STIEFEL-WHITNEY CLASSES OF REPRESENTATIONS OF SOME FINITE GROUPS OF LIE TYPE

A thesis

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Neha Malik

Reg. ID: 20152033

IISER PUNE

Department of Mathematics
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE
Pashan, Pune 411008, India

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Dedicated to
My Parents

Certificate

Certified that the work incorporated in the thesis entitled "*Stiefel-Whitney Classes of Representations of Some Finite Groups of Lie Type*", submitted by *Neha Malik* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Steven Spallone

Date: September 9, 2022

Prof. Steven Spallone
Thesis Supervisor

Declaration

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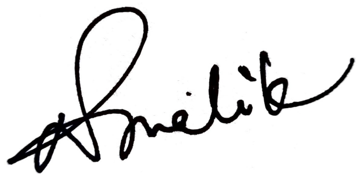

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Neha Malik
Reg. ID: 20152033

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Contents

Abstract

Orthogonal representations π of a finite group G have invariants $w_i(\pi)$ living in the ith degree cohomology group $H^i(G, \mathbb{Z}/2\mathbb{Z})$, called *Stiefel-Whitney Classes* (SWCs). Their sum $w(\pi) = 1 + w_1(\pi) + w_2(\pi) + \dots$ is called the *total SWC* of π.

There do not seem to have many explicit calculations in the literature of SWCs for the non-abelian groups. In this thesis we present the total SWCs for orthogonal representations of several finite groups of Lie type, namely symplectic groups $\mathrm{Sp}(2n, q)$ and special linear groups $\mathrm{SL}(2n+1, q)$ when q is odd. We also describe the SWCs for $\mathrm{SL}(2, q)$ for even q. All our formulas for SWCs are in terms of character values at certain diagonal involutions.

List of Symbols

$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{F}_q$	Finite field with q elements
$\mathbb{F}_q^\times$	$\mathbb{F}_q \backslash \{0\}$
$\mathbb{Z}$	Ring of all integers
C_n	Cyclic group of order n
S_n	Symmetric group of degree n
Q	Quaternion group of order 8
$\cong$	isomorphism
$\boxtimes$	External tensor product
$\widehat{G}$	Character group of G
G^n	n-fold product of group G
G_2	Sylow 2-subgroup of G
$H \leqslant G$	H is a subgroup of G

$\mathrm{Aut}(G)$	Group of all automorphisms of G
$V^\vee$	Dual space to vector space V
χ_π	Character of group representation π
$\mathrm{res}^G_H \pi$ or $\pi\vert_H$	Restriction of π from G to a subgroup H
$\deg \pi$	Degree of linear representation π
$\mathrm{GL}(n,q)$	General linear group of degree n over $\mathbb{F}_q$
$\mathrm{SL}(n,q)$	Special linear group of degree n over $\mathbb{F}_q$
$\mathrm{Sp}(2n,q)$	Symplectic group of degree $2n$ over $\mathbb{F}_q$
$\mathbb{1}$	The identity matrix
$\mathrm{diag}(a_1,\ldots,a_n)$	Diagonal matrix with diagonal entries $a_1,\ldots,a_n$ in this exact order
$H^*(G)$	Cohomology ring $H^*(G,\mathbb{Z}/2\mathbb{Z})$

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Introduction

The theory of Stiefel-Whitney classes (SWCs) of vector bundles is an old unifying concept in geometry. Real representations of a finite group G give rise to *flat* vector bundles over the classifying space BG. Via this construction (see [2] or [14] for instance), to a real representation ρ, one associates cohomology classes $w_i^{\mathbb{R}}(\rho) \in H^i(G, \mathbb{Z}/2\mathbb{Z})$, called *Stiefel-Whitney classes* (SWCs).

Let (π, V), with V a complex finite-dimensional vector space, be an orthogonal representation of G. There is a representation (π_0, V_0), with V_0 a real vector space, so that $\pi_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \pi$. Moreover, π_0 is unique up to isomorphism. (See Proposition 2.2 below.) We prefer to work with orthogonal complex representations, thus define

$$w_i(\pi) := w_i^{\mathbb{R}}(\pi_0) \quad ; \quad 0 \leq i \leq \deg \pi.$$

Their sum

$$w(\pi) = w_0(\pi) + w_1(\pi) + \ldots \in H^*(G, \mathbb{Z}/2\mathbb{Z})$$

is known as the *total Stiefel-Whitney class* of π.

Formulas for SWCs of cyclic groups are well-known; we review this in Section 2.3.3. The second SWCs of representations of S_n and related groups were found in [12]. The case of $\mathrm{GL}(n, q)$ with q odd has been completed recently in [10] and [11]; their results are analogous to ours. In this thesis, we give formulas for the SWCs of all finite symplectic groups $\mathrm{Sp}(2n, q)$ when q is odd as well as special linear groups $\mathrm{SL}(2n+1, q)$ for odd q.

We have announced some of the preliminary results of this thesis for low ranks in [19]. Also our work on the SWCs of special linear groups $\mathrm{SL}(2, q)$ can be found in [20].

A notion of "detection by subgroups" underlies our calculations. Let $H^*_{\mathrm{SW}}(G, \mathbb{Z}/2\mathbb{Z})$ be the subalgebra of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ generated by SWCs $w_i(\pi)$ of orthogonal representations π of G. Let $H \leqslant G$ be a subgroup. We say H *detects the mod* 2 *cohomology* of G when the restriction map

$$H^*(G, \mathbb{Z}/2\mathbb{Z}) \to H^*(H, \mathbb{Z}/2\mathbb{Z})$$

is an injection. Whereas we say H *detects SWCs* of G if this map is injective on $H^*_{\mathrm{SW}}(G, \mathbb{Z}/2\mathbb{Z})$. Suppose one of these is true, and that π is an orthogonal representation of G. Then $w(\pi)$ is specified by its image in $H^*(H, \mathbb{Z}/2\mathbb{Z})$, which is actually the SWC of the restriction of π to H. In our formulas, when there is a detecting subgroup, we simply write $w^H(\pi)$ for this image.

Chapter 2 reviews all such basic definitions and concepts from the theory of representations and characteristic classes.

Chapter 3 is dedicated to the generalized quaternions, particularly the quaternion group Q of order 8. These groups are featured in the detection of $\mathrm{SL}(2, q)$ when q is odd. Here we determine many SWCs of orthogonal representations of Q.

Lemma 1.1. *Let π be an orthogonal representation of Q with character χ_π. Put $r_\pi = \frac{1}{8}(\chi_\pi(1) - \chi_\pi(-1))$. Then,*

$$w_{4i}(\pi) = \binom{r_\pi}{i} e^i \quad ; \quad 0 \leq 4i \leq \deg \pi$$

where e is the non-zero element in $H^4(Q, \mathbb{Z}/2\mathbb{Z})$.

Chapter 4 contains a complete calculation of the SWCs of representations of $\mathrm{SL}(2, q)$. This has been achieved via detection results. For instance:

Theorem 1.2. *Let q be odd. Then the center Z detects SWCs of* $\mathrm{SL}(2, q)$.

This detection leads to:

Theorem 1.3. *Let $G = \mathrm{SL}(2, q)$ with q odd. Let π be an orthogonal representation of G.*

Then the total SWC of π is,

$$w(\pi) = (1 + \mathfrak{e})^{r_\pi}$$

where $\mathfrak{e}$ is the non-zero element in $H^4(\mathrm{SL}(2,q), \mathbb{Z}/2\mathbb{Z})$ and $r_\pi = \frac{1}{8}(\chi_\pi(\mathbb{1}) - \chi_\pi(-\mathbb{1}))$. (Here $\mathbb{1}$ is the identity matrix.)

Using [13, Theorem 1], if π is irreducible orthogonal, then $\chi_\pi(-\mathbb{1}) = \chi_\pi(\mathbb{1})$, and so $r_\pi = 0$. Therefore:

Corollary 1.3.1. *Let q be odd. Let π be an irreducible orthogonal representation of $\mathrm{SL}(2,q)$. Then $w(\pi) = 1$.*

On the other hand, let ϖ be an irreducible symplectic representation of $\mathrm{SL}(2,q)$. (By this, we mean ϖ is an irreducible representation on a complex vector space V admitting a non-degenerate $\mathrm{SL}(2,q)$-invariant *antisymmetric* $B : V \times V \to \mathbb{C}$.) Its double $\varpi \oplus \varpi$ is orthogonal. Again [13, Theorem 1] allows simplification for $\varpi \oplus \varpi$:

$$w(\varpi \oplus \varpi) = (1 + e)^{\frac{1}{2} \deg \varpi}.$$

Now let q be even, say $q = 2^r$. Let N be the subgroup of upper unitriangular matrices in $\mathrm{SL}(2,q)$. Then N detects the mod 2 cohomology of $\mathrm{SL}(2,q)$.

Set $n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in N$, and $s_\pi = \frac{1}{q}(\chi_\pi(\mathbb{1}) - \chi_\pi(n_0))$.

Theorem 1.4. *Let $q = 2^r$. Let π be an orthogonal representation of $\mathrm{SL}(2,q)$. The total SWC of π is*

$$w^N(\pi) = \Bigg(\prod_{v \in H^1(N,\mathbb{Z}/2\mathbb{Z})} (1 + v) \Bigg)^{s_\pi} \in H^*(N, \mathbb{Z}/2\mathbb{Z}).$$

The expansion of this product is well-known. We have

$$\prod_{v \in H^1(N,\mathbb{Z}/2\mathbb{Z})} (1 + v) = 1 + \sum_{i=0}^{r-1} d_{r,i} \in H^*(N, \mathbb{Z}/2\mathbb{Z}), \tag{1.1}$$

where $d_{r,i}$ are the Dickson invariants described in [27]. We review them in Section 4.2.1.

These results have some interesting corollaries. We first characterize the representations π with non-zero "top SWC" $w_{\deg(\pi)}(\pi)$.

Corollary 1.5. *Let π be an orthogonal representation of* $\mathrm{SL}(2,q)$. *The top SWC of π is non-zero precisely when:*

(i) $\pi(-\mathbb{1}) = -1$, *for q odd.*

(ii) *π is cuspidal, for q even.*

Secondly, we have:

Corollary 1.6. *The subalgebra* $H^*_{\mathrm{SW}}(\mathrm{SL}(2,q),\mathbb{Z}/2\mathbb{Z})$ *is:*

(i) $\mathbb{Z}/2\mathbb{Z}[\mathfrak{c}]$, *for q odd,*

(ii) *generated by the Dickson invariants of $\mathbb{F}_q$, for q even.*

We describe the *obstruction degree* of an orthogonal π, meaning the least $k > 0$ with $w_k(\pi) \neq 0$. (If $w(\pi)$ is trivial, then this degree is infinite.) Write $\mathrm{ord}_2(n)$ for the highest power of 2 which divides an integer n.

Corollary 1.7. *The obstruction degree of π is:*

(i) 2^{t+2}, *where* $t = \mathrm{ord}_2(r_\pi)$, *for q odd,*

(ii) 2^{r+s-1}, *where* $s = \mathrm{ord}_2(s_\pi)$ *and* $q = 2^r$.

We also describe the subgroup of the complete cohomology ring $\widehat{H}^*(G,\mathbb{Z}/2\mathbb{Z})$ (see Section 2.3.5) generated by $w(\pi)$, as π varies over orthogonal representations.

Moving ahead, Chapter 5 determines the total SWC for the symplectic groups $\mathrm{Sp}(2n,q)$ when q is odd. Our work on $\mathrm{SL}(2,q)$ is the stepping stone for these groups. We have a subgroup X of $\mathrm{Sp}(2n,q)$, described in Section 5.1.1, which is isomorphic to the n-fold product $\mathrm{SL}(2,q)^n$ and gives a detection:

Lemma 1.8 ([1], Chapter VII, Lemma 6.2)**.** *The subgroup X detects the mod 2 cohomology of* $\mathrm{Sp}(2n,q)$ *with odd q.*

The n-fold product Z^n is the subgroup of $\mathrm{Sp}(2n,q)$ consisting of diagonal matrices with eigenvalues ± 1. We generalize Theorem 1.2 to:

Theorem 1.9. *Let q be odd. The subgroup Z^n detects the SWCs of* $\mathrm{Sp}(2n,q)$.

This detection is the key to unlocking an explicit formula for SWCs of these symplectic groups. Here, we describe the SWCs for $\mathrm{Sp}(4,q)$ as an instance. For a general result, please refer to Theorem 5.11.

Consider the projections $\mathrm{pr}_i : \mathrm{SL}(2,q)^n \to \mathrm{SL}(2,q)$, and let $\mathfrak{e}_i = \mathrm{pr}_i^*(\mathfrak{e})$. Put $d_1 = \mathrm{diag}(1,-1,-1,1) \in \mathrm{Sp}(4,q)$. (As usual, this means the diagonal matrix with these diagonal entries.)

Theorem 1.10. *Let q be odd. Let π be an orthogonal representation of $\mathrm{Sp}(4,q)$. Then the total SWC of π is*

$$w^X(\pi) = ((1+\mathfrak{e}_1)(1+\mathfrak{e}_2))^{r_\pi}(1+\mathfrak{e}_1+\mathfrak{e}_2)^{s_\pi},$$

where

$$\begin{aligned} r_\pi &= \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(-\mathbb{1})\Big), \\ s_\pi &= \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) + \chi_\pi(-\mathbb{1}) - 2\chi_\pi(d_1)\Big). \end{aligned}$$

Again from Theorem 1 of [13], when π is irreducible orthogonal, we deduce

$$w^X(\pi) = (1+\mathfrak{e}_1+\mathfrak{e}_2)^{s_\pi},$$

where

$$s_\pi = \frac{1}{8}\left(\deg\pi - \chi_\pi(d_1)\right).$$

The final chapter 6 is concerned with the SWCs of special linear groups $\mathrm{SL}(2n+1,q)$ when q is odd. Once more, we use a detection for this. From a theorem of [23] one can deduce:

Lemma 1.11. *When n and q are odd, the diagonal subgroup T of $\mathrm{SL}(n,q)$ detects its mod 2 cohomology.*

Let $T[2]$ be the subgroup of diagonal matrices with 1 or -1 on the diagonal.

Proposition 1.12. *Let q be odd. Then the subgroup $T[2]$ detects SWCs of $\mathrm{SL}(2n+1,q)$.*

This detection leads to the calculation of SWCs for these groups. Please see Theorems 6.6 and 6.11 for the formulas.

For simplicity, we illustrate our results with $\mathrm{SL}(3,q)$ and $\mathrm{SL}(5,q)$. The detecting subgroups T and $T[2]$ are cyclic; their SWCs use the well-known generators v_i, t_i of the cohomology of cyclic groups. (See Section 2.3.3.)

Proposition 1.13. *Let q be odd, and put $c_1 = \mathrm{diag}(-1,-1,1) \in \mathrm{SL}(3,q)$. The total SWC of an orthogonal representation π of $\mathrm{SL}(3,q)$ is*

$$\begin{aligned} w^T(\pi) &= \Big((1+t_1)(1+t_2)(1+t_1+t_2)\Big)^{m_\pi/2}, \quad \textit{when } q \equiv 1 \pmod 4 \\ w^{T_2}(\pi) &= \Big((1+v_1)(1+v_2)(1+v_1+v_2)\Big)^{m_\pi}, \quad \textit{when } q \equiv 3 \pmod 4 \end{aligned}$$

where $m_\pi = \frac{1}{4}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(c_1)\Big)$.

Let q be odd, and put

$$\begin{aligned} c_1 &= \mathrm{diag}(-1,-1,1,1,1), \\ c_2 &= \mathrm{diag}(-1,-1,-1,-1,1) \in \mathrm{SL}(5,q). \end{aligned}$$

Proposition 1.14. *Let π be an orthogonal representation of $\mathrm{SL}(5,q)$. Then the total SWC of π is*

$$w^T(\pi) = \Big(\prod_{i=1}^{4}(1+t_i)(1+\sum_{i=1}^{4} t_i)\Big)^{m_\pi/2}\Big(\prod_{1\le i<j\le 4}(1+t_i+t_j)\prod_{1\le i<j<k\le 4}(1+t_i+t_j+t_k)\Big)^{n_\pi/2},$$

for $q \equiv 1 \pmod 4$, and

$$w^{T_2}(\pi) = \Big(\prod_{i=1}^{4}(1+v_i)(1+\sum_{i=1}^{4} v_i)\Big)^{m_\pi}\Big(\prod_{1\le i<j\le 4}(1+v_i+v_j)\prod_{1\le i<j<k\le 4}(1+v_i+v_j+v_k)\Big)^{n_\pi},$$

for $q \equiv 3 \pmod 4$.

Moreover the exponents m_π, n_π are in terms of character values of π:

$$\begin{aligned} m_\pi &= \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) + 2\chi_\pi(c_1) - 3\chi_\pi(c_2)\Big) \\ n_\pi &= \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) - 2\chi_\pi(c_1) + \chi_\pi(c_2)\Big). \end{aligned}$$

2

Preliminaries

This chapter collects the concepts from representation theory and group cohomology which are being used throughout this thesis. We begin with a review on the orthogonal representations of a finite group G, aiming to naturally transform a real representation into a complex orthogonal representation of G. This enables us to define Stiefel-Whitney Classes (SWCs) for orthogonal representations from the well-known theory of SWCs of real vector bundles. We also discuss some other characteristic classes like Chern classes, symplectic classes and the Euler class.

2.1 Orthogonal Representations

In this section we review the theory of orthogonal representations, referring to [3, Chapter II, Section 6] for proofs. All our representations are finite-dimensional.

Let G be a finite group, and $F = \mathbb{R}$ or $\mathbb{C}$. Let $\mathrm{Rep}(G, F)$ be the category of G-representations on finite-dimensional F-vector spaces.

Definition 2.1. For $(\pi, V) \in \mathrm{Rep}(G, F)$, the *dual representation* $\pi^\vee$ is defined on the dual space $V^\vee$ via contragradient map * as follows.

$$\pi^\vee(g) = \pi(g^{-1})^* \text{ for all } g \in G.$$

One calls (π, V) *self-dual* if $(\pi^\vee, V^\vee) \cong (\pi, V)$.

A representation $(\pi, V) \in \mathrm{Rep}(G, \mathbb{C})$ is self-dual if and only if there exists a a non-degenerate G-invariant bilinear form $B : V \times V \to \mathbb{C}$. Moreover, we say (π, V) is *orthogonal* (or *symplectic*), provided there exists a non-degenerate G-invariant symmetric (resp. non-symmetric) bilinear form $B : V \times V \to \mathbb{C}$.

For a complex representation π of G, there is a well-known number $\varepsilon(\pi)$, called the *Frobenius-Schur indicator* of π. Write $\chi_\pi(g)$ for the character of π at an element $g \in G$. Then it may be computed as the sum

$$\varepsilon(\pi) = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g^2).$$

If π is irreducible, $\varepsilon(\pi)$ determines whether or not π is self-dual (orthogonal, symplectic) due to the following:

$$\varepsilon(\pi) = \begin{cases} 0, & \pi \text{ is not self-dual} \\ 1, & \pi \text{ is orthogonal} \\ -1, & \pi \text{ is symplectic.} \end{cases} \tag{2.1}$$

We now state a proposition to characterize complex orthogonal representations of G.

Proposition 2.2 ([3], Chapter II, Proposition 6.4)**.** *Let* $(\pi, V) \in \mathrm{Rep}(G, \mathbb{C})$*. Then the following statements are equivalent:*

(i) (π, V) *is orthogonal.*

(ii) (π, V) *has a real structure, meaning there is a conjugate-linear* G*-map* $j : V \to V$ *with* $j^2 = \mathrm{Id}_V$ *.*

(iii) *There exists a representation* (π_0, V_0)*, with* V_0 *a real vector space, such that* $\pi \cong \pi_0 \otimes_{\mathbb{R}} \mathbb{C}$.

Let $\mathrm{ORep}(G, \mathbb{C})$ be the category of orthogonal G-representations on finite-dimensional $\mathbb{C}$-vector spaces, and let $(\pi, V) \in \mathrm{ORep}(G, \mathbb{C})$. By the Proposition above, (π, V) possesses a real structure j. One defines

$$V_0 := V^j = \{v \in V \mid j(v) = v\},$$

a real vector space, considered as a representation of G, via the restriction

$$\pi_0 := \pi|_{V_0} : G \to \mathrm{GL}(V_0).$$

This gives a functor

$$\begin{aligned} s_0 : \mathrm{ORep}(G, \mathbb{C}) &\to \mathrm{Rep}(G, \mathbb{R}) \\ (\pi, V) &\rightsquigarrow (\pi_0, V_0) \end{aligned}$$

such that $(\pi_0 \otimes_{\mathbb{R}} \mathbb{C}, V_0 \otimes_{\mathbb{R}} \mathbb{C}) \cong (\pi, V)$.

Conversely, a real representation (ρ, U) can be extended via tensoring to get $(\rho \otimes_{\mathbb{R}} \mathbb{C}, U \otimes_{\mathbb{R}} \mathbb{C}) \in \mathrm{Rep}(G, \mathbb{C})$. Then a structure map on $U \otimes_{\mathbb{R}} \mathbb{C}$ defined as $j(u \otimes z) = (u \otimes \bar{z})$ makes $\rho \otimes_{\mathbb{R}} \mathbb{C}$ orthogonal. Therefore we have another functor

$$\begin{aligned} e_0 : \mathrm{Rep}(G, \mathbb{R}) &\to \mathrm{ORep}(G, \mathbb{C}) \\ (\rho, U) &\rightsquigarrow (\rho \otimes_{\mathbb{R}} \mathbb{C}, U \otimes_{\mathbb{R}} \mathbb{C}). \end{aligned} \tag{2.2}$$

The compositions $e_0 s_0$ and $s_0 e_0$ are naturally equivalent to the identity in the respective categories. We now consider a pair of functors defined as follows. One can view a complex representation (π, V) as real by thinking of V as an $\mathbb{R}$-vector space with the same G-action. Denote this forgetful functor by

$$r_{\mathbb{R}}^{\mathbb{C}} : \mathrm{Rep}(G, \mathbb{C}) \to \mathrm{Rep}(G, \mathbb{R}).$$

Second, there is the extension functor

$$\begin{aligned} e_{\mathbb{R}}^{\mathbb{C}} : \mathrm{Rep}(G, \mathbb{R}) &\to \mathrm{Rep}(G, \mathbb{C}) \\ (\rho, U) &\rightsquigarrow (\rho \otimes_{\mathbb{R}} \mathbb{C}, U \otimes_{\mathbb{R}} \mathbb{C}). \end{aligned}$$

Let (π, V), (π', V') be two isomorphic complex orthogonal G-representations. We use s_0 to get the corresponding real representations (π_0, V_0), (π_0', V_0') with

$$(\pi_0 \otimes_{\mathbb{R}} \mathbb{C}, V_0 \otimes_{\mathbb{R}} \mathbb{C}) \cong (\pi_0' \otimes_{\mathbb{R}} \mathbb{C}, V_0' \otimes_{\mathbb{R}} \mathbb{C}). \tag{2.3}$$

From [3, Chapter II, Proposition 6.1], we have

$$r_{\mathbb{R}}^{\mathbb{C}} \circ e_{\mathbb{R}}^{\mathbb{C}} = 2, \tag{2.4}$$

which means for real G-modules U, there exists a natural isomorphism $r_{\mathbb{R}}^{\mathbb{C}} \circ e_{\mathbb{R}}^{\mathbb{C}}(U) \cong U \oplus U$.

So, applying $r_{\mathbb{R}}^{\mathbb{C}}$ on (2.3) gives

$$(\pi_0 \oplus \pi_0, V_0 \oplus V_0) \cong (\pi_0' \oplus \pi_0', V_0' \oplus V_0'),$$

which implies (π_0, V_0) and (π_0', V_0') have the same character, and therefore $(\pi_0, V_0) \cong (\pi_0', V_0')$. This means π_0 (in Proposition 2.2) is unique up to isomorphism.

In fact, $[(\pi, V)] \mapsto [(\pi_0, V_0)]$ is a bijection between the sets of isomorphism classes of complex orthogonal G-representations and real G-representations. Here $[(\pi, V)]$ means the isomorphism class of (π, V).

Theorem 2.3 ([3], Chapter II, Section 6)**.** *Let ρ be an irreducible real representation of G. Then, exactly one of the following must be true:*

(i) *$e_{\mathbb{R}}^{\mathbb{C}}\rho \cong \pi$ where π is a complex irreducible orthogonal representation of G, or*

(ii) *$\rho \cong r_{\mathbb{R}}^{\mathbb{C}}\varphi$, for some irreducible, non-orthogonal, complex representation φ of G.*

Proof. Let $\pi = \rho \otimes_{\mathbb{R}} \mathbb{C}$, the extension of ρ. Then π is orthogonal due to Proposition 2.2. If π is irreducible, then (i) is satisfied. If not, then it can be decomposed into complex irreducible representations of G.

For a representation φ, let $I(\varphi)$ be the number of its irreducible constituents. Note that $I(\varphi) = 1$ if and only if φ is irreducible.

Suppose $I(\pi) = n > 1$. Then, we can write

$$\pi = e_{\mathbb{R}}^{\mathbb{C}}\rho \cong \bigoplus_{i=1}^{n} \varphi_n,$$

where $\varphi_1, \ldots, \varphi_n$ are (complex) irreducible. Applying $r_{\mathbb{R}}^{\mathbb{C}}$ to this and using Equation (2.4) give

$$r_{\mathbb{R}}^{\mathbb{C}}\pi \cong \bigoplus_{i=1}^{n} r_{\mathbb{R}}^{\mathbb{C}}\varphi_n \cong \rho \oplus \rho. \tag{2.5}$$

Since $I(\rho \oplus \rho) = 2$ and $n > 1$, we have $n = 2$ and both $r_{\mathbb{R}}^{\mathbb{C}}\varphi_1$, $r_{\mathbb{R}}^{\mathbb{C}}\varphi_2$ are irreducible. Therefore φ_1, φ_2 must be irreducible. Moreover $\{\varphi_1^\vee, \varphi_2^\vee\} = \{\varphi_1, \varphi_2\}$, up to equivalence, due to self-duality of π. Thus,

$$\rho \cong r_{\mathbb{R}}^{\mathbb{C}}\varphi_1 \cong r_{\mathbb{R}}^{\mathbb{C}}\varphi_2,$$

where either φ_1, φ_2 are both self-dual or $\varphi_2^\vee \cong \varphi_1$.

