-1}, t_0, t_1 \in \widetilde{F}_\alpha.$$

Let SU_3 be the special unitary $\widetilde{F}_{\alpha+\alpha'}$-group associated to the Hermitean form $\mathfrak{h}$. Then, $\mathrm{Res}_{\widetilde{F}_{\alpha+\alpha'}/F} SU_3$ (the Weil restriction of SU_3 over $\widetilde{F}_{\alpha+\alpha'}/F$) is isomorphic to G^a. Similarly, using the covering map π, one can show that the inclusion of unipotent subgroups: $\widetilde{\mathbf{U}}_\alpha \widetilde{\mathbf{U}}_{\alpha+\alpha'} \widetilde{\mathbf{U}}_{\alpha'} \subset \mathbf{U}_a \times_F \widetilde{F}$ induces a canonical F-isomorphism $\pi' : \mathrm{Res}_{\widetilde{F}_{\alpha+\alpha'}/F}(\widetilde{\mathbf{U}}_\alpha \widetilde{\mathbf{U}}_{\alpha+\alpha'} \widetilde{\mathbf{U}}_{\alpha'}) \simeq \mathbf{U}_a$.

Define a subset $H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'})$ of $\widetilde{F}_\alpha \times \widetilde{F}_\alpha$ by

$$H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'}) := \{(u, v) \in \widetilde{F}_\alpha \times \widetilde{F}_\alpha \mid v + \sigma(v) = \sigma(u)u\},$$

with the following action: $H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'}) \times H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'}) \to H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'})$ defined by $((u_1, v_1), (u_2, v_2)) \mapsto (u_1 + u_2, v_1 + v_2 + \sigma(u_1)u_2)$. Then $H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'})$ becomes a $\widetilde{F}_{\alpha+\alpha'}$-group scheme. In particular, one can identify $H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'})$ with a closed subgroup of SU_3. Now using Chevalley-Steinberg system $(\widetilde{x}_\alpha)_{\alpha \in \Phi_{\widetilde{F}}}$, one can define the $\widetilde{F}_{\alpha+\alpha'}$-group isomorphism $\widetilde{y}_{\alpha+\alpha'} : H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'}) \to \widetilde{\mathbf{U}}_\alpha \widetilde{\mathbf{U}}_{\alpha+\alpha'} \widetilde{\mathbf{U}}_{\alpha'}$ defined by,

$$\widetilde{y}_{\alpha+\alpha'}(u,v) = \widetilde{x}_\alpha(u)\widetilde{x}_{\alpha+\alpha'}(v)\widetilde{x}_{\alpha'}(\sigma(u)).$$

Consider the F-isomorphism

$$y_a = \mathrm{Res}_{\widetilde{F}_{\alpha+\alpha'/F}}(\widetilde{y}_{\alpha+\alpha'}) : \mathrm{Res}_{\widetilde{F}_{\alpha+\alpha'/F}}(H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'})) \to \mathrm{Res}_{\widetilde{F}_{\alpha+\alpha'/F}}(\widetilde{\mathbf{U}}_\alpha \widetilde{\mathbf{U}}_{\alpha+\alpha'} \widetilde{\mathbf{U}}_{\alpha'}).$$

Then the composition $\pi' \circ y_a$ map induces the F-isomorphism

$$x_a = \pi' \circ y_a : \mathrm{Res}_{\widetilde{F}_{\alpha+\alpha'/F}} H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'}) \to \mathbf{U}_a.$$

This isomorphism x_a is called the parametrization of $\mathbf{U}_a$. These isomorphisms induces ismorphism x_a on F-points as:

$$x_a = (\pi' \circ y_a)(F) : \mathrm{Res}_{\widetilde{F}_{\alpha+\alpha'/F}}(H_0(\widetilde{F}_\alpha, \widetilde{F}_{\alpha+\alpha'}))(F) \to \mathbf{U}_a(F).$$

Note that the root subgroup $\mathbf{U}_{2a}$ corresponding to the root $2a$ is the subgroup of the group $\mathbf{U}_a$ given by the image of $x_a(0,v)$.

Now, the parametrization x_a allows us to define the valuation $\varphi_a : \mathbf{U}_a(F) \to \mathbb{R} \cup \{\infty\}$ of root groups $\mathbf{U}_a(F)$ as follows:

$$\varphi_a(u) = \frac{1}{2}\mathrm{val}_F(v') \text{ for } x_a(v,v') = u,$$

and the valuation $\varphi_{2a} : \mathbf{U}_{2a}(F) \to \mathbb{R} \cup \{\infty\}$ of root group $\mathbf{U}_{2a}(F)$ is defined by

$$\varphi_{2a}(u) = \mathrm{val}_F(v) \text{ for } x_a(0,v) = u.$$

Set $\Gamma_a := \varphi_a(\mathbf{U}_a(F) \setminus \{1\}) \subset \mathbb{R}$, $\Gamma'_a := \{\varphi_a(u) \mid u \in \mathbf{U}_a(F) \setminus \{1\}$ and $\varphi_a(u) = \mathrm{Sup}\, \varphi_a(u\mathbf{U}_{2a}(F))\}$ and $\Gamma_{2a} = \Gamma'_{2a} := \varphi_{2a}(\mathbf{U}_{2a}(F) \setminus \{1\}) \subset \mathbb{R}$. For each $l \in \mathbb{R}$, define $U_{a,l} := \varphi_a^{-1}([l,\infty)), U_{2a,l} := \varphi_{2a}^{-1}([l,\infty))$ with $U_{a,\infty} = \{1\}, U_{a,-\infty} = \mathbf{U}_a(F), U_{2a,\infty} = \{1\}, U_{2a,-\infty} = \mathbf{U}_{2a}(F), U_{a,l+} := \cup_{k>l} U_{a,k}$, and $U_{2a,l+} := \cup_{k>l} U_{2a,k}$. Then, $(U_{a,l})_{l\in\mathbb{R}}$ (resp. $(U_{2a,l})_{l\in\mathbb{R}}$) defines a filtration of $\mathbf{U}_a(F)$ (resp. $\mathbf{U}_{2a}(F)$) and according to [Lan1, Lemma 4.15], these filtrations are independent of the choice of $\alpha, \alpha' \in \Phi^a_{\widetilde{F}}$.

3.4.3 Affine roots

Now, we recall the apartment $A_0 = A_0(\mathbf{G}, \mathbf{S}, F)$ associated to the tuple $(\mathbf{G}, \mathbf{S}, F)$. According to [BT1, Section 6], the valuation of root groups i.e., the families of maps $(\varphi_a : \mathbf{U}_a(F) \to \mathbb{R} \cup \{\infty\})_{a \in \Phi_F}$ corresponds to a special point x_0 in $A_0 = A(\mathbf{G}, \mathbf{S}, F)$. Let $a \in \Phi_K$. An affine function $\theta : A_0 \to \mathbb{R}$ is called an *affine root* with direction a, if there exists $l \in \Gamma'_a$ such that $\theta(y) = a(y - x_0) + l$. The affine root θ will be denoted by $a + l$, when $\theta(y) = a(y - x_0) + l$ for some $a \in \Phi_F$ and $l \in \Gamma'_a$. Let Φ_F^{aff} be the set $\{\theta \equiv a + l \mid a \in \Phi_F, l \in \Gamma'_a\}$ of all affine roots on A_0.

For each affine root $\theta \in \Phi_F^{\text{aff}}$, define the affine hyperplane $H_\theta = \{x \in A_0 \mid \theta(x) = 0\}$. The set $\{H_\theta \mid \theta \in \Phi_F^{\text{aff}}\}$ of affine hyperplanes defines a poly-simplicial structure on the apartment A_0 in the following way: One can define an equivalence relation $\sim$ on A_0 by $x \sim y$ if for any affine hyperplane H_θ, either both $x, y \in H_\theta$, or both x, y are in the same connected component of $A_0 \setminus H_\theta$. The equivalence classes are called the *facets* in the apartment A_0 and the singleton facet is called a *vertex*. A facet with maximal dimension is called an *alcove*.

For every affine root $\theta \in \Phi_F^{\text{aff}}$ with its gradient $a_\theta \in \Phi_F$, one can define the subgroup U_θ of $\mathbf{U}_{a_\theta}(F)$ by

$$U_\theta := \{u \in \mathbf{U}_{a_\theta}(F) \mid u = 1 \text{ or } \varphi_{a_\theta}(u) \geq \theta(x_0)\}.$$

Then U_θ will be a compact open subgroup of $\mathbf{U}_{a_\theta}(F)$ and for any root $a \in \Phi_F$, the root subgroup $\mathbf{U}_a(F)$ will admit a filtration by these subgroups U_θ indexed by those affine roots $\theta \in \Phi_F^{\text{aff}}$ whose gradients are the root a.

Note that if the affine rooot $\theta \equiv a_\theta + l$ for some $l \in \mathbb{R}$ and $a_\theta \in \Phi_F$, from [Lan1, Proposition 7.7] we have $U_\theta = U_{a_\theta, l}$, where $U_{a_\theta, l}$ is the filtration subgroup of $\mathbf{U}_{a_\theta}(F)$ as defined in §3.4.2.

For any (non-empty) subset Ω of A_0, define a map $f_\Omega : \Phi_F \to \mathbb{R} \cup \{\pm\infty\}$ by

$$f_\Omega(a) = \inf\{l \in \Gamma_a \mid a(x) + l \geq 0 \text{ for all } x \in \Omega\}.$$

For any (non-empty) subset Ω of A_0, define U_Ω to be the subgroup of $\mathbf{G}(F)$ generated by the filtration subgroups $U_{a, f_\Omega(a)}$ for $a \in \Phi_F^{\text{red}}$. Define $N_\Omega := \{n \in \mathcal{N}(F) \mid \nu(n)x =$

x, for all $x \in \Omega\}$, which is a subgroup normalizes U_Ω. Set $P_\Omega := N_\Omega U_\Omega = \langle U_\Omega, N_\Omega \rangle$, then P_Ω will be a subgroup of $\mathbf{G}(F)$. If Ω is a singleton set $\{x\}$, then the subgroups $U_{\{x\}}$, $N_{\{x\}}$, and $P_{\{x\}}$ will be denoted by U_x, N_x, and P_x respectively.

3.5 The Building

For a connected reductive group $\mathbf{G}$ defined over a local field F, Bruhat and Tits in [BT1,BT2] associated to $\mathbf{G}_{\mathrm{der}}(F)$ an affine building $\mathcal{B}_0(\mathbf{G}, F)$, called the *reduced Bruhat-Tits building*, where $\mathbf{G}_{\mathrm{der}}(F)$ is the F-points group of the derived subgroup $\mathbf{G}_{\mathrm{der}}$ of $\mathbf{G}$, and also to $\mathbf{G}(F)$, they associated another building $\mathcal{B}(\mathbf{G}, F)$, called the *extended Bruhat-Tits building*. This section contains a brief discussion about these buildings associated to the pair $(\mathbf{G}, F)$.

3.5.1 The reduced Bruhat-Tits building

The *reduced Bruhat-Tits building* associated to $\mathbf{G}$ defined over F will be denoted by $\mathcal{B}_0(\mathbf{G}, F)$. The building $\mathcal{B}_0(\mathbf{G}, F)$ is the set of equivalence classes

$$\mathcal{B}_0(\mathbf{G}, F) := \mathbf{G}(F) \times A_0(\mathbf{G}, \mathbf{S}, F)/\sim,$$

where the equivalence relation '$\sim$' on $\mathbf{G}(F) \times A_0(\mathbf{G}, \mathbf{S}, F)$ is defined by the following way: for any $g, h \in \mathbf{G}(F)$ and $x, y \in A_0(\mathbf{G}, \mathbf{S}, F)$

$$(g, x) \sim (h, y) \text{ if there exists an element} n \in \mathcal{N}(F) \text{ such that } n \cdot x = y \text{ and } g^{-1}hn \in U_x.$$

Via the map $\nu : \mathcal{N}(F) \to \mathrm{Aff}(A_0)$, we have a $\mathcal{N}(F)$-action on $A_0(\mathbf{G}, \mathbf{S}, F)$. Now one can extend this action to an action of $\mathbf{G}(F)$ on $\mathcal{B}_0(\mathbf{G}, F)$. Consider the map

$$\begin{aligned} \mathbf{G}(F) \times [\mathbf{G}(F) \times A_0(\mathbf{G}, \mathbf{S}, F)] &\to \mathbf{G}(F) \times A_0(\mathbf{G}, \mathbf{S}, F) \\ (g, (h, x)) &\mapsto (gh, x), \end{aligned}$$

where $g \in \mathbf{G}(F)$ and $(h, x) \in \mathbf{G}(F) \times A_0(\mathbf{G}, \mathbf{S}, F)$. According to the definition of the equivalence relation '$\sim$', the above map induces an action of $\mathbf{G}(F)$ on $\mathcal{B}_0(\mathbf{G}, F)$ from the left side, extending the $\mathcal{N}(F)$-action on $A_0(\mathbf{G}, \mathbf{S}, F)$.

One can identify $A_0(\mathbf{G}, \mathbf{S}, F)$ with its canonical image $\mathcal{A}_0$ in $\mathcal{B}_0(\mathbf{G}, F)$ via the map $A_0(\mathbf{G}, \mathbf{S}, F) \hookrightarrow \mathcal{B}_0(\mathbf{G}, F)$ given by $x \mapsto [(1, x)]$, which is a $\mathcal{N}(F)$-equivarient embedding.

A subset $\mathcal{A}'_0 \subset \mathcal{B}_0(\mathbf{G}, F)$ is called an *apartment* in $\mathcal{B}_0(\mathbf{G}, F)$ if there exists an element $g \in \mathbf{G}(F)$ such that $\mathcal{A}'_0 = g.\mathcal{A}_0$. Similarly, if $\mathcal{F}_0$ is the image (in $\mathcal{B}_0(\mathbf{G}, F)$) of a facet of $A_0(\mathbf{G}, \mathbf{S}, F)$ under the above canonical embedding, the subset of the form $g \cdot \mathcal{F}_0$ for each $g \in \mathbf{G}(F)$ is called *facet* in $\mathcal{B}_0(\mathbf{G}, F)$.

Now from [BT1, §7.4], we will briefly recall some important properties of $\mathcal{B}_0(\mathbf{G}, F)$ based on the $\mathbf{G}(F)$-action.

(1) The *apartment* $\mathcal{A}'_0 = g.\mathcal{A}_0$ can be identified with the apartment corresponding to the maximal F-split torus ${}^g\mathbf{S}$ for $g \in \mathbf{G}(F)$. Therefore, we have a one-to-one correspondence between the set of maximal F-split tori of $\mathbf{G}$, and the set of apartments of $\mathcal{B}_0(\mathbf{G}, F)$.

(2) If $\Omega \subset \mathcal{A}_0$ is a non-empty subset, then P_Ω (as defined in §3.4.3) is the subgroup of $\mathbf{G}(F)$ that fixes all points in Ω i.e., $P_\Omega = \{g \in \mathbf{G}(F) \mid gx = x \text{ for all } x \in \Omega\}$.

(3) For any $g \in \mathbf{G}(F)$, there exists an element $n \in \mathcal{N}(F)$ such that $g \cdot x = n \cdot x$ for all $x \in \mathcal{A}_0 \cap g^{-1}.\mathcal{A}_0$.

(4) Let $\Omega \subset \mathcal{A}_0$ be a non-empty subset. Then the group U_Ω (as defined in §3.4.3) acts on the set of all apartments containing Ω transitively.

(5) For any two facets (resp. points) in the building $\mathcal{B}_0(\mathbf{G}, F)$, there exists an apartment in the building which contains both the facets (resp. points).

3.5.2 The extended Bruhat-Tits building

The extended (a.k.a. enlarged) *Bruhat-Tits building* $\mathcal{B}(\mathbf{G}, F)$ of $\mathbf{G}$ defined over F, is obtained by gluing together the extended apartments (which are affine spaces defined over real vector spaces $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$, for maximal F-split torus $\mathbf{S}$) is defined by all maximal F-split torus of $\mathbf{G}$. The primary purpose to study the extended building is that when the center $Z(\mathbf{G})$ of of the group $\mathbf{G}$ is of positive split rank, the stabilizer P_x of a point x in the reduced building $\mathcal{B}_0(\mathbf{G}, F)$ is no longer a compact open subgroup of $\mathbf{G}(F)$. To solve this issue, we will now introduce a larger building $\mathcal{B}(\mathbf{G}, F)$ of $\mathbf{G}$ following [LN, §3] and [Bou1, §2.7].

