On Cancellation Properties in Affine Geometry

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This is to certify that this dissertation entitled On Cancellation Properties in Affine Geometrytowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents original research carried out by Mohammad Munaif Iqbal Ahmedat Indian Institute of Science Education and Research under the supervision of Dr. S.M. Bhatwadekar, Visiting Professor, Maths Department , during the academic year 2014-2015.

Dr. S.M. Bhatwadekar

Committee:
Dr. S.M. Bhatwadekar
Dr. Rabeya Basu

To my parents and brother who have always supported me

Declaration

I hereby declare that the matter embodied in the report entitled On Cancellation Properties in Affine Geometry are the results of the investigations carried out by me at the Maths Department, Name of the Institute, under the supervision of Dr. S.M. Bhatwadekar and the same has not been submitted elsewhere for any other degree.

Mohammad Munaif Iqbal Ahmed

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Abstract

Let k be an algebriacally closed field and A, B be finitely generated (affine) k-domains. A is said to be stably isomorphic to B if the polynomial algebra $A[X]$(in one variable) is k-isomorphic to the polynomial algebra $B[Y]$. Keeping this definition in mind, we now state one of the important problems in affine geometry.

Question: Let A, B be affine domains such that A is stably isomorphic to B. Is A isomorphic to B as k-domains?

This problem has been investigated by many well-known mathematicians. In this thesis, we report on the progress achieved so far regarding this problem.

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Chapter 1

Introduction

Let k be an algebraically closed field and let A, B be affine k-domains.A is said to be stably isomorphic to B if $A[X]$(in one variable) is k-isomorphic to $B[Y]$. Keeping this definition in mind, we now state one of the important problems in affine geometry.

Question: Let A, B be affine domains such that A is stably isomorphic to B. Is A isomorphic to B as k-domains?

Note that if A is a polynomial algebra over k then the above problem is known as the "Zariski Cancellation Problem".

Abhyankar, Eakin and Heinzer have shown that if dim $A = 1$, then the above question has an affirmative answer ([1]).

If dim$A = 2$, then Danielewski (in his unpublished paper) has shown by an example that the answer to this question is not always affirmative even if k is a field of complex numbers.

However if k is a field of characteristic 0, and $A = k[X, Y]$ a well known result of Miyanishi-Sugie and Fujita says that the question has an affirmative answer ([5], [8]).

Recently, Neena Gupta has shown by an example ([6]) that the above question does not have an affirmative answer in general if characteristic of k is $p(p > 0)$ and $A = k[X, Y, Z]$.

In this thesis, we first present the original proof of Abhyankar, Eakin and Heinzer theorem (A-E-H theorem). We also give an alternative proof to the same. The original proof of Miyanishi-Sugie and Fujita of the cancellation theorem for $k[X, Y]$ is highly geometric and needs deep results from projective geometry. However, Crachiola and Makar-Limanov([3]) have given an algebraic proof which we have presented here.

The main ingredient of our alternative proof of A-E-H theorem, the algebraic proof of Crachiola and Makar-Limanov (on the cancellation of $k[X, Y]$) and Neena Gupta's result is the notion of exponential map which has been studied by Crachiola and Makar-Limanov in their paper ([3]).
Let A, B be affine k-domains such that $A[X] \cong B[Y]$. Let $\sigma : A[X] \to B[Y]$ be an isomorphism. Then $\sigma(A)[\sigma(X)] = B[Y]$. Therefore, by identifying A with $\sigma(A)$ and X with $\sigma(X)$, we can formulate the above question as follows,
Question: Let R be an affine domain of dimension $n \geq 1$ over a field k and let A, B be affine k-subdomains of dimension $n - 1$ such that $A[X] = R = B[Y]$.Is $A \simeq B$ as k-domains?
The layout of this thesis is as follows. In Chapter 2, we give relevant definitions and results. In Chapter 3, we present original as well as an alternative proof of A-E-H theorem. Chapter 4 is devoted to presenting the algebraic proof given by Crachiola and Makar-Limanov (of cancellation for $A = k[X, Y]$). Chapter 5 deals with the example given by Danielewski. Chapter 6 deals with the result of Neena Gupta.

Chapter 2

Preliminaries

In this section we define the terms used in the thesis, recall a few well-known results, and prove a few lemmas for later use. Throughout this thesis all rings considered are commutative with identity.

For a ring C, $C^{[n]}$ will denote a polynomial ring in n variables over C.

We begin with the following lemma.

Lemma 2.1. Let A be a domain containing a field k. Assume that $\mathrm{tr}_k(A) = 1$. Then $\dim(A) \leq 1$.

Proof. Assume that $\dim(A) \geq 2$. Then there exists a non-zero prime ideal P of A which is not maximal. Let $\mathfrak{m}$ be a maximal ideal of A containing P. Since P is non-zero $\exists\, x \in P$ such that $x \neq 0$. Since $P \neq \mathfrak{m}$ $\exists\, y \in \mathfrak{m} \setminus P$. Let $C = k[x, y]$ be the k-subalgebra of A generated by x, y.

Let $Q_1 = C \cap P$ and $Q_2 = C \cap \mathfrak{m}$. Then, as $x \in Q_1$, Q_1 is a non-zero prime ideal of C such that $Q_1 \subset Q_2$. Note that Q_2 is a prime ideal of C. Moreover, by choice of y, $y \in Q_2 \setminus Q_1$. This shows that the non-zero prime ideal Q_1 is properly contained in the prime ideal Q_2. Hence $\dim(C) \geq 2$.

Since C is an affine domain over k, $tr_k(C) = \dim(C) \geq 2$. Therefore, as $k \subset C \subset A$, we have $1 = tr_k(A) \geq tr_k(C) \geq 2$ which is a contradiction. Hence $\dim(A) \leq 1$. ♣

The following proposition is an easy consequence of the above lemma and is used very often.

Proposition 2.2. Let A, B be unique factorization domains such that $A \subset B \subset A^{[n]}$. Assume that $\mathrm{tr}_A(B) = 1$. Then $B = A^{[1]}$.

Proof. Let us write $A[T_1, \cdots, T_n]$ for $A^{[n]}$. Let $M = (T_1, \cdots, T_n)$ be a prime ideal of $A[T_1, \cdots, T_n]$ and $P = M \cap B$. Then it is easy to see that $P \cap A = 0$ and $A \subset B/P \subset A[T_1, \cdots, T_n]/M = A$. Thus $B/P = A$ and hence, as $\mathrm{tr}_A B = 1$, P is a non-zero prime ideal of B.

Let $S = A \setminus (0)$. Then $S^{-1}(A) = K$ is the quotient field of A and we have the inclusion of rings $K \subset S^{-1}(B) \subset K[T_1, \cdots, T_n]$. Since $\mathrm{tr}_K S^{-1}(B) = \mathrm{tr}_A(B) = 1$, by Lemma 2.1, $\dim S^{-1}(B) = 1$. Since $P \cap A = 0$ and P is non-zero, P is a prime ideal of B of height one. Therefore, as B is UFD, P is a principal ideal of B.
Let $P = (G)$. Now we prove that $B = A[G]$ $(A^{[1]})$.
Let $b \in B$. We show that $b \in A[G]$ by induction on the degree l of b as an element of $A[T_1, \cdots, T_n]$.
If $l = 0$ then $b \in A$ and hence $b \in A[G]$. Now assume that $l \geq 1$. Since $A = B/P = B/(G)$, there exists $a \in A$ such that $b - a \in (G)$. Let $b - a = cG, \quad c \in B$. Note that $G \in (T_1, \cdots, T_n)$ and hence degree of G (as an element of $A[T_1, \cdots, T_n]$) is positive and hence degree of $c < l$. Therefore, by induction hypothesis, $c \in A[G]$ and hence $b \in A[G]$. ♣

2.1 Conductor Ideal

Definition 2.3. Let $A \hookrightarrow B$ be an inclusion of domains. Let $M = B/A$, a module over A. The conductor ideal of B over A is defined to be $\mathrm{Ann}_A M$.