Also φ_1, φ_2 can not be orthogonal. If orthogonal, there exist $\sigma_i \in \mathrm{Rep}(G, \mathbb{R})$ such that $\varphi_i \cong \sigma_i \otimes_{\mathbb{R}} \mathbb{C}$. This gives $\rho \otimes_{\mathbb{R}} \mathbb{C} \cong (\sigma_1 \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\sigma_2 \otimes_{\mathbb{R}} \mathbb{C})$.Applying $r_{\mathbb{R}}^{\mathbb{C}}$ and using Equation (2.4) again give

$$\rho \oplus \rho \cong (\sigma_1 \oplus \sigma_1) \oplus (\sigma_2 \oplus \sigma_2).$$

This implies $\rho \cong \sigma_1 \oplus \sigma_2$ giving a contradiction to the irreducibility of ρ. Hence $\rho \cong r_{\mathbb{R}}^{\mathbb{C}} \varphi_1$, where φ_1 is irreducible, non- orthogonal and (ii) holds. □

2.1.1 Orthogonally Irreducible Representations (OIRs)

Let (π, V) be a complex representation of G. We have the following two operations to get a real representation from π, depending whether π is orthogonal or not. First, if π is orthogonal, then from Proposition 2.2 there exists a unique real representation (π_0, V_0), up to equivalence, so that $\pi_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \pi$. Second, irrespective of orthogonality, we apply the functor $r_{\mathbb{R}}^{\mathbb{C}}$ on π from above.

Given (π, V) (maybe non-orthogonal), we can symmetrize it to get an orthogonal representation of G by defining

$$S(\pi) := \pi \oplus \pi^{\vee}$$

on the vector space $V \oplus V^{\vee}$. Now, we give a symmetric G-invariant bilinear map B on $V \oplus V^{\vee}$ as,

$$B((v, \alpha), (w, \beta)) = \alpha(w) + \beta(v).$$

We call $S(\pi)$ the *symmetrization* of π.

From [3, Chapter II, Proposition 6.1], we have

$$e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}}(\pi) \cong S(\pi)$$

for a complex representation π. Also $S(\pi) \cong e_{\mathbb{R}}^{\mathbb{C}} S(\pi)_0$ from the equivalence of $e_0 s_0$ to the identity. Therefore both $e_{\mathbb{R}}^{\mathbb{C}} S(\pi)_0$ and $e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}}(\pi)$ are isomorphic to $S(\pi)$, which leads to a relation between the two operations above:

$$S(\pi)_0 \cong r_{\mathbb{R}}^{\mathbb{C}} \pi. \tag{2.6}$$

Definition 2.4. We say π is an *orthogonally irreducible representation* (OIR) provided π is orthogonal, and can not be decomposed into a direct sum of orthogonal representations.

Suppose π is irreducible. Then π is orthogonally irreducible if and only if it is orthogonal. Moreover for irreducible non-orthogonal φ, its symmetrization $S(\varphi)$ is an OIR.

Let Π be an orthogonal complex representation of G. Along with Theorem 2.3, we use the natural equivalence between $\mathrm{ORep}(G, \mathbb{C})$ and $\mathrm{Rep}(G, \mathbb{R})$ to have the following decomposition of Π into OIRs:

$$\Pi \cong \bigoplus_i \pi_i \oplus \bigoplus_j S(\varphi_j), \tag{2.7}$$

such that each π_i is irreducible orthogonal and φ_j are irreducible non-orthogonal representations of G. This decomposition establishes that all the OIRs of G are either irreducible orthogonal π, or of the form $S(\varphi)$ with φ irreducible and non-orthogonal.

2.1.2 OIRs of a Direct Product of Groups

Let $G_1, \ldots, G_n$ be finite groups. We consider their direct product

$$G = G_1 \times \cdots \times G_n.$$

Given G_i-representations (π_i, V_i), one can form their *external tensor product* $\pi_1 \boxtimes \cdots \boxtimes \pi_n$ from the action of the product group G on the tensor space $V_1 \otimes \cdots \otimes V_n$ as,

$$(g_1, \ldots, g_n)(v_1 \otimes \cdots \otimes v_n) = g_1 v_1 \otimes \cdots \otimes g_n v_n$$

for $(g_1, \ldots, g_n) \in G$ and $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes \cdots \otimes V_n$.

Let $\mathrm{Irr}(G)$ be the set of isomorphism classes of complex irreducible representations of G, and let $\pi \in \mathrm{Irr}(G)$. We can write

$$\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_n,$$

where $\pi_i \in \mathrm{Irr}(G_i)$ for each $1 \leq i \leq n$. To see when is π orthogonal, we use the Frobenius-Schur indicator $\varepsilon(\pi)$ expressed as the product of $\varepsilon(\pi_1), \ldots, \varepsilon(\pi_n)$.

$$
\begin{aligned}
\varepsilon(\pi) &= \varepsilon(\pi_1 \boxtimes \cdots \boxtimes \pi_n) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{\pi_1 \boxtimes \cdots \boxtimes \pi_n}(g^2) \\
&= \frac{1}{|G_1| \ldots |G_n|} \sum_{(g_1, \ldots, g_n) \in G} \chi_{\pi_1}(g_1^2) \ldots \chi_{\pi_n}(g_n^2) \\
&= \left(\frac{1}{|G_1|} \sum_{g_1 \in G_1} \chi_{\pi_1}(g_1^2) \right) \ldots \left(\frac{1}{|G_n|} \sum_{g_n \in G_n} \chi_{\pi_n}(g_n^2) \right) \\
&= \varepsilon(\pi_1)\varepsilon(\pi_2) \ldots \varepsilon(\pi_n).
\end{aligned} \tag{2.8}
$$

Write $F(\pi)$ for the multiset $\{\pi_1, \pi_2, \ldots, \pi_n\}$. Then with the help of equalities (2.1) and (2.8), we can list the OIRs of $G = G_1 \times \cdots \times G_n$ in terms of irreducible representations of G_i as follows:

1. (Irreducible orthogonal representations of G)

$$\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_n,$$

   where $\pi_i \in \mathrm{Irr}(G_i)$ are self-dual for each i and an even number of representations in $F(\pi)$ are symplectic. In particular when all π_i are orthogonal, so is π.

2. (Symmetrization of irreducible non-orthogonal representations of G)

$$S(\varphi) = S(\varphi_1 \boxtimes \cdots \boxtimes \varphi_n),$$

   where $\varphi_i \in \mathrm{Irr}(G_i)$ for $1 \leq i \leq n$ and exactly one of the following holds:

   (a) At least one of φ_i is not self-dual.

   (b) Each φ_i is self-dual and there is an odd number of symplectic representations in $F(\varphi)$.

2.2 Group Cohomology

Let G be a finite group. Let R be a commutative ring. We can think of R as a trivial G-module and by $H^*(G, R)$, we mean the usual group cohomology ring.

2.2.1 Coefficient Maps

Let R, S be two commutative rings and $\phi : R \to S$ be a ring homomorphism. Consider the map induced by ϕ on cohomology

$$\kappa(\phi) : H^*(G, R) \to H^*(G, S). \tag{2.9}$$

This map κ is called the *coefficient map of cohomology*. (We drop ϕ from the notation if there is no confusion.) We will later use that κ is a ring homomorphism. For the lack of a reference, we include a proof, though probably it is well-known. We begin by quoting a general result for all G-modules M, N.

Proposition 2.5 ([24], Chapter VIII, Proposition 5)**.** *Let G be a finite group. Given G-modules M, N, there exists a unique family of homomorphisms*

$$\cup_0^{p,q} : H^p(G, M) \otimes H^q(G, N) \to H^{p+q}(G, M \otimes_{\mathbb{Z}} N)$$

for each $(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that these are natural transformations of functors when we consider the two sides of the arrow as bifunctors covariant in (M, N).

Let us rewrite this in more detail. Consider the two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows. The cateogory $\mathcal{C}_1$ has the pairs of G-modules (M, N) as objects and a morphism is a pair of G-module homomorphisms (ϕ, ψ) between two objects (M, N) and (M', N'). Let $\mathcal{C}_2$ be the category of abelian groups.

Consider the identity map Id: $G \to G$ and a G-module homomorphism $\phi : M \to M'$. The pair (Id, ϕ) is compatible (in the sense of [24, Chapter VII, Section 5]) which for every $p \geq 0$, defines a homomorphism

$$\phi_*^p : H^p(G, M) \to H^p(G, M').$$

Note that these together give a map $\phi_* : H^*(G, M) \to H^*(G, M')$. Now using these homomorphisms, we define the functors F, F' for a pair of non-negative integers (p, q) as follows.

$$F, F' : \mathcal{C}_1 \to \mathcal{C}_2$$

are such that for each (M, N) in $\mathcal{C}_1$,

$$F(M,N) = H^p(G,M) \otimes_{\mathbb{Z}} H^q(G,N)$$
$$F'(M,N) = H^{p+q}(G, M \otimes_{\mathbb{Z}} N).$$

To a morphism $(\phi, \psi) = \left\{ \begin{array}{c} M \xrightarrow{\phi} M' \\ N \xrightarrow{\psi} N' \end{array} \right\}$ in $\mathcal{C}_1$, these functors associate the following morphisms in $\mathcal{C}_2$:

$$F(\phi,\psi) = \phi_*^p \otimes \psi_*^q$$
$$F'(\phi,\psi) = (\phi \otimes \psi)_*^{p+q}$$

where $\phi_*^p \otimes \psi_*^q : H^p(G,M) \otimes_{\mathbb{Z}} H^q(G,N) \to H^p(G,M') \otimes_{\mathbb{Z}} H^q(G,N')$ is the tensor product of homomorphisms ϕ_*^p and ψ_*^q, and $(\phi \otimes \psi)_*^{p+q} : H^{p+q}(G, M \otimes_{\mathbb{Z}} N) \to H^{p+q}(G, M' \otimes_{\mathbb{Z}} N')$ is induced by the tensor product $\phi \otimes \psi : M \otimes_{\mathbb{Z}} N \to M' \otimes_{\mathbb{Z}} N'$.

Now, Proposition 2.5 says that for each $(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, the following diagram commutes:

$$\begin{tikzcd}
H^p(G,M) \otimes_{\mathbb{Z}} H^q(G,N) \arrow[r, "\phi_*^p \otimes \psi_*^q"] \arrow[d, "\cup_0^{p,q}"'] & H^p(G,M') \otimes_{\mathbb{Z}} H^q(G,N') \arrow[d, "\cup_0^{p,q}"] \\
H^{p+q}(G, M \otimes_{\mathbb{Z}} N) \arrow[r, "(\phi \otimes \psi)_*^{p+q}"] & H^{p+q}(G, M' \otimes_{\mathbb{Z}} N').
\end{tikzcd}$$

Using the homomorphisms $\cup_0^{p,q}$, we get a map on cohomology groups

$$\cup_0 : H^*(G,M) \otimes H^*(G,N) \to H^*(G, M \otimes_{\mathbb{Z}} N).$$

Corollary 2.5.1. *Let $\phi : R \to S$ be a ring homomorphism. Then the induced map $\kappa(\phi) : H^*(G,R) \to H^*(G,S)$ is a ring homomorphism.*

Proof. From [24, Chapter VII, §5], we can establish that $\kappa(\phi) = \phi_*$ is a group homomorphism as (Id, ϕ) is a compatible pair. To show that κ preserves the multiplication structure, we consider

$$\begin{aligned} m^R : R \otimes_{\mathbb{Z}} R &\to R \\ r_1 \otimes r_2 &\mapsto r_1 r_2 \end{aligned}$$

which induces the map m^R_* on cohomology. Now the composition

$$\cup^R := m^R_* \circ \cup^R_0 : H^*(G,R) \otimes H^*(G,R) \xrightarrow{\cup^R_0} H^*(G, R \otimes_{\mathbb{Z}} R) \xrightarrow{m^R_*} H^*(G,R).$$

is the multiplication in the ring $H^*(G,R)$. We intend to show that $\kappa(x \cup y) = \kappa(x) \cup \kappa(y)$ for all $x, y \in H^*(G,R)$, which is the same as

$$\phi_*(m^R_*(x \cup^R_0 y)) = m^S_*(\phi_*(x) \cup^S_0 \phi_*(y)).$$

In other words, we would like to prove that the outer square in the diagram below commutes.

$$\begin{tikzcd}
H^*(G,R) \otimes_{\mathbb{Z}} H^*(G,R) \arrow[r, "\phi_* \otimes \phi_*"] \arrow[d, "\cup^R_0"'] \arrow[dr, phantom, "1"] & H^*(G,S) \otimes_{\mathbb{Z}} H^*(G,S) \arrow[d, "\cup^S_0"] \\
H^*(G, R \otimes_{\mathbb{Z}} R) \arrow[r, "(\phi \otimes \phi)_*"] \arrow[d, "m^R_*"'] \arrow[dr, phantom, "2"] & H^*(G, S \otimes_{\mathbb{Z}} S) \arrow[d, "m^S_*"] \\
H^*(G,R) \arrow[r, "\phi_*"] & H^*(G,S)
\end{tikzcd}$$

The commutativity of square 1 follows from Proposition 2.5. For the second square, we consider

$$\begin{tikzcd}
R \otimes_{\mathbb{Z}} R \arrow[r, "\phi \otimes \phi"] \arrow[d, "m^R"'] & S \otimes_{\mathbb{Z}} S \arrow[d, "m^S"] \\
R \arrow[r, "\phi"] & S
\end{tikzcd}$$

which is commutative as ϕ is a ring homomorphism. Since the diagram induced by it on cohomology is exactly the square 2, we have the proof. □

Let G, H be finite groups. A group homomorphism $\varphi : H \to G$ induces a map $\varphi^* : H^*(G, R) \to H^*(H, R)$ on cohomology, which is in fact a ring homomorphism. (See [24, Chapter VII, §5] for details.) Now, we mostly work with $R = \mathbb{Z}/2\mathbb{Z}$ and are particularly interested in φ^* when H is a subgroup of G and φ is the inclusion of H into G. Such φ^* are known as *restriction homomorphisms* and are extensively used in the concept of "detection". (See Section 2.3.1.)

2.3 Stiefel-Whitney Classes (SWCs)

Let E be a d-dimensional real vector bundle over a paracompact base space B. From [22], there is a sequence of cohomology classes

$$w_1(E), \dots, w_d(E)$$

with each $w_i(E) \in H^*(B, \mathbb{Z}/2\mathbb{Z})$.

For every finite group G there is a classifying space BG with a universal principal G-bundle EG, unique up to homotopy. From a real representation (ρ, U) of G, one can form the associated vector bundle $EG[U] = EG \times_G U$ over BG. Then one puts

$$w_i^{\mathbb{R}}(\rho) = w_i(EG[U]), \tag{2.10}$$

see for instance [2] or [14]. Moreover the singular cohomology $H^*(BG, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to the group cohomology $H^*(G, \mathbb{Z}/2\mathbb{Z})$. From this point, we simply write $H^*(G)$ for $H^*(G, \mathbb{Z}/2\mathbb{Z})$ unless mentioned otherwise.

Hence, to a real representation ρ, we can associate cohomology classes

$$w_i^{\mathbb{R}}(\rho) \in H^i(G) \quad ; \quad i = 0, 1, 2, \dots$$

called *Stiefel-Whitney Classes* (SWCs). We prefer to work with complex orthogonal representations. This can be done due to the equivalence between $\mathrm{Rep}(G, \mathbb{R})$ and $\mathrm{ORep}(G, \mathbb{C})$. An orthogonal complex representation π comes from a unique real representation π_0 by Proposition 2.2. We can thus define the following:

Definition 2.6. Let π be an orthogonal complex representation of G. We put

$$w_i(\pi) := w_i^{\mathbb{R}}(\pi_0) \quad ; \quad i = 0, 1, 2, \ldots$$

Then, the *total Stiefel-Whitney class* of π is defined to be the sum

$$w(\pi) = w_0(\pi) + w_1(\pi) + w_2(\pi) + \ldots \in H^*(G).$$

In fact $w_i(\pi) = 0$ for all $i > \deg \pi$ which makes $w(\pi)$ a finite sum. There are nice interpretations for the first few $w_i(\pi)$. Firstly, $w_0(\pi) = 1 \in H^0(G)$ and the first SWC, applied to linear characters $G \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$, is the well-known isomorphism

$$w_1 : \mathrm{Hom}(G, \pm 1) \xrightarrow{\cong} H^1(G). \tag{2.11}$$

More generally, if π is an orthogonal representation, then $w_1(\pi) = w_1(\det \pi)$, where $\det \pi$ is simply the composition of π with the determinant map. When $\det \pi = 1$, then $w_2(\pi)$ vanishes if and only if π lifts to the corresponding spin group. (See [12] for details.)

The SWCs are *functorial* or *natural* in the following sense: Given a homomorphism $\varphi : G_1 \to G_2$ of groups and π an orthogonal representation of G_2, we have

$$\varphi^*(w(\pi)) = w(\pi \circ \varphi). \tag{2.12}$$

where $\varphi^* : H^*(G_2) \to H^*(G_1)$ is the map induced on cohomology.

The SWCs are also *multiplicative*. This means if π_1 and π_2 are both orthogonal, then

$$w(\pi_1 \oplus \pi_2) = w(\pi_1) \cup w(\pi_2), \tag{2.13}$$

which can also be expressed as,

$$w_k(\pi_1 \oplus \pi_2) = \sum_{i=0}^{k} w_i(\pi_1) \cup w_{k-i}(\pi_2).$$

This is known as the *Whitney Product Theorem.*

Now we make an observation for later use as a lemma below. But its proof would require "Chern classes", which are another characteristic classes discussed in Section 2.4.1 later.

Lemma 2.7. *Let G be a finite group. Let (π, V) be a complex representation of G with* $\det(\pi) = 1$. *Then, we have*

$$w_i(S(\pi)) = 0 \text{ for } i = 1, 2, 3.$$

Proof. Proposition 2.16 relates SWCs and Chern classes via coefficient map $\kappa : H^*(G, \mathbb{Z}) \to H^*(G)$. That is:

$$\kappa(1 + c_1(\pi) + c_2(\pi) + \ldots) = 1 + w_1(S(\pi)) + w_2(S(\pi)) + \ldots.$$

Since Chern classes live in even degrees, all the odd SWCs of $S(\pi)$ vanish. Moreover $c_1(\pi) = 0$ due to $\det(\pi) = 1$ hypothesis . This implies $w_2(S(\pi)) = 0$. □

2.3.1 Detection by Subgroups

Let H be a subgroup of G. We now define the *detection by a subgroup*:

Definition 2.8. We say H *detects the mod* 2 *cohomology of* G, provided the restriction map, induced by inclusion $i : H \hookrightarrow G$, on cohomology is an injection. That is,

$$i^* : H^*(G) \hookrightarrow H^*(H).$$

Let $H^*_{\mathrm{SW}}(G)$ be the subalgebra of $H^*(G)$ generated by SWCs $w_i(\pi)$ of orthogonal representations π of G. We can define a weaker form of detection:

Definition 2.9. We say H *detects* SWCs of G provided the restriction map i^* is injective on $H^*_{\mathrm{SW}}(G)$, meaning

$$i^* : H^*_{\mathrm{SW}}(G) \hookrightarrow H^*(H).$$

Moreover it is easy to see that the image

$$\mathrm{Im}(i^*) \subseteq H^*(H)^{N_G(H)}, \tag{2.14}$$

where $N_G(H)$ is the normalizer of H in G.

Such detection results for G are very useful in calculating its SWCs. Let π be an orthogonal representation of G, and let H be a *nice* subgroup. (By “nice”, we mean the

SWCs of representations of H are well understood.) By the naturality of SWCs, we have

$$i^*(w(\pi)) = w(\pi \circ i) = w(\operatorname{res}_H^G \pi).$$

($\operatorname{res}_H^G \pi$, also denoted by $\pi|_H$, means the restriction of π to H.)

Therefore when H is a detecting subgroup of G, we can identify $w(\pi)$ with $w(\operatorname{res}_H^G \pi)$. We will write $w^H(\pi)$ for the image $i^*(w(\pi))$. An instance of detection is:

Lemma 2.10 ([1], Chapter II, Corollary 5.2)**.** *Let H be a subgroup that contains a Sylow 2-subgroup of G. Then, H detects the mod 2 cohomology of G.*

2.3.2 External Tensor Products

Let G_1, G_2 be two finite groups. Given orthogonal G_i-representations (π_i, V_i), their external tensor product $(\pi_1 \boxtimes \pi_2, V_1 \otimes V_2)$ is also orthogonal (from Section 2.1.2). In this section we define $w(\pi_1 \boxtimes \pi_2)$ and give its description in terms of SWCs of π_1 and π_2. Please see [22] for details.

To orthogonal representations π_1, π_2 correspond the real representations $(\pi_1^{\mathbb{R}}, V_1^{\mathbb{R}})$, $(\pi_2^{\mathbb{R}}, V_2^{\mathbb{R}})$ respectively, which are unique up to isomorphism. (Here $\pi_i^{\mathbb{R}} = (\pi_i)_0$ and $V_i^{\mathbb{R}} = (V_i)_0$ in the sense of Proposition 2.2.) Associated to $\pi_i^{\mathbb{R}}$ are the vector bundles

$$E_i = EG_i[V_i^{\mathbb{R}}] \xrightarrow{\Pi_i} BG_i.$$

Now we consider the projection maps $p_i : BG_1 \times BG_2 \to BG_i$ and let $p_i^* E_i$ be the pullback of E_i by p_i consisting of elements $((b_1, b_2), e_i) \in (BG_1 \times BG_2) \times E_i$ such that

$$\Pi_i(e_i) = p_i(b_1, b_2) = b_i.$$

We will have the following commutative diagram:

$$\begin{tikzcd}
p_i^* E_i \arrow[r, "p_i^E"] \arrow[d, "p_i^B"] & E_i \arrow[d, "\Pi_i"] \\
BG_1 \times BG_2 \arrow[r, "p_i"] & BG_i
\end{tikzcd}$$

where p_i^B and p_i^E are the projections from $p_i^* E_i$ onto $BG_1 \times BG_2$ and E_i respectively:

$$p_i^B((b_1, b_2), e_i) = (b_1, b_2)$$
$$p_i^E((b_1, b_2), e_i) = e_i.$$

This way $p_1^* E_1$ and $p_2^* E_2$ are vector bundles over the same base space $BG_1 \times BG_2$, and we can construct their internal tensor product. Put

$$E_1 \boxtimes E_2 := p_1^* E_1 \otimes p_2^* E_2,$$

which is again a vector bundle over $BG_1 \times BG_2$. In fact, $E_1 \boxtimes E_2$ associates to $\pi_1 \boxtimes \pi_2$, being isomorphic to the vector bundle

$$(EG_1 \times EG_2) \times_{G_1 \times G_2} (V_1 \otimes V_2) \tag{2.15}$$

over the classifying space $B(G_1 \times G_2)$. Therefore, by $w(\pi_1 \boxtimes \pi_2)$, we mean $w(E_1 \boxtimes E_2)$.

Proposition 2.11. *Let (π_1, V_1), (π_2, V_2) be orthogonal representations of G_1, G_2 with respective degrees m, n. Then,*

$$w(\pi_1 \boxtimes \pi_2) = p_{m,n}(w_1(\pi_1), \ldots, w_m(\pi_1), w_1(\pi_2), \ldots, w_n(\pi_2)),$$

where $p_{m,n}$ is a polynomial in $m+n$ variables specified as follows. Let $\epsilon_1, \ldots, \epsilon_m$ be the elementary symmetric polynomials in indeterminates $t_1, \ldots, t_m$ and $\varepsilon_1, \ldots, \varepsilon_n$ be the elementary symmetric polynomials in $s_1, \ldots, s_n$. Then,

$$p_{m,n}(\epsilon_1, \ldots, \epsilon_m, \varepsilon_1, \ldots, \varepsilon_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + t_i + s_j). \tag{2.16}$$

Proof. We understand

$$w(\pi_1 \boxtimes \pi_2) = w(E_1 \boxtimes E_2) = w(p_1^* E_1 \otimes p_2^* E_2),$$

and use [22, Chapter 7, Problem 7-C] for the total SWC of an internal tensor product.

We get the polynomial $p_{m,n}$ defined in Equation (2.16) with

$$\begin{aligned} w(p_1^*E_1 \otimes p_2^*E_2) &= p_{m,n}(w_1(p_1^*E_1), \dots, w_m(p_1^*E_1), w_1(p_2^*E_2), \dots, w_n(p_2^*E_2)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(as an element of } H^*(G_1 \times G_2)) \\ &= p_{m,n}(w_1(E_1), \dots, w_m(E_1), w_1(E_2), \dots, w_n(E_2)) \\ &\qquad\qquad\qquad\qquad\qquad \text{(as an element of } H^*(G_1) \otimes H^*(G_2)) \\ &= p_{m,n}(w_1(\pi_1), \dots, w_m(\pi_1), w_1(\pi_2), \dots, w_n(\pi_2)), \end{aligned}$$

with the understanding that any product of $w_i(\pi_1)$ and $w_j(\pi_2)$ for $1 \leq i \leq m$, $1 \leq j \leq n$ appearing in the polynomial $p_{m,n}$ is actually their cross product (in the sense of [22, Appendix A]). □

Corollary 2.11.1. *Let π_1, π_2 be as above with $w(\pi_2) = 1$. Then, we have*

$$w(\pi_1 \boxtimes \pi_2) = w(\pi_1)^n.$$

In this equality $w(\pi_1)^n \in H^*(G_1 \times G_2)$ when thought as $p_1^*(w(\pi_1))^n$.

Proof. Given that $w(\pi_2) = 1$ means $w_i(\pi_2) = 0$ for all $i > 0$. This means $\varepsilon_j = 0$ for all $1 \leq j \leq n$ in Equation (2.16), which implies $s_j(w_1(\pi_2), w_2(\pi_2), \dots) = 0$ for all j. The double product thus simplifies

$$\begin{aligned} \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + t_i + s_j) &= \prod_{j=1}^{n} \Big(\prod_{i=1}^{m} (1 + t_i) \Big) \\ &= \prod_{j=1}^{n} (1 + \epsilon_1 + \dots + \epsilon_m), \end{aligned}$$

and we obtain

$$\begin{aligned} w(\pi_1 \boxtimes \pi_2) &= \prod_{j=1}^{n} (1 + w_1(\pi_1) + \dots + w_m(\pi_1)) \\ &= \prod_{j=1}^{n} w(\pi_1) \\ &= w(\pi_1)^n. \end{aligned}$$

□

Remark 2.12. *Similarly, we have:*
Let π_1, π_2 be as in Proposition 2.11 above with $w(\pi_1) = 1$. Then,

$$w(\pi_1 \boxtimes \pi_2) = w(\pi_2)^m.$$

2.3.3 Cyclic Groups

Let n be even, and $G = C_n$, the cyclic group of order n. Let g be a generator of G. We write $g^{n/2} = -1$, the unique order 2 element of G.

Let ψ be a linear character of G. We call ψ *quadratic* provided $\psi^2 = 1$. Also we say ψ is *even* if $\psi(-1) = 1$, and *odd* if $\psi(-1) = -1$. We put $\epsilon_\psi = 0$ when ψ is even, and $\epsilon_\psi = 1$ when ψ is odd. Note that ψ is non-orthogonal if and only if non-quadratic.