Extended Apartment:

We recall the standard apartment $A_0(\mathbf{G}, \mathbf{S}, F)$ corresponding to the tuple $(\mathbf{G}, \mathbf{S}, F)$, which is an affine space under the $\mathbb{R}$-vector space

$$V = V_0/V' = X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}/X_*(\mathcal{Z}_{c,s}) \otimes_{\mathbb{Z}} \mathbb{R}$$

together with a homomorphism $\nu : \mathcal{N}(F) \to \mathrm{Aff}(A_0(\mathbf{G}, \mathbf{S}, F))$. There exists a unique group homomorphism $\nu_0' : \mathbf{G}(F) \to X_*(\mathcal{Z}_{c,s}) \otimes_{\mathbb{Z}} \mathbb{R} \subset V'$ characterized by

$$\left\langle \nu_0'(g), \chi|_{\mathcal{Z}_{c,s}} \right\rangle = -\mathrm{val}_F(\chi(g)), \text{ for all } g \in \mathbf{G}(F) \text{ and } \chi \in X_F^*(\mathbf{G}).$$

We denote $\mathbf{G}(F)^1 := \ker(\nu_0')$, the kernel of the group homomorphism ν_0'. Then, the quotient group $\mathbf{G}(F)/\mathbf{G}(F)^1$ is a finitely generated abelian group and there exists an isomorphism

$$(\mathbf{G}(F)/\mathbf{G}(F)^1) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X_*(\mathcal{Z}_{c,s}) \otimes_{\mathbb{Z}} \mathbb{R} = V'.$$

Now consider the real vector space $V' = X_*(\mathcal{Z}_{c,s}) \otimes_{\mathbb{Z}} \mathbb{R}$ together with a $\mathbf{G}(F)$-action on it via the above isomorphism. Fix an affine space A' under the real vector space V' with a morphism $\nu' : \mathbf{G}(F) \to \mathrm{Aff}(A')$ where for each $g \in \mathbf{G}(F)$, $\nu'(g)$ is the translation defined by $\nu'(g)(a) = \nu_0'(g) + a$.

We can now define the *extended standard apartment* $A = A(\mathbf{G}, \mathbf{S}, F)$ associated to the tuple $(\mathbf{G}, \mathbf{S}, F)$ by

$$A(\mathbf{G}, \mathbf{S}, F) := A_0(\mathbf{G}, \mathbf{S}, F) \times A'$$

together with a homomorphism $\widetilde{\nu} : \mathcal{N}(F) \to \mathrm{Aff}(A(\mathbf{G}, \mathbf{S}, F))$ defined by

$$\widetilde{\nu}(n) = \nu(n) \oplus \nu'(n) \text{ for all } n \in \mathcal{N}(F).$$

From the direct sum decomposition of the vector space $V_0 = V \oplus V'$, it can be shown that the extended apartment $A(\mathbf{G}, \mathbf{S}, F)$ is an affine space under the vector space $V_0 = X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ together with the map $\widetilde{\nu}$.

According to Bruhat and Tits [BT2], to each point x in the extended apartment

$A(\mathbf{G}, \mathbf{S}, F)$, one can associate a smooth affine $\mathcal{O}_F$-group scheme $\mathbf{G}_{x,0}$ such that its generic fibre $\mathbf{G}_{x,0} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(F)$ is equal to $\mathbf{G}$ and its special fibre $\mathcal{G}_x := \mathbf{G}_{x,0} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(k_F)$ is a connected reductive algebraic group defined over the finite residue class field k_F. For each $x \in A(\mathbf{G}, \mathbf{S}, F)$, the group $\mathbf{G}_{x,0}$ is called a *parahoric subgroup* of $\mathbf{G}$. Let $\mathbf{G}(F)_{x,0} := \mathbf{G}_{x,0}(\mathcal{O}_F)$ be the group of $\mathcal{O}_F$-points of $\mathbf{G}_{x,0}$. Then $\mathbf{G}(F)_{x,0}$ is a compact open subgroup of $\mathbf{G}(F)$ for each $x \in A(\mathbf{G}, \mathbf{S}, F)$.

Extended Building

Definition 17. *The extended Bruhat-Tits building $\mathcal{B}(\mathbf{G}, F)$ associated to $\mathbf{G}$ defined over F is the set of equivalence classes*

$$\mathcal{B}(\mathbf{G}, F) := \mathbf{G}(F) \times A(\mathbf{G}, \mathbf{S}, F)/\sim,$$

where the equivalence relation '$\sim$' on $\mathbf{G}(F) \times A(\mathbf{G}, \mathbf{S}, F)$ is defined by the following way: for any $g, h \in \mathbf{G}(F)$ and $x, y \in A(\mathbf{G}, \mathbf{S}, F)$

$(g, x) \sim (h, y)$ *if there exists an elementn $\in \mathcal{N}(F)$ such that $n \cdot x = y$ and $g^{-1}hn \in \mathbf{G}(F)_{x,0}$.*

We can also identify the extended building $\mathcal{B}(\mathbf{G}, F)$ with the product of two $\mathbf{G}(F)$-sets as

$$\mathcal{B}(\mathbf{G}, F) = \mathcal{B}_0(\mathbf{G}, F) \times V'.$$

Via the group homomorphism ν_0' and the $\mathbf{G}(F)$-action on $\mathcal{B}_0(\mathbf{G}, F)$, one can define a $\mathbf{G}(F)$-action on the extended building $\mathcal{B}(\mathbf{G}, F)$ in the following way: for $g \in \mathbf{G}(F)$ and $(x, v) \in \mathcal{B}_0(\mathbf{G}, F) \times V'$,

$$g \cdot (x, v) = (g \cdot x, \nu_0'(g) + v)$$

Let $\mathcal{A}$ be the image of $A(\mathbf{G}, \mathbf{S}, F)$ under the inclusion map $A(\mathbf{G}, \mathbf{S}, F) \hookrightarrow \mathcal{B}(\mathbf{G}, F)$ given by $x \mapsto [(1, x)]$. Then $\mathcal{A} = \mathcal{A}_0 \times V'$ for some apartment $\mathcal{A}_0$ in the reduced building $\mathcal{B}_0(\mathbf{G}, F)$. The apartments of the building $\mathcal{B}(\mathbf{G}, F)$ are of the forms $g \cdot \mathcal{A}$ for $g \in \mathbf{G}(F)$, where $g \cdot \mathcal{A}$ is the extended apartment corresponding to the torus ${}^g\mathbf{S}$ and there exists some apartment $\mathcal{A}_0' \subset \mathcal{B}_0(\mathbf{G}, F)$ such that $g \cdot \mathcal{A} = \mathcal{A}_0' \times V'$. Similarly, a *facet* $\mathcal{F}$ in the extended building $\mathcal{B}(\mathbf{G}, F)$ is a product of a facet $\mathcal{F}_0$ in the reduced building $\mathcal{B}_0(\mathbf{G}, F)$ and V' i.e., $\mathcal{F} = \mathcal{F}_0 \times V'$. Note that for any two points $x, y \in \mathcal{B}(\mathbf{G}, F)$, there exists an extended apartment $\mathcal{A}' \subset \mathcal{B}(\mathbf{G}, F)$ such that $\mathcal{A}'$ contains both the points x, y.

3.5.3 Galois descent

When the reductive group $\mathbf{G}$ is not F-quasi-split, then also one can define extended building of $\mathbf{G}$ in the following method. Let E be a fixed maximal unramified extension of F with its ring of integers $\mathcal{O}_E$ and residue class field k_E. Then k_E is an algebraic closure of k_F and the Galois group $\Gamma = \mathrm{Gal}(E/F)$ can be identified with the Galois group $\mathrm{Gal}(k_E/k_F)$. Consider a maximal F-split torus $\mathbf{S}$ of G. According to [BT2, §5.1.12], there exists a maximal E-split torus $\mathbf{T}$ of $\mathbf{G}_E = \mathbf{G} \times_F E$ defined over F and containing $\mathbf{S}$ as a subtorus. Since k_E is algebraically closed, $\mathbf{G}_E$ is quasi-split over E. As usual, we can now define extended building $\mathcal{B}(\mathbf{G}_E, E)$ with its extended standard apartment $A(\mathbf{G}_E, \mathbf{T}, E)$. This apartment $A(\mathbf{G}_E, \mathbf{T}, E)$ is equipped with an action of the Galois group Γ such that the Γ-fixed point set $A(\mathbf{G}_E, \mathbf{T}, E)^\Gamma$ is equal to the extended apartment $A(\mathbf{G}, \mathbf{S}, F)$. Also there exists a natural Γ action on the group of E-rational points $\mathbf{G}_E(E)$ of $\mathbf{G}_E$. Combining both these Γ-actions, one can define an action of Γ on the building $\mathcal{B}(\mathbf{G}_E, E) = \mathbf{G}_E(E) \times A(\mathbf{G}_E, \mathbf{T}, E)/\sim$. Then the extended building $\mathcal{B}(\mathbf{G}, F)$ of $\mathbf{G}$ over F will be the Γ-fixed point set of $\mathcal{B}(\mathbf{G}_E, E)$ i.e.,

$$\mathcal{B}(\mathbf{G}, F) = \mathcal{B}(\mathbf{G}_E, E)^\Gamma.$$

3.5.4 Stabilizer and Parahoric subgroups:

For each point x (resp. each facet $\mathcal{F}$) in $\mathcal{B}(\mathbf{G}, F)$, let us define their stabilizer subgroups $\mathrm{Stab}_{G(F)}(x)$ (resp. $\mathrm{Stab}_{G(F)}(\mathcal{F})$) by $\mathrm{Stab}_{G(F)}(x) := \{g \in \mathbf{G}(F) \mid g \cdot x = x\}$ and $\mathrm{Stab}_{G(F)}(\mathcal{F}) := \{g \in \mathbf{G}(F) \mid g \cdot x = x \ \forall x \in \mathcal{F}\}$ respectively. Then $\mathrm{Stab}_{G(F)}(x)$ (resp. $\mathrm{Stab}_{G(F)}(\mathcal{F})$) is a compact subgroup of $\mathbf{G}(F)$. To each point x (resp. each facet $\mathcal{F}$) in $\mathcal{B}(\mathbf{G}, F)$, Bruhat and Tits [BT2] associated a unique smooth affine $\mathcal{O}_F$-group scheme $\mathbf{G}_x$ (resp. $\mathbf{G}_\mathcal{F}$) with finite component group such that its generic fiber $\mathbf{G}_x \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(F)$ (resp. $\mathbf{G}_\mathcal{F} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(F)$) is equal to $\mathbf{G}$ and its $\mathcal{O}_F$-points group $\mathbf{G}(F)_x := \mathbf{G}_x(\mathcal{O}_F)$ (resp. $G_\mathcal{F} := \mathbf{G}_\mathcal{F}(\mathcal{O}_F)$) is equal to $\mathrm{Stab}_{G(F)}(x)$ (resp. $\mathrm{Stab}_{G(F)}(\mathcal{F})$). Let $\mathbf{G}_x^\circ$ (resp. $\mathbf{G}_\mathcal{F}^\circ$) denotes the identity component of $\mathbf{G}_x$ (resp. $\mathbf{G}_\mathcal{F}$). If x is contained in some extended apartment $\mathcal{A} \subset \mathcal{B}(\mathbf{G}, F)$, then $\mathbf{G}_x^\circ = \mathbf{G}_{x,0}$ (as defined in §3.5.2)

Definition 18 (Parahoric subgroup)**.** *The group $\mathbf{G}(F)_{x,0} := \mathbf{G}_x^\circ(\mathcal{O}_F)$ (resp. $\mathbf{G}_{\mathcal{F}}^\circ(\mathcal{O}_F)$) of $\mathcal{O}_F$-points of the connected component $\mathbf{G}_x^\circ$ (resp. $\mathbf{G}_{\mathcal{F}}^\circ$) of $\mathbf{G}_x$ (resp. $\mathbf{G}_{\mathcal{F}}$) is called a parahoric subgroup of $\mathbf{G}(F)$ for each $x \in \mathcal{B}(\mathbf{G}, F)$ (resp. each facet $\mathcal{F} \subset \mathcal{B}(\mathbf{G}, F)$). When the facet $\mathcal{F} \subset \mathcal{B}(\mathbf{G}, F)$ is an alcove i.e., $\mathcal{F}$ has maximal dimension, then the parahoric subgroup $\mathbf{G}_{\mathcal{F}}^\circ(\mathcal{O}_F)$ is called an Iwahori subgroup of $\mathbf{G}(F)$.*

Therefore, the parahoric subgroups are not the stabilizer subgroups on the building. From the identification $\mathcal{B}(\mathbf{G}, F) = \mathcal{B}_0(\mathbf{G}, F) \times V'$, we have a natural projection map $\mathcal{B}(\mathbf{G}, F) \twoheadrightarrow \mathcal{B}_0(\mathbf{G}, F)$ and we will denote the image of a point $x \in \mathcal{B}(\mathbf{G}, F)$ in the reduced building $\mathcal{B}_0(\mathbf{G}, F)$ by $[x]$. Then for any $x, y \in \mathcal{B}(\mathbf{G}, F)$, $\mathbf{G}_x = \mathbf{G}_y$ if and only if $[x] = [y]$. For a given $x \in \mathcal{B}(\mathbf{G}, F)$, the stabilizer subgroup $\mathbf{G}(F)_{[x]} := \mathrm{Stab}_{\mathbf{G}(F)}([x])$ of the point $[x] \in \mathcal{B}_0(\mathbf{G}, F)$ is a compact modulo center subgroup of $\mathbf{G}(F)$, whose maximal compact subgroup coincides with $\mathbf{G}(F)_x = \mathrm{Stab}_{\mathbf{G}(F)}(x)$. Infact, the subgroup $\mathbf{G}(F)_{[x]}$ is equal to the normalizer $N_{\mathbf{G}(F)}(\mathbf{G}(F)_x)$ of the subgroup $\mathbf{G}(F)_x$ in $\mathbf{G}(F)$.

3.6 Moy-Prasad Filtration

Fix a point $x \in \mathcal{B}(\mathbf{G}, F)$ and choose an apartment $A = A(\mathbf{G}, \mathbf{S}, F)$ containing the point x, where $\mathbf{S}$ is a maximal F-split torus in $\mathbf{G}$ corresponding to the apartment A. Consider the parahoric subgroup $\mathbf{G}(F)_{x,0}$ associated to the point $x \in A$. In [MP3, MP4], Moy and Prasad defined certain filtration $\{\mathbf{G}(F)_{x,r} \mid r \in \mathbb{R}_{\geq 0}\}$ of the group $\mathbf{G}(F)_{x,0}$ by smaller open normal subgroups $\mathbf{G}(F)_{x,r}$. Here, we will now recall the *Moy-Prasad filtration* subgroup $\mathbf{G}(F)_{x,r}$ associated to the tuple $(\mathbf{G}, x, r)$. We also assume here that $\mathbf{G}$ is a connected reductive quasi-split group defined over F. Let $\mathcal{Z} = Z_{\mathbf{G}}(\mathbf{S})$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$. Since $\mathbf{G}$ is F-quasi-split, $\mathcal{Z}$ is a (maximal) torus defined over F. There exists a unique smooth affine $\mathcal{O}_F$-group scheme $\mathcal{Z}_b$ (which is a lft-Neron model of $\mathcal{Z}$) such that its generic fibre $\mathcal{Z}_b \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(F)$ is equal to $\mathcal{Z}$. Then, the group $\mathcal{Z}_b(\mathcal{O}_F)$ of $\mathcal{O}_F$-points of $\mathcal{Z}_b$ is the maximal bounded subgroup of $\mathcal{Z}(F)$. Let $\mathcal{Z}(F)_0$ be the group $\mathcal{Z}_b^\circ(\mathcal{O}_F)$ of $\mathcal{O}_F$-points of the connected component $\mathcal{Z}_b^\circ$ of $\mathcal{Z}_b$. Then $\mathcal{Z}(F)_0$ is called *Iwahori* (i.e., parahoric) subgroup of $\mathcal{Z}(F)$, where the index of $\mathcal{Z}(F)_0$ in $\mathcal{Z}_b(\mathcal{O}_F)$ is finite.