Lemma 2.4. Let $A \hookrightarrow B$ be an inclusion of domains. Let I be the conductor ideal of B over A. Then $IB = I$

Proof. Since I is the conductor ideal, it is easy to see that $I \subset IB \subset A$. Let $IB = J$. Now J is an ideal of A such that $JB = IBB = IB = J$. Hence J annihilates the A-module B/A. Thus $J \subset \mathrm{Ann}_A(B/A)$.Therefore $I = J = IB$ ♣

Lemma 2.5. Let $A \hookrightarrow B$ be an inclusion of domains and let I be the conductor ideal of B over A. Assume that A is Noetherian. Then $I \neq 0 \Leftrightarrow B$ is a finite A-module and B and A have the same quotient field.

Proof. Let $0 \neq a \in I$. Then $(0) \subsetneq aB \subset A$. Thus aB is an ideal of A and hence is a finitely generated A-module as A is Noetherian. Since $aB \cong B$ as an A-module, B is a finite A-module. Moreover, for every $b \in B$, $c = ab \in A$. Hence $b = c/a$. Thus B and A have the same quotient field.
Conversely, assume that B is a finite A-module and B and A have the same quotient field K. Let $B = \sum_{i=1}^{n} Ax_i$. Since $x_i \in B \subset K$, we have $x_i = y_i/a_i$ such that $y_i, a_i \in A$ and $a_i \neq 0$. Let $a = \prod a_i$. Then $a \neq 0$ and $ax_i \in A, \forall\, i\ (1 \leq i \leq n)$. Therefore, $aB \subset A$ and hence $a \in I = \mathrm{Ann}_A B/A$ ♣

Remark 2.6. Let k be a field and let A be an affine domain over k. Let $\bar{A}$ be the integral closure of A in its quotient field K. Then $\bar{A}$ is a finite A-module having quotient field K. Therefore the conductor ideal I of $\bar{A}$ over A is a non zero ideal of A.

Lemma 2.7. Let A be an affine domain over a field k with quotient field K and let $\bar{A}$ be its integral closure in K. Let I be the conductor ideal of $\bar{A}$ over A. Let $R = A[X]$ and $\bar{R} = \bar{A}[X]$. Then the conductor ideal of $\bar{R}$ over R is $I\bar{R} = IR$ (i.e. $I\bar{A}[X] = IA[X]$).

Proof. Let $J \subset \bar{A}[X]$ be an ideal such that $J \subset A[X]$. We need to show $J \subset I\bar{A}[X] = IA[X]$.
Let f$= \sum_{0 \leq l \leq n} a_l X^l \in J$. Then $a_l \in A\ \forall\, l$. Let $b \in \bar{A}$. Then $b.f = \sum_{0 \leq l \leq n} b.a_l X^l \in J \subset A[X]$. Therefore $b.a_l \in A$ for $\forall\, l$. Thus, for $\forall\, b \in \bar{A}$, $b.a_l \in A$. This implies $a_l\bar{A} \subset A$ i.e. $a_l \in I$. Hence $J \subset IA[X]$. Thus $IA[X] = I\bar{A}[X]$ is the conductor ideal of $\bar{R}$ over R. ♣

2.2 Locally Nilpotent derivation, Locally finite iterative higher derivation and Exponential Map

2.2.1 Local Nilpotent Derivation (LND)

Definition 2.8. Let B be a domain. A group homomorphism $d : B \to B$ is called locally nilpotent derivation(LND) if

1. $d(ab) = ad(b) + bd(a) \qquad a, b \in B$.

2. For every $b \in B$, $\exists$ a positive integer n(depending upon b) such that $d^n(b) = 0$.

2.2.2 Locally finite iterative higher derivation

Definition 2.9. Let B be a domain. A locally finite iterative higher derivation on B is a sequence $d_i : B \to B$ $(0 \le i < \infty)$ of functions satisfying following properties.

1. $d_0 = id$.
2. $d_n(a+b) = d_n(a) + d_n(b) \qquad a, b \in B$.
3. $d_n(a.b) = \sum\limits_{i=0}^{n} d_i(a)d_{n-i}(b)$.
4. $d_i d_j = \binom{i+j}{i} d_{i+j}$.
5. For $b \in B$ $\exists$ $n_0 \in \mathbb{N}$ (depending on b) such that $d_n(b) = 0$ for $\forall$ $n \ge n_0$

2.2.3 Exponential Map

Definition 2.10. Let B be a domain and $d_i : B \to B$ $(0 \le i < \infty)$ be a locally finite iterative higher derivation. A ring morphism $\phi : B \to B[T]$ defined by

$$\phi(b) = \sum d_i(b)T^i$$

is called the exponential map (associated to a sequence d_i $(0 \le i < \infty)$).

2.2.4 Ring of ϕ-invariants

Definition 2.11. Let B be a domain and let $\phi : B \to B[T]$ be the exponential map associated to a locally finite iterative higher derivation $d_i(0 \le i < \infty)$. Let $B^\phi = \{a \mid a \in B, \phi(a) = a\}$ (equivalently $B^\phi = \{a \mid \ d_i(a) = 0 \ \forall \ i \ge 1\}$) Then B^ϕ is called the ring of ϕ-invariants. We say ϕ is non-trivial if $B^\phi \ne B$

Remark 2.12. Let B be a domain and d_i $(0 \le i < \infty)$ be a locally finite iterative higher derivation on B. Then d_1 is a locally nilpotent derivation on B. Morever, if B contains the field of rationals, then $d_i = \frac{d_1^i}{i!}$ $(\forall\ i \ge 1)$. Conversely, if B contains the field of rationals and d is a locally nilpotent derivation, then $d_i = d^i/i!$ is a locally finite iterative higher derivation on B. Thus, if B contains the field of rationals then every locally nilpotent derivation $d : B \to B$ gives rise to the exponential map ϕ on B such that $B^\phi = \ker(d)$.

Let d_i $(0 \leq i < \infty)$ be a locally finite iterative higher derivation on B and let $\phi : B \to B[T]$ be the corresponding exponential map Assume that ϕ is non-trivial. Let $b \in B$ be such that the $\deg_T \phi(b) = n$ $(n \geq 1)$. Let $\phi(b) = \sum_{i=0}^{n} d_i T^i$. Then $d_n(b) \neq 0$ and $d_n(b) \in B^{\phi}$.

Lemma 2.13. Let $\phi : B \to B[T]$ be an exponential map and let $A = B^{\phi}$. Then

1. A is factorially closed in B i.e. for $b, c \in B - \{0\}$, if $bc \in A$ then $b \in A$ and $c \in A$.
2. $A^* = B^*$
3. For $a \in A$ $\;A \cap aB = aA$
4. If $a \in A$ is a prime element of B then a is a prime element of A also.
5. A is algebraically closed in B.
6. If B is UFD, then A is UFD.

Proof. Properties 2, 3, 4, 5 and 6 follow easily from 1.
Since $A = \phi^{-1}(B)$, and B is factorially closed in $B[T]$, the result follows. ♣

Lemma 2.14. Let B be a domain and $\phi : B \to B[T]$ be an exponential map. Let A be the ring of ϕ-invariants. Suppose $\exists\, b \in B$ such that $\phi(b) = b + T$. Then $B = A[b]$ $(A^{[1]})$. In particular, if B contains the field of rationals and $d : B \to B$ is a locally nilpotent derivation with $A = \ker(d)$ such that $d(b) = 1$ for $b \in B$, then $B = A[b]$ $(A^{[1]})$.

