Modular forms, Bianchi groups, torsion growth, and exploring the Bergeron-Venkatesh conjecture

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by

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Certificate

This is to certify that this dissertation entitled Modular forms, Bianchi groups, torsion growth, and exploring the Bergeron-Venkatesh conjecture towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Tanuj Mathur at Indian Institute of Science Education and Research under the supervision of Dr Debargha Banerjee, Associate Professor, Department of Mathematics , during the academic year 2022-2023.

Debargha Banerjee

Dr Debargha Banerjee

Committee:

Dr Debargha Banerjee

Dr Manish Mishra

To my family.

Declaration

I hereby declare that the matter embodied in the report entitled Modular forms, Bianchi groups, torsion growth, and exploring the Bergeron-Venkatesh conjecture are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr Debargha Banerjee and the same has not been submitted elsewhere for any other degree.

Tanuj Mathur

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Abstract

We discuss several aspects of Bianchi groups, particularly from a computational point of view, in an attempt to gain acumen around the Bergeron-Venkatesh conjecture. To get there, we come across a variety of concepts and problems from the fields of modular forms, group theory and hyperbolic geometry.

Contents

Introduction

The thesis describes a journey - A journey attempting to explore the relatively unexplored study of the geometric and number theoretic aspects of a certain class of arithmetic groups, the Bianchi groups.

Luigi Bianchi, an Italian mathematician from the late 19th and early 20th century, noticed that the projective groups over the ring of integers of imaginary number fields acted rather nicely on the three-dimensional hyperbolic space. In 1892[BL1], he computed the fundamental domains of some small groups of that kind, hence giving them the name, Bianchi Groups.

Bianchi Groups are an important subclass of the larger Kleinian Groups which have been studied regularly since the 1960s, driven by the exploration into generalized modular forms by Serre. But the next breakthrough in the study of Bianchi Groups came through Swan[RS1] in 1971. Swan provided a geometric method of finding the fundamental domains and finite presentations for most Bianchi Groups. Swan's method is still in use for the exploration of such groups, and plays a very important role in this thesis too.

Since then, the study of Galois representations and (co)homologies of these groups have been an important part of the study of certain aspects of the Langlands program.

Unfortunately, most of the fundamental questions of the area still remain conjectures. A lack of theoretical development has driven computational research in the field. The development has been extensive but the work by Cremona and Sengun particularly stands out.

The work done on the abelianization of the groups by Sengun[MHS1] has converged into the study of torsion in the homology of arithmetic groups. It is a topic of interest that has been gaining traction lately and it's driven by Bergeron and Venkatesh.
A conjecture by Bergeron and Venkatesh suggests that, for Bianchi groups and congruence subgroups, torsion in the homology should grow exponentially with respect to the volume, and there have been a variety of ways of presenting it.
We are looking at a particular expression of looking at this growth given as -

$$\frac{\log\left|\left(\Gamma_n^{ab}\right)_{tor}\right|}{\mathrm{vol}\left(\Gamma_n\backslash\mathbb{H}\right)}$$

Bergeron and Venketesh conjecture that the following holds true -

Conjecture 0.0.1. *Let $\{\Gamma_n\}_n$ be a sequence of finite index congruence subgroups of some fixed Bianchi group. Then*

$$\lim_{n\to\infty}\frac{\log\left|\left(\Gamma_n^{ab}\right)_{tor}\right|}{\mathrm{vol}\left(\Gamma_n\backslash\mathbb{H}\right)}=\frac{1}{6\pi}$$

The thesis can be roughly divided into three sections -

- In the first part, we provide the theoretical background covered during the project, omitting most of the proofs involved.
- In the second part, we introduce the methods studied during the project and the algorithms used.
- In the third part, we give examples of the computations performed under the project

with the implementation of these methods, alongside visualizations to provide insight to the reader.

The project primarily used [EGM] and [MR] as the references and the books should cover all the theoretical aspects mentioned in the thesis.

0.1 Original Work

The field still remains fairly undiscovered and that leads to a lot of scope of original work. The thesis provides an algorithm and the computation of the covolume of congruence subgroups of Bianchi groups, which still lack a closed-form version. Using that, we test the Bergeron-Venkatesh conjecture for a new class of groups. On top of this, we provide a variety of illustrations relevant to the field previously absent in the literature.

0.2 Notations and Definitions

- We interchangeably use $\mathbb{H}$ for two-dimensional and three-dimensional hyperbolic space, given the context. There is no conflict between the overlapping notation in this thesis.
- We interchangeably refer to $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$ as the modular group, given the context. There is no conflict between the overlapping notation in this thesis.
- $P^{n-1}(F)$ is the projective space associated with the vector space F^n over F, where F is a field.
- $\mathcal{H}$ refers to the space of quaternion defined as $\{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} | a, b, c, d \in \mathbb{R}\}$ along with the standard multiplication rules.

- C_n refers to the cyclic group of order n.

- If G, H are groups, then $G * H$ refers to the free product of the groups G and H.

- $M(n, R)$ refers to the ring of $n \times n$ matrices taking entries from a ring R.

- If G is a group, then $G = \langle X \mid R \rangle$ refers to the presentation of the group G.

  $X \subset G$ refers to the generators of the group and R is the set of equations between the elements of X.

- The Riemann-Zeta function $\zeta(s)$ is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, where $s \in \mathbb{C}$ and $Re(s) > 1$.

- Let K be a number field. $f : K \to \mathbb{C}$ is called an embedding of K if f is a field homomorphism. If f can be restricted to $\mathbb{R}$, then it is a real embedding. Otherwise, it's a complex embedding.

- A conjugate pair of complex embeddings is called a complex place while every real embedding is referred to as a real place of a number field.

- Let L be an extension of a number field K. We say that a prime ideal in the ring of integers of K is ramified in L if the image of the prime ideal in the ring of integers in L factorizes into prime ideals non-trivially.

- If $\mathfrak{i}$ is an ideal in the ring of integers in a number field $\mathcal{O}_K$, the norm of the ideal $\mathfrak{i}$ is defined as $N(\mathfrak{i}) = \left|\mathcal{O}_K/\mathfrak{i}\right|$.

  If $\mathfrak{p}$ is a prime ideal, then the norm of the ideal is a prime power p^n, where $\mathfrak{p}$ is a prime ideal lying over p.

Chapter 1

Preliminaries

This chapter deals with the preliminaries needed for the thesis. We start off by describing the well-known theory of two-dimensional and three-dimensional hyperbolic spaces, followed by an introduction to modular forms. That'll precede an introduction to Kleinian groups, and the particular case of Bianchi Groups.

1.1 The Projective Groups

The section contains an introduction to projective groups and their actions relevant to our study.

When $\mathrm{GL}(n, F)$ and $SL(n, F)$ are quotiented by their respective centres, we get the projective linear group and projective special linear group.

$$\mathrm{PGL}(n,F) = \frac{\mathrm{GL}(n,F)}{Z(\mathrm{GL}(n,F))} = \frac{\mathrm{GL}(n,F)}{F^\times}$$
$$\mathrm{PSL}(n,F) = \frac{\mathrm{SL}(n,F)}{Z(\mathrm{SL}(n,F))} = \frac{\mathrm{SL}(n,F)}{\{\alpha \in F^\times \mid \alpha^n = 1\}}$$

Theorem 1.1.1. *$\mathrm{PGL}(n,F)$ and $\mathrm{PSL}(n,F)$ are isomorphic iff every element of F has an nth root in F.*

Proof. The determinant map from $\mathrm{PGL}(n,F)$ into the scalers of F splits in the following fashion -

$$\mathrm{PSL}(n,F) \hookrightarrow \mathrm{PGL}(n,F) \to F^\times / \left(F^\times\right)^n$$

The split sequence gives rise to the following isomorphism

$$\mathrm{PSL}(n,F) \rtimes F^\times / \left(F^\times\right)^n \cong \mathrm{PGL}(n,F)$$

This isomorphism is enough to prove our assertion. □

So, we can conclude that $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{PGL}(2,\mathbb{C})$. This group is called the Mobius group and will have a canonical action on the hyperbolic space. These groups act on the $n-1$ dimensional projective space $P^{n-1}(F)$ using

Lemma 1.1.2. *Every element of $P^{n-1}(F)$ is stabilized by the multiples of I_n in $\mathrm{GL}(n,F)$ and by no other member of the group.*

Here, $P^{n-1}(F)$ represents the set of one-dimensional subspaces of F^n. Now, we know that the scalars are exactly the centre of $GL(n,\mathbb{F})$, so this lemma ends up inducing a **faithful** action by $\mathrm{PGL}(n, \mathbb{F})$ onto the projective space.

Definition 1.1.1. *The action of a group G on a set X is called doubly transitive if G acts transitively on the set of all ordered pairs of distinct elements of X. In other words, given any $x_1, x_2, y_1, y_2 \in X$ so that $x_1 \neq x_2$ and $y_1 \neq y_2$ there is a $g \in G$ so that $gx_i = y_i$ for $i = 1, 2$.*

Lemma 1.1.3. *If $n \geq 2$, the action of $\mathrm{SL}(n, F)$ and thus of $\mathrm{PSL}(n, F)$ on projective space is doubly transitive.*

Proof. Please refer to the first section in Chapter 1 in [EGM]. □

1.2 Hyperbolic models and Modular Forms

1.2.1 The Upper Half-Space Model

The upper half-space $\mathbb{H}$ provides an intuitive model of 3-dimensional hyperbolic space as the model has a variety of properties analogous to the upper half-plane as a model of plane hyperbolic geometry.

It is defined as -

$$\mathbb{H} := \{(x, y, r) \quad | \quad x, y, r \in \mathbb{R}, \quad r > 0\}$$

For the sake of ease of computation, this space $\mathbb{H}$ can be considered a subset of the quaternion space $\mathcal{H}$. Under the usual $\mathbb{R}$-basis of $\mathcal{H}$, we can consider the elements of $\mathbb{H}$ as -

$$P = (z, r) = (x, y, r) = z + rj \ ,$$
$$\text{where } z = x + iy \text{ and } \quad j = (0, 0, 1)$$

There exists a generalized Riemannian Metric for $\mathbb{H}$ as well. It is defined as -

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$$

There's a generalization of the Laplace operator, **Laplace–Beltrami operator**

$$\Delta = \frac{1}{\sqrt{g}} \sum_{\mu,\nu} \frac{\partial}{\partial x^\mu} \sqrt{g} g^{\mu\nu} \frac{\partial}{\partial x^\nu}, \text{ where } g = \det\left(g_{\mu\nu}\right) \text{ and } \left(g^{\mu\nu}\right) = \left(g_{\mu\nu}\right)^{-1} .$$

defined as $\Delta = r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r \frac{\partial}{\partial r}$

Theorem 1.2.1. *The following are the equivalent definitions of the Riemann Sphere -*

- *The complex projective line $P^1(\mathbb{C})$ i.e. All the one dimensional subspaces of $\mathbb{C}^2$.*
- *The extended complex numbers $\mathbb{C} \cup \{\infty\}$*
- $S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ *sitting in* $\mathbb{R}^3$

Proof. This is a standard result in geometry and they follow each other using the natural projection map and treating one-dimensional subspaces as the scalars of the field. $\square$

Using the second definition, it is clear to see how the Riemann Sphere sits inside the upper half space model. In fact, the Riemann square acts like the border of the upper half space model.

The actions of $mathrmPSL(2, \mathbb{C})$ on $\mathbb{H}$ and on its boundary $\mathbb{P}^1\mathbb{C}$ can be described by simple formulas.