To every real number $r > 0$, Moy and Prasad attached an open subgroup $\mathcal{Z}(F)_r$ of

$\mathcal{Z}(F)_0$, defined by

$$\mathcal{Z}(F)_r := \{z \in \mathcal{Z}(F)_0 \mid \mathrm{val}_F(\chi(z) - 1) \geq r \text{ for all } \chi \in X^*(\mathcal{Z})\}.$$

This gives a decreasing sequence $\{\mathcal{Z}(F)_r \mid r \in \mathbb{R}_{\geq 0}\}$ of subgroups, called Moy-Prasad filtration of $\mathcal{Z}(F)_0$.

From §3.4.3, we recall the compact open subgroup U_θ attached to each affine root $\theta \in \Phi_F^{\mathrm{aff}}$. Then for real number $r \geq 0$, the *Moy-Prasad filtration* subgroup $\mathbf{G}(F)_{x,r}$ of $\mathbf{G}(F)_{x,0}$ is the group generated by $\mathcal{Z}(F)_r$ and those U_θ with $\theta \in \Phi_F^{\mathrm{aff}}$ such that $\theta(x) \geq r$ i.e.,

$$\mathbf{G}(F)_{x,r} := \left\langle \mathcal{Z}(F)_r, U_\theta \mid \theta \in \Phi_F^{\mathrm{aff}} \text{ with } \theta(x) \geq r \right\rangle.$$

In [Yu3], Yu showed that for each $x \in \mathcal{B}(\mathbf{G}, F)$ and $r \in \mathbb{R}_{\geq 0}$, there exists a smooth affine $\mathcal{O}_F$-group scheme $\mathbf{G}_{x,r}$ such that its generic fiber $\mathbf{G}_{x,r} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(F)$ is equal to $\mathbf{G}$ and its $\mathcal{O}_F$-points group $\mathbf{G}_{x,r}(\mathcal{O}_F)$ is equal to the Moy-Prasad subgroup $\mathbf{G}(F)_{x,r}$. Therefore, these Moy-Prasad subgroups $\mathbf{G}(F)_{x,r}$ are also schematic. We set

$$\mathbf{G}(F)_{x,r+} := \bigcup_{s>r} \mathbf{G}(F)_{x,s} \text{ together with } \mathbf{G}(F)_{x,r:r+} := \mathbf{G}(F)_{x,r}/\mathbf{G}(F)_{x,r+}.$$

Then, the quotient $\mathbf{G}(F)_{x,0:0+}$ can be identified with the k_F-points group $\mathcal{G}_x(k_F)$ of the finite reductive group $\mathcal{G}_x$. For any $r, s \in \mathbb{R}_{\geq 0}$, $\mathbf{G}(F)_{x,r+s}$ contains the commutator subgroup $[\mathbf{G}(F)_{x,r}, \mathbf{G}(F)_{x,s}]$. Therefore, the quotient group $\mathbf{G}(F)_{x,r:r+}$ is an abelian group for each positive real number r.

Remark 19. When $\mathbf{G}$ is not quasi-split over F, using Galois descent, one can define Moy-Prasad subgroups $\mathcal{Z}(F)_r, \mathbf{G}(F)_{x,r}$ etc.

We will now recall the definition of *depth* of a representation, which is mentioned in the main theorem of Moy-Prasad theory developed in [MP3, MP4].

Theorem 20 (Moy and Prasad)**.** *Suppose π be an irreducible admissible representation of $\mathbf{G}(F)$ on a complex vector-space $\mathcal{V}_\pi$. Then the infimum $\rho(\pi)$ of the set*

$$\{r \in \mathbb{R}_{\geq 0} \mid \mathcal{V}_\pi^{\mathbf{G}(F)_{x,r+}} \neq 0 \textit{ for } x \in \mathcal{B}(\mathbf{G}, F)\}$$

can be achieved for some $x \in \mathcal{B}(\mathbf{G}, F)$, where $\mathcal{V}_\pi^{\mathbf{G}(F)_{x,r+}}$ denotes the $\mathbf{G}(F)_{x,r+}$-fixed vectors of $\mathcal{V}_\pi$. Moreover, $\rho(\pi)$ is a rational number.

Definition 21 (Depth of a representation)**.** *If π is an irreducible admissible representation of $\mathbf{G}(F)$, then the number $\rho(\pi)$ (as mentioned in the above Theorem of Moy and Prasad) is called the* depth *of the representation π.*

4

Representations of a p-adic group

4.1 Bernstein decomposition

Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$ defined over F and $\mathbf{M}(F)$ denotes the group of F-rational points of $\mathbf{M}$. In short, we say $\mathbf{M}(F)$ is an F-Levi subgroup of $\mathbf{G}(F)$. Let $\mathrm{X}_F(\mathbf{M}(F))$ be the group of F-rational characters $\phi : \mathbf{M}(F) \to F^\times$ of $\mathbf{M}(F)$. For each $\phi \in \mathrm{X}_F(\mathbf{M}(F))$ and $r \in \mathbb{C}$, one can define a smooth character $\chi_{\phi,r} : \mathbf{M}(F) \to \mathbb{C}^\times$ of $\mathbf{M}(F)$ by $\chi_{\phi,r}(g) = ||\phi(g)||_F^r$ for $g \in \mathbf{M}(F)$, where $|| \cdot ||_F$ denotes the normalized absolute value on F. Let $\mathrm{X}_{ur}(\mathbf{M}(F))$ denote the group generated by those smooth characters $\chi_{\phi,r}$ for $\phi \in \mathrm{X}_F(\mathbf{M}(F))$ and $r \in \mathbb{C}$. Then the elements of $\mathrm{X}_{ur}(\mathbf{M}(F))$ are called *unramified quasicharacters* of $\mathbf{M}(F)$.

We consider a cuspidal pair $(\mathbf{M}(F), \sigma)$, where $\mathbf{M}(F)$ is an F-Levi subgroup of $\mathbf{G}(F)$ and σ is an irreducible supercuspidal representation of $\mathbf{M}(F)$. Define an equivalence relation (called *inertial equivalence*) on the set of all cuspidal pairs $(\mathbf{M}(F), \sigma)$ as: two cuspidal pairs $(\mathbf{M}_1(F), \sigma_1)$ and $(\mathbf{M}_2(F), \sigma_2)$ are called *inertially equivalent* if there exist $g \in G$ and $\chi \in \mathrm{X}_{ur}(\mathbf{M}_2(F))$ such that $(\mathbf{M}_2(F), \sigma_2 \otimes \chi) = ({}^g\mathbf{M}_1(F), {}^g\sigma_1)$, where ${}^g\sigma_1$ is defined by ${}^g\sigma_1(x) = \sigma_1(gxg^{-1})$, for $x \in {}^g\mathbf{M}_1(F) = g^{-1}\mathbf{M}_1(F)g$. Let $[\mathbf{M}(F), \sigma]_{\mathbf{G}(F)}$ be the *inertial equivalence class* corresponding to the cuspidal pair $(\mathbf{M}(F), \sigma)$. Denote $\mathfrak{B}(\mathbf{G}(F))$ for the Bernstein spectrum of $\mathbf{G}(F)$, which is the set of all inertial equivalence classes in $\mathbf{G}(F)$. We say that a smooth irreducible representation $(\pi, \mathcal{V})$ has *inertial* *support* $[\mathbf{M}(F), \sigma]_{\mathbf{G}(F)}$ if $(\pi, \mathcal{V})$ appears as a subquotient of a representation parabolically induced from some element of $[\mathbf{M}(F), \sigma]_{\mathbf{G}(F)}$.

Let $\mathfrak{R}(\mathbf{G}(F))$ be the category of all smooth complex representations of $\mathbf{G}(F)$. For each $\mathfrak{s} := [\mathbf{M}(F), \sigma]_{\mathbf{G}(F)} \in \mathfrak{B}(\mathbf{G}(F))$, one can define a full subcategory $\mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F))$ (called the *Bernstein block* corresponding to $\mathfrak{s} = [\mathbf{M}(F), \sigma]_{\mathbf{G}(F)}$) of $\mathfrak{R}(\mathbf{G}(F))$ such that $\mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F))$ consists of those smooth complex representations $(\pi, \mathcal{V}) \in \mathfrak{R}(\mathbf{G}(F))$ whose each irreducible subquotient has inertial support $\mathfrak{s} = [\mathbf{M}(F), \sigma]_{\mathbf{G}(F)}$.

Theorem 22 (Bernstein)**.** *The Bernstein decomposition gives a direct product decomposition of $\mathcal{R}(\mathbf{G}(F))$ into indecomposable subcategories $\mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F))$:*

$$\mathfrak{R}(\mathbf{G}(F)) = \prod_{\mathfrak{s} \in \mathfrak{B}(\mathbf{G})} \mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F)),$$

where $\mathfrak{s}$ runs over the spectrum $\mathfrak{B}(\mathbf{G}(F))$.

Concretely, if $(\pi, \mathcal{V}) \in \mathcal{R}(\mathbf{G}(F))$, then for each $\mathfrak{s} \in \mathfrak{B}(\mathbf{G})$, $\mathcal{V}$ has a unique maximal $\mathbf{G}(F)$-subspace $\mathcal{V}_{\mathfrak{s}} \in \mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F))$ and

$$\mathcal{V} = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(\mathbf{G})} \mathcal{V}_{\mathfrak{s}}.$$

Moreover, if $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{B}(\mathbf{G})$ with $\mathfrak{s} \neq \mathfrak{s}'$, then $\mathrm{Hom}_{\mathbf{G}(F)}(\mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F)), \mathfrak{R}^{\mathfrak{s}'}(\mathbf{G}(F))) = 0$.

4.2 Hecke algebra

Let (τ, V) be an irreducible smooth representation of a compact open subgroup J of $\mathbf{G}(F)$. The *Hecke algebra* $\mathcal{H}(\mathbf{G}(F), (J, \tau))$ is the space of all compactly supported functions $f : \mathbf{G}(F) \to \mathrm{End}_{\mathbb{C}}(V^{\vee})$ satisfying

$$f(j_1 g j_2) = \tau^{\vee}(j_1) f(g) \tau^{\vee}(j_2) \text{ for all } j_1, j_2 \in J \text{ and } g \in \mathbf{G}(F).$$

Here $(\tau^{\vee}, V^{\vee})$ denotes the contragredient representation of (τ, V) . The standard *convolution operation* '$*$' defined by

$$f_1 * f_2(g) = \int_{\mathbf{G}(F)} f_1(x) f_2(x^{-1} g)\, dx, \text{ for } f_1, f_2 \in \mathcal{H}(\mathbf{G}(F), (J, \tau)),$$

gives the space $\mathcal{H}(\mathbf{G}(F),(J,\tau))$ a structure of an associative $\mathbb{C}$-algebra with identity. Whenever the compact open subgroup J of $\mathbf{G}(F)$ is explicitly mentioned, we will denote $\mathcal{H}(\mathbf{G}(F),\tau)$ for the *Hecke algebra* $\mathcal{H}(\mathbf{G}(F),(J,\tau))$ by dropping the notation J.

An element $g \in \mathbf{G}(F)$ is said to *intertwine* τ if the space $\mathrm{Hom}_{J\cap {}^gJ}(\tau, {}^g\tau)$ is non-zero. Equivalently, an element $g \in \mathbf{G}(F)$ intertwines τ if and only if there exists a function $f \in \mathcal{H}(\mathbf{G}(F),\tau)$ whose support contains the element g. We denote $\mathcal{I}_{\mathbf{G}(F)}(\tau)$ for the set of element $g \in \mathbf{G}(F)$ such that g intertwines τ. This set $\mathcal{I}_{\mathbf{G}(F)}(\tau)$ of intertwining serves as an essential object in the study of the Hecke algebra $\mathcal{H}(\mathbf{G}(F),\tau)$.

If $(\pi,\mathcal{V})$ is a smooth complex representation of $\mathbf{G}(F)$, then the *τ-isotypic subspace* $\mathcal{V}^\tau$ of $\mathcal{V}$ is the sum of all irreducible J-subspaces of $\mathcal{V}$, which are equivalent to τ. Let $\mathfrak{R}_\tau(\mathbf{G}(F))$ denote the subcategory of $\mathfrak{R}(\mathbf{G}(F))$ whose objects are the smooth complex representations $(\pi,\mathcal{V})$ of $\mathbf{G}(F)$ generated by the τ-isotypic subspace $\mathcal{V}^\tau$ of $\mathcal{V}$. There is a functor

$$\mathcal{M}_\tau : \mathfrak{R}_\tau(\mathbf{G}(F)) \to \mathcal{H}(\mathbf{G}(F),\tau)\text{-Mod}, \tag{4.1}$$

given by

$$\pi \mapsto \mathrm{Hom}_J(\tau,\pi).$$

Here $\mathcal{H}(\mathbf{G}(F),\tau)$-Mod denotes the category of unital left-modules over the *Hecke algebra* $\mathcal{H}(\mathbf{G}(F),\tau)$.

4.3 Notion of Types

For each $\mathfrak{s} \in \mathfrak{B}(\mathbf{G}(F))$, Bushnell and Kutzko [BK] described the abelian category $\mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F))$ in terms of a class of pairs (J,τ) consisting of irreducible smooth representations τ of compact open subgroups J of $\mathbf{G}(F)$. For any finite subset $\mathfrak{S} \subset \mathfrak{B}(\mathbf{G}(F))$, we write $\mathfrak{R}^{\mathfrak{S}}(\mathbf{G}(F)) = \prod_{\mathfrak{s}\in\mathfrak{S}} \mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F))$ i.e., $\mathfrak{R}^{\mathfrak{S}}(\mathbf{G}(F))$ is the full subcategory of $\mathfrak{R}(\mathbf{G}(F))$ such that $\mathfrak{R}^{\mathfrak{S}}(\mathbf{G}(F))$ consists of those smooth complex representations $(\pi,\mathcal{V}) \in \mathfrak{R}(\mathbf{G}(F))$ whose each irreducible subquotient has *inertial support* contained in $\mathfrak{S}$.

Definition 23. *Suppose τ be an irreducible smooth representation of a compact open subgroup J of $\mathbf{G}(F)$. For a finite subset $\mathfrak{S}$ of $\mathfrak{B}(\mathbf{G}(F))$, the pair (J,τ) is called an $\mathfrak{S}$-type in $\mathbf{G}(F)$ if for any irreducible smooth representation $(\pi,\mathcal{V})$ of $\mathbf{G}(F)$, we have $(\pi,\mathcal{V}) \in \mathfrak{R}^{\mathfrak{S}}(\mathbf{G}(F))$ if and only if* $\mathrm{Hom}_J(\tau,\pi) \neq 0$.

Equivalently, the pair (J,τ) is an $\mathfrak{S}$-*type* in $\mathbf{G}(F)$ if $\mathfrak{R}_\tau(\mathbf{G}(F)) = \mathfrak{R}^{\mathfrak{S}}(\mathbf{G}(F))$ as subcategories of $\mathfrak{R}(\mathbf{G}(F))$. When $\mathfrak{S}$ is a singleton set $\{\mathfrak{s}\}$, we will write '$\mathfrak{s}$-*type*' in place of $\{\mathfrak{s}\}$-*type*. From [BK, Theorem 4.3] and [LN, §4.2], we now review some properties of types. If $\mathfrak{S}$ is a finite subset of $\mathfrak{B}(\mathbf{G}(F))$ and the pair (J,τ) is consisting of an irreducible smooth representation τ of a compact open subgroup J of $\mathbf{G}(F)$, then the followings happen.

(1) The pair (J,τ) is an $\mathfrak{S}$-*type* in $\mathbf{G}(F)$ if and only if $\mathfrak{R}_\tau(\mathbf{G}(F)) = \mathfrak{R}^{\mathfrak{S}}(\mathbf{G}(F))$ as subcategories of $\mathfrak{R}(\mathbf{G}(F))$.