Proof. Since $b \in B - A$, by Lemma 2.13, b is transcendental over A. Let $c \in B$. We prove the result by induction on $\deg_T \phi(c)$.
If $\deg_T \phi(c) = 0$, then $c \in A$ and hence $c \in A[b]$.
Assume $\deg_T \phi(c) \geq 1$. Let $\phi(c) = \sum_{i=0}^{n} d_i(c) T^i$. Then $d_n(c) \in A$. Therefore,

$$\begin{aligned}\phi(c - d_n(c) b^n) &= \phi(c) - \phi(d_n(c))\phi(b^n)\\ &= \phi(c) - d_n(c)(\phi(b))^n\\ &= \phi(c) - d_n(c)(b + T)^n\end{aligned}$$

Clearly, $\phi(c - d_n(c)b^n)$ is a polynomial such that $\deg_T \phi(c - d_n(c)b^n) \leq n - 1$. Therefore, by induction hypothesis, $c - d_n(c)b^n \in A[b]$. Let $c' = c - d_n(c)b^n$. Then $c = c' + d_n(c)b^n$. As $d_n(c) \in A$ and $c' \in A[b]$, we see that $c \in A[b]$. Thus $B = A[b]$ ♣

The next lemma is proved in [3]

Lemma 2.15. Let $\phi : B \to B[T]$ be a non-trivial exponential map on a domain B. Let A be the ring of ϕ-invariants. Then there exists a non-trivial exponential map $\phi_1 : B \to B[T]$ and $b \in B$ such that $B^{\phi_1} = A$ and $\phi_1(b) = b + aT \quad a \neq 0$. As a consequence, $a \in A$ and $B[1/a] = A[1/a][b]$ $(A[1/a]^{[1]})$. Hence $\mathrm{tr}_A B = 1$. Therefore, if B is an affine domain of dimension n over k, then $\mathrm{tr}_k A = n - 1$.

Lemma 2.16. Let B be a domain and let $\phi : B \to B[T]$ be the exponential map corresponding to a locally finite iterative higher derivation $(d_i,\ 0 \leq i < \infty)$ on B. Let A be the ring of ϕ-invariants. Let $z \in A$ be such that z is a prime element of B (and hence of A). Assume that for some $j \geq 1$, $d_j(B) \not\subset zB$. Then ϕ induces a non-trivial exponential map $\bar{\phi}$ on B/zB.

Proof. Since $z \in A$ we have $d_i(z) = 0\ \forall\ i \geq 1$. Therefore d_i induces a map $\delta_i : B/zB \to B/zB$ for $i \geq 1$. Let δ_0 be the identity map of B/zB. As $z \in A$, it is easy to see that the sequence $(\delta_i,\ 0 \leq i < \infty)$ defines a locally finite iterative higher derivation on B/zB and hence gives rise to an exponential map $\bar{\phi}$ on B/zB. Note that, $\delta_j \neq 0$ as $d_j(B) \not\subset zB$. Hence $\bar{\phi}$ is non-trivial. ♣

Notation

Let $R = \bigoplus_{-\infty < i < \infty} R_i$ be a graded domain. For $g = \sum_{m \leq i \leq n} g_i,\ g_i \in R_i,\ g_n \neq 0$ we say that g_n is the highest weight form of g and we denote it by $\mathrm{hwf}(g)$.

Keeping this notation in mind, we state the following version of a result of H.Derksen, O.Hadas and L.Makar-Limanov as presented in [4](Theorem 2.6) (See also [6], Theorem 2.3).

Theorem 2.17. Let $R = \bigoplus_{-\infty < i < \infty} R_i$ be a graded domain containing a field k. Assume that R and R_0 are affine domains over k. let $\phi : R \to R[T]$ be a non-trivial exponential map. Then ϕ induces a non-trivial exponential map $\psi : R \to R[T]$ such that if $A = R^{\psi}$ then

1. $A = \bigoplus_{-\infty < i < \infty} (A \cap R_i)$

2. If $g \in R^{\phi}$ then $\text{hwf}(g) \in A$.

As a consequence we have the following result.

Proposition 2.18. Let k be an infinite field and let B be an affine domain over k. Let $R = B[Y]$. Assume that B does not admit any non-trivial exponential map. Let $\phi : R \to R[T]$ be an exponential map. Then $B \subset R^{\phi}$. In particular, if ϕ is non-trivial then $R^{\phi} = B$.

Proof. If ϕ is trivial then $R^{\phi} = R$ and hence we are through.

Assume that ϕ is non-trivial. If $R^{\phi} \neq B$ then, as R^{ϕ} is algebraically closed in R and transcendence degree of R over R^{ϕ} is one, $R^{\phi} \not\subset B$. Hence $\exists g \in R^{\phi} \setminus B$. Therefore $\deg_Y g = n \geq 1$.

Note that $B[Y] = R = \bigoplus_{0 \leq i < \infty} R_i$ where $R_0 = B$ and $R_i = BY^i$ $(i \geq 1)$. Therefore, by Theorem 2.17, there exists a non-trivial exponential map $\psi : R \to R[T]$ such that the ring A of ψ-invariants contains the element $\text{hwf}(g)$. Since $\deg_Y(g) = n \geq 1$, it follows that Y divides $\text{hwf}(g)$. Hence $Y \in A$. Therefore $Y - \lambda \in A$ $(\forall\ \lambda \in k)$. Since ψ is non-trivial there exists $F \in R$ such that $\deg_T \psi(F) = m \geq 1$. Let $f \in R$ be the leading coefficient of $\psi(F)$. Since R is Noetherian and k is infinite, $\exists\ \beta \in k$ such that f is not divisible by the prime element $Z = Y - \beta$ (of R). Therefore, as $Z \in A$, by (Lemma 2.16), ψ induces a non-trivial exponential map on $B = R/ZR$ which is contradiction. Hence $R^{\phi} = B$. ♣

2.2.5 Makar-Limanov Invariant

Definition 2.19. Let B be a domain and let $EXP(B)$ denote the set of all **non-trivial** exponential maps on B. We define the Makar-Limanov invariant $(AK(B))$, or the ring of absolute constants of B as $AK(B) = \bigcap_{\phi \in EXP(B)} B^{\phi}$ where B^{ϕ} is the ring of ϕ-invariants.

Definition 2.20. Let B be a domain. Then the Derksen Invariant of B $(DK(B))$ is define to be the subalgebra of B generated by the sets B^{ϕ} such that ϕ is a non-trivial exponential map.

Remark 2.21. Let B be a domain which contains the field of rationals. Let LND denote the set of all non-zero locally nilpotent derivations on B. Then, it is easy to see that (by Remark 2.12), that $AK(B) = \bigcap_{d \in LND} ker(d)$

Remark 2.22. We can restate Proposition 2.18 as follows. Let B be a domain such that the set $EXP(B) = \emptyset$ (i.e. B does not admit any non-trivial exponential map). Then $AK(B^{[1]}) = B$.

Lemma 2.23. Let k be a field and let $B = k[X_1, \cdots, X_n]$ be a polynomial algebra in n variables over k. Then $AK(B) = k$ and $DK(B) = B$.

Proof. For $1 \leq i \leq n$, let $\phi_i : B \to B[T]$ be an exponential map defined as $\phi(X_j) = X_j$ for $j \neq i$ and $\phi_i(X_i) = X_i + T$. Let A_i denote the ring of ϕ_i-invariant. Then it is easy to see that $A_i = k[X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n]$. Hence $AK(B) = k$ and $DK(B) = B$. ♣

Chapter 3

Cancellation results for dimension 1

Lemma 3.1. Let k be a field and let $f \in k[X]$ be a monic polynomial in X of positive degree. Then f is transcendental over k and the polynomial algebra $k[X]$ is a finite module over $k[f]$.

Proof. Since $f \in k[X]$ is a monic polynomial in X of positive degree n, f is of the form

$$f(X) = X^n ... + a_1 X + a_0 \tag{3.0.1}$$

Hence f is transcendental over k. To show that $k[X]$ is a finite module over $k[f]$, we will show that X is integral over $k[f]$. Let

$$G(T) = T^n ... + a_1 T + a_0 - f \tag{3.0.2}$$

$G(T)$ is a monic polynomial in T with coefficients in $k[f]$ and $G(X) = 0$. Therefore, X is integral over $k[f]$. Hence, $k[X]$ is a finitely generated $k[f]$ module. ♣

Lemma 3.2. Let $k \hookrightarrow A \hookrightarrow k[X]$ and $k \neq A$. Then A is an affine k-algebra. Moreover $k[X]$ is a finite A-module.

