Eigenvalue distribution of families of regular graphs

A Thesis

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by

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Certificate

This is to certify that this dissertation titled "Eigenvalue distribution of families of regular graphs" towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by A. Dileep Kumar at Indian Institute of Science Education and Research under the supervision of Dr. Kaneenika Sinha, Assistant Professor, Department of Mathematics, during the academic year 2016-2017.

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This thesis is dedicated to all nice and genuine people

Declaration

I hereby declare that the matter embodied in the report entitled " Eigenvalue distribution of families of regular graphs ", is the result of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research (IISER) Pune, under the supervision of Dr. Kaneenika Sinha and the same has not been submitted elsewhere for any other degree.

A. Dileep Kumar

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Abstract

This thesis presents an exposition of a result of Serre about the asymptotic distribution of eigenvalues of families of regular graphs. This result is part of a paper published by Serre in 1997 titled “the equidistribution of eigenvalues of Hecke operators”. Then, we discuss a specific example of a family of Ramanujan graphs given by Lubotzky, Phillips and Sarnak in their 1988 paper on Ramanujan graphs, and calculate this limiting distribution measure of the eigenvalues of that family using Serre's result. We also give an alternate way of computing the measure using a result published by B.D.McKay in 1981 about the limiting distribution measure of the eigenvalues of a family of regular graphs satisying certain properties. We then discuss a similar result for a family of cycle graphs.

Contents

Chapter 1

Equidistribution Theory

Let x be a real number. Denote by $[x]$ the greatest integer less than or equal to x; let $\{x\}$ be the fractional part of x. Consider a sequence $w = (x_n)_{n\geq 1}$ of real numbers.

Let I be the unit interval $[0, 1)$. For a fixed positive integer N and a subset $E \subset I$. We define the counting function $A(E, N, w)$ as:

$$A(E, N, w) = \#\{1 \leq n \leq N : x_n \in E\}$$

Definition 1.1. *(Uniform distribution mod 1). The sequence $w = (x_n)_{n\geq 1}$ is said to be uniformly distributed modulo 1 if for all a and b satisfying $0 \leq a < b < 1$, we have*

$$\lim_{N\to\infty} \frac{A([a, b); N; w)}{N} = b - a. \tag{1.1}$$

Define the characteristic function of $[a, b)$ as follows:

$$c_{[a,b)}(x) = \begin{cases} 1 & \text{for } x \in [a, b) \\ 0 & \text{otherwise.} \end{cases}$$

The above equation can also be written in terms of $c_{[a,b)}$.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{[a,b)}(x_n) = \int_0^1 c_{[a,b)}(x)dx. \tag{1.2}$$

This equation can further be extended to all real valued continuous functions on I.

Theorem 1.1. *The sequence $(x_n)_{n\geq 1}$, of real numbers is u.d. mod 1 iff for every real valued continuous function f defined on $[0,1]$, we have:*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx. \tag{1.3}$$

Proof. Form (1.2), the theorem holds true for all step functions on I of the form $f(x) = \sum_{i=0}^{k-1} d_i c_{[a_i,a_{i+1})}(x)$ where $0 < a_0 < a_1 < \ldots < a_k = 1$. Now consider a continuous function f defined on I. For any $\epsilon > 0$, we can find two step functions f_1 and f_2 such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in I$ and $\int_0^1 (f_2(x) - f_1(x)) \leq \epsilon$ (as f is Riemann integrable). Now,

$$\begin{aligned}
\int_0^1 f(x)dx - \epsilon \leq \int_0^1 f(x) - (f_2(x) - f_1(x))dx &\leq \int_0^1 f_1(x)dx \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_1(\{x\}) \text{ (as } f_1 \text{ is a step function)} \\
&\leq \underline{\lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) \\
&\leq \overline{\lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) \\
&\leq \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_2(\{x_n\}) \\
&= \int_0^1 f_2(x)dx \\
&= \int_0^1 (f_2(x) - f(x))dx + \int_0^1 f(x)dx \\
&\leq \int_0^1 f(x)dx + \epsilon
\end{aligned}$$

As ϵ is arbitrary,

$$\int_0^1 f(x)dx = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\})$$

For the converse, we need to show that for a sequence (x_n), if (1.3) holds for all continuous functions on I, then (x_n) is u.d. mod 1. Consider an interval $[a,b) \subset I$. Given any $\epsilon > 0$, there exist two continuous functions g_1 and g_2 such that $g_1(x) \leq c_{[a,b)} \leq g_2(x) < \epsilon$ for $x \in I$ with the property that $\int_0^1 (g_2(x) - g_1(x))dx$.

$$\begin{aligned}
b - a - \epsilon = \int_0^1 c_{[a,b)} dx - \epsilon &\leq \int_0^1 g_2(x)dx - \epsilon \\
&\leq \int_0^1 g_1(x)dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g_1(\{x_n\}) \\
&\leq \varliminf_{N\to\infty} \frac{A([a,b);N)}{N} \\
&\leq \varlimsup_{N\to\infty} \frac{A([a,b);N)}{N} \\
&\leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g_2(\{x_n\}) \\
&= \int_0^1 g_2(x)dx \leq \int_0^1 g_1(x)dx + \epsilon \leq b - a + \epsilon
\end{aligned}$$

□

Theorem 1.2. *The sequence $(x_n)_{n\geq 1}$ is u.d. mod 1 iff for every complex-valued continuous function f on $\mathbb{R}$ with period 1 we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx. \tag{1.4}$$

Proof. Suppose the sequence (x_n) is u.d. mod 1. Consider a complex valued continuous function f. Apply Theorem 3.1 on the real and imaginary part of f separately to get,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx$$

But f has a period 1, so $f(\{x_n\}) = f(x_n)$. So we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx$$

For the converse, assume that (3.4) is true for all complex valued continuous functions with period 1. In the proof of the converse of Theorem 1.1, while choosing g_1 and g_2, put an additional condition that $g_1(0) = g_1(1)$ and $g_2(0) = g_2(1)$. Then the definitions of g_1 and g_2 can be extended periodically to $\mathbb{R}$ and so (3.4) is satisfied by g_1 and g_2. Now use the same argument as in the proof for the converse of Theorem 2.1. □

Theorem 1.3 (Weyl's criterion)**.** *The sequence* $(x)_n$ *is u.d. mod* 1 *iff*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \text{ for all integers } h \neq 0. \tag{1.5}$$

Proof. Suppose a sequence $(x)_n$ is u.d. mod 1. Then take $f(x) = e^{2\pi i h x}$ in Theorem 3.2 to get

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x_n} dx = 0.$$

For the converse, suppose (x_n) satisfies (3.5). Let f be a complex valued continuous function with period 1. If we can show that Theorem 3.2 holds for f, then we're done. Let ϵ be an arbitrary positive number. By Weierstrass approximation theorem, there exists a trigonometric polynomial g(x) (i.e. linear combination of function of the form $e^{2\pi i h x_n}$, $h \in \mathbb{Z}$) satisfying

$$\sup_{0\leq x\leq 1} |f(x) - g(x)| \leq \epsilon. \tag{1.6}$$

$$\begin{aligned} |\int_0^1 f(x)dx - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)| &\leq |\int_0^1 (f(x)-g(x))dx| + |\int_0^1 g(x)dx - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)| \\ &\leq |\int_0^1 (f(x)-g(x))dx| + |\int_0^1 g(x)dx - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g(x_n)| \\ &+ |\frac{1}{N}\sum_{n=1}^{N}(f(x_n) - g(x_n))| \end{aligned}$$

The first term and third term are less than ϵ (as $\sup\limits_{0\leq x\leq 1} |f(x) - g(x)| \leq \epsilon$).

Now, $\lim_{N\to\infty} \dfrac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0$. So if we take N large enough, we will have

$|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}| \leq$ any arbitrarily small number. By choosing this "arbitrarily small number" suitably, we can show that the second term is less than ϵ. So, theorem 3.2 holds for f and (x_n) is u.d. mod 1. □

Till now, we have looked at sequences which are uniformly distributed with respect to the Lebesgue measure. But, it could be uniformly distributed with respect to a more general measure.

Definition 1.2. *We say that a sequence $(x_n), n = 1, 2, \ldots$ is μ-equidistributed if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)d\mu \tag{1.7}$$

We also refer to $\mu(x)$ as the limiting probability density function of (x_n).

In this report, we will look at a sequence of families. The below definition is a version of the equidistribution criterion for a sequence of families.