(2) If the pair (J,τ) is an $\mathfrak{S}$-*type* in $\mathbf{G}(F)$, the functor $\mathcal{M}_\tau$ as mentioned in Eq. (4.1) gives an equivalence of the categories:

$$\mathcal{M}_\tau : \mathfrak{R}_\tau(\mathbf{G}(F)) \xrightarrow{\sim} \mathcal{H}(\mathbf{G}(F),\tau)\text{-Mod}.$$

(3) The pair (J,τ) is an $\mathfrak{S}$-*type* in $\mathbf{G}(F)$ if and only if the pair $({}^gJ, {}^g\tau)$ is an $\mathfrak{S}$-*type* in $\mathbf{G}(F)$ for any $g \in \mathbf{G}(F)$.

(3) Let $\mathfrak{s} = [\mathbf{G}(F),\pi]_{\mathbf{G}(F)}$ be an inertial equivalence class of a supercuspidal representation π of $\mathbf{G}(F)$ and ρ is an irreducible smooth representation of a compact open subgroup K of $\mathbf{G}(F)$ with $J \subset K$ such that the space $\mathrm{Hom}_J(\tau,\rho) \neq 0$. Then the pair (J,τ) is an $\mathfrak{s}$-*type* in $\mathbf{G}(F)$ implies the pair (K,ρ) is an $\mathfrak{s}$-*type* in $\mathbf{G}(F)$.

4.3.1 G-cover

Let $\mathbf{M}$ be a proper Levi subgroup of $\mathbf{G}$ defined over F and $\mathbf{M}(F)$ denotes the group of F-points of $\mathbf{M}$.

Definition 24. *Consider the pair $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ consisting of a compact open subgroup $K_{\mathbf{M}(F)}$ of $\mathbf{M}(F)$ and an irreducible smooth representation $\rho_{\mathbf{M}(F)}$ of the subgroup $K_{\mathbf{M}(F)}$. Let K be a compact open subgroup of $\mathbf{G}(F)$ and ρ be an irreducible smooth representation of K. The pair (K,ρ) is called a $\mathbf{G}(F)$-cover of $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ if for any opposite pair of F-parabolic subgroups $\mathbf{P} = \mathbf{MN}$ and $\mathbf{P}^- = \mathbf{MN}^-$ with common Levi factor $\mathbf{M}$ and unipotent radicals $\mathbf{N}$ and $\mathbf{N}^-$ respectively, the pair (K,ρ) satisfies the following properties:*

(1) K decomposes with respect to $(\mathbf{N}(F), \mathbf{M}(F), \mathbf{N}^{-}(F))$ i.e.,

$$K = (K \cap \mathbf{N}(F))(K \cap \mathbf{M}(F))(K \cap \mathbf{N}^{-}(F)).$$

(2) $K_{\mathbf{M}(F)} = K \cap \mathbf{M}(F)$, $\rho|_{K_{\mathbf{M}(F)}} = \rho_{\mathbf{M}(F)}$ and $K \cap \mathbf{N}(F), K \cap \mathbf{N}^{-}(F)$ is contained in the kernel $\ker(\rho)$ of ρ.

(3) For any smooth representation $(\pi, \mathcal{V})$ of $\mathbf{G}(F)$; the natural projection $\mathcal{V}$ to the Jacquet module $\mathcal{V}_{\mathbf{N}(F)}$ induces an injection on $\mathcal{V}^{\rho}$.

See [Blo, Theorem 1] for this reformulation of the original definition of $\mathbf{G}(F)$-*cover* due to Bushnell and Kutzko [BK, §8.1] (see also [Roc, §4.2]). The transitivity property of cover follows from [BK, Proposition 8.5]. Explicitly, if $\mathbf{L}$ is an F-Levi subgroup of $\mathbf{G}$ such that $\mathbf{L}$ is contained in $\mathbf{M}$ and the pair $(K_{\mathbf{L}(F)}, \rho_{\mathbf{L}(F)})$ is consisting of a compact open subgroup $K_{\mathbf{L}(F)}$ of $\mathbf{L}(F)$ together with an irreducible smooth representation $\rho_{\mathbf{L}(F)}$ of the subgroup $K_{\mathbf{L}(F)}$, then the followings hold:

(1) If (K, ρ) is a $\mathbf{G}(F)$-*cover* of $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ and $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ is a $\mathbf{M}(F)$-*cover* of $(K_{\mathbf{L}(F)}, \rho_{\mathbf{L}(F)})$, then (K, ρ) is a $\mathbf{G}(F)$-*cover* of $(K_{\mathbf{L}(F)}, \rho_{\mathbf{L}(F)})$.

(2) Suppose (K, ρ) is a $\mathbf{G}(F)$-*cover* of $(K_{\mathbf{L}(F)}, \rho_{\mathbf{L}(F)})$. If we denote $K \cap \mathbf{M}(F) = K_{\mathbf{M}(F)}$ and $\rho|_{K_{\mathbf{M}(F)}} = \rho_{\mathbf{M}(F)}$, then (K, ρ) is a $\mathbf{G}(F)$-*cover* of $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ and $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ is a $\mathbf{M}(F)$-*cover* of $(K_{\mathbf{L}(F)}, \rho_{\mathbf{L}(F)})$.

Let $\mathfrak{B}(\mathbf{M}(F))$ denote the Bernstein spectrum of $\mathbf{M}(F)$. Consider any finite subset $\mathfrak{S}_{\mathbf{M}(F)} = \{[L_i, \sigma_i]_{\mathbf{M}(F)} : i = 1, ..., n\} \subset \mathfrak{B}(\mathbf{M}(F))$, where L_i is an F-Levi subgroup of $\mathbf{M}(F)$ and σ_i is an irreducible supercuspidal representation of L_i. Each class $[L_i, \sigma_i]_{\mathbf{M}(F)}$ uniquely determines an element $[L_i, \sigma_i]_{\mathbf{G}(F)}$ in $\mathfrak{B}(\mathbf{G}(F))$. Therefore, every finite subset $\mathfrak{S}_{\mathbf{M}(F)} = \{[L_i, \sigma_i]_{\mathbf{M}(F)} : i = 1, ..., n\} \subset \mathfrak{B}(\mathbf{M}(F))$ corresponds to a finite subset $\mathfrak{S}_{\mathbf{G}(F)} = \{[L_i, \sigma_i]_{\mathbf{G}(F)} : i = 1, ..., n\}$ of $\mathfrak{B}(\mathbf{G}(F))$. Using the notion of *cover*, Bushnell and Kutzko [BK, §8] showed that every $\mathfrak{S}_{\mathbf{M}(F)}$-*type* in $\mathbf{M}(F)$ induces an $\mathfrak{S}_{\mathbf{G}(F)}$-*type* in $\mathbf{G}(F)$. One key property of $\mathbf{G}(F)$-*cover* is the following:

Proposition 25. *[BK, Theorem 8.3] Suppose the pair $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ is an $\mathfrak{S}_{\mathbf{M}(F)}$-type in $\mathbf{M}(F)$. If (K, ρ) is a $\mathbf{G}(F)$-cover of $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$, then (K, ρ) is an $\mathfrak{S}_{\mathbf{G}(F)}$-type in $\mathbf{G}(F)$.*

In particular, if $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$ is an $[\mathbf{M}(F), \sigma]_{\mathbf{M}(F)}$-type in $\mathbf{M}(F)$ and (K, ρ) is a $\mathbf{G}(F)$-cover of $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$, then (K, ρ) is an $[\mathbf{M}(F), \sigma]_{\mathbf{G}(F)}$-type in $\mathbf{G}(F)$, and the representation $c\text{-Ind}_K^{\mathbf{G}(F)}\rho$ decomposes as a direct sum of some finite-length admissible representations of $\mathbf{G}(F)$, where each one is isomorphic to $c\text{-Ind}_{\mathbf{P}(F)}^{\mathbf{G}(F)}\sigma'$ for some parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with its Levi factor $\mathbf{M}$ and some irreducible supercuspidal representation σ' of $\mathbf{M}(F)$ contained in the inertial support of σ [LN, Lemma 4.3].

Suppose (K, ρ) is a $\mathbf{G}(F)$-*cover* of $(K_{\mathbf{M}(F)}, \rho_{\mathbf{M}(F)})$, then for any F-parabolic subgroup $\mathbf{P}' = \mathbf{M}\mathbf{N}'$ with its Levi factor $\mathbf{M}$, and its unipotent radical $\mathbf{N}'$, there is a $\mathbb{C}$-algebra embedding [BK, §8.3]

$$t_{\mathbf{P}'} : \mathcal{H}(\mathbf{M}(F), \rho_{\mathbf{M}(F)}) \to \mathcal{H}(\mathbf{G}(F), \rho), \tag{4.2}$$

with the property that for any smooth representation Υ of $\mathbf{G}(F)$,

$$\mathcal{M}_{\rho_{\mathbf{M}(F)}}(\Upsilon_{\mathbf{N}'(F)}) \cong t^*_{\mathbf{P}'}(\mathcal{M}_\rho(\Upsilon) \text{ as } \mathcal{H}(\mathbf{M}(F), \rho_{\mathbf{M}(F)})\text{-mod} \tag{4.3}$$

Here $\Upsilon_{\mathbf{N}'(F)}$ is the Jacquet-module of Υ at the unipotent radical $\mathbf{N}'(F)$, and $t^*_{\mathbf{P}'}$: $\mathcal{H}(\mathbf{G}(F), \rho)$-mod $\to$ $\mathcal{H}(\mathbf{M}(F), \rho_{\mathbf{M}(F)})$-mod is the restriction functor induced by $t_{\mathbf{P}'}$ (see [BK, Eq. 7.10]).

Kim and Yu [KY], using Kim's work [Kim], showed that Yu's construction of supercuspidals [Yu2] can be used to produce $\mathbf{G}(F)$-*covers* of $[\mathbf{M}(F), \sigma]_{\mathbf{M}(F)}$-*types* for all $[\mathbf{M}(F), \sigma]_{\mathbf{G}(F)} \in \mathfrak{B}(\mathbf{G}(F))$, assuming F has characteristic 0 and the residue characteristic p of F is suitably large. Using an approach different from Kim, recently in [Fin], Fintzen has shown the construction of *types* for all Bernstein blocks without any restriction on the characteristic of F, and assuming only that the order of the Weyl group of $\mathbf{G}$ can not be divided by p.

Part II

Depth of a representation of a p-adic group:

5

Notion of Depth

Let $G = \mathrm{Gal}(L/F)$ be the Galois group of the field extension L/F, where L and F are local fields and let M be a G-module. Define the depth of M as

$$\mathrm{dep}_G(M) = \inf\{r \geq 0 \mid M^{G^s} \neq 0 \;\forall\; s > r\}.$$

Define the depth of a co-cycle as $\varphi \in \mathrm{H}^1(G, M)_{\mathrm{adm}} := \bigcup_{r \geq 0} \mathrm{H}^1(G/G^r, M^{G^r})$ to be:

$$\mathrm{dep}_G(\varphi) = \inf\{r \geq 0 \mid G^s \subset \ker(\varphi) \;\forall s > r\}.$$

5.1 Depth change under induction

Let $G = \mathrm{Gal}(L/F)$ be the Galois group of the extension L/F and $H = \mathrm{Gal}(L/K)$ be the Galois group of the extension L/K, where $F \subseteq K \subseteq L$ and K/F is finite Galois extension.

Let N be an H-module. We define induction of N to G by

$$\mathrm{Ind}_H^G N := \mathrm{Hom}_H(G, N),$$

i.e. $\mathrm{Ind}_H^G N$ consists of all functions $f : G \to N$ on G satisfying $f(hg) = hf(g)$ for all $g \in G, h \in H$. The G-action on $\mathrm{Ind}_H^G N$ is given by $(g \cdot f)(x) = f(xg)$ for $g, x \in G$ and $f \in \mathrm{Ind}_H^G N$.

Let M be a G-module. Then using Frobenius reciprocity theorem $\mathrm{Hom}_G(M, \mathrm{Ind}_H^G N) \cong \mathrm{Hom}_H(\mathrm{Res}_H M, N)$, we have a natural isomorphism

$$(\mathrm{Ind}_H^G N)^G \cong N^H \tag{5.1}$$

here Res_H denotes the restriction functor to the subgroup H.

Proposition 26. *If N be an H-module then,*

$$\mathrm{dep}_H(N) = \psi_{K/F}(\mathrm{dep}_G(\mathrm{Ind}_H^G N)).$$

Proof. By Mackey theory, the restriction $\mathrm{Res}_{G^r}(\mathrm{Ind}_H^G N)$ of the G-module $\mathrm{Ind}_H^G N$ to a G^r-module decomposes as a direct sum

$$\mathrm{Res}_{G^r}(\mathrm{Ind}_H^G N) = \bigoplus_{\bar{g} \in G^r \backslash G/H} \mathrm{Ind}_{G^r \cap {}^g H}^{G^r} N^g,$$

where g is a representative of the double coset $\bar{g}$, ${}^g H$ denotes the subgroup $g^{-1}Hg$ and N^g denotes the g-twisted module N with $(G^r \cap {}^g H)$-action defined by $x \cdot N^g = (gxg^{-1}) \cdot N$ for $x \in G^r \cap {}^g H$, which only depends on the coset $\bar{g}$ but not on the chosen coset representative g. By Proposition 10, $G^r \cap {}^g H = ({}^g H)^{\psi_{K/F}(r)}$. Thus for any $r \geq 0$

$$\begin{aligned}
(\mathrm{Ind}_H^G N)^{G^r} \neq 0 \quad &\iff \quad (\mathrm{Ind}_{({}^g H)^{\psi_{K/F}(r)}}^{G^r} N^g)^{G^r} \neq 0 \text{ for some } \bar{g} \in G^r \backslash G/H \\
&\iff \quad (N^g)^{({}^g H)^{\psi_{K/F}(r)}} \neq 0 \text{ (follows from Eq.5.1)} \\
&\iff \quad (N)^{H^{\psi_{K/F}(r)}} \neq 0.
\end{aligned}$$

Hence, the result follows. □

5.2 Depth change under Shapiro's isomorphism

Again $G = \mathrm{Gal}(L/F)$ and $H = \mathrm{Gal}(L/K)$ where $F \subseteq K \subseteq L$ and K/F is any finite extension and let N be an H-module.

Shapiro's lemma states that the map

$$\mathrm{Sh} : \mathrm{H}^1(G, \mathrm{Ind}_H^G N) \to \mathrm{H}^1(H, N)$$

defined by

$$\gamma \mapsto (h \mapsto \gamma(h)(1))$$

is an isomorphism. We wish to relate the depth of co-cycles under this isomorphism. We first observe the following:

Lemma 27. *Let A be a group, B and C subgroups of A with C being normal in A. Let M be a B-module. Then there is a canonical isomorphism of A/C-modules:*

$$(\mathrm{Ind}_B^A M)^C \cong \mathrm{Ind}_{B/B\cap C}^{A/C} M^{B\cap C}.$$

Proof. For each $f \in \left(\mathrm{Ind}_B^A M\right)^C$, define $\widetilde{f} : A/C \to M$ by $\widetilde{f}(aC) = f(a)$. We will show that $\widetilde{f} \in \mathrm{Ind}_{B/B\cap C}^{A/C} M^{B\cap C}$. Put $D = B \cap C$. Since M is B-module, we have M^D is B/D-module. Since $f \in \left(\mathrm{Ind}_B^A M\right)^C$, we have $d \cdot \widetilde{f}(aC) = d \cdot f(a) = f(a) = \widetilde{f}(aC)$ for all $aC \in A/C$ and $d \in D$. Therefore, image of $\widetilde{f}$ lies in $M^{B\cap C}$. Define a map $\Psi : \left(\mathrm{Ind}_B^A M\right)^C \to \mathrm{Ind}_{B/B\cap C}^{A/C} M^{B\cap C}$ by $f \mapsto \widetilde{f}$. Clearly, Ψ is well-defined A/C-linear map.