Definition 1.2.1 (Poincaré action)**.** *If* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{C})$*, we define the* γ *acting on* $\mathbb{H}$

as $\gamma \cdot (z, \zeta) = (z', \zeta')$, *where*

$$\zeta' = \frac{|\det \gamma|\zeta}{|cz-d|^2 + \zeta^2|c|^2},$$
$$z' = \frac{(\overline{d-cz})(az-b) - \zeta^2 \bar{c}a}{|cz-d|^2 + \zeta^2|c|^2}.$$

Theorem 1.2.2. *The Hyperbolic Metric defined using the standard line element is* $\mathrm{PSL}(2, \mathbb{C})$-*invariant. So, the hyperbolic volume and distance are invariant under action from* $\mathrm{PSL}(2, \mathbb{C})$ *as well.*

Theorem 1.2.3. *There exists an isomorphism between* $PSL(2, \mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$ *and the group of isometries of* $\mathbb{H}$, $\mathrm{Iso}(\mathbb{H})$. *The non-zero element of* $\mathbb{Z}/2\mathbb{Z}$ *takes elements of* $\mathrm{PSL}(2, \mathbb{C})$ *to their complex conjugation.*

Hyperbolic Distance

Definition 1.2.2. *For* $P = z + rj, P' = z' + r'j \quad (z, z' \in \mathbb{C}, r, r' > 0)$ *the hyperbolic distance* $d(P, P')$ *is given by -*

$$\cosh d(P, P') = \delta(P, P')$$

where $\delta(P, P') := \frac{|z-z'|^2 + r^2 + r'^2}{2rr'}$

1.2.2 The Unit Ball Model

The aforementioned model is mainly used in the context of linear fractional transformations which are merely the translation maps where the fixed point is mapped to infinity. But when queries concerning rotational symmetry are examined, the unit ball model of hyperbolic geometry is suitable. Consider the unit ball and the line element

$$\mathbb{B} = \{ u = u_0 + u_1 i + u_2 j \in \mathcal{H} \quad | \; \|u\|^2 < 1 \}$$

$$ds^2 = 4 \cdot \frac{du_0^2 + du_1^2 + du_2^2}{(1 - u_0^2 - u_1^2 - u_2^2)^2}$$

The space with the metric(from the line element) defines the unit ball model.

The distance of point u from the origin is -

$$d(0, u) = 2 \int_0^{\rho} \frac{dt}{1 - t^2} = \log \frac{1 + \rho}{1 - \rho}$$

where ρ is the norm of u.

Theorem 1.2.4. *For $P \in \mathbb{H}$ the quaternion $-jP + 1$ has an inverse such that $(P - j) \cdot (-jP + 1)^{-1} \in \mathbb{B}$. The map $\eta_0 : \mathbb{H} \to \mathbb{B}, \eta_0(P) := (P - j) \cdot (-jP + 1)^{-1}$ is an isometry.*

Proof. Please refer to the section on The Unit Ball Model in [EGM], Chapter 1. □

Definition 1.2.3. *We define the matrix group* $\mathrm{SB}(2, \mathcal{H})$ *as -*

$$\mathbf{SB}(2, \mathcal{H}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathcal{H}) \mid \quad d = a', b = c', a\bar{a} - c\bar{c} = 1 \right\}$$

Theorem 1.2.5. *The following relationships between* $\mathrm{SB}(2, \mathcal{H})$, $\mathrm{SL}(2, \mathbb{C})$ *and Iso* $^+(\mathbb{B})$ *hold -*

1. *Consider $g := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}$. Then the function $\eta : A \mapsto \bar{g} \cdot A \cdot g$ can be used to make a isomorphism between the groups $\eta : \mathrm{SL}(2, \mathbb{C}) \to \mathbf{SB}(2, \mathcal{H})$.*

2. *Take $u \in \mathbb{B}$ and $f = \begin{pmatrix} a & c' \\ c & a' \end{pmatrix} \in \mathbf{SB}(2, \mathcal{H})$, we know that the quaternion $cu + a'$ has an inverse by definition.*

Now, the maps defined as $f \cdot u := (au + c') \cdot (cu + a')^{-1}$ *lead to isometries of* $\mathbb{B}$*. We can, hence, describe an action of* $\mathbf{SB}(2, \mathcal{H})$ *on* $\mathbb{B}$*.*

3. *The action defined in the aforementioned list leads to the following natural exact sequence*

$$1 \to \{1, -1\} \to \mathbf{SB}(2, \mathcal{H}) \to \mathbf{Iso}^{+}(\mathbb{B}) \to 1$$

Proof. The proof is described in Chapter 1 of [EGM], in the Unit Ball Model section. □

1.2.3 Modular Forms over the Modular Group

Definition 1.2.4 (Modular Group)**.**

$$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2, \mathbb{Z}) : ad - bc = 1 \right\}.$$

By exploiting the condition of the elementary Bezout's lemma, we can just pick a pair of co-prime numbers (a, b) to get a corresponding pair of $(-c, d)$ that satisfies $ad - bc = 1$

Theorem 1.2.6. *The matrices* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ *generate* $\mathrm{SL}(2, \mathbb{Z})$*.*

Proof. The first chapter of [FD] provides a neat algebraic proof of the assertion. □

Using Theorem 1.2.6, we can conclude

$$\mathrm{SL}(2, \mathbb{Z}) \cong \left\langle S, T \mid S^2 = I, (ST)^3 = I \right\rangle$$

Now, that condition implies that we've got two torsion-free presentation rules for the group, suggesting that the group is isomorphic to $C_2 * C_3$.

Definition 1.2.5 (Modular Form of weight k over the Modular Group)**.** *Let $k \in \mathbb{Z}$. A modular form of weight k for* $\mathrm{SL}(2,\mathbb{Z})$ *is a function $f : \mathbb{H} \to \mathbb{C}$ such that -*

- *f is holomorphic on $\mathbb{H}$*
- $f\left(\dfrac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ *and all* $\tau \in \mathbb{H}$
- *the values $f(\tau)$ are bounded as* $\operatorname{Im}\tau \to \infty$.

The second condition is called the **modularity condition** of Modular forms. Clearly, the zero(trivial) map can be considered a modular form regardless of the weight.

1.2.4 Exploiting the Modularity Condition

We can get 3 crucial properties of Modular Forms by applying simple elements from $\mathrm{SL}(2,\mathbb{Z})$ onto the Modularity Condition.

The Periodicity Property For the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, the modularity condition suggests $f(\tau + 1) = f(\tau)\ \forall\ \tau \in \mathbb{H}$. Notice that there is no relevance of the weight k here.

Second Property For the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, we can use the condition to show that $f(-1/\tau) = \tau^k f(\tau)\ \forall\ \tau \in \mathbb{H}$. In this property, there is a prominent presence of the weight k.

Third Property For the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, we can use the condition to show that $f(\tau) = (-1)^k f(\tau)\ \forall\ \tau \in \mathbb{H}$. Now, whenever k is odd, we can conclude that f is zero.

Hence, the sole modular form for any weight $k = 2m + 1$ for $\mathrm{SL}(2, \mathbb{Z})$ is equivalent to the zero map.

Definition 1.2.6 (Linear Fractional Transformations). *For τ in $\mathbb{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}^+(2, \mathbf{R})$, we define the group action -*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$$

1.2.5 Computation of Modular Forms

Eisenstein Series

Definition 1.2.7 (Weight k Eisenstein Series)**.** *We define the weight k Eisenstein series as*

$$G_k(\tau) := \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}$$

Theorem 1.2.7. *The Eisenstein series G_k is always a modular form over $\mathrm{SL}(2, \mathbb{Z})$, having weight k, if $k \geq 4$ and k is even,*

Proof. The only non-trivial verification of the three properties needed is the holomorphicity, which requires absolute convergence of the series. The verification has been covered in [FD], Chapter 1. □

Of course, the theorem is true for odd weight as well, but it'll end up being the zero function anyway.

Periodicity Condition and Standard Form

We previously saw that every modular form satisfies the periodicity condition. The function $f(\tau) = e^{2\pi i\tau}$ also satisfies $f(\tau+1) = f(\tau)$. We use this to define a formal style of describing modular forms via a power series in $e^{2\pi i\tau}$.

Lemma 1.2.8 (Power Series lemma)**.** *If a function $f : \mathbb{H} \to \mathbb{C}$ satisfies the following conditions -*

- *f is holomorphic*
- *$f(\tau+1) = f(\tau)\ \forall\ \tau$*
- *As $\tau \to \infty$, f must be bounded*

then there are complex coefficients $a_n \in \mathbb{C}$ for $n \geq 0$ such that

$$f(\tau) = \sum_{n\geq 0} a_n e^{2\pi i n\tau}$$

$\forall\ \tau \in \mathbb{H}$. We also know that, as $\tau \to i\infty$, $f(\tau)$ has a limit.

Proof. Please refer to [FD], chapter 1. □

Q-Expansion

Definition 1.2.8. *The q-expansion of any m.f. $f(\tau)$ is defined as $\sum_{n\geq 0} a_n q^n$ for which $f(\tau) = \sum_{n\geq 0} a_n e^{2\pi i n\tau}$.*
The complex coefficients $a_n \in \mathbb{C}$ in the q-expansion are colloquially known as the Fourier coefficients of f.

Now, we must note that q-expansion is more than just some abstract entity: the expression $f(\tau) = \sum_{n\geq 0} a_n e^{2\pi i n\tau}$ is, in fact, analytic on LHS and RHS, with the RHS being convergent regardless of the element $\tau \in \mathbb{H}$.

If we are describing any modular form $f(\tau)$ with its q-expansion, it is a standard abuse of notation to describe the map as $f(q)$, where f is being carried over from the original definition and the changed domain input is $q = e^{2\pi i\tau}$.

Q-expansion of an Eisenstein Series

Definition 1.2.9 (Cusp Form)**.** *Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a modular form of weight k. Now, f is a cusp form if $a_0 = 0$ in the Fourier expansion of $f[\alpha]_k$ for all $\alpha \in \mathrm{SL}(2,\mathbb{Z})$,*

Theorem 1.2.9. *For even $k \geq 4$, the q-expansion of $G_k(\tau)$ is*

$$G_k(\tau) = 2\sum_{n\geq 1} 1/n^k + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$

Proof. The result is a consequence of the Power Series Lemma, and an application of Poisson Summation of the series. The proof is covered in the section on Fourier Analysis and Poisson Summation in the 4th chapter of [**?**, FD] □

In the previous theorem, we've seen that the zero-degree term in the series is $2\sum_{n\geq 1} 1/n^k$, which means that the constant term of $G_k(\tau)$ is $2\zeta(k)$. We already know that the zeta function is really nice for even positive integers, being a normalized multiple of π^k

Bernoulli numbers form

This is the Euler formula for Zeta function for positive, even integers

$$\zeta(k) = \frac{(2\pi)^k(-1)^{k/2+1}}{k!}\frac{B_k}{2} = -\frac{(2\pi i)^k}{(k-1)!}\frac{B_k}{2k}$$

where B_k is the k^{th} Bernoulli number.

Definition 1.2.10 (Bernoulli number)**.** *We define B_k, the k^{th} Bernoulli number, as the coefficient of x^k in the following series.*

$$\frac{x}{e^x - 1} = \sum_{k\geq 0} \frac{B_k}{k!}x^k = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

Plugging it into the last theorem, we get -

$$G_k(\tau) = 2\zeta(k) - \frac{4k\zeta(k)}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

We realize that a normalized version of the Eisenstein Series(constant term 1) can be computed just with the knowledge of Bernoulli Numbers.

1.2.6 Congruence Subgroups of $\mathrm{SL}(2, \mathbb{Z})$

Definition 1.2.11 (Principle Congruence Subgroup)**.**

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

is called the principle congruence subgroup of level N.

Definition 1.2.12 (Congruence Subgroup of level N). *A congruence subgroup is defined as a subgroup* Γ *of* $\mathrm{SL}(2, \mathbb{Z})$ *such that* $\Gamma(N) \subset \Gamma$ *for a particular* $N \in \mathbb{Z}^+$.

Based on the value of N*, we say that* Γ *is a congruence subgroup of level* N.

Special subgroups

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} (\bmod N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} (\bmod N) \right\}$$

Clearly, $\Gamma_0(N)$ is just the elements of $\Gamma(N)$ with $c = 0 \bmod N$.

$\Gamma_1(N)$ is just the elements of $\Gamma_0(N)$ with $a, d = 1 \bmod N$.