Proof. Since $k \neq A$, $\exists\, f \in A - k$. Since f is not in k, as an element of $k[X]$, $\deg_X f = n \geq 1$. Let

$$f = a_n X^n ... + a_1 X + a_0 \qquad (n \geq 1,\, a_n \neq 0) \tag{3.0.3}$$

Since $a_n \in k^*$, changing f by $a_n^{-1} f$, we can assume that f is a monic polynomial in X of degree $n \geq 1$. As $f \in A$, we have the following inclusion of k-algebras.

$$k \hookrightarrow k[f] \hookrightarrow A \hookrightarrow k[X] \tag{3.0.4}$$

By Lemma 3.1, $k[X]$ is a finite module over a Noetherian ring $k[f]$. Therefore $k[f]$-submodule A of $k[X]$ is a finite module over $k[f]$ and $k[X]$ is a finite A-module. Hence A is an affine k-algebra. ♣

Proposition 3.3. Let $k \subsetneq A \subset k[X]$ be an inclusion of k-algebras. Suppose A is normal. Then $A = k^{[1]}$.

Proof. Let L be the field of fractions of A. Then we have the inclusion $k \subsetneq L \subset k(X)$. By Lüroth's Theorem, $L = k(f)$ for some $f \in k(X)$ such that f is transcendental over k. Let $R = k[1/X]_{(1/X)}$ Then R is a discrete valuation ring such that $k[X] \not\subset R$. Therefore, as $k[X]$ is integral over A, $A \not\subset R$.
Let $S = R \cap L$. Then clearly, $A \not\subset S$ (since $A \not\subset R$). Also S is a subring of R contained in L.

Claim: S is a valuation ring with the field of fractions L.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} R \arrow[r, hook] & k(X) \\ S = R \cap L \arrow[u, hook] \arrow[r, hook] & L \arrow[u] \end{tikzcd}$$

It can be clearly seen from above that S is a valuation ring of L, since we know that R is a valuation ring of $k(X)$. Also note that S is a proper subring of L. If not, then $S = L$ and $A \subset L$, therefore $A \subset S$ which is a contradiction. ♣

Claim: $\exists\ u \in L$ such that

1. $u \notin S$

2. $k(u) = L = k(f)$

Proof. If $f \notin S$, then we take $f = u$. If $f \in S$ and $f^{-1} \notin S$, then we take $f^{-1} = u$. Suppose f,$f^{-1} \in S$. Therefore $f \in S^*$. Let $\mathfrak{m}(R)$ denote the maximal ideal of R. Then clearly, $\mathfrak{m}(R) \cap S$ is the maximal ideal of S. Moreover $k \subset S/\mathfrak{m}(R) \cap S \hookrightarrow R/\mathfrak{m}R$. Now from the construction of R, it is clear that $R/\mathfrak{m}(R) = k$. Therefore $k = S/\mathfrak{m}(R) \cap S$.
Let $\bar{f}$ be the image of f in k. As $f \in S^*$, $\bar{f} \in k^*$. Let $\bar{f} = \lambda$. Then clearly, $f - \lambda \in \mathfrak{m}(R) \cap S$. Therefore, $1/(f-\lambda) \notin S$ as $f - \lambda$ not a unit in S. Now take $1/(f-\lambda) = u$. Then clearly, $L = k(f) = k(f-\lambda) = k(1/(f-\lambda)) = k(u)$. ♣

Thus S is a valuation ring with the field of fractions L such that $k[u] \not\subset S$. In fact, it is the discrete valuation ring such that above condition is satisfied.
Since A is normal, A is a dedekind domain. Therefore, for every maximal ideal $\mathfrak{n}$ of A, the local ring $A_{\mathfrak{n}}$ is a discrete valuation ring.
Since $A \not\subset S$, $A_{\mathfrak{n}} \neq S$ for any maximal ideal $\mathfrak{n}$ of A. As $k[u] \not\subset S$, we have $k[u] \subset A_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of A. Therefore, $k[u] \subset A = \bigcap\limits_{\mathfrak{n} \in max(A)} A_{\mathfrak{n}}$. Thus, we have

$$k[u] \hookrightarrow A \hookrightarrow k[X] \qquad \deg_X u \geq 1$$

Then by Lemma 3.1, $k[X]$ is integral over $k[u]$. Hence A is integral over $k[u]$ and have the same field of fractions L. Since $k[u]$ is integrally closed in L, $A = k[u]$. Thus $A = k^{[1]}$.
♣

Theorem 3.4 (**A-E-H** Theorem)**.** Let $k = \bar{k}$. Let A, B be affine domains of dimension 1 over k such that $A^{[1]} = B^{[1]} = R$. Let $A[X] = R = B[Y]$. Then

1. If $A = k^{[1]} \Rightarrow B = k^{[1]}$
2. If $A \neq k^{[1]} \Rightarrow A = B$

Proof. Suppose $A = k^{[1]}$, then $A^{[1]} = R = k^{[2]}$. Therefore R is a UFD. This implies that B is a UFD. Therefore, as $k \subset B \subset k^{[2]}$ and $\text{tr.}_k(B) = 1$, by Proposition 2.2 $B = k^{[1]}$.
Now suppose $A \neq k^{[1]}$.
Case 1: Assume A is normal
Clearly, $B \neq k^{[1]}$. Let $\mathfrak{m}$ be a maximal ideal of B. Let $\mathfrak{m}R = P$. Then since $\mathfrak{m}$ is a height 1 prime ideal, P is a height 1 prime ideal in R. Let $\mathfrak{n} = P \cap A$. Clearly, $\mathfrak{n}$ is a prime ideal

in A. Therefore $\mathfrak{n}$ is either zero ideal or maximal. Now we have the following inclusion

$$A/\mathfrak{n} \hookrightarrow R/P = B/\mathfrak{m}^{[1]} = k^{[1]}$$

Therefore,since $A \neq k^{[1]}$ and A is normal, $\mathfrak{n}$ must be a maximal ideal of A and $\mathfrak{n}R \subset P$. As $\mathfrak{n}R$ and P are prime ideals of R of height 1, we have $\mathfrak{n}R = P$. Therefore, we have $\mathfrak{n}R = P = \mathfrak{m}R$.

Now we know that A is normal. Therefore B is also normal. Since B is of dimension 1, B is a Dedekind domain. Let $b \in B$ such that $b \notin B^*,\ b \neq 0$. Then we have the following,

$$\begin{aligned} bB &= \prod_{l_i} \mathfrak{m}_i^{l_i} \qquad (\mathfrak{m}_{\mathfrak{i}} \in \max(B)) \\ bR &= \prod_{l_i} (\mathfrak{m}R)^{l_i} \\ bR &= \prod_{l_i} (\mathfrak{n}_i R)^{l_i} \qquad (\mathfrak{n}_i = \mathfrak{m}_i R \cap A) \\ bR &= \prod_{l_i} \mathfrak{n}_i^{l_i} R \end{aligned} \tag{3.0.5}$$

Let $J = \prod_{l_i} \mathfrak{n}_i^{l_i}$. Then since $JR = bR$, JR is principal and thus J is a principal ideal of A. Let $J = aA$. Then $bR = aR$. Therefore, $b = \lambda a$ where $\lambda \in R^* = A^* = B^*$. Thus $b \in A \cap B$ and hence $B \subset A$. One can similarly show $A \subset B$. Thus $A = B$.

Case-2: Assume A is not normal

Let $\bar{A}$ and $\bar{B}$ be the normalizations of A and B respectively. Let $\bar{R}$ be the normalization of R. Then clearly, $\bar{R} = \bar{A}[X] = \bar{B}[Y]$.

Sub-case-1: $\bar{A} \neq k^{[1]}$

From case-1, it is clear that $\bar{A} = \bar{B}$. We will now show that $A = B$. Since $R = B[Y]$ and $\bar{R} = \bar{B}[Y]$, we have $A[X] = R = B[Y]$ and $\bar{A}[X] = \bar{R} = \bar{B}[Y] = \bar{A}[Y]$.Then $A = R \cap \bar{A} = R \cap \bar{B} = B$.

Sub-case-2: $\bar{A} = k[u](k^{[1]})$

Then $\bar{B} = k[t]$. Hence $k[u, X] = \bar{R} = k[t, Y]$

Let I be the conductor ideal of $\bar{A}$ over A. Let J be the conductor ideal of $\bar{B}$ over B.Then the conductor ideal of $\bar{R}$ over R is $I\bar{A}[X] = J\bar{B}[Y]$ (by Lemma 2.7). Clearly, $I \neq 0$, $J \neq 0$.(by Lemma 2.5).

