# Self-Complementarity and The Erdős-Hajnal Conjecture

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# Certificate

This is to certify that this dissertation entitled 'Self-Complementarity and the Erdős-Hajnal Conjecture' towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Mihir Sanjeev Neve at Indian Institute of Science Education and Research under the supervision of Dr. Soumen Maity, Associate Professor, Department of Mathematics, IISER Pune and Prof. Saket Saurabh, Professor, Department of Theoretical Computer Science, IMSc Chennai, during the academic year 2022-2023.

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This thesis is dedicated to
Aaji, Vidhu, Mummy and Pappa

# Declaration

I hereby declare that the matter embodied in the report entitled 'Self-Complementarity and the Erdős-Hajnal Conjecture' are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Soumen Maity and Prof. Saket Saurabh, and the same has not been submitted elsewhere for any other degree.

MihirSN

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# Abstract

In this thesis, we take a closer look at the Erdős-Hajnal Conjecture. A Graph $H$ is said to have the Erdős-Hajnal (EH) property, if for some constant $\gamma(H)$, every sufficiently large $H$-free graph $G$ has a homogeneous set of size at least $|G|^{\gamma(H)}$. The Erdős-Hajnal Conjecture claims that every finite graph has the EH-property. It is known that the substitution operation preserves the EH-property, so it suffices to focus our attention only on substitution-prime graphs. We begin by studying the techniques used to prove the EH-property for the few known cases, namely $P_4$, $C_5$ and the Bull.

We extend some of these techniques, to show that in order to prove the EH-property for the smallest open case $P_5$, it suffices to look for large homogeneous sets in *dense* $P_5$-free graphs. We then ask whether these large homogeneous sets can be found in a dense $P_5$-free graph $G$ on making it $P_4$-free, by removing at most $c|G|$ number of vertices. We answer this question in the negative using a construction involving the substitution operation.

Finally, we note the role of 'Self-complementarity' in most of the known proofs of the EH-property and ask whether it is possible to further reduce the conjecture to proving the EH-property for a class of substitution prime self-complementary graphs. We show that this is possible by proving the following results about the self-complementary Paley graphs: Every graph is an induced subgraph of some primitive Paley graph, and all Paley graphs are substitution prime. Thus, we further reduce the Erdős-Hajnal Conjecture, by showing that it suffices to prove the EH-property for primitive Paley graphs. We also prove some simple upper bounds on $\gamma(H)$ for substitution prime graphs $H$.

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# Introduction

Given a graph $G$, a *clique (an independent set)* in $G$ is a subset of vertices which are all pairwise adjacent (non-adjacent) to each other. Cliques and independent sets in graphs constitute the most homogeneous and extreme form of subgraphs of a graph. Roughly, they denote the subset of nodes of a network with either the highest or lowest degrees of connectivity and thus, are valued subgraphs of a graph. In 1930, F. Ramsey [30] showed that all sufficiently large graphs, with at least $R(r,r)$ vertices, contained a homogeneous set on $r$ vertices, for all $r \in \mathbb{N}$. It is natural to ask that given any graph $G$, what is the largest-sized homogeneous subgraph to be guaranteed in the graph $G$.

Erdős and Szekeres [17] gave a lower bound by proving that the Ramsey number $R(r,r) \leq 2^{2r-3}$. In other words, every graph has a homogeneous set of size $\mathcal{O}(\log|G|)$. On the other hand, Erdös [15] showed the existence of certain random graphs which have no homogenous sets of size larger than $\mathcal{O}(\log|G|)$. Thus, over the class of all finite graphs, only the existence of homogeneous sets of size $\mathcal{O}(\log|G|)$ can be guaranteed; however, several graphs have homogeneous sets of size much larger than $\mathcal{O}(\log|G|)$ and hence, in order to search for a better bound, one must restrict oneself to a proper subfamily of all finite graphs.

In 1989, Erdős and Hajnal conjectured that it is possible to increase these logarithmic bounds dramatically, by restricting to the particular subclass of $H$-free graphs (see [16]). A graph $G$ is said to be $H$-free if $G$ does not contain $H$ as an induced subgraph. Then, a graph $H$ is said to have the EH-property if there exists a constant $c(H) > 0$ such that every $H$-free graph $G$ has a homogeneous set of size at least $|G|^{c(H)}$. The Erdős-Hajnal conjecture claims that all graphs have the EH-property.

The primary form of attacking this conjecture is to consider each graph $H$ separately, and ask whether a non-zero constant $c(H)$ exists or not. The key idea is that an induced subgraph specifies the positions of both edges and non-edges in the parent graph $G$. Hence, forbidding $H$ as an induced subgraph adds restrictions on the structure of $G$, and can induce some nice properties on the class of $H$-free graphs. Thus, in some sense, the Erdős-Hajnal Conjecture is really a statement on the relation between local and global properties of a graph. This induced structure on $G$ is, roughly, more restrictive for smaller graphs $H$, as they occur more frequently in graphs of a given size. Consequently, the problem becomes progressively difficult for graphs of larger sizes.

Indeed, it is extremely difficult to consider each graph one at a time, and hence a graph operation that preserves the EH-property is highly desired. In [16], Erdős and Hajnal show that the join and union of graphs are two such graph operations. Later, Alon et al [1] showed that the *substitution operation* (also called the graph replacement operation) also preserves the EH-property. Consequently, we focus our attention on *substitution prime graphs*, those which cannot be obtained from smaller graphs by a sequence of substitution operations. $P_4$, $C_5$, Bull, $P_5$ and $\overline{P_5}$ are the smallest prime graphs on at most five vertices. To give an idea of the progress of the conjecture, note that presently the EH-property is known only for $K_1, K_2, I_2, P_4, C_5$, the bull, and all those graphs which can be obtained through a sequence of substitution operations from these graphs.

A desirable property, which was crucially used in the proofs of the EH-property for $P_4$, $C_5$ and the Bull, is the self-complementarity of $H$. Note that the conjecture inherently behaves symmetrically with the complement operation, for a clique and an independent set switch roles on taking complements, and both $hom(G)$ and perfectness are invariant under the complement operation. Thus, the freedom in switching from $G$ to its complement for an $H$-free graph $G$, given a self-complementary graph $H$, is often helpful, as it can be used to add more properties to the graph $G$. We later prove some results which incorporate self-complementarity with the Erdős-Hajnal conjecture.

## Organisation of the thesis

The aim of this thesis is twofold. First, we compile some of the known results of the conjecture, in an attempt to study common features in each of the individual cases under a uniform framework; and second, we try to obtain some further results and reductions on the conjecture as a whole, or on the EH-property for some graph.

Chapter 1 covers some basic graph theoretic concepts and notation which shall be used throughout the thesis.

Chapter 2 introduces the conjecture and looks at its known relations with the substitution operation. In this chapter, we tried to formalise the statement of the conjecture by introducing the notion of $\gamma(H)$ and $\Gamma(H)$. This helped in proving various properties with ease and precision. We see how the substitution operation preserves the EH-property, and examine some properties of substitution prime graphs. We end the chapter with a side note on the relation between the asymptotic and non-asymptotic versions of the conjecture. We show that this relation becomes useful in getting some trivial upper bounds on $\gamma(H)$ for all prime graphs $H$.

Chapter 3 talks about a useful result by Rödl [31], which emphasises how forbidding a graph $H$ as an induced subgraph affects the distribution of edges, thereby giving rise to sufficiently large induced subgraphs with extremal edge densities. This theorem proves to be useful in multiple instances [9, 4], specifically when paired with a class of graphs closed under complements and induced subgraphs.

Chapter 4 presents the proofs of the EH-property known for three prime graphs, $P_4$, $C_5$ [9], and the bull [8]. The proof for $P_4$ is a classical result in graph theory, while the proof for $C_5$ is used later to get some results on the EH-property for $P_5$. Finally, the case of the bull becomes interesting as its proof incorporates almost all the major ideas related to the conjecture, namely restrictions due to forbidding the bull, self-complementarity, perfectness, and the substitution operation.

Chapter 5 presents some original progress towards the EH-property for $P_5$, the smallest prime graph for which the EH-property is not known to be true. We show that in order to prove the EH-property for $P_5$, it suffices to look for a large homogeneous set in *dense* $P_5$-free graphs. We then consider whether a dense $P_5$-graph has a small subset $S \subset V(G)$ such that $G[V(G)\backslash S]$ is $P_4$-free, for then, we can hope to find large homogeneous sets as required. However, we show that this is not possible.

Finally, in Chapter 6, we focus on the issue of self-complementarity and examine the self-complementary class of *Paley* graphs. We develop some required background in quadratic residues, finite fields, and prove some properties of *Paley* graphs. We then show that all graphs are induced subgraphs of some primitive *Paley* graph, and all *Paley* graphs are substitution primes. As a result, we reduce the Erdős-Hajnal Conjecture by showing that it suffices to prove the EH-property for the self-complementary, substitution prime family of primitive *Paley* graphs.

# Original Contributions

The Sections 2.4, 5.1, 5.2, and 6.2 of the thesis comprise entirely original work. These include the results on relations between asymptotic and non-asymptotic versions of the conjecture, the progress on the EH-property for $P_5$ and the reduction of the conjecture to *Paley* graphs. Section 6.1 provides an exposition of results on quadratic residues which have been extended from integers to finite fields of any order.