Definition 1.3. *Consider a sequence of families $(x_\lambda)_{\lambda\geq 1}$. Denote by $|x_\lambda|$ the number of elements in the family x_λ. We assume that $|x_\lambda| \to \infty$ as $\lambda \to \infty$. We say that the sequence $(x_\lambda)_{\lambda\geq 1}$ is μ-equidistributed if*

$$\lim_{\lambda\to\infty} \frac{1}{|x_\lambda|}\sum_{n=1}^{|x_\lambda|} f(x_n) = \int_0^1 f(x)d\mu \tag{1.8}$$

Chapter 2

Basic Notions in Graph theory

Definition 2.1. *A graph G is a pair (V, E), where V is the set of vertices and $E \subseteq V \times V$ along with two functions:*
(i) "origin", $o : E \to V$ defined as follows: for $y = (a, b) \in E$, o(y)=a.
(ii) "inverse", $E \to E$ defined as follows:

$$y \mapsto \overline{y} \text{ where, for } y = (a, b),\ \overline{y} := (b, a)$$

We define $|G|$ = number of vertices of G.

Definition 2.2. *We say that G is a regular graph of degree k if for every $x \in V$, the set of edges with origin x has k elements.*

Definition 2.3. *A walk of length $r \geq 1$ is an alternating sequence of vertices and edges, $\{v_1, e_1, v_2, e_2, \ldots, e_r, v_{r+1}\}$. The origin of a walk is taken to be the first vertex, that is v_1. A walk is said to be closed if $v_1 = v_{r+1}$. A walk is said to be "without back-tracking" if $e_{i+1} \neq \overline{e_i}$ for $1 \leq i \leq r$.*

Definition 2.4. *A trail is a walk in which all edges are distinct. A path is a trail in which all vertices are distinct (except possibly the start and end vertices). A cycle is a closed path. An r-cycle is a cycle of length r. The length of the smallest cycle is called the girth of the graph.*

Definition 2.5. *A circuit is a closed walk without back-tracking and $e_r \neq \overline{e_1}$. An r-circuit is a circuit of length r. Notice that a cycle is a circuit but not vice-versa.*

Definition 2.6. *An acyclic graph is a graph with no cycles.*

Definition 2.7. *Let $A \subseteq V$. The subgraph induced by $A \subset V$ is the set of vertices in A together with the set of edges whose endpoints are both in A.*

Definition 2.8. *The adjacency matrix of a graph is defined as $[a_{ij}]_{|G|\times|G|}$, where a_{ij} is the number of edges between v_i and v_j. The eigenvalues of this matrix are taken to be the eigenvalues of the graph.*

Definition 2.9. *Chebyshev polynomials: Let $\Omega = [-2, 2]$. If $x \in \Omega$, we can write x uniquely in the form*

$$x = 2\cos\phi,\ \ 0 \leq \phi \leq \pi$$

If r is an integer ≥ 0, we define:

$$X_r(x) = e^{in\phi} + e^{i(n-2)\phi} + \ldots + e^{-in\phi} = \frac{\sin(r+1)\phi}{\sin\phi}$$

$X_r(x)$ is called the r-th Chebyshev polynomial. $X_n s$ are polynomials in x:

$$X_0(x) = 1, X_1(x) = x, X_2(x) = x^2 - 1, X_3(x) = x^3 - 2x, \ldots\ldots$$

Define $Y_{n,q} = X_n - \frac{1}{q}X_{n-2}$ (assuming that $X_m(x) = 0$ if $m < 0$.)

Remark 1. *X_n's form a basis of the set of all polynomials on Ω.*

Chapter 3

Outline of the thesis

This chapter contains some of the important theorems which I have studied. The proofs of these theorems are presented in later sections.

3.1 Eigenvalue distribution of k-regular graphs

In 1981, BD McKay [5] proved the following result about the asymptotic distribution of eigenvalues of a certain family of k-regular graphs.

Theorem 3.1. *Consider a sequence of k-regular graphs E_λ's such that $|E_\lambda| \to \infty$ as $n \to \infty$. Let $C_{r,\lambda}$ be the number of r-cycles in E_λ.*
If $\lim_{n\to\infty} C_{r,n}/|E_\lambda| = 0 \ \forall \ r \geq 3$, then the limiting probability density function (or in other words, the distribution measure) of the eigenvalues of E_λ is given by:

$$f(x) = \begin{cases} k\dfrac{\sqrt{4(k-1)-x^2}}{2\pi(k^2-x^2)} & \textit{for } |x| \leq 2\sqrt{k-1} \\ 0 & \textit{otherwise.} \end{cases} \tag{3.1}$$

3.2 A more general theorem

Serre, in [7] proved a far more general result. He proved the following:

Theorem 3.2. *Consider a family of k-regular graphs (E_λ) for which $|E_\lambda| \to \infty$ as $\lambda \to \infty$. Let $c_{r,\lambda}$ be the number of r-circuits in E_λ and (x_λ) be the family of eigenvalues of E_λ. Let $k = q + 1$.*
1) The following two properties are equivalent:
(i) There exists a measure μ on $\Omega_q = [-(q^{1/2} + q^{-1/2}), +(q^{1/2} + q^{-1/2})]$ such that (x_λ) are μ-equidistributed.
(ii) $\forall r \geq 1$, $c_{r,\lambda}/|E_\lambda|$ has a limit when $\lambda \to \infty$.
2) Suppose (i) and (ii) are satisfied, and let:

$$\gamma_r = \lim_{\lambda \to \infty} c_{r,\lambda}/|E_\lambda|, \text{ for } r = 1, 2, \dots. \tag{3.2}$$

then we have $\mu = \mu_q + \nu$, where μ_q is a measure on $\Omega = [-2, 2]$ defined as

$$\mu_q := \frac{(q+1)}{\pi[(q^{1/2} + q^{-1/2})^2 - x^2]} \sqrt{1 - \frac{x^2}{4}}$$

and ν is a measure on Ω_q, characterised by:

$$\int_{\Omega_q} Y_{r,1}(x)\nu(x)dx = \begin{cases} 0, & \text{if } r = 0 \\ \gamma_r q^{-r/2} & \text{if } r > 0 \end{cases}$$

where $Y_{r,1} = X_r - X_{r-2}$ and X_r's are Chebyshev polynomials (see Definition 2.9).

Note. From here on, we will denote $Y_{r,1}$ by Y_r.

The above theorem provides a recipe to compute the distribution measure [1] for any family of graphs in which the limits $\lim_{\lambda \to \infty} c_{r,\lambda}/|E_\lambda|$ exist whereas B.D.McKay looked at sequences for which $\lim_{\lambda \to \infty} C_{r,\lambda}/|E_\lambda| = 0$.
We present the proof of this theorem in Section 4.

[1] By computing the distribution measure, we mean the asymptotic distribution measure of the family of regular graphs considered.

3.3 Finding families that satisfy Serre's condition and computing their measure

Once we understand Serre's result, it is natural to look for families of k-regular graphs and compute the distribution measures for them. We look at a type of Ramanujan graphs defined by Lubotzky, Phillips and Sarnak in their paper on Ramanujan graphs [4]. They compute the asymptotic distribution measure for any family of k-regular graphs, $X_{n,k}$'s for which the girth $g_{X_{n,k}}$ also tends to infinity as $n \to \infty$. This is relevant to the family of Ramanujan graphs which they consider as they show in [4] that the girth of that family tends to infinity as $n \to \infty$.

Consider a sequence of k-regular graphs $(X_{n,k})$ for which $n \to \infty$ and the girth of $X_{n,k}$, $g_{X_{n,k}} \to \infty$ as $n \to \infty$. Associate with each graph in the family a measure $\mu_{X_{n,k}}$ supported on $[-k, k]$ which puts point masses $1/n$ at each of its eigenvalues. They prove the following:

Theorem 3.3 (Prop 4.3, [4])**.**

$$\lim_{\substack{n\to\infty \\ g_{X_{n,k}}\to\infty}} \mu_{X_{n,k}} = \mu_k$$

where

$$d\mu_k(t) = \begin{cases} \dfrac{\sqrt{k-1-t^2/4}}{\pi k(1-(t/k)^2)}dt & \text{if } |t| \le 2\sqrt{k-1} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We have proved the above theorem using both Serre's and B.D.McKay's result.

We then show that for a family of cycle graphs (C_n) such that $|C_n| \to \infty$ as $n \to \infty$, the limiting distribution measure for the eigenvalues is given by:

$$\mu(t) = \begin{cases} \dfrac{1}{\pi\sqrt{4-t^2}}dt & \text{if } |t| \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Details are presented in section 5.

Chapter 4

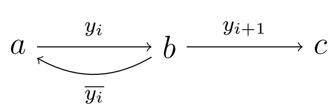
Distribution of eigenvalues

4.1 Regular graphs of degree $q + 1$

In the following, q is a fixed integer ≥ 1.

All the graphs considered are finite regular graphs of degree $q + 1$. These graphs need not be simple graphs, that is, they may contain loops and multiple edges. They also need not be connected.
We quickly review the notions of walks and circuits.