Let $\psi_\bullet$ be the linear character of G with $\psi_\bullet(g) = e^{\frac{2\pi i}{n}}$. Then $\psi_\bullet^{n/2}$ is the linear character of order 2 with $\psi_\bullet^{n/2}(g) = -1$. We write $\psi_\bullet^{n/2} = \mathrm{Sgn}$ when $n \equiv 0 \pmod 4$, and $\psi_\bullet^{n/2} = \mathrm{sgn}$ when $n \equiv 2 \pmod 4$. Both 'Sgn' and 'sgn' denote the unique non-trivial quadratic character of G.

It is known [18] that

$$H^*(C_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z}[s,t]/(s^2), & n \equiv 0 \pmod 4 \\ \mathbb{Z}/2\mathbb{Z}[v], & n \equiv 2 \pmod 4 \end{cases}$$

where $s = w_1(\mathrm{Sgn})$, $t = w_2(S(\psi_\bullet))$ for $n \equiv 0 \pmod 4$, and $v = w_1(\mathrm{sgn})$ when $n \equiv 2 \pmod 4$.

Being so simple, the cyclic case is our first example of expressing SWCs in terms of character values.

When $n \equiv 2 \pmod 4$, C_2 is the Sylow 2-subgroup of G and therefore detects the mod 2 cohomology of G due to Lemma 2.10. (Note that every representation of C_2 is orthogonal.) Such a detection is not available to us when $n \equiv 0 \pmod 4$. So we first find the SWCs of $S(\psi)$ for non-orthogonal linear characters ψ in this case.

Lemma 2.13. *Let $n \equiv 0 \pmod 4$. Let ψ be a non-quadratic linear character of G. Then we have*

$$w(S(\psi)) = 1 + \epsilon_\psi t.$$

Again this proof requires Proposition 2.16 from Section 2.4.1 on Chern classes.

Proof. We can write $\psi = \psi_\bullet^j$ for some $1 \leq j < n$ and $j \neq n/2$. From Proposition 2.16, we have

$$\begin{aligned} w(S(\psi)) &= \kappa(c(\psi)) \\ &= \kappa(1 + c_1(\psi_\bullet^j)) \\ &= \kappa(1 + jc_1(\psi_\bullet)) \\ &= 1 + jw_2(S(\psi_\bullet)) \\ &= 1 + jt. \end{aligned}$$

Now $j \equiv \epsilon_\psi \pmod 2$ because ψ is even if and only if j is even, which completes the proof. $\square$

We now calculate SWCs for any orthogonal π of G:

Proposition 2.14. *Let π be an orthogonal representation of G. Put $b_\pi = \frac{1}{2}(\deg \pi - \chi_\pi(-1))$.*

(i) *If $n \equiv 2 \pmod 4$, then*
$$w^{C_2}(\pi) = (1+v)^{b_\pi}.$$

(ii) *If $n \equiv 0 \pmod 4$, then*
$$w(\pi) = (1 + \delta_\pi s)(1+t)^{b_\pi/2},$$
where
$$\delta_\pi = \begin{cases} 0, & \det \pi = 1 \\ 1, & \det \pi = -1. \end{cases}$$

Proof. Let $n \equiv 2 \pmod 4$. Since C_2 is the detecting subgroup, it is enough to work with $\operatorname{res}^G_{C_2} \pi$. This restriction looks like

$$\operatorname{res}^G_{C_2} \pi \cong a1 \oplus b(\mathrm{sgn})$$

for non-negative integers a, b. From the multiplicativity of SWCs, we obtain

$$w^{C_2}(\pi) = w(\operatorname{res}^G_{C_2} \pi) = (1+v)^b.$$

To express b in the character values, consider the equations:

$$\chi_\pi(1) = a + b$$
$$\chi_\pi(-1) = a - b$$

giving $b = \frac{1}{2}(\chi_\pi(1) - \chi_\pi(-1)) = b_\pi$.

Let $n \equiv 0 \pmod 4$. An orthogonal representation π of C_n has the form

$$\pi = a1 \oplus b(\mathrm{Sgn}) \oplus \bigoplus_{\substack{\{\psi,\psi^{-1}\} \\ \psi^2 \neq 1}} m_\psi S(\psi),$$

where a, b, m_ψ are all non-negative integers. The multiplicativity of SWCs along with Lemma 2.13 leads to

$$w(\pi) = (1+s)^b \prod_{\substack{\{\psi,\psi^{-1}\} \\ \psi^2 \neq 1}} (1 + \epsilon_\psi t)^{m_\psi}.$$

Let us take

$$m_0 = \sum_{\substack{\{\psi,\psi^{-1}\} \\ \psi^2 \neq 1,\ \epsilon_\psi = 0}} m_\psi, \quad \text{and} \quad m_1 = \sum_{\substack{\{\psi,\psi^{-1}\} \\ \psi^2 \neq 1,\ \epsilon_\psi = 1}} m_\psi.$$

The expression for $w(\pi)$ then reduces to

$$w(\pi) = (1 + bs)(1+t)^{m_1}.$$

We solve the following equations to express m_1 in terms of character values:

$$\chi_\pi(1) = a + b + 2m_0 + 2m_1$$
$$\chi_\pi(-1) = a + b + 2m_0 - 2m_1.$$

This gives $m_1 = b_\pi/2$. Moreover $\det \pi = 1$ if and only if $w_1(\pi) = 0$, which happens if and only if b is even. Therefore $\delta_\pi \equiv b \pmod 2$, and we get

$$w(\pi) = \begin{cases} (1+t)^{b_\pi/2}, & \det \pi = 1 \\ (1+s)(1+t)^{b_\pi/2}, & \det \pi = -1. \end{cases}$$

$\square$

Let C_n^r be the r-fold product of C_n, with projection maps $\mathrm{pr}_i : C_n^r \to C_n$ for $1 \leq i \leq r$. By Künneth, we have

$$H^*(C_n^r) = \begin{cases} \mathbb{Z}/2\mathbb{Z}[s_1, \ldots, s_r, t_1, \ldots, t_r]/(s_1^2, \ldots, s_r^2), & n \equiv 0 \pmod 4 \\ \mathbb{Z}/2\mathbb{Z}[v_1, \ldots, v_r], & n \equiv 2 \pmod 4 \end{cases}$$

where we put $s_i = w_1(\mathrm{Sgn} \circ \mathrm{pr}_i)$ and $t_i = w_2(S(\psi_\bullet) \circ \mathrm{pr}_i)$ for $n \equiv 0 \pmod 4$, and $v_i = w_1(\mathrm{sgn} \circ \mathrm{pr}_i)$, for $n \equiv 2 \pmod 4$. (Here $1 \leq i \leq r$.)

2.3.4 Steenrod Squares

For $n, i \geq 0$, there are operations on cohomology, called *Steenrod Squares*

$$\mathrm{Sq}^i : H^n(G) \to H^{n+i}(G),$$

which can be characterized axiomatically from the following properties:

(i) These are additive homomorphisms and Sq^0 is the identity.

(ii) Steenrod operations are *functorial*, meaning for a group homomorphism $\varphi : G \to G'$, we have
$$\varphi^*(\mathrm{Sq}^i\, y) = \mathrm{Sq}^i(\varphi^*(y)) \text{ for all } y \in H^i(G').$$

(iii) They satisfy $\mathrm{Sq}^i(x) = x \cup x$ for $i = \deg(x)$, and $\mathrm{Sq}^i(x) = 0$ for $i > \deg(x)$.

(iv) For $x, y \in H^*(G)$,
$$\mathrm{Sq}^n(x \cup y) = \sum_{i+j=n} (\mathrm{Sq}^i\, x) \cup (\mathrm{Sq}^j\, y).$$

This is famously known as *Cartan Formula.*

We now state the well-known *Wu formula*:

Proposition 2.15 ([21], Chapter 23, Section 6)**.** *Let π be an orthogonal representation of G. Then, the cohomology class $\mathrm{Sq}^i(w_j(\pi))$ can be expressed as a polynomial in $w_1(\pi), \ldots, w_{i+j}(\pi)$:*

$$\mathrm{Sq}^i(w_j(\pi)) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t}(\pi) w_{j+t}(\pi).$$

For instance, with $i = 1$, $j = 2$ in the formula above, we can express $w_3(\pi)$ in terms of w_1, w_2 as follows:

$$w_3(\pi) = w_1(\pi) \cup w_2(\pi) + \mathrm{Sq}^1(w_2(\pi)). \tag{2.17}$$

In particular if $w_1(\pi) = w_2(\pi) = 0$ for some orthogonal π, then $w_3(\pi) = 0$.

2.3.5 SWCs for Virtual Representations

Let G be a finite group. A virtual representation of G can be thought as a difference $\pi = \pi_1 \ominus \pi_2$, for representations π_1, π_2. When the π_i are orthogonal, one may define the SWC of π as

$$w(\pi) = w(\pi_1) \cup w(\pi_2)^{-1},$$

but we must "complete" the cohomology ring so that the inversion makes sense.

More formally, let $\mathrm{RO}(G)$ be the free abelian group on the isomorphism classes of OIRs (following [3, Chapter II, Section 7]). The members of $\mathrm{RO}(G)$ are called *virtual orthogonal representations* of G. Let $\mathrm{RO}^+(G)$ be the set of (isomorphism classes of) orthogonal representations of G. The total SWC may be regarded as a map $w : \mathrm{RO}^+(G) \to H^*(G)$.

Let $\widehat{H}^*(G)$ be the complete cohomology ring

$$\widehat{H}^*(G) = \prod_i H^i(G),$$

consisting of all formal infinite series $\alpha_0 + \alpha_1 + \cdots$, with $\alpha_i \in H^i(G)$. (Please see [26, page 44].) Each $w(\pi)$ is invertible in this ring. It is now clear that we may "extend" w to a group homomorphism to the units of $\widehat{H}^*(G)$, i.e.,

$$w : \mathrm{RO}(G) \to \widehat{H}^*(G)^\times. \tag{2.18}$$

We find the image of this map w for certain groups in the later chapters.

2.4 Other Characteristic Classes

This section quickly reviews *Chern classes* and *symplectic classes* for a finite group G. We are particularly interested in their relation to SWCs. The significance of such a relationship has already been seen in proving Lemmas 2.7 and 2.13.

We also mention one more characteristic class, called the *Euler class*, and see how it relates to the "top SWC". This class detects the non-triviality of tangent bundle of spheres. Please refer to [22] for details.

2.4.1 Chern Classes

Associated to a complex representation π of G are the cohomology classes

$$c_i(\pi) \in H^{2i}(G, \mathbb{Z}),$$

known as *Chern classes*. Their sum

$$c(\pi) = c_0(\pi) + c_1(\pi) + c_2(\pi) + \ldots \in H^*(G, \mathbb{Z})$$

is called the *total Chern class* of π. As with SWCs, we have $c_0(\pi) = 1$, and $c_i(\pi) = 0$ for $i > \deg(\pi)$. The first Chern class, applied to linear characters, gives the well-known isomorphism

$$c_1 : \mathrm{Hom}(G, S^1) \xrightarrow{\cong} H^2(G, \mathbb{Z}).$$

More generally, $c_1(\pi) = c_1(\det \pi)$ for a complex representation π.

These classes are also *functorial*, meaning for a group homomorphism $\varphi : G_1 \to G_2$ and a complex representation π of G_2, we have

$$\varphi^*_{\mathbb{Z}}(c(\pi)) = c(\pi \circ \varphi),$$

where $\varphi^*_{\mathbb{Z}} : H^*(G_2, \mathbb{Z}) \to H^*(G_1, \mathbb{Z})$ is the map induced by φ.

Chern classes are *multiplicative* too. For π_1, π_2 complex, we have

$$c(\pi_1 \oplus \pi_2) = c(\pi_1) \cup c(\pi_2).$$

Now consider the coefficient map of cohomology from Section 2.2:

$$\kappa : H^*(G, \mathbb{Z}) \to H^*(G, \mathbb{Z}/2\mathbb{Z}). \tag{2.19}$$

For a complex representation π, we understand $S(\pi)_0 \cong r^{\mathbb{C}}_{\mathbb{R}}\pi$ from (2.6). Then [22], Problem 14-B] gives:

Proposition 2.16. *For π complex, we have*

$$\kappa(c(\pi)) = w(S(\pi)).$$

External Tensor Products. Let G_1, G_2 be finite groups, and let π_i be complex representations of G_i. Then the external tensor product $\pi_1 \boxtimes \pi_2$ is a complex representation of $G_1 \times G_2$. One can think of the vector bundle associated to $\pi_1 \boxtimes \pi_2$ as an internal tensor product of "pullback bundles" associated to π_i. This way $c(\pi_1 \boxtimes \pi_2)$ can be defined in the sense of [15] or [22]. (Please see Section 2.3.2 for an analogous construction for real vector bundles.)

When π_1, π_2 both have degree one, we have

$$c(\pi_1 \boxtimes \pi_2) = 1 + c_1(\pi_1) + c_2(\pi_2). \tag{2.20}$$

(See [15, Section 3.1] for instance.)

More generally, there is an easily comprehensible description for the total Chern class $c(\pi_1 \boxtimes \pi_2)$ when π_1, π_2 are direct sum of degree one complex representations. Suppose

$$\pi_1 = \bigoplus_{i=1}^{m} \phi_i \quad \text{and} \quad \pi_2 = \bigoplus_{j=1}^{n} \psi_j$$

where ϕ_i, ψ_j have degrees one for all i, j.

Proposition 2.17. *For π_1, π_2 as above, we have*

$$c(\pi_1 \boxtimes \pi_2) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + c_1(\phi_i) + c_1(\psi_j)).$$

Proof. It follows from the multiplicativity of Chern classes and Equation (2.20). □

General explicit formulas can be quite tedious, but the Splitting principle [15, Section 3.1] establishes that:
For complex representations π_1, π_2 with m, n as their respective degrees, there is a polynomial P in Chern classes of π_1, π_2 such that

$$c(\pi_1 \boxtimes \pi_2) = P(c_1(\pi_1), \ldots, c_m(\pi_1), c_1(\pi_2), \ldots, c_n(\pi_2)). \tag{2.21}$$

2.4.2 Euler Class

Let π be an n-dimensional orthogonal representation of G with $w_1(\pi) = 0$. To such π is associated a cohomology class

$$e(\pi) \in H^n(G, \mathbb{Z}),$$

called the *Euler class* of π. The "top SWC" $w_n(\pi)$ is the reduction of $e(\pi)$ mod 2. That is:

$$\kappa(e(\pi)) = w_n(\pi)$$

where κ is the coefficient map of cohomology from (2.19).

In Chapter 4 we determine which representations of special linear groups $\mathrm{SL}(2, q)$ have non-trivial mod 2 Euler class.

2.4.3 Symplectic Classes

Let ϖ be a quaternionic representation of G, meaning $\varpi : G \to \mathrm{Sp}(W)$ where W is an $\mathbb{H}$-module. Associated to ϖ are cohomology classes

$$k_i^{\mathbb{H}}(\varpi) \in H^{4i}(G, \mathbb{Z})$$

called *symplectic classes*. (Please see [4, Chapter 4] for details.)

Given (ϖ, W), there exists a complex representation $(\varpi_{\mathbb{C}}, W_{\mathbb{C}})$ when W is considered as a $\mathbb{C}$-vector space by restricting scalars from $\mathbb{H}$ to $\mathbb{C}$. Furthermore $\varpi_{\mathbb{C}}$ is symplectic and unique up to equivalence. In fact, every complex symplectic representation comes from a unique quaternionic representation. (Please see [3, Chapter II, Section 6] for proofs.) Moreover from [4, Chapter 4, Corollary 4.2], we have

$$\begin{aligned} c_i(\varpi_{\mathbb{C}}) &= 0 \text{ if } i \text{ is odd}, \\ c_{2i}(\varpi_{\mathbb{C}}) &= (-1)^i k_i^{\mathbb{H}}(\varpi). \end{aligned} \tag{2.22}$$

These facts together allow us to have the following definition:

Definition 2.18. Let π be a symplectic complex representation of G. Then, we put

$$k_i(\pi) := (-1)^i c_{2i}(\pi), \quad ; \quad i = 0, 1, 2, \dots,$$

These we call the *symplectic classes* of π, and their sum

$$k(\pi) = k_0(\pi) + k_1(\pi) + k_2(\pi) + \ldots \in H^*(G, \mathbb{Z})$$

is the *total symplectic class* of π. Again $k_0(\pi) = 1$ and $k_i(\pi) = 0$ for $i > \deg(\pi)$.

Lemma 2.19. *For π symplectic, we have*

$$w_i(S(\pi)) = \begin{cases} \kappa(k_m(\pi)), & \textit{when } i = 4m \\ 0, & \textit{otherwise} . \end{cases}$$

Proof. The odd SWCs of $S(\pi)$ vanish for any complex π. If π is symplectic, then we can use Equation (2.22) in $w(S(\pi)) = \kappa(c(\pi))$ and by comparison of degrees, we obtain

$$\begin{aligned} w_{4m+2}(S(\pi)) &= \kappa(c_{2m+1}(\pi)) = 0, \\ w_{4m}(S(\pi)) &= \kappa(c_{2m}(\pi)) = \kappa(k_m(\pi)) \end{aligned}$$

for $m \geq 0$. Therefore $w_i(S(\pi)) = 0$ unless $4|i$ and $w_i(S(\pi)) = \kappa(k_m(\pi))$ when $i = 4m$, as desired. $\square$

For the special linear groups $\mathrm{SL}(2, q)$ when q is odd, almost all the representations that matter to us will be $S(\pi)$ with π symplectic. Therefore, the theory of symplectic classes shows that we will only have SWCs in degrees divisible by 4. Ultimately we will see this directly.

3

Quaternion Groups

The quaternion group Q is an order 8, non-abelian group with the familiar presentation

$$Q = \langle i, j \mid i^2 = j^2,\ i^4 = 1,\ jij^{-1} = i^{-1} \rangle.$$

This chapter reviews the group cohomology of Q, and the so-called "generalized quaternions" Q_{2^n}. These groups play an important role in the detection theorem and the calculations of SWCs for the special linear groups $\mathrm{SL}(2,q)$. Here we also determine some SWCs of orthogonal representations of Q.

3.1 Character table of Q

Let $[Q,Q]$ be the derived subgroup of Q, which is $\{\pm 1\}$. Then the quotient $Q/[Q,Q]$ is isomorphic to the Klein-4 group $C_2 \times C_2$. Therefore there are exactly 4 one-dimensional representations of Q and are all orthogonal because such representations factor through $Q/[Q,Q]$. We denote them by 1, χ_1, χ_2, $\chi_3 = \chi_1 \otimes \chi_2$.

The group Q also possesses a unique 2-dimensional irreducible representation

$$\rho : Q \to \mathrm{SL}(2, \mathbb{C})$$

defined by

$$\rho(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \rho(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\mathbb{H}$ be the usual division algebra of quaternions over real numbers. It can be represented as the algebra of complex matrices of the form

$$\begin{pmatrix} s+ti & u+vi \\ -u+vi & s-ti \end{pmatrix} \text{ where } s,t,u,v \in \mathbb{R}.$$

Since both $\rho(a)$, $\rho(b)$ are of this form, ρ is an injection of Q into $\mathbb{H}^1$, the subgroup of norm 1 real quaternions. Therefore, ρ is symplectic. This makes $S(\rho) = \rho \oplus \rho$ the only OIR of Q, which is not irreducible.

The character table of Q is now given below (with $k = ij$).

Q	1	-1	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
1	1	1	1	1	1
χ_1	1	1	-1	1	-1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_ρ	2	-2	0	0	0

Table 3.1: Character Table of Q

3.2 The Cohomology ring $H^*(Q)$

The SWCs of orthogonal representations of Q are certain elements in $H^*(Q)$. Here we describe the cohomology ring $H^*(Q)$ with [1, Chapter IV] as our reference.

Recall that w_1 gives an isomorphism between $\mathrm{Hom}(Q, \pm 1) = \{1, \chi_1, \chi_2, \chi_3\}$ and $H^1(Q)$. We define

$$x := w_1(\chi_1), \text{ and}$$
$$y := w_1(\chi_2).$$

The cohomology group $H^4(Q)$ is one-dimensional over $\mathbb{Z}/2\mathbb{Z}$, and we write 'e' for its nonzero element.

Proposition 3.1 ([1], Section V.1)**.** *The cohomology ring of Q is*

$$H^*(Q) \cong \mathbb{Z}/2\mathbb{Z}[x, y, e]/(xy + x^2 + y^2, x^2y + xy^2).$$

The first few cohomology groups of Q are as follows:

$$\begin{aligned} H^0(Q) &= \{0, 1\} \\ H^1(Q) &= \{0, x, y, x + y\} \\ H^2(Q) &= \{0, x^2, y^2, x^2 + y^2\} \\ H^3(Q) &= \{0, x^2y\} \\ H^4(Q) &= \{0, e\}. \end{aligned}$$

Note that $x^3 = y^3 = 0$.

Definition 3.2. A finite group G is called *periodic* with period $n > 0$, provided that

$$H^i(G, \mathbb{Z}) \cong H^{i+n}(G, \mathbb{Z}) \text{ for all } i \geq 1$$

where the G-action on $\mathbb{Z}$ is trivial.

In fact, G is periodic only if its $\mathbb{Z}/p\mathbb{Z}$-cohomology is periodic for all p. (See [1, Section IV.6] for proof.) The quaternion group Q is periodic with period 4. Therefore the higher cohomology groups are obtained by the cup product with e. This means the map

$$\begin{aligned} H^i(Q) &\to H^{i+4}(Q) \\ z &\mapsto z \cup e \end{aligned}$$

is an isomorphism of groups for $i \geq 1$.

3.3 SWCs of Representations of Q

We first compute the total SWCs of OIRs of Q. For the linear orthogonal representations of Q, it is clear that $w(1) = 1$, $w(\chi_1) = 1+x$, $w(\chi_2) = 1+y$, and $w(\chi_3) = 1+w_1(\chi_1 \otimes \chi_2) = 1 + x + y$ with $x, y \in H^1(Q)$ from above.

Lemma 3.3. *The total SWC of $S(\rho)$ is*

$$w(S(\rho)) = 1 + e,$$

where e is the non-trivial element of $H^4(Q)$.

Proof. From Lemma 2.7, we have

$$w_i(S(\rho)) = 0 \text{ for } i = 1, 2, 3.$$

This gives $w(S(\rho)) = 1 + w_4(S(\rho))$. We now prove that $w_4(S(\rho)) = e \in H^4(Q)$.

Let Z be the center of Q, which is $\{\pm 1\}$. Since

$$\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

we have $\operatorname{res}^Q_Z \rho = \operatorname{sgn} \oplus \operatorname{sgn}$, where $\operatorname{sgn} : Z \to \mathbb{C}^\times$ is the linear character of order 2.

From Section 2.3.3, we have $H^*(Z) \cong \mathbb{Z}/2\mathbb{Z}[v]$ where $v = w_1(\operatorname{sgn})$. Then, the multiplicativity of SWCs gives

$$w(\operatorname{res}^Q_Z S(\rho)) = w(\operatorname{sgn})^4 = (1 + v)^4 = 1 + v^4,$$

which implies $w_4(S(\rho)) \neq 0$. But the only non-trivial element in $H^4(Q)$ is e. Therefore, $w_4(S(\rho)) = e$. □

From the above proof, we note that the restriction map, induced by the inclusion of Z into Q, is an isomorphism on $H^4(Q)$. The periodicity of Q then gives

$$\begin{aligned} \operatorname{res}^* : H^{4i}(Q) &\to H^{4i}(Z) \\ e^i &\mapsto v^{4i}. \end{aligned} \tag{3.1}$$

is an isomorphism of groups for all $i \geq 0$.

Let π be an orthogonal representation of Q. From (2.7), we can write

$$\pi \cong m_0 1 \oplus m_1 \chi_1 \oplus m_2 \chi_2 \oplus m_3 \chi_3 \oplus m_4 S(\rho),$$

where m_i are non-negative integers.

Lemma 3.4. *Let π be as above. Then, for $0 \leq 4i \leq \deg \pi$, we have*

$$w_{4i}(\pi) = \binom{m_4}{i} e^i,$$

where $m_4 = \frac{1}{8}(\chi_\pi(1) - \chi_\pi(-1))$.

Proof. Let us consider

$$S = \mathbb{Z}/2\mathbb{Z}[x, y]/(xy + x^2 + y^2, x^2y + xy^2),$$

which is a 6-dimensional subalgebra of $H^*(Q)$. By the multiplicativity of SWCs, we have

$$\begin{aligned} w(\pi) &= w(\chi_1)^{m_1} \cup w(\chi_2)^{m_2} \cup w(\chi_3)^{m_3} \cup w(S(\rho))^{m_4} \\ &= \underbrace{(1+x)^{m_1}(1+y)^{m_2}(1+x+y)^{m_3}}_{\in S}(1+e)^{m_4}. \end{aligned}$$

Therefore we can write $(1+x)^{m_1}(1+y)^{m_2}(1+x+y)^{m_3}$ as a polynomial of the form $P(x, y) = 1 + Ax + By + Cx^2 + Dy^2 + Exy^2$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$, and

$$\begin{aligned} w(\pi) &= P(x, y)(1+e)^{m_4} \\ &= P(x, y) \sum_{i=0}^{m_4} \binom{m_4}{i} e^i \end{aligned}$$

where $e^i \in H^{4i}(Q)$ for each i. From the comparison of degrees, we obtain

$$w_{4i} = \binom{m_4}{i} e^i.$$

To express m_4 in terms of character values, we evaluate χ_π at 1 and -1 using the character table 3.1 of Q:

$$\begin{aligned} \chi_\pi(1) &= m_0 + m_1 + m_2 + m_3 + 4m_4, \\ \chi_\pi(-1) &= m_0 + m_1 + m_2 + m_3 - 4m_4. \end{aligned}$$

and so $m_4 = \frac{1}{8}(\chi_\pi(1) - \chi_\pi(-1))$. □

3.4 Generalized Quaternions

The construction of the quaternion group Q generalizes to give a family of non-abelian groups which have the presentation

$$Q_{2^n} := \langle a, b \mid a^{2^{n-2}} = b^2, b^4 = 1, bab^{-1} = a^{-1} \rangle \ ; \ n \geq 3.$$

These groups are called *generalized quaternion groups* and have order 2^n.

We have the derived subgroup $[Q_{2^n}, Q_{2^n}] = \langle a^2 \rangle$ and, the quotient $Q_{2^n}/[Q_{2^n}, Q_{2^n}]$ is isomorphic to $C_2 \times C_2$. Again there are 4 linear (orthogonal) representations of Q_{2^n}, say 1, ψ_1, ψ_2, ψ_3. These can be defined on the generators of Q_{2^n} as:

$$\begin{aligned} \psi_1(a) &= -1 \quad , \quad \psi_1(b) = \ \ 1 \\ \psi_2(a) &= 1 \quad \ \ , \quad \psi_2(b) = -1 \\ \psi_3(a) &= -1 \quad , \quad \psi_3(b) = -1. \end{aligned} \tag{3.2}$$

Let $\zeta = e^{2\pi i/2^{n-1}}$. There is an irreducible 2-dimensional representation of Q_{2^n}:

$$\varrho : Q_{2^n} \to \mathrm{SL}(2, \mathbb{C})$$

defined by

$$\varrho(a) = \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix} \quad , \quad \varrho(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As before, ϱ maps Q_{2^n} into $\mathbb{H}^1$, so it is a symplectic representation.