Further, in Section 2.3, we derive some properties of substitution prime graphs and use them to generate all possible primes on up to seven vertices. Figure 2.4 lists all prime graphs on six vertices, as generated by us. To the best of our knowledge, such a roster has not been published before.

Finally, we also introduce the notions of $\gamma(H)$, $\Gamma(H)$, and $\delta_G$ to aid the discussion on the conjecture and allow for proving various properties with ease and precision. All the remaining sections present known results with an occasional modification or rewriting of some proofs.

# Chapter 1

# Preliminaries

We define some basic concepts and notations used in the thesis. We refer to [13] for any further properties or definitions.

## Basic Graph Theoretic Terminology

**Definition 1.1** (Graph)**.** *A* graph $G(V, E)$ *is characterised by a pair of sets $V$, $E$; where $E \subset [V]^2$, the set of 2-element subsets of $V$.*

Graphs can be used to diagrammatically represent symmetric relations on a set of objects $V(G)$. The elements of $V(G)$ are called the *vertices (or nodes)* of the graph $G$, and any pair $x, y$ of vertices are joined by an *edge* $(x, y)$ iff $\{x, y\} \in E(G)$. All our graphs are assumed to be *finite* (with $|V(G)| < \infty$) and, by definition, *simple* (without any self-loops or multi-edges).

By a slight abuse of notation, we use $(x, y) \in E(G)$ to mean $\{x, y\} \in E(G)$. Then, two vertices $x, y \in V(G)$ are said to be *adjacent* if $(x, y) \in E(G)$, and *non-adjacent* otherwise. The vertices $x, y$ are called the *ends* of the edge $(x, y)$, and conversely, the edge $(x, y)$ is said to be *incident* to both vertices $x$ and $y$.

**Notation 1.1.** *The set of all finite, simple graphs is denoted by $\mathcal{G}$. For any graph $G \in \mathcal{G}$, the* size *of the graph, defined as the number of vertices $|V(G)|$ in $G$, is denoted by $|G|$.*

*For $A, B \subset V(G)$, $E[A, B]$ denotes the set of edges in $E(G)$ with one end in $A$ and the other end in $B$. Similarly, $E[A]$ represents the set of edges with both ends in $A \subset V(G)$.*

**Definition 1.2** (Neighbours)**.** *Given a vertex $v \in V(G)$, the vertices adjacent to $x$ are called the* neighbours *of $v$. The set of neighbours of $v$, also called the* open neighbourhood *of $v$, is denoted by $N(v)$. The* closed neighbourhood *of $v$ is defined as $N[v] \coloneqq N(v) \cup \{v\}$. The set of* non-neighbours *of $v$ is given by the set $V(G) \backslash N[v]$.*

*Extending this notion to $S \subset V(G)$, We define $N(S)$ as the union of neighbours of $v \in S$, and similarly, the closed neighbourhood $N[S] \coloneqq N(S) \cup S$.*

**Definition 1.3** (Degree of a Vertex)**.** *The* degree *of a vertex $v \in V(G)$ is defined as $d(v) \coloneqq |N(v)|$. Vertices with degree zero are said to be* isolated vertices*, and vertices with degree one are called* leaf nodes.

**Definition 1.4** (Max/Min Degree of a Graph)**.** *Given a graph $G \in \mathcal{G}$, the* maximum degree *$\Delta(G)$ and the* minimum degree *$\delta(G)$ of the graph are defined as:*

$$\Delta(G) = \max_{v \in V(G)} d(v), \quad \delta(G) = \min_{v \in V(G)} d(v)$$

Some basic examples of graphs are as follows: The *complete graph* $K_n$ is a graph on $n$ vertices such that every pair of vertices are adjacent. Similarly, the *null graph* $I_n$ is a graph on $n$ vertices where every pair of vertices are non-adjacent. We denote by $C_k$ and $P_k$, the *cycle graph* and the *path graph* on $k$ vertices, respectively. Note that:

$$|E(K_n)| = \binom{n}{2}, \quad |E(I_n)| = 0, \quad |E(C_n)| = n, \text{ and } |E(P_n)| = n - 1$$

**Definition 1.5** (Paths in a Graph)**.** *A* path *$P$ in a graph $G$ is a sequence $v_1 e_1 v_2 e_2 \ldots e_{k-1} v_k$ of non-repeating vertices and edges, such that $e_i = (v_i, v_{i+1})$. We say that $P$ is a path joining $x_1$ to $x_k$, of* size *$k$ and* length *$k - 1$.*

**Definition 1.6** (Connected Graph)**.** *A graph $G$ is said to be* connected*, if for every pair of vertices $x, y \in V(G)$, there is a path joining $x$ to $y$. Otherwise, $G$ is said to be* disconnected.

**Definition 1.7** (Complete/Anti-complete)**.** *Given two disjoint subsets $A, B \subset V(G)$ of vertices of a graph $G$, we say $A$ is* complete (anti-complete) *to $B$ if every $a \in A$ is adjacent (non-adjacent) to every $b \in B$.*

## Graph Isomorphisms

Given two graphs $G, H$, a function $f : V(G) \to V(H)$ is said to *preserve adjacencies* if, for all pair of vertices $x, y \in V(G)$, we have $(f(x), f(y)) \in E(H) \iff (x, y) \in E(G)$.

**Definition 1.8** (Isomorphism, Automorphism). *A function $F : V(G) \to V(H)$ is an* isomorphism from $G$ to $H$ *if it is bijective and preserves adjacencies. An isomorphism from a graph $G$ to itself is called an* automorphism *on $G$.*

Two graphs $G$, $H$ are defined to be *isomorphic*, denoted by $G \simeq H$, if there exists an isomorphism from $G$ to $H$. Note that isomorphic graphs have essentially the same structure and differ only in terms of vertex labelling. Thus, we predominantly look at graphs *up to isomorphism*, by considering all isomorphic graphs to be indistinguishable, and thus equal. Conceptually, a set of isomorphic graphs gets replaced by a representative graph, called the *abstract graph* - which preserves the adjacency structure present in the isomorphic graphs.

## Subgraphs and Induced Subgraphs

**Definition 1.9** (Subgraphs). *Given a graph $G(V, E)$, we say that the graph $H(V', E')$ is a* subgraph *of $G$ if $V' \subset V$ and $E' \subset E$. The subgraph relation is denoted by $H \subset G$.*

**Definition 1.10** (Induced Subgraphs). *Given a graph $G(V, E)$, we say that the graph $H(V', E')$ is an* induced subgraph *of $G$ if $V' \subset V$ and $E' = E[V']$. Thus, an induced subgraph can be uniquely determined by the choice of $V' \subset V$.*

*We say that $V'$* induces *the subgraph $H$ in $G$, and we denote it as $H = G[V']$. The induced subgraph relation is usually denoted by $H \leq G$.*

Note that the induced subgraph relation is transitive: If $H_1 \leq H_2$ and $H_2 \leq H_3$, then $H_1 \leq H_3$. Some special induced subgraphs include the following: An *independent set (or stable Set)* $S \subset V(G)$ is a set of vertices which induces the null graph $I_{|S|}$ in $G$. Similarly, A *clique (or complete set)* $S \subset V(G)$ is a set of vertices which induces the complete graph $K_{|S|}$ in $G$. Finally, a maximal connected induced subgraph of $G$ is called a *component (or connected component)* of $G$. A connected graph has only one component: the graph itself; while disconnected graphs have no less than two connected components.

**Definition 1.11** ($H$-free Graphs)**.** *Given a graph $H \in \mathcal{G}$, a graph $G$ is said to be $H$-free if it does not contain any induced copies of $H$. Formally, $\forall S \subset V(G)$, we have $H \not\cong G[S]$. If $\mathcal{H} \subset \mathcal{G}$ is a set of graphs, then a graph $G$ is said to be $\mathcal{H}$-free if $G$ is $H$-free for all $H \in \mathcal{H}$.*

Note that by the definition above and by transitivity of induced subgraphs, The class of $\mathcal{H}$-free graphs is closed under induced subgraphs. Conversely too, any subset of graphs $\mathcal{A} \subset \mathcal{G}$ has a forbidden induced subgraph characterisation if $\mathcal{A}$ is closed under induced subgraphs. The latter follows from a simple constructive proof, similar to the proof of the existence of a basis for vector spaces. In essence, the forbidden graphs $H_1, H_2, \ldots$ are picked from $\mathcal{G} \backslash \mathcal{A}$, such that $H_i \not\leq H_j$, $\forall i \neq j$, until exhaustion.

Further, If $H_1 \leq H_2$, then the transitivity of induced subgraphs implies that $H_1 \leq G$ if $H_2 \leq G$. By contraposition, $G$ being $H_1$-free implies that $G$ is $H_2$-free whenever $H_1 \leq H_2$.

## Graph Invariants

Any function with domain $\mathcal{G}$ is said to be a *graph invariant* if isomorphic graphs are mapped to equal elements in the co-domain. For example, $|G|$, $|E(G)|$, $\delta(G)$, and $\Delta(G)$ are all graph invariants, while a map ordering the vertices of a graph is not invariant. The value of a graph invariant can be obtained from the abstract graph and does not depend on the labelling of vertices. The following are some commonly used graph invariants, for a graph $G$:

- $\alpha(G)$ (*Independence Number*): The size of the largest independent set in $G$.
- $\omega(G)$ (*Clique Number*): The size of the largest clique in $G$.
- $hom(G) \coloneqq \max\{\alpha(G), \omega(G)\}$: The size of the largest homogeneous set in $G$.
- $\chi(G)$ (*Chromatic Number*): The minimum number of colours required to colour elements of $V(G)$, such that no vertices of the same colour are adjacent.