Walks and circuits. r is an integer ≥ 1. A *walk* of length r is a sequence in E, $y = (y_1, y_2, \ldots, y_r)$ consisting of r edges such that $t(y_i) = o(y_{i+1})$ for $1 \leq i \leq r$. We define $o(y) = o(y_1)$, $t(y) = t(y_r)$.
A walk is *closed* if it's origin and tail are the same i.e. $o(y) = t(y)$.
A walk is said to be "without back-tracking" if $y_{i+1} \neq \overline{y_i}$ for $1 \leq i < r$.

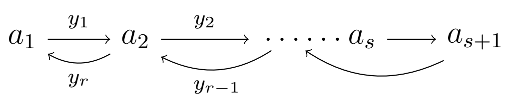


We say that a walk $\mathbf{y}$ is a *circuit* if:
(i) it is closed
(ii) without back-tracking" and

(iii) if $y_r \neq \overline{y_1}$

((ii) and (iii) can be combined to have just one condition: if $y_{i+1} \neq \overline{y_i}$ $\forall i \in \mathbb{Z}/r\mathbb{Z}$.)

This is how it would look like if $y_r = \overline{y_1}$ (for some $1 < s < r$):



$y \cdot y'$ consists of two walks y and y' such that $t(y) = o(y')$. It is defined as follows: Assume that there is an edge y between a and b and an edge y ′ between b and c.

$a \xrightarrow{y} b \xrightarrow{y'} c$

Then $y \cdot y'$ looks like, $y \cdot y'$:

$a \longrightarrow b \longrightarrow c$

In particular, we can talk about powers z^s $(s = 1, 2, \ldots)$ of a closed walk z. A circuit y is said to be <u>*primitive*</u> if it is not equal to any of the powers z^s, with $s > 1$, where z is a circuit.

Lemma 4.1. *Any circuit can be written uniquely as a power of a primitive circuit.*

Proof. Suppose y is not a primitive circuit. Then, it follows from the definition of primitive circuits that $y = z^s$ where z is a closed circuit. Without loss of generality, we can assume that z is primitive. So, we are done. And if y is a primitive circuit, just take $y = z$.

Uniqueness: If y is a primitive circuit. Say $y = z^s$ where z is a primitive circuit. This contradicts the fact that y is a primitive circuit. If y is not a primitive circuit, in that case suppose $y = {z_1}^s = {z_2}^l$ where z_1 and z_2 are primitive circuits and s, l are positive integers. If two closed walks are equal, it means that the sequence of vertices and edges in them is the same. In that case, ${z_1}^s = {z_2}^l$ is possible only when either z_1 is a power of z_2 or vice-versa. Without loss of generality, assume $z_1 = z_2^r$. This contradicts the fact that z_1 is a primitive circuit. So, $z_1 = z_2$. □

Number of circuits. Let f_r be the number of closed walks without back-tracking of length r, and c_r (resp ${c_r}^o$) be the number of circuits (resp. primitive circuits) of length r.

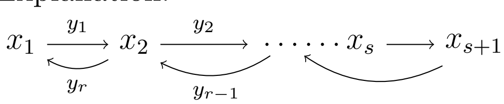
Lemma 4.2. $c_r = \sum_{s|r} {c_s}^o$.

Proof. Every circuit (**y**) can be written as a power of a primitive circuit i.e. $y = z^s$, where z is a primitive circuit. Let number of edges in $\mathbf{y}$ be r, number of edges in z be r'. $\implies r = r' \cdot s$. So, counting circuits of a certain length (here, r) is equivalent to counting the number of primitive circuits of length which is a factor of r. $\square$

Lemma 4.3.

$$f_r - c_r = \sum_{1 \le i < r/2} (q-1)q^{i-1} c_{r-2i} = (q-1)c_{r-2} + (q-1)qc_{r-4} + \ldots \tag{4.1}$$

Proof. $f_r - c_r =$ number of closed walks without back-tracking s.t. $y_r = \overline{y_1}$. This formula is demonstrated by noting that a closed walk without back-tracking, $y = (y_1, y_2, \ldots, y_r)$ which is not a circuit, is of the form $y_1 \cdot z \cdot \overline{y_1}$, where $z = (y_2, \ldots, y_{r-1})$ and a closed walk without back-tracking of length $r-2$.
For a fixed z,
(i) there are $q-1$ choices for y_1 if z is a circuit.
Explanation:



As x_2 is already part of 2 edges, one to x_3 and the other one, edge that comes back to x_2 so that z is a circuit).]
(ii) q choices when z is not a circuit. This means, $y_{r-1} = \overline{y_2}$.
x_2 is just part of one edge in this case. The edge from x_2 to x_3 is used twice, once while going towards x_3 from x_2 and the other time while completing the closed walk z. So, we have:

$$\begin{aligned}
f_r - c_r &= (q-1)c_{r-2} + q(f_{r-2} - c_{r-2}) \\
&= (q-1)c_{r-2} + q((q-1)c_{r-4} + q(f_{r-4} - c_{r-4})) \\
&= (q-1)c_{r-2} + q(q-1)c_{r-4} + q^2(f_{r-4} - c_{r-4}) \\
&= (q-1)c_{r-2} + q(q-1)c_{r-4} + q^2(q-1)c_{r-6} + q^3(f_{r-6} - c_{r-6})) \\
&\vdots \\
&= (q-1)[c_{r-2} + qc_{r-4} + q^2c_{r-6} + \ldots + q^{\frac{r-2}{2}}(f_2 - c_2)] \\
&= (q-1)[c_{r-2} + qc_{r-4} + q^2c_{r-6} + \ldots + q^{\frac{r-4}{2}}c_2 + q^{\frac{r-2}{2}}[(q-1)c_0 + q(f_0 - c_0)] \\
&= (q-1)[c_{r-2} + qc_{r-4} + q^2c_{r-6} + \ldots + q^{\frac{r-4}{2}}c_2 + q^{\frac{r-2}{2}}[(q-1)c_0 + q(f_0 - c_0)] \\
&= \sum_{1 \le i < r/2} (q-1)q^{i-1} c_{r-2i}
\end{aligned}$$

□

Remark 2. *The above lemma can be proved using induction also.*

4.2 The operators T and Θ_r.

Let G be as above. We note C_G to be the group of 0-chains of G i.e. the $\mathbb{Z}$-module of functions on $V(G)$ with values in $\mathbb{Z}$. If $x \in V(G)$, we define:

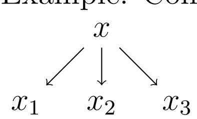
$$e_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

Easy to see that e_x forms a basis for C_G. (for $f \in C_G$, $f = \sum_{v \in V(G)} f(v) \cdot e_v$)

The operator T. Let T be an endomorphism of C_G defined as:

$$T(e_x) = \sum_{y \in E : o(y) = x} e_{t(y)} \tag{4.2}$$

Example: Consider a graph which looks as follows:



Then, $T(e_x) = e_{x_1} + e_{x_2} + e_{x_3}$. Seen as a correspondence on $V(G)$, T transforms a vertex to the sum of neighbours of the vertex.

Correspondence between T and the adjacency matrix of G.
$(i,j)^{th}$ entry of $[T]$ = coefficient of e_{x_j} in $T(e_{x_i})$

(1) If there's an edge between x_i and x_j:
$T(e_{x_i}) = \ldots + e_{x_j} + \ldots$
coefficient of e_{x_j} in $T(e_{x_i}) = 1$

(2) If there's no edge between x_i and x_j:
In $T(e_{x_i})$, there's no e_{x_j} term in that case.

So, $(i, j)^{th}$ entry of $[T] = 0$.
So, the matrix of T w.r.t the basis e_x is the adjacency matrix of G.

We are interested in the distribution of its eigenvalues in $\mathbb{R}$.

The operators Θ_r*.* The definition of T is generalised in the following way: for all $r \geq 1$, we define $\Theta_r \in End(C_G)$ as:

$$\Theta_r(e_x) = \sum_{\mathbf{y}} e_{t(y)} \tag{4.3}$$

or the sum over the walks without backtracking, $\mathbf{y} = (y_1, y_2, \ldots, y_r)$ with origin x and length r. It is clear that one has $\Theta_1 = T$.

The definition is completed by putting $\Theta_0 = I$.