$\Gamma(N)$ is just the elements of $\Gamma_1(N)$ with $b = 0 \bmod N$.

We get -

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}(2, \mathbb{Z})$$

This is called the chain of standard congruence groups.

Theorem 1.2.10. $\Gamma(N)$ *has a finite index in* $\mathrm{SL}(2, \mathbb{Z})$

Proof. Please refer to [FD], chapter 1. □

Modular Forms wrt congruence groups

Definition 1.2.13 (Factor of automorphy). *For* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ *and* $\tau \in \mathbb{H}$*, we define a factor of automorphy as -*

$$j(\gamma, \tau) = c\tau + d$$

Definition 1.2.14 (Weight k operator). *For $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and any integer k, weight- k operator $[\gamma]_k$ on functions $f : \mathbb{H} \longrightarrow \mathbb{C}$ is defined as -*

$$(f[\gamma]_k)(\tau) = j(\gamma,\tau)^{-k} f(\gamma(\tau)), \quad \tau \in \mathbb{H}.$$

Definition 1.2.15 (Modular Form wrt a congruence group). *Pick a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and call it Γ. Now we define a modular form of weight k with respect to Γ as a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ if*

- *f is holomorphic,*
- *f is weight-k invariant under Γ,*
- *$f[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}(2,\mathbb{Z})$.*

1.3 Fundamental Domains for Modular Forms

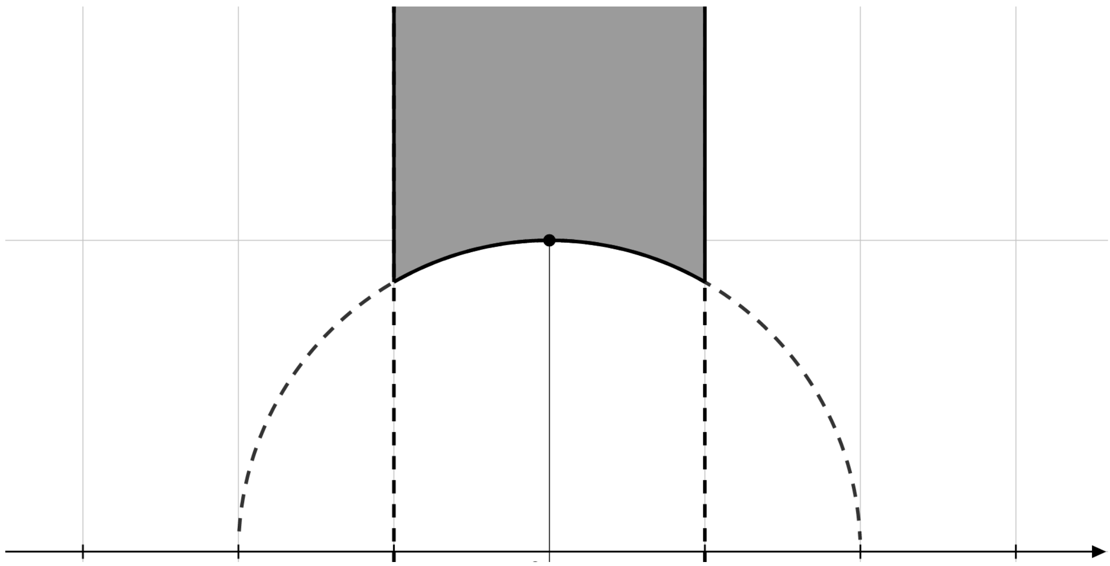
Definition 1.3.1 (Fundamental Domain for congruence groups). *We pick a congruence subgroup Γ to define the fundamental domain for. We call a closed subset of $\mathbb{H}$ a fundamental domain F if -*

- *every element of $\mathbb{H}$ can be taken to a point in F by acting an element of Γ onto the point.*
- *no two points in the interior of F can go to the same point under an action from an element of Γ*

There is a classical fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ is given by -

$$F = \left\{ z : z \in \mathbb{H} \text{ and } |z| \geq 1 \text{ and } |Re(z)| \leq \frac{1}{2} \right\}$$

Figure 1.1: The classical fundamental domain for the Modular Group made using tikz



We use this domain to create fundamental domains for congruence groups in the following fashion -

Theorem 1.3.1. *Take* Γ *as a congruence subgroup of* $\mathrm{SL}(2, \mathbf{Z})$ *and write the full group as a union of the various cosets of the congruence subgroup*

$$\mathrm{SL}(2, \mathbf{Z}) = \coprod_{i=1}^{n} \alpha_i \Gamma$$

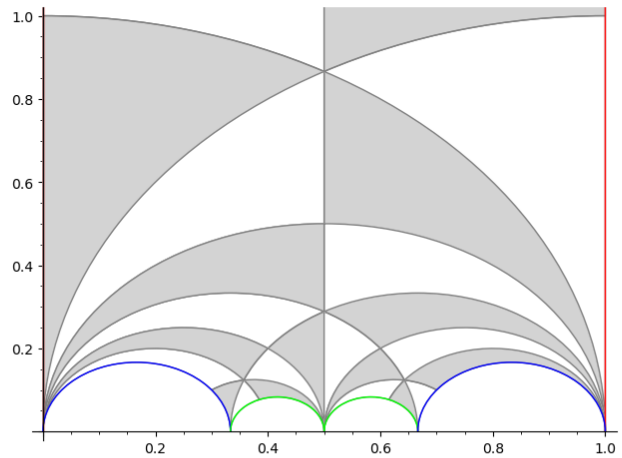
Now, one appropriate fundamental domain for Γ *can be given by* $F_\Gamma := \coprod_{i=1}^{n} \alpha_i^{-1} F$

Cusps on a Congruence Group

Definition 1.3.2. *Extension of the two-dimensional hyperbolic space*

$$\overline{\mathbb{H}} = \mathbb{H} \cup \{i\infty\} \cup \mathbf{Q}$$

Figure 1.2: Fundamental domain for $\Gamma_0(11)$ made using Faray Symbols in SageMath



Definition 1.3.3. *The points added through the extension are distributed into the equivalence classes by Γ and the representatives of those classes are called the cusps of Γ.*

$\mathrm{SL}(2,\mathbb{Z})$ **has a unique cusp.** We previously saw how we can create an element of $\mathrm{SL}(2,\mathbb{Z})$ with any lowest form fraction $\frac{a}{c}$. This gives us a map from $i\infty$ to each element in $\mathbf{Q}$, making the cusp unique.

Intuitively, we can say that the cusps are the points of intersection between the newly-added boundary of $\mathbb{H}$ and the boundary of a fundamental domain of the group.

1.4 Kleinian Groups

1.4.1 $\mathrm{PSL}(2,\mathbb{C})$ and its subgroups.

Definition 1.4.1. *If all the elements of a group Γ have a shared fixed point in their action on the Riemann Sphere, the the group is called reducible.*

Otherwise, the group is irreducible.

Definition 1.4.2. *A discrete subgroup of* $\mathrm{PSL}(2, \mathbb{R})$ *is called a Fuchisian group.*

Definition 1.4.3. *If a group* Γ *has a finite orbit in its action on* $\mathbb{H}$ *and its boundary(i.e. the Riemann Sphere), then the group is called elementary .*
If not, the group is considered non-elementary.

Definition 1.4.4. *A discrete subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *is called a Kleinian group.*

Classification of elements of a Kleinian group

Definition 1.4.5. *If* $\gamma \in \mathrm{PSL}(2.\mathbb{C}), \gamma \neq \pm I$ *we call* γ

- γ *is elliptic if* $tr(\gamma) \in \mathbb{R}$ *and* $\mid tr(\gamma) \mid < 2$.
- γ *is parabolic if* $tr\ \gamma = \pm 2$.
- *If* γ *is neither elliptic or parabolic, it is called loxodromic.*
- *If* γ *is loxodromic and the trace of* γ *is real, then* γ *is called hyperbolic.*

Definition 1.4.6. *We call a Kleinian group* Γ *finite covolumed if it has a fundamental domain* F *having a finite hyperbolic volume. The covolume of* Γ *is given by*

$$\mathrm{Covol}(\Gamma) = \int_F dV$$

Theorem 1.4.1. *Any two domains of finite covolume groups will have the same hyperbolic volume.*

Proof. The proof is covered by [MR] in Section 1.2 of Chapter 1. □

Definition 1.4.7. *If $\Gamma_1, \Gamma_2 \subset \mathrm{PSL}(2, \mathbb{C})$, we call the pair of subgroups directly commensurable if their intersection has got a finite index in Γ_1 and Γ_2.*

Definition 1.4.8. *We say that the pair are commensurable if Γ_1 and a conjugate of Γ_2 are directly commensurable.*

1.5 Fundamental Polyhedrons and relevant notations

Definition 1.5.1. *Elements $(\mu, \lambda) \in \mathcal{O}_F^2$ are called unimodular if $\mu\mathcal{O}_F + \lambda\mathcal{O}_F = \mathcal{O}_F$*

Definition 1.5.2. *We define a polygon $P \subset \mathbb{H}$ as a closed connected subset of a plane of $\mathbb{H}$ such that -*

- *The boundary of P can be written as a countable union of $s_i \cap \mathbb{H}$. Here, s_i are the hyperbolic segments in the extended $\mathbb{H}$ space.*
- *The family $\{s_i \cap \mathbb{H}\}$ is locally finite.*

The sets $s_i \cap \mathbb{H}$ are called the edges of the polygon.
The finite endpoints of these edges are called the vertices of the polygon.

Definition 1.5.3. *We define a polyhedron $\mathcal{F} \in \mathbb{H}$ as a connected open set such that -*

- *The boundary of the polyhedron $\mathcal{F}$ can be written as a countable union of polygons. These polygons are called the faces of $\mathcal{F}$.*
- *The intersection of any two faces can be fit in a hyperbolic geodesic and the family of the faces is locally finite.*

Definition 1.5.4. *We define a fundamental polyhedron for any subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *as a fundamental domain for the said subgroup that also satisfies the conditions of a polyhedron that we've defined above.*

Definition 1.5.5 (Pairing transformations)**.** *If* $\mathcal{F}$ *is a polyhedron, and* F *is the set of faces of the polyhedron* $\mathcal{F}$*, we define a face pairing of the polyhedron as a function*
$^* \times g : F \to F \times \mathrm{PSL}(2,\mathbb{C})$ *such that*
(a) $g(f) \cdot f = f^*$
(b) $\cdot^* : F \to F$ *is an involution map*
(c) Each f *has a neighborhood* V *which satisfies* $(g(f) \cdot (V \cap \mathcal{F})) \cap \mathcal{F} = \emptyset$.
where f^* *is the final face and* $g(f) \in \mathrm{PSL}(2,\mathbb{C})$ *is the map used for the transformation. The elements of* $\mathrm{PSL}(2,\mathbb{C})$ *that take* f *to* f^* *are called pairing transformations.*

Definition 1.5.6 (Complete Polyhedron)**.** *Given a face pairing,* $\overline{\mathcal{F}}$ *gets partitioned into equivalence classes by a given pairing transformation.*
Call it $\mathcal{F}^* = \overline{\mathcal{F}}/\sim$*, with the natural map* $\pi : \overline{\mathcal{F}} \to \mathcal{F}^*$.
Pick any two $x, y \in \mathcal{F}^*$*, and define* $\mathrm{d}^*(x,y) = \inf \sum_{i=1}^{n} \mathrm{d}\left(z_i, w_i\right)$ *where we find the infimum over* $\left(z_i, w_i\right)_i$ *of* $\overline{\mathcal{F}}$ *such that* $\pi\left(z_1\right) = x, z_{i+1} \sim w_i$ *and* $\pi\left(w_n\right) = y$.
Now, we say that the polyhedron is complete if
(a) for every $x \in \overline{\mathcal{F}}, \pi^{-1}(x)$ *is finite, in which case* d^* *is a metric on* $\mathcal{F}^*$*, and*
(b) $\mathcal{F}^*$ *is complete for this metric.*

1.6 Constructing Kleinian Groups from Quaternion Algebra

Throughout this section, F will be a field with char $\neq 2$ unless stated otherwise.