Note that, If $G$ is given a vertex colouring as above, then vertices of the same colour form an independent set in $G$. Thus, we have: $\chi(G)\alpha(G) \geq |G|$. It is also known that $\chi(G) \leq \Delta(G) + 1$. Combining both these relations, we get the following useful bound on $\alpha(G)$:

$$\alpha(G) \;\geq\; \frac{|G|}{1 + \Delta(G)}$$

## Graph Operations

A graph operation acts on finitely many graphs to return a new graph. It is interesting to study how graph invariants behave with graph operations. The following are some examples:

**Definition 1.12** (Vertex Deletion Operation)**.** *Given a graph $G$ and a subset $S \subset V(G)$, the vertex deletion operation returns the graph $G \backslash S \coloneqq G[V(G)\backslash S]$. If $v \in V(G)$, we denote by $G\backslash v$, the graph $G\backslash\{v\}$.*

**Definition 1.13** (Union Operation)**.** *The* union *(or disjoint union) of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ with $V_1 \cap V_2 = \phi$, returns the graph $G(V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The union operation is denoted by $G = G_1 \sqcup G_2$.*

**Definition 1.14** (Join Operation)**.** *The* join *of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ with $V_1 \cap V_2 = \phi$, returns the graph $G(V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup (V_1 \times V_2)$. The join operation is denoted by $G = G_1 + G_2$.*

Stated otherwise, the union operation returns disjoint copies of the input graphs. On the other hand, The join of two graphs $G_1, G_2$ is formed by taking $G_1 \sqcup G_2$ and joining every vertex of $V_1$ to every vertex of $V_2$ via an edge. For example, $I_2 \sqcup I_2 = I_4$ and $I_2 + I_2 = C_4$. The following properties hold true:

$$\alpha(G_1 + G_2) = \max\{\alpha(G_1), \alpha(G_2)\}, \quad \alpha(G_1 \sqcup G_2) = \alpha(G_1) + \alpha(G_2)$$

$$\omega(G_1 \sqcup G_2) = \max\{\omega(G_1), \omega(G_2)\}, \quad \omega(G_1 + G_2) = \omega(G_1) + \omega(G_2)$$

**Notation 1.2.** *Given a graph $G$, the graph $nG$ denotes $G \sqcup G \sqcup \ldots \sqcup G$, taken $n$ times. These operations can* sometimes *be useful to describe graphs. For example, the graph represented by a triangle with a leaf node can be described concisely as $G = \overline{P_3 \sqcup K_1}$*

## Complement of a Graph

Given a graph $G$, the complement of $G$, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = [V(G)]^2 \backslash E(G)$. In other words, $(x, y)$ is an edge in $G$ if and only if $(x, y)$ is not an edge in $\overline{G}$. The following are some easily observable properties of complements:

**Lemma 1.1.** *Given a graph $G$ and its complement $\overline{G}$, the following properties hold:*

- (Self-inverse operation) $\overline{(\ \overline{G}\ )} = G$.
- (Relation with cliques and independent sets) *A clique in $G$ is an independent set in $\overline{G}$, and vice-versa. Thus, $\alpha(G) = \omega(\overline{G})$ and $\omega(G) = \alpha(\overline{G})$.*
- (Invariance under $hom(G)$) *By the above relations, $hom(G) = hom(\overline{G})$.*
- (Relation with joins and unions) $\overline{G_1 + G_2} = \overline{G_1} \sqcup \overline{G_2}$ *and* $\overline{G_1 \sqcup G_2} = \overline{G_1} + \overline{G_2}$.
- (Relation with $H$-freeness) *$G$ is $H$-free iff $\overline{G}$ is $\overline{H}$-free.*
- (Relation with connectedness) *The complement of a disconnected graph is connected.*

**Definition 1.15** (Self-Complementary Graphs)**.** *A graph $G$ is said to be* self-complementary *if $\overline{G} \simeq G$. For instance, $K_1$, $P_4$, and $C_5$ are examples of self-complementary graphs.*

Note that for a self-complementary graph $G$, the isomorphism from $G$ to $\overline{G}$ acts as an automorphism on $G$. Further, if $|G| = n$, then $|E(G)| = \frac{1}{2}\binom{n}{2}$, and so, $|G| \equiv 0$ or $1$ mod 4. Finally, the class of $H$-free graphs is closed under complements if and only if $H$ is self-complementary.

## Perfect Graphs and their Properties

A graph $G$ is said to be *perfect* if, for every induced subgraph $H \leq G$, we have $\chi(H) = \omega(H)$. Note that perfect graphs are closed under induced subgraphs and hence have a forbidden induced subgraph characterisation. Perfect graphs enjoy the following properties:

- (*The Weak Perfect Graph Theorem* [28]) A graph $G$ is perfect iff $\overline{G}$ is perfect.
- (*The Strong Perfect Graph Theorem* [7]) A graph $G$ is perfect if and only if it contains no odd holes or odd anti-holes with more than five vertices. An induced copy of $C_k$ ($\overline{C_k}$) for $k \geq 4$ is called a *hole (anti-hole)* in $G$.
- (*Relation with Clique and Independent Sets*) If $G$ is a perfect graph, then $\alpha(G)\omega(G) = \alpha(G)\chi(G) \geq |G|$, and thus $G$ has either a clique or an independent set of size $\sqrt{|G|}$.

# Chapter 2

# The Erdős-Hajnal Conjecture

In this chapter, we introduce the Erdős-Hajnal conjecture, which claims the existence of polynomial-sized homogeneous sets in the class of $H$-free graphs. We look at some simple examples and relate the conjecture to the substitution operation. Finally, we study the properties of substitution prime graphs, and establish a relation between the asymptotic and non-asymptotic versions of the conjecture for such graphs.

## 2.1 The Erdős-Hajnal Conjecture

In 1989, Paul Erdős and András Hajnal showed that restricting to the class of $H$-free graphs, for any graph $H$ could help improve lower bounds on the size of homogeneous sets. Particularly, in their paper "Ramsey-type Theorems" [16], they proved the following theorem:

**Theorem 2.1.1.** *For constants $d$ and $c < \frac{1}{2d}$, there exists a natural number $N = N(c, d)$ such that for every graph $G$ with $n > N$ vertices and for $k \leq e^{c\sqrt{\log(n)}}$, either $G$ has a homogeneous set of size $k$ or contains all graphs on $d$ vertices as induced subgraphs.*

Stated otherwise, the above theorem proves that for every graph $H$, there is a constant $c(H)$ such that all large $H$-free graphs have a homogeneous set of size at least $e^{c(H)\sqrt{\log(|G|)}}$. While this was an improvement to the $\mathcal{O}(\log(|G|)$ bound, Erdős and Hajnal postulated a further surprising improvement in the same 1989 paper. They conjectured the existence of polynomial-sized homogeneous sets over the class of $H$-free graphs as follows:

**Conjecture 2.1.2.** *For every graph $H$, there exist constants $c(H) > 0$ and $N(H) \in \mathbb{N}$ such that every $H$-free graph $G$ with no less than $N$ vertices has a homogeneous set of size $|G|^{c(H)}$*

This is commonly called *The Erdős-Hajnal Conjecture*. In order to prove the conjecture, it suffices to consider each graph $H$ separately, and find the constants as required in the statement above. For ease of discussion, we define $\gamma(H)$ to be the supremum of all constants $c(H)$, which satisfy the definition above. More precisely:

**Definition 2.1.1.** *Given a graph $H$, define the set $\Gamma_H \subset [0, 1]$ as:*

$$\Gamma_H := \{c \in [0, 1] \mid \exists n_0 \in \mathbb{N} \text{ s.t. } hom(G) \geq |G|^c \text{ for all } H\text{-free graphs } G \text{ with } |G| \geq n_0\}$$

*Let $\gamma(H) := \sup \Gamma_H$ be the largest constant in $\Gamma_H$.*

Note that $0 \in \Gamma_H$ for every graph $H$ as any vertex in a graph is a homogeneous set of size 1. It follows that $\gamma(H)$ exists for all graphs $H$, as the set $\Gamma_H$ is non-empty and bounded above by 1. We wish to remark that it is not very clear whether $\gamma(H) \in \Gamma_H$ as well. However, if $\gamma(H) > 0$, we can use it interchangeably with $\gamma(H) - \varepsilon$ for some small enough $\varepsilon > 0$. Further, if $H$ is a substitution prime graph (see Section 2.3), then it can be shown that $\gamma(H) \in \Gamma_H$, as we shall do in Section 2.4.