Let $n > 3$. We use the isomorphism

$$\begin{aligned} w_1 : \mathrm{Hom}(Q_{2^n}, \pm 1) &\xrightarrow{\cong} H^1(Q_{2^n}) \\ \{1, \psi_1, \psi_2, \psi_3\} &\leftrightarrow \{0, X, Y, X+Y\} \end{aligned}$$

to define $X = w_1(\psi_1)$ and $Y = w_1(\psi_3)$.

[1, Chapter IV, Lemma 2.11] describes the mod 2 cohomology ring of generalized quaternions Q_{2^n} for $n > 3$ as follows: With non-zero $E \in H^4(Q_{2^n})$,

$$H^*(Q_{2^n}) \cong \mathbb{Z}/2\mathbb{Z}[X, Y, E]/(XY, X^3 + Y^3). \tag{3.3}$$

Lemma 3.5. *We have* $E = w_4(S(\varrho))$.

Proof. This can be proved precisely as in Lemma 3.3; again $w(S(\varrho)) = 1 + E$. $\square$

Remark 3.6. *Let* $n > 3$. *Let* $i : Q \to Q_{2^n}$ *be any group homomorphism. Then,*

$$i^* : H^3(Q_{2^n}) \to H^3(Q)$$

is the zero map.

Proof. From (3.3), we can deduce $H^3(Q_{2^n}) = \{0, X^3\}$ and since i^* is a ring homomorphism, we have the image $i^*(X) \in H^1(Q) = \{0, x, y, x+y\}$. But $x^3 = y^3 = (x+y)^3 = 0$ in $H^*(Q)$. This implies

$$i^*(X^3) = (i^*(X))^3 = 0.$$

$\square$

Consider $Q^{(1)} = \langle a^{2^{n-3}}, b\rangle \leqslant Q^{2^n}$. This subgroup is isomorphic to Q by $i \leftrightarrow a^{2^{n-3}}$ and $j \leftrightarrow b$. With this identification, and

$$\begin{aligned} \varrho(a^{2^{n-3}}) &= \rho(i), \\ \varrho(b) &= \rho(j) \end{aligned}$$

we have $\operatorname{res}^{Q_{2^n}}_{Q} \varrho = \rho$. Write ι_Q for this inclusion of Q into Q_{2^n}. Now

$$\begin{aligned} \iota_Q^*(E) &= \iota_Q^*(w_4(\varrho)) \\ &= w_4(\operatorname{res}^{Q_{2^n}}_{Q} \varrho) \\ &= w_4(\rho) \\ &= e. \end{aligned}$$

Also, Q_{2^n} is 4-periodic (see [1, Chapter IV, 2.10-2.12]) which finally leads to:

Proposition 3.7. *With notations as above, we have isomorphisms*

$$\begin{aligned} \iota_Q^* : H^{4i}(Q_{2^n}) &\to H^{4i}(Q) \\ E^i &\mapsto e^i \end{aligned} \tag{3.4}$$

for all $i \geq 0$.

4

Special Linear Group $\mathrm{SL}(2, q)$

Let p be a prime and $q = p^r$. Let $\mathbb{F}_q$ be the finite field with q elements. Let

$$G = \mathrm{SL}(2, q) := \{A \in \mathrm{GL}(2, q) : \det(A) = 1\}.$$

This chapter is dedicated to calculating the SWCs of the orthogonal representations of these special linear groups. We deal with these groups in two cases: (i) when q is odd, and (ii) when q is even.

4.1 Case of q odd

Let $G = \mathrm{SL}(2, q)$ with q odd throughout this section. We begin with a description of the mod 2 cohomology of G.

Proposition 4.1 ([7], Chapter VI, Sec. 5)**.** *Let q be odd. The group $\mathrm{SL}(2, q)$ is periodic with period 4 and its mod 2 cohomology ring is*

$$H^*(\mathrm{SL}(2, q)) \cong \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}] \otimes \mathbb{Z}/2\mathbb{Z}[\mathfrak{b}]/(\mathfrak{b}^2)$$

with $\deg(\mathfrak{b}) = 3$, $\deg(\mathfrak{e}) = 4$.

4.1.1 Detection

Write '$\mathbb{1}$' for the identity matrix in G. Let Z be the center of G which is $\{\pm\mathbb{1}\}$. We have:

Theorem 4.2. *The center Z detects SWCs of G.*

To prove this theorem, we require the following result: Let $n = \operatorname{ord}_2 |G|$, meaning n is the largest integer such that 2^n divides $|G|$.

Lemma 4.3 ([7], Chapter VI, Lemma 5.1)**.** *A Sylow 2-subgroup of G is isomorphic to the generalized quaternion group Q_{2^n}.*

Proof of Theorem 4.2. We deduce from Proposition 4.1 that

$$H^m(G) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & m \equiv 0,3 \pmod 4 \\ 0, & \text{otherwise.} \end{cases}$$

Let π be an orthogonal representation of G. Let $m = 4k+3$ for some k. With $i = 1$, $j = m-1$ such that $i + j = m$, we apply Wu formula from Proposition 2.15:

$$\mathrm{Sq}^1(w_{m-1}(\pi)) = w_1(\pi) \cup w_{m-1}(\pi) + \binom{m-2}{1} w_0(\pi) \cup w_m(\pi).$$

Here, $\mathrm{Sq}^1(w_{m-1}(\pi)) = 0$ because $w_{m-1}(\pi) \in H^{4k+2}(G)$ is zero and Sq^1 is a homomorphism. Also, $w_1(\pi) = 0$ and $(m-2)$ is odd which implies

$$w_m(\pi) = 0 \text{ when } m \equiv 3 \pmod 4.$$

Therefore, the non-zero SWCs of representations of G can occur only in degrees that are multiples of 4, which means

$$H^*_{\mathrm{SW}}(G) \subseteq \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}].$$

Since Q_{2^n} is a Sylow 2-subgroup of G, it detects the mod-2 cohomology of G due to Lemma 2.10. Consider the quaternion subgroup $Q \cong Q^{(1)} \leqslant Q_{2^n}$. Note that the center of Q is Z. By the isomorphisms (3.1) and (3.4), we obtain the following sequence of inclusions for each $i \geq 0$:

$$\begin{array}{ccccccc} H^{4i}(G) & \hookrightarrow & H^{4i}(Q_{2^n}) & \hookrightarrow & H^{4i}(Q) & \hookrightarrow & H^{4i}(Z) \\ \mathfrak{e}^i & \mapsto & E^i & \mapsto & e^i & \mapsto & v^{4i}. \end{array}$$

Therefore, the subalgebra $\mathbb{Z}/2\mathbb{Z}[\mathfrak{e}] \leqslant H^*(G)$, containing the SWCs of G, injects into $H^*(Z)$, which completes the proof. □

Let T_1, T_2 be the maximal split and elliptic tori of G respectively. That is:

$$T_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_q^\times \right\} \cong C_{q-1},$$

and with a chosen generator ϵ of the cyclic group $\mathbb{F}_q^\times$,

$$T_2 = \left\{ \begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix} : x^2 - \epsilon y^2 = 1 \right\} \cong C_{q+1}.$$

Corollary 4.2.1. *Both T_1 and T_2 detect SWCs of G.*

Proof. We have the inclusions $Z \hookrightarrow T_i \hookrightarrow G$ such that the composition

$$H^*_{\mathrm{SW}}(G) \to H^*(T_i) \to H^*(Z)$$

is injective by Theorem 4.2. Hence, the restriction maps $H^*_{\mathrm{SW}}(G) \to H^*(T_i)$ must be injective for $i = 1, 2$. □

4.1.2 Formulas for SWCs

We now give an explicit formula for the total SWCs of orthogonal representations of SL$(2, q)$ in terms of character values.

Theorem 4.4. *Let $G = \mathrm{SL}(2, q)$ with q odd. Let π be an orthogonal representation of G. Then the total SWC of π is,*

$$w(\pi) = (1 + \mathfrak{e})^{r_\pi} \tag{4.1}$$

where $\mathfrak{e}$ is the non-zero element in $H^4(G)$, and $r_\pi = \frac{1}{8}(\chi_\pi(\mathbb{1}) - \chi_\pi(-\mathbb{1}))$.

Proof. To find $w(\pi)$, it is enough to work with $\mathrm{res}_Z^G\,\pi$ due to the detection Theorem 4.2. We do this restriction in two steps. We first restrict π to the quaternion subgroup $Q \leqslant G$, and then further from Q to Z. With notations from Section 3.1, we can write

$$\mathrm{res}_Q^G\,\pi \cong m_0 1 \oplus m_1\chi_1 \oplus m_2\chi_2 \oplus m_3\chi_3 \oplus m_4(S(\rho))$$

where m_i are non-negative integers. From the character table (3.1) of Q, we can see that

$$\begin{aligned} \mathrm{res}_Z^Q\,\chi_i &= 1 \text{ for } i = 1,2,3 \\ \mathrm{res}_Z^Q\,\rho &= \mathrm{sgn} \oplus \mathrm{sgn}\,. \end{aligned} \tag{4.2}$$

Therefore, the further restriction of $\mathrm{res}_Q^G\,\pi$ to the center Z will be

$$\mathrm{res}_Z^G\,\pi = \mathrm{res}_Z^Q\,\mathrm{res}_Q^G\,\pi \cong (m_0 + m_1 + m_2 + m_3)1 \oplus 4m_4(\mathrm{sgn}).$$

By Proposition 2.14, we then obtain

$$w^Z(\pi) = (1+v)^{4m_4} = (1+v^4)^{m_4},$$

with $4m_4 = \frac{1}{2}(\chi_\pi(\mathbb{1}) - \chi_\pi(-\mathbb{1}))$. From the proof of Theorem 4.2, we have $i_Z^*(\mathfrak{e}) = v^4$ where i_Z is the inclusion of Z into G. Therefore,

$$w(\pi) = (1+\mathfrak{e})^{r_\pi},$$

where $r_\pi = m_4$ is the multiplicity of $S(\rho)$ in $\mathrm{res}_Q^G\,\pi$. □

From the proof above, it is clear that if an orthogonal representation π of G has $w(\pi) \neq 1$, then $\mathrm{res}_Q^G\,\pi$ must have $S(\rho)$ as a component.

We now quote a result due to R. Gow, which helps in extracting more information about the SWCs of irreducible orthogonal representations of $\mathrm{SL}(2,q)$.

Theorem 4.5 ([13], Theorem 1)**.** *Let $G = \mathrm{SL}(2,q)$ with q odd. Let π be an irreducible self-dual representation of G with central character ω_π. Then, the Frobenius-Schur Indicator $\varepsilon(\pi)$ equals $\omega_\pi(-\mathbb{1})$.*

We simply call this equality *Gow's formula.*

This leads to the following:

Corollary 4.4.1. *Let π be an irreducible orthogonal representation of G. Then, its total SWC $w(\pi) = 1$.*

Proof. For π irreducible orthogonal, we have $\varepsilon(\pi) = 1$ due to Equation (2.1). Therefore $r_\pi = 0$ for all such π by Gow's formula:

$$\begin{aligned} r_\pi &= \frac{1}{8}(\chi_\pi(\mathbb{1}) - \chi_\pi(-\mathbb{1})) \\ &= \frac{\chi_\pi(\mathbb{1})}{8}(1 - \omega_\pi(-\mathbb{1})) \\ &= 0. \end{aligned}$$

□

Let π be an irreducible, non-orthogonal representation of G. Then, for the representation $S(\pi) = \pi \oplus \pi^\vee$, we have

$$\begin{aligned} r_{S(\pi)} &= \frac{1}{8}\Big(\chi_{S(\pi)}(\mathbb{1}) - \chi_{S(\pi)}(-\mathbb{1})\Big) \\ &= \frac{1}{4}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(-\mathbb{1})\Big) \\ &= \frac{\chi_\pi(\mathbb{1})}{4}\Big(1 - \omega_\pi(-\mathbb{1})\Big). \end{aligned}$$

Furthermore, for symplectic π, it turns out to be

$$r_{S(\pi)} = \frac{\chi_\pi(\mathbb{1})}{2} = \frac{\deg(\pi)}{2} \tag{4.3}$$

due to Gow's formula.

Corollary 4.4.2. *Let $G = \mathrm{SL}(2, q)$ with q odd. Then the image of w in (2.18) is*

$$\{(1 + \mathfrak{e})^n \mid n \in \mathbb{Z}\}.$$

Proof. For $q = 3$, there exists a unique irreducible symplectic representation π_0 of G with degree 2. From Equation (4.3), we obtain $r_{S(\pi_0)} = 1$, and so $w(S(\pi_0)) = 1 + \mathfrak{e}$.

When $q > 3$, there exist irreducible symplectic representations π_1 and π_2 of G of degrees $q + 1$ and $q - 1$ respectively. For convenience, we recall the construction of these principal series and cuspidal representations. Please refer to [9] and [5] for an explicit description of all the irreducible representations of G.

Let $\widetilde{T}$ be the diagonal subgroup, isomorphic to $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$, contained in the standard Borel subgroup $\widetilde{B}$ of upper-triangular matrices of $\widetilde{G} = \mathrm{GL}(2,q)$. When α, β are linear characters of $\mathbb{F}_q^\times$, we will write $\alpha \boxtimes \beta$ for the corresponding linear character of $\widetilde{T}$. Choose $\alpha : \mathbb{F}_q^\times \to \mathbb{C}^\times$ satisfying

$$\alpha(-1) = -1$$
$$\alpha^2 \neq 1.$$

One can inflate $\alpha \boxtimes 1$ from $\widetilde{T}$ to $\widetilde{B}$ and then consider the usual complex parabolic induction $\pi_\alpha := \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}(\alpha \boxtimes 1)$. This is an irreducible principal series representation of $\widetilde{G}$ of degree $q+1$.

We take $\pi_1 = \mathrm{res}_G^{\widetilde{G}} \pi_\alpha$. This restriction is self-dual and irreducible. By Gow's formula, π_1 is symplectic.

Let $\widetilde{T}_e$ be an elliptic torus of $\widetilde{G}$, thus isomorphic to $\mathbb{F}_{q^2}^\times$. Let $\widetilde{Z}$ be the center and N be the upper unitriangular subgroup of $\widetilde{G}$. Choose a linear character χ of $\widetilde{T}_e$ such that

$$\chi^q \neq \chi, \;\; \chi^2 \neq 1$$
$$\chi(-1) = -1.$$

We fix a nontrivial character φ of N, and define a linear character of $\widetilde{Z}N$ as $\chi_\varphi(zn) = \chi(z)\varphi(n)$. Set

$$\pi_\chi = \mathrm{Ind}_{\widetilde{Z}N}^{\widetilde{G}} \chi_\varphi - \mathrm{Ind}_{\widetilde{T}_e}^{\widetilde{G}} \chi.$$

This is an irreducible, cuspidal representation of $\widetilde{G}$ of dimension $q-1$. When restricted to G, it remains irreducible.

Define $\pi_2 = \mathrm{res}_G^{\widetilde{G}} \pi_\chi$. Again one sees that π_2 is symplectic by Gow's formula .
From Equation (4.3), we have

$$r_{S(\pi_1)} = \frac{q+1}{2} \;,\; r_{S(\pi_2)} = \frac{q-1}{2}$$

which are co-prime. So by Bézout's Identity, there exist integers a, b such that $a\left(\frac{q+1}{2}\right) + b\left(\frac{q-1}{2}\right) = 1$.

Therefore, there is a virtual representation $\pi \in \mathrm{RO}(G)$ with $r_\pi = 1$ such that

$$w(\pi) = 1 + \mathfrak{e}.$$

Hence, the result follows from the multiplicativity for SWCs. $\square$

It is already known from Theorem 4.2 that $H^*_{\mathrm{SW}}(G) \subseteq \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}]$. Here we make a stronger statement:

Corollary 4.4.3. *Let $G = \mathrm{SL}(2,q)$ with q odd. Then,*

$$H^*_{\mathrm{SW}}(G) = \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}].$$

Proof. For equality, we construct an orthogonal representation η of G such that $w_4(\eta) = \mathfrak{e}$:

When $q > 3$, we consider the irreducible symplectic representations π_1, π_2 of G from Corollary 4.4.2. Let $\eta = S(\pi_1) \oplus S(\pi_2)$. It can easily seen that $r_{\pi\oplus\pi'} = r_\pi + r'_\pi$ for any orthogonal π, π'. Therefore, we have

$$r_\eta = r_{S(\pi_1)\oplus S(\pi_2)} = r_{S(\pi_1)} + r_{S(\pi_2)} = \frac{q+1}{2} + \frac{q-1}{2} = q.$$

Since q is odd, we have

$$w(\eta) = (1+\mathfrak{e})^q = 1 + \mathfrak{e} + \dots.$$

Therefore, $w_4(\eta) = \mathfrak{e}$.

Moreover, we already have $w_4(S(\pi_0)) = \mathfrak{e}$ for $q = 3$ which completes the proof. $\square$

Recall that for π orthogonal with $\det \pi = 1$, the mod 2 Euler class of π is the top SWC $w_{\deg \pi}(\pi)$. Our next result provides a nonvanishing condition for the top SWC.

Corollary 4.4.4. *Let π be an orthogonal representation of G. Then $w_{\deg \pi}(\pi)$ is non-zero if and only if $\pi(-\mathbb{1}) = -1$, meaning $\pi(-\mathbb{1})$ acts by the scalar -1.*

Proof. We want to show that $w_{\deg \pi}(\pi) \neq 0$ if and only if $\pi = S(\varphi)$, where every irreducible constituent φ_i of φ has central character satisfying $\omega_{\varphi_i}(-\mathbb{1}) = -1$.

We first assume $\pi = S(\varphi) = \bigoplus\limits_{i=1}^{m} S(\varphi_i)$ is such that $\omega_{\varphi_i}(-\mathbb{1}) = -1$ for all $1 \leq i \leq m$.

Clearly $\chi_\pi(\mathbb{1}) = 2\chi_\varphi(\mathbb{1})$, and since $\chi_{\varphi_i}(-\mathbb{1}) = \omega_{\varphi_i}(-\mathbb{1})\chi_{\varphi_i}(\mathbb{1}) = -\chi_{\varphi_i}(\mathbb{1})$, we have

$$\begin{aligned}\chi_\pi(-\mathbb{1}) &= 2\sum_{i=1}^{m}\chi_{\varphi_i}(-\mathbb{1})\\ &= -2\sum_{i=1}^{m}\chi_{\varphi_i}(\mathbb{1})\\ &= -2\chi_\varphi(\mathbb{1}).\end{aligned}$$

This gives

$$r_\pi = \frac{1}{2}\chi_\varphi(\mathbb{1}) = \frac{1}{4}\chi_\pi(\mathbb{1}).$$

Therefore we have $w(\pi) = (1+\mathfrak{e})^{\frac{1}{4}\deg\pi}$, implying

$$w_{\deg\pi}(\pi) = \mathfrak{e}^{\frac{1}{4}\deg\pi} \neq 0.$$

For the converse, suppose $w_{\deg\pi}(\pi) \neq 0$. Being orthogonal, we can write π as

$$\pi = \bigoplus_i \rho_i \oplus \bigoplus_j S(\phi_j) \oplus \bigoplus_k S(\varphi_k)$$

such that each ρ_i is irreducible orthogonal, whereas ϕ_j, φ_k are irreducible non-orthogonal with $\omega_{\phi_j}(-\mathbb{1}) = 1$ for each j, and $\omega_{\varphi_k}(-\mathbb{1}) = -1$ for each k.
From Theorem 4.4 and Corollary 4.4.1, we obtain

$$\begin{aligned}w(\pi) &= \prod_k (1+\mathfrak{e})^{\frac{\deg\varphi_k}{2}}\\ &= (1+\mathfrak{e})^{\frac{\deg\varphi}{2}}, \text{ where } \varphi = \bigoplus_k \varphi_k.\end{aligned}$$

Now, the condition $w_{\deg\pi}(\pi) \neq 0$ implies $\deg\pi = 4\cdot\frac{\deg\varphi}{2}$. Therefore,

$$\begin{aligned}\sum_i \deg\rho_i + 2\sum_j \deg\phi_j + 2\sum_k \deg\varphi_k &= 2\deg\varphi\\ \sum_i \deg\rho_i + 2\sum_j \deg\phi_j &= 0,\end{aligned}$$

which means ρ_i and $S(\phi_j)$ don't appear in π. Hence $\pi = \bigoplus_i S(\varphi_i)$, with each φ_i irreducible and $\omega_{\varphi_i}(-\mathbb{1}) = -1$. $\square$

For π orthogonal, let k_0 be the least $k > 0$ such that $w_k(\pi) \neq 0$. Then $w_{k_0}(\pi)$ is known as the *obstruction class* of π, following [10].

Corollary 4.4.5. *Let π be an orthogonal representation of G. Put $t = \mathrm{ord}_2(r_\pi)$. Then the obstruction class of π is $w_{2^{t+2}}(\pi) = \mathfrak{e}^{2^t}$.*

Proof. By Theorem 4.4,

$$w_k(\pi) = \begin{cases} \binom{r_\pi}{i}\mathfrak{e}^i, & 0 \leq k = 4i \leq \deg \pi \\ 0, & \text{otherwise.} \end{cases}$$

A consequence of Lucas Theorem ([8]) is: $\binom{r_\pi}{a} \equiv 0 \pmod 2$ for $a < 2^t$. Thus $k_0 \geq 4 \cdot 2^t$.

The result [10, Proposition 3] says that : For a non-negative integer n, $\mathrm{ord}_2(n) = k$ if and only if $\binom{n}{2^0}, \binom{n}{2^1}, \ldots, \binom{n}{2^{k-1}}$ are all even, but $\binom{n}{2^k}$ is odd.

Therefore, $\binom{r_\pi}{2^t}$ is odd, which means $w_k(\pi) \neq 0$ for $k = 4 \cdot 2^t$, giving the corollary. □

4.2 Case of q even

Let $G = \mathrm{SL}(2, q)$ with $q = 2^r$ for this section. Consider its subgroup of upper unitriangular matrices

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\} \cong (\mathbb{F}_q, +).$$

It is easy to see that N is a Sylow 2-subgroup of G. Therefore N detects the mod 2 cohomology of G due to Lemma 2.10.

4.2.1 Formulas for SWCs

The subgroup N is an elementary abelian 2-group of rank r. So by Section 2.3.3, the mod 2 cohomology ring of N is

$$H^*(N) \cong H^*(C_2^r) \cong \mathbb{Z}/2\mathbb{Z}[v_1, v_2, \ldots, v_r].$$

Let T be the subgroup of diagonal matrices in G. That is

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_q^\times \right\}.$$

Consider the set

$$\widehat{N} := \{\chi : N \to \mathbb{C}^\times \text{ is a group homomorphism}\}.$$

This is in fact an abelian group, called the *character group* of N. More generally one can have a character group of any abelian group.

The diagonal subgroup T acts on $\widehat{N}$ via conjugation:

$$\begin{aligned} T \times \widehat{N} &\to \widehat{N} \\ (t, \chi) &\mapsto {}^t\chi \end{aligned}$$

where ${}^t\chi : n \mapsto \chi(tnt^{-1})$ for all $n \in N$.

The conjugation action of T on N is equivalent to the action of $\mathbb{F}_q^\times$ on $\mathbb{F}_q$ through multiplication by squares. Since $\mathbb{F}_q^\times$ has odd order, this action is transitive on $\mathbb{F}_q - \{0\}$. An isomorphism between $\mathbb{F}_q$ and $\widehat{\mathbb{F}}_q$ then leads to:

Lemma 4.6. *T acts transitively on the non-trivial linear characters of N.*

From the Lemma, the T-orbits of $\widehat{N}$ are: $\{1\}$, $\{\chi : \chi \neq 1\}$.

Let π be an orthogonal representation of G. In fact, all representations of G are orthogonal by the main result in [25]. Now to find $w(\pi)$, it is enough to work with

$$w(\operatorname{res}_N^G \pi) \in H^*(N)$$

due to detection by N. Since π is T-invariant, so is $\operatorname{res}_N^G \pi$. Therefore, it is of the form

$$\begin{aligned} \operatorname{res}_N^G \pi &\cong \ell_\pi 1 \oplus m_\pi \Big(\bigoplus_{\chi \neq 1} \chi \Big) \\ &= (\ell_\pi - m_\pi) 1 \oplus m_\pi \operatorname{reg}(N) \end{aligned} \tag{4.4}$$

where ℓ_π, m_π are non-negative integers and $\operatorname{reg}(N)$ is the regular representation of N.

Lemma 4.7. *Let π be a non-trivial irreducible representation of G. Then, $m_\pi = 1$.*

Proof. For $r \geq 2$, it is known that G has no non-trivial normal subgroups. For non-trivial π, we must have $m_\pi \geq 1$ because $\ker(\pi) = \mathbb{1}$ and $N \nleq \ker(\pi)$. Now from (4.4), we have

$$\deg \pi = \ell_\pi + m_\pi(q-1).$$

If $m_\pi > 1$, then the sum $\ell_\pi + m_\pi(q-1)$ would be greater than the highest possible degree for irreducible representations of G, which is $(q+1)$. That gives a contradiction. Hence, m_π must be 1.

For $r = 1$, we have $\mathrm{SL}(2,2) \cong S_3$ with only two non-trivial irreducible representations. That is π is either the 'sgn' representation or the 2-dimensional standard representation of S_3. Here we can see from direct calculations that $m_\pi = 1$. $\square$

It follows that $m_\pi = \dim_{\mathbb{C}}(V/V^G)$, where V denotes the representation space of π and V^G is the G-fixed vectors in V. Also it is the number of non-trivial irreducible constituents in π.

We now describe the SWCs of representations of G. Since N is a detecting subgroup, we may and will identify $w(\pi)$ with its image in $H^*(N)$.

Set $n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in N$.

Theorem 4.8. *Let $q = 2^r$. Let π be a representation of* $\mathrm{SL}(2,q)$*. Then, the total SWC of π is*

$$w(\pi) = \Big(\prod_{v \in H^1(N)} (1+v)\Big)^{m_\pi},$$

with $m_\pi = \frac{1}{q}(\chi_\pi(\mathbb{1}) - \chi_\pi(n_0))$.