Let $f \in \mathrm{Ker}(\Psi)$. Therefore $\widetilde{f} \equiv \mathbf{0} \Rightarrow \widetilde{f}(aC) = 0$ for all $a \in A \Rightarrow f(a) = 0$ for all $a \in A \Rightarrow f \equiv \mathbf{0}$. Therefore, $\mathrm{Ker}(\Psi) = \{\mathbf{0}\}$ and Ψ is injective.

Let $\tau \in \mathrm{Ind}_{B/B\cap C}^{A/C} M^{B\cap C}$. Therefore τ is a map from A/C to $M^{B\cap C}$, where $\tau(bD.aC) = bD.\tau(aC)$. Define a map $f : A \mapsto M$ by $f(a) = \tau(aC)$. Now for $b \in B$ and $a \in A$ we have $f(ba) = \tau(baC) = \tau(bD \cdot aC) = bD \cdot \tau(aC) = bf(a)$ and, $(c \cdot f)(a) = f(ac) = \tau(acC) = \tau(aC) = f(a)$ for each $c \in C$ and $a \in A$. So, $f \in \left(\mathrm{Ind}_B^A M\right)^C$ and $\Psi(f) = \tau$. Hence the surjectivity of Ψ follows.

□

Lemma 28. *For $r \geq 0$, Shapiro's lemma induces an isomorphism* $\mathrm{H}^1(G/G^r, (\mathrm{Ind}_H^G N)^{G^r}) \cong \mathrm{H}^1(H/H^{\psi_{K/F}(r)}, N^{H^{\psi_{K/F}(r)}})$.

Proof. For $r \geq 0$, we have

$$\begin{aligned}
\mathrm{H}^1(G/G^r, (\mathrm{Ind}_H^G N)^{G^r}) &\cong \mathrm{H}^1(G/G^r, \mathrm{Ind}_{H/G^r\cap H}^{G/G^r} N^{G^r\cap H}) \\
&\cong \mathrm{H}^1(G/G^r, \mathrm{Ind}_{H/H^{\psi_{K/F}(r)}}^{G/G^r} N^{H^{\psi_{K/F}(r)}}) \\
&\cong \mathrm{H}^1(H/H^{\psi_{K/F}(r)}, N^{H^{\psi_{K/F}(r)}}).
\end{aligned}$$

The first isomorphism follows from Lemma 27, second from Proposition 10 and the last from Shapiro's lemma. $\square$

Write $\mathrm{H}^1(G, \mathrm{Ind}_H^G N)_{\mathrm{adm}} = \bigcup_{r\geq 0} \mathrm{H}^1(G/G^r, (\mathrm{Ind}_H^G N)^{G^r})$. In the next result, we will give a depth comparison formulae under Shapiro's isomorphism.

Corollary 29. *If* $\lambda \in \mathrm{H}^1(G, \mathrm{Ind}_H^G N)_{\mathrm{adm}}$, *then* $\mathrm{dep}_G(\lambda) = \varphi_{K/F}(\mathrm{dep}_H(\mathrm{Sh}(\lambda))$.

Proof. Let $\mathrm{dep}_G(\lambda) = r$. Then $G^s \subset \ker(\lambda)$ if $s > r$. By Lemma 28, this implies $H^{\psi_{K/F}(s)} \subset \ker(\mathrm{Sh}(\lambda))$ if $s > r$. Therefore $\mathrm{dep}_H(\mathrm{Sh}(\lambda)) \leq \psi_{K/F}(\mathrm{dep}_G(\lambda))$. The argument is reversible showing that $\mathrm{dep}_H(\mathrm{Sh}(\lambda)) \geq \psi_{K/F}(\mathrm{dep}_G(\lambda))$. Therefore $\mathrm{dep}_H(\mathrm{Sh}(\lambda)) = \psi_{K/F}(\mathrm{dep}_G(\lambda))$.

$\square$

6

Local Langlands correspondence for tori

First we briefly discuss the history of the *local Langlands correspondence* (in short LLC) for tori, which was observed by Robert P Langlands. Then we will review here the statement of the *local Langlands correspondence* (LLC) for tori as stated and proved in [Yu1].

6.1 Local class field theory

Let $W_F^{\mathrm{ab}} := W_F/\overline{[W_F, W_F]}$ be the maximal abelian quotient of the Weil group W_F of the local field F relative to F^{sep}. Recall the following well known result from the local class field theory:

Theorem 30 (Artin reciprocity)**.** *There exists a unique natural isomorphism of topological groups*

$$\tau_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}. \tag{6.1}$$

Let $\mathrm{Irr}(F^\times)$ be the collection of all irreducible smooth complex characters of $F^\times$. Then the Artin reciprocity map τ_F gives us a bijection

$$\mathrm{Irr}(F^\times) = \mathrm{Hom}(F^\times, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}(W_F^{\mathrm{ab}}, \mathbb{C}^\times) = \mathrm{Hom}(W_F, \mathbb{C}^\times). \tag{6.2}$$

(Here 'Hom' means smooth homomorphism of topological groups.) Interpreting the bijection $\lambda_{\mathbf{G}_m} : \mathrm{Irr}(F^\times) \xrightarrow{\sim} \mathrm{Hom}(W_F, \mathbb{C}^\times)$ as a statement for $GL_1(F)$, Langlands generalized this idea to a beautiful correspondence (LLC) for tori.

6.2 LLC for split tori:

Let $\mathbf{T}$ be an F-split torus and $T = \mathbf{T}(F)$ denotes the group of F-rational points of $\mathbf{T}$. Let $X^*(\mathbf{T})$ (resp. $X_*(\mathbf{T})$) be the lattice of of algebraic characters $\mathbf{T} \to \mathbf{G}_m$ (resp. algebraic co-characters $\mathbf{G}_m \to \mathbf{T}$). Then $\mathbf{T}(F) \cong X_*(\mathbf{T}) \otimes_{\mathbb{Z}} F^\times$ and $T^\vee := X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ be the complex dual torus of $T = \mathbf{T}(F)$ with the identifications $X^*(\mathbf{T}) = X_*(T^\vee)$ and $X_*(\mathbf{T}) = X^*(T^\vee)$. Let $\mathrm{Irr}(\mathbf{T}(F))$ be the collection of all irreducible smooth complex representations (which are characters) of $\mathbf{T}(F)$. Using the Hom-tensor duality and the Artin reciprocity map τ_F, we have the following isomorphism

$$\begin{aligned}
\mathrm{Irr}(\mathbf{T}(F)) = \mathrm{Hom}(\mathbf{T}(F), \mathbb{C}^\times) &= \mathrm{Hom}(X_*(T) \otimes_{\mathbb{Z}} F^\times, \mathbb{C}^\times) \\
&\cong \mathrm{Hom}(F^\times, X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times) \\
&\cong \mathrm{Hom}(W_F^{ab}, T^\vee) \\
&= \mathrm{Hom}(W_F, T^\vee).
\end{aligned}$$

A *Langlands parameter* for $\mathbf{T}(F)$ is defined to be an element of $\mathrm{Hom}(W_F, T^\vee)$. The collection of such parameters is denoted by $\Phi(\mathbf{T}(F))$, so we have a natural bijection

$$\lambda_{\mathbf{T}} : \mathrm{Irr}(\mathbf{T}(F)) \xrightarrow{\sim} \Phi(\mathbf{T}(F)), \tag{6.3}$$

called the local Langlands correspondence for split tori. The Langlands parameter of $\chi : \mathbf{T}(F) \to \mathbb{C}^\times$ is the homomorphism $\varphi_\chi : W_F \to T^\vee$ defined by

$$\gamma(\varphi_\chi(x)) = \chi(\gamma(\tau_F(x))) \tag{6.4}$$

for all $\gamma \in X_*(\mathbf{T}) = X^*(T^\vee)$ and $x \in W_F$.

6.3 LLC for induced tori: a special case

Let $\mathbf{T} = \mathrm{Res}_{F'/F}\mathbb{G}_m$, where F' is a finite separable extension of F contained in F^{sep} and $\mathrm{Res}_{F'/F}$ denotes the Weil restriction. Then, the group of F-rational points $\mathbf{T}(F) = F'^{\times}$ and the group of characters $X^*(\mathbf{T})$ is canonically a free $\mathbb{Z}$-module with basis $W_F/W_{F'}$ where W_F (resp. $W_{F'}$) denotes the Weil group of F (resp. F') and $W_{F'}$ is an open subgroup of W_F of index $[F' : F]$. From this, it follows that the complex dual $T^\vee = \widehat{\mathbf{T}}(\mathbb{C})$ of $\mathbf{T}(F)$ is canonically isomorphic to $\mathrm{Ind}_{W_{F'}}^{W_F}\mathbb{C}^\times$, which is a complex torus of dimension $[F' : F]$ and the group $T^\vee$ is endowed with a canonical action of W_F (see [ABPS2, Lan2] for more details). We get,

$$\begin{aligned}
\mathrm{Irr}(\mathbf{T}(F)) = \mathrm{Hom}(\mathbf{T}(F), \mathbb{C}^\times) &\cong \mathrm{Hom}(F'^{\times}, \mathbb{C}^\times) \\
&\cong \mathrm{Hom}(W_{F'}, \mathbb{C}^\times) \qquad (6.5) \\
&\cong \mathrm{H}^1(W_{F'}, \mathbb{C}^\times) \\
&\cong \mathrm{H}^1(W_F, \mathrm{Ind}_{W_{F'}}^{W_F}\mathbb{C}^\times) \qquad (6.6) \\
&\cong \mathrm{H}^1(W_F, T^\vee).
\end{aligned}$$

The isomorphism (6.5) follows from the Artin reciprocity map $\tau_{F'}$ and the isomorphism (6.6) by Shapiro's lemma. Therefore, we have a natural bijection

$$\lambda_{\mathbf{T}} : \mathrm{Irr}(\mathbf{T}(F)) \xrightarrow{\sim} \mathrm{H}^1(W_F, T^\vee), \tag{6.7}$$

called the local Langlands correspondence for induced tori. Here the space of Langlands parameters $\Phi(\mathbf{T}(F))$ is $\mathrm{H}^1(W_F, T^\vee)$.

6.4 The LLC for tori in general

Langlands showed that the correspondence like $\lambda_{\mathbf{T}}$ is actually possible for every (non-split) torus $\mathbf{T}$. From [Yu1, §7.5], we will recall the statement of the local Langlands correspondence for tori in general.

Theorem. *[Lan2]There is a unique family of homomorphisms*

$$\lambda_{\mathbf{T}} : \mathrm{Hom}(\mathbf{T}(F), \mathbb{C}^{\times}) \to \mathrm{H}^1(W_F, T^{\vee})$$

with the following properties:

1. *$\lambda_{\mathbf{T}}$ is additive functorial in $\mathbf{T}$, i.e., it is a morphism between two additive functors from the category of tori over F to the category of abelian groups;*

2. *For $\mathbf{T} = \mathrm{Res}_{F'/F}\mathbb{G}_m$, where F'/F is a finite separable extension, $\lambda_{\mathbf{T}}$ is the isomorphism described in Section 6.3.*

6.5 Depth change under LLC for tori

We keep the notations as mentioned above. Let L be a local field. Recall that $L^{\times}$ admits a filtration $\{L_r^{\times}\}_{r\geq 0}$ where $L_0^{\times}$ is the units of the ring of integers and for $r > 0$, $L_r^{\times} := \{x \in L \mid \mathrm{val}_L(x-1) \geq r\}$. Here val_L is the valuation of L normalised so that $\mathrm{val}_L(L^{\times}) = \mathbb{Z}$. Under local class field theory isomorphism

$$L_r^{\times} \cong (W_L^r)^{\mathrm{ab}}.$$

We recall that $\mathbf{T}(F)$ carries a Moy-Prasad filtration $\{\mathbf{T}(F)_r\}_{r\geq 0}$ (see §3.6). The depth $\mathrm{dep}_{\mathbf{T}}(\chi)$ of a character $\chi : \mathbf{T}(F) \to \mathbb{C}^{\times}$ is defined to be

$$\inf\{r \geq 0 \mid \mathbf{T}(F)_s \subset \ker(\chi) \text{ for } s > r\}.$$

The group $\mathbf{T}(F)_0$ is called the Iwahori subgroup of $\mathbf{T}(F)$. It is a subgroup of finite index in the maximal compact subgroup of $\mathbf{T}(F)$. When $\mathbf{T} = \mathrm{Res}_{F'/F}\mathbb{G}_{\mathrm{m}}$, then for $r > 0$,

$$\begin{aligned} \mathbf{T}(F)_r &= \{x \in \mathbf{T}(F) = F'^{\times} \mid \mathrm{val}'_F(x-1) \geq r\} && (6.8)\\ &= \{x \in F'^{\times} \mid \mathrm{val}_{F'}(x-1) \geq er\} && (6.9)\\ &= F'^{\times}_{er}. \end{aligned}$$

Here val'_F is the valuation on F' normalised so that $\mathrm{val}'_F(F^{\times}) = \mathbb{Z}$, and e is the ramification index of F'/F. The equality 6.8 follows from [Yu3, Sec. 4.2] and the equality 6.9 follows from the fact that $\mathrm{val}_{F'}(\alpha) = e \cdot \mathrm{val}'_F(\alpha)$ for all $\alpha \in F^\times$.

Proposition 31. *[MP1, Theorem 6] Let $\mathbf{T} = \mathrm{Res}_{F'/F}\mathbb{G}_{\mathrm{m}}$, where F'/F is a finite separable extension of local fields of ramification index e. Then for $r \geq 0$, the local Langlands correspondence for tori induces an isomorphism:*

$$\mathrm{Hom}(\mathbf{T}(F)/\mathbf{T}(F)_r, \mathbb{C}^\times) \cong \mathrm{H}^1(W_F/W_F^{\varphi_{F'/F}(er)}, T^{\vee W_F^{\varphi_{F'/F}(er)}}).$$

Proof. The case $r = 0$ is a special case of [Mis, Theorem 7]. For $r > 0$, this follows by

$$\begin{aligned} \mathrm{Hom}(\mathbf{T}(F)/\mathbf{T}(F)_r, \mathbb{C}^\times) &\cong \mathrm{Hom}(F'^\times/F'^\times_{er}, \mathbb{C}^\times) \\ &\cong \mathrm{Hom}(W_{F'}/W_{F'}^{er}, \mathbb{C}^\times) && (6.10) \\ &\cong \mathrm{H}^1(W_{F'}/W_{F'}^{er}, \mathbb{C}^\times) \\ &\cong \mathrm{H}^1(W_F/W_F^{\varphi_{F'/F}(er)}, (\mathrm{Ind}_{W_{F'}}^{W_F}\mathbb{C}^\times)^{W_F^{\varphi_{F'/F}(er)}}) && (6.11) \\ &\cong \mathrm{H}^1(W_F/W_F^{\varphi_{F'/F}(er)}, (T^\vee)^{W_F^{\varphi_{F'/F}(er)}}). \end{aligned}$$

Here, the isomorphism 6.11 follows from Lemma 28. □

Theorem 32. *Let $\mathbf{T} = \mathrm{Res}_{F'/F}\mathbb{G}_{\mathrm{m}}$, where F'/F is a finite separable extension of local fields of ramification index e. Then we have the following depth changing formulae under LLC for tori:*

$$\varphi_{F'/F}(e \cdot \mathrm{dep}_{\mathbf{T}}(\chi)) = \mathrm{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi)).$$

Proof. Suppose, $\mathrm{dep}_{\mathbf{T}}(\chi) = r$ for some non-negative real number r. Then, $\mathbf{T}(F)_s \subset \ker(\chi)$ for $s > r$. Now, from Proposition 31, it follows that $W_F^{\varphi_{F'/F}(er)} \subset \ker(\lambda_{\mathbf{T}}(\chi))$ for $s > r$. Therefore, we have

$$\mathrm{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi)) \leq \varphi_{F'/F}(er) = \varphi_{F'/F}(e \cdot \mathrm{dep}_{\mathbf{T}}(\chi))$$

The argument is reversible showing that

$$\mathrm{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi)) \geq \varphi_{F'/F}(e \cdot \mathrm{dep}_{\mathbf{T}}(\chi))$$

Therefore, $\varphi_{F'/F}(e \cdot \mathrm{dep}_{\mathbf{T}}(\chi)) = \mathrm{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi))$. □

Remark 33. The slope of the map $r \mapsto \varphi_{F'/F}(er)$ at a differentiable point r is $\frac{e}{(G_0:G_r)} \geq 1$. Thus, when F'/F is a wildly ramified extension, $\varphi_{F'/F}(er) > r$ and consequently $\operatorname{dep}_{\mathbf{T}}(\chi) < \operatorname{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi))$.