We shall say that a graph $H$ has the *Erdős-Hajnal property* (EH-property) if $\gamma(H) > 0$. Consequently, we have an equivalent version of the Conjecture:

**Conjecture 2.1.3.** *Every graph has the Erdős-Hajnal Property*

We begin by making the following observation:

**Lemma 2.1.4.** *Suppose the graph $H$ has the EH-property, then*

- *The complement graph $\overline{H}$ has the EH-property with $\gamma(\overline{H}) = \gamma(H)$.*
- *All induced subgraphs $H' \leq H$ have the EH-Property with $\gamma(H') \geq \gamma(H)$.*

The first observation follows directly by noting that $hom(G) = hom(\overline{G})$ and that any graph $G$ is $H$-free if and only if $\overline{G}$ is $\overline{H}$-free. For the latter, note that if $H' \leq H$ is an induced subgraph of $H$, then any $H'$-free graph is $H$-free as well. Hence, $\Gamma_H \subset \Gamma_{H'}$, and thus $\gamma(H) \leq \gamma(H')$. It follows that $H'$ has the EH-property too.

Let us consider some simple examples:

**Example 1: Graphs with at most two vertices**

Let us begin with some trivial cases. There are three graphs with at most two vertices, namely $K_1$ (a vertex), $K_2$ (an edge), and $I_2$ (two independent vertices). $K_1$ has the EH-property vacuously with $\gamma(K_1) = 1$, as no graph is $K_1$-free. All $K_2$-free graphs have no edges and thus are independent sets themselves. Thus, $\gamma(K_2) = 1$. Likewise, $I_2$-free graphs are complete graphs, and so $\gamma(I_2) = 1$ too. Thus all graphs on at most two vertices have the EH-property.

**Example 2: Complete and Null Graphs**

**Theorem 2.1.5.** *Every large $K_r$-free graph $G$ has an independent set of size at least $|G|^{1/(r-1)}$.*

*Proof.* We prove this by induction. For the base case, note that every $K_2$-free graph has $\alpha(G) = |G|$. Now, fix some $r > 2$, and suppose the theorem is true for $K_{r-1}$-free graphs. Let $G$ be a $K_r$-free graph. If $G$ has a vertex $v$ of degree $d(v) \geq |G|^{(r-2)/(r-1)}$, then the neighbours $N(v)$ form a $K_{r-1}$-free graph. By the induction hypothesis, $N(v)$, and hence $G$, has an independent set of size at least $|N(v)|^{1/(r-2)} \geq |G|^{1/(r-1)}$, as required. If no such vertex exists, then $G$ has maximum degree $\Delta(G) < |G|^{(r-2)/(r-1)}$. Let $|G|$ be large enough so that $|G|^{(r-2)/(r-1)} \approx 1 + |G|^{(r-2)/(r-1)}$, then, the theorem follows by the following independence number bound for $G$:

$$\alpha(G) \geq \frac{|G|}{\Delta(G)+1} \geq \frac{|G|}{|G|^{(r-2)/(r-1)}+1} \approx \frac{|G|}{|G|^{(r-2)/(r-1)}} = |G|^{1/(r-1)}$$

$\square$

Thus, by the above theorem and Lemma 2.1.4, all complete graphs and null graphs have the EH-property with $\gamma(K_r) = \gamma(I_r) \geq \dfrac{1}{r-1}$.

**Example 3: Graphs with three vertices**

There are precisely four graphs on three vertices up to isomorphism, namely $K_3$ (The triangle), $I_3$ (Independent set on three vertices), $P_3$ (The Path on three vertices), and its complement $\overline{P_3}$. Consider first, the case of $P_3$. Recall that a *cluster graph* is defined as a disjoint union of cliques. The class of cluster graphs admits a forbidden induced subgraph characterisation as follows:

**Theorem 2.1.6.** *A graph $G$ is a cluster graph if and only if it is $P_3$-free.*

*Proof.* It is easy to see that cluster graphs contain no induced $P_3$ as any two vertices are adjacent in a clique and $P_3$ is connected. For the reverse direction, let $G$ be a $P_3$-free graph, and $C$ be any component of $G$ with at least three vertices ($G$ is a cluster graph if such a $C$ does not exist). If $C$ has two non-adjacent vertices, $x, y$, then by the connectedness of $C$, there exists a path $P = (x = x_0), x_1, x_2, \dots x_k, y$ in $C$ connecting $x$ and $y$. As $(x_0, y) \notin E(G)$ and $(x_k, y) \in E(G)$, there exists an $i \leq k$, for which $(x_{i-1}, y) \notin E(G)$ and $(x_i, y) \in E(G)$. Then, $S = \{x_{i-1}, x_i, y\}$ induces a $P_3$ in $G$. By contradiction, each connected component $C$ is a complete graph, and so $G$ is a cluster graph. □

If $G$ is a cluster graph, then each component has size at most $\omega(G)$. Further, $G$ has exactly $\alpha(G)$ many components as any independent set of $G$ contains at most one vertex from each clique. Thus, we have that $|G| \leq \alpha(G)\omega(G)$, and so $G$ has a homogeneous set of size at least $\sqrt{|G|}$. It follows that $P_3$ has the EH-property with $\gamma(P_3) \geq 1/2$. To see the tightness of this bound, note that every graph $G$ from the set $S = \{nK_n \ : \ n \in \mathbb{N}\}$ has $hom(G) = n$ and $|G| = n^2$ for some $n \in \mathbb{N}$. Hence, $hom(G) = \sqrt{|G|}$ for all graphs in the infinite family $S$, thereby showing that $\gamma(P_3) = 1/2$.

The EH-property for $K_3$ follows from Theorem 2.1.5, with $\gamma(K_3) \geq 1/2$. In 1995, J.H. Kim [23] proved that for every large enough $n \in \mathbb{N}$, there exists a $K_3$-free graph with $\alpha(G) = hom(G) \leq 9\sqrt{|G| \log |G|}$. From some real analysis, we see that for every $d > 1/2$, there exists $N \in \mathbb{N}$ such that $81 \log(N) = N^{2d-1}$ and $9\sqrt{nlog(n)} \leq n^d$, $\forall n \geq N$. Hence, by the existence of the graphs by Kim, we have $\gamma(K_3) \leq 1/2$, thereby establishing the tightness of the bound.

By Lemma 2.1.4, it follows that both $I_3$ and $\overline{P_3}$ also have the EH-property with $\gamma(I_3) = \gamma(\overline{P_3}) = 1/2$. Hence, every graph on three vertices has the EH-property.

## 2.2 The Substitution Operation

We saw that all graphs with at most three vertices, complete graphs, and null graphs are some examples of graphs which have the EH-property. However, verifying the truthfulness of the property for each graph $H$ can indeed be a daunting task. In such a situation, it would be helpful to find an operation that preserves the EH-property, for it helps in reducing the family of graphs for which the property must be verified. We shall explore such operations in this section.

Note that an operation on a set of graphs is said to *preserve a graph property* if the satisfaction of the property on the input graphs implies that the output graphs satisfy the property too. For instance, Lemma 2.1.4 shows that the operations of taking graph complements and induced subgraphs preserve the EH-property.

In their 1989 paper [16], Erdős and Hajnal showed that the join and union operations preserve the EH-property. More precisely, the EH-property for graphs $H_1$ and $H_2$ implies that the graphs $H_1 + H_2$ and $H_1 \sqcup H_2$ have the EH-property too. The substitution operation, also known as the graph replacement operation, generalises this notion of joins and unions.

**Definition 2.2.1.** *Given two graphs $H_1$, $H_2$ and a vertex $v \in V(H_1)$, the substitution operation returns a graph $H$, denoted by $S(H_1, v, H_2)$, which satisfies the following properties:*

1. $V(H) = V(H_2) \cup V(H_1) \setminus \{v\}$.
2. $H[V(H_2)] \simeq H_2$
3. $H[V(H_1)\setminus\{v\}] \simeq H_1\setminus v$
4. *For all vertices $w \in V(H_2) \subset V(H)$ and $u \in V(H_1)\setminus\{v\}$, the graph $H$ has $(u, w)$ as an edge if and only if $u \in N_{H_1}(v)$.*

Roughly speaking, the substitution operation embeds a graph into some vertex $v$ of another graph, while preserving the adjacencies of $v$. See Figure 2.1 for an example. Further, properties 3 and 4 in the definition above imply that the induced subgraph $H[V(H_1) \cup \{w\}\setminus\{v\}]$ is isomorphic to the graph $H_1$ for any vertex $w \in V(H_2) \subset V(H)$. As a result, at least $|H_2|$ copies of the graph $H_1$ are induced in $H = S(H_1, v, H_2)$. However, note that the statement $H[V(H_1) \cup \{w\}\setminus\{v\}] \simeq H_1 \ \forall w \in (H_2) \subset V(H_2)$ is not equivalent to properties 3.