Proof. The restriction $\mathrm{res}_N^G \pi$, from (4.4), is of the form

$$\mathrm{res}_N^G \pi \cong (\ell_\pi - m_\pi)1 \oplus m_\pi \,\mathrm{reg}(N).$$

Since $\widehat{N} = \mathrm{Hom}(N, \pm 1)$ and $\mathrm{reg}(N) = \sum_{\chi \in \widehat{N}} \chi$, we use the isomorphism between $\widehat{N}$ and $H^1(N)$ to have

$$w(\mathrm{reg}(N)) = \prod_{v \in H^1(N)} (1+v).$$

Again by the multiplicativity of SWCs, we obtain

$$\begin{aligned} w^N(\pi) &= w(\mathrm{reg}(N))^{m_\pi} \\ &= \Big(\prod_{v\in H^1(N)}(1+v)\Big)^{m_\pi}. \end{aligned}$$

To get the character formula for m_π, we have the following equations:

$$\begin{aligned} \chi_\pi(\mathbb{1}) &= \ell_\pi + (2^r-1)m_\pi \\ \chi_\pi(n_0) &= \ell_\pi - m_\pi. \end{aligned}$$

Now, the result follows. □

The expansion of the product above is well-known. We have

$$\prod_{v\in H^1(N)}(1+v) = 1 + \sum_{i=0}^{r-1} d_{r,i}(\bar{v}) \in H^*(N), \tag{4.5}$$

where $d_{r,i}(\bar{v})$ are *Dickson invariants* in the generators $\{v_1, v_2, \dots, v_r\}$ of polynomial algebra $H^*(N)$. (We use shorthand $d_{r,i}(\bar{v})$ for $d_{r,i}(v_1,\dots,v_r)$.)

We digress for a moment to recount the theory of Dickson invariants for mod 2 spaces. Let E be an $\mathbb{F}_2$-vector space. The ring of polynomials from E to $\mathbb{F}_2$ can be identified with the symmetric algebra $S[E^\vee]$ on the dual space $E^\vee$. The linear group $\mathrm{GL}(E)$ acts on $S[E^\vee]$ via the contragradient map. It is natural to look for the invariants under the action.

Theorem 4.9 ([27]). *Suppose* $\dim(E) = r$. *Then the ring of invariants* $S[E^\vee]^{\mathrm{GL}(E)}$ *is a polynomial algebra generated by elements* $d_{r,i}$ *called Dickson invariants, for* $0 \le i < r$. *The polynomials* $\{d_{r,i}\}$ *have degrees* $\{2^r - 2^i\}$. *Moreover,*

$$\prod_{v\in E^\vee}(1+v) = 1 + \sum_{i=0}^{r-1} d_{r,i} \in S[E^\vee].$$

These invariants are described explicitly in terms of certain determinants. (See [27], or [1, Chapter III] for instance.)

Let us illustrate with some examples. If $E = \mathbb{F}_2$, then $S[E^\vee]$ is a polynomial algebra with only one generator v, that is $S[E^\vee] \cong \mathbb{Z}/2\mathbb{Z}[v]$. Here we have $d_{1,0}(v) = v$, and the ring of invariants

$$S[E^\vee]^{\mathrm{GL}(E)} \cong \mathbb{Z}/2\mathbb{Z}[v]^{\mathrm{GL}(1,2)} = \mathbb{Z}/2\mathbb{Z}[v].$$

Theorem 4.8 for $G = \mathrm{SL}(2,2)$ and $\pi \neq 1$ irreducible gives

$$w(\pi) = 1 + v = 1 + d_{1,0}(v).$$

Next, suppose $\dim E = 2$. Then $S[E^\vee]$ is a polynomial algebra with two generators, say v_1, v_2 such that $S[E^\vee] \cong \mathbb{Z}/2\mathbb{Z}[v_1, v_2]$. Here the ring of invariants $S[E^\vee]^{\mathrm{GL}(E)} \cong \mathbb{Z}/2\mathbb{Z}[v_1, v_2]^{\mathrm{GL}(2,2)}$ is generated by

$$\begin{aligned} d_{2,1}(\bar{v}) &= v_1^2 + v_2^2 + v_1 v_2, \\ d_{2,0}(\bar{v}) &= v_1 v_2 (v_1 + v_2). \end{aligned}$$

Theorem 4.8 says that: For $G = \mathrm{SL}(2,4)$, the total SWC of a non-trivial irreducible representation π of G is

$$w(\pi) = 1 + d_{2,1}(\bar{v}) + d_{2,0}(\bar{v}).$$

Now we let $d_r(\bar{v}) = \sum_{i=0}^{r-1} d_{r,i}(\bar{v})$ be the sum of Dickson invariants, so that we can succinctly write

$$w(\pi) = (1 + d_r(\bar{v})^{m_\pi}.$$

Corollary 4.8.1. Let $G = \mathrm{SL}(2,q)$ with even q. Then the image of w in (2.18) is

$$\{(1 + d_r(\bar{v}))^n : n \in \mathbb{Z}\}.$$

Proof. Let π be a non-trivial irreducible representation of G. Then, its total SWC is

$$w(\pi) = 1 + d_r(\bar{v}),$$

since $m_\pi = 1$ from Lemma 4.7. With the virtual representation $n\pi$, we obtain $(1 + d_r(\bar{v}))^n$ in the image of w for each $n \in \mathbb{Z}$. □

The above proof also gives:

Corollary 4.8.2. Let $G = \mathrm{SL}(2,q)$ with $q = 2^r$. Then,

$$H^*_{\mathrm{SW}}(G) = \mathbb{Z}/2\mathbb{Z}[d_{r,0}(\bar{v}), \ldots, d_{r,r-1}(\bar{v})].$$

We now describe the representations of $\mathrm{SL}(2,q)$ which have non-zero top SWC in the case of q even.

Corollary 4.8.3. Let $G = \mathrm{SL}(2,q)$ with $q = 2^r$. Let π be a representation of G. Then, $w_{\deg \pi}(\pi) \neq 0$ if and only if π is cuspidal.

Proof. For trivial π, it is obvious that $w_{\deg \pi}(\pi) = 0$. Suppose π is non-trivial irreducible. Then,

$$w(\pi) = 1 + \sum_{i=0}^{r-1} d_{r,i}(\bar{v})$$

with $\deg(d_{r,i}) = 2^r - 2^i$. Therefore, $w_{\deg \pi}(\pi) \neq 0$ if and only if $\deg \pi = 2^r - 2^0 = q - 1$. We know that only cuspidal irreducible representations of G have degrees $q-1$.

Let us suppose $\pi = \bigoplus_{i=1}^{m} \pi_i$ with each π an irreducible cuspidal representation of G. We have, $\deg \pi = m(q-1)$. From Theorem 4.8, we obtain

$$w_{\deg \pi}(\pi) = d_{r,0}^m(\bar{v}) \neq 0.$$

Conversely, let π be a representation of G with $w_{\deg \pi}(\pi) \neq 0$. From Theorem 4.8, we can deduce that the largest k such that $w_k(\pi) \neq 0$ is always $m_\pi(q-1)$, where m_π is the number of non-trivial irreducible constituents of π. Therefore, $\deg \pi$ must be $m_\pi(q-1)$ for $w_{\deg \pi}(\pi) \neq 0$.

Suppose at least one of its irreducible constituents of π is not cuspidal. The other possible degrees for such a constituent are 1, q or $(q+1)$. Let a, b, c, d be the number of constituents in π with degrees 1, $(q-1)$, q and $(q+1)$ respectively. Then, the condition $\deg \pi = m_\pi(q-1)$ implies

$$a + b(q-1) + cq + d(q+1) = (b+c+d)(q-1) = m_\pi(q-1).$$

This gives $a + 2d + c = 0$. Therefore the above condition holds only if $b = m_\pi$ and $a = c = d = 0$. Hence, the result follows. $\square$

Corollary 4.8.4. Let $G = \mathrm{SL}(2,q)$ with $q = 2^r$. Let π be a representation of G. Put $s = \mathrm{ord}_2(m_\pi)$. Then, the obstruction class of π is equal to $w_{2^{r+s-1}}(\pi) = d_{r,r-1}^{2^s}(\bar{v})$.

Proof. From Theorem 4.8, we have

$$w(\pi) = \sum_{i=0}^{m_\pi} \binom{m_\pi}{i} d_r^i(\bar{v}).$$

As in Corollary 4.4.5, we can say that $\binom{m_\pi}{2^s}$ is the first odd binomial coefficient appearing in the above sum. By expanding the term

$$\binom{m_\pi}{2^s} d_r^{2^s}(\bar{v}) = \binom{m_\pi}{2^s} (d_{r,r-1}^{2^s}(\bar{v}) + \ldots + d_{r,0}^{2^s}(\bar{v})),$$

we can deduce $\binom{m_\pi}{2^s} d_{r,r-1}^{2^s}(\bar{v})$ has the least degree, which is $(2^{r-1} \cdot 2^s)$. Therefore, the least $k > 0$ such that $w_k(\pi) \neq 0$ is 2^{r+s-1} as claimed. $\square$

5

Symplectic groups $\mathrm{Sp}(2n, q)$

Let q be an odd prime power. Let V be a $2n$-dimensional symplectic vector space over $\mathbb{F}_q$ with Ω a non-degenerate skew-symmetric bilinear form. Then the symplectic group $\mathrm{Sp}(V)$ is defined as the group of $\mathbb{F}_q$-linear transformations of V that preserve Ω.

$$\mathrm{Sp}(V) = \{g \in \mathrm{GL}(V) : \Omega(gv_1, gv_2) = \Omega(v_1, v_2) \text{ for all } v_1, v_2 \in V\}.$$

Being a symplectic space, V possesses a symplectic basis. We can choose this basis to be $\mathcal{B} = \{e_1, e_2 \ldots, e_n, f_n, f_{n-1} \ldots, f_1\}$ so that the matrix for Ω with respect to $\mathcal{B}$ is

$$J = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ -1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Upon fixing the basis, we can think of $\mathrm{Sp}(V)$ as the group of $2n \times 2n$ symplectic matrices over $\mathbb{F}_q$ denoted by $\mathrm{Sp}(2n, q)$. That is:

$$\mathrm{Sp}(2n, q) = \{A \in \mathrm{GL}(2n, q) : A^t J A = J\}.$$

We have found the SWCs for $\mathrm{SL}(2,q)$ in the previous chapter, which is the simplest symplectic group with $n=1$. Here we generalize its detection and SWC formula to all symplectic groups $\mathrm{Sp}(2n,q)$ with $n \geq 1$.

5.1 Some Subgroups

Let q be odd, and $G = \mathrm{Sp}(2n,q)$ from this point. Write 'Id' for the identity map. In this section, we discuss some subgroups of G which appear in the detection results.

5.1.1 The Direct product $\mathrm{SL}(2,q)^n$

We begin by considering the subspaces H_i of V spanned by $\{e_i, f_i\}$ for each $1 \leq i \leq n$. Then V has the orthogonal decomposition $V = \mathbb{F}_q^{2n} = H_1 \oplus \ldots \oplus H_n$. Also $\mathrm{Sp}(H_i)$ is the group of isometries of $(H_i, \Omega|_{H_i})$.

We define
$$X = \bigcap_{i=1}^{n} \mathrm{Stab}_G(H_i),$$
where $\mathrm{Stab}_G(H_i) = \{g \in \mathrm{Sp}(V) : g(H_i) \subseteq H_i\}$ for each i. Consider the following homomorphism by restriction:

$$\begin{aligned} \mathrm{res} : X &\to \mathrm{Sp}(H_1) \times \mathrm{Sp}(H_2) \times \ldots \times \mathrm{Sp}(H_n) \\ g &\mapsto (g|_{H_1}, g|_{H_2}, \ldots, g|_{H_n}). \end{aligned}$$

Suppose $g \in \mathrm{Sp}(V)$ is such that $g|_{H_i} = \mathrm{Id}$ for each i. Then g must be the identity on V due to the decomposition $V = \bigoplus_{i=1}^{n} H_i$. This says the map 'res' is injective.

We can also see that 'res' is surjective as follows. Given $(g_1, \ldots, g_n) \in \mathrm{Sp}(H_1) \times \ldots \times \mathrm{Sp}(H_n)$, we construct $g \in \mathrm{Sp}(V)$ by defining it on $v = h_1 + \ldots + h_n \in V$ as,

$$g(v) = \sum_{i=1}^{n} g_i(h_i).$$

For every $h_i \in H_i$, we have $g(h_i) = g_i(h_i) \in H_i$. This implies $g \in \mathrm{Stab}_G(H_i)$ for all i, and therefore $g \in X$.

This shows that res is a group isomorphism.

Moreover, if we define

$$X_i = \{g \in X : g|_{H_j} = \mathrm{Id} \text{ for all } j \neq i\},$$

then it is easy to see that $X_i \cong \mathrm{Sp}(H_i)$ for each $1 \leq i \leq n$. Therefore,

$$X \cong X_1 \times \ldots \times X_n \cong \mathrm{Sp}(H_1) \times \ldots \times \mathrm{Sp}(H_n).$$

In terms of our symplectic basis $\mathcal{B}$, X is the subgroup of matrices in G, all of whose nonzero entries lie either on the diagonal or the antidiagonal. That is

$$X = \begin{pmatrix} \square & 0 & 0 & & \ldots & & 0 & 0 & \square \\ 0 & \ddots & & & & & & \iddots & 0 \\ 0 & & \square & & & & \square & & 0 \\ \vdots & & \vdots & & \boxed{}_{2\times 2} & & \vdots & & \vdots \\ 0 & & \square & & & & \square & & 0 \\ 0 & \iddots & & & & & & \ddots & 0 \\ \square & 0 & 0 & & \ldots & & 0 & 0 & \square \end{pmatrix}.$$

Note that X is isomorphic to the direct product of n copies of $\mathrm{SL}(2, q)$.

5.1.2 Symmetric group S_n

Let W, W' be the subspaces of V defined as:

$$W = \mathrm{Span}_{\mathbb{F}_q}\{e_1, e_2, \ldots, e_n\},$$
$$W' = \mathrm{Span}_{\mathbb{F}_q}\{f_1, f_2, \ldots, f_n\}.$$

We can express V as their direct sum (this sum is not orthogonal). Also both W, W' are maximal isotropic, meaning $\Omega|_W = \Omega|_{W'} = 0$.

We define

$$Y = \mathrm{Stab}_G(W) \cap \mathrm{Stab}_G(W').$$

Consider $\Omega|_{W\times W'} : W \times W' \to \mathbb{F}_q$. Being non-degenerate, this map is a perfect pairing. Therefore W' is isomorphic to the dual space W^*, which in turn gives $\mathrm{GL}(W') \cong \mathrm{GL}(W^*)$. This leads to:

Given $h \in \mathrm{GL}(W)$, there is a unique $h' \in \mathrm{GL}(W')$ defined via

$$\Omega(hw, w') = \Omega(w, h'w') \;\text{ for all }\; w \in W,\; w' \in W'.$$

Let $g = h \oplus h^* \in \mathrm{GL}(V)$ where $h^* = (h')^{-1}$. We can check that $g \in \mathrm{Sp}(V)$ as follows. Consider $v_1, v_2 \in V$. We can write $v_i = w_i + w_i'$ such that $w_i \in W$ and $w_i' \in W'$. Then, we have

$$\begin{aligned}
\Omega(gv_1, gv_2) &= \Omega(g(w_1 + w_1'), g(w_2 + w_2')) \\
&= \Omega(hw_1 + h^*w_1', hw_2 + h^*w_2') \\
&= \Omega(hw_1, (h')^{-1}w_2') + \Omega((h')^{-1}w_1', hw_2) \\
&= \Omega(h^{-1}hw_1, w_2') + \Omega(w_1', h^{-1}hw_2) \\
&= \Omega(w_1, w_2') + \Omega(w_1', w_2) \\
&= \Omega(w_1 + w_1', w_2 + w_2') \\
&= \Omega(v_1, v_2).
\end{aligned}$$

This shows g preserves Ω. From the construction, it is also clear that $g \in Y$ and is unique for $h \in \mathrm{GL}(W)$. Therefore, Y is isomorphic to $\mathrm{GL}(W)$. In particular, Y contains S_n as a subgroup.

Let σ be a permutation in Y, meaning $\sigma \in S_n \leqslant \mathrm{GL}(W)$. Since Ω is preserved by $\sigma \oplus \sigma^*$, we have

$$\Omega(\sigma e_i, \sigma^* f_j) = \Omega(e_i, f_j) = \begin{cases} 0, & i \neq j \\ (-1)^{i+1}, & i = j \end{cases}.$$

Clearly $\sigma(e_i) = e_{\sigma(i)}$ for all $e_i \in W$. Also $\sigma^* f_j$ is an element of W' for each j because $\sigma \oplus \sigma^*$ stabilizes both W, W'.

Fix i and let $\sigma^* f_i = \sum_{k=1}^{n} c_k f_k$ with $c_k \in \mathbb{F}_q$.

Then, we have

$$\begin{aligned}(-1)^{i+1} = \Omega(\sigma e_i, \sigma^* f_i) &= \Omega\Big(e_{\sigma(i)}, \sum_{k=1}^{n} c_k f_k\Big) \\ &= c_k \sum_{k=1}^{n} \Omega(e_{\sigma(i)}, f_k) \\ &= c_{\sigma(i)}(-1)^{\sigma(i)+1}.\end{aligned}$$

Similarly for each $j \neq i$, we can use the equality $\Omega(\sigma e_j, \sigma^* f_i) = 0$ to finally have

$$c_k = \begin{cases} 0 & k \neq \sigma(i) \\ (-1)^{i+\sigma(i)} & k = \sigma(i) \end{cases}$$

which gives $\sigma^* f_i = (-1)^{i+\sigma(i)} f_{\sigma(i)}$ for each $1 \leq i \leq n$.

Putting all this together, we can say that σ permutes the subspaces H_i:

$$\sigma \cdot H_i = H_{\sigma(i)} \text{ for all } i,$$

implying $\sigma \cdot X = \bigcap_{i=1}^{n} \mathrm{Stab}_G(H_{\sigma(i)}) = X$.

Let Z be the center of $\mathrm{SL}(2, q)$, which is $\{\pm 1\}$. Then, the n-fold product Z^n is a normal subgroup of X. It is implied that S_n acts on Z^n by permuting and in fact S_n normalizes Z^n in G.

5.2 Detecting SWCs

Let $K = \mathrm{SL}(2, q)$ for this section. From above, there is a subgroup X of $G = \mathrm{Sp}(2n, q)$, isomorphic to the n-fold product K^n. So there are projections

$$\mathrm{pr}_j : K^n \to K \quad ; \quad 1 \leq j \leq n,$$

and by Künneth we have

$$H^*(X) \cong \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n] \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}[\mathfrak{b}_1, \ldots, \mathfrak{b}_n]/(\mathfrak{b}_1^2, \ldots, \mathfrak{b}_n^2), \tag{5.1}$$

where $\mathfrak{e}_j = \mathrm{pr}_j^*(\mathfrak{e})$ and $\mathfrak{b}_j = \mathrm{pr}_j^*(\mathfrak{b})$ with $\mathfrak{b}, \mathfrak{e}$ from Proposition 4.1.

Moreover $\mathfrak{e}_j = w_4(\eta_j)$ for each j, where η_j are described as follows:

Consider the orthogonal representation η of K, from Corollary 4.4.3, with $w_4(\eta) = \mathfrak{e}$. We define

$$\eta_j = \eta \circ \mathrm{pr}_j \quad ; \quad 1 \leq j \leq n.$$

We can also write η_j as the external tensor product:

$$\eta_j = 1 \boxtimes \cdots \boxtimes 1 \boxtimes \underbrace{\eta}_{j\text{th position}} \boxtimes 1 \boxtimes \cdots \boxtimes 1, \tag{5.2}$$

with η at jth position and 1 everywhere. This way we have $w(\eta_j) = \mathrm{pr}_j^*(w(\eta))$ from Corollary 2.11.1. Therefore,

$$w_4(\eta_j) = \mathrm{pr}_j^*(w_4(\eta)) = \mathrm{pr}_j^*(\mathfrak{e}) = \mathfrak{e}_j$$

as claimed.

There is a known detection for $G = \mathrm{Sp}(2n, q)$ when q is odd:

Lemma 5.1 ([1], Chapter VII, Lemma 6.2)**.** *For odd q, the subgroup X detects the mod 2 cohomology of G.*

Consider the center Z of $\mathrm{SL}(2, q)$. We have seen that Z^n is a subgroup of G normalized by S_n. From Section 2.3.3, the mod 2 cohomology of Z^n is

$$H^*(Z^n) \cong H^*(C_2^n) \cong \mathbb{Z}/2\mathbb{Z}[v_1, \ldots, v_n].$$

We now have the following detection for SWCs of G:

Theorem 5.2. *Let $G = \mathrm{Sp}(2n, q)$ with q odd. The subgroup Z^n detects the SWCs of G. More precisely,*

$$i_Z^* : H^*_{\mathrm{SW}}(G) \hookrightarrow \mathbb{Z}/2\mathbb{Z}[v_1^4, \ldots, v_n^4]^{S_n}$$

where i_Z is the inclusion of Z^n into G.

The proof of this theorem requires a number of results. We begin a lemma:

Lemma 5.3. *Let $K = \mathrm{SL}(2, q)$ with q odd. Let $k \equiv 0 \pmod 4$. Then, the Steenod square $\mathrm{Sq}^2(x) = 0$ for each $x \in H^k(K \times K)$.*

Proof. The description of cohomology in (5.1) gives

$$H^k(K \times K) \subseteq \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2] \text{ for all } k \equiv 0 \pmod 4.$$

From [7, Chapter VI, Proposition 5.7], whenever $i \not\equiv 0 \pmod 4$, we have

$$\mathrm{Sq}^i(\mathfrak{e}) = 0 \text{ in } H^*(K).$$

Then from the naturality of Steenrod operations, we obtain

$$\begin{aligned}\mathrm{Sq}^i(\mathfrak{e}_j) &= \mathrm{Sq}^i(\mathrm{pr}_j^* \mathfrak{e}) \\ &= \mathrm{pr}_j^*(\mathrm{Sq}^i \mathfrak{e}) \\ &= 0\end{aligned}$$

for $i = 1, 2, 3$ and $j = 1, 2$. Now by the repeated application of Cartan's formula, we can see that

$$\mathrm{Sq}^2(\mathfrak{e}_1^s \mathfrak{e}_2^t) = 0$$

for any monomial $\mathfrak{e}_1^s \mathfrak{e}_2^t \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2]$. Therefore,

$$\mathrm{Sq}^2(x) = 0 \text{ for any } x \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2],$$

and the result follows. □

Proposition 5.4. *Let $K = \mathrm{SL}(2, q)$ with q odd. We have*

$$H^*_{\mathrm{SW}}(K \times K) = \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2].$$

Proof. Let π be an orthogonal representation of $K \times K$. Again we begin by observing from (5.1) that

$$H^k(K \times K) = \{0\} \text{ for all } k \equiv 1 \pmod 4.$$

It is then enough to establish that $w_k(\pi) = 0$ whenever $k \equiv 2, 3 \pmod 4$.

Let $m \equiv 2 \pmod 4$. We use Wu formula from Proposition 2.15 with $i = 2$, $j = m - 2$ such that $i + j = m$.

This gives

$$\begin{aligned}\mathrm{Sq}^2(w_{m-2}(\pi)) &= \binom{m-5}{0} w_2(\pi)w_{m-2}(\pi) + \binom{m-4}{1} w_1(\pi)w_{m-1}(\pi) + \binom{m-3}{2} w_0(\pi)w_m(\pi) \\ &= \frac{(m-3)(m-4)}{2} w_m(\pi)\end{aligned}$$

where we have second equality because $H^i(K \times K) = \{0\}$ for $i = 1, 2$. Also $\frac{(m-3)(m-4)}{2}$ is odd for $m \equiv 2 \pmod 4$, and $\mathrm{Sq}^2(w_{m-2}(\pi)) = 0$ due to Lemma 5.3. Therefore,

$$w_m(\pi) = 0 \text{ for all } m \equiv 2 \pmod 4.$$

Let $m' \equiv 3 \pmod 4$. We use Wu formula with $i = 1$, $j = m' - 1$ such that $i + j = m'$ and as in the proof of Theorem 4.2, we obtain

$$w_{m'}(\pi) = 0 \text{ for } m' \equiv 3 \pmod 4.$$

Therefore, the non-zero SWCs of an orthogonal representation of $K \times K$ lie only in the degrees divisible by 4, which implies

$$H^*_{\mathrm{SW}}(K \times K) \subseteq \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2].$$

For the equality, we have representations η_1, η_2 of $K \times K$ with $w_4(\eta_1) = \mathfrak{e}_1$ and $w_4(\eta_2) = \mathfrak{e}_2$. □

We can generalize the above result to:

Theorem 5.5. *Let $K = \mathrm{SL}(2, q)$ with q odd. Then we have*

$$H^*_{\mathrm{SW}}(K^n) = \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n].$$

Proof. Since SWCs are multiplicative, it is enough to show that the SWCs of all OIRs of K^n lie in the subalgebra $\mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n]$. We have the description of OIRs of a direct product in Section 2.1.2.

We begin with $\varphi = \varphi_1 \boxtimes \cdots \boxtimes \varphi_n$ irreducible non-orthogonal. Then an OIR of K^n is:

$$S(\varphi) = S(\varphi_1 \boxtimes \cdots \boxtimes \varphi_n).$$

By Proposition 2.16, we have

$$\begin{aligned} w(S(\varphi)) &= \kappa(c(\varphi)) \\ &= \kappa(c(\varphi_1 \boxtimes \cdots \boxtimes \varphi_n)). \end{aligned}$$

From (2.21), we can infer that $c(\varphi_1 \boxtimes \cdots \boxtimes \varphi_n)$ is a polynomial P in the Chern classes of $\varphi_1, \ldots, \varphi_n$. Since κ is a ring homomorphism, we then obtain

$$\begin{aligned} w(S(\varphi)) &= \kappa(P(c_1(\varphi_1), \ldots, c_{\deg \varphi_1}(\varphi_1), \ldots, c_1(\varphi_n), \ldots, c_{\deg \varphi_n}(\varphi_n))) \\ &= P(\kappa(c_1(\varphi_1)), \ldots, \kappa(c_{\deg \varphi_1}(\varphi_1)), \ldots, \kappa(c_1(\varphi_n)), \ldots, \kappa(c_{\deg \varphi_n}(\varphi_n))) \\ &= P(w_2(S(\varphi_1)), \ldots w_{\deg S(\varphi_1)}(S(\varphi_1)), \ldots, w_2(S(\varphi_n)), \ldots, w_{\deg S(\varphi_n)}(S(\varphi_n))), \end{aligned}$$

where each $S(\varphi_i)$ is an orthogonal representation of K.

In the last equality above, we understand $w_2(S(\varphi_1)) \in H^*(K^n)$ by thinking it as $\mathrm{pr}_1^*(w_2(S(\varphi_1)))$, $w_2(S(\varphi_2))$ as $\mathrm{pr}_2^*(w_2(S(\varphi_2)))$ and so on. Here pr_i^* are maps on cohomology induced by the projections $\mathrm{pr}_i : K^n \to K$.