Definition 1.6.1 (Quaternion Algebra)**.** *Let* $a, b \in F^\times$*.*

$$H = \left\langle i, j \mid i^2 = a, j^2 = b, ij = -ji \right\rangle$$

We call it H *a quaternion algebra over* F *and denote it by* $\left(\frac{a,b}{F}\right)$*.*

Example 1. *The* 2×2 *Matrix ring over a field* F *is a quaternion algebra over* F *with the follwoing notation*

$$\mathcal{M}(2, F) \cong \left(\frac{1,1}{F}\right)$$

The algebra is generated by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *as* i *and* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *as* j*.*

Definition 1.6.2 (Quasi-totally Real Field, Kleinian Quaternion Algebra)**.** *Let* F *be a number field.* F *is considered a quasi-totally real field or QTR if the field has exactly two conjugate complex embeddings or exactly one complex place. We define a Kleinian quaternion algebra as a quaternion algebra over a quasi-totally real number field, which is ramified at every real place.*

Every quadratic imaginary number field is QTR.

Definition 1.6.3 (Order)**.** *Let* $\mathbb{Z}_F$ *be the ring of integers of a number field* F*. An Order in* B*, an algebra in* F*, is an f.g.* $\mathbb{Z}_F$*-submodule* $\mathcal{O} \subset B$ *and* $B = F\mathcal{O}$*.*

Definition 1.6.4. *Define* $\mathcal{O}_1^\times = \{x \in \mathcal{O}^\times \mid x\bar{x} = 1\}$*.*

Definition 1.6.5. *Choose a Kleinian group* Γ*. We define the group as arithmetic if it is commensurable with some* $P\rho\left(\mathcal{O}_1^\times\right)$*, where* $\mathcal{O}$ *is an order in a quaternion algebra over* F*, ramified at every real place of a QTR number field* F*, and* ρ *is a discrete embedding* $\rho : \mathcal{O}_1^\times \hookrightarrow \mathrm{SL}(2, \mathbb{C})$*.*

Theorem 1.6.1. *Let* F *be a QTR number field of degree* n*,* B *a Kleinian quaternion algebra over* F*, and* $\mathcal{O}$ *an order in* B*. Let* $\Gamma = P\rho\left(\mathcal{O}_1^\times\right)$ *where* ρ *is a discrete embedding* $\rho : \mathcal{O}_1^\times \hookrightarrow \mathrm{SL}(2, \mathbb{C})$*. Then* Γ *has finite covolume.*

Proof. The theorem is covered in the 11th Chapter of [MR] in detail. □

1.7 Bianchi Groups

Definition 1.7.1 (Bianchi Group)**.** *Let F be an imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$ and $\mathcal{O}_F$ be its ring of integers. A Kleinian group* $\mathrm{PSL}\,(2, \mathcal{O}_F)$ *is called a Bianchi group.*

Theorem 1.7.1.

$$\mathcal{O}_F = \mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\},$$
$$\textit{where } \omega = \begin{cases} \frac{-1+\sqrt{-d}}{2}, & \textit{if } d \equiv 3(\mathrm{mod}4); \\ \sqrt{-d}, & \textit{otherwise.} \end{cases}$$

Definition 1.7.2 (Principal Congruent Subgroup)**.** *Suppose $\mathfrak{i} \neq 0$ is an ideal in $\mathcal{O}_F$.*

$$\Gamma(\mathfrak{i}) := \{\gamma \in \mathrm{PSL}\,(2, \mathcal{O}_F) : \gamma \equiv 1 \mod \mathfrak{i}\}$$

is called the principal congruent subgroup of $\mathrm{PSL}\,(2, \mathcal{O}_F)$ *of level i.*

Definition 1.7.3 (Congruence subgroup)**.** *A finite index subgroup of* $\mathrm{PSL}\,(2, \mathcal{O}_F)$ *containing a principal congruent subgroup is called a congruent subgroup.*

Definition 1.7.4. *Let $\mathfrak{n}$ be an ideal in* $\mathrm{PSL}\,(2, \mathcal{O}_F)$*, we define*

$$\Gamma_0(\mathfrak{n}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \in \mathfrak{n}\}$$

We need the following Proposition to assure us that the class of groups in question throughout the thesis are finite indexed and we can use that to our advantage.

Proposition 1.7.2. *The index of $\Gamma_0(\mathfrak{a})$ in* $\mathrm{PSL}\,(2, \mathcal{O}_K)$ *is given by the multiplicative func-*

tion, where N is the ideal norm in the ring of integers.

$$\iota(\mathfrak{a}) = \mathrm{N}(\mathfrak{a}) \prod_{\mathfrak{p} | \mathfrak{a}} \left(1 + \frac{1}{\mathrm{N}(\mathfrak{p})} \right).$$

Proof. The proof is covered in Chapter 7 by [EGM]. $\square$

Chapter 2

Fundamental domain and presentation for Bianchi Groups

This section deals with the method devised by Swan[RS1] in 1971. Swan provided a geometric technique for finding the fundamental domains and finite presentations for most Bianchi Groups. The fundamental domain evaluation will be crucial to our study of the covolume of the Bianchi groups in question.

2.1 Swan's method

Definition 2.1.1. *$S_{\mu,\lambda} \subset \mathbb{H}$ is the hemisphere satisfying $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$, where (μ, λ) is a given unimodular pair. We define $B := \{(z,\zeta) \in \mathbb{H} : |\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ is satisfied for all u.p. $(\mu, \lambda) \in \mathcal{O}^2$ with $\mu \neq 0\}$.*

Lemma 2.1.1 (Swan)**.** *B includes at least one representative from the orbits of all the points given the natural action of $\mathrm{SL}(2, \mathcal{O})$ on $\mathbb{H}$.*

Consider the stabiliser group Γ_∞ of the point at infinity $\in \partial\mathbb{H}$. Then,

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathcal{O} \right\},$$

Proposition 2.1.2. *A fundamental domain for Γ_∞ in the complex plane is given by*

$$D_0 := \begin{cases} \{a + b\sqrt{-m} \in \mathbb{C} \mid 0 \leqslant a \leqslant 1, 0 \leqslant b \leqslant 1\}, & m \equiv 1 \textit{ or } 2 \mod 4, \\ \left\{a + b\sqrt{-m} \in \mathbb{C} \mid \frac{-1}{2} \leqslant a \leqslant \frac{1}{2}, 0 \leqslant b \leqslant \frac{1}{2}\right\}, & m \equiv 3 \mod 4. \end{cases}$$

And a fundamental domain for Γ_∞ in $\mathbb{H}$ is given by

$$D_\infty := \{(z, \zeta) \in \mathbb{H} \mid z \in D_0\}$$

Definition 2.1.2. *The Bianchi fundamental polyhedron*

$$D := D_\infty \cap B.$$

From Lemma 2.1.1, we get $\Gamma \cdot B = \mathbb{H}$, and as $\Gamma_\infty \cdot D_\infty = \mathbb{H}$ implies $\Gamma_\infty \cdot D = B$, we get that $\Gamma \cdot D = \mathbb{H}$.

Swan showed that there are only finitely many u.p (λ, μ) where the intersection of $S_{\mu,\lambda}$ with the B.F.P is non-empty. From this, we can infer that there are only finitely many points on the border of the polyhedron.

Another corollary was the fact that there are only finitely more matrices that meet the polyhedron non-trivially.

Picking the polyhedron While B was picked from infinitely many elements, we need to ensure that we get a good approximate by using finitely many pairs. Swan came up with a criterion that ensures precise calculation of B using finitely many elements. The approach goes as follows -

Make a selection of n hemispheres, where the i-th hemisphere is given by $S(\alpha_i)$, where α_i is its centre, $\alpha_i = \frac{\lambda_i}{\mu_i}$ in the number field F. Here, we require the ideal (λ_i, μ_i) to be the whole ring of integers $\mathcal{O}$.

Definition 2.1.3. *Define* $B(\alpha_1, \ldots, \alpha_n) := \{(z, \zeta) \in \mathbb{H} :$ *The inequality* $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ *is true for all u.p* $(\mu, \lambda) \in \mathcal{O}^2$ *with* $\frac{\lambda}{\mu} = \alpha_i + \gamma$, *for some* $\gamma \in \mathcal{O}\}$. *Then* $B(\alpha_1, \ldots, \alpha_n)$ *is the set of all points in* $\mathbb{H}$ *lying above all hemispheres* $S(\alpha_i + \gamma)$; *for any* $\gamma \in \mathcal{O}$.

Claim 1. $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ *with the fundamental domain* D_∞ *for the translation group* Γ_∞, *is equal to the Bianchi fundamental polyhedron.*

Definition 2.1.4. *The hemisphere* $S_{\mu,\lambda}$ *is considered strictly below the hemisphere* $S_{\beta,\alpha}$ *at a point* $z \in \mathbb{C}$ *if the following inequality is satisfied:*

$$\left|z - \frac{\alpha}{\beta}\right|^2 - \frac{1}{|\beta|^2} < \left|z - \frac{\lambda}{\mu}\right|^2 - \frac{1}{|\mu|^2}$$

2.1.1 Dealing with cusps

It is well known that a given cusp $\frac{\lambda}{\mu}$ is in the $\mathrm{SL}_2(\mathcal{O})$-orbit of another cusp $\frac{\lambda'}{\mu'}$, iff the ideals (λ', μ') and (λ, μ) are members of the same ideal class. Clearly, the point at infinity represents the principal ideals. We use this result by Swan, to find representatives of non-principal ideals -

Lemma 2.1.3 (Swan)**.** *The singular points of* $F \bmod \mathcal{O}$ *are*

$$\frac{p(r + \sqrt{-m})}{s}$$

Where -

- *If* $m \equiv 1$ *or* $2 \bmod 4$, $s \neq 1, s \mid r^2 + m$, *the numbers* p *and* s *are coprime, and* p *is modulo* $\bmod s$*;*

- *- if* $m \equiv 3 \bmod 4$, s *is even,* $s \neq 2, 2s \mid r^2 + m$, *the numbers* p *and* $\frac{s}{2}$ *are coprime;* p *is modulo* $\frac{s}{2}$.

- $p, r, s \in \mathbb{Z}, s > 0, \quad \frac{-s}{2} < r \leqslant \frac{s}{2}, \quad s^2 \leqslant r^2 + m,$

Definition 2.1.5. $L\left(\frac{\alpha}{\beta}, \frac{\lambda}{\mu}\right)$ *is the set of* $z \in \mathbb{C}$ *over which neither* $S_{\beta,\alpha}$ *is strictly below* $S_{\mu,\lambda}$ *nor vice versa.*

This set can either be the entirety of $\mathbb{C}$ or a line trisecting the space into two open half-planes.

Theorem 2.1.4. *(Swan criterion)* $B(\alpha_1, \ldots, \alpha_n) = B$ *iff no vertex of* $\partial B(\alpha_1, \ldots, \alpha_n)$ *can be strictly below any hemisphere* $S_{\mu,\lambda}$.

This shows that it's enough to compute the endpoints of the polyhedron, to get the fundamental domain.

2.2 Applying the method

Definition 2.2.1. $S_{\mu,\lambda}$ *is everywhere below* $S_{\beta,\alpha}$ *when:*

$$\left|\frac{\lambda}{\mu} - \frac{\alpha}{\beta}\right| \leqslant \frac{1}{|\beta|} - \frac{1}{|\mu|}$$

Theorem 2.2.1. *Let* $S(\alpha_n)$ *be everywhere below* $S(\alpha_i)$, *where* $i \in \{1, \ldots, n-1\}$. *Then* $B(\alpha_1, \ldots, \alpha_n) = B(\alpha_1, \ldots, \alpha_{n-1})$.

Proof. Using the definition 2.1.3, we need to prove -

Given $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 < 1$ holds, $|\tau z - \theta|^2 + |\tau|^2\zeta^2 < 1$ is true.