When F'/F is a tamely ramified extension, $\varphi_{F'/F}(r) = \frac{r}{e}$. Therefore in this case, Theorem 32 simplifies to,

$$\operatorname{dep}_{\mathbf{T}}(\chi) = \operatorname{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi)).$$

This is a special case of Depth-preservation Theorem of Yu for tamely ramified tori [Yu2, Theorem 7.10].

6.5.1 Case of a tamely induced tori

Recall that a F-torus is called *induced* if it is of the form $\Pi_{i=1}^{n}\mathrm{Res}_{L_i/F}\mathbb{G}_{\mathrm{m}}$, where L_i are finite separable extensions of F. A F-torus $\mathcal{T}$ is called *tamely induced* it $\mathcal{T} \otimes_F F_t$ is an induced torus for some tamely ramified extension F_t of F. In this section, we compare depths under LLC for such tori following the proof in [Yu2, Sec. 7.10].

Let $\mathbf{T}$ be a tamely induced F-torus. Then there exists an induced torus $\mathbf{T}' = \prod_{i=1}^{n}\mathrm{Res}_{F_i'/F}\mathbb{G}_{\mathrm{m}}$ such that $\mathbf{T}' \twoheadrightarrow \mathbf{T}$ and $C_0 := \ker(\mathbf{T}' \to \mathbf{T})$ is connected. Further $\mathbf{T}'(F)_r \twoheadrightarrow \mathbf{T}(F)_r\ \forall r > 0$ (see proof in [Yu3, Lemma 4.7.4]). Let $\chi \in \mathrm{Hom}(\mathbf{T}(F), \mathbb{C}^\times)$ and let χ' denote its lift to $\mathbf{T}'(F)$. Then

$$\operatorname{dep}_{\mathbf{T}}(\chi) = \operatorname{dep}_{\mathbf{T}'}(\chi') = \sup\{\operatorname{dep}_{\mathbf{T}_i'}(\chi_i') \mid 1 \leq i \leq n\}. \tag{6.12}$$

Here $\mathbf{T}_i'$ denotes $\mathrm{Res}_{F_i'/F}\mathbb{G}_{\mathrm{m}}$ and $\chi_i = \chi|_{\mathbf{T}_i'(F)}$. By functoriality, $\lambda_{\mathbf{T}'}(\chi')$ is the image of $\lambda_{\mathbf{T}}(\chi)$ under $\mathrm{H}^1(W_F, T^\vee) \to \mathrm{H}^1(W_F, (T')^\vee)$ and therefore $\operatorname{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi)) = \operatorname{dep}_{W_F}(\lambda_{\mathbf{T}'}(\chi'))$. But

$$\begin{aligned}
\operatorname{dep}_{W_F}(\lambda_{\mathbf{T}'}(\chi')) &= \sup\{\operatorname{dep}_{W_F}(\lambda_{\mathbf{T}_i'}(\chi_i')) \mid 1 \leq i \leq n\} \\
&= \sup\{\varphi_{F_i'/F_i}(e_i \cdot \operatorname{dep}_{\mathbf{T}_i'}(\chi_i')) \mid 1 \leq i \leq n\}. \\
&\geq \sup\{\operatorname{dep}_{\mathbf{T}_i'}(\chi_i') \mid 1 \leq i \leq n\}.
\end{aligned}$$

Here e_i denotes the ramification index of F_i'/F. Thus

$$\mathrm{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi)) \geq \mathrm{dep}_{\mathbf{T}}(\chi). \tag{6.13}$$

Now assume $\mathbf{T}$ is wildly ramified. We will now produce a character of $\mathbf{T}(F)$ for which the inequality (6.13) is strict. We can assume without loss of generality that $\mathbf{T}_0 := \mathrm{Res}_{F_1'/F}\mathbb{G}_{\mathrm{m}}$ is wildly ramified. Let χ_0' be a positive depth character of $\mathbf{T}_0(F)$ which is trivial on $C_0 \cap \mathbf{T}_0(F)$. Extend χ_0' trivially to a character χ' of $\mathbf{T}'(F)$. Then since $\mathbb{C}^\times$ is divisible, the character χ' lifts to a character χ of $\mathbf{T}(F)$. By Remark 33, $\mathrm{dep}_{\mathbf{T}_0}(\chi_0') < \mathrm{dep}_{W_F}(\lambda_{\mathbf{T}_0}(\chi_0'))$. Since $\mathrm{dep}_T(\chi) = \mathrm{dep}_{\mathbf{T}_0}(\chi_0')$ and $\mathrm{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi)) = \mathrm{dep}_{W_F}(\lambda_{\mathbf{T}_0}(\chi_0'))$, it follows that the inequality (6.13) is strict for this choice of χ.

6.5.2 Example

Let $F = \mathbb{Q}_p$, $L = F(\zeta_{p^n})$, where ζ_{p^n} denotes a primitive p^nth root of unity, $n \geq 1$. Then L/F is a totally ramified extension of degree $(p-1)p^{n-1}$. Consider the intermediate extension $K = F(\zeta_p)$ of F of degree $p-1$ over F. Then, L/K is a wildly ramified extension. Write $G = \mathrm{Gal}(L/F)$ and $H = \mathrm{Gal}(L/K)$.

Lemma 34. *For $r \geq 1$, we have $\varphi_{L/K}(r) = (p-1)\varphi_{L/F}(r)$.*

Proof. We first note that since we considering abelian extensions, the jumps in filtration occur at integer values. We have for $r \geq 1$,

$$\begin{aligned}
\varphi_{L/F}(r) &= \int_0^r \frac{dt}{(G_0 : G_t)} \\
&= \int_0^1 \frac{dt}{(G_0 : G_t)} + \int_1^r \frac{dt}{(G_0 : G_t)} \\
&= \frac{1}{p-1} + \int_1^r \frac{dt}{(G_0 : G_t)} \\
&= \frac{1}{p-1} + \int_1^r \frac{(H_0 : H_t)}{(G_0 : G_t)} \frac{dt}{(H_0 : H_t)} \\
&= \frac{1}{p-1} + \int_1^r \frac{(G_t : H_t)}{(G_0 : H_0)} \frac{dt}{(H_0 : H_t)} \\
&= \frac{1}{p-1} + \frac{1}{p-1} \int_1^r \frac{dt}{(H_0 : H_t)}.
\end{aligned}$$

The last equality holds because $G_t = H_t$ for $t \geq 1$ and $(G_0 : H_0) = p - 1$. Thus

$$\begin{aligned}\varphi_{L/F}(r) &= \frac{1}{p-1} + \frac{1}{p-1}(\varphi_{L/F}(r) - \int_0^1 \frac{1}{(H_0 : H_t)}dt) \\ &= \frac{1}{p-1} + \frac{1}{p-1}(\varphi_{L/K}(r) - 1) \\ &= \frac{\varphi_{L/K}(r)}{(p-1)}.\end{aligned}$$

□

Let us recall the ramification subgroups $\{G_u\}_{u\geq 0}$ of G from [Ser, Chap IV]. Write $m = p^n$ and let $G(m) = (\mathbb{Z}/m\mathbb{Z})^\times$. By [Ser, Chap IV, Prop. 17], $G = G(m)$. Define

$$G(m)^s := \{a \in G(m) \mid a \equiv 1 \bmod p^s\}.$$

Then $G(m)^s = \mathrm{Gal}(L/F(\zeta_{p^s}))$. The ramification groups G_u of G are [Ser, Chap IV, Prop. 18]:

$$\begin{array}{rl} & G_0 = G \\ \text{if } 1 \leq u \leq p-1 & G_u = G(m)^1 \\ \text{if } p \leq u \leq p^2 - 1 & G_u = G(m)^2 \\ \vdots & \vdots \\ \text{if } p^{n-1} \leq u & G_u = 1. \end{array}$$

We now calculate the Hasse-Herbrand function $\varphi_{L/K}$.

Proposition 35. *The Hasse-Herbrand function of the wildly ramified extension L/K is given by*

$$\varphi_{L/K}(r) = \begin{cases} k(p-1) + \frac{r-p^k+1}{p^k} & \text{if } p^k - 1 < r \leq p^{k+1} - 1 \text{ with } 0 \leq k < n-1 \\ (n-1)(p-1) + \frac{r-p^{n-1}+1}{p^{n-1}} & r > p^{n-1} - 1 \end{cases} \tag{6.14}$$

Proof. We consider various cases:

- **Case** $0 < r \leq 1$

$$\begin{aligned}
\varphi_{L/K}(r) &= \int_0^r \frac{dt}{(H_0 : H_t)} \\
&= \frac{1}{(H_0 : H_1)} \int_0^r dr \\
&= r.
\end{aligned}$$

- **Case** $1 < r \leq p-1$

$$\begin{aligned}
\varphi_{L/F}(r) &= \int_0^r \frac{dt}{(G_0 : G_t)} \\
&= \int_0^1 \frac{dt}{(G_0 : G_1)} + \int_1^r \frac{dt}{(G_0 : G_t)} \\
&= \frac{1}{p-1} + \int_1^r \frac{dt}{(G_0 : G(m)^1)} \\
&= \frac{r}{p-1}.
\end{aligned}$$

Therefore, $\varphi_{L/K}(r) = r$.

- **Case** $p^k - 1 < r \leq p^{k+1} - 1$ with $1 \leq k < n-1$

$$\begin{aligned}
\varphi_{L/F}(r) &= \int_0^r \frac{dt}{(G_0 : G_t)} \\
&= \sum_{i=0}^{k-1} \int_{(p^i-1)}^{(p^{i+1}-1)} \frac{dt}{(G_0 : G_t)} + \int_{p^k-1}^r \frac{dt}{(G_0 : G_t)} \\
&= \int_0^1 \frac{dt}{(G_0 : G_1)} + \int_1^{p-1} \frac{dt}{(G_0 : G(m)^1)} + \sum_{i=1}^{k-1} \int_{p^i-1}^{p^{i+1}-1} \frac{dt}{(G_0 : G(m)^{i+1}} \\
&\quad + \int_{p^k-1}^r \frac{dt}{(G_0 : G(m)^{k+1}} \\
&= \frac{1}{p-1} + \frac{p-2}{p-1} + \sum_{i=1}^{k-1} \frac{p^{i+1} - p^i}{(p-1)p^i} + \frac{r - p^k + 1}{(p-1)p^k} \\
&= k + \frac{r - p^k + 1}{(p-1)p^k}.
\end{aligned}$$

Therefore, $\varphi_{L/K}(r) = k(p-1) + \frac{r-p^k+1}{p^k}$.

- **Case** $r > p^{n-1} - 1$

$$\begin{aligned}\varphi_{L/F}(r) &= \int_0^r \frac{dt}{(G_0 : G_t)} \\ &= \int_0^{p^{n-1}-1} \frac{dt}{(G_0 : G_1)} + \int_{p^{n-1}-1}^r \frac{dt}{(G_0 : G_t)} \\ &= (n-1) + \frac{r - p^{n-1} + 1}{(p-1)p^{n-1}}.\end{aligned}$$

Therefore, $\varphi_{L/K}(r) = (n-1)(p-1) + \frac{r-p^{n-1}+1}{p^{n-1}}$.

□

Now write $\mathbf{T} = \mathrm{Res}_{L/K}\mathbb{G}_{\mathrm{m}}$ and let $\lambda_{\mathbf{T}}$ be as denoted in Sec. 6.3. Then the following result immediately follows from Proposition 35:

Lemma 36. *For $r > 0$, we have*

$$\varphi_{L/F}(p^{n-1}r) > r.$$

Consequently, for all positive depth character $\chi \in \mathrm{Hom}(\mathbf{T}(F), \mathbb{C}^\times)$, we have $\mathrm{dep}_T(\chi) < \mathrm{dep}_{W_F}(\lambda_{\mathbf{T}}(\chi))$.

6.6 Further results

In this section, we will give a brief description of a recent work [AP] of Aubert and Plymen, who have generalized our Theorem 32 by giving a depth depth changing formula under the enhanced local Langlands correspondence for the groups coming from Weil-restriction.

Let $F \subset F'$ be non-archimedean local fields such that the field extension F'/F is a finite Galois extension. If $\mathbf{G}$ is a connected reductive algebraic group defined over the field F', the Weil-restriction of scalars from $\mathbf{G}$, which we will denote by $\mathbf{H} := \mathrm{Res}_{F'/F}\mathbf{G}$, is again a reductive algebraic group defined over F and we have an isomorphism $\iota : \mathbf{G}(F') \to \mathbf{H}(F)$ between the F'-rational points of $\mathbf{G}$ and the F-rational points of $\mathbf{H}$. Consider $\mathbf{G}(F')$ admits LLC $\lambda_{\mathbf{G}} : \mathrm{Irr}(\mathbf{G}(F')) \to \Phi(\mathbf{G}(F'))$. Then one can define $\lambda_{\mathbf{H}}$ as the unique map $\lambda_{\mathbf{H}} : \mathrm{Irr}(\mathbf{H}(F)) \to \Phi(\mathbf{H}(F))$ such that the following diagram :

$$\begin{tikzcd}
\mathrm{Irr}(\mathbf{H}(F)) \arrow[r, "\lambda_{\mathbf{H}}"] \arrow[d, "\iota^*"'] & \Phi(\mathbf{H}(F)) \arrow[d, "\widetilde{\mathrm{Sh}}"] \\
\mathrm{Irr}(\mathbf{G}(F')) \arrow[r, "\lambda_{\mathbf{G}}"] & \Phi(\mathbf{G}(F'))
\end{tikzcd}$$

commutes. Here $\widetilde{\mathrm{Sh}} : \Phi(\mathbf{H}(F)) \xrightarrow{\sim} \Phi(\mathbf{G}(F'))$ is the canonical bijection as defined in [Bor2, Proposition 8.4] using Shapiro's isomorphism Sh and ι^* is the natural map coming from ι. Anne-Marie Aubert and Roger Plymen investigate how the depth can change in the transition from $\mathbf{G}(F')$ to $\mathbf{H}(F)$. They have shown in [AP, Theorem 1.2] that if $\lambda_{\mathbf{G}}$ preserves depth i.e., $\mathrm{dep}_{\mathbf{G}}(\pi') = \mathrm{dep}_{W_{F'}}(\lambda_{\mathbf{G}}(\pi'))$ for each $\pi' \in \mathrm{Irr}(\mathbf{G}(F'))$, then the depth changing formula under $\lambda_{\mathbf{H}}$ is

$$\varphi_{F'/F}(e \cdot \mathrm{dep}_{\mathbf{H}}(\pi)) = \mathrm{dep}_{W_F}(\lambda_{\mathbf{H}}(\pi)),$$

for each $\pi \in \mathrm{Irr}(\mathbf{H}(F))$.
In particular, they get the followings:

(i) $\lambda_{\mathbf{H}}$ preserves depth if and only if F'/F is a tamely ramified extension. This reproduces the depth-preservation Theorem [Yu2, Theorem 7.10] of Yu for tamely ramified induced tori.

(ii) For each $\pi \in \mathrm{Irr}(\mathbf{H}(F))$ with $\mathrm{dep}_{\mathbf{H}}(\pi) > 0$, we have $\mathrm{dep}_{\mathbf{H}}(\pi) < \mathrm{dep}_{W_F}(\lambda_{\mathbf{H}}(\pi))$ if F'/F is a wildly ramified extension. This fact implies our main Theorem 32, if one puts $\mathbf{G} = \mathrm{GL}_1$.