Let $I = f\bar{A}$ and $J = g\bar{B}$ where $f \in A$, $f \notin A^*$, and $g \in B$, $g \notin B^*$. As stated above, $I\bar{A}[X] = J\bar{B}[Y]$, therefore $f\bar{R} = g\bar{R}$. Thus $f = \lambda g$, where $\lambda \in \bar{R}^*$. But $\bar{R}^* = \bar{A}^* = \bar{B}^* = k^*$ as $\bar{A} = k^{[1]}$. Hence $\lambda \in k^*$. Therefore, $f \in B$. Thus $f \in A \cap B$. Therefore, $\deg_Y f = 0$. Since $f \in \bar{A} = k[u]$ and $f \notin k$, we get $\deg_Y u = 0$. Therefore $u \in \bar{B}$. Hence, $\bar{A} = k[u] \subset \bar{B}$. Similarly, $\bar{B} = k[t] \subset \bar{A}$. Hence $\bar{A} = \bar{B}$. Therefore, $A = R \cap \bar{A} = R \cap \bar{B} = B$. ♣

Now we give another proof of **A-E-H** theorem using the notion of *exponential map*

Lemma 3.5. Let k be an algebraically closed field and let A be an affine domain of dimension 1 over k. Then $A \simeq k[U]$ if and only if A admits a non-trivial exponential map. As a consequence , if $R = A[X]$ $(A^{[1]})$ then $AK(R) = k$ if and only if $A = k^{[1]}$.

Proof. It is easy to see that a k algebra homomorphism $\psi : k[U] \to k[U][T]$ such that $\psi(U) = U + T$ is an exponential map.

Suppose $\phi : A \to A[T]$ is a non-trivial exponential map. Let C denote the ring of ϕ-constants. Then $k \subset C$ and, by (Lemma 2.15), $\exists\, c(\neq 0) \in C$ such that $A[1/c] = C[1/c]^{[1]}$. Hence $\mathrm{tr}_k C = \dim(A) - 1 = 0$. Therefore, as k is algebraically closed, $k = C$ and $c \in k^*$ Therefore $A = k^{[1]}$.

Now if $A = k^{[1]}$ then $R = k^{[2]}$ and therefore, by (Lemma 2.23), $AK(R) = k$. Conversely, if $AK(R) = k$, then A must admit a non-trivial exponential map (see Remark 2.22) and hence $A = k^{[1]}$.

♣

Theorem 3.6 (**A-E-H** Theorem)**.** Let k be an algebraically closed field and let A and B be affine domains of dimension 1 over k. Suppose $A[X] = B[Y]$. Then $A \simeq B$. Moreover if $A{\neq}k^{[1]}$ then $A = B$.

Proof. Let $A[X] = R = B[Y]$. Assume that $A = k^{[1]}$. Then $R = k^{[2]}$ and hence, by (Lemma 2.23), $AK(R) = k$. Therefore , as $R = B[Y]$, by (3.5), $B = k^{[1]}$. Thus $A \simeq B$.

Now suppose $A \neq k^{[1]}$. Then, by (Lemma 3.5), A does not admit any non-trivial exponential map. Therefore, by (Proposition 2.18), $AK(R) = A$. Since $R = B[Y]$ and $AK(R) \neq k$, by (Lemma 3.5), $B{\neq}k^{[1]}$ and hence B does not admit any non-trivial exponential map. Therefore again ,by (Proposition 2.18), $AK(R) = B$. Thus $A = AK(R) = B$. ♣

Chapter 4

Cancellation theorem for $k[X, Y]$

In this section we prove the following theorem. This theorem was originally proved by Miyanishi-Sugie and Fujita using geometric methods ([5], [8]). The (algebraic) proof presented here is due to Crachiola and Makar-Limanov([3]).
For the proof of this theorem, we need the following lemma

Lemma 4.1. Let B be a domain containing the field of rationals. Let $d : B \to B$ be a non-zero locally nilpotent derivation and let $A = ker(d)$. Suppose for $a \in B,\ a \neq 0$, $d(a) \in aB$, then $a \in A$.

Proof. Since B contains the field of rationals, by Remark 2.12, d gives rise to a non-trivial exponential map $\phi : B \to B[T]$ such that A is the ring of ϕ-invariants.
Suppose $a \notin A$. Then, as d is locally nilpotent ,$\exists\ n \geq 1$ such that $d^n(a) \neq 0$ but $d^{n+1}(a) = 0$. Therefore $d^n(a)$ is a non-zero element of A. Since $d(a) \in aB$, it is easy to see, $d^n(a) \in aB$. Let $d^n(a) = ac,\ c \in B$. Therefore, as A is factorially closed in B, $a \in A$ which is a contradiction.

♣

Theorem 4.2. Let k be an algebraically closed field of characteristic 0. Let B be an affine domain of dimension 2 over k such that $B^{[1]} = k^{[3]}$. Then $B = k^{[2]}$.

Proof. Let $B[W] = R = k[X, Y, Z]$. Then $AK(R) = k$ (by Lemma 2.23). Since $R = B[W]$, and $AK[R] \neq B$, by Proposition 2.18, B admits a non-trivial exponential map ϕ. Since B contains the field of rationals, by Remark 2.12, ϕ gives rise to a non-zero locally nilpotent

derivation $d : B \to B$ such that $B^{\phi} = \ker(d) = A$.
Claim: $A = k^{[1]}$.

Proof. Since $B[W] = k[X,Y,Z]$, B is a UFD and hence, by Lemma 2.13, A is a UFD such that $\text{tr}_k A = \text{tr}_k B - 1 = 1$(by Lemma 2.15). Since $k \subset A \subset k[X,Y,Z]$, $A = k[U]$ ($k^{[1]}$) by Proposition 2.2. ♣

Therefore, to prove $B = k^{[2]}$, it is enough to prove that $B = A^{[1]}$.
Note that since $B[W] = k[X,Y,Z]$, B is an affine domain over k. Let $B = k[z_1, z_2 ... z_n]$ and let $a = \gcd(d(z_1), ... d(z_n))$. Then it is easy to see that $d(B) \subset aB$. Hence , by 4.1, $a \in A$. Let $d_1 = a^{-1}d$. Then, as $d(B) \subset aB$ and $a \in A = \ker(d)$, it is easy to see that $d_1 : B \to B$ is a (non-zero) locally nilpotent derivation of B such that $A = \ker(d_1)$. Therefore, replacing d by d_1, we assume without loss of generality that $\gcd(d(z_1), ..d(z_n)) = 1$.
Note that $A = ker(d)$, B is UFD and $A = k[U]$. Hence $\forall\ \lambda \in k$, $(U - \lambda)$ is a prime element in B such that $d((U - \lambda)B)) \subset (U - \lambda)B$. Hence d induces a locally nilpotent derivation $\bar{d}_\lambda : B/(U - \lambda)B \to B/(U - \lambda)B$. Since $U - \lambda \nmid d(z_i)$ for some i, $\bar{d}_\lambda$ is a non-zero locally nilpotent derivation. Hence, as $\dim(B/(U - \lambda)B) = 1$, by Lemma 3.5, $B/(U - \lambda)B = k^{[1]}$.

Let $I = d(B) \cap A$. Note that, as $A = \ker(d)$, $d \neq 0$, I is a non-zero ideal of A. In view Lemma 2.14, to prove that $B = A^{[1]}$, it is enough to show that $I = A$. Since $A = k[U]$, $I = f(U)A$. Therefore, to prove that $I = A$, it is enough to show that $\text{degree}_U(f) = 0$.

Assume that $\text{degree}_U(f) = n \geq 1$. But then , since k is algebraically closed, $\exists \lambda \in k$ such that $U - \lambda \mid f$. Let $W \in B$ be such that $d(W) = f$. Let $\bar{W}$ denote the image of W in $B/(U - \lambda)B$. Then $\bar{W} \in \ker(\bar{d}_\lambda)$. By the choice of d, $\bar{d}_\lambda$ is a non-zero locally nilpotent derivation on $B/(U - \lambda)B = k^{[1]}$. Therefore $\bar{W} \in k$. This implies that $\exists \beta \in k$ such that $W - \beta = (U - \lambda)V$ for some $V \in B$. Therefore $f(U) = d(W) = (U - \lambda)d(V)$. Hence $d(V) \in d(B) \cap A = I = f(U)A$ which leads to a contradiction.