Expression of Θ_r *as a function of* T*.* Θ_r are written as polynomials in T.
$\Theta_0 = I$, $\Theta_1 = T$, $\Theta_2 = T^2 - (q+1)$
Let us consider the $r = 2$ case.

$$\begin{aligned}
\Theta_2(e_x) &= T^2(e_x) - (q+1)e_x \\
&= T(T(e_x)) - (q+1)e_x \\
&= T(\sum_{o(y)=x} e_{t(y)}) - (q+1)e_x \\
&= \sum_{o(y)=x} T(e_{t(y)}) - (q+1)e_x \text{ } (\because \text{T is an endomorphism \& hence a homormorphism}) \\
&= \sum_{o(y)=x} \sum_{o(z)=t(y)} e_{t(z)} - (q+1)e_x
\end{aligned}$$

The indices x, y, z satisfying $o(y) = x$ and $o(z) = t(y)$ correspond to this picture:
$x_1 \overset{y}{—} x_2 \overset{z}{—} x_3$

But it also includes walks with back-tracking. That's why we have the second term.

$\Theta_3 = T^3 - (2q+1)T$
Again, the second term is due to the extra condition of "without back-tracking". Everything else is similar to the case that one gets by taking powers of adjacency matrix or powers of

T.

$x_1 \overset{y_1}{\text{———}} x_2 \overset{y_2}{\text{———}} x_3 \overset{y_3}{\text{———}} x_4$

$\Theta_3(e_x) = T(\Theta_2(e_x)) - qT(e_x)$

One of the ways to see it is to demonstrate the formula:

$$T\Theta_r = \Theta_{r+1} + \begin{cases} q+1 & \text{, if } r = 1 \\ q\Theta_{r-1} & \text{, if } r > 1 \end{cases} \tag{4.4}$$

Proof. There's a correspondence between $[T]$ and adjacency matrix of G, which gives us that composing Θ_r with T is the same as adding an edge to the already existing walk of length r.

For $r = 1 : T\Theta_1 = \Theta_2 + (q+1)$
has already been shown as one of the examples.
Assume it is true for $r = k(> 1)$ i.e. $T\Theta_k = (\Theta_{k+1}) + q\Theta_{k-1}$
$\implies T^2\Theta_k = (T\Theta_{k+1}) + qT(\Theta_{k-1})$
$\implies T\Theta_{k+1} = \Theta_{k+2} + q\Theta_k$.
So, it is true for k+1 also. □

We deduce the generator series:

$$\sum_{r=0}^{\infty} \Theta_r t^r = \frac{1-t^2}{1-tT+qt^2} \tag{4.5}$$

Proof. We have from (4.4),

$$T\Theta_r = \Theta_{r+1} + \begin{cases} q+1 & \text{, if } r = 1 \\ q\Theta_r & \text{, if } r > 1 \end{cases}$$

$$\implies T\Theta_{r-1} = \Theta_r + \begin{cases} q+1 & \text{, if } r = 2 \\ q\Theta_{r-2} & \text{, if } r > 2 \end{cases}$$

$$\implies T\Theta_r = \Theta_{r-1} - \begin{cases} q+1 & \text{, if } r = 2 \\ q\Theta_{r-2} & \text{, if } r > 2 \end{cases}$$

$$\sum_{r=0}^{\infty} \Theta_r t^r = \sum_{r=0}^{\infty} \Theta_r t^r = \Theta_0 t^0 + \Theta_1 t + \Theta_2 t^2 + \sum_{r=3}^{\infty} \Theta_r t^r$$
$$= 1 + Tt + (T\Theta_1 - (q+1)t^2) + \sum_{r=3}^{\infty} q\Theta_{r-2} t^r$$

$$\text{But, } \sum_{r=0}^{\infty} T\Theta_r t^r = T\sum_{r=0}^{\infty} \Theta_r t^r \ (\because \text{ T is an endomorphism.})$$
$$= 1 + Tt + T^2t^2 - (q+1)t^2 + Tt\sum_{r=3}^{\infty} \Theta_{r-1} t^{r-1} - qt^2 \sum_{r=3}^{\infty} \Theta_{r-2} t^{r-2}$$
$$= 1 + Tt + T^2t^2 - (q+1)t^2 + Tt\sum_{r=2}^{\infty} \Theta_r t^r - qt^2 \sum_{r=1}^{\infty} \Theta_r t^r$$

Let $\sum_{r=0}^{\infty} \Theta_r t^r = f$. Then,

$$f = 1 + Tt + T^2t^2 - (q+1)t^2 + Tt(f - Tt - 1) - qt^2(f-1)(here, f = \sum_{r=0}^{\infty} \Theta_r t^r)$$
$$\implies f = 1 + Tt + T^2t^2 - (q+1)t^2 + Ttf - T^2t^2 - Tt - qt^2 f + qt^2$$
$$\implies f - Ttf + qt^2 f = 1 - t^2$$
$$\implies f(1 - Tt + qt^2) = 1 - t^2$$
$$\implies f = \frac{1-t^2}{1 - Tt + qt^2}$$

□

If we set:

$$T' = T/q^{1/2} \text{ and } {\Theta_r}' = \Theta_r/q^{r/2} \tag{4.6}$$

Using the same method as in the above proof, the formula (4.5) can be rewritten as:

$$\sum_{r=0}^{\infty} \Theta_r' t^r = \frac{1 - t^2/q}{1 - T't + t^2} \tag{4.7}$$

Lemma 4.4 (eq. 23, [7]).

$$\sum_{n=0}^{\infty} Y_{n,q}(x) t^n = \frac{1 - t^2/q}{1 - xt + t^2} \tag{4.8}$$

Comparing (4.7) with (4.8), we can deduce that

$$\Theta_r' = Y_{r,q}(T') \tag{4.9}$$

where $Y_{r,q} = X_r - q^{-1} X_{r-2}$.
In other words:

$$\Theta_r = q^{r/2} Y_{r,q}(T/q^{r/2}) \tag{4.10}$$

Trace of Θ_r. If $r \geq 1$, it is clear that $Tr\ \Theta_r = f_r$ (f_r is the number of closed walks without back-tracking of length r). Hence, from (4.1):

$$Tr \Theta_r = c_r + \sum_{1 \leq i \leq r/2} (q-1) q^{i-1} c_{r-2i} (r \geq 1) \tag{4.11}$$

Thus, the knowledge of $Tr\ \Theta_r$, for $r = 1, 2, \ldots$, is equivalent to that of c_r. (By solving the equation for one r at a time, starting with $r = 1$.) From (4.8), it follows that, for any polynomial P, the trace of $P(T')$ can be expressed as a linear combination of c_r and $|G| = Tr\ I$. This follows from the below:

Note: $Y_{n,q}$'s also form a basis for the set of polynomials on Ω, like X_n's.

We'll need to look at the special case where P is a polynomial $Y_r = X_r - X_{r-2}$

Lemma 4.5. *If* $r \geq 1$, *we have:*

$$Tr\ Y_r(T') = c_r q^{-r/2} - \begin{cases} (q-1) q^{-r/2} |E| & , \textit{if } r \textit{ is even} \\ 0 & , \textit{if } r \textit{ is odd} \end{cases} \tag{4.12}$$

Proof. From (4.9) & (4.10), we have (for $r \geq 1$):

$$\begin{aligned} Tr(\Theta_r) &= Tr(q^{r/2} Y_{r,q}(T')) \\ &= c_r + \sum_{1 \leq i < r/2} (q-1) q^{i-1} c_{r-2i} \\ \implies q^{r/2} Tr(Y_{r,q}(T')) &= c_r + \sum_{1 \leq i < r/2} (q-1) q^{i-1} c_{r-2i}, \text{ if } r \geq 1 \end{aligned}$$

Note: for $r = 0$, $Tr(\Theta_0) = Tr(I) = |G|$.

So, $q^{r/2} \cdot Tr(Y_{r,q}(T')) = \begin{cases} |E| & \text{, if } r = 0 \\ c_r + \sum_{1 \leq i < r/2} (q-1) q^{i-1} c_{r-2i} & \text{, if } r \geq 1 \end{cases}$

As $Y_{r,q} = X_r - \frac{1}{q} X_{r-2}$, we can deduce by induction on r:

$$q^{r/2} Tr X_r(T') = \sum_{0 \leq i < r/2} q^i c_{r-2i} + \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} \tag{4.13}$$

For example, for $r = 2$ we have from (4.13):
$q \cdot Tr(X_{2,q}(T')) = q \cdot Tr(X_2(T')) - \frac{1}{q} \cdot Tr(X_0(T')) = (c_2 + (q-1)c_0)$

$\implies qTr(X_2(T')) = (c_2 + (q-1)c_0) + Tr(X_0(T')) = c_2 + Tr(I) = c_2 + |E|$.

Proof.[For 4.13] We'll use induction on r.

For $r = 1$:

$q^{1/2} Tr(Y_{1,q}(T')) = c_1$ (using (4.10))

$\implies q^{1/2} Tr(X_1(T') - \frac{1}{q} X_{-1}(T')) = c_1$

$\implies q^{1/2} Tr(X_1(T')) = c_1$

RHS of equality, (4.13), $\sum_{0 \leq i < 1/2} q^i c_{r-2i} = q^0 c_1 = c_1$.