This is a simple consequence of triangular inequality after dividing both sides of the said inequality by $|\mu|^2$.

$$\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2}$$

Now, from Definition 2.2.1, we know $\left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \frac{1}{|\mu|} \leq \frac{1}{|\tau|}$

Plugging that into the inequality and squaring, we get the desired result

$$\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2 < \frac{1}{|\tau|^2}$$

□

Initialization We initialize the process by picking an element such that the norm of $\mu \in \mathcal{O}$ can take the minimum value, namely 1. Hence, μ is a unit in $\mathcal{O}$, and for any $\lambda \in \mathcal{O}$, it forms a u.p. We thus obtain unit hemispheres (of radius 1), centred at the integers $\lambda \in \mathcal{O}$. We choose the centre points which reside in the Bianchi fundamental polyhedron for the action of Γ_∞ on the complex plane.

Increasing the norm Iterate through a finite set of μ with increasing norm, and a finite set of τ for each μ while ensuring -

- $\frac{\lambda}{\mu}$ lies in the fundamental rectangle.
- Ideal sum of μ and τ gives the whole ring.
- The hemisphere formed is not strictly lower than the preceding hemisphere.

Fulfilling the criteron and terminating the algorithm Calculate $L\left(\frac{\alpha}{\beta}, \frac{\lambda}{\mu}\right)$ for all the $S_{\beta,\alpha}, S_{\mu,\lambda}$ in the list which touch one another. Then, for every $S_{\beta,\alpha}$ present, we find the intersection of all such lines $L\left(\frac{\alpha}{\beta}, \frac{\lambda}{\mu}\right)$ and $L\left(\frac{\alpha}{\beta}, \frac{\theta}{\tau}\right)$ referring to $\frac{\alpha}{\beta}$.

We erase the intersection points at which $S_{\beta,\alpha}$ is strictly below another hemisphere in the list.

Next, we remove the ones from our list, for which two or fewer intersection points exist.

Now, the vertices of $B(\alpha_1, \dots, \alpha_n) \cap D_\infty$ are the lifts of the remaining intersection points.

Check for the termination criterion now, by choosing the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathbb{H}$ is the lift of a remaining intersection point z.

If $\zeta \geqslant \frac{1}{|\mu|}$, the criterion is satisfied and we have found the Bianchi fundamental polyhedron.

Otherwise, set ζ as the updated expected value for $\frac{1}{|\mu|}$.

Repeat **Increasing the norm** until $|\mu|$ reaches $\frac{1}{\zeta}$ and then proceed with termination again.

Chapter 3

The numerator - Abelianization

3.1 Abelianization of a group

Definition 3.1.1 (Commutator subgroup)**.** *Let G be a group and $a, b \in G$. Choose $M = \{aba^{-1}b^{-1} : a, b \in G\}$ and denote $[G, G]$ as the subgroup of G generated by M. $[G, G]$ is called the commutator subgroup of G.*

Definition 3.1.2. *The commutator subgroup is a normal subgroup and the associated quotient group is called the abelianization of G.*

3.1.1 Dirichlet Domain

Definition 3.1.3. *Choose a point $p \in \mathcal{B}$ such that its stabilizer is trivial in Γ. We define the Dirichlet domain centered at p as*

$$D_p(\Gamma) = \{x \in \mathcal{B} \mid \text{ for all } \gamma \in \Gamma \backslash \{1\}, \mathrm{d}(x, p) < \mathrm{d}(\gamma x, p)\}$$

Theorem 3.1.1. *Let $D_p(\Gamma)$ be a Dirichlet Domain. Then the domain meets the conditions of a fundamental domain for Γ. In fact, the domain is a convex fundamental polyhedron.*

Proof. The proof is described in Section 3.5 in [AM]. □

Theorem 3.1.2. *If a Kleinian group Γ is finite covolumed, then any associated Dirichlet Domain to the group will have only finitely many faces.*

Proof. The proof is described in Section 3.5 in [AM]. □

Definition 3.1.4. *We define a Kleinian group Γ to be geometrically finite if any Dirichlet domain associated with the group has got only finitely many faces.*

Theorem 3.1.3 (Presentation)**.** *Take a Dirichlet domain of a Kleinian group Γ where it is centred around 0 and the group geometrically finite.*
Then the group can be presented using the face pairing transformations of the group Γ, with the rules derived from the cyclic and reflective relations of the transformations.

Proof. This is the main theorem of Section 3.5 from [AM] and the presentation of the group is derived explicitly and in detail. We request the reader to refer to the book for the proof. □

We can use the said presentation to get the generators of the group through a Reidemeister-Schreier Rewriting Process.

In our case, we were generously provided with a more effective algorithm for abelianization calculations.

Chapter 4

The denominator - Volume

4.1 Using Swan's fundamental polyhedron

Proposition 4.1.1. *The integral*

$$-\int_0^\theta \ln|2\sin u|du$$

converges for $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$ *and admits a continuous extension to* $\mathbb{R}$*, which is odd and periodic with period* π.

Definition 4.1.1. *The aforementioned extension is called the Lobachevsky function* $\mathcal{L}(\theta)$.

Proposition 4.1.2. *The Lobachevsky function admits a power series expansion:*

$$\mathcal{L}(\theta) = \theta\left(1 - \ln(2\theta) + \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{2n(2n+1)!}\theta^{2n}\right)$$

where the B_n *are the Bernoulli numbers.*

Theorem 4.1.3 (Uniqueness of tetrahedron and volume approximation). *Let $T_{\alpha,\gamma}$ be the tetrahedron in $\mathbb{H}$ where 1 of the vertices is at infinity and the rest P, Q, R are on the hemisphere of radius one such that they project vertically onto P', Q', R' in $\mathbb{C}$ with $P' = 0$ to form a Euclidean triangle, with angles $\frac{\pi}{2}$ at Q' with and α and γ as the other angles. Then $T_{\alpha,\gamma}$ is unique up to isometry and*

$$\mathrm{Vol}\left(T_{\alpha,\gamma}\right) = \frac{1}{4}\left[\mathcal{L}(\alpha+\gamma) + \mathcal{L}(\alpha-\gamma) + 2\mathcal{L}\left(\frac{\pi}{2} - \alpha\right)\right].$$

Proof. Please refer to [MR], chapter 11. □

4.2 Algorithm

The algorithm was implemented in SageMath[SGM] using the vertices derived through Swan's Method.

It was designed to partition the fundamental polyhedron into a finite set of tetrahedrons with one point at infinity and the other three on the unit sphere of the hyperbolic space. Using Theorem 4.1.3, we know that the tetrahedrons described are pullbacks of triangles in the complex plane and we create the tetrahedrons by splitting the vertices into triangles recovered from Swan's method.

The volume for each tetrahedron is approximated using the power series mentioned in Proposition 4.1.2, picking a cutoff of 0.01 between consecutive terms. The finite termination of Swan's Method alongside this cutoff ensures that the algorithm is feasible for actual calculations.

Algorithm 1 Volume of the Fundamental Polyhedron

Input: Vertices recovered from Swan's Method = F
Output: Vol(F)

$0 \leftarrow Vol(F)$ ▷ Initialize Volume
$[] \leftarrow Tetra$ ▷ Initialize Set of Tetrahedrons
$[] \leftarrow Triangle$ ▷ Initialize Set of Triangles

$f_0 = 0$ ▷ Pick the point at infinity
for $f_1 \in F$ **do**
    for $f_2 \in F$ **do**

        append (f_1, f_2, f_0) to $Triangle$

for $t \in Triangle$ **do**

    append pullback(t) to $Tetra$ ▷ Tetra is now the set of all the polyhedrons
$[] \leftarrow UnitTetra$
▷ Initialize Set of Tetrahedrons at Unit Sphere
for $t \in Tetra$ **do**

    if t is on the unit sphere **then**
        append t to $UnitTetra$
for $t \in UnitTetra$ **do**
    Fix a point P
    if The first vertex is not P **then**
        remove t from $UnitTetra$ ▷ Fix one vertex of $UnitTetra$
for $t \in UnitTetra$ **do**
    Compute the angles of the triangles between the vertices not at infinity.
    if There isn't a right angle on the side angles **then**
        remove t from $UnitTetra$ ▷ Fix right angle triangles

We now have a partition of the polyhedron and we have a volume approximation for each of them.

for t in $UnitTetra$ **do**
    Vol(F) = Vol(F) + Vol(t)
return Vol(F)

Chapter 5

Computations

This section contains the various computation results acquired during the project. While most of the results in sections 5.1 and 5.2 are present in the literature already[MHS1][SGM][AP], we believe that certain computations in sections 5.4 and 5.5 are original and absent from any literature.

5.1 Fundamental domains of classical Modular Forms

5.1.1 Farey Symbols and the computational algorithm

In 1991, Ravi Kulkarni[RK1] presented a compututationally feasible method for finding the fundamental domains of certain subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. The method used the classical sequence of reduced fractions between 0 and 1, and was hence named Farey Symbols. The method revolved around describing the an arithmetic subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ using corrosponding elements of a Farey sequence and joining these elements in the two-dimensional hyperbolic space. The theory can be further explored from [RK1].

5.1.2 Farey Sequence

We define a generalized Farey Sequence here as follows -

Definition 5.1.1.

$$S = \left\{ \frac{-1}{0}, t_0, \ldots, t_n, \frac{1}{0} \right\}$$

We say that a set S is a Farey Sequence if -

- *Each t_i is a rational number and denote them as $t_i = \frac{p_i}{q_i}$ in their reduced form.*
- *If we let $p_{-1} = -1, q_{-1} = 0, p_{n+1} = 1,$ and $q_{n+1} = 0$ then*

  $$p_{i+1}q_i - p_i q_{i+1} = 1$$

  for each $0 \leq i \leq n$
- *$t_i = 0$ for some $0 \leq i \leq n$*

For the sake of completion of the definition, we call the first and last term as t_{-1} and t_{n+1}.

Now, Farey Symbols are a special subclass of Farey Sequences with further conditions on the adjacent pair of elements of the sequences.

The Farey Symbols provide us with the following information about the group.

- Minimal generators of the group.

- Index and level of the group in $\mathrm{PSL}_2(\mathbb{Z})$.

- Coset representatives of the group in $\mathrm{PSL}_2(\mathbb{Z})$.

- Number of cusps of the group in $\mathrm{PSL}_2(\mathbb{Z})$.

- Genus of the surface $\Gamma\backslash\mathbb{H}$.

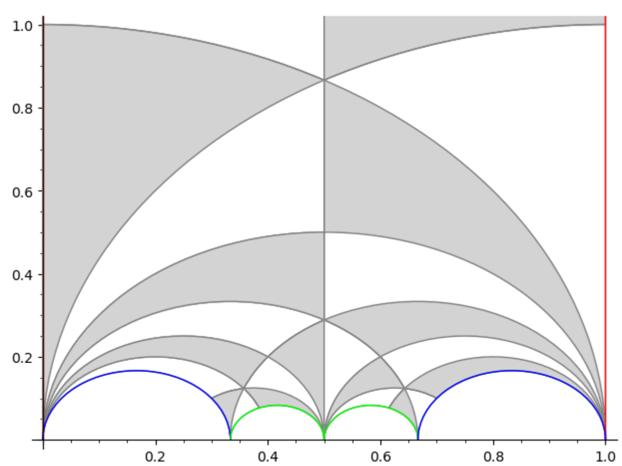
- Fundamental Domain of the group.

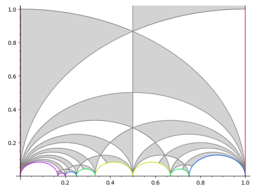
5.1.3 Examples and limitations

We used Farey Symbols to compute all the aforementioned features of the group for the three well-known classes of congruence subgroups - $\Gamma_0(n)$, $\Gamma_1(n)$ and $\Gamma(n)$. The following three subsections cover the data generated.