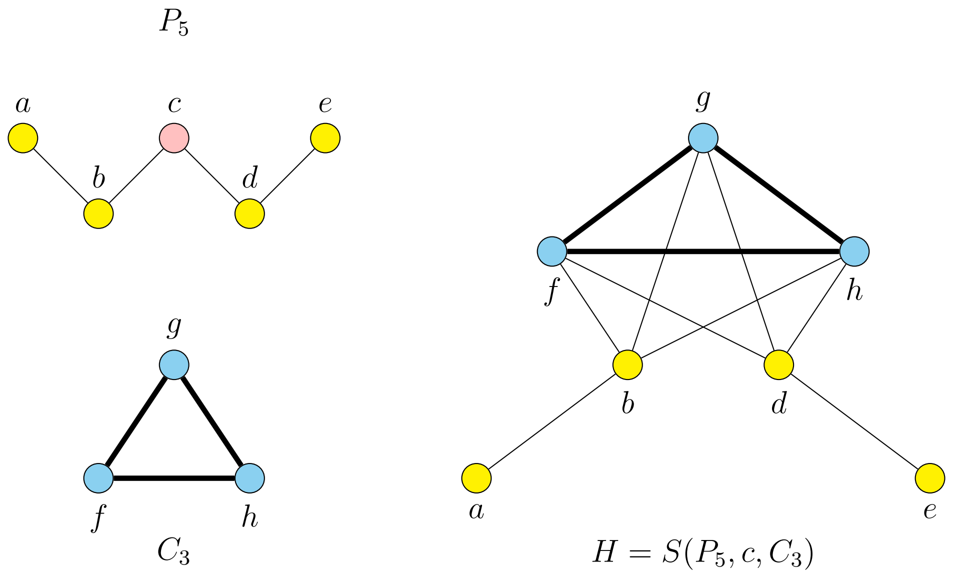


Figure 2.1: An example of the substitution operation

and 4. above, and hence cannot be replaced in the definition of the substitution operation. For instance, the graph $H = \overline{P_5}$ satisfies the former statement with $H_1 = K_1 + \overline{P_3}$, $H_2 = K_2$ and $v$ being the leaf node in $H_1$; but $H$ fails to satisfy property 4 in the definition above (Note that $H$ is substitution a prime graph too!).

Moreover, both the join and union operations can be performed via a sequence of two substitutions. For instance, the join of two graphs is equivalent to substituting both these graphs on the endpoints of an edge. More precisely, if $K_2$ is the complete graph on two vertices, with vertex set $\{x, y\}$, and graphs $H_1$, $H_2$ are given, then their join can be obtained by: 1. $S_1 \coloneqq S(K_2, x, H_1)$ and 2. $H_1 + H_2 = S(S_1, y, H_2)$. Similarly, the union graph $H_1 \cup H_2$ can be obtained by using $I_2$ instead of $K_2$ above. Thus, the union and join operations are indeed generalised by the notion of substitutions. However, the converse is not true. Consider, for example, the graph $H$ in Figure 2.1. If $H$ can be obtained by a sequence of union and join operations, then, it would contain a subset $S$ of vertices, such that $S$ is either complete or anti-complete to $\overline{S}$. One can check that no such subset exists in the graph $H$.

In 2001, Noga Alon, Jànos Pach, and József Solymosi [1] extended the results of Erdős and Hajnal, by proving that the substitution operation preserves the EH-property. Their proof, similar to the 1989 paper [16], makes use of counting arguments to provide a lower bound for $\gamma(S(H_1, v, H_2))$ in terms of $\gamma(H_1)$ and $\gamma(H_2)$. We present their proof here:

**Theorem 2.2.1.** *[1] Let $H_1$ and $H_2$ be two graphs having the EH-property, and let $v \in V(H_1)$ be any vertex of $H_1$, then the graph $H = S(H_1, v, H_2)$ has the EH-property with*

$$\gamma(H) \;\geq\; \frac{\gamma(H_1)\;\gamma(H_2)}{\gamma(H_1) + |H_1|\gamma(H_2)}$$

*Proof.* Suppose graphs $H_1, H_2$, and $H = S(H_1, v, H_2)$ are given as above. Let $H_0$ be the graph isomorphic to $H \backslash \{v\}$ and let $\mu$ be the right-hand side of the inequality above. Fix a large enough natural $N_0$, such that $N_0^{\mu/\gamma(H_1)} > |H_1|$ and let $G$ be any $H$-free graph on $n \geq N_0$ vertices. We shall show that $G$ has a homogeneous set of size at least $|G|^{\mu}$.

Suppose, for contradiction, that $hom(G) < n^{\mu}$. Let $m = \lceil n^{\mu/\gamma(H_1)} \rceil$ ($m > |H_1|$) and let $U$ be a subset of vertices of $G$ of size $m$. Then, $G[U]$ contains an induced copy of $H_1$: Else, $G[U]$ would be $H_1$-free and, by the EH-property for $H_1$, would contain a homogeneous set of size at least $|U|^{\gamma(H_1)} \geq n^{\mu}$, which is not possible. Thus every induced subgraph of $G$ of size $m$ induces a copy of $H_1$. Further, every such induced copy of $H_1$ lies in exactly $\binom{n-|H_1|}{m-|H_1|}$ induced subgraphs of $G$ of size $m$. Thus, $G$ induces at least $\binom{n}{m}/\binom{n-|H_1|}{m-|H_1|}$ copies of $H_1$.

By using a simple counting argument, one can show that $G$ has at most $n(n-1)\ldots(n-|H_1|+2)$ induced copies of $H_0$. Further, every induced copy of $H_1$ also induces a copy of $H_0$ in $G$. By the pigeon-hole principle, there is some copy of $H_0$ in $G$, which is contained in at least $M$ copies of $H_1$, where for large enough graph $G$,

$$M = \frac{\binom{n}{m}/\binom{n-|H_1|}{m-|H_1|}}{n(n-1)\ldots(n-|H_1|+2)} \;=\; \frac{n-|H_1|+1}{m(m-1)\ldots(m-|H_1|+1)} \;\geq\; \frac{n}{m^{|H_1|}}$$

Here, the last inequality follows by replacing each term of the denominator with $m$, and by assuming $n$ to be large enough so that $n - |H_1| + 1 \approx n$. The above result states that $G$ has a subset $X \subset V(G)$ of size $|H_1| - 1$ and a set $Y \subset V(G)$ of size at least $M$, such that $G[X] \simeq H_0$ and $G[X \cup \{w\}] \simeq H_1$, $\forall w \in Y$. Note that $G[Y]$ must be $H_2$-free, for otherwise, $G$ will contain an induced copy of $H$. Thus, by the EH-property for $H_2$, we have $hom(G[Y]) \geq |Y|^{\gamma(H_2)} \geq M^{\gamma(H_2)}$. As $hom(G) \geq hom(G[Y])$ and $hom(G) < n^{\mu}$, we obtain the following:

$$n^{\mu} > M^{\gamma(H_2)} \;\geq\; \left(\frac{n}{m^{|H_1|}}\right)^{\gamma(H_2)} \;\geq\; n^{\gamma(H_2)-\mu|H_1|\frac{\gamma(H_2)}{\gamma(H_1)}}$$

Comparing the exponents of $n$, we have:

$$\mu > \gamma(H_2) - \mu|H_1|\frac{\gamma(H_2)}{\gamma(H_1)} \quad \iff \quad \mu > \frac{\gamma(H_1)\,\gamma(H_2)}{\gamma(H_1) + |H_1|\gamma(H_2)} = \mu$$

Thereby, leading to a contradiction. Thus, every large enough $H$-free graph $G$ has a homogeneous set of size at least $|G|^\mu$, and so $\mu \in \Gamma_H$. The theorem now follows by noting that $\gamma(H) = \sup \Gamma_H$. $\square$

## 2.3 Substitution Prime Graphs

From the previous section, it is evident that in order to prove the Erdős-Hajnal Conjecture, it suffices to prove the property for just those graphs which can't be obtained from smaller graphs via the substitution operation. Such graphs are termed *substitution prime graphs*. More formally,

**Definition 2.3.1.** *A graph $H$ with $|H| \geq 3$ is said to be substitution prime, if $H \not\simeq S(H_1, v, H_2)$ for any pair of graphs $H_1, H_2$ with $2 \leq |H_1|, |H_2| < |H|$.*

We shall occasionally drop the term 'substitution' while describing substitution prime graphs if it calls for no confusion. The restriction $|H| \geq 3$ has been added to set aside the few degenerate cases in favour of attaining uniformly applicable properties for prime graphs. The structure of a prime graph can be better understood by the well-known concept of modules of a graph:

**Definition 2.3.2.** *Given a graph $G$, a subset $S \subset V(G)$ is called a Module of $G$, if $V(G)\backslash S$ can be partitioned into two sets $K, N$ such that $S$ is complete to $K$ and anti-complete to $N$.*

*A module $S$ is said to be* trivial *if $|S| \leq 1$ or $|S| = |G|$.*

It is easy to see that all singleton sets $\{v\} \in V(G)$ are modules with $K = N(v)$ and $N = N[v]^c$. Further, both $\phi$ and $V(G)$ are modules trivially. As stated above, all these sets are trivial modules for any graph $G$. On the other hand, *twin pairs* in a graph $H$ are precisely the non-trivial modules of $H$ of size two. Recall that two vertices $x, y \in V(H)$ are said to be twin primes if $N(x)\backslash\{y\} = N(y)\backslash\{x\}$. Then, $S = \{x, y\}$ is a module of $H$ with $K = N(x)\backslash\{y\}$ and $N = N[x]^c\backslash\{y\}$.