For $r = 2$:

$q^{2/2}Tr(X_{2,q}(T')) = c_2$

$\implies q^{2/2}Tr(X_2(T')) - \frac{1}{q}TrX_0(T') = c_2$

$\implies q^{2/2}Tr(X_2(T')) = Tr(X_0(T')) + c_2 = |E| + c_2$

RHS of equality $= \sum_{0\leq i<1} q^i c_{r-2i} + |E| = q^0 c_2 + |E|$

Assume the given statement is true for $r = k$:

$$q^{k/2}TrX_k(T') = \sum_{0\leq i<k/2} q^i c_{k-2i} + \begin{cases} |E| & \text{, if k is even} \\ 0 & \text{, if k is odd} \end{cases}$$

To show: the equality holds for $k+2$.

From (4.13),

$$q^{(k+2)/2}Tr(X_{k+2,q}(T')) = q^{(k+2)/2}Tr(X_{k+2}(T') - \frac{1}{q}X_k(T')) + \begin{cases} |E| & \text{, if } r = 0 \\ c_r + \sum_{1\leq i<k/2+1} q^i c_{k+2-2i} & \text{, if } r \geq 1 \end{cases}$$

$$\implies q^{(k+2)/2}Tr(X_{k+2}(T')) - q^{k/2}Tr(X_k(T')) = \begin{cases} |E| & \text{, if } k+2 = 0 \\ c_{k+2} + \sum_{1\leq i<k/2+1} q^i c_{k+2-2i} & \text{, if } k+2 \geq 1 \end{cases} \tag{4.14}$$

$$\implies q^{(k+2)/2}Tr(X_{k+2}(T')) = q^{k/2}Tr(X_k(T')) + c_{k+2} + \sum_{1\leq i<k/2+1} q^i c_{k+2-2i}$$

$$= \sum_{1\leq i<k/2+1} q^i c_{k+2-2i} + \begin{cases} |E| & \text{, if k is even} \\ 0 & \text{, if k is odd} \end{cases} + c_{k+2} + \sum_{1\leq i<k/2} (q-1)q^{i-1} c_{k+2-2i}$$

$$= \sum_{1\le i<k/2} q^{i-1}c_{k-2(i-1)} + c_{k+2} + \sum_{1\le i<k/2} (q-1)q^{i-1}c_{k+2-2i}$$
$$+ \begin{cases} |E| & \text{, if k is even} \\ 0 & \text{, if k is odd} \end{cases}$$

$$= \sum_{1\le i<k/2} q^{i-1}c_{k+2-2i}[1+(q-1)] + c_{k+2} + \begin{cases} |E| & \text{, if k is even} \\ 0 & \text{, if k is odd} \end{cases}$$
$$= \sum_{1\le i<k/2} q^{i}c_{k+2-2i} + c_{k+2} + \begin{cases} |E| & \text{, if k is even} \\ 0 & \text{, if k is odd} \end{cases}$$
$$= \sum_{1\le i<k/2} q^{i}c_{k+2-2i} + \begin{cases} |E| & \text{, if k is even} \\ 0 & \text{, if k is odd} \end{cases}$$

Case 1: k is even

$$\implies q^{(k+2)/2}TrX_{k+2}(T') = \sum_{1\le i<(k+2)/2} q^{i}c_{k+2-2i} + |E|$$
So, the equality holds.

Case 2: k is odd.

$$\implies q^{(k+2)/2}TrX_{k+2}(T') = \sum_{1\le i<(k+2)/2} q^{i}c_{k+2-2i} + 0$$

So we have,

$$q^{(k+2)/2}Tr(X_{k+2}(T')) = \sum_{1\le i\le k/2} q^{i}c_{k+2-2i} + \begin{cases} |E| & \text{, if k+2 is even} \\ 0 & \text{, if k+2 is odd} \end{cases}$$
$$\implies q^{r/2}TrX_r(T') = \sum_{1\le i<r/2} q^{i}c_{r-2i} + \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} \quad \forall r\ge 1$$

□

Coming back to the proof of Lemma 4.5 again,

$$\implies TrX_r(T') = q^{-r/2} \sum_{1 \le i < r/2} q^i c_{r-2i} + q^{-r/2} \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases}$$

$$\implies TrY_r(T') = TrX_r(T') - TrX_{r-2}(T')$$

$$= q^{-r/2} \sum_{1 \le i < r/2} q^i c_{r-2i} + q^{-r/2} \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} -$$

$$q^{-(r-2)/2} \sum_{1 \le i < (r-2)/2} q^i c_{r-2i} + q^{-(r-2)/2} \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases}$$

$$\implies TrY_r(T') = q^{-r/2} \sum_{-1 \le i < r/2-1} q^{i+1} c_{r-2(i+1)} + q^{-r/2}(1-q) \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases}$$

$$- q^{-(r-2)/2} \sum_{-1 \le i < r/2-1} q^i c_{r-2-2i}$$

$$= q^{-r/2} \cdot q \cdot \sum_{-1 \le i < r/2-1} q^i c_{r-2-2i)} - q^{-r/2+1} \cdot \sum_{0 \le i < r/2-1} q^i c_{r-2-2i}$$

$$- q^{-r/2}(q-1) \cdot \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases}$$

$$= q^{-r/2+1} \cdot q^{-1} \cdot c_{r-2+2} - q^{-r/2}(q-1) \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases}$$

$$= q^{-r/2} c_r - q^{-r/2}(q-1) \begin{cases} |E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases}$$

□

4.3 Equidistribution of eigenvalues of T'

$\{E_\lambda\}$ is a family of graphs of the above type (i.e. finite, non-empty & regular graphs of degree $q+1$). Let $c_{r,\lambda}$ (respectively $c_{r,\lambda}{}^o$) be the number of circuits (respectively primitive circuits) in E_λ of length r. For each λ, the adjacency matrix T_λ of E_λ is a symmetric matrix whose coefficients are ≥ 0 & sum is $q+1$ (on each row). This results in the fact that the eigenvalues of T are real and absolute values $\leq q+1$. To be more specific, this follows from the below two lemmas:

Lemma 4.6. *The eigenvalues of a real symmetric matrix are real.*

Lemma 4.7. *For a row stochastic matrix, absolute value of eigenvalues is less than equal to sum of each row.*

So from the above lemma, Eigenvalues of T_λ lie in $[-(q+1),(q+1)]$. This is true as the adjacency matrix of T_λ is a row stochastic matrix with row sum equal to $q+1$. As in the preceding, it is convenient to divide T_λ by $q^{1/2}$, which gives a matrix whose eigenvalues belong to the interval: $\Omega_q = [-\omega_q, +\omega_q]$ where $\omega_q = q^{1/2} + q^{-1/2}$ So, eigenvalues of $q^{-1/2}$. T_λ would lie in $\left[\dfrac{-(q+1)}{q^{1/2}}, \dfrac{(q+1)}{q^{1/2}}\right] = [-\omega_q, +\omega_q]$.) This interval contains the interval $\Omega = [-2, 2]$ used hitherto. ($\because q^{1/2} + q^{-1/2} \geq 2 \;\forall\; q > 1$.)

In particular, any measure on Ω is identified with a measure on Ω_q whose support is contained in Ω. Let (x_λ) be the family of eigenvalues of T'_λ, viewed as family of points in the space Ω_q.

Theorem 4.8. *1) The following two properties are equivalent:*
(i) There exists a measure μ on Ω_q such that x_λ are μ-equidistributed.
(ii) $\forall r \geq 1$, $c_{r,\lambda}/|E_\lambda|$ has a limit when $\lambda \to \infty$.
2) Suppose (i) & (ii) are satisfied, and let:

$$\gamma_r = \lim_{\lambda\to\infty} c_{r,\lambda}/|E_\lambda|, \;\textit{for}\; r = 1, 2, \ldots. \tag{4.15}$$

then we have $\mu = \mu_q + \nu$, where μ_q is a measure on Ω defined as

$$\mu_q := \frac{(q+1)}{\pi[(q^{1/2}+q^{-1/2})^2 - x^2]}\sqrt{1 - \frac{x^2}{4}}$$

and ν is a measure on Ω_q, characterised by:

$$\int_{\Omega_q} Y_r(x)\nu(x)dx = \begin{cases} 0, & \textit{if } r = 0 \\ \gamma_r q^{-r/2} & \textit{if } r > 0 \end{cases}$$

where $Y_r = X_r - X_{r-2}$ and X_r's are Chebyshev polynomials (see Chapter 3).

We will need the following lemma to prove Theorem 4.1.