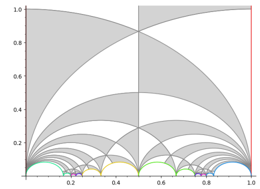
$\Gamma_0(n)$

Figure 5.1: Fundamental domain for $\Gamma_0(11)$ made using Faray Symbols in SageMath





(a) Fundamental domain for $\Gamma_0(12)$



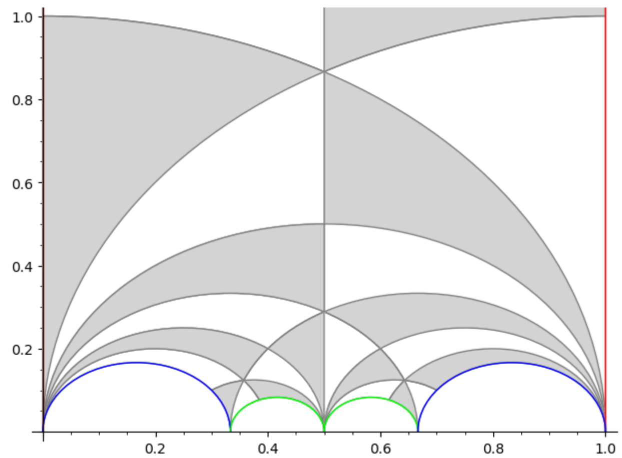
(b) Fundamental domain for $\Gamma_0(18)$

n	Generators	Farey Symbols
1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	[0]
2	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$	[0, 1]
3	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	[0, 1]
4	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	[0, 1/2, 1]
5	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$	[0, 1/2, 1]

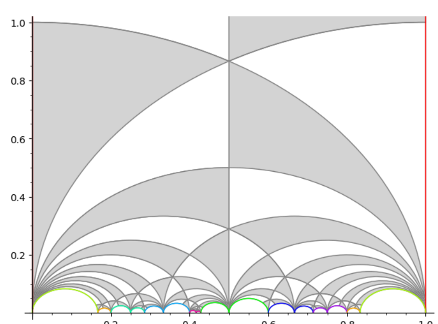
n	Number of generators	Index	Number of Cusps	Genus
1	2	1	1	0
2	2	3	2	0
3	2	4	2	0
4	3	6	3	0
5	3	6	2	0
6	4	12	4	0
7	3	8	2	0
8	4	12	4	0
9	4	12	4	0
10	5	18	4	0
11	4	12	2	1
12	6	24	6	0
13	5	14	2	0
14	6	24	4	1
15	6	24	4	1
16	6	24	6	0
17	5	18	2	1
18	8	36	8	0
19	5	20	2	1
20	8	36	6	1
21	7	32	4	1
22	8	36	4	2
23	6	24	2	2
24	10	48	8	1
25	7	30	6	0
26	9	42	4	2
27	8	36	6	1
28	10	48	6	2
29	7	30	2	2
30	14	72	8	3
31	7	32	2	2

$\Gamma_1(n)$

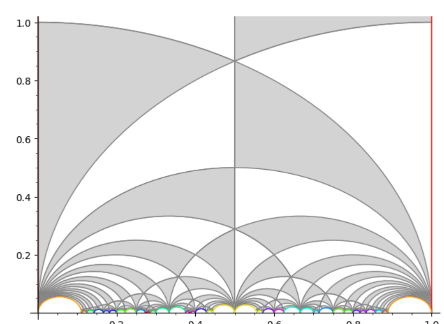
Figure 5.3: Fundamental domain for $\Gamma_1(11)$ made using Faray Symbols in SageMath



n	Generators	Farey Symbols
1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	[0]
2	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$	[0, 1]
3	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	[0, 1]
4	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}$	[0, 1/2, 1]
5	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -4 & 1 \\ -5 & 1 \end{pmatrix}, \begin{pmatrix} 11 & -4 \\ 25 & -9 \end{pmatrix}$	[0, 1/3, 2/5, 1/2, 1]



(a) Fundamental domain for $\Gamma_1(12)$

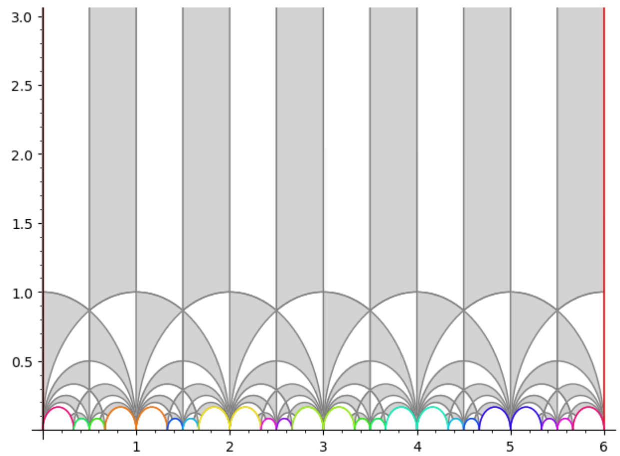


(b) Fundamental domain for $\Gamma_1(18)$

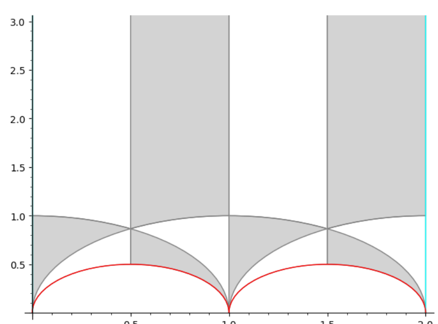
n	Number of generators	Index	Cusp Size	Genus
1	2	1	1	0
2	2	3	2	0
3	2	4	2	0
4	2	6	3	0
5	3	12	4	0
6	3	12	4	0
7	5	24	6	0
8	5	24	6	0
9	7	36	8	0
10	7	36	8	0
11	11	60	10	1
12	9	48	10	0
13	15	84	12	2
14	13	72	12	1
15	17	96	16	1
16	17	96	14	2
17	25	144	16	5
18	19	108	16	2
19	31	180	18	7
20	25	144	20	3
21	33	192	24	5
22	31	180	20	6
23	45	264	22	12
24	33	192	24	5
25	51	300	28	12
26	43	252	24	10
27	55	324	30	13
28	49	288	30	10
29	71	420	28	22
30	49	288	32	9
31	81	480	30	26

$\Gamma(n)$

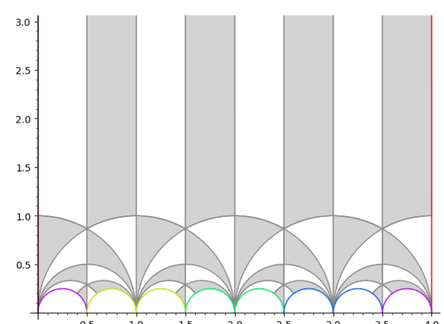
Figure 5.5: Fundamental domain for $\Gamma(6)$ made using Faray Symbols in SageMath



n	Generators	Farey Symbols
1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	[0]
2	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$	[0, 1]
3	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	[0, 1]
4	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	[0, 1/2, 1]
5	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$	[0, 1/2, 1]



(a) Fundamental domain for $\Gamma(2)$



(b) Fundamental domain for $\Gamma(4)$

n	Number of generators	Index	Number of Cusps	Genus
1	2	1	1	0
2	2	3	2	0
3	2	4	2	0
4	3	6	3	0
5	3	6	2	0
6	4	12	4	0
7	3	8	2	0
8	4	12	4	0
9	4	12	4	0
10	5	18	4	0
11	4	12	2	1
12	6	24	6	0
13	5	14	2	0
14	6	24	4	1
15	6	24	4	1
16	6	24	6	0
17	5	18	2	1
18	8	36	8	0
19	5	20	2	1
20	8	36	6	1
21	7	32	4	1
22	8	36	4	2
23	6	24	2	2
24	10	48	8	1
25	7	30	6	0
26	9	42	4	2
27	8	36	6	1
28	10	48	6	2
29	7	30	2	2
30	14	72	8	3
31	7	32	2	2

Limitations The main limitation of the method is that it is only applicable on finite index subgroups of the Modular group.

5.2 Torsion in Bianchi Groups and certain subgroups

Pick $S = \{\mathfrak{p}_n\}_n$ as a sequence of prime ideals in $\mathcal{O}_d$. Then we look at various $\Gamma_0(\mathfrak{p}_\mathfrak{n})$ for 4 different values of d. We define the torsion as the product of the abelian invariants of $\Gamma_0(\mathfrak{p}_\mathfrak{n})$. The next 3 subsections cover those computations.

5.2.1 Case 1

For the first case, we consider the zeroth example of a quadratic number field i.e. $K = \mathbb{Q}(i)$, with the congruence groups constructed by prime ideals $\mathfrak{p}$ in the ring of integers of the field i.e. Gaussian integers or $\mathbb{Z}[i]$. The norm of the prime ideals $\mathfrak{p}$ is given by p.

In the table on the following page, we are comparing the norm of the ideals with the logarithm of the torsion rather than the torsion itself.

This is a statement of the growth of torsion in the sequence of groups, as we increase the norm associated with the prime ideals.

p	log(Tor)		p	log(Tor)		p	log(Tor)
13	3.17		277	10.80		601	14.68
17	3.46		281	13.46		613	14.15
29	5.12		293	11.12		617	14.23
37	4.27		313	12.67		641	21.13
41	5.07		317	9.93		653	16.16
53	4.64		337	12.34		661	19.28
61	4.78		349	11.36		673	13.70
73	4.96		353	13.47		677	13.90
89	7.56		373	14.20		701	19.28
97	6.86		389	12.92		709	15.30
101	8.12		397	11.22		733	21.29
109	5.37		401	15.98		757	15.31
113	6.79		409	13.91		761	19.09
137	7.10		421	11.28		769	22.47
149	7.63		433	11.58		773	15.21
157	8.22		449	15.84		797	19.82
173	7.44		457	11.57		809	18.74
181	10.41		461	13.16		821	20.20
193	8.71		509	12.36		829	21.14
197	9.26		521	19.41		853	13.74
229	12.35		541	15.78		857	23.38
233	8.21		557	16.10		877	21.32
241	9.11		569	17.56		881	24.39
257	7.68		577	16.36		929	16.52
269	11.44		593	18.87		937	22.86

5.2.2 Case 2

This case is concerned with the number field $K = \mathbb{Q}(\sqrt{-2})$ and the corresponding prime ideals $\mathfrak{p}$ of $\mathcal{O}_{-2}$ with norm p.