Thus $I = A$ and hence $B = A^{[1]} = k^{[2]}$. ♣

Chapter 5

Example of Danielewski

In this and subsequent section we present some examples to show that the (cancellation) question mentioned in the introduction does nor have an affirmative answer in general. These examples involve rings of the type $C[U,V]/(U^2V-g)$ where C is a polynomial algebra over a field k and $g \in C \setminus k$.

For simplicity of notation we denote $C[U,V]/(U^2V-g)$ by B and by u,v images of U,V in B. Then $B = \bigoplus_{-\infty<n<\infty} B_n$ where $B_0 = C,\ B_{-i} = Cu^i$ $(i \geq 1)$, $B_{2j-1} = Cuv^j$ $(j \geq 1)$, $B_{2r} = Cv^r$ $(r \geq 1)$. Moreover u is transcendental over C and $B[1/u] = C[u,u^{-1}]$.

Now we proceed to give an example of Danielewski. k will denote an algebraically closed field of characteristic 0.

Let $B = k[U,V,W]/(U^2V-W^2-1)$. We denote by u,v,w images of U,V,W in B. It is easy to see that B is an affine domain of dimension 2 over k and k-subalgebras $k[u,v], k[u,w], k[v,w]$ of B are polynomial algebras in two variables over k.

We first prove some properties of locally nilpotent derivations on B.

Lemma 5.1. Let d be a non-zero locally nilpotent derivation on B. Then $\ker(d) \cap k[w] = k = \ker(d) \cap k[v]$.

Proof. Note that since $\ker(d)$ is factorially closed in B, $k \subset \ker(d)$ and k is algebraically closed, $\ker(d) \cap k[w] \neq k$ would imply that $\ker(d) \cap k[w] = k[w]$. But then, since $u^2v = w^2+1$, $u,v \in \ker(d)$ contradicting the fact that d is non-zero. Hence $\ker(d) \cap k[w] = k$.

Note that, since k is algebraically closed and char.$(k) = 0$, for every $\lambda \in k \setminus (0)$ $B/(v-\lambda) \simeq k[u,w]/(\lambda u^2 - w^2 - 1) \simeq k[t,t^{-1}]$ (Laurent polynomial algebra in one variable over k). Therefore, for $\lambda \in k \setminus (0)$, $v - \lambda$ is a prime element of B such that $B/(v - \lambda)$ does not admit any non-zero locally nilpotent derivation.

Since d is non-zero, $\exists\, b \in B$ such that $d(b) \neq 0$ and hence $v - \beta$ does not divide $d(b)$ for some $\beta \in k \setminus (0)$. If $\ker(d) \cap k[v] \neq k$, then arguing as above we see that $k[v] \subset \ker(d)$. Therefore $d(v - \beta) = 0$. Therefore, by Lemma 2.16, we see that d induces a non-zero locally nilpotent derivation on $B/(v - \beta)$; a contradiction.

Hence $\ker(d) \cap k[v] = k$.

♣

Lemma 5.2. Let d be a derivation on B given by $d(u) = 0, \quad d(w) = u^2, \quad d(v) = 2w$. Then d is a non-zero locally nilpotent derivation on B such that $\ker(d) = k[u]$.

Proof. It is easy to see that d is locally nilpotent and $k[u] \subset \ker(d)$. Since $B[1/u] = k[u, u^{-1}][w](= k[u, u^{-1}]^{[1]})$ and $k[u] = B \cap k[u, u^{-1}]$, $k[u]$ is algebraically closed in B and hence $k[u] = \ker(d)$.

♣

Proposition 5.3. Let $B = k[U, V, W]/(U^2V - W^2 - 1)$. The Makar-Limanov invariant $\mathrm{AK}(B) = k[u]$.

Proof. In view of Lemma 5.2, we have $k \subset \mathrm{AK}(B) \subset k[u]$. To complete the proof it is enough to show that if D is a non-zero locally nilpotent derivation then $k[u] \subset \ker(D)$. Since D is non-zero and $\mathrm{tr.deg_k(B)} = 2$, we get that $\mathrm{tr.deg}_k(\ker(D)) = 1$. Since $k[u]$ is algebraically closed in B, $k[u] \subset \ker(D)$ if and only if $k[u] = \ker(D)$. Since $\ker(D)$ is algebraically closed in B, $k[u] = \ker(D)$ if and only if $\ker(D) \subset k[u]$. Now we proceed to prove that $\ker(D) \subset k[u]$.

Recall that $B = \bigoplus_{-\infty<n<\infty} B_n$ where $B_0 = k[w], B_{-i} = k[w]u^i \ (i \geq 1),\ B_{2j-1} = k[w]uv^j \ (j \geq 1),\ B_{2r} = k[w]v^r \ (r \geq 1)$.

Suppose $\exists\, g \in \ker(D)$ such that $g \notin k[u]$. Then $\mathrm{hwf}(g) \in B_n, \quad n \geq 0$ and $\mathrm{hwf}(g) \notin k$. By Theorem 2.17, there exists a non-zero derivation $\bar{D}$ over B such that if $C = \ker(\bar{D})$ then

1. $C = \bigoplus_{-\infty<i<\infty} C \cap B_i$
2. $\mathrm{hwf}(g) \in C \cap B_n$.

Since $\text{hwf}(g) \notin k$ and $n \geq 0$ either $\text{hwf}(g) \in k[w] \setminus k$ or it is divisible by v. Therefore, since C is factorially closed in B, we get that either $k[w] \subset C$ or $k[v] \subset C$ which is a contradiction in view of Lemma 5.1.

Therefore $\ker(D) \subset k[u]$ and hence $\text{AK}(B) = k[u]$.

♣

Proposition 5.4. Let $B = k[U,V,W]/(U^2V - W^2 - 1)$ and $R = B[Y]$ $(B^{[1]})$. Let d be a locally nilpotent derivation on R defined as: $d(u) = 0,\ \ d(w) = u^2,\ \ d(v) = 2w,\ \ d(Y) = u$ and let $A = \ker(d)$. Then

1. $R = A^{[1]}$ and

2. $\text{AK}(A) = k$

Proof. . Let $f = uY - w \in R$. Then clearly, $d(f) = 0$. Note that

$$\begin{aligned} f^2 + 1 &= u^2Y^2 + w^2 - 2uwY + 1 \\ &= u^2Y^2 + u^2v - 2uwY \\ &= u(uY^2 + uv - 2wY) \end{aligned} \tag{5.0.1}$$

Let $g = uY^2 + uv - 2wY$. Then $f^2 + 1 = ug$. Since $d(f) = 0$, $d(f^2 + 1) = 2d(f) = 0$. Thus, $d(ug) = 0$ and hence, $u, g \in A$.

To show that $R = A^{[1]}$ it is enough to show that $\exists\ X \in R$ such that $d(X) = 1$. Since $ug = f^2 + 1$, u, f are co-maximal. Let $X_1 = (Y^2 - v)/2$. Then we have

$$\begin{aligned} d(X_1) &= Y d(Y) - d(v)/2 \\ &= uY - w \\ &= f \end{aligned} \tag{5.0.2}$$

Now consider $X = gY - fX_1$, then $d(X) = gu - f^2 = 1$. Thus, by Lemma 2.14 $R = A[X]$ $(A^{[1]})$

Now we proceed to prove that $\text{AK}(A) = k$.

Claim: $A = k[u, g, f]$

Proof. Since $u, f, g \in A$, $k[u, g, f] \subset A$. Let C denote the k-subalgebra $k[u, g, f]$ of A.

Note that $B[1/u] = k[u, u^{-1}][w]$ $(k[u, u^{-1}]^{[1]})$. Therefore

$$\begin{aligned} R[1/u] &= k[u.u^{-1}][w, Y] \\ &= k[u, u^{-1}][f, w] \\ &= C[1/u][w] \text{ (i.e. } C[1/u]^{[1]}) \end{aligned} \tag{5.0.3}$$

On the other hand, since $d(Y) = u$ and $A = \ker(d)$, we have $R[1/u] = A[1/u][Y]$ $(A[1/u]^{[1]})$. Therefore, as $C \subset A$, we have $C[1/u] = A[1/u]$. Hence to prove the claim it is enough to show that $C \cap uA = uC$. Since $A \cap uR = uA$ (A being kernel of d), it enough to show that $C \cap uR = uC$ i.e. the canonical homomorphism $C/uC \to R/uR$ is injective.