Lemma 4.9 ([1], Chapter 3)**.** *Let X be a locally compact space. Denote by $C(X;E)$ the vector space of continuous functions from X to E. We shall denote by $K(X;E)$ the subspace of $C(X;E)$ formed by the continuous functions with compact support. Let V be a linear subspace of $K(X;R)$ having the following property: For every compact subset K of X, there exists a function $f \in V$ such that $f \geq 0$ and $f(x) > 0 \ \forall \ x \in K$. Under these conditions, every positive linear form on V for the ordering induced by that of $K(X;R)$ may be extended to a positive measure on X (which is unique when V is dense in $K(X;R)$).*

Proof.[Theorem 4.1] Define $\langle Y_r, \nu \rangle = \int\limits_{-\omega_q}^{\omega_q} Y_r(x)\nu(x)dx$: the integral covers the whole interval Ω_q. Let δ_{x_λ} be the discrete measure on Ω_q defined by the family x_λ. According to (4.11),

$$TrY_r(T_{\lambda'}) = c_r q^{-r/2} - \begin{cases} (q-1)q^{-r/2}|E| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} \tag{4.16}$$

$$\begin{aligned} \langle Y_r, \delta_{x_\lambda} \rangle &= \int_{-\omega_q}^{\omega_q} Y_r(x).\delta_{x_\lambda}(x) \\ &= \frac{1}{E_\lambda} \sum_{i \in I_\lambda} Y_r(x_{i,\lambda}) \end{aligned}$$

where, I_λ is an indexing set for $Spec(T'_\lambda)$.

Claim: $TrY_r(T'_\lambda) = \sum_{i \in I_\lambda} Y_r(x_{i,\lambda})$
Fact: $Tr(A^n) = \sum\limits_{Spec(A)} {\lambda_i}^n$. (here, Spec(A) or the spectrum of A denotes the set of eigenvalues of A).

Let $Y_r(x) = \sum_{j=0}^{r} c_j x^j$. Then,

$$\begin{aligned}
Y_r(T'_\lambda) &= \sum_{j=0}^{r} c_j Tr[(T_\lambda)^j] \\
&= \sum_{j=0}^{r} c_j \sum_{Spec(T'_\lambda)} {x_\lambda}^j \\
&= \sum_{Spec(T'_\lambda)} \sum_{j=0}^{r} c_j {x_\lambda}^j \text{ (finite summations, so can be interchanged)} \\
&= \sum_{Spec(T'_\lambda)} Y_r(x_\lambda) \\
&= \sum_{i \in I_\lambda} Y_r(x_{i,\lambda}) \text{ (just writing it in terms of the indexing set)}
\end{aligned}$$

We have

$$\begin{aligned}
\langle Y_r, \delta_{x_\lambda} \rangle &= \frac{1}{|E_\lambda|} \sum_{i \in I_\lambda} Y_r(x_{i,\lambda}) \\
&= \frac{1}{|E_\lambda|} Tr Y_r(T'_\lambda)
\end{aligned}$$

But from (4.1), we have

$$\langle Y_r, \delta_{x_\lambda} \rangle = \frac{1}{|E_\lambda|} c_{r,\lambda} q^{-r/2} - \frac{1}{|E_\lambda|} \begin{cases} (q-1)q^{-r/2}|E_\lambda| & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} \tag{4.17}$$

$$= \frac{c_{r,\lambda} q^{-r/2}}{|E_\lambda|} - \begin{cases} (q-1)q^{-r/2} & \text{, if r is even} \\ 0 & \text{, if r is odd.} \end{cases} \tag{4.18}$$

If $(x)_\lambda$ are μ-equidistributed i.e. $\delta_{x,\lambda} \to \mu$, then

$$\lim_{\lambda \to \infty} \langle Y_r, \delta_{x_\lambda} \rangle = \langle Y_r, \mu \rangle \tag{4.19}$$

$$\text{Thus, } \lim_{\lambda \to \infty} \frac{c_{r,\lambda} q^{-r/2}}{|E_\lambda|} - \begin{cases} (q-1)q^{-r/2} & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} = \langle Y_r, \mu \rangle \tag{4.20}$$

$$\implies \lim_{\lambda\to\infty} \frac{c_{r,\lambda} q^{-r/2}}{|E_\lambda|} - \begin{cases} (q-1)q^{-r/2} & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} \text{ exists.}$$

$$\implies \lim_{\lambda\to\infty} \frac{c_{r,\lambda}}{|E_\lambda|} \text{ exists } \forall r > 0.$$

Proof of the converse: Suppose $\lim_{\lambda\to\infty} \frac{c_{r,\lambda}}{|E_\lambda|}$ exists for every $r \geq 1$.

Then, $\langle Y_r, \delta_{x_\lambda}\rangle$ has a limit (from (4.18)).

For $r = 0$, $\langle Y_r, \delta_{x_\lambda}\rangle = \frac{1}{|E_\lambda|} \sum_{i\in I_\lambda} Y_0(x_{i,\lambda}) = \frac{1}{|E_\lambda|} \sum\limits_{i\in I_\lambda} 1 = \frac{|E_\lambda|}{|E_\lambda|} = 1$.

One can check that, $\{Y_r, r = 0, 1, \dots n\}$ forms a basis of set of all polynomials of degree $\leq n$. So by linearity, $\langle P, \delta_{x_\lambda}\rangle$ has a limit $\forall$ polynomials P on Ω. Let $\lim\limits_{\lambda\to\infty} \langle P, \delta_{x_\lambda}\rangle = \mu(P)$. We have,

$$\begin{aligned} \mu(1) &= \lim_{\lambda\to\infty} \langle 1, \delta_{x_\lambda}\rangle \\ &= \lim_{\lambda\to\infty} \frac{1}{|E_\lambda|} \sum_{i\in I_\lambda} 1 = 1 \\ \mu(P) &= \lim_{\lambda\to\infty} \langle P, \delta_{x_\lambda}\rangle \\ &= \lim_{\lambda\to\infty} \frac{1}{|E_\lambda|} \sum_{i\in I_\lambda} P(x_\lambda) \end{aligned}$$

$\implies$ If $P \geq 0$ on Ω, $\mu(P) \geq 0$.

Let $X = \Omega$, $E = \mathbb{R}$, $K(X;\mathbb{R}) = K(\Omega;\mathbb{R}) = C(\Omega;\mathbb{R})$ and V = set of all real-valued polynomials on Ω. X is a locally compact space. So from Lemma 4.9, we can extend μ to a positive measure on Ω. Also, this measure is unique as V is dense in $C(\Omega;\mathbb{R})$. This follows from the Stone-Weierstrass theorem.

Hence, $\mu(f) = \lim\limits_{\lambda\to\infty} \langle f, \delta_{x_\lambda}\rangle$ exists for all continuous functions f on Ω. This is nothing but an equivalent condition for (x_λ) to be equidistributed.

This ends the proof of statement (1) of the theorem.

Proof of (2): Let $\lim_{\lambda\to\infty} \dfrac{c_{r,\lambda}}{|E_\lambda|} = \gamma_r$.

$$\implies \lim_{\lambda\to\infty} \frac{c_{r,\lambda} q^{-r/2}}{|E_\lambda|} - \begin{cases} (q-1)q^{-r/2} & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} = \langle Y_r, \mu \rangle \text{ (from (4.19))}. \tag{4.21}$$

According to (90, [7]),

$$\langle Y_r, \mu_q \rangle = \begin{cases} -(q-1)q^{-r/2} & \text{, if r is even} \\ 0 & \text{, if r is odd} \end{cases} \tag{4.22}$$

$$\begin{aligned} \implies \langle Y_r, \mu \rangle &= \gamma_r . q^{-r/2} + a_r(\mu_q) \\ &= \gamma_r . q^{-r/2} + \langle Y_r, \mu_q \rangle \end{aligned}$$

$$\begin{aligned} &\implies \langle Y_r, \mu - \mu_q \rangle = \gamma_r . q^{-r/2} \\ &\implies \langle Y_r, \nu \rangle = \gamma_r . q^{-r/2} \text{ where } \nu = \mu - \mu_q \end{aligned}$$

So, we're done. $\square$

Chapter 5

Ramanujan Graphs

Definition 5.1. *Let $X = X_{n,k}$ be a k-regular graph on n vertices. Let $\lambda(X)$ be the absolute value of its largest eigenvalue (distinct from $\pm k$). A graph $X_{n,k}$ is called a Ramanujan graph if*

$$\lambda(X) \leq 2\sqrt{k-1}.$$

Lubotzky, Phillips and Sarnak, [4], constructed a particular family of Ramanujan graphs (discussed in sections 5.1 and 5.2). They compute the distribution measure for families of regular graphs for which the girth asymptotically tends to infinity. This result is relevant to the family of graphs constructed by them as the girth for this family also asymptotically tends to infinity.