Further examples of non-trivial modules can be found in non-prime graphs. Consider any non-prime graph $H = S(H_1, v, H_2)$, where $H_1, H_2$ are chosen such that $2 \leq |H_1|, |H_2| < |H|$, by the virtue of Definition 2.3.1. Then, $V(H_2)$ is a non-trivial module of the graph $H$ with $K = N_{H_1}(v)$ and $N = \overline{N_{H_1}[v]}$, where the non-triviality follows from the bounds on $|H_1|$ and $|H_2|$. Conversely, any graph $H$ with a non-trivial module $S$ can be obtained by the substitution operation $H = S(H_1, v, H_2)$ from strictly smaller graphs $H_1 = H[(V(H)\backslash S) \cup \{v\}]$ and $H_2 = H[S]$, for some $v \in S \subset V(H)$. Clearly, $H$ is not a prime graph. Based on the discussion above, we get the following relation between prime graphs and modules:

**Theorem 2.3.1.** *A graph $H$ is substitution prime if and only if all modules of $H$ are trivial.*

Theorem 2.3.1 allows us to express the idea of primes graphs in terms of a more workable notion of modules. We shall indeed see that modules become very useful while proving properties of prime graphs, especially for proofs by contradiction. Let us look at a simple class of prime graphs.

**Theorem 2.3.2.** *All paths $P_k,\ k \geq 4$ and cycles $C_k,\ k \geq 5$ are substitution prime.*

*Proof.* Let $H$ be a connected graph with maximum degree $\Delta(H) = 2$. Suppose that $H$ is not prime. Then, $H$ has a non-trivial module $S \subset V(H)$, with sets $K, N$ as defined in Definition 2.3.2. If $K = \phi$, then based on whether $N$ is empty or not, either $|S| = |H|$ or $H$ is disconnected, both of which are not possible by the non-triviality of $S$ and connectedness of $H$. Thus, $K \neq \phi$, and hence $S \subset N(K)$ implies that $|S| \leq \Delta(H) = 2$. By non-triviality of $S$, we conclude that $|S| = 2$, say $\{x, y\}$.

Note that, any vertex of $K$ has all its $\Delta(H) = 2$ neighbours in $S$, and thus $N = \phi$, for otherwise $H$ would be disconnected. Further, as $K \subset N(x)$, we have $|K| \leq 2$. This is sufficient to describe the structure of $H$. If $|K| = 1$, then $H$ is either $P_3$ or $C_3$ based on whether $x, y$ are adjacent or not. Likewise, $H = C_4$ if $|K| = 2$. Thus, if $H$ is a connected non-prime graph with maximum degree two, then $H$ is one of $P_3, C_3$, or $C_4$. The theorem follows by setting $H = P_k$ for $k \geq 4$ or $H = C_k$ for $k \geq 5$. $\square$

Note that all graphs on three vertices are not substitution primes as they are precisely the graphs attained via substitution from any pair of graphs from $\{K_2, I_2\}$ taken with repetition. Thus, $P_4$, the path on four vertices is the smallest prime graph. Unfortunately, not much is known about larger prime graphs.

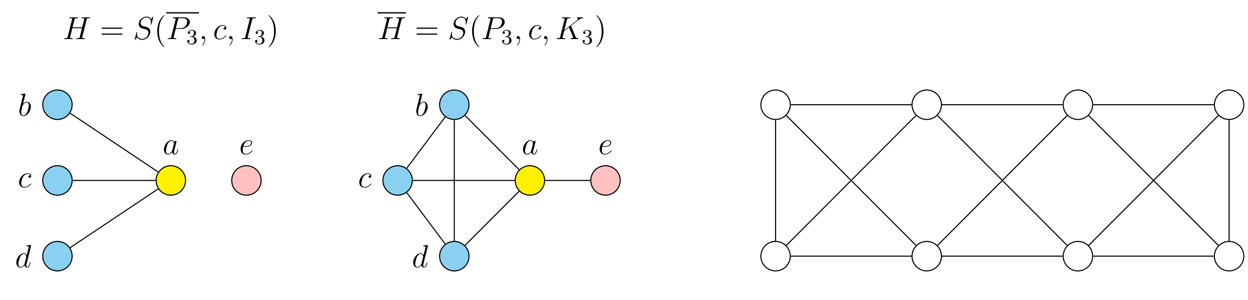


Figure (a) Figure (b)

Figure 2.2: (a) Example for the complement operation distributes over the substitution operation. (b) A self-complementary non-prime graph on eight vertices

Let us note some simple properties of prime graphs:

1. If $H$ is a prime graph, then $H$ is connected. For otherwise, $H$ is a disjoint union of two smaller graphs and hence can be obtained via the substitution operation, as seen in the previous section.

2. The class of substitution prime graphs is closed under complements. This follows directly by observing that any module $S$ of a graph $H$ also forms a module of its complement $\overline{H}$ by switching the roles of $K$ and $N$.

3. Every prime graph contains $P_4$ as an induced subgraph. As we shall see in a later chapter, any $P_4$-free graph can be obtained from $K_1$ via a series of join and union operations, and thus are not substitution prime.

4. A prime graph $H$ has no twin pairs, for twin pairs are non-trivial modules of size two.

Adding to point 2 above, we see that the class of non-prime graphs is closed under complements too. It further can be proved that the complement operation distributes over the substitution operation to give a nice relation between the structure of a non-prime graph $H$ and its complement. More precisely, $S(\overline{H_1}, v, \overline{H_2}) = \overline{S(H_1, v, H_2)}$. See Figure 2.2 (a) for an example.

Combining the properties above, we obtain the following theorem summarising some necessary properties of prime graphs:

**Theorem 2.3.3.** *If $H$ is a prime graph, then the following hold:*

- *Both $H$ and $\overline{H}$ are connected.*
- *$H$ has an induced $P_4$.*
- *$H$ has no twin pairs of vertices.*

$P_4$ is the only prime graph on four vertices; and the bull, $C_5$, $P_5$, and $\overline{P_5}$ are the four prime graphs on five vertices (see Figure 2.3). However, the number of prime graphs on $n$ vertices increases rapidly with $n$. We used the above necessary conditions as a sieve to narrow down our search for prime graphs, and then searched for non-trivial modules by brute force, to find larger prime graphs. We were able to find all the 26 prime graphs on six vertices and all 260 prime graphs on seven vertices (See Figure 2.4 for all primes on six vertices). Presently, the enumeration of prime graphs on $n$ vertices upto isomorphism is only known till $n = 11$ [29], and the sequence is seen to rise very rapidly:

$$0,\ 0,\ 0,\ 1,\ 4,\ 26,\ 260,\ 4670,\ 145870,\ 8110356,\ 804203096$$

Returning to the Erdős-Hajnal Conjecture, we saw that it is sufficient to prove the EH-property for prime graphs. Among prime graphs, the EH-property is presently known to be true only for $P_4$, the bull, and $C_5$. Thus all graphs on at most five vertices, other than $P_5$ and its complement, are known to have the EH-property. We shall look at the proofs of the EH-property for the known graphs and make some progress on $P_5$ in the upcoming chapters. We end this section by noting that even though $P_4$, $C_5$, and the bull are primes, it is not true that each self-complementary graph is prime as well. See Figure 2.2 (b) for an example of a non-prime self-complementary graph. However, we shall return to an important class of substitution prime self-complementary graphs in a later chapter.

## 2.4 Upper Bounds and the Non-Asymptotic Version

We take a brief detour to investigate the non-asymptotic version of Conjecture 2.1.2 and use it to find some trivial upper bounds on $\gamma(H)$ for some graphs $H$. Note that Conjecture 2.1.2 asked for polynomial-sized homogeneous sets in large enough $H$-free graphs, where the number of vertices exceeded a certain threshold. We can forego the latter requirement by