Therefore it follows from Theorem 4.2 that

$$w(S(\varphi)) \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n].$$

Next we consider an irreducible orthogonal representation of K^n of the form

$$\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_n,$$

where each π_i is irreducible orthogonal. Again from Theorem 4.2, we have $w(\pi_i) \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}]$ for each i. Now Proposition 2.11 expresses $w(\pi)$ as a certain polynomial in the SWCs of $\pi_1, \ldots, \pi_n$. This leads to

$$w(\pi) \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n].$$

Now we take another irreducible orthogonal representation of K^n which has the form

$$\varpi = \varpi_1 \boxtimes \varpi_2 \boxtimes \cdots \boxtimes \varpi_{2r-1} \boxtimes \varpi_{2r} \boxtimes \pi_{2r+1} \boxtimes \cdots \boxtimes \pi_n, \tag{5.3}$$

where $\varpi_1, \ldots, \varpi_{2r}$ are symplectic with $r > 0$ and $\pi_{2r+1}, \ldots, \pi_n$ are orthogonal.

We think

$$(\varpi_1 \boxtimes \varpi_2), \ldots, (\varpi_{2r-1} \boxtimes \varpi_{2r})$$

as the representations of $K \times K$. Each one is orthogonal.

By Proposition 5.4, we have

$$w(\varpi_{2j-1} \boxtimes \varpi_{2j}) \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2] \text{ for all } 1 \leq j \leq r.$$

Also each π_j is an irreducible orthogonal representation of K. So,

$$w(\pi_j) \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}] \text{ for all } 2r < j \leq n.$$

This way ϖ is an external tensor product of $(n - r)$ orthogonal representations and therefore we apply Proposition 2.11 to obtain

$$w(\varpi) \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n].$$

In a more general setting, let $\psi = \boxtimes_{i=1}^{n} \psi_i$ be an irreducible orthogonal representation of K^n with $2r > 0$ symplectic representations in the multiset $\{\psi_1, \psi_2, \ldots, \psi_n\}$. Then there exists an element $\sigma \in S_n$ such that $\psi = \sigma \cdot \varpi$, where the action is by permuting, and ϖ is of the form (5.3). This gives

$$w(\psi) = w(\sigma \cdot \varpi) = \sigma^*(w(\varpi)).$$

Since $\mathfrak{e}_j = w_4(\eta_j)$ for representations η_j of K^n defined in (5.2) and $\sigma \cdot \eta_j = \eta_{\sigma^{-1}(j)}$, we have

$$\sigma^*(\mathfrak{e}_j) = \mathfrak{e}_{\sigma^{-1}(j)}.$$

Thus, σ^* maps $\mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n]$ into itself, implying

$$w(\psi) \in \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n].$$

This proves that the result holds for all OIRs of K^n and therefore, for all orthogonal representations of K^n. The equality is due to η_j's. $\square$

Now we are ready to prove our main detection theorem.

Proof of Theorem 5.2. Let i_K is the inclusion of K^n into G (by identifying K^n with the subgroup X of G). The naturality of SWCs with Lemma 5.1 gives

$$i_K^* : H^*_{\mathrm{SW}}(G) \hookrightarrow H^*_{\mathrm{SW}}(K^n).$$

Now we consider the inclusion $i_{Z,K} : Z^n \hookrightarrow K^n$ which induces

$$i_{Z,K}^* : H^*(K^n) \to H^*(Z^n) \cong \mathbb{Z}/2\mathbb{Z}[v_1, \ldots, v_n]. \tag{5.4}$$

By Theorem 5.5, we have $H^*_{\mathrm{SW}}(K^n) = \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n]$, and we are interested in the restriction of $i_{Z,K}^*$ to this subalgebra.

For each $1 \le j \le n$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{e} \arrow[d, mapsto] & H^4(K) \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}_j^*"'] & H^4(Z) \arrow[d, "\mathrm{pr}_j^*"] & v^4 \arrow[d, mapsto] \\
\mathfrak{e}_j & H^4(K^n) \arrow[r, "i_{Z,K}^*"] & H^4(Z^n) & v_j^4
\end{tikzcd}$$

The maps pr_j^* are induced by the projections, and the top isomorphism is from the proof of Theorem 4.2. By following the diagram, we obtain

$$i_{Z,K}^*(\mathfrak{e}_j) = v_j^4 \text{ for each } 1 \le j \le n$$

which leads to the isomorphism:

$$i_{Z,K}^*\Big|_{H^*_{\mathrm{SW}}(K^n)} : \mathbb{Z}/2\mathbb{Z}[\mathfrak{e}_1, \ldots, \mathfrak{e}_n] \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}[v_1^4, \ldots, v_n^4]. \tag{5.5}$$

(5.4) and (5.5) together give a sequence of inclusions:

$$H^*_{\mathrm{SW}}(G) \hookrightarrow H^*_{\mathrm{SW}}(K^n) \hookrightarrow \mathbb{Z}/2\mathbb{Z}[v_1^4, \ldots, v_n^4] \subset H^*(Z^n)$$

implying Z^n detects SWCs of G.

Consider $S_n \leqslant N_G(Z^n)$. From (2.14), we have $i_Z^*(H^*_{\mathrm{SW}}(G)) \subseteq H^*(Z^n)^{S_n}$ which implies

$$i_Z^* : H^*_{\mathrm{SW}}(G) \hookrightarrow \mathbb{Z}/2\mathbb{Z}[v_1^4, \ldots, v_n^4]^{S_n}.$$

$\square$

Let π be an orthogonal representation of $G = \mathrm{Sp}(2n, q)$. To calculate $w(\pi)$, it is enough to work with $\pi|_{Z^n}$ due to the detection above. Also being a G-representation, π is S_n-invariant, and then so will be $\pi|_{Z^n}$. We, therefore, ask the following:

Question. What are the SWCs of S_n-invariant representations of C_2^n?

This has been answered in [11]. Their methodology involves the theory of supercharacters, introduced by Isaacs and Diaconis in [6]. We take a digression to talk about this in the next section.

5.3 A Supercharacter Theory

Let H be a finite group. For $h \in H$, write $[h]$ for its conjugacy class. Write $\mathcal{C}(H)$ for the set of all the conjugacy classes in H, and $\mathrm{Irr}(H)$ for the set of isomorphism classes of irreducible characters of H.

Let $\mathrm{Aut}(H)$ be the group of automorphisms, and A be its subgroup. There is a usual action of A on $\mathcal{C}(H)$. For $\alpha \in A$, $h \in H$, we have

$$\begin{aligned} A \times \mathcal{C}(H) &\to \mathcal{C}(H) \\ (\alpha, [h]) &\mapsto [\alpha(h)]. \end{aligned}$$

The group A also acts on $\mathrm{Irr}(H)$ as follows. With $\alpha \in A$, $\chi \in \mathrm{Irr}(H)$, we have

$$\begin{gathered} A \times \mathrm{Irr}(H) \to \mathrm{Irr}(H) \\ (\alpha, \chi) \mapsto \chi^\alpha \end{gathered}$$

where $\chi^\alpha : h \mapsto \chi(\alpha^{-1}(h))$. More generally, A acts on the space of complex-valued class functions of H with the same action.

The following is known due to Brauer:

Lemma 5.6 ([16], Cor. 6.33)**.** *The number of orbits for the action of A on the sets $\mathcal{C}(H)$ and $\mathrm{Irr}(H)$ is equal.*

Say this number is n. We write $\mathcal{O}_1, \ldots, \mathcal{O}_n$ for the A-orbits in $\mathrm{Irr}(H)$, and $\mathrm{O}_A(h_1), \ldots, \mathrm{O}_A(h_n)$ for the A-orbits in $\mathcal{C}(H)$, where h_i are the representatives. We call $\mathrm{O}_A(h_i)$ *A-conjugacy classes* of H.

We have a particular interest in $H = C_2^n$. So, let H be abelian as we move forward. Set

$$\chi_i = \sum_{\chi \in \mathbb{O}_i} \chi \quad ; \quad 1 \leq i \leq n.$$

Let ζ be an A-invariant character of H. Then for each i, every irreducible character χ in $\mathbb{O}_i$ must have the same multiplicity in ζ. This means ζ is of the form

$$\zeta = \sum_{i=1}^{n} m_i \chi_i$$

where m_i are non-negative integers.

Let $\langle\ ,\ \rangle$ be the standard inner product on the space of complex-valued class functions of H. For ζ, ζ' A-invariant characters of H, we have

$$\begin{aligned}
\langle \zeta, \zeta' \rangle &= \frac{1}{|H|} \sum_{h \in H} \zeta(h) \overline{\zeta'(h)} \\
&= \frac{1}{|H|} \sum_{h_i \in H/A} |\,\mathrm{O}_A(h_i)| \zeta(h_i) \overline{\zeta'(h_i)}.
\end{aligned}$$

In particular,

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 0 & i \neq j \\ |\mathbb{O}_i| & i = j. \end{cases}$$

This is because the characters χ_i and χ_j have no irreducible component in common when $i \neq j$. Whereas for $i = j$, each $\chi \in \mathbb{O}_i$ contributes 1 to the inner product $\langle \chi_i, \chi_i \rangle$:

$$\begin{aligned}
\langle \chi_i, \chi_i \rangle &= \Big\langle \sum_{\chi \in \mathbb{O}_i} \chi, \sum_{\chi \in \mathbb{O}_i} \chi \Big\rangle \\
&= \sum_{\chi \in \mathbb{O}_i} \langle \chi, \chi \rangle \\
&= \sum_{\chi \in \mathbb{O}_i} 1 \\
&= |\mathbb{O}_i|.
\end{aligned}$$

Therefore the set $\{\chi_i : 1 \leq i \leq n\}$ forms an orthogonal basis for the space of A-invariant class functions of H. We call χ_i *A-irreducible characters* of H.

In fact, one can form a "character table type" matrix with A-irreducible characters

χ_i and representatives h_j of A-conjugacy classes of H.

Let M be the $n \times n$ matrix whose (i,j)-entry is $\chi_i(h_j)$ for all $1 \leq i, j \leq n$. It can be depicted as the following table:

-	h_1	h_2	$\dots$	h_n
χ_1	$\chi_1(h_1)$	$\chi_1(h_2)$	$\dots$	$\chi_1(h_n)$
χ_2	$\chi_2(h_1)$	$\chi_2(h_2)$	$\dots$	$\chi_2(h_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_n	$\chi_n(h_1)$	$\chi_n(h_2)$	$\dots$	$\chi_n(h_n)$

Table 5.1: Table for A-irreducible characters of H

We now prove its invertibility:

Proposition 5.7. *The matrix M is invertible with*

$$|\det(M)| = |H|^{n/2} \sqrt{\frac{\prod_{i=1}^{n} |\mathcal{O}_i|}{\prod_{j=1}^{n} |\operatorname{O}_A(h_j)|}}.$$

Proof. Set $\mu_k = \frac{|\operatorname{O}_A(h_k)|}{|H|}$ for each $1 \leq k \leq n$. We apply the following row operations on the matrix $\overline{M}^t$:

$$R_k \to \mu_k R_k \quad ; \quad 1 \leq k \leq n$$

to get a new matrix

$$N = \begin{pmatrix} \mu_1 \overline{\chi}_1(h_1) & \dots & \mu_1 \overline{\chi}_n(h_1) \\ \vdots & \ddots & \vdots \\ \mu_n \overline{\chi}_1(h_n) & \dots & \mu_n \overline{\chi}_n(h_n) \end{pmatrix}.$$

Basically $N = \mu \overline{M}^t$, where $\mu = \operatorname{diag}(\mu_1, \dots, \mu_n)$.

Now, the (i, j)-entry of MN is:

$$\begin{aligned}(MN)_{ij} &= \frac{1}{|H|}\sum_{k=1}^{n} |\,\mathrm{O}_A(h_k)|\chi_i(h_k)\overline{\chi}_j(h_k) \\ &= \langle \chi_i, \chi_j\rangle \\ &= \begin{cases} 0 & i \neq j \\ |\mathcal{O}_i| & i = j. \end{cases}\end{aligned}$$

Let $\mathcal{O}$ =diag$(|\mathcal{O}_1|, \ldots, |\mathcal{O}_n|)$. Then we write

$$MN = M\mu\overline{M}^t = \mathcal{O}.$$

Since the determinant is multiplicative and $\det(\overline{M}^t) = \overline{\det(M)}$, we obtain

$$|\det(M)|^2 = |H|^n \frac{\prod_{i=1}^{n} |\mathcal{O}_i|}{\prod_{j=1}^{n} |\,\mathrm{O}_A(h_j)|}$$

thereby completing the proof. □

Remark. The above orbit decompositions $\{\mathcal{O}_i : 1 \leq i \leq n\}$ of Irr(H) and $\{\mathrm{O}_A(h_j) : 1 \leq j \leq n\}$ of $\mathcal{C}(H)$ is a non-trivial "supercharacter theory" for H. (See [6] for details.) Generally, these A-orbits in $\mathcal{C}(H)$ are known as *superclasses*, and the functions $\chi_i = \sum_{\chi\in\mathcal{O}_i} \chi(1)\chi$ as *supercharacters*.

We use this example of supercharacter theory with $H = C_2^n$ to calculate the SWCs of its S_n-invariant representations.

5.3.1 SWCs of S_n-invariant Representations of C_2^n

Let $H = C_2^n$ with $C_2 = \{\pm 1\}$. Let $A = S_n$, the symmetric group which acts on H by permuting. For an abelian group, the conjugacy classes are singleton sets. So $\mathcal{C}(H) = H$.

We consider

$$d_k = (\underbrace{-1, \ldots, -1}_{k}, \underbrace{1, \ldots, 1}_{n-k}) \in C_2^n \quad ; \quad 0 \leq k \leq n.$$

It is easy to see that under the action of A, there are $(n+1)$ orbits in H with d_k as the representatives.

Again since H is abelian, all its irreducible characters are linear, and $\mathrm{Irr}(H)$ is the character group $\widehat{H}$. (We therefore use the words "characters" and "representations" synonymously for C_2^n without creating any confusion.)

To understand the A-action on $\widehat{H}$, we first list the linear characters of H (with notations from Section 2.3.3).

Let X_n be the set of binary vectors $\vec{x} = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ of length n. To each $\vec{x} \in X_n$, we associate a linear character $\mathrm{sgn}_{\vec{x}} = \boxtimes_{i=1}^n \mathrm{sgn}^{x_i}$. We have

$$\widehat{H} = \{\mathrm{sgn}_{\vec{x}} : \vec{x} \in X_n\}.$$

Let $B_{k,n} = \{\vec{x} = (x_1, \ldots, x_n) : \vec{x} \in X_n,\ \sum\limits_{i=1}^{n} x_i = k\}$. Then the A-orbits in $\widehat{H}$ are:

$$\mathbb{O}_k = \{\mathrm{sgn}_{\vec{x}} : \vec{x} \in B_{k,n}\} \quad ; \quad 0 \leq k \leq n.$$

This makes

$$\sigma_k = \bigoplus_{\vec{x} \in B_{k,n}} \mathrm{sgn}_{\vec{x}} \quad ; \quad 0 \leq k \leq n \tag{5.6}$$

the A-irreducible representations of H. We now find $w(\sigma_k)$.

Again from Section 2.3.3, we have

$$H^*(H) \cong \mathbb{Z}/2\mathbb{Z}[v_1, \ldots, v_n],$$

where $v_i = w_1(\mathrm{sgn}_{0\cdots010\cdots0})$ with 1 at the ith position in $(0, \ldots, 0, 1, 0, \ldots, 0) \in B_{1,n}$.

Consider a linear character $\mathrm{sgn}_{\vec{x}}$ where $\vec{x} \in B_{k,n}$ with 1 at positions $i_1, i_2, \ldots, i_k$. It is straightforward from Proposition 2.11 that

$$w(\mathrm{sgn}_{\vec{x}}) = 1 + v_{i_1} + v_{i_2} + \ldots + v_{i_k}.$$

Obviously $w(\sigma_0) = w(1) = 1$. Otherwise by the multiplicativity of SWCs, we have

$$w(\sigma_k) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} (1 + v_{i_1} + v_{i_2} + \ldots + v_{i_k}) \quad ; \quad 1 \leq k \leq n.$$

Let φ be an S_n-invariant representation of H. We can write it as:

$$\varphi \cong \bigoplus_{k=0}^{n} m_k \sigma_k \tag{5.7}$$

where m_k are non-negative integers. Then we obtain

$$\begin{aligned} w(\varphi) &= \prod_{k=1}^{n} w(\sigma_k)^{m_k} \\ &= \prod_{k=1}^{n} \Big(\prod_{1 \le i_1 < \ldots < i_k \le n} (1 + v_{i_1} + v_{i_2} + \ldots + v_{i_k}) \Big)^{m_k}. \end{aligned}$$

Moreover, we have the matrix equation:

$$\begin{pmatrix} \chi_\varphi(d_0) \\ \chi_\varphi(d_1) \\ \vdots \\ \chi_\varphi(d_n) \end{pmatrix} = \underbrace{\begin{pmatrix} \chi_{\sigma_0}(d_0) & \chi_{\sigma_1}(d_0) & \cdots & \chi_{\sigma_n}(d_0) \\ \chi_{\sigma_0}(d_1) & \chi_{\sigma_1}(d_1) & \cdots & \chi_{\sigma_n}(d_1) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{\sigma_0}(d_n) & \chi_{\sigma_1}(d_n) & \cdots & \chi_{\sigma_n}(d_n) \end{pmatrix}}_{\text{Call it `}S\text{'}} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{pmatrix},$$

where the matrix S is invertible by Proposition 5.7. Therefore, we can write the coefficients m_k in terms of character values $\chi_\varphi(d_i)$ by inverting S.

In fact, there is a nice description of these coefficients in [11, Propositions 2-3]. All this has been summed up as the following proposition:

Proposition 5.8 ([11]). *Let φ be an S_n-invariant representation of C_2^n as in (5.7). Then, we have*

$$w(\varphi) = \prod_{k=1}^{n} \Big(\prod_{\vec{x} \in B_{k,n}} (1 + \vec{v} \cdot \vec{x}) \Big)^{m_k},$$

where $\vec{v} \cdot \vec{x} = \sum_{i=1}^{n} v_i x_i$ is the dot product of $\vec{v} = (v_1, v_2, \ldots, v_n)$ with $\vec{x} = (x_1, \ldots, x_n)$ and

$$m_k = \frac{1}{2^n} \sum_{i=0}^{n} \chi_{\sigma_i}(d_k) \chi_\varphi(d_i).$$

In addition, the character value $\chi_{\sigma_i}(d_k)$ is the coefficient of y^i in the expression $(1-y)^k(1+y)^{n-k}$.

5.4 SWCs of Representations of $\mathrm{Sp}(2n, q)$

Let $G = \mathrm{Sp}(2n, q)$, and π be an orthogonal representation of G. To find $w(\pi)$, we work with

$$w(\pi|_{Z^n}) \in \mathbb{Z}/2\mathbb{Z}[v_1^4, \ldots, v_n^4]^{S_n}$$

due to the detection in Theorem 5.2. Being S_n-invariant, $\pi|_{Z^n}$ has its total SWC described by Proposition 5.8. But we can say more about the exponents m_k appearing in $w(\pi|_{Z^n})$ because $\pi|_{Z^n}$ is coming from a representation of the bigger group G.

We begin with the quaternion subgroup $Q \leqslant \mathrm{SL}(2, q)$ from the proof of Theorem 4.2. Clearly Z is also the center of Q. We then have a sequence of inclusions (with appropriate identifications):

$$Z^n \hookrightarrow Q^n \hookrightarrow \mathrm{SL}(2, q)^n \hookrightarrow G.$$

(Here we have identified $\mathrm{SL}(2, q)^n$ with the subgroup X of G and Z^n with the subgroup of diagonal matrices in G which have 1 or -1 on the diagonal.)

Since Z^n detects SWCs of G, we can infer that Q^n also detects the SWCs of G. Let's now spend some time discussing Q^n and its representations, which can help to improve the SWCs for G.

An irreducible representation ϕ of Q^n has the form

$$\phi \cong \phi_1 \boxtimes \cdots \boxtimes \phi_n \tag{5.8}$$

where each ϕ_i is an irreducible representation of Q.

Recall from Section 3.1 that Q has five irreducible representations: $1, \chi_1, \chi_2, \chi_3, \rho$.

Lemma 5.9. *Let ϕ be an irreducible representation of Q^n as above. Suppose $r = \#\{i : \phi_i \cong \rho\}$. Then, ϕ is orthogonal if and only if r is even.*

Proof. Consider the Frobenius-Schur indicator $\varepsilon(\phi)$. Since ρ is symplectic and $1, \chi_1, \chi_2, \chi_3$ are all orthogonal, we have

$$\begin{aligned}\varepsilon(\phi) &= \varepsilon(\phi_1)\varepsilon(\phi_2)\ldots\varepsilon(\phi_n) \\ &= (-1)^r\end{aligned}$$

from Equations (2.1) and (2.8). Therefore, $\varepsilon(\phi) = 1$ if and only if r is even. $\square$

Lemma 5.10. *Let ϱ be an orthogonal representation of Q^n. Let θ be a non-trivial linear character of Z^n. Then, the multiplicity of θ in $\varrho|_{Z^n}$ is divisible by 4.*

We write $m\langle\theta, \varrho|_{Z^n}\rangle$ for the multiplicity of θ in $\varrho|_{Z^n}$.

Proof. Let ϕ be an irreducible representation of Q^n; it has the form (5.8). We now consider the multiset $F(\phi) = \{\phi_1, \phi_2, \ldots, \phi_n\}$.

Suppose ρ has muliplicity r in $F(\phi)$ and it appears at $i_1, i_2, \ldots, i_r$ positions in the tensor representation ϕ. If $r = 0$, then ϕ restricts to the trivial representation of Z^n due to Equation (4.2). We thus take $r > 0$.

Now the restriction $\phi|_{Z^n}$ is a non-trivial linear character θ_ϕ with multiplicity 2^r:

$$\phi|_{Z^n} \cong 2^r\theta_\phi$$

where θ_ϕ is the n-external tensor product

$$\theta_\phi = 1 \boxtimes \underbrace{\mathrm{sgn}}_{i_1^{\text{th}}\text{position}} \boxtimes \cdots \boxtimes \underbrace{\mathrm{sgn}}_{i_r^{\text{th}}\text{position}} \boxtimes \cdots 1$$

with

$$\begin{cases} \mathrm{sgn} & \text{at positions } i_1, \ldots, i_r \\ 1 & \text{everywhere else.} \end{cases}$$

This means $m\langle\theta_\phi, \phi|_{Z^n}\rangle = 2^r$.

If ϕ is irreducible orthogonal, then r is even by Lemma 5.9 and therefore 4 divides $m\langle\theta_\phi, \phi|_{Z^n}\rangle$. Whereas if ϕ is irreducible symplectic, then $S(\phi)$ is an OIR of Q^n and

$$m\langle\theta_\phi, S(\phi)|_{Z^n}\rangle = 2^r + 2^r = 2^{r+1},$$

which is again divisible by 4 for $r > 0$. Therefore the result holds for OIRs of Q^n.

Consider an orthogonal representation ϱ of Q^n. It will be of the form

$$\varrho \cong \bigoplus_j b_j\varrho_j,$$

where b_j are non-negative integers and each ϱ_j is an OIR of Q^n such that $\varrho_j|_{Z^n}$ is a non-trivial character θ_{ϱ_j} of Z^n with multiplicity ℓ_j. Also from above, it follows that $4|\ell_j$ for all j. Therefore $m\langle\theta_{\varrho_j}, \varrho|_{Z^n}\rangle = b_j l_j$ is divisible by 4. This proves our claim. □

Let π be an orthogonal representation of G. Clearly $\operatorname{res}^G_{Z^n} \pi = \operatorname{res}^{Q^n}_{Z^n} \operatorname{res}^G_{Q^n} \pi$ and is S_n-invariant. Therefore the description (5.7) and Lemma 5.10 provide

$$\pi|_{Z^n} \cong \bigoplus_{k=0}^{n} m_k \sigma_k$$

where σ_k are given by Equation (5.6) and all m_k are divisible by 4.

We can now obtain $w(\pi)$ as its image in $H^*(X)$ with the help of Proposition 5.8 (for $\varphi = \pi|_{Z^n}$) and by identifying $v_i^4 \in H^*(Z^n)$ with $\mathfrak{e}_i \in H^*(X)$ for $i = 1, 2, \ldots, n$:

Theorem 5.11. *Let $G = \mathrm{Sp}(2n, q)$ with q odd. Let π be as above. Then the total SWC of π is*

$$w^X(\pi) = \prod_{k=1}^{n} \Big(\prod_{\vec{x} \in B_{k,n}} (1 + \vec{\mathfrak{e}} \cdot \vec{x}) \Big)^{m_k/4}$$

where $\vec{\mathfrak{e}} \cdot \vec{x} = \sum_{i=1}^{n} \mathfrak{e}_i x_i$ is the dot product of $\vec{\mathfrak{e}} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_n)$ with $\vec{x} = (x_1, \ldots, x_n)$ and m_k are described in Proposition 5.8.

5.4.1 Application of Gow's Formula

Write $\mathbb{1}$ for the identity matrix. The Gow's formula in Theorem 4.5 generally holds for all symplectic groups $G = \mathrm{Sp}(2n, q)$ with q odd. That is:

Theorem 5.12 ([13], Theorem 1)**.** *Let $G = \mathrm{Sp}(2n, q)$ with q odd. Let π be an irreducible self-dual representation of G with central character ω_π. Then, we have*

$$\varepsilon(\pi) = \omega_\pi(-\mathbb{1}). \tag{5.9}$$

In other words, π is orthogonal if and only if $-\mathbb{1} \in \ker(\pi)$.

It means

$$\chi_\pi(\mathbb{1}) = \chi_\pi(-\mathbb{1})$$

for irreducible orthogonal π of G. This is same as $\chi_\pi(d_0) = \chi_\pi(d_n)$, in the notations from Section 5.3.1, which leads to

$$\chi_\pi(d_i) = \chi_\pi(d_{n-i}) \quad ; \quad 0 \le i \le n. \tag{5.10}$$

The exponents m_k in Proposition 5.8 are given in terms of these character values $\chi_\pi(d_i)$. Here we use the above equalities to have a more refined expression for m_k. This simplifies the SWC formula for the irreducible orthogonal representations of G.

We begin with the following:

Definition 5.13. Let f be a polynomial of degree n with $f(0) \neq 0$. We define the *reverse* of f to be the function

$$\widetilde{f}(y) = y^n f(1/y).$$

We say f is *symmetric* if $\widetilde{f} = f$, and *anti-symmetric* if $\widetilde{f} = -f$.

Write $[f]_i$ for the coefficient of y^i in $f(y)$. For each $0 \leq i \leq n$, it is clear that

$$[f]_{n-i} = \begin{cases} [f]_i, & f \text{ is symmetric,} \\ -[f]_i, & f \text{ is anti- symmetric.} \end{cases}$$

Example 1. The polynomial $f(y) = (1+y)^n$ is symmetric:

$$\begin{aligned} \widetilde{f}(1/y) &= y^n(1+1/y)^n \\ &= (y+1)^n = f(y). \end{aligned}$$

Example 2. Let $g(y) = (1-y)^m$. Then,

$$\begin{aligned} \widetilde{g}(1/y) &= y^m(1-1/y)^m \\ &= (y-1)^m \\ &= (-1)^m g(y). \end{aligned}$$

Therefore $g(y)$ is symmetric when m is even and anti-symmetric when m is odd.