Part III

Genericity of representations of a p-adic group:

7

Gelfand-Graev spaces

7.1 Non-degenerate characters:

Let $\mathbf{G}$ be a connected reductive algebraic group defined over a non-archimedean local field F with residue field k_F. If $\mathbf{H}$ is a (Zariski-) closed subgroup of $\mathbf{G}$ defined over F, we will denote the group of F-points of $\mathbf{H}$ by $\mathbf{H}(F)$. Fix a maximal F-split torus $\mathbf{S}$ in $\mathbf{G}$ and let $\mathbf{T} = Z_{\mathbf{G}}(\mathbf{S})$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$. Then $\mathbf{T}$ is the Levi factor of a minimal F-parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ defined over F. We denote the unipotent radical of $\mathbf{B}$ by $\mathbf{U}$ and $\mathbf{B}$-opposite F-parabolic subgroup of $\mathbf{G}$ by $\mathbf{B}^-$ with its unipotent radical $\mathbf{U}^-$ i.e., $\mathbf{B} = \mathbf{TU}$ and $\mathbf{B}^- = \mathbf{TU}^-$.

Consider a smooth character $\psi : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$. Since $\mathbf{U}(F)$ is normalized by $\mathbf{S}(F)$, for each $s \in \mathbf{S}(F)$, one can define a smooth character $\psi^s : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$ by $x \mapsto \psi(sxs^{-1})$. Therefore, there is an action of $\mathbf{S}(F)$ on the space $\widehat{\mathbf{U}(F)}$ of all smooth characters of $\mathbf{U}(F)$ defined by

$$s \cdot \psi = \psi^s \text{ for } s \in \mathbf{S}(F) \text{ and } \psi \in \widehat{\mathbf{U}(F)}.$$

Definition 37. *A smooth character $\psi : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$ is called non-degenerate (a.k.a. generic, or principal) if the stabilizer $\{s \in \mathbf{S}(F) \mid \psi^s = \psi\}$ of ψ in $\mathbf{S}(F)$ lies in the center $Z_{\mathbf{G}(F)}$ of $\mathbf{G}(F)$.*

Let $\Phi = \Phi(\mathbf{G}, \mathbf{S}, F)$ be the relative root system of $\mathbf{G}$ with respect to the maximal F-split torus $\mathbf{S}$ in $\mathbf{G}$. The minimal F-parabolic subgroup $\mathbf{B}$ determines a subset $\Phi^+ \subset \Phi$ of positive roots in Φ and a subset $\Delta \subset \Phi^+$ of simple roots in Φ. For each root $\alpha \in \Phi$, we denote the corresponding root subgroup by $\mathbf{U}_\alpha$ defined over F. Then an alternative definition of non-degenerate character can be given from the result below:

Proposition 38. *[BH, Prop. 1.2] A smooth character $\psi : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$ is non-degenerate if and only if the restriction of ψ to each root subgroup $\mathbf{U}_\alpha(F)$ is non-trivial for every $\alpha \in \Delta$.*

We fix a non-degenerate character $\psi : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$. The Gelfand-Graev representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$ of $\mathbf{G}(F)$ (associated to the generic character ψ) is provided by the space of right $\mathbf{G}(F)$-smooth compactly supported modulo $\mathbf{U}(F)$ functions $f : \mathbf{G}(F) \to \mathbb{C}$ satisfying:

$$f(ug) = \psi(u)f(g), \forall u \in \mathbf{U}(F), g \in \mathbf{G}(F).$$

In short, we will write GGR for Gelfand-Graev representation. Through Bernstein decomposition (Theorem 22), the Gelfand-Graev representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$ decomposed into the direct sum of certain representations $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)_{\mathfrak{s}} \in \mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F))$ for $\mathfrak{s} \in \mathfrak{B}(\mathbf{G}(F))$. In [BH, Theorem 4.2], Bushnell and Henniart showed that the representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)_{\mathfrak{s}}$ is finitely generated over $\mathbf{G}(F)$ for each $\mathfrak{s} \in \mathfrak{B}(\mathbf{G}(F))$. In this chapter, we mainly focus on Bernstein component $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)_{\mathfrak{s}}$ corresponding to the inertial equivalence class $\mathfrak{s} = [\mathbf{T}(F), \chi]_{\mathbf{G}(F)}$, where $\mathbf{T}$ is a minimal F-Levi subgroup of $\mathbf{G}$ and χ is a smooth character of $\mathbf{T}(F)$.

7.2 Jacquet modules of GGR

Fix a minimal F-parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with its Levi factor $\mathbf{T}$, and unipotent radical $\mathbf{U}$. Let $\mathbf{M}$ be a $(\mathbf{B}, \mathbf{T})$-standard F-Levi subgroup of an F-parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$ with unipotent radical $\mathbf{N}$, i.e., $\mathbf{M}$ contains $\mathbf{T}$ and $\mathbf{P}$ contains $\mathbf{B}$. Then $\mathbf{B} \cap \mathbf{M}$ is a minimal F-parabolic subgroup of $\mathbf{M}$ with Levi factor $\mathbf{T}$ and unipotent radical $\mathbf{U}_{\mathbf{M}} := \mathbf{U} \cap \mathbf{M}$. Fix a non-degenerate character $\psi : \mathbf{U}(F) \to \mathbb{C}^\times$ of $\mathbf{U}(F)$. Let $\psi_{\mathbf{M}(F)} := \psi|_{\mathbf{U}_{\mathbf{M}}(F)}$ denote the restriction of ψ to $\mathbf{U}_{\mathbf{M}}(F)$. Then $\psi_{\mathbf{M}(F)} : \mathbf{U}_{\mathbf{M}}(F) \to \mathbb{C}^\times$ is also a non-degenerate character of $\mathbf{U}_{\mathbf{M}}(F)$ (see [BH, Proposition 2.2]).

Now, consider the Gelfand-Graev representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$ of $\mathbf{G}(F)$ and the Gelfand-Graev representation $c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)})$ of $\mathbf{M}(F)$ associated to the non-degenerate characters ψ and $\psi_{\mathbf{M}(F)}$ respectively. As before, denote by $\mathbf{P}^- = \mathbf{M}\mathbf{N}^-$, the $\mathbf{P}$-opposite F-parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{N}^-$. We write $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))_{\mathbf{N}^-(F)}$ for the Jacquet module of the smooth representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$ relative to the unipotent radical $\mathbf{N}^-(F)$. We will consider the $\mathbf{M}(F)$-representation $c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)})$ as a $\mathbf{P}^-(F)$-representation, on which the unipotent radical $\mathbf{N}^-(F)$ acts trivially. Then we have the following isomorphism of $\mathbf{M}(F)$ representations:

Theorem 39. *[BH, Theorem 2.2] There is a unique $\mathbf{P}^-(F)$-homomorphism*

$$r_{\mathbf{P}^-} : c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi) \to c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)}),$$

which induces an isomorphism of $\mathbf{M}(F)$ representations:

$$c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)}) \cong (c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))_{\mathbf{N}^-(F)}. \tag{7.1}$$

7.3 Isotypic component of GGR

In this section, we assume that $\mathbf{G}$ is a connected reductive algebraic group defined over a local field F such that $\mathbf{G}$ is split over a tamely ramified field extension of F and p, the residual characteristic of F does not divide the order of the Weyl group of $\mathbf{G}(\bar{F})$.

Now let χ be a smooth character of $\mathbf{T}(F)$. Consider a pair $(K_{\mathbf{T}(F)}, \rho_{\mathbf{T}(F)})$ consisting of a compact open subgroup $K_{\mathbf{T}(F)}$ of $\mathbf{T}(F)$ and an irreducible smooth representation $\rho_{\mathbf{T}(F)}$ of $K_{\mathbf{T}(F)}$ such that $(K_{\mathbf{T}(F)}, \rho_{\mathbf{T}(F)})$ be a $[\mathbf{T}(F), \chi]_{\mathbf{T}(F)}$-type in $\mathbf{T}(F)$. Let the (K, ρ) consists of a a compact open subgroup K of $\mathbf{G}(F)$ and an irreducible smooth representation ρ of K such that (K, ρ) is a $\mathbf{G}(F)$-cover of $(K_{\mathbf{T}(F)}, \rho_{\mathbf{T}(F)})$. Then, (K, ρ) is a $[\mathbf{T}(F), \chi]_{\mathbf{G}(F)}$-type in $\mathbf{G}(F)$. Under the assumption that the residue characteristic p does not divide the order of the Weyl group of $\mathbf{G}$, J. Fintzen [Fin] shows that the pair (K, ρ) exists . In that case, $\mathfrak{R}^{\mathfrak{s}}(\mathbf{G}(F)) = \mathfrak{R}_{\rho}(\mathbf{G}(F))$ as subcategories of $\mathfrak{R}(\mathbf{G}(F))$ and the Bernstein component $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)_{\mathfrak{s}}$ associated to the inertial equivalence class $\mathfrak{s} = [\mathbf{T}(F), \chi]_{\mathbf{G}(F)}$ is generated by the ρ-isotypic components $\left(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)\right)^{\rho}$ of the smooth representation $c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)$

of $\mathbf{G}(F)$. Now, we will show that the $\mathcal{H}(\mathbf{G}(F),\rho)$-module $\left(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)\right)^{\rho}$ is cyclic. For that, let $\mathbf{B}^- = \mathbf{T}\mathbf{U}^-$ denote the $\mathbf{B}$-opposite Borel subgroup of $\mathbf{G}$. View $\mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)})$ as a subalgebra of $\mathcal{H}(\mathbf{G}(F),\rho)$ via the embedding:

$$t_{\mathbf{B}'} : \mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)}) \to \mathcal{H}(\mathbf{G}(F),\rho), \tag{7.2}$$

of Equation (4.2).

Theorem 40. *[MP2, Theorem 1] There is an isomorphism*

$$(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho} \cong \mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)})$$

of $\mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)})$-modules. Consequently, $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho}$ is a cyclic $\mathcal{H}(\mathbf{G}(F),\rho)$-module.

Proof. Putting $\mathbf{M} = \mathbf{T}$ in Equation (7.1) and observing that in this case, $\mathbf{U_M} = 1$, we get an isomorphism of $\mathbf{T}(F)$-representations

$$\begin{aligned}(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))_{U^-} &\cong c\text{-ind}_{1}^{\mathbf{T}(F)}(\mathbb{C})\\ &\cong C_c^{\infty}(\mathbf{T}(F)).\end{aligned}$$

Consequently, this isomorphism of $\mathbf{T}(F)$-representations induces an isomorphism between their $\rho_{\mathbf{T}(F)}$-isotypic components

$$\begin{aligned}(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))_{U^-}^{\rho_{\mathbf{T}(F)}} &\cong C_c^{\infty}(\mathbf{T}(F))^{\rho_{\mathbf{T}(F)}}\\ &\cong \mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)})\end{aligned}$$

as $\mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)})$-modules. Now by Equation (4.3),

$$\begin{aligned}(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho} &\cong (c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))_{U^-}^{\rho_{\mathbf{T}(F)}}\\ &\cong \mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)})\end{aligned}$$

as $\mathcal{H}(\mathbf{T}(F),\rho_{\mathbf{T}(F)})$-modules. The result follows. □

7.4 Principal series Hecke algebra

In this subsection, we summarize some results of Roche in [Roc]. Let the notations be as in Section §7.1. We assume further that $\mathbf{S} = \mathbf{T}$, so that $\mathbf{B} = \mathbf{TU}$ is now an F-Borel subgroup of $\mathbf{G}$ containing the maximal F-split torus $\mathbf{T}$. The pair $(\mathbf{B}, \mathbf{T})$ determines a based root datum $\Psi = (X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$. Here X (resp. $X^\vee$) is the character (resp. co-character) lattice of $\mathbf{T}$ and Δ (resp. $\Delta^\vee$) is a basis (resp. dual basis) for the set of roots $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ (resp.$\Phi^\vee$) of $\mathbf{T}$ in $\mathbf{G}$. For the results of this section, we assume that F has characteristic 0 and the residue characteristic p of F satisfies the following hypothesis.

Hypothesis 41. *If Φ is irreducible, p is restricted as follows:*

1. *for type $A_n, p > n+1$*
2. *for type $B_n, C_n, D_n, p \neq 2$*
3. *for type $F_4, p \neq 2, 3$*
4. *for types $G_2, E_6, p \neq 2, 3, 5$*
5. *for types $E_7, E_8, p \neq 2, 3, 5, 7$*

If Φ is not irreducible, then p excludes primes attached to each of its irreducible factors.

We let $\mathbf{T}(F)_0 = \mathbf{T}(\mathcal{O}_F)$ denote the maximal compact subgroup of $\mathbf{T}(F)$, $N_{\mathbf{G}}(\mathbf{T})$ to be the normalizer of $\mathbf{T}$ in $\mathbf{G}$ and $W = W(\mathbf{G}, \mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} = N_{\mathbf{G}}(\mathbf{T})(F)/\mathbf{T}(F)$ denote the Weyl group of $\mathbf{G}$.

Let $\chi^\#$ be a smooth character of $\mathbf{T}(F)$ and put $\chi = \chi^\#|_{\mathbf{T}(F)_0}$ be the restriction of $\chi^\#$ to $\mathbf{T}(F)_0$, where $\mathbf{T}(F)_0$ denotes the maximal compact subgroup of $\mathbf{T}(F)$. Then $(\mathbf{T}(F)_0, \chi)$ is a $[\mathbf{T}(F), \chi^\#]_{\mathbf{T}(F)}$-type in $\mathbf{T}(F)$.

Let $N_{\mathbf{G}}(\mathbf{T})(F)_\chi$ (resp. $N_{\mathbf{G}}(\mathbf{T})(\mathcal{O}_F)_\chi$, resp. W_χ) denote the subgroup of $N_{\mathbf{G}}(\mathbf{T})(F)$ (resp. $N_{\mathbf{G}}(\mathbf{T})(\mathcal{O}_F)$, resp. W) which fixes χ. The group $N_{\mathbf{G}}(\mathbf{T})(F)_\chi$ contains $\mathbf{T}(F)$ and we have $W_\chi = N_{\mathbf{G}}(\mathbf{T})(F)_\chi/\mathbf{T}(F)$. Denote by $\widetilde{W} = \widetilde{W}(\mathbf{G}, \mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})(F)/\mathbf{T}(F)_0$, the Iwahori-Weyl group of $\mathbf{G}$. There is an identification $N_{\mathbf{G}}(\mathbf{T})(F) = X^\vee \rtimes N_{\mathbf{G}}(\mathbf{T})(\mathcal{O}_F)$ given by the choice of a uniformizer of F. Since $N_{\mathbf{G}}(\mathbf{T})(\mathcal{O}_F)/\mathbf{T}(F)_0 = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, this identification gives us an identification $\widetilde{W} = X^\vee \rtimes W$. Let $\widetilde{W}_\chi = X^\vee \rtimes W_\chi$ be the subgroup of $\widetilde{W}$ which fixes χ.

Let

$$\Phi' := \{\alpha \in \Phi \mid \chi \circ \alpha^\vee|_{\mathcal{O}_F^\times} = 1\}$$

Then Φ' is a closed subroot system of Φ. Let s_α denotes the reflection on the space $\mathcal{A} = X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ associated to a root $\alpha \in \Phi$ and write $W' = \langle s_\alpha \mid \alpha \in \Phi' \rangle$ to be the associated Weyl group. Let Φ^+ (resp. Φ^-) be the system of positive (resp. negative) roots determined by the choice of the Borel $\mathbf{B}$ and let $\Phi'^+ = \Phi^+ \cap \Phi'$. Then Φ'^+ is a positive system in Φ'. Put

$$C_\chi = \{w \in W_\chi \mid w(\Phi'^+) = \Phi'^+\}.$$

Then we have,

$$W_\chi = W' \rtimes C_\chi$$

The character χ extends to a W_χ-invariant character $\widetilde{\chi}$ of $N_{\mathbf{G}}(\mathbf{T})(\mathcal{O}_F)_\chi$. Denote by $\widetilde{\chi}$, the character of $N_{\mathbf{G}}(\mathbf{T})(F)_\chi$ extending $\widetilde{\chi}$ trivially on $X^\vee$.