Note that C is an affine domain of dimension 2 over k. Let $\theta : k[T_1, T_2, T_3] \to C$ be a surjective homomorphism given by

$$\begin{aligned} \theta(T_1) &= u \\ \theta(T_2) &= g \\ \theta(T_3) &= f. \end{aligned} \tag{5.0.4}$$

Since $ug = f^2 + 1$, $\ker(\theta) = (T_1T_2 - {T_3}^2 - 1)$ i.e $C \simeq k[T_1, T_2, T_3]/(T_1T_2 - {T_3}^2 - 1)$. Therefore $k[f]/(f^2 + 1)[g] = C/uC \simeq k[T_3]/({T_3}^2 + 1)[T_2]$ (a polynomial algebra in one variable over $k[T_3]/({T_3}^2 + 1)$).

Now $R/uR = k[w]/(w^2 + 1)[v, Y]$ (a polynomial algebra in two variables v, Y over $k[w]/(w^2+1)$) and under the canonical homomorphism $C/uC \to R/uR$ image of f = image of $-w$ in R/uR and image of $g = -2wY$. Therefore the homomorphism $C/uC \to R/uR$ is injective and hence $C = A$.

Thus the claim is proved

♣

Since $A \simeq k[T_1, T_2, T_3]/(T_1T_2 - {T_3}^2 - 1)$, it easy to see that there exists two non-zero locally nilpotent derivations d_1, d_2 on A such that $d_1(u) = 0 = d_2(g)$. As $k[u]$ and $k[g]$ are algebraically closed in A we have $\ker(d_1) = k[u]$ and and $\ker(d_2) = k[g]$. Hence $k \subset \mathrm{AK}(A) \subset k[u] \cap k[g] = k$.

Thus $\mathrm{AK}(A) = k$.

♣

Corollary 5.4.1. Let B, R, A be as in Proposition 5.4. Then A is stably isomorphic to B but $A \not\cong B$.

Proof. Since $A[X] = R = B[Y]$, A is stably isomorphic to B. Since $\mathrm{AK}(B) = k[u]$ and $\mathrm{AK}(A) = k$, A is not isomorphic to B ♣

Chapter 6

Example of Neena Gupta

In this section we present an example (due to Neena Gupta) of a three dimensional affine domain R over an algebraically closed field k of *positive characteristic* such that $R^{[1]} = k^{[4]}$ but $R \neq k^{[3]}$, thus showing that even Zariski cancellation problem does not have an affirmative answer in general in the case the base field has a positive characteristic. The proof of the fact that $R \neq k^{[3]}$ is quite involved and we need series of results which will be stated without proof.

In what follows k will denote an algebraically closed field of characteristic $p > 0$. The following result is due to Russell and Sathaye ([10])

Lemma 6.1. Let $G \in k[Z,T]$ be a prime element. Suppose $\exists\ h \in k[G]$ such that $k[Z,T,1/h] = k[G,1/h]^{[1]}$ then $k[Z,T] = k[G]^{[1]}$ i.e. G is a variable in $k[Z,T]$.

Next proposition is an easy consequence of the above lemma.

Proposition 6.2. Let ϕ be a non-trivial exponential map on $k[Z,T](k^{[2]})$. Let A denote the ring of ϕ-invariants. Then $A = k[G]$ and $k[Z,T] = A^{[1]}$ (i.e. G is a variable in $k[Z,T]$).

Proof. Since $k[Z,T]$ is a UFD, by Lemma 2.13, A is also UFD. Since ϕ is non-trivial, $\mathrm{tr}_k(A) = 1$ and $k \subset A \subset k[Z,T]$. Hence $A = k^{[1]}$ by Proposition 2.2 Let $A = k[G]$. Since A is the ring of ϕ-invariants, by Lemma 2.15, $\exists h \in k[G]$ such that $k[Z,T,1/h] = k[G,1/h]^{[1]}$. Hence, by Lemma 6.1, G is a variable in $k[Z,T]$.

♣

A proof of following result can be found in Nagata's book entitled "Automorphism group of $k[Z,T]$"([9])

Proposition 6.3. Let k be an algebraically closed field of positive characteristic p and let $q = p + 1$. Let $\sigma : k[Z, T] \to k[U]$ be a k-algebra homomorphism defined as: $\sigma(Z) = U + U^{pq}, \sigma(T) = U^{p^2}$. Let $F = T + T^{pq} - Z^{p^2}$. Then

1. $\ker(\sigma) = (F)$
2. σ is surjective and hence $k[Z, T]/(F) = k[U]$
3. F is not a variable in $k[Z, T]$.

The following result is due to Asanuma ([2], Theorem 5.1 and Corollary 5.3.).

Theorem 6.4. Let $F \in k[Z, T]$ be as in Proposition 6.3. Let $R = k[U, V, Z, T]/(U^2V - F)$. Then $R^{[1]} = k^{[4]}$.

In what follows we assume that k is an algebraically closed field of positive characteristic $p \geq 3$, $q = p + 1$, $F = T + T^{pq} - Z^{p^2} \in k[Z, T]$ and $R = k[U, V, Z, T]/(U^2V - F)$. u, v will denote images of U, V in R. We regard $k[Z, T]$ as a k-subalgebra of R

Note that, by Proposition 6.3, $k[Z, T]/(F) = k^{[1]}$ but F is not a variable in $k[Z, T]$. Moreover, by Theorem 6.4, $R^{[1]} = k^{[4]}$.

The following lemma is about non-trivial exponential maps on R.

Lemma 6.5. Let ψ be a non-trivial exponential map on R. Then $F \notin R^{\psi}$ (the ring of ψ-invariants).

Proof. . Note that for every non-zero $\lambda \in k$, the canonical map $k[Z, T] \to R/(u - \lambda)$ is an isomorphism sending F to the image of $\lambda^2 v$. This shows that $u - \lambda$ is a prime element of R for every non-zero $\lambda \in k$.

Let C denote R^{ψ}. Since C is factorially closed in R, $F \in C$ implies that $u, v \in C$. Since ψ is non-trivial, it is easy to see that $\exists\, \beta \in k \setminus (0)$ such that ψ induces a non-trivial exponential map $\bar{\psi}$ on $R/(u - \beta) = k[Z, T]$. Moreover, $v \in C$ implies $\bar{\psi}(F) = F$, thus showing that F is an element of the ring of $\bar{\psi}$-invariants. By Proposition 6.2, $k[G]$ is the ring of $\bar{\psi}$-invariants for some variable $G \in k[Z, T]$. Hence $F \in k[G]$. Now F is a prime element of $k[Z, T]$ and k is algebraically closed. Therefore $k[G] = k[F]$. This means that F is also a variable in $k[Z, T]$ which is a contradiction in view of Proposition 6.3.

Thus $F \notin R^{\psi}$. ♣

Lemma 6.6. Let k be an algebraically closed field such that char.$k \neq 2$. Let $f \in k[W]$ be such that $k[U,W]/(U^2 - f) = k^{[1]}$. Then f is a variable in $k[W]$.

Proof. Let $k[U,W]/(U^2 - f) = k[X]$. Then $k[W] \subset k[X]$ and $k[X]$ is a free $k[W]$-module of rank two. This show that $\deg_X(W) = 2$.

Let $W = \alpha_0 + \alpha_1 X + \alpha_2 X^2$ with $\alpha_i \in k$ and α_2 is non-zero. Without loss of generality we assume that $\alpha_2 = 1$. Recall that char.$(k) \neq 2$. Let $X_1 = X + 1/2\alpha_1$. Then $W = {X_1}^2 + \beta$ for some $\beta \in k$. Let $W_1 = W - \beta$.