Example 3. Let f, g be both symmetric polynomials with degrees n, m respectively. Then, $f \cdot g$ is also symmetric:

$$\begin{aligned} (\widetilde{f} \cdot \widetilde{g})(y) &= y^{n+m}(\widetilde{f} \cdot \widetilde{g})(1/y) \\ &= \big(y^n \widetilde{f}(1/y)\big)\big(y^m \widetilde{g}(1/y)\big) \\ &= (f \cdot g)(y). \end{aligned}$$

Similarly, the product of a symmetric and anti-symmetric polynomial is anti-symmetric.

Let m_k, d_k be as in Proposition 5.8. We have:

Lemma 5.14. *Let φ be an S_n-invariant representation of C_2^n such that $\chi_\varphi(d_i) = \chi_\varphi(d_{n-i})$ for all $0 \leq i \leq n$. Then, we have*

$$m_k = \begin{cases} 0 & \textit{when } k \textit{ is odd} \\ \dfrac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-1}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) & \textit{when } k \textit{ is even, } n \textit{ is odd} \\ \dfrac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) + \dfrac{1}{2^n}\Big(\chi_{\sigma_{\frac{n}{2}}}(d_k)\chi_\varphi(d_{\frac{n}{2}})\Big) & \textit{when } k, n \textit{ both are even} \end{cases}$$

for $1 \leq k \leq n$.

Proof. For each k, let $f_k(y) = (1-y)^k(1+y)^{n-k}$. From Proposition 5.8, note that

$$[f_k]_i = \chi_{\sigma_i}(d_k) \quad ; \quad 0 \leq i \leq n.$$

By the above examples, f_k is anti-symmetric when k is odd, otherwise symmetric. This implies

$$[f_k]_{n-i} = (-1)^k [f_k]_i \quad ; \quad 0 \leq i \leq n$$

which is is same as

$$\chi_{\sigma_{n-i}}(d_k) = (-1)^k \chi_{\sigma_i}(d_k) \quad ; \quad 0 \leq i \leq n.. \tag{5.11}$$

Let n be even. We have

$$\begin{aligned} m_k &= \frac{1}{2^n} \sum_{i=0}^{n} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) \\ &= \frac{1}{2^n}\Big(\sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) + \chi_{\sigma_{\frac{n}{2}}}(d_k)\chi_\varphi(d_{\frac{n}{2}}) + \sum_{i=\frac{n+2}{2}}^{n} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i)\Big) \\ &= \frac{1}{2^n}\Big(\sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) + \chi_{\sigma_{\frac{n}{2}}}(d_k)\chi_\varphi(d_{\frac{n}{2}}) + \sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_{n-i}}(d_k)\chi_\varphi(d_{n-i})\Big). \end{aligned}$$

This last equality is by replacing i by $n-i$ in the second summation. Moreover the middle term is zero when k is odd. This is because the coefficients $[f_k]_{\frac{n}{2}} = 0$ for odd k. We now use the hypothesis and (5.11) to get

$$\begin{aligned} m_k &= \frac{1}{2^n}\Big(\sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) + \chi_{\sigma_{\frac{n}{2}}}(d_k)\chi_\varphi(d_{\frac{n}{2}}) + (-1)^k \sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i)\Big) \\ &= \begin{cases} 0 & \text{for odd } k \\ \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) + \frac{1}{2^n}\Big(\chi_{\sigma_{\frac{n}{2}}}(d_k)\chi_\varphi(d_{\frac{n}{2}})\Big) & \text{for even } k. \end{cases} \end{aligned}$$

When n is odd, we can again decompose the summation formula for m_k into two parts as above. There is no middle term involving $\frac{n}{2}$ in this case. We do the same calculations to have

$$m_k = \begin{cases} 0 & \text{for odd } k \\ \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-1}{2}} \chi_{\sigma_i}(d_k)\chi_\varphi(d_i) & \text{for even } k \end{cases}$$

as desired. ☐

This lemma simplifies Theorem 5.11 when π is irreducible orthogonal:

Corollary 5.11.1. *Let q be odd. Let π be an irreducible orthogonal representation of* $\mathrm{Sp}(2n, q)$*. Then the total SWC of π is*

$$w^X(\pi) = \prod_{k=1}^{n} \Big(\prod_{\vec{x} \in B_{k,n}} (1 + \vec{\mathfrak{e}} \cdot \vec{x}) \Big)^{m_k/4}$$

where

$$m_k = \begin{cases} 0 & \textit{when } k \textit{ is odd} \\ \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-1}{2}} \chi_{\sigma_i}(d_k)\chi_\pi(d_i) & \textit{when } k \textit{ is even, } n \textit{ is odd} \\ \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-2}{2}} \chi_{\sigma_i}(d_k)\chi_\pi(d_i) + \frac{1}{2^n}\Big(\chi_{\sigma_{\frac{n}{2}}}(d_k)\chi_\pi(d_{\frac{n}{2}})\Big) & \textit{when } k, n \textit{ both are even.} \end{cases}$$

5.4.2 Some Examples

We illustrate our results for $\mathrm{Sp}(2n, q)$ with $n = 1, 2, 3$.

Example 1. Let $G = \mathrm{Sp}(2, q)$.

Let π be an orthogonal representation of G. Theorem 5.11 applied for $n = 1$ gives the total SWC of π:

$$w(\pi) = (1 + \mathfrak{e})^{m_1/4}$$

where m_1 can be expressed in terms of character values at $\pm\mathbb{1}$ as follows. Use Proposition 5.8 with $k = 1$, $d_0 = \mathbb{1}$, $d_1 = -\mathbb{1}$ and $\varphi = \pi|_Z$ to obtain

$$\begin{aligned} m_1 &= \frac{1}{2}\Big(\chi_{\sigma_0}(-\mathbb{1})\chi_\pi(\mathbb{1}) + \chi_{\sigma_1}(-\mathbb{1})\chi_\pi(-\mathbb{1})\Big) \\ &= \frac{1}{2}\Big(\chi(\mathbb{1}) - \chi(-\mathbb{1})\Big). \end{aligned}$$

The second equality is because $\chi_{\sigma_0}(-\mathbb{1})$ is the constant term and $\chi_{\sigma_1}(-\mathbb{1})$ is the coefficient of y in the polynomial $(1 - y)$.

Moreover when π is irreducible orthogonal, $m_1 = 0$ by Corollary 5.11.1.

With $r_\pi = m_1/4$, this description of $w(\pi)$ is consistent with our previous Theorem 4.4 and its Corollary 4.4.1 about the SWCs of representations of $\mathrm{SL}(2, q)$.

Example 2. Let $G = \mathrm{Sp}(4, q)$.

With $n = 2$ in Theorem 5.11, the total SWC of an orthogonal representation π of G is:

$$\begin{aligned} w^X(\pi) &= \Big(\prod_{\vec{x} \in B_{1,2}} (1 + \vec{\mathfrak{e}} \cdot \vec{x})\Big)^{m_1/4} \Big(\prod_{\vec{x} \in B_{2,2}} (1 + \vec{\mathfrak{e}} \cdot \vec{x})\Big)^{m_2/4} \\ &= \Big((1 + \mathfrak{e}_1)(1 + \mathfrak{e}_2)\Big)^{m_1/4} \Big(1 + \mathfrak{e}_1 + \mathfrak{e}_2\Big)^{m_2/4}. \end{aligned}$$

where m_1, m_2 can be described using Proposition 5.8 as follows.

Write $\mathbf{1}$ for the identity matrix in $\mathrm{SL}(2, q)$. We understand Z^2 sits inside the subgroup $X(\cong \mathrm{SL}(2, q)^2)$ of G as depicted in Section 5.1.1. This way d_i are certain diagonal elements of G. We have

$$\begin{aligned} d_0 &= (\mathbf{1}, \mathbf{1}) = \mathbb{1}, \\ d_1 &= (-\mathbf{1}, \mathbf{1}) = \mathrm{diag}(1, -1, -1, 1), \\ d_2 &= (-\mathbf{1}, -\mathbf{1}) = -\mathbb{1}. \end{aligned}$$

Now to get the character values $\chi_{\sigma_i}(d_k)$, we expand the polynomials $(1-y)^k(1+y)^{2-k}$ for $k = 1, 2$ and look at the coefficients of y^i.

$$\begin{aligned} k=1: \quad (1-y)(1+y) &= 1 + 0y - y^2 = \chi_{\sigma_0}(d_1) + \chi_{\sigma_1}(d_1)y + \chi_{\sigma_2}(d_1)y^2, \\ k=2: \quad (1-y)^2 &= 1 - 2y + y^2 = \chi_{\sigma_0}(d_2) + \chi_{\sigma_1}(d_2)y + \chi_{\sigma_2}(d_2)y^2. \end{aligned}$$

We obtain

$$\begin{aligned} m_1 &= \frac{1}{4}\Big(\chi_{\sigma_0}(d_1)\chi_\pi(d_0) + \chi_{\sigma_1}(d_1)\chi_\pi(d_1) + \chi_{\sigma_2}(d_1)\chi_\pi(d_2)\Big) \\ &= \frac{1}{4}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(-\mathbb{1})\Big), \text{ and} \end{aligned}$$

$$\begin{aligned} m_2 &= \frac{1}{4}\Big(\chi_{\sigma_0}(d_2)\chi_\pi(d_0) + \chi_{\sigma_1}(d_2)\chi_\pi(d_1) + \chi_{\sigma_2}(d_2)\chi_\pi(d_2)\Big) \\ &= \frac{1}{4}\Big(\chi_\pi(\mathbb{1}) - 2\chi_\pi(d_1) + \chi_\pi(-\mathbb{1})\Big). \end{aligned}$$

Therefore we have a formula for the SWCs of Sp$(4, q)$ in terms of character values at diagonal involutions.

Furthermore when π is irreducible orthogonal, Corollary 5.11.1 (which is a result of Gow's formula) leads to the simplification:

$$w^X(\pi) = \Big(1 + \mathfrak{e}_1 + \mathfrak{e}_2\Big)^{m_2/4}$$

where

$$m_2 = \frac{1}{2}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(d_1)\Big).$$

Example 3. Let $G = \mathrm{Sp}(6, q)$.

Once more we apply Theorem 5.11 for $n = 3$ to have the total SWC of an orthogonal representation π of G:

$$\begin{aligned} w^X(\pi) &= \Big(\prod_{\vec{x}\in B_{1,3}}(1 + \vec{\mathfrak{e}}\cdot\vec{x})\Big)^{m_1/4}\Big(\prod_{\vec{x}\in B_{2,3}}(1 + \vec{\mathfrak{e}}\cdot\vec{x})\Big)^{m_2/4}\Big(\prod_{\vec{x}\in B_{3,3}}(1 + \vec{\mathfrak{e}}\cdot\vec{x})\Big)^{m_3/4} \\ &= \Big((1+\mathfrak{e}_1)(1+\mathfrak{e}_2)(1+\mathfrak{e}_3)\Big)^{m_1/4}\Big(\prod_{1\le i<j\le 3}(1 + \mathfrak{e}_i + \mathfrak{e}_j)\Big)^{m_2/4}\Big((1 + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3)\Big)^{m_3/4}, \end{aligned}$$

where m_1, m_2, m_3 are described using Proposition 5.8 as follows.

Again $\mathbf{1}$ is the identity matrix in $\mathrm{SL}(2,q)$ and by viewing Z^3 as a subgroup of G as shown in Section 5.1.1, we have

$$\begin{aligned}
d_0 &= (\mathbf{1},\mathbf{1},\mathbf{1}) = \mathbb{1},\\
d_1 &= (-\mathbf{1},\mathbf{1},\mathbf{1}) = \mathrm{diag}(1,1,-1,-1,1,1),\\
d_2 &= (-\mathbf{1},-\mathbf{1},\mathbf{1}) = \mathrm{diag}(1,-1,-1,-1,-1,1),\\
d_3 &= (-\mathbf{1},-\mathbf{1},-\mathbf{1}) = -\mathbb{1}.
\end{aligned}$$

We obtain the character values $\chi_{\sigma_i}(d_k)$ by expanding the polynomials $(1-y)^k(1+y)^{3-k}$ for $k = 1, 2, 3$:

$$\begin{aligned}
k=1: \quad & (1-y)(1+y)^2 = 1 + y - y^2 - y^3 = \sum_{i=0}^{3} \chi_{\sigma_i}(d_1) y^i\\
k=2: \quad & (1-y)^2(1+y) = 1 - y - y^2 + y^3 = \sum_{i=0}^{3} \chi_{\sigma_i}(d_2) y^i\\
k=3: \quad & (1-y)^3(1+y)^0 = 1 - 3y + 3y^2 - y^3 = \sum_{i=0}^{3} \chi_{\sigma_i}(d_3) y^i.
\end{aligned}$$

We replace y^i by $\chi_\pi(d_i)$ in the above summations and divide by 2^3 to finally have

$$\begin{aligned}
m_1 &= \frac{1}{8}\Big(\chi_\pi(\mathbb{1}) + \chi_\pi(d_1) - \chi_\pi(d_2) - \chi_\pi(-\mathbb{1})\Big),\\
m_2 &= \frac{1}{8}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(d_1) - \chi_\pi(d_2) + \chi_\pi(-\mathbb{1})\Big),\\
m_3 &= \frac{1}{8}\Big(\chi_\pi(\mathbb{1}) - 3\chi_\pi(d_1) + 3\chi_\pi(d_2) - \chi_\pi(-\mathbb{1})\Big).
\end{aligned}$$

This completes the calculations for $\mathrm{Sp}(6,q)$. Once again the application of Gow's formula through Corollary 5.11.1 allows simplification for irreducible orthogonal π:

$$w^X(\pi) = \Big(\prod_{1\le i<j\le 3} (1 + \mathfrak{e}_i + \mathfrak{e}_j)\Big)^{m_2/4},$$

where

$$m_2 = \frac{1}{4}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(d_1)\Big).$$

6

Special Linear Groups $\mathrm{SL}(2n+1,q)$

Let p be an odd prime and $q = p^r$. Let n be a positive integer, and $G = \mathrm{SL}(2n+1, q)$ throughout. In this chapter, we compute the SWCs of orthogonal representations of these special linear groups in terms of character values at diagonal involutions.

6.1 Detection

As usual, we first find a detecting subgroup for G. We do this through a famous result of Quillen [23, Theorem 3] saying: The mod 2 cohomology of $\mathrm{GL}(n,q)$ is detected by its diagonal subgroup, when q is odd.

Isomorphic to the general linear group $\mathrm{GL}(2n,q)$, we consider the following subgroup of G:

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix} : A \in \mathrm{GL}(2n,q) \right\}.$$

Let T be the diagonal subgroup of $\mathrm{GL}(2n,q)$. When viewed as a subgroup of G, the elements of T have the form

$$\begin{pmatrix} a_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & a_{2n} & 0 \\ 0 & \dots & 0 & a_1^{-1}a_2^{-1}\dots a_{2n}^{-1} \end{pmatrix} \longleftrightarrow (a_1, a_2, \dots, a_{2n}) \in (\mathbb{F}_q^\times)^{2n}.$$

Let $W = N_G(T)/C_G(T)$ be the Weyl group of G. We now have the following detection:

Lemma 6.1. *Let $G = \mathrm{SL}(2n+1, q)$ with q odd. Let i_T be the inclusion of T into G. Then the restriction map*

$$i_T^* : H^*(G) \hookrightarrow H^*(T)^W$$

is injective.

Proof. We observe that

$$\begin{aligned} \frac{|\,\mathrm{SL}(2n+1,q)|}{|\,\mathrm{GL}(2n,q)|} &= \frac{(q^{2n+1}-1)(q^{2n+1}-q)\dots(q^{2n+1}-q^{2n})}{(q-1)(q^{2n}-1)(q^{2n}-q)\dots(q^{2n}-q^{2n-1})} \\ &= q^{2n}(1+q+q^2+\dots+q^{2n}) \\ &\equiv 1 \pmod 2 \text{ for odd } q. \end{aligned}$$

This is $\mathrm{GL}(2n, q)$ has an odd index in G. A subgroup with odd index in a group contains a Sylow 2-subgroup. Thus $\mathrm{GL}(2n, q)$ detects the mod 2 cohomology of G by Lemma 2.10. By combining this with [23, Theorem 3], we have the injectivity of i_T^*:

$$i_T^* : H^*(G) \hookrightarrow H^*(\mathrm{GL}(2n,q)) \hookrightarrow H^*(T).$$

Moreover the conjugation by $g \in N_G(T)$ induces an action on $H^*(T)$. From (2.14), we understand that the image of i_T^* is invariant under this action. In fact, the action is trivial if $g \in C_G(T)$. Therefore,

$$\mathrm{Im}(i_T^*) \subseteq H^*(T)^W.$$

□

Remark. The above argument fails for $\mathrm{SL}(n,q)$ when n or q is even. This is because $|\mathrm{SL}(n,q)|/|\mathrm{GL}(n-1,q)|$ becomes even, and therefore we don't have such a detection for these cases.

We have, in fact, a stronger detection for G. Let $T[2]$ be the subgroup of G consisting of diagonal matrices with 1 or -1 on the diagonal. This subgroup is isomorphic to C_2^{2n} and turns out to be a detecting subgroup for SWCs of G. We prove this below.

6.1.1 When $q \equiv 3 \pmod 4$

$T[2]$ is the Sylow 2-subgroup of T when $q \equiv 3 \pmod 4$. Again by Lemma 2.10, $T[2]$ detects the mod 2 cohomology of T. Therefore by a sequence of inclusions, $H^*(G)$ is detected by $T[2]$ in this case:

$$i_{T[2]}^* : H^*(G) \hookrightarrow H^*(T[2])^W, \tag{6.1}$$

where $i_{T[2]}$ is the inclusion of $T[2]$ into G.

6.1.2 When $q \equiv 1 \pmod 4$

Consider the detecting subgroup T of G, which is isomorphic to C_{q-1}^{2n}.

Set $q-1=m$. We use the notations from Section 2.3.3: Let g be the generator of the cyclic group C_m, and $\psi_\bullet$ be the linear character with $\psi_\bullet(g)=e^{\frac{2\pi i}{m}}$. Write 'Sgn' for the linear character of C_m of order 2. One has $\mathrm{res}_{C_2}^{C_m}\psi_\bullet = \mathrm{sgn}$, where 'sgn' is the non-trivial linear character of C_2.

Let $\mathbf{x}_i$ be the vector $(0,\dots 0,1,0\dots,0)$ of length $2n$ with 1 at the ith position. For each $1 \le i \le 2n$, we define

$$\begin{aligned}
\mathrm{Sgn}_{\mathbf{x}_i} &= 1 \boxtimes \cdots \boxtimes 1 \boxtimes \underbrace{\mathrm{Sgn}}_{i\text{th position}} \boxtimes \cdots \boxtimes 1,\\
\psi_{\mathbf{x}_i} &= 1 \boxtimes \cdots \boxtimes 1 \boxtimes \underbrace{\psi_\bullet}_{i\text{th position}} \boxtimes \cdots \boxtimes 1.
\end{aligned}$$

We note $\widehat{T} = \underbrace{\langle\psi_\bullet\rangle \times \langle\psi_\bullet\rangle \times \ldots \times \langle\psi_\bullet\rangle}_{2n}$ is the character group of T, and generally write

$\psi_{j_1 j_2 \cdots j_{2n}} := (\psi_\bullet)^{j_1} \boxtimes \ldots \boxtimes (\psi_\bullet)^{j_{2n}}$ for the elements of $\widehat{T}$. Now one has

$$H^*(T) = \mathbb{Z}/2\mathbb{Z}[s_1, \ldots, s_{2n}, t_1, \ldots, t_{2n}]/(s_1^2, s_2^2, \ldots, s_{2n}^2),$$

where $s_i = w_1(\mathrm{Sgn}_{\mathbf{x}_i})$ and $t_i = w_2(S(\psi_{\mathbf{x}_i}))$ for each i.

Similarly, we define

$$\mathrm{sgn}_{\mathbf{x}_i} := 1 \boxtimes \cdots \boxtimes 1 \boxtimes \underbrace{\mathrm{sgn}}_{i\text{th position}} \boxtimes \cdots \boxtimes 1 \quad , \quad 1 \leq i \leq 2n$$

such that

$$H^*(T[2]) \cong \mathbb{Z}/2\mathbb{Z}[v_1, v_2, \ldots, v_{2n}],$$

with $v_i = w_1(\mathrm{sgn}_{\mathbf{x}_i})$ for each i.

We now prove the detection:

Proposition 6.2. *Let $G = \mathrm{SL}(2n+1, q)$ with $q \equiv 1 \pmod 4$. The subgroup $T[2]$ detects SWCs of G.*

Proof. Let π be an orthogonal representation of G. One has $\det(\pi) = 1$ due to the perfectness of G. Now by thinking of π as an extension of a representation of $\mathrm{GL}(2n, q)$ to G, we can apply [11, Theorem 1] to obtain

$$w(\mathrm{res}_T^G \pi) = 1 + P(t_1, \ldots, t_{2n}),$$

where P is a polynomial in $t_1, \ldots, t_{2n}$. This means

$$i_T^* : H^*_{\mathrm{SW}}(G) \hookrightarrow \mathbb{Z}/2\mathbb{Z}[t_1, t_2, \ldots, t_{2n}].$$

Further we consider the restriction map from $H^*(T)$ to $H^*(T[2])$. This is an injection on the subalgebra $\mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_{2n}]$ as follows. Since

$$\mathrm{res}^T_{T[2]} S(\psi_{\mathbf{x}_i}) = \mathrm{sgn}_{\mathbf{x}_i} \oplus \mathrm{sgn}_{\mathbf{x}_i},$$

we understand t_i maps to v_i^2 for each i due to the naturality of SWCs. Therefore,

$$i_{T[2]}^* : H^*_{\mathrm{SW}}(G) \hookrightarrow \mathbb{Z}/2\mathbb{Z}[t_1, t_2, \ldots, t_{2n}] \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}[v_1^2, \ldots, v_{2n}^2] \subset H^*(T[2])$$

completing the proof. $\square$

Let q be odd, and π be an orthogonal representation of $G = \mathrm{SL}(2n+1,q)$. To find $w(\pi)$, it is enough to work with $\pi|_{T[2]}$ due to the detection above. Moreover being a G-representation, π is W-invariant and then so is $\pi|_{T[2]}$. We, therefore, focus our attention on the W-invariant representations of $T[2]$.

6.2 W-invariant representations of $T[2]$

As usual $G = \mathrm{SL}(2n+1,q)$. The group $T[2]$ is isomorphic to C_2^{2n}; we thus follow the notations from Section 5.3.1. Consider the character group

$$\widehat{T[2]} \cong \{\mathrm{sgn}_{\vec{x}} : \vec{x} \in X_{2n}\},$$

and the Weyl group W of G, isomorphic to the symmetric group S_{2n+1}. There is an action of W on $\widehat{T[2]}$ via conjugation:

$$\begin{aligned} W \times \widehat{T[2]} &\to \widehat{T[2]} \\ (\omega,\chi) &\mapsto {}^{\omega}\chi \end{aligned} \tag{6.2}$$

where ${}^{\omega}\chi$ sends t to $\omega t \omega^{-1}$.

Consider the following subgroup of C_2^{2n+1}:

$$H = \{(a_1, a_2, \ldots, a_{2n}, a_1^{-1}a_2^{-1}\ldots a_{2n}^{-1}) : a_i \in C_2 \ \forall\, i\} \cong C_2^{2n}. \tag{6.3}$$

The symmetric group S_{2n+1} acts on H by permuting, which induces an action of S_{2n+1} on $\widehat{H}$. This induced action is equivalent to the above action of W on $\widehat{T[2]}$. We thus use the language and notations involving S_{2n+1} and C_2^{2n} for all our proofs about W and $T[2]$.

Lemma 6.3. *The orbits in $\widehat{T[2]}$ under the action of W are:*
$\mathcal{O}_0 = \{\mathrm{sgn}_{\vec{0}}\}$,
$\mathcal{O}_k = \{\mathrm{sgn}_{\vec{x}} : \vec{x} \in B_{k,2n}\} \cup \{\mathrm{sgn}_{\vec{y}} : \vec{y} \in B_{2n-k+1,2n}\}$ *for each* $1 \le k \le n$.

Proof. Clearly every element in S_{2n+1} sends $\mathrm{sgn}_{\vec{0}}$ to itself. We let

$$\vec{1}_k = (\underbrace{1,\ldots,1}_{k},\underbrace{0,\ldots,0}_{2n-k}) \quad ; \quad 1 \le k \le 2n.$$

Fix k and consider a vector $\vec{x}_k \in B_{k,2n}$ with 1 at the positions $i_1, i_2, \dots, i_k$. There exists a permutation $g \in S_{2n}$ which acts on C_2^{2n} by sending the coordinates j to i_j for all $1 \leq j \leq k$. That is $g \cdot \vec{1}_k = \vec{x}_k$, giving $B_{k,2n}$ as the orbit of $\vec{1}_k$ under the action of S_{2n}. Also g^{-1} acts on $\mathrm{sgn}_{\vec{1}_k}$ in the following way:

$$\begin{aligned} g^{-1} \cdot \mathrm{sgn}_{\vec{1}_k}(a_1, a_2, \dots, a_{i_1}, \dots, a_{2n}) &= \mathrm{sgn}_{\vec{1}_k}(a_{i_1}, a_{i_2}, \dots, a_{i_k}, \dots, a_n) \\ &= \mathrm{sgn}(a_{i_1})\,\mathrm{sgn}(a_{i_2})\,\mathrm{sgn}(a_3)\dots\mathrm{sgn}(a_{i_k}) \\ &= \mathrm{sgn}_{\vec{x}_k}(a_1, a_2, \dots, a_{i_1}, \dots, a_{2n}) \end{aligned}$$

for all $(a_1, \dots, a_{2n}) \in C_2^{2n}$. This means the set $\{\mathrm{sgn}_{\vec{x}} : \vec{x} \in B_{k,2n}\}$ forms the orbit of $\mathrm{sgn}_{\vec{1}_k}$ under the S_{2n}-action.