In this chapter, we try to apply Serre's result to compute the distribution measure for such families. We also compute the measure using a result published by B.D.McKay [5].

5.1 Cayley Graphs

Let G be a finite group and S a k-element subset of G. A set $S \subset G$ is said to be a symmetric set if $s \in S$ implies $s^{-1} \in S$.

We can construct the graph $X(G, S)$ using G and S as follows: Take the vertex set to be the elements of G. For any vertices x, y in G, (x, y) is an edge if and only if $xy^{-1} \in S$.

Claim: $X(G, S)$ is a k-regular graph.
For every vertex $x \in G$, we need to look for $y \in G$ such that xy^{-1} lies in S i.e. $xy^{-1} = s$ for some $s \in S$. So, $y = s^{-1}x$. There are k choices for s^{-1}. So, x is connected to k vertices.

Lemma 5.1. *If the symmetric subset S does not generate the entire group G, then the Cayley Graph $X(G, S)$ is not connected.*

Proof. Let us assume that $X(G, S)$ is connected. We are given that S does not generate G. Let $x \in G$ be an element not generated by S. As x is connected, there is a path x and any other vertex (say y) in G. Choose y to be in S. Let the vertex sequence of the path be $x, x_1, x_2, \ldots x_r, y$. x_r and y share an edge. So, $x_r = s_r y$ for some $s_r \in S$. Similarly, we can show that $x_{r-1} = s_{r-1}s_r y$ and so on. This gives $x = s_1 s_2 \ldots y$ which means that x lies in S, contradicting our assumption. □

5.2 Lubotzky-Phillips-Sarnak's Construction of Ramanujan Graphs

This is a construction of an explicit Ramanujan graph given by Lubotzky, Phillips, Sarnak ([4]):
Let p, q be two unequal primes congruent to 1 mod 4. Let i be an integer satisying $i^2 \equiv -1 \pmod{q}$.

- Consider the equation ${a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2 = p$. There are $8(p + 1)$ solutions $\alpha = (a_0, a_1, a_2, a_3)$ to this equation. Out of these, there are $(p + 1)$ solutions with $a_0 > 0$ and odd and a_j even for $j = 1, 2, 3$.

- To each solution associate the matrix $\tilde{\alpha}$ in $PGL(2, \mathbb{Z}/q\mathbb{Z})$

$$\tilde{\alpha} = \begin{bmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{bmatrix}$$

- Form the Cayley graph of $PGL(2, \mathbb{Z}/q\mathbb{Z})$ by taking the set of above $p + 1$ solutions as the symmetric subset.

- The graph obtained is a $(p+1)$ regular graph with $n = q(q^2-1)$ vertices.
  $|PGL(2,\mathbb{Z}/q\mathbb{Z})| = |GL(2,\mathbb{Z}/q\mathbb{Z})|/|Z(GL(2,\mathbb{Z}/q\mathbb{Z}))|$

$$Z(GL(2,\mathbb{Z}/q\mathbb{Z})) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \text{ where } \lambda \in (\mathbb{Z}/q\mathbb{Z})^*$$

  So, $|PGL(2,\mathbb{Z}/q\mathbb{Z})| = (q^2-1)(q^2-q)/(q-1)$.

- $PSL(2,\mathbb{Z}/q\mathbb{Z})$ is an index two subgroup of $PGL(2,\mathbb{Z}/q\mathbb{Z})$. Let Z_S & Z_G be the centres of $SL(2,\mathbb{Z}/q\mathbb{Z})$ & $GL(2,\mathbb{Z}/q\mathbb{Z})$ respectively. $PSL(2,\mathbb{Z}/q\mathbb{Z}) = SL(2,\mathbb{Z}/q\mathbb{Z})/Z_S$ $PGL(2,\mathbb{Z}/q\mathbb{Z}) = GL(2,\mathbb{Z}/q\mathbb{Z})/Z_G$

$$Z_G = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \text{ where } \lambda \in (\mathbb{Z}/q\mathbb{Z})^*$$

  and

$$Z_S = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \text{ where } \lambda \in (\mathbb{Z}/q\mathbb{Z})^* \text{ such that } \lambda^2 = 1.$$

  Let $g \in SL(2,\mathbb{Z}/q\mathbb{Z})$. $g.Z_S \in PSL(2,\mathbb{Z}/q\mathbb{Z})$& $g.Z_G \in PGL(2,\mathbb{Z}/q\mathbb{Z})$. But, $Z_S \subsetneq Z_G$. So, $gZ_S \neq gZ_G$. $PSL(2,\mathbb{Z}/q\mathbb{Z})$ is not a subgroup of $PGL(2,\mathbb{Z}/q\mathbb{Z})$ in the usual sense. We define a homomorphism from $PSL(2,\mathbb{Z}/q\mathbb{Z})$ to $PGL(2,\mathbb{Z}/q\mathbb{Z})$ as follows:

$$PSL(2,\mathbb{Z}/q\mathbb{Z}) \to PGL(2,\mathbb{Z}/q\mathbb{Z})$$
$$gZ_S \hookrightarrow gZ_G$$

- If $(\dfrac{p}{q}) = 1$, all $\tilde{\alpha}$'s lie inside $PSL(2,\mathbb{Z}/q\mathbb{Z})$).

  - *Claim:* Any matrix in $PGL(2,\mathbb{Z}/q\mathbb{Z})$ lies in $PSL(2,\mathbb{Z}/q\mathbb{Z})$ if its determinant is a square mod q.
  - Every element in $PSL(2,\mathbb{Z}/q\mathbb{Z})$ is of the form $g.Z_S$, which is mapped to $g.Z_G$. determinant of (representative of) $g.Z_G = det(g).det(Z_G) = 1.\lambda^2$ for some $\lambda \in \mathbb{Z}/q\mathbb{Z}$.
    So, the determinant of every element in $PSL(2,\mathbb{Z}/q\mathbb{Z})$ is a square.
  - If $\alpha \in PGL(2,\mathbb{Z}/q\mathbb{Z})$ and $\det(\alpha)$ is a square, say x^2 for some $x \in \mathbb{Z}/q\mathbb{Z}$. We

want to show that $\alpha = g.Z_G$ for some $g \in SL(2, \mathbb{Z}/q\mathbb{Z})$. Take $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ as the representative of Z_G. It is clear that $g = \alpha.\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}^{-1}$ works. ($g \in SL(2, \mathbb{Z}/q\mathbb{Z})$ as $det(g) = det(\alpha).\frac{1}{det\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}}$

  – $Det(\tilde{\alpha}) = p$. If $(\frac{p}{q}) = 1$, $p \equiv x^2 (mod q)$ for some $x \in \mathbb{Z}/q\mathbb{Z}$. So, $Det\tilde{\alpha}$ is a square and hence $\tilde{\alpha}$ lies in $PGL(2, \mathbb{Z}/q\mathbb{Z})$.

- It follows from Lemma 5.1 that the graph is disconnected.
- So, they define the graph ($X^{p,q}$) to the above Cayley graph as above if $(\frac{p}{q}) = -1$ and to be the Cayley graph of $PSL(2, \mathbb{Z}/q\mathbb{Z})$ if $(\frac{p}{q}) = 1$.

5.3 Asymptotic eigenvalue distribution in a family of $X_{n,k}$'s

Consider a sequence of k-regular graphs $X_{n,k}$'s for which $n \to \infty$ and the girth (of $X_{n,k}$) $g_{X_{n,k}} \to \infty$ as $n \to \infty$. Lubotkzy, Phillips and Sarnak (LPS) show in their paper that the Cayley graphs $X^{p,q}$ have the property that $g_{X_{p,q}} \to \infty$ as $q \to \infty$. So, the condition on the girth tending to infinity is relevant to their construction.

Associate with each graph in the family a measure $\mu_{X_{n,k}}$ supported on $[-k, k]$ which puts point masses $1/n$ at each of its eigenvalues. LPS show in [4] the following proposition about the asymptotic distribution measure of the eigenvalues of this family of graphs:

Theorem 5.2 (Prop 4.3, [4])**.**

$$\lim_{\substack{n \to \infty \\ g_{X_{n,k}} \to \infty}} \mu_{X_{n,k}} = \mu_k$$

where

$$d\mu_k(t) = \begin{cases} \frac{\sqrt{k-1-t^2/4}}{\pi k(1-(t/k)^2)} dt & \text{if } |t| \le 2\sqrt{k-1} \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

We now present alternate proofs of Theorem 5.2 using a result of B.D.McKay and then using Serre's result (Theorem 4.1).