On the table on the following table, notice how the torsion growth is not only astronomical with respect to the norm but also orders of magnitude higher than the case of Gaussian Integers.

p	log(Tor)		p	log(Tor)		p	log(Tor)
3	1.10		257	19.63		571	31.05
11	1.61		281	18.19		577	42.03
17	2.77		283	16.99		587	30.26
89	7.56		401	17.46		673	41.14
97	12.20		409	24.19		683	44.92
19	3.29		307	16.41		593	35.82
41	3.69		313	22.08		601	33.85
43	4.14		331	21.03		617	39.13
59	3.36		337	21.22		619	28.65
67	7.07		347	16.89		641	35.25
73	9.01		353	18.17		643	42.20
83	7.17		379	27.08		659	38.2
107	3.97		419	19.78		691	40.53
113	8.87		433	25.72		739	46.61
131	5.78		443	30.25		761	40.88
137	11.14		449	31.65		769	48.86
139	10.49		457	31.25		787	44.64
163	13.01		467	21.89		809	37.28
179	9.01		491	22.69		811	52.93
193	18.14		499	32.95		827	44.22
211	12.20		521	26.74		857	48.97
227	13.47		523	30.29		859	38.67
233	15.91		547	35.94		881	53.28
241	16.66		563	32.02		883	52.90
251	18.39		569	31.77		907	53.42

5.2.3 Case 3

This is the case of the number field $K = \mathbb{Q}(\sqrt{-3})$. The ring of integers associated with the field K are the famous Eisenstein Integers i.e.

$$\mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]$$

The notation follows from the last two cases and the growth rate is still rapid, but notice that the torsion is significantly smaller than the last case and slightly smaller than case 1. One reason that can be conjectured for the observation is that the torsion growth has a correlation with the absolute value of the discriminant of the number field.
We know that the discriminant for $K = \mathbb{Q}(\sqrt{-3})$ is -3, while the discriminant for $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(i)$ are -8 and -4 respectively.

p	log(Tor)		p	log(Tor)		p	log(Tor)
3	2.20		241	4.50		577	7.68
7	2.20		271	9.30		601	10.82
13	2.89		277	10.46		607	12.81
19	3.29		283	7.84		613	9.67
31	3.81		307	9.62		619	12.14
37	3.99		313	9.24		631	11.68
43	4.14		331	6.20		643	13.11
61	4.50		337	7.61		661	8.50
67	6.20		349	10.40		673	10.38
73	2.89		367	9.08		691	11.97
79	4.76		373	12.04		709	11.47
97	5.66		379	7.72		727	11.00
103	6.41		397	11.75		733	16.60
109	5.09		409	10.57		739	12.96
127	5.24		421	11.61		751	13.22
139	6.43		433	11.32		757	9.11
151	5.41		439	11.01		769	15.28
157	8.23		457	8.60		787	12.42
163	7.89		463	10.30		811	10.39
181	5.60		487	10.60		823	12.28
193	7.05		499	10.17		829	13.45
199	6.79		523	8.74		853	16.86
211	5.75		541	10.77		859	14.88
223	7.75		547	12.51		877	15.32
229	5.83		571	12.95		883	13.72

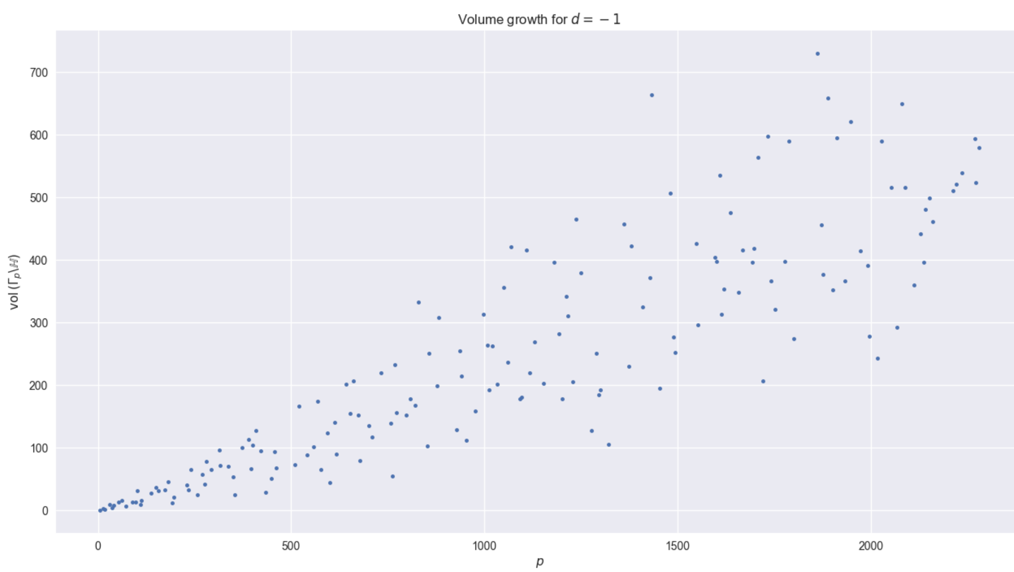
5.3 Covolume computation using fundamental domains

We compute the volume for the congruence subgroups using the algorithm defined in Section 4. We also learnt that the covolume of the space is identical to the volume of the fundamental polyhedron and we exploit that fact to our advantage. We'll use these volume computations in Section 5.4 to study the conjecture for the class of subgroups discussed.

5.3.1 Case 1

The case corresponds to the first case from the torsion calculation with $K = \mathbb{Q}(-i)$ and groups constructed from prime ideals from the Gaussian Integers.

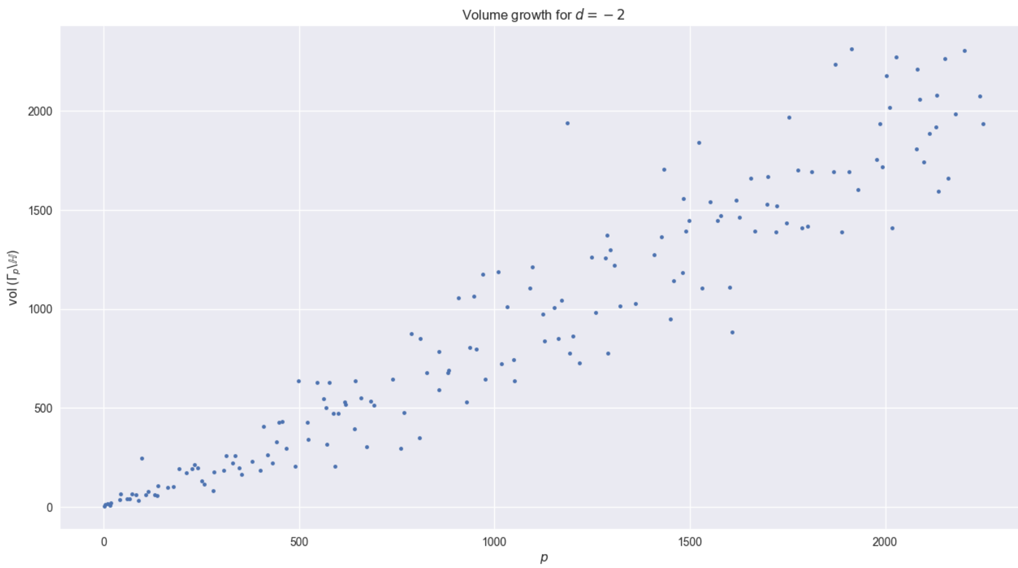
Figure 5.7: Growth in the covolume



5.3.2 Case 2

In this case, we estimate the covolume for the congruent subgroups defined as $\Gamma_0(\mathfrak{p})$ where $d = -2$ and $\mathfrak{p}$ are prime ideals of $\mathcal{O}_d$

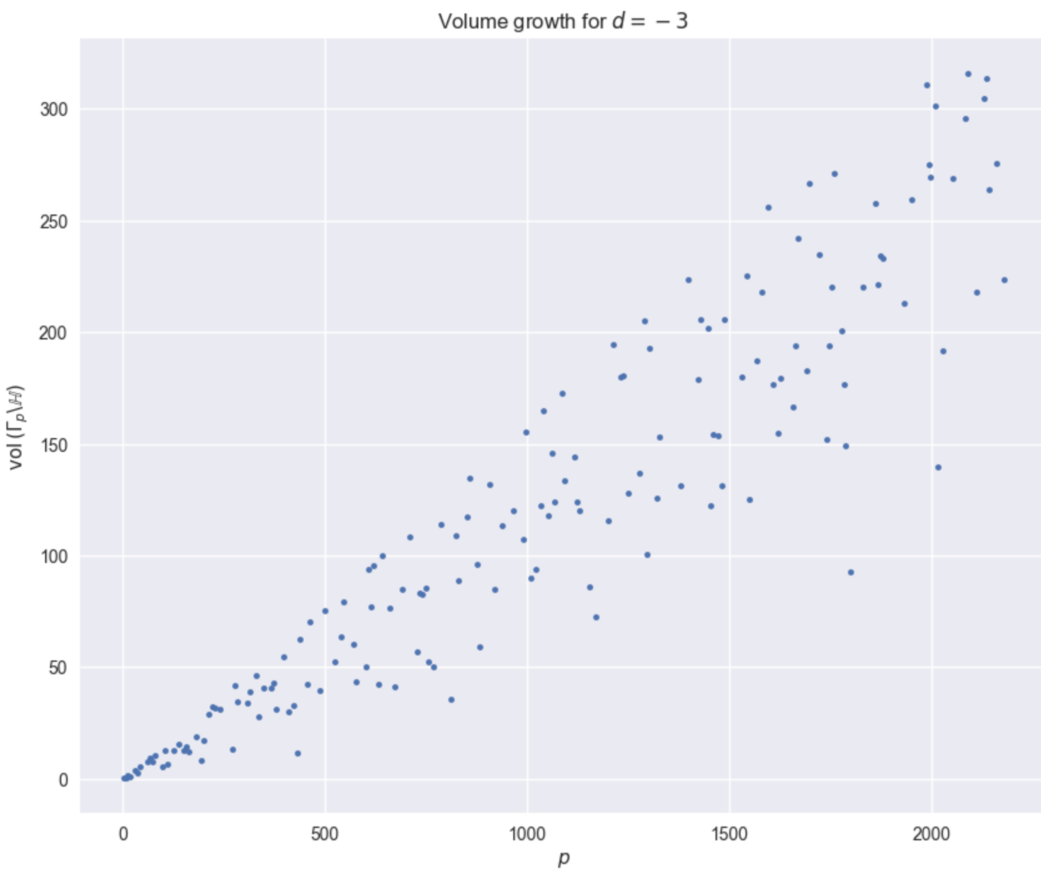
Figure 5.8: Growth in the covolume



5.3.3 Case 3

This case corresponds to the case of $K = \mathbb{Q}(\sqrt{-3})$ and the Eisenstein Integers as their ring of integers. The covolume of the quotient space $\mathrm{vol}\,(\Gamma_0(\mathfrak{p})\backslash\mathbb{H})$ is plotted against the norm of prime ideals $\mathfrak{p}$.

Figure 5.9: Growth in the covolume



5.4 The conjecture

In Section 5.2, we observed that, for Bianchi groups and congruence subgroups, torsion in the integral homology i.e. the abelianization should grow exponentially with respect to the norm, and in Section(5.3), we observe that volume roughly grows linearly with the norm. We are looking at a particular expression of looking at this growth given as -

$$\frac{\log\left|\left(\Gamma_n^{ab}\right)_{tor}\right|}{\mathrm{vol}\left(\Gamma_n\backslash\mathbb{H}\right)}$$

Bergeron and Venketesh conjecture that the following holds true -

Conjecture 5.4.1. *Let $\{\Gamma_n\}_n$ be a sequence of finite index congruence subgroups of some fixed Bianchi group. Then*

$$\lim_{n\to\infty}\frac{\log\left|\left(\Gamma_n^{ab}\right)_{tor}\right|}{\mathrm{vol}\left(\Gamma_n\backslash\mathbb{H}\right)}=\frac{1}{6\pi}$$

In the upcoming section, we attempt to verify the conjecture computationally for the three cases mentioned in sections 5.2 and 5.3.

We are plotting the ratio $\frac{\log\left|\left(\Gamma_0(\mathfrak{p})^{ab}\right)_{tor}\right|}{\mathrm{vol}\left(\Gamma_0(\mathfrak{p}\backslash\mathbb{H}\right)}$ against the norm of the ideal $\mathfrak{p}$

We describe the conjecture for each of the sequences below. For each of the cases, we hit a cutoff of the maximum difference between consecutive terms of $\delta = 0.025$.