Roche's construction produces a $[\mathbf{T}(F), \chi^\#]_{\mathbf{G}(F)}$-type (K, ρ). The pair (K, ρ) depends on the choice of $\mathbf{B}, \mathbf{T}, \chi$ but not on the extension $\chi^\#$ of χ. Denote by $\mathcal{I}_{\mathbf{G}(F)}(\rho)$, the set of elements in $\mathbf{G}(F)$ which intertwine ρ. Equivalently, $g \in \mathcal{I}_{\mathbf{G}(F)}(\rho)$ iff the double coset KgK supports a non-zero function in $\mathcal{H}(\mathbf{G}(F), \rho)$. We have an equality

$$\mathcal{I}_{\mathbf{G}(F)}(\rho) = K\widetilde{W}_\chi K. \tag{7.3}$$

For an element $w \in \widetilde{W}_\chi$, choose any representative n_w of w in $N_{\mathbf{G}}(\mathbf{T})(F)_\chi$ and let $\mathcal{T}_{\widetilde{\chi},w}$ be the unique element of the Hecke algebra $\mathcal{H}(\mathbf{G}(F), \rho)$ supported on Kn_wK and taking value $q^{-\mathfrak{l}(w)/2}\widetilde{\chi}(n_w)^{-1}$ at n_w . Here $\mathfrak{l}$ is the length function on the affine Weyl group $\widetilde{W}$. The functions $\mathcal{T}_{\widetilde{\chi},w}$ for $w \in \widetilde{W}_\chi$ form a basis for the $\mathbb{C}$-vector space $\mathcal{H}(\mathbf{G}(F), \rho)$.

Definition 42 ($\mathcal{H}_{W_\chi}$)**.** *Define $\mathcal{H}_{W_\chi}$ to be the subalgebra of $\mathcal{H}(\mathbf{G}(F), \rho)$ generated by $\{\mathcal{T}_{\widetilde{\chi},w} \mid w \in W'\}$.*

Also, identify $\mathcal{H}(\mathbf{T}(F), \chi)$ as a subalgebra of $\mathcal{H}(\mathbf{G}(F), \rho)$ using the embedding $t_{\mathbf{B}}$. When $\chi \neq 1$, we further assume that $\mathbf{G}$ has connected center. Then, assuming Hypothesis 41, we have $C_\chi = 1$ and so $W_\chi = W'$. In that case, $\mathcal{H}_{W_\chi}$ and $\mathcal{H}(\mathbf{T}(F), \chi)$ together generate the full Hecke algebra $\mathcal{H}(\mathbf{G}(F), \rho)$. In particular, there exists a connected reductive F-split group $\mathbf{H}$ and a certain Iwahori subgroup $\mathcal{I}_H$ of the F-points group $\mathbf{H}(F)$ of $\mathbf{H}$ such that the corresponding Iwahori-Hecke algebra $\mathcal{H}(\mathbf{H}(F), 1_{\mathcal{I}_H})$ of $\mathbf{H}(F)$ and the algebra $\mathcal{H}(\mathbf{G}(F), \rho)$ are isomorphic via a family of support preserving $\mathbb{C}$-algebra isomorphisms (see [Roc, §8] for more detail). In [Ros], Sean Rostami showed that the Iwahori-Hecke algebra $\mathcal{H}(\mathbf{H}(F), 1_{\mathcal{I}_H})$ has Bernstein type presentation, the kind of presentation of the Hecke algebra Lusztig gave for affine Hecke algebra in [Lus]. Therefore, in our situation, Roche's Hecke algebra $\mathcal{H}(\mathbf{G}(F), \rho)$ also has a Bernstein type presentation and we can expressed it as

$$\mathcal{H}(\mathbf{G}(F), \rho) \cong \mathcal{H}_{W_\chi} \otimes_{\mathbb{C}} \mathcal{H}(\mathbf{T}(F), \chi),$$

where the tensor product relation comes from the Bernstein relation as described in [Ros, §5].

7.5 Principal series component of GGR

We continue to assume that $\mathbf{G}$ is F-split. Extend the triple $(\mathbf{G}, \mathbf{B}, \mathbf{T})$ to a Chevalley-Steinberg pinning of $\mathbf{G}$. This determines a hyperspecial point x in the Bruhat-Tits building $\mathcal{B}(\mathbf{G}(F))$, which gives $\mathbf{G}(F)$ the structure of a Chevalley group. With this identification, $(\mathbf{G}, \mathbf{B}, \mathbf{T})$ are defined over $\mathcal{O}_F$ and the hyperspecial subgroup $\mathbf{G}(F)_{x,0}$ at x is $\mathbf{G}(\mathcal{O}_F)$. Let $\mathbf{G}(F)_{x,0+}$ denote the pro-unipotent radical of $\mathbf{G}(F)_{x,0}$. Then $\mathbf{G}(F)_{x,0}/\mathbf{G}(F)_{x,0+} \cong \mathbf{G}(\mathbb{F}_q)$. We say that ψ is of generic depth-zero at x if $\psi|_{\mathbf{U}(F)\cap\mathbf{G}(F)_{x,0}}$ factors through a generic character ψ_q of $\mathbf{U}(\mathbb{F}_q) \cong \mathbf{U}(F) \cap \mathbf{G}(F)_{x,0}/\mathbf{U}(F) \cap \mathbf{G}(F)_{x,0+}$ (see [DR, §1], for the more general definition). Note that if $\mathbf{G}$ has connected center, then all generic characters of $\mathbf{U}(F)$ form a single orbit under the action of $\mathbf{T}(F)$.

Let sgn denote the one dimensional representation of $\mathcal{H}_{W_\chi}$ in which $\mathcal{T}_{\widetilde{\chi},w}$ acts by the scalar $(-1)^{\mathfrak{l}'(w)}$. Here $\mathfrak{l}'$ denotes the length function on W'.

Theorem 43. *[MP2, Theorem 3] If $\chi = 1$, then assume that the $\mathbf{T}(F)$-orbit of ψ contains a character of generic depth zero at x. If $\chi \neq 1$, then assume that the center of $\mathbf{G}$ is connected. If χ has positive depth, then assume further that F has characteristic 0 and the residue characteristic satisfies Hypothesis 41. Then $\mathcal{H}(\mathbf{G}(F), \rho)$-*

module $\left(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi)\right)^{\rho}$ *is isomorphic to*

$$\mathcal{H}(\mathbf{G}(F),\rho)\otimes_{\mathcal{H}_{W_\chi}}\mathrm{sgn}.$$

Proof. To give a proof of this theorem, we take the following steps: when χ has positive depth, then we will reduce the result to a depth zero character situation. Later, we will give a proof of the result, when χ is a depth-zero character.

Reduction to depth zero: It follows from the the proof of [Roc, Theorem 4.15], (see also loc. cit., page 385, 2nd last paragraph), that there exists a standard F-Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ which is the Levi factor of a standard parabolic $\mathbf{P}=\mathbf{M}\mathbf{N}$ of $\mathbf{G}$ and which is minimal with the property that

$$I_{\mathbf{G}(F)}(\rho)\subset K\mathbf{M}(F)K. \tag{7.4}$$

Put $(K_{\mathbf{M}(F)},\rho_{\mathbf{M}(F)})=(K\cap\mathbf{M}(F),\rho|_{K_{\mathbf{M}(F)}})$. From [BK, Theorem 7.2(ii)], it follows that (K,ρ) satisfies the requirements [BK, §8.1], of being $\mathbf{G}(F)$-cover of $(K_{\mathbf{M}(F)},\rho_{\mathbf{M}(F)})$. It also follows from [BK, Theorem 7.2(ii)], that there is a support preserving Hecke algebra isomorphism

$$\Psi^{\mathbf{M}}:\mathcal{H}(\mathbf{M}(F),\rho_{\mathbf{M}(F)})\xrightarrow{\simeq}\mathcal{H}(\mathbf{G}(F),\rho) \tag{7.5}$$

By Equation (7.1), we have an isomorphism of $\mathcal{H}(\mathbf{M}(F),\rho_{\mathbf{M}(F)})$-modules

$$(c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)}))^{\rho_{\mathbf{M}(F)}}\cong((c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))_{\mathbf{N}^{-}(F)})^{\rho_{\mathbf{M}(F)}} \tag{7.6}$$

And by Equation (4.3), we have a $\Psi^{\mathbf{M}}$-equivariant isomorphism

$$((c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))_{\mathbf{N}^{-}(F)})^{\rho_{\mathbf{M}(F)}}\cong(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho} \tag{7.7}$$

Combining Equations (7.6) and (7.7), we get a $\Psi^{\mathbf{M}}$-equivariant isomorphism

$$(c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)}))^{\rho_{\mathbf{M}(F)}}\cong(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho}.$$

Also it is shown in the proof of [Roc, Theorem 4.15], that for such an $\mathbf{M}$, there is a character χ_1 of $\mathbf{M}(F)$ such that $\chi\chi_1$ viewed as a character of $\mathbf{T}(F)_0$ is depth-zero. We then have an isomorphism

$$\Psi_{\chi_1} : \mathcal{H}(\mathbf{M}(F), \rho_{\mathbf{M}(F)}) \xrightarrow{\simeq} \mathcal{H}(\mathbf{M}(F), \rho_{\mathbf{M}(F)}\chi_1) \tag{7.8}$$

given by $f \mapsto f\chi_1$. This gives a Ψ_{χ_1}-equivariant isomorphism

$$(c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)}))^{\rho_{\mathbf{M}(F)}} \cong (c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)}))^{\rho_{\mathbf{M}(F)}\chi_1}.$$

We thus have a $\Psi^{\mathbf{M}} \circ \Psi_{\chi_1}^{-1}$-equivariant isomorphism

$$(c\text{-ind}_{\mathbf{U}_{\mathbf{M}}(F)}^{\mathbf{M}(F)}(\psi_{\mathbf{M}(F)}))^{\rho_{\mathbf{M}(F)}\chi_1} \cong (c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\rho}. \tag{7.9}$$

By Equations (7.2) and (7.4), it follows that $\Psi^{\mathbf{M}}$ restricts to an algebra isomorphism

$$\mathcal{H}_{W(\mathbf{G},\mathbf{T})_\chi} \xrightarrow{\simeq} \mathcal{H}_{W(\mathbf{M},\mathbf{T})_\chi} \tag{7.10}$$

From the proof of [Roc, Theorem 4.15], $W(\mathbf{M},\mathbf{T})_\chi = W(\mathbf{M},\mathbf{T})_{\chi\chi_1}$ and therefore Ψ_{χ_1} restricts to an isomorphism

$$\mathcal{H}_{W(\mathbf{M},\mathbf{T})_\chi} \cong \mathcal{H}_{W(\mathbf{M},\mathbf{T})_{\chi\chi_1}}.$$

Thus $\Psi^{\mathbf{M}} \circ \Psi_{\chi_1}^{-1}$ restricts to an isomorphism

$$\mathcal{H}_{W(\mathbf{G},\mathbf{T})_\chi} \xrightarrow{\simeq} \mathcal{H}_{W(\mathbf{M},\mathbf{T})_{\chi\chi_1}}. \tag{7.11}$$

Note that if $\mathbf{G}$ has connected center, then so does $\mathbf{M}$ (see proof of [Car, Propositon 8.1.4], for instance for this fact). Thus, from Equations (7.9) and (7.11), it follows that to prove Theorem 43, we can and do assume without loss of generality that χ has depth-zero.

Remark 44. For a much more general statement of the isomorphism $\Psi^{\mathbf{M}} \circ \Psi_{\chi_1}^{-1}$, see [AM, §8].

Proof in depth-zero case: For results of this section, no restriction on characteristic or residue characteristic is imposed. Let I be the Iwahori subgroup of $\mathbf{G}(F)$ which is in good position with respect to $(\mathbf{B}^-(F), \mathbf{T}(F))$ i.e., I is the inverse image of $\mathbf{B}^-(\mathcal{O}_F/\mathcal{P}_F)$

under the map $\mathbf{G}(\mathcal{O}_F) \to \mathbf{G}(\mathcal{O}_F/\mathcal{P}_F)$ (note here that we are taking opposite Borel) and let I_{0+} denote its pro-unipotent radical. Put $\mathbf{T}(F)_{0+} = I_{0+} \cap \mathbf{T}(F)_0$. Then $I/I_{0+} \cong \mathbf{T}(F)_0/\mathbf{T}(F)_{0+}$. Since χ is depth-zero, it factors through $\mathbf{T}(F)_0/\mathbf{T}(F)_{0+}$ and consequently lifts to a character of I which we denote by ρ. The pair (I,ρ) is then a $\mathbf{G}(F)$-cover of $(\mathbf{T}(F)_0,\chi)$

By conjugating with $\mathbf{T}(F)$ if required, we can assume that ψ is of generic depth-zero at x. Define $\phi : \mathbf{G}(F) \to \mathbb{C}$ to be the function supported on $\mathbf{U}(F).(I \cap \mathbf{B}^-(F))$ such that $\phi(ui) = \psi(u)\chi(i)$ for $u \in \mathbf{U}(F)$ and $i \in (I \cap \mathbf{B}^-(F))$. There is a surjection of $\mathbf{G}(\mathbb{F}_q)$-spaces

$$(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^{\mathbf{G}(F)_{x,0+}} \twoheadrightarrow \text{ind}_{\mathbf{U}(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}(\psi_q). \tag{7.12}$$

Under this surjection, ϕ maps to a function $\phi_q : \mathbf{G}(\mathbb{F}_q) \to \mathbb{C}$ which is supported on $\mathbf{U}(\mathbb{F}_q).\mathbf{B}^-(\mathbb{F}_q)$ and such that $\phi_q(ub) = \psi_q(u)\chi(b)$ for $u \in \mathbf{U}(\mathbb{F}_q)$ and $b \in \mathbf{B}^-(\mathbb{F}_q)$. There is a unique irreducible common $\mathbf{G}(\mathbb{F}_q)$-constituent σ of both the representations $\text{ind}_{\mathbf{B}^-(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}\chi$ and $\text{ind}_{\mathbf{U}(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}(\psi_q)$. In particular, σ^χ is an irreducible $\mathcal{H}_{W_\chi}$-module isomorphic to the χ-isotypical component of the irreducible ψ_q-generic constituent of $\text{ind}_{\mathbf{B}^-(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}\chi$. Observe that $\phi_q \in \sigma^\chi \subset \sigma$. If χ is trivial, then σ^χ corresponds to the Steinberg constituent of $\text{ind}_{\mathbf{B}^-(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}\chi$. If $\mathbf{G}$ has connected center, then it is shown in [Ree, §7.2, 2nd last paragraph] that as $\mathcal{H}_{W_\chi}$-modules $\sigma^\chi \cong \text{sgn}$. Thus in either case, the 1-dimensional space spanned by ϕ_q affords the sgn representation of $\mathcal{H}_{W_\chi}$. Consequently, the 1-dimensional space spanned by ϕ affords the sgn representation of $\mathcal{H}_{W_\chi}$. It is readily checked that ϕ maps to 1 under the isomorphism of Theorem 40. It follows that ϕ is a generator of the cyclic $\mathcal{H}(\mathbf{G}(F),\rho)$-module $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^\rho$. Using Frobenius reciprocity, $\text{Hom}_{\mathcal{H}_{W_\chi}}(\text{sgn}, (c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^\rho)$ is isomorphic to

$$\text{Hom}_{\mathcal{H}(\mathbf{G}(F),\rho)}(\mathcal{H}(\mathbf{G}(F),\rho) \otimes_{\mathcal{H}_{W_\chi}} \text{sgn}, (c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^\rho).$$

This isomorphism sends $1 \mapsto \phi$ to the element $1 \otimes 1 \mapsto \phi$. Since $\mathcal{H}_{W_\chi}$ and $\mathcal{H}(\mathbf{T}(F),\chi)$ together generate the full Hecke algebra $\mathcal{H}(\mathbf{G}(F),\rho)$, as a $\mathcal{H}(\mathbf{T}(F),\chi)$-module $\mathcal{H}(\mathbf{G}(F),\rho)\otimes_{\mathcal{H}_{W_\chi}}\text{sgn}$ is free and generated by $1\otimes 1$. Theorem 43 now follows from the fact that $\mathcal{H}(\mathbf{G}(F),\rho) \otimes_{\mathcal{H}_{W_\chi}} \text{sgn}$ and $(c\text{-ind}_{\mathbf{U}(F)}^{\mathbf{G}(F)}(\psi))^\rho)$ are free $\mathcal{H}(\mathbf{T}(F),\chi)$-modules generated by $1 \otimes 1$ and ϕ respectively.

□

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