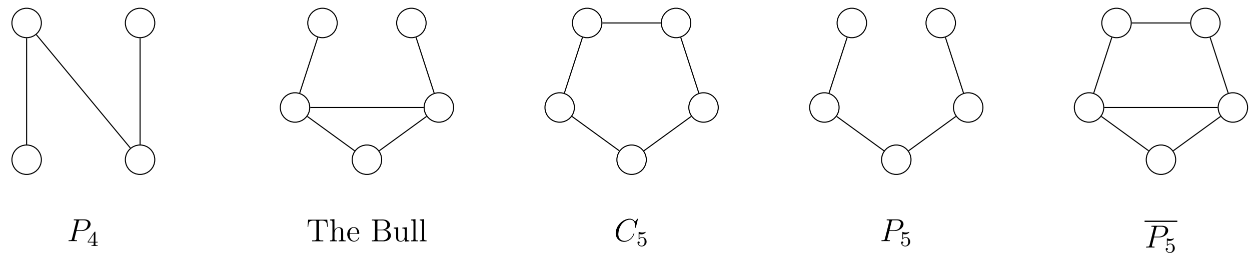


Figure 2.3: Substitution prime graphs on at most five vertices

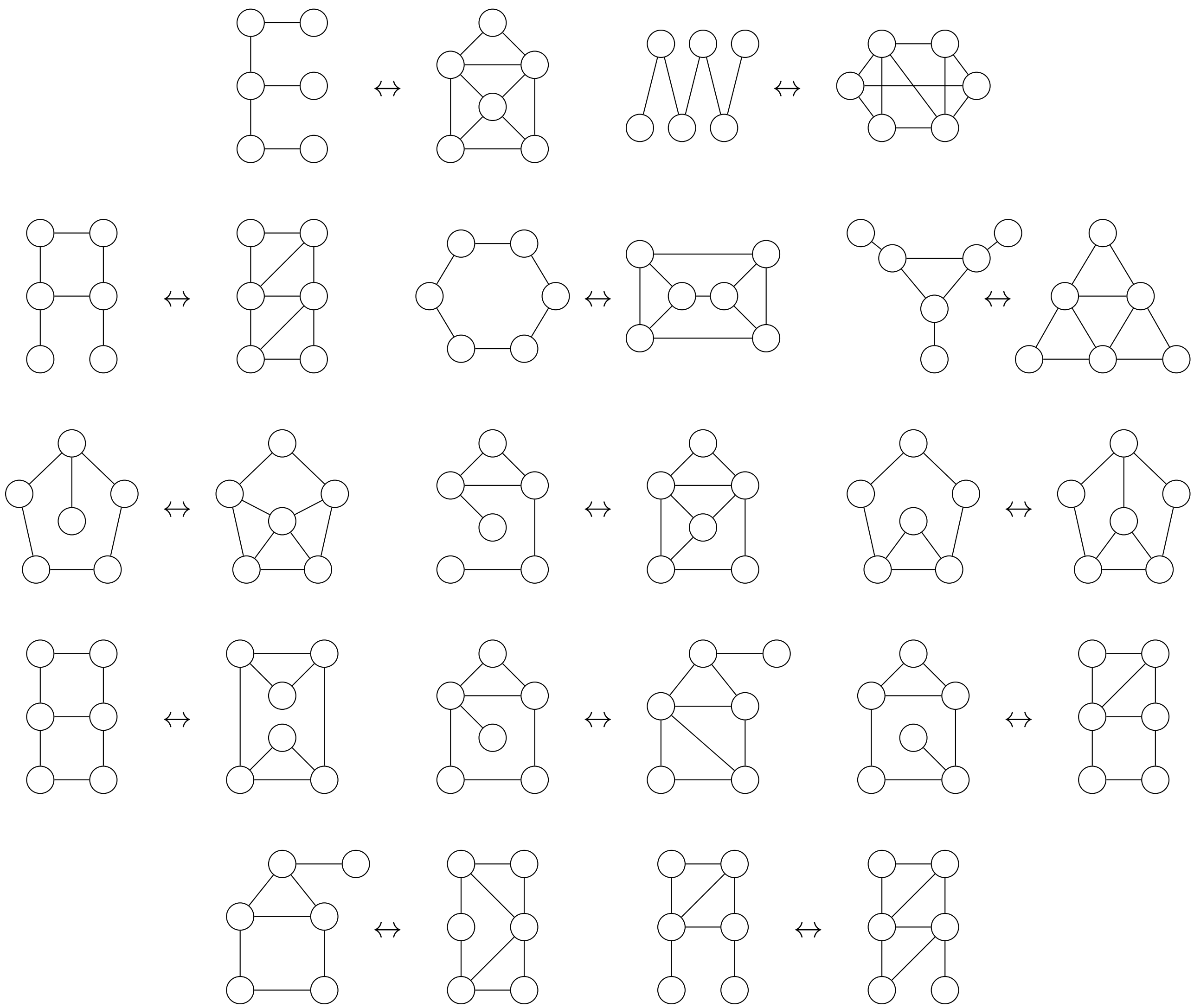


Figure 2.4: The 26 substitution prime graphs on six vertices (paired with complements)

considering all $H$-free graphs instead. This non-asymptotic version of the conjecture can be stated as follows:

**Conjecture 2.4.1.** *For every graph $H$, there exists a constant $c(H) > 0$, such that every $H$-free graph $G$ has a homogeneous set of size $|G|^{c(H)}$*

Similar to our approach in Section 2, we can define the constant $\gamma^*(H)$ for the non-asymptotic version as follows:

**Definition 2.4.1.** *Given a graph $H$, Define the set $\Gamma^*_H \subset [0,1]$ as:*

$$\Gamma^*_H \;:=\; \{c \in [0,1] \mid hom(G) \geq |G|^c \text{ for all } H\text{-free graphs } G\}$$

*Let $\gamma^*(H) := \sup \Gamma^*_H$ be the largest constant in $\Gamma^*_H$.*

Using some analysis, it can be shown that the constant $\gamma^*(H) \in \Gamma^*(H)$. Let us try to understand both constants $\gamma(H)$ and $\gamma^*(H)$ by looking at them from a slightly different perspective, as we shall explain now. For any graph $G$, we define the constant $\delta_G$ as the logarithm of $hom(G)$ taken with the base $|G|$. More precisely,

$$\delta_G \;:=\; \frac{\log(hom(G))}{\log |G|}, \qquad \forall G \in \mathcal{G},\; |G| \geq 2$$

Note that the maximum-sized homogeneous set in any graph $G$ has size $|G|^{\delta_G}$. Keeping this in mind, we can extend the definition of $\delta_G$ to the single vertex graph $K_1$, by defining $\delta_{K_1} = 1$. It can be observed that $\delta_G$ is non-zero for every graph $G$, as all graphs on at least two vertices have $hom(G) \geq 2$, and $\delta_{K_1} = 1$. Further, If $G$ is an $H$-free graph, then every homogeneous set must have a size no more than $|G|^{\delta_G}$, and thus, $\gamma^*(H) \leq \delta_G$ for every $H$-free graph. More specifically, $\gamma^*(H)$ is related to $\delta_G$ by the following formulation:

**Lemma 2.4.2.** *For every graph $H$, we have that $\gamma^*(H) \;=\; \inf\,\{\delta_G : \; G \text{ is } H\text{-free}\}$.*

*Proof.* Let $H$ be a given graph. Define $D_H := \{\delta_G : \; G$ is $H$-free$\}$ and let $\lambda_H \; := \inf D_H$. That $\gamma^*(H) \leq \lambda_H$ follows from the fact that for every $H$-free graph $G$, $\gamma^*(H) \leq \delta_G$, as explained above. For the reverse inequality, assume for contradiction that $\gamma^*(H) < \lambda_H$. The definition of $\gamma^*(H) = \sup \Gamma^*_H$ implies that $\lambda_H \notin \Gamma^*_H$, and hence, there exists an $H$-free graph

$G$, for which $hom(G) < |G|^{\lambda_H}$. Consequently, $\delta_G < \lambda_H$, leading to a contradiction. Thus, $\gamma^*(H) \geq \delta_G$, thereby completing the proof. $\square$

To obtain some trivial upper bounds on $\gamma^*(H)$, we can take inspiration from the theorem above, and define a sequence $\Delta = \{\delta_m\}$, where $\delta_m := \inf_{|G|=m} \delta_G$ for every natural number $m \in \mathbb{N}$. One can observe that the sequence reduces monotonically between two diagonal Ramsey numbers, and might increase at each $m = R(k,k)$. This happens because if $m$ satisfies $R(k,k) \leq m \leq R(k+1,k+1)-1$, then every graph on $m$ vertices has a homogeneous set of size $k$, but there are graphs on $m$ vertices with neither $K_{k+1}$ nor $I_{k+1}$. For instance, The first few values of $\Delta$ are approximately given by $1,\ 1,\ 0.63,\ 0.5,\ 0.43,\ 0.613,\ \ldots$. Here, an increase can be observed at $\delta_6 = 0.613$ which then reduces monotonically till $\delta_{17} = 0.38$, where $R(3,3) = 6$ and $R(4,4) = 18$. Note that Erdős' lower bound on $R(k,k)$ implies that $\Delta \to 0$ as $m \to \infty$.

For any given graph $H$, all graphs with less than $|H|$ vertices are $H$-free. Thus, by Lemma 2.4.2, the inequality $\gamma^*(H) \leq \delta_m\ \forall m < |H|$ gives a simple upper-bound on the value of $\gamma^*(H)$. Further, graphs on $|H|$ vertices, other than $H$, can also be used to provide an additional upper bound. However, as opposed to $\gamma(H)$, upper bounds on $\gamma^*(H)$ need not necessarily be representative of the behaviour of homogeneous sets in $H$-free graphs ad infinitum. For instance, if $\gamma^*(H) \leq x$, then it is possible that there are only finitely many graphs with $\delta_G = x$, and thus the upper bound would provide no information for all large $H$-free graphs above a certain threshold number of vertices.

In essence, finding upper bounds on $\gamma(H)$ is more desirable, but often more difficult, than finding upper bounds on $\gamma^*(H)$. In such a scenario, it would be useful to find conditions when the implication $\gamma^*(H) \leq x \Rightarrow\ \gamma(H) \leq x$ holds true. We shall prove such a condition in the remainder of this section. Let's begin by making some observations on the relation between $\gamma(H)$ and $\gamma^*(H)$. Let $H$ be any given graph along with constants $\gamma(H)$ and $\gamma^*(H)$. By definition, there is a threshold $n_0$, such that $hom(G) \geq |G|^{\gamma(H)}$ whenever $G$ is $H$-free with at least $n_0$ vertices. Then, by Lemma 2.4.2 and its analogue for $\gamma(H)$, we have:

$$\gamma^*(H) \quad = \quad \min\left\{ \min_{|G|<n_0} \ \delta_G \ , \ \gamma(H) \ \right\}$$

It follows from the above relation that:

1. The constant $\gamma^*(H) > 0$ if and only if $\gamma(H) > 0$. This follows by observing that there are finitely many graphs with at most $n_0$ vertices, and as $\delta_G > 0 \ \forall G \in \mathcal{G}$, the first term on the right-hand side above is always non-zero. Equivalently, a graph $H$ has the EH-property if and only if it has the non-asymptotic EH-property too.