5.4 Alternate proofs of Proposition 4.3

5.4.1 Proof using B.D.McKay's result

Consider a sequence of k-regular graphs $X_{n,k}$'s such that $|X_{n,k}| \to \infty$ as $n \to \infty$. Let $C_{r,n}$ be the number of r-cycles in $X_{n,k}$.
B.D.McKay proved the following result in 1981:

Theorem 5.3. *If* $\lim_{n\to\infty} C_{r,n}/|X_{n,k}| = 0$ *for every* $r \geq 3$*, then the limiting probability density function (or the distribution measure) of the eigenvalues of* $X_{n,k}$ *is given by:*

$$f(x) = \begin{cases} k\dfrac{\sqrt{4(k-1)-x^2}}{2\pi(k^2-x^2)} & \text{for } |x| \leq 2\sqrt{k-1} \\ 0 & \text{otherwise.} \end{cases}$$

In Theorem 5.2, we are looking at $X_{n,k}$'s such that $g_{X_{n,k}} \to \infty$ as $n \to \infty$. If $g_{X_{n,k}} \to \infty$ as $n \to \infty$, it is easy to see that $\lim_{n\to\infty} C_{r,n} = 0 \ \forall \ r > 0$. So, $\lim_{n\to\infty} C_{r,n}/|X_{n,k}| = 0 \ \forall \ r \geq 3$ and hence using B.D.McKay's result we can obtain the limiting distribution measure for the eigenvalues of this family of graphs. The measure obtained here is the same as that in Theorem 5.2.

Remark 3. *B.D. McKay's paper was published in 1981, whereas Lubotzky, Phillips, Sarnak's paper on Ramanujan Graphs came out in 1988.*

5.4.2 Proof using Serre's result

Like in the previous subsection, consider sequence of k-regular graphs $X_{n,k}$'s such that $|X_{n,k}| \to \infty$ as $n \to \infty$. Let $(x_{n,k})$ be the family of eigenvalues of $X_{n,k}$. We also have that $g_{X_{n,k}} \to \infty$ as $n \to \infty$. So again, $\lim_{n\to\infty} C_{r,n} = 0 \ \forall \ r > 0$.

Lemma 5.4. *Every closed walk contains a cycle.*

Proof. Consider a closed walk W.

Case I. Let W be of odd length. We'll use induction on length of W to prove our result. Base step: length of $W = 1$ i.e. W is a loop. A loop is a cycle. So, we're done. Assume our result is true for all closed walks of length $\leq 2r - 1$. Let W be a closed walk of length $2r + 1$. If there are no repeating vertices in W, then W is a cycle in which case we're done. If there are repeating vertices, consider the first vertex which repeats itself, say $v_i = v_j$. Now the circuit can be broken down into two closed walks. The first closed walk starts from the origin of W and goes till v_i and comes back to the origin (there's a way to come back to the origin as W is a closed walk). The other closed walk is the walk from v_i to $v_j (= v_i)$. Now, one of the walks has to be odd length. So by induction, W contains a cycle.

Case II. Let W be of even length. We'll again use induction on the walk length. Base step is clear (length 2 closed walks are multiple edges). Assume result is true for all closed walks of length $\leq 2r$. Now consider a closed walk of length $2r$. Split W into two closed walks as above. Either both of them are of odd length or both are of even length. If both are of odd length, it follows from Case I that both of them contain a cycle. If both of them are of even length, if follows from induction that both the walks contain a cycle. So, W contains a cycle.

□

Now, a circuit is also a closed walk. So, every circuit contains a cycle. But we have, $\lim_{n\to\infty} C_{r,n} = 0 \; \forall \; r > 0$. Let $c_{r,n}$ be the number of circuits in $X_{n,k}$. So, $\lim_{n\to\infty} c_{r,n} = 0 \; \forall \; r > 0$. Otherwise, there'll be a circuit of length (say) r as $n \to \infty$. And it'll contain a cycle which contradicts the fact that $\lim_{n\to\infty} C_{r,n} = 0 \; \forall \; r > 0$.

So, we have $\lim_{n\to\infty} c_{r,n}/|X_{n,k}| = 0 \; \forall \; r > 0$. Let $k = q+1$. It follows from Theorem 4.1 that $x_{n,k}$'s are equidistributed with respect to the measure μ given by:

$$\mu = \mu_q + \nu$$

and ν is a measure on Ω_q, characterised by:

$$\int_{\Omega_q} Y_r(x)\nu(x)dx = 0 \; \forall \; r.$$

It follows that $\nu(x) = 0$.

$$\implies \mu = \mu_q$$
$$\implies \mu(x) = \frac{(q+1)}{\pi[(q^{1/2}+q^{-1/2})^2 - x^2]}\sqrt{1-\frac{x^2}{4}}dx, \text{ for } |x| \leq 2.$$

Define $y = x.\sqrt{q}$.

$$\implies \frac{dy}{dx} = \sqrt{q} \implies dx = \frac{dy}{\sqrt{q}}$$

$$\implies \mu(x)dx = \frac{k}{\pi\left[\left(\frac{q+1}{\sqrt{q}}\right)^2 - \left(\frac{y}{\sqrt{q}}\right)^2\right]} \cdot \sqrt{1-\frac{y^2}{4q}}\frac{dy}{\sqrt{q}} \quad , \text{ for } |y| \leq 2\sqrt{q}$$
$$= \frac{k}{\frac{\pi}{q}[(q+1)^2 - (y)^2]} \cdot \sqrt{1-\frac{y^2}{4q}}\frac{dy}{\sqrt{q}} \quad , \text{ for } |y| \leq 2\sqrt{q}$$
$$= \frac{kq}{\pi[k^2 - (y)^2]} \cdot \sqrt{1-\frac{y^2}{4q}}\frac{dy}{\sqrt{q}} \quad , \text{ for } |y| \leq 2\sqrt{q}$$
$$= \frac{k\sqrt{q}}{\pi[k^2 - (y)^2]} \cdot \sqrt{1-\frac{y^2}{4q}}dy \quad , \text{ for } |y| \leq 2\sqrt{k-1}$$
$$= \frac{\sqrt{q}}{\pi k[1 - (\frac{y}{k})^2]} \cdot \sqrt{1-\frac{y^2}{4q}}dy \quad , \text{ for } |y| \leq 2\sqrt{k-1}$$
$$= \frac{\sqrt{q\big(1-\frac{y^2}{4q}\big)}}{\pi k\big[1 - (\frac{y}{k})^2\big]}dy \quad , \text{ for } |y| \leq 2\sqrt{k-1}$$
$$= \frac{\sqrt{q-\frac{y^2}{4}}}{\pi k\big[1 - (\frac{y}{k})^2\big]}dy \quad , \text{ for } |y| \leq 2\sqrt{k-1}$$
$$= \frac{\sqrt{k-1-\frac{y^2}{4}}}{\pi k\big[1 - (\frac{y}{k})^2\big]}dy \quad , \text{ for } |y| \leq 2\sqrt{k-1}$$

which is what we had in Theorem 5.2.

Remark 4. *In general, for any sequence of regular graphs* $(X_{n,k})$ *such that* $|C_n| \to \infty$ *as* $n \to \infty$ *and* $g_{C_n} \to \infty$ *as* $n \to \infty$*, the limiting distribution measure of the eigenvalues is given by:*

$$\mu(x) = \mu_q(x) = \begin{cases} \dfrac{\sqrt{1 - t^2/4}}{k\pi(1 - (t/k)^2)} dt & \textit{if } |t| \leq 2\sqrt{k-1} \\ 0 & \textit{otherwise.} \end{cases}$$

5.5 Other regular graphs: Cycle graph

Consider a family of cycle graphs C_n (i.e. a cycle on n vertices). We can see that $|C_n| \to \infty$ as $n \to \infty$. Let g_{C_n} be the girth of C_n. Clearly, $g_{C_n} \to \infty$ as $n \to \infty$. So, the limiting distribution measure of the eigenvalues of C_n's is given by (see Remark 3):

$$\mu_k(t) = \begin{cases} \dfrac{\sqrt{k - 1 - t^2/4}}{2\pi(1 - (t/k)^2)} dt & \text{if } |t| \leq 2\sqrt{k-1} \\ 0 & \text{otherwise.} \end{cases}$$

As $k = 2$ here,

$$\mu_2(t) = \begin{cases} \dfrac{1}{\pi\sqrt{4 - t^2}} dt & \text{if } |t| \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 6

Conclusion

We compute the distribution measure for families of regular graphs for which the girth asymptotically goes to infinity. It would be interesting to determine the error terms for them. In other words, we would like to investigate the discrepancy between the expected and actual number of eigenvalues that lie in an interval (say $[a, b]$), that is, $|A([a, b), N, (x_n)) - N \int_I \mu|$ (assuming μ is the limiting distribution measure for that family).

In this report, we looked at the limiting distribution of the eigenvalues as the number of vertices goes to infinity. We would also like to study the case when the degree of the graph, that is, k also goes to infinity.

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