Now $k[X] = k[X_1]$, $k[W] = k[W_1]$ and ${X_1}^2 = W_1$. Hence $k[X]$ is a free module over $k[W]$ with a basis $(1, X_1)$ Since $k[X] = k[U,W]/(U^2 - f)$, $(1,u)$ is also basis of $k[X]$ over $k[W]$. Since $u^2 = f \in k[W]$ and ${X_1}^2 = W_1$ and char.$(k) \neq 2$ we see that $u = \lambda X_1$ for some non-zero $\lambda \in k$. Therefore $u^2 = f = \lambda^2 W_1$. Hence f is a variable in $k[W]$.

♣

The following proposition of Neena Gupta is very crucial in proving that $R \neq k^{[3]}$.

Proposition 6.7. Let k be an algebraically closed field of positive characteristic $p \geq 3$, $q = p+1$, $F = T + T^{pq} - Z^{p^2} \in k[Z,T]$ and $R = k[U,V,Z,T]/(U^2V - F)$. Let ϕ be a non-trivial exponential map on R. Then the ring R^{ϕ} of ϕ-invariants is a k-subalgebra of $k[u,Z,T]$.

Proof. We begin our proof with some observations.

Recall that $R = \bigoplus_{-\infty < n < \infty} R_n$ where $R_0 = k[Z,T]$, $R_{-i} = R_0 u^i$ $(i \geq 1)$, $R_{2j-1} = R_0 u v^j$ $(j \geq 1)$, $R_{2r} = R_0 v^r$ $(r \geq 1)$. Therefore every element of R_n $(n \geq 1)$ is divisible by v. Moreover $\bigoplus_{-\infty < n \leq 0} R_n = k[u,Z,T]$ $(k^{[3]})$ and $\bigoplus_{0 \leq 2r < \infty} R_{2r} = k[v,Z,T]$ $(k^{[3]})$.

Let ϕ be a non-trivial exponential map on R. We prove the proposition by showing that we get a contradiction if the ring R^{ϕ} of ϕ-invariants is not contained in $k[u,Z,T]$.

Now suppose R^{ϕ} is not contained in $k[u,Z,T]$. Then there exists $g \in R^{\phi}$ such that $\mathrm{hwf}(g) \in R_n$ for some $n \geq 1$ and hence is divisible by v. Moreover, by Theorem 2.17, ϕ induces a non-trivial exponential map $\psi : R \to R[T]$ such that if $A = R^{\psi}$ then

1. $A = \bigoplus_{-\infty < n < \infty} (A \cap R_n)$

2. $\mathrm{hwf}(g) \in A$.

Since v divides hwf(g) and A is factorially closed in R, $v \in A$. But, by 6.5, $F \notin A$ and hence $u \notin A$. This shows that, since A is a factorially closed graded k- subdomain of R, $A = \bigoplus_{0 \leq 2r < \infty} (A \cap R_{2r})$ and hence $k[v] \subset A \subset k[v, Z, T]$. Note that A is UFD and $\mathrm{tr}_k(A) = 2 = \mathrm{tr}_{k[v]}(A) + 1$. Therefore, by Proposition 2.2, $A = k[v]^{[1]}$.

Claim $A = k[v, G]$ for some $G \in k[Z, T]$.

Proof. Note that A is graded, factorially closed in R, $A \subset k[v, Z, T]$ and $R_{2r} = k[Z, T]v^r$ if $r \geq 0$. Therefore , if $C = A \cap k[Z, T]$, then $A = \bigoplus_{0 \leq 2r < \infty} Cv^r$. As A is factorially closed in R, C is factorially closed in $k[Z, T]$ and hence C is a UFD. Since $\mathrm{tr}_k(A) = 2$ and $F \notin A$, we get that $k \subset C \subset k[Z, T]$ and $\mathrm{tr}_k(C) = 1$. Therefore, by Proposition 2.2, $C = k[G](k^{[1]})$. Thus claim is proved. ♣

Now we show that $k[Z, T] = k[G]^{[1]}$.

Since $A = R^{\psi}$, by Lemma 2.15, $\exists\ h \in A$ such that $R[1/h] = A[1/h]^{[1]}$. Since $A[1/h] \subset k[v, Z, T][1/h] \subset R[1/h]$ and $k[v, Z, T] = k^{[3]}$ is UFD, by Proposition 2.2, $k[v, Z, T][1/h] = A[1/h]^{[1]}$. Let $\beta \in k^*$ be such that $v - \beta$ does not divide h in A and hence in R as A is factorially closed in R. Note that, as $R/(v-\beta) = k[U, Z, T]/(\beta U^2 - F)$ we have $k[Z, T] \subset R/(v-\beta)R$. This shows that $(v-\beta)R \cap k[v, Z, T] = (v-\beta)k[v, Z, T]$ and hence we get that $k[G] = A/(v-\beta)A \subset k[v, Z, T]/(v-\beta)k[v, Z, T] = k[Z, T] \subset R/(v-\beta)R$.

For simplicity of notation we denote $R/(v-\beta)R$ by B. Recall that $R[1/h] = A[1/h]^{[1]}$. Therefore, if $\bar{h}$ denotes the image of h in $A/(v-\beta)A = k[G]$, we get that $k[G][1/\bar{h}] \subset k[Z, T][1/\bar{h}] \subset B[1/\bar{h}] = k[G, 1/\bar{h}]^{[1]}$. Hence, by Proposition 2.2, $k[Z, T][1/\bar{h}] = k[G, 1/\bar{h}]^{[1]}$. Therefore, by Lemma 6.1, $k[Z, T] = k[G]^{[1]}$. Let $k[Z, T] = k[G, W]$.

Now we consider F as a polynomial in W with coefficients in $k[G]$. Since $F \notin A$ and $k[G] \subset A$, we get that $F \notin k[G]$. Therefore $\deg_W(F) = n \geq 1$. Let $\alpha(G)$ be the leading coefficient of F (as a polynomial in W). It is easy to see that $\exists\ \lambda \in k$ such that $G - \lambda$ does not divide $\bar{h}$ as well as $\alpha(G)$. Then, if f denote the image of F in $k[G, W]/(G-\lambda) = k[W]$ we get that $\deg_W(F) = n = \deg_W(f)$. Moreover, as $\bar{h} \in k[G]$, the image of $\bar{h}$ in $k[G]/(G-\lambda)$ is an element of k^*.

As $R[1/h] = A[1/h]^{[1]}$, h is not divisible by $v - \beta$ (in R) and $v - \beta \in A$, we have $R[1/h]/(v-\beta) = A[1/h]/(v-\beta)^{[1]}$ i.e. $B[1/\bar{h}] = k[G][1/\bar{h}]^{[1]}$.

Note that $R/(v-\beta) = B = k[U,Z,T]/(\beta U^2 - F) = k[U,G,W]/(\beta U^2 - F)$ and $k[G] = A/(v-\beta)A \subset k[v,W,G]/(v-\beta)k[v,W,G] = k[G,W] \subset B$. $B[1/\bar{h}] = k[G,1/\bar{h}]^{[1]}$. Therefore, using the fact that $G-\lambda$ is co-maximal to $\bar{h}$, we get that $k[U,W]/(\beta U^2 - f) = B/(G-\lambda) = k^{[1]}$. Therefore, by Lemma 6.6, f is a variable in $k[W]$ and hence $\deg_W(F) = n = \deg_W(f) = 1$.

Thus $F = \theta(G) + \alpha(G)W.$. Since $k[Z,T]/(F) = k^{[1]}$, $\alpha(G) \in k^*$. Therefore F must be a variable in $k[G,W] = k[Z,T]$ which is a contradiction.

Therefore $R^{\phi} \subset k[u,Z,T]$. Thus the proposition is proved.

♣

As a consequence of the above proposition we have the following theorem ([6]) which shows that Zariski cancellation problem does not have an affirmative answer in general in the case the base field has a positive characteristic.

Theorem 6.8. Let $k, k[Z,T], F, R$ be as in Proposition 6.7. Then $R^{[1]} = k^{[4]}$ but $R \neq k^{[3]}$.

Proof. By Theorem 6.4, $R^{[1]} = k^{[4]}$. On the other hand, by Proposition 6.7, the Derksen Invariant $DK(R)$ is a k-subalgebra of $k[u,Z,T]$ and hence $DK(R) \neq R$. Hence $R \neq k^{[3]}$, as $DK(k^{[3]}) = k^{[3]}$ (see Lemma 2.23). ♣

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