Case 1 -

Number Field	$\mathbb{Q}(i)$
Discriminant of the Field	-4
Ring of Integers	$\mathbb{Z}[i]$
Ideals considered	Prime ideals $\{\mathfrak{p}_n\}_n$ lying over prime p_n
Norm of the ideal	$N(\mathfrak{p}_\mathfrak{n}) = p_n$
Groups constructed	$\Gamma_0(\mathfrak{p}_\mathfrak{n}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, Z[i]) \mid c \in \mathfrak{p}_\mathfrak{n}\}$

Case 2 -

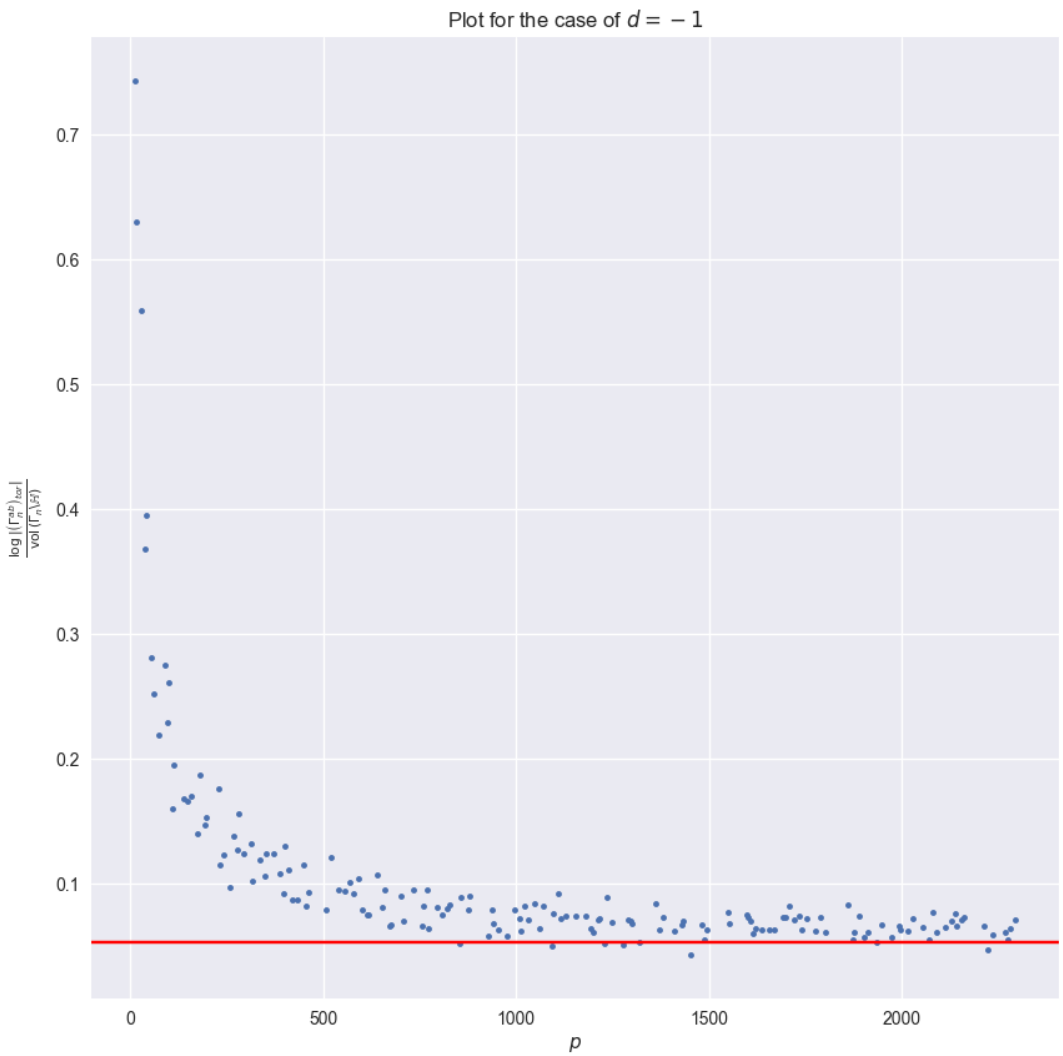
Number Field	$\mathbb{Q}(\sqrt{-2})$
Discriminant of the Field	-8
Ring of Integers	$\mathbb{Z}[\sqrt{-2}]$
Ideals considered	Prime ideals $\{\mathfrak{p}_n\}_n$ lying over prime p_n
Norm of the ideal	$N(\mathfrak{p}_\mathfrak{n}) = p_n$
Groups constructed	$\Gamma_0(\mathfrak{p}_\mathfrak{n}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, Z[i]) \mid c \in \mathfrak{p}_\mathfrak{n}\}$

Case 3 -

Number Field	$\mathbb{Q}(\sqrt{-3})$
Discriminant of the Field	-3
Ring of Integers	$\mathbb{Z}[\frac{-1+i\sqrt{3}}{2}]$
Ideals considered	Prime ideals $\{\mathfrak{p}_n\}_n$ lying over prime p_n
Norm of the ideal	$N(\mathfrak{p}_\mathfrak{n}) = p_n$
Groups constructed	$\Gamma_0(\mathfrak{p}_\mathfrak{n}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, Z[i]) \mid c \in \mathfrak{p}_\mathfrak{n}\}$

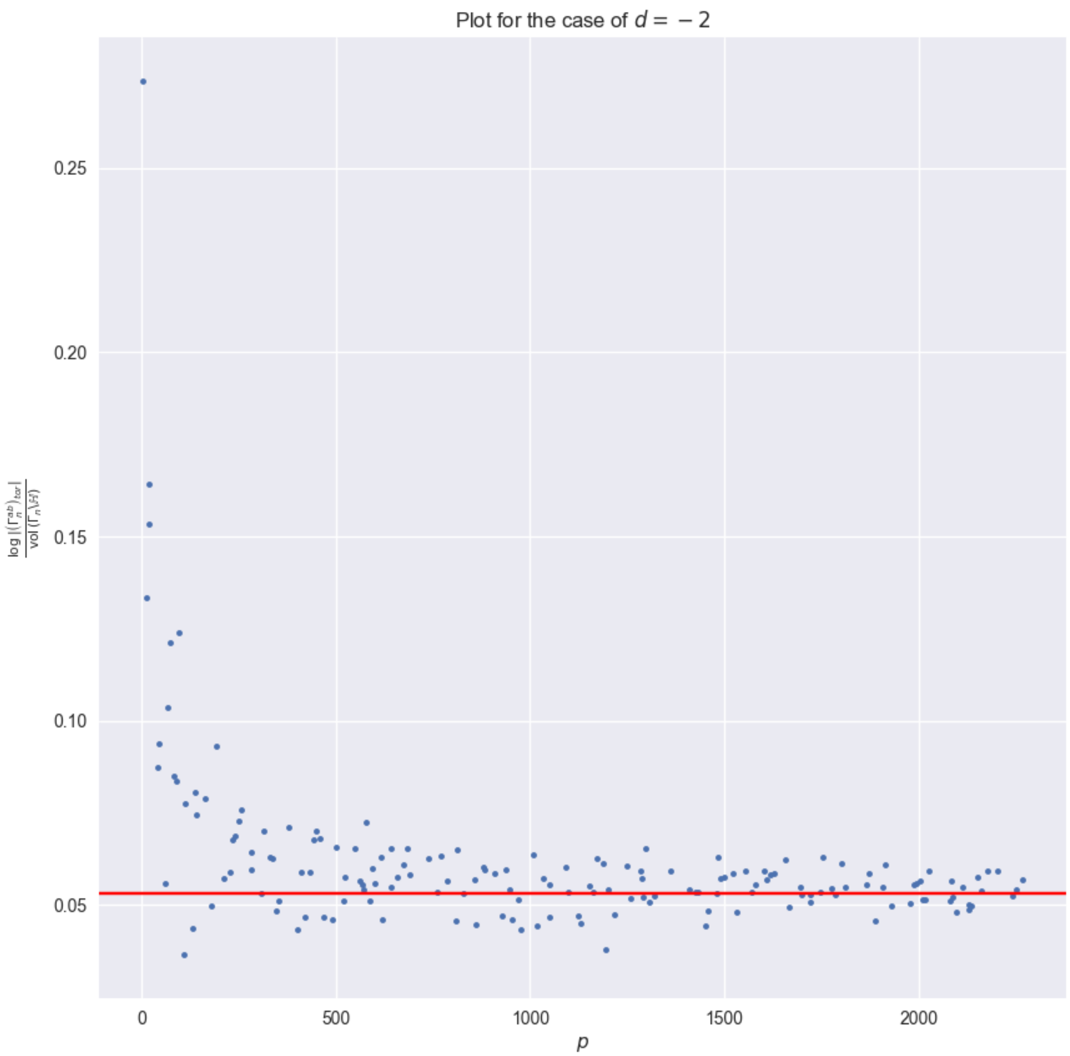
5.4.1 Case 1

Figure 5.10: Convergence of the ratio plotted with $y = \frac{1}{6\pi}$



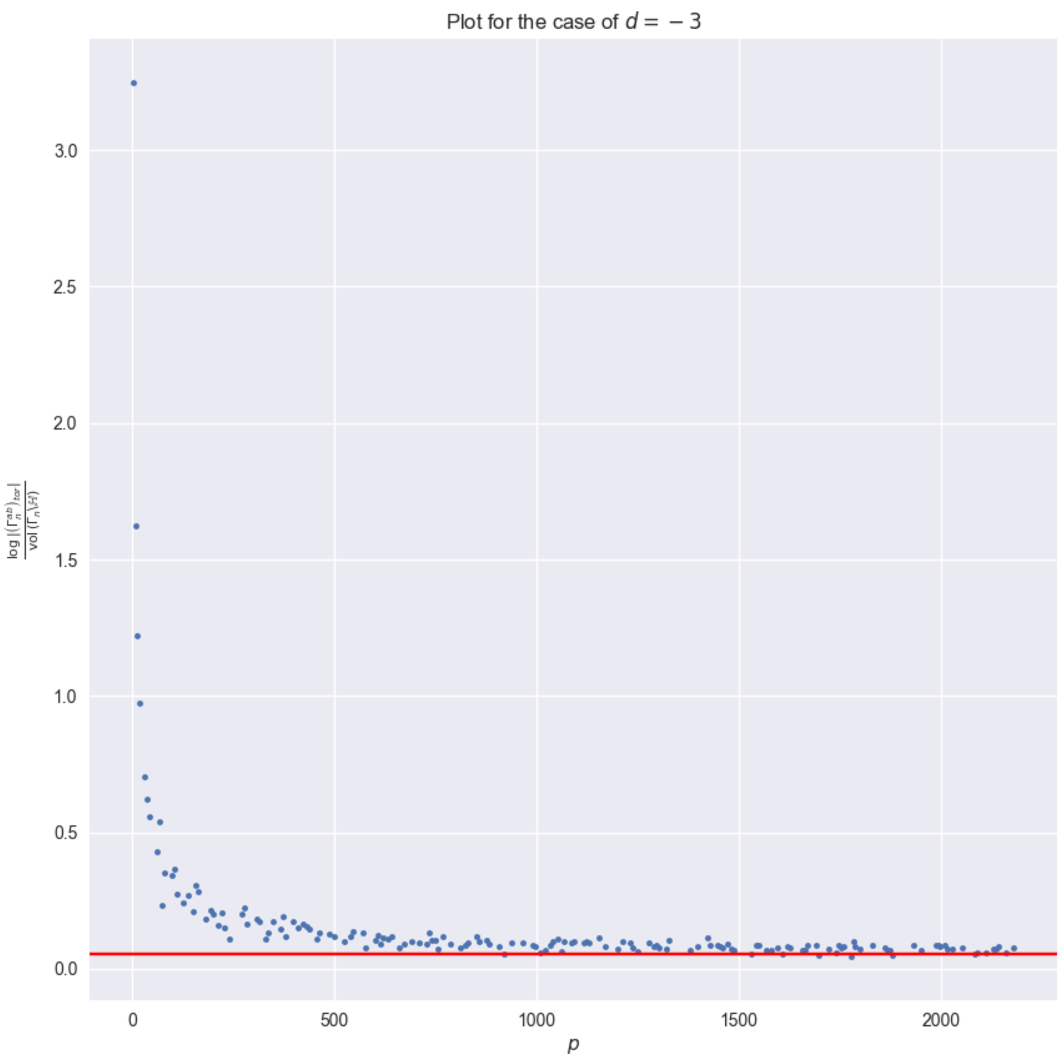
5.4.2 Case 2

Figure 5.11: Convergence of the ratio plotted with $y = \frac{1}{6\pi}$



5.4.3 Case 3

Figure 5.12: Convergence of the ratio plotted with $y = \frac{1}{6\pi}$



Limitations It has been observed that the torsion part of the group grows exponentially as we increase the index of the subgroup of a Bianchi group and as we increase the discriminant associated with the number field. That makes deriving any asymptotic conclusions from the program a time-consuming, memory consuming and at times, computationally infeasible, process.

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