Consider $h = (1, 2n+1) \in S_{2n+1}$, which acts on $\mathrm{sgn}_{\vec{1}_k}$ as follows:

$$\begin{aligned} h \cdot \mathrm{sgn}_{\vec{1}_k}(a_1, a_2, \dots, a_{2n}) &= \mathrm{sgn}_{\vec{1}_k}(a_1^{-1}a_2^{-1}\dots a_{2n}^{-1}, a_2, \dots, a_{2n}) \\ &= \mathrm{sgn}(a_1^{-1})\,\mathrm{sgn}(a_2^{-1})\dots\mathrm{sgn}(a_{2n}^{-1})\,\mathrm{sgn}(a_2)\dots\mathrm{sgn}(a_k) \\ &= \mathrm{sgn}(a_1)\,\mathrm{sgn}(a_{k+1})\dots\mathrm{sgn}(a_{2n}) \\ &= \mathrm{sgn}_{\vec{x}_{2n-k+1}}(a_1, a_2, \dots, a_{2n}) \end{aligned}$$

where $\vec{x}_{2n-k+1} \in B_{2n-k+1,2n}$ with 1 at the positions $1, k+1, k+2, \dots, 2n$.

Therefore the sets $\{\mathrm{sgn}_{\vec{x}} : \vec{x} \in B_{k,2n}\}$ and $\{\mathrm{sgn}_{\vec{y}} : \vec{y} \in B_{2n-k+1,2n}\}$ both are contained in the orbit of $\mathrm{sgn}_{\vec{1}_k}$ under the combined action of h and S_{2n} on $\widehat{C_2^{2n}}$. Since S_{2n+1} is generated by the transpositions $(1,2), (1,3), \dots, (1, 2n+1)$, we get the orbit of $\mathrm{sgn}_{\vec{1}_k}$ under S_{2n+1}-action as:

$$\mathrm{O}_{S_{2n+1}}(\mathrm{sgn}_{\vec{1}_k}) = \{\mathrm{sgn}_{\vec{x}} : \vec{x} \in B_{k,2n}\} \cup \{\mathrm{sgn}_{\vec{y}} : \vec{y} \in B_{2n-k+1,2n}\} =: \mathbb{O}_k$$

as claimed. □

6.2.1 SWCs of W-irreducible representations

With notations from above, we define

$$\pi_k := \bigoplus_{\chi \in \mathbb{O}_k} \chi = \Big(\bigoplus_{\vec{x} \in B_{k,2n}} \mathrm{sgn}_{\vec{x}} \Big) \oplus \Big(\bigoplus_{\vec{y} \in B_{2n-k+1,2n}} \mathrm{sgn}_{\vec{y}} \Big) \quad ; \quad 1 \leq k \leq n. \tag{6.4}$$

Also let $\pi_0 = \mathrm{sgn}_{\vec{0}}$. These π_k are W-irreducible representations of $T[2]$ (in sense of Section 5.3). We now aim to find $w(\pi_k)$.

It is obvious that $w(\pi_0) = w(1) = 1$. For $\vec{x} \in B_{k,2n}$ with 1 at the positions $i_1, i_2, \ldots, i_k$, it follows from Proposition 2.11 that

$$w(\mathrm{sgn}_{\vec{x}}) = 1 + v_{i_1} + v_{i_2} + \ldots + v_{i_k}. \tag{6.5}$$

This equality along with the multiplicativity of SWCs leads to:

Lemma 6.4. *Let π_k be as defined in (6.4). Then the total SWC of π_k is,*

$$w(\pi_k) = \prod_{1 \leq i_1 < i_2 < \ldots < i_k \leq 2n} (1+v_{i_1}+v_{i_2}+\ldots+v_{i_k}) \prod_{1 \leq j_1 < \ldots < j_{2n-k+1} \leq 2n} (1+v_{j_1}+v_{j_2}+\ldots+v_{j_{2n-k+1}}).$$

Let ϖ be a W-invariant representation of $T[2]$. We can write

$$\varpi \cong \bigoplus_{k=0}^{n} m_k \pi_k, \tag{6.6}$$

where m_k are non-negative integers. Then the total SWC of ϖ is

$$\begin{aligned} w(\varpi) &= \prod_{k=1}^{n} w(\pi_k)^{m_k} \\ &= \prod_{k=1}^{n} \Big(\prod_{1 \leq i_1 < \ldots < i_k \leq 2n} (1 + v_{i_1} + \ldots + v_{i_k}) \prod_{1 \leq j_1 < \ldots < j_{2n-k+1} \leq 2n} (1 + v_{j_1} + \ldots + v_{j_{2n-k+1}}) \Big)^{m_k}. \end{aligned}$$

Next we would like to find the character formulas for the coefficients m_k. But do such formulas exist for $T[2]$? If yes, which elements of $T[2]$ appear in these formulas? We answer such questions below via the theory of supercharacters from Section 5.3.

6.2.2 A Character Table Type Matrix

Here we adhere to the notations of Section 5.3 with $H = C_2^{2n}$ and $A = S_{2n+1}$, where A is a subgroup of $\mathrm{Aut}(H)$. The action of A is via permutations by thinking H as in (6.3).

Consider

$$d_k = (\underbrace{-1, \ldots, -1}_{k}, \underbrace{1, \ldots, 1}_{2n-k}) \in H \quad ; \quad 0 \leq k \leq 2n. \tag{6.7}$$

Lemma 6.5. *Let A act on H as above. The A-orbits in H are $(n+1)$ in number with $\{d_{2k} : 0 \leq k \leq n\}$ as the set of representatives.*

Proof. From Section 5.3.1, the orbits in H under the action of S_{2n} are:

$$\mathrm{O}_{S_{2n}}(d_k) = \{(a_1, a_2, \ldots, a_{2n}) \in H : a_i \in \{\pm 1\} \text{ and exactly } k \text{ number of } a_i \text{ are } -1\}.$$

Let k be even. Then in the sense of (6.3), d_k as a element of C_2^{2n+1} looks like

$$(\underbrace{-1, \ldots, -1}_{k}, \underbrace{1, \ldots, 1}_{2n-k}, 1)$$

and the transposition $(1, 2n+1) \in A$ acts on d_k giving

$$(1, 2n+1) \cdot d_k = (1, \underbrace{-1, \ldots, -1}_{k-1}, \underbrace{1, \ldots, 1}_{2n-k}) \in \mathrm{O}_{S_{2n}}(d_{k-1}).$$

We thus understand $d_{k-1} \in \mathrm{O}_A(d_k)$ and

$$\mathrm{O}_A(d_{2i}) = \mathrm{O}_A(d_{2i-1}) \quad ; \quad 1 \leq i \leq n.$$

Since A is generated by S_{2n} and the transposition $(1, 2n+1)$, therefore

$$\{\mathrm{O}_A(d_{2k}) : 0 \leq k \leq n\}$$

is the set of A-orbits in H. □

We have the induced action of A on $\widehat{H}$ equivalent to the action described by (6.2). Therefore, Lemma 6.3 has the A-orbits in $\widehat{H}$ and π_k defined in (6.4) are the A-irreducible characters of $\widehat{H}$.

Let $c_k = d_{2k}$ for $0 \leq k \leq n$. We can form a“supercharacter table” matrix M whose (i,j)-entry is $\chi_{\pi_{i-1}}(c_{j-1})$ for all $1 \leq i, j \leq n+1$.

Now an A-invariant representation ϖ of H is of the form

$$\varpi = \bigoplus_{k=0}^{n} m_k \pi_k, \tag{6.8}$$

which gives rise to the matrix equation:

$$\begin{pmatrix} \chi_{\varpi}(c_0) \\ \chi_{\varpi}(c_1) \\ \vdots \\ \chi_{\varpi}(c_n) \end{pmatrix} = \underbrace{\begin{pmatrix} \chi_{\pi_0}(c_0) & \chi_{\pi_1}(c_0) & \dots & \chi_{\pi_n}(c_0) \\ \chi_{\pi_0}(c_1) & \chi_{\pi_1}(c_1) & \dots & \chi_{\pi_n}(c_1) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{\pi_0}(c_n) & \chi_{\pi_1}(c_n) & \dots & \chi_{\pi_n}(c_n) \end{pmatrix}}_{\text{Matrix } M^t} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{pmatrix}. \tag{6.9}$$

With constant entries, the matrix M^t is invertible by Proposition 5.7. Therefore by simply inverting M^t, we get character formulas for the coefficients m_k.

We thus obtain a description for the SWCs of SL$(2n+1, q)$ in terms of character values. This has been stated below in two cases: (i) $q \equiv 3 \pmod 4$, and $q \equiv 1 \pmod 4$. We also work out a few examples.

6.3 The $q \equiv 3 \pmod 4$ Case

Let $G = \mathrm{SL}(2n+1, q)$ with $q \equiv 3 \pmod 4$ for this section. Let π be an orthogonal representation of G. To find $w(\pi)$, we simply work with $\pi|_{T[2]}$ due to (6.1). Since $\pi|_{T[2]}$ is W-invariant, we have

$$\pi|_{T[2]} \cong \bigoplus_{k=0}^{n} m_k \pi_k,$$

where π_k are W-irreducible representations of $T[2]$ from Section 6.2.1. We give $w(\pi)$ by its image in $H^*(T[2])$ using Lemma 6.4 and Equation (6.9) as follows:

Theorem 6.6. *Let π be as above. Then the total SWC of π is,*

$$w^{T[2]}(\pi) = \prod_{k=1}^{n} \Big(\prod_{1 \le i_1 < \ldots < i_k \le 2n} (1 + v_{i_1} + \ldots + v_{i_k}) \prod_{1 \le j_1 < \ldots < j_{2n-k+1} \le 2n} (1 + v_{j_1} + \ldots + v_{j_{2n-k+1}}) \Big)^{m_k},$$

where

$$(m_0, m_1, \ldots, m_n) = \Big(\chi_\pi(c_0),\ \chi_\pi(c_1), \ldots, \chi_\pi(c_n) \Big) \cdot M^{-1}.$$

The exponents m_k are given in terms of character values of π at diagonal involutions $c_k = d_{2k} \in T[2]$ from (6.7).

6.3.1 Examples

We now illustrate our results for $\mathrm{SL}(2n+1,q)$ with $q \equiv 3 \pmod 4$ and $n = 1, 2$.

Example 1. Let $G = \mathrm{SL}(3,q)$.

Let π be an orthogonal representation of G. The detecting subgroup $T[2]$ is the Klein 4-group. Being W-invariant, the restriction of π to $T[2]$ looks like

$$\pi|_{T[2]} \cong m_0 1 \oplus m_1 \underbrace{(\mathrm{sgn}_{10} \oplus \mathrm{sgn}_{01} \oplus \mathrm{sgn}_{11})}_{\pi_1}.$$

Therefore by the multiplicativity of SWCs, we have

$$w^{T[2]}(\pi) = ((1+v_1)(1+v_2)(1+v_1+v_2))^{m_1}.$$

We use (6.9) for m_1 which gives the following matrix equation:

$$\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \begin{pmatrix} 1 & \chi_{\pi_1}(1,1) \\ 1 & \chi_{\pi_1}(-1,-1) \end{pmatrix}^{-1} \begin{pmatrix} \chi_{\pi}(1,1) \\ \chi_{\pi}(-1,-1) \end{pmatrix},$$

where

$$\chi_{\pi_1}(1,1) = (\mathrm{sgn}_{10} \oplus \mathrm{sgn}_{01} \oplus \mathrm{sgn}_{11})(1,1) = 3$$
$$\chi_{\pi_1}(-1,-1) = (\mathrm{sgn}_{10} \oplus \mathrm{sgn}_{01} \oplus \mathrm{sgn}_{11})(-1,-1) = -1.$$

With viewing $(1,1)$ and $(-1,-1)$ as the elements in G, we then have

$$\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ \mathrm{diag}(-1,-1,1) \end{pmatrix},$$

where $\mathbb{1}$ is the identity matrix. This results into:

Proposition 6.7. *Let $G = \mathrm{SL}(3,q)$ with $q \equiv 3 \pmod 4$ and $c_1 = \mathrm{diag}(-1,-1,1) \in G$. Let π be an orthogonal representation of G. Then,*

$$w^{T[2]}(\pi) = ((1+v_1)(1+v_2)(1+v_1+v_2))^{m_\pi},$$

where $m_\pi = \frac{1}{4}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(c_1)\Big)$.

Remark. The action of Weyl group $W \cong S_3$ on $H^*(T[2])$ is equivalent to the natural action of GL$(2,2)$ on $\mathbb{Z}/2\mathbb{Z}[v_1, v_2]$. Due to this, we again encounter the Dickson product from Theorem (4.9):

$$(1+v_1)(1+v_2)(1+v_1+v_2) = 1 + d_{2,1}(\bar{v}) + d_{2,0}(\bar{v}) = 1 + d_2(\bar{v}),$$

making $w(\pi)$ an element of Dickson algebra $\mathbb{Z}/2\mathbb{Z}[d_{2,1}(\bar{v}), d_{2,0}(\bar{v})]$.

Corollary 6.7.1. *Let π be an orthogonal representation of G as above. Let $r = \mathrm{ord}_2(m_\pi)$. Then the obstruction class of π is*

$$w_{2^{r+1}}(\pi) = d_{2,1}^{2^r}(\bar{v}) = v_1^{2^{r+1}} + v_2^{2^{r+1}} + v_1^{2^r} v_2^{2^r}.$$

Proof. From Proposition 6.7, we have

$$w(\pi) = \sum_{i=0}^{m_\pi} \binom{m_\pi}{i} d_2^i(\bar{v}).$$

As in Corollary 4.4.5, we obtain $\binom{m_\pi}{2^r}$ is the first odd binomial coefficient appearing in the above sum. From the term

$$\binom{m_\pi}{2^r} d_2^{2^r}(\bar{v}) = \binom{m_\pi}{2^r} (d_{2,0}^{2^r}(\bar{v}) + d_{2,1}^{2^r}(\bar{v})),$$

we can imply $\binom{m_\pi}{2^r} d_{2,1}^{2^r}(\bar{v})$ has the least degree, which is $(2 \cdot 2^r)$ and

$$\begin{aligned} w_{2^{r+1}}(\pi) &= (v_1^2 + v_2^2 + v_1 v_2)^{2^r} \\ &= v_1^{2^{r+1}} + v_2^{2^{r+1}} + v_1^{2^r} v_2^{2^r} \end{aligned}$$

as claimed. □

Example 2. Let $G = \mathrm{SL}(5, q)$.

Let π be an orthogonal representation of G. From Theorem 6.6, the total SWC of π is

$$w^{T[2]}(\pi) = \Big(\prod_{i=1}^{4}(1+v_i)(1+\sum_{i=1}^{4} v_i)\Big)^{m_1} \Big(\prod_{1\le i<j\le 4}(1+v_i+v_j) \prod_{1\le i<j<k\le 4}(1+v_i+v_j+v_k)\Big)^{m_2}.$$

The exponents m_1, m_2 in terms of character values are given by the matrix equation:

$$\begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 & \chi_{\pi_1}(1,1,1,1) & \chi_{\pi_2}(1,1,1,1) \\ 1 & \chi_{\pi_1}(-1,-1,1,1) & \chi_{\pi_2}(-1,-1,1,1) \\ 1 & \chi_{\pi_1}(-1,-1,-1,-1) & \chi_{\pi_2}(-1,-1,-1,-1) \end{pmatrix}^{-1} \begin{pmatrix} \chi_\pi(1,1,1,1) \\ \chi_\pi(-1,-1,1,1) \\ \chi_\pi(-1,-1,-1,-1) \end{pmatrix}$$

where

$$\begin{aligned} \pi_1 &= \Big(\bigoplus_{\vec{x} \in B_{1,4}} \mathrm{sgn}_{\vec{x}} \Big) \oplus \mathrm{sgn}_{1111} \\ \pi_2 &= \Big(\bigoplus_{\vec{x} \in B_{2,4}} \mathrm{sgn}_{\vec{x}} \Big) \oplus \Big(\bigoplus_{\vec{y} \in B_{3,4}} \mathrm{sgn}_{\vec{y}} \Big). \end{aligned}$$

By doing the calculations and viewing $(1,1,1,1)$, $(-1,-1,1,1)$, $(-1,-1,-1,-1)$ as the elements of G, we obtain

$$\begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 1 & 10 & 5 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ \mathrm{diag}(-1,-1,1,1,1) \\ \mathrm{diag}(-1,-1,-1,-1,1) \end{pmatrix}. \tag{6.10}$$

With $m_\pi = m_1$ and $n_\pi = m_2$, we sum up:

Proposition 6.8. *Let $q \equiv 3 \pmod 4$. Let $G = \mathrm{SL}(5,q)$ with $c_1 = \mathrm{diag}(-1,-1,1,1,1)$, $c_2 = \mathrm{diag}(-1,-1,-1,-1,1) \in G$. For π orthogonal, the total SWC is*

$$w^{T[2]}(\pi) = \Big(\prod_{i=1}^{4} (1+v_i)(1+\sum_{i=1}^{4} v_i) \Big)^{m_\pi} \Big(\prod_{1 \le i < j \le 4} (1+v_i+v_j) \prod_{1 \le i < j < k \le 4} (1+v_i+v_j+v_k) \Big)^{n_\pi},$$

where

$$\begin{aligned} m_\pi &= \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) + 2\chi_\pi(c_1) - 3\chi_\pi(c_2) \Big) \\ n_\pi &= \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) - 2\chi_\pi(c_1) + \chi_\pi(c_2) \Big). \end{aligned}$$

6.4 The $q \equiv 1 \pmod 4$ case

For this section, let $G = \mathrm{SL}(2n+1,q)$ with $q \equiv 1 \pmod 4$. Let π be an orthogonal representation of G. Again it is enough to work with $\pi|_{T[2]}$ due to Proposition 6.2. In fact Theorem 6.6 describes $w(\pi)$ in $H^*(T[2])$ for this case too. Now the purpose of this section is to have a stronger formula for $w(\pi)$ by its image in $H^*(T)$, where T is the subgroup of diagonal matrices in G.

We begin by considering the character group $\widehat{T}$. Let $\widehat{T}_{\mathrm{orth}}$ be the set of orthogonal linear characters in $\widehat{T}$. A linear character $\psi = \boxtimes_{i=1}^{2n} \psi_i \in \widehat{T} - \widehat{T}_{\mathrm{orth}}$ is not self-dual.

Lemma 6.9. *Let $\psi \in \widehat{T} - \widehat{T}_{orth}$ be as above. Then we have*

$$w(S(\psi)) = 1 + \sum_{i=1}^{2n} \epsilon_{\psi_i} t_i$$

where $\epsilon_{\psi_i} = 1$ if ψ_i is odd, otherwise 0.

Proof. By Propositions 2.16 and 2.17, we have

$$\begin{aligned} w(S(\psi)) &= \kappa(c(\psi)) \\ &= \kappa(1 + \sum_{i=1}^{2n} c_1(\psi_i)) \\ &= 1 + \sum_{i=1}^{2n} w_2(S(\psi_i)) \\ &= 1 + \sum_{i=1}^{2n} \epsilon_{\psi_i} t_i. \end{aligned}$$

The last equality is by Lemma 2.13. (Note that $w_2(S(\psi_i))$ in the above sum means $\mathrm{pr}_i^*(w_2(S(\psi_i))$ where $\mathrm{pr}_i : C_m^{2n} \to C_m$ are projection maps.) $\square$

We define a relation $\sim$ on $\widehat{T} - \widehat{T}_{\mathrm{orth}}$ by $\psi \sim \psi^{-1}$, and set $\widetilde{T} = (\widehat{T} - \widehat{T}_{\mathrm{orth}})/\sim$. Then an orthogonal representation φ of T has the form

$$\varphi = \bigoplus_{\chi \in \widehat{T}_{\mathrm{orth}}} m_\chi \chi \oplus \bigoplus_{\psi \in \widetilde{T}} m_\psi S(\psi),$$

where m_χ, m_ψ are all non-negative integers.

Lemma 6.10. *Let π be an orthogonal representation of G, and $\theta \in \widehat{T[2]}$ be non-trivial. Then the multiplicity of θ in $\pi|_{T[2]}$ is even.*

Proof. We first restrict π to T which takes the form

$$\pi|_T \cong \bigoplus_{\chi \in \widehat{T}_{\mathrm{orth}}} m_\chi \chi \oplus \bigoplus_{\psi \in \widetilde{T}} m_\psi S(\psi).$$

It is easy to see that

$$\chi|_{T[2]} = 1 \text{ for all } \chi \in \widehat{T}_{\mathrm{orth}}.$$

If $\psi \in \widetilde{T}$ is such that $\psi = \boxtimes_{i=1}^{2n} \psi_i$ where $\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_k}$ are odd, then

$$\mathrm{res}^T_{T[2]} \psi = \theta_\psi,$$

where

$$\theta_\psi := 1 \boxtimes \underbrace{\mathrm{sgn}}_{i_1^{\text{th}}\text{position}} \boxtimes \cdots \boxtimes \underbrace{\mathrm{sgn}}_{i_r^{\text{th}}\text{position}} \boxtimes \cdots 1.$$

Also $\theta_\psi = \theta_{\psi^{-1}}$, which makes

$$S(\psi)|_{T[2]} = 2\theta_\psi.$$

Therefore when $\pi|_T$ is further restricted to $T[2]$, we obtain

$$\pi|_{T[2]} \cong m_0 1 \oplus \bigoplus_{\psi \in \widetilde{T}} 2m_\psi \theta_\psi,$$

where $m_0 = \sum_{\chi \in \widehat{T}_{\mathrm{orth}}} m_\chi$. This shows that every non-trivial linear character in $\pi|_{T[2]}$ has even multiplicity as claimed. $\square$

Consider the decomposition

$$\pi|_{T[2]} \cong \bigoplus_{k=0}^{n} m_k \pi_k,$$

where π_k are W-irreducible representations of $T[2]$ from Section 6.2.1. We have all the coefficients m_k even by the lemma above.

We thus have $w(\pi)$ as its image in $H^*(T)$ from Theorem 6.6 by identifying $v_i^2 \in H^*(T[2])$ with $t_i \in H^*(T)$ for each i:

Theorem 6.11. *Let $G = \mathrm{SL}(2n+1, q)$ with $q \equiv 1 \pmod 4$. Let π be as above. The total SWC of π is*

$$w^T(\pi) = \prod_{k=1}^{n} \Big(\prod_{1 \leq i_1 < \ldots < i_k \leq 2n} (1 + t_{i_1} + \ldots + t_{i_k}) \prod_{1 \leq j_1 < \ldots < j_{2n-k+1} \leq 2n} (1 + t_{j_1} + \ldots + t_{j_{2n-k+1}}) \Big)^{m_k/2},$$

where

$$(m_0, m_1, \ldots, m_n) = \Big(\chi_\pi(c_0),\ \chi_\pi(c_1), \ldots, \chi_\pi(c_n)\Big) \cdot M^{-1}.$$

Recall M is the invertible matrix from (6.9) and the integer-valued exponents $m_k/2$ are given in the character values of π at involutions $c_k = d_{2k} \in T$ from (6.7).

6.4.1 Examples

For $q \equiv 1 \pmod 4$, we again illustrate our results with $\mathrm{SL}(3, q)$ and $\mathrm{SL}(5, q)$.

Example 1. Let $G = \mathrm{SL}(3, q)$.

Theorem 6.11 with Example 1 in Section 6.3.1 give:

Proposition 6.12. *Let $G = \mathrm{SL}(3, q)$ with $q \equiv 1 \pmod 4$ and $c_1 = \mathrm{diag}(-1, -1, 1) \in G$. For orthogonal π of G, we have*

$$w^T(\pi) = \Big((1 + t_1)(1 + t_2)(1 + t_1 + t_2)\Big)^{m_\pi/2},$$

where $m_\pi = \dfrac{1}{4}\Big(\chi_\pi(\mathbb{1}) - \chi_\pi(c_1)\Big)$.

We recall the detecting subgroup T of G is the bicyclic group $C_{q-1} \times C_{q-1}$ with

$$H^*(T) \cong \mathbb{Z}/2\mathbb{Z}[s_1, s_2, t_1, t_2]/(s_1^2, s_2^2).$$

From the proposition above, we have $H^*_{\mathrm{SW}}(G) \subset \mathbb{Z}/2\mathbb{Z}[t_1, t_2]$. Now the Weyl Group $W \cong S_3$ acts on $\mathbb{Z}/2\mathbb{Z}[t_1, t_2]$ by sending

$$t_1 \xmapsto{(1,2)} t_2 \xmapsto{(2,3)} t_1 + t_2.$$

This action is equivalent to the natural action of $\mathrm{GL}(2, 2)$ on $\mathbb{Z}/2\mathbb{Z}[t_1, t_2]$. That's why we again have the Dickson product

$$(1+t_1)(1+t_2)(1+t_1+t_2) = 1 + d_{2,1}(\bar{t}) + d_{2,0}(\bar{t})$$

appearing in $w(\pi)$. Therefore:

Corollary 6.12.1. *Let $G = \mathrm{SL}(3,q)$ with $q \equiv 1 \pmod 4$. We have*

$$H^*_{\mathrm{SW}}(G) \subseteq \mathbb{Z}/2\mathbb{Z}[d_{2,1}(\bar{t}), d_{2,0}(\bar{t})].$$

We observe $H^*_{\mathrm{SW}}(G) \neq \mathbb{Z}/2\mathbb{Z}[d_{2,1}(\bar{t}), d_{2,0}(\bar{t})]$ due to the following reason:

The linear groups $\mathrm{SL}(2n+1,q)$ are perfect for odd q, and have trivial Schur multiplier. Therefore π must be spinorial by [17, Proposition 6]. These facts along with Wu formula imply

$$w_1(\pi) = w_2(\pi) = w_3(\pi) = 0.$$

Hence, $d_{2,1}(\bar{t}), d_{2,0}(\bar{t})$ don't belong to $H^*_{\mathrm{SW}}(G)$.

Corollary 6.12.2. *Let π be an orthogonal representation of G. Let $r = \mathrm{ord}_2(m_\pi)$. Then the obstruction class of π is,*

$$w_{2^{r+2}}(\pi) = t_1^{2^{r+1}} + t_2^{2^{r+1}} + t_1^{2^r} t_2^{2^r}.$$

Proof. The proof is analogous to that of Corollary 6.7.1. □

Example 2. Let $G = \mathrm{SL}(5,q)$.

Again from Theorem 6.11 and Example 2 in Section 6.3.1, we have:

Proposition 6.13. *Let $q \equiv 1 \pmod 4$. Let $G = \mathrm{SL}(5,q)$ with $c_1 = \mathrm{diag}(-1,-1,1,1,1)$, $c_2 = \mathrm{diag}(-1,-1,-1,-1,1) \in G$. The total SWC of an orthogonal π is*

$$w^T(\pi) = \Big(\prod_{i=1}^{4}(1+t_i)(1+\sum_{i=1}^{4} t_i)\Big)^{m_\pi/2}\Big(\prod_{1\le i<j\le 4}(1+t_i+t_j)\prod_{1\le i<j<k\le 4}(1+t_i+t_j+t_k)\Big)^{n_\pi/2},$$

where

$$m_\pi = \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) + 2\chi_\pi(c_1) - 3\chi_\pi(c_2)\Big),$$
$$n_\pi = \frac{1}{16}\Big(\chi_\pi(\mathbb{1}) - 2\chi_\pi(c_1) + \chi_\pi(c_2)\Big).$$

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