2. For all graphs $H \in \mathcal{G}$, we have $\gamma^*(H) \leq \gamma(H)$. This also follows from the fact that the usual EH-property considers only a subset of the graphs covered by the non-asymptotic version, and if $S \subset T$, then $\inf_{x \in S} f(x) \geq \inf_{x \in T} f(x)$.

3. It is not true that $\gamma(H) = \gamma^*(H)$ for every graph $H$. A simple counter-example comes from $H = K_3$. We saw that $\gamma(K_3) = 1/2$. However, $C_5$ is a triangle-free graph with $hom(C_5) = 2$ and $\delta_{C_5} = 0.43$, and hence, $\gamma^*(K_3) \leq 0.43$ does not equal $\gamma(K_3)$.

Thus, the implication $\gamma^*(H) \leq x \implies \gamma(H) \leq x, \ \ \forall x > 0$ holds only for those graphs for which $\gamma(H) = \gamma^*(H)$. We showed that if $H$ is a connected graph with girth $> 4$ and is such that no leaves share a common neighbour, then $\gamma(H) = \gamma^*(H)$. However, taking a hint from [20], we were able to extend the proof to show that $\gamma^*(H) = \gamma(H)$ for every prime graph $H$. Before proving this result, we require the following construction:

**Definition 2.4.2.** *For any graph $G$, define the sequence of graphs $S_G = \{G_k\}_k$ recursively as follows: Set $G_0 = G$. The graph $G_{k+1}$ is obtained by substituting a copy of $G_k$ at every vertex of the graph $G$.*

For example, The sequence $S_{K_r} = \{K_{r^n}\}_n$ and it can be shown that $S_{\overline{G}} = \overline{S_G}$, where the latter denotes the sequence of complements of graphs in $S_G$. Note that the graph $G_{k+1}$ is obtained by substituting $G_k$ at each vertex of $V(G)$. Hence, for simplicity, we shall refer to elements of $V(G)$ as *bags* in $G_{k+1}$ containing a copy of $V(G_k)$. Further, note that by the definition of substitution operation, picking exactly one vertex from each bag induces the graph $G$ in $G_{k+1}$. We look at some properties of $S_G$, beginning with invariance under $H$-freeness, for prime graphs $H$.

**Lemma 2.4.3.** *If $H$ is a substitution prime graph and $G$ is $H$-free, then all graphs in $S_G$ are $H$-free as well.*

*Proof.* Let $H$ be a prime graph, $G$ be $H$-free and consider the sequence of graphs $S_G = \{G_k\}_k$. We prove by induction that $G_k$ is $H$-free for all $k \geq 0$. The base case holds true as $G_0 = G$

is given to be $H$-free. Assume that $G_k$ is $H$-free.

If $G_{k+1}$ contains a subset of vertices $X \subset V(G_{k+1})$ which induces a copy of $H$, then each vertex of $X$ cannot be in separate bags of $G_{k+1}$, for otherwise, it will induce a copy of $H$ in the original graph $G$. Thus, there exists a bag $u_G \in V(G)$ which contains at least two vertices of $X$. Let $X_u$ be the set of these vertices. Note that $X \neq X_u$, for otherwise the graph $G_k$ embedded in $u_G$ induces the graph $H$, contradicting the induction hypothesis. Further, By definition of $G_{k+1}$, any vertex $v \in V(G_{k+1})$ outside the bag $u_G$ is either complete or anti-complete to $X_u$. Hence $X_u$ is a non-trivial module in $X$, making the graph $G_{k+1}[X] \simeq H$ a non-prime graph. Thus, by contradiction, the graph $G_{k+1}$ is $H$-free. □

We can compare some graph invariants of $G$ with those of $G_k \in S_G$. For instance, as $G_k$ is formed by substituting copies of $G_{k-1}$ in every vertex of $G$, the number of vertices in $G_k$ satisfy the relation: $|G_k| = |G| \cdot |G_{k-1}|$. By solving the recurrence, we obtain, $|G_k| = |G|^{k+1}\ \forall k \geq 0$. Next, consider the invariant $\alpha(G_k)$, and let $X \subset V(G_k)$ be the maximum independent set in $G_k$. Then, the bags containing $X$ must form an independent set in $G$, and the vertices of $X$ in each bag must form an independent set in $G_{k-1}$. Thus, we have that $\alpha(G_k) \leq \alpha(G)\alpha(G_{k-1})$, where equality can be attained by selecting $X$ as follows: Pick bags in $V(G)$ corresponding to the maximum independent set in $G$; and from each bag, pick a maximum independent set for $G_{k-1}$, to form an independent set of size $\alpha(G)\alpha(G_{k-1}) = \alpha(G_k)$. Once again, solving this recurrence relation, we obtain that $\alpha(G_k) = \alpha(G)^{k+1}$. Repeating a similar analysis for maximum cliques, we get $\omega(G_k) = \omega(G)^{k+1}$, and therefore, $hom(G_k) = hom(G)^{k+1}$. We now have the following lemma:

**Lemma 2.4.4.** *Let $G$ be a graph and $S_G = \{G_k\}_k$, Then $\delta_{G_k} = \delta_G$ for every $k \geq 0$.*

*Proof.* For every $k \geq 0$, we have:

$$\delta_{G_k} = \frac{\log(hom(G_k))}{\log(|G_k|)} = \frac{\log(hom(G)^{k+1})}{\log(|G|^{k+1})} = \frac{(k+1)\log(hom(G))}{(k+1)\log(|G|)} = \delta_G$$

□

Now that we have developed the required machinery, we turn to the main theorem. Note that the following theorem also implies that $\gamma(H) \in \Gamma_H$ for substitution prime graphs $H$. We now present the theorem as follows:

**Theorem 2.4.5.** *If $H$ is substitution prime, then $\gamma^*(H) = \gamma(H)$*

*Proof.* Let $H$ be substitution prime. It suffices to show that $\gamma(H) \leq \gamma^*(H)$. Suppose not! Then, $\gamma(H) > \gamma^*(H)$ and there exists a constant $\gamma^*(H) < c < \gamma(H)$. Note that $c \notin \Gamma^*_H$ as $\gamma^*(H) = \sup \Gamma^*_H$, and hence there exists a graph $G$ for which $hom(G) < |G|^c$. We know that $hom(G) = |G|^{\delta_G}$, and so $\delta_G < c$. Further, as $c \in \Gamma_H$, by definition of $\Gamma_H$, there exists a threshold $N_0 \in \mathbb{N}$ such that every $H$-free graph with at least $N_0$ vertices has $hom(G) \geq |G|^c$. Choose $k \in \mathbb{N}$, such that $G_k \in S_G$ has $|G|_k \geq N_0$ vertices. Then, by Lemma 2.4.3, $G_k$ is an $H$-free graph with at least $N_0$ vertices, and hence $hom(G_k) \geq |G_K|^c$. It follows that $\delta_{G_k} \geq c > \delta_G$, thereby contradicting Lemma 2.4.4. By Contradiction, $\gamma(H) \leq \gamma^*(H)$, and hence, they are equal. □

Thus, we can now use upper bounds given by $\delta_m$ values to *asymptotically* upper bound the value of $\gamma(H)$ for prime graphs $H$. For Instance, if $H = P_5$, then $\gamma(P_5) = \gamma^*(P_5) \leq \min\{\delta_1, \delta_2, \delta_3, \delta_4\} = 1/2$. Further, as the graph $C_5$ is $P_5$-free, we see that $\gamma^*(P_5) \leq \delta_{C_5} = 0.43$, and hence, $\gamma(P_5) \leq 0.43$.

We also note that the methods used in this section show that working with the non-asymptotic version of the conjecture is equivalent to the asymptotic version, and thus, it is possible to switch between the version of the problem for ease of proof. For instance, the EH-property for $K_r$ requires the asymptotic version as we assume $|G|$ to be large enough for $\Delta(G) \approx \Delta(G) + 1$. On the other hand, we shall see that the proof of EH-property for $C_5$ is by contradiction, and considers a minimal counter-example, making the non-asymptotic version crucial. In either case, it is possible to switch between constants from both versions in a manner similar to the methods used in this section.

# Chapter 3

# Rödl's Theorem and Related Results

The typical properties of graphs can be studied using the notion of random graphs. A random graph on $n$ vertices can be generated via the Erdős-Rényi model, where an edge is added between a pair of vertices with probability $p$, independent of the addition of other edges. Smaller values of $p$ generate sparser graphs with higher probability and $p = 0.5$ ensures that all graphs are generated with equal probability. We say that *almost all graphs* have a property $P$ if the probability of a random graph on $n$ vertices having the property $P$ tends to one as $n \to \infty$. In the $p = 0.5$ case, this probability is simply the fraction of graphs having the property on $n$ vertices.

If $H$ is a fixed graph, it can be shown using probabilistic methods that almost all graphs contain $H$ as an induced subgraph (See Chapter 11 [13] for a proof). As there are finitely many graphs on $k$ vertices for any fixed integer $k$, the previous theorem extends to show that almost all graphs are $k$-universal. Here, a graph is said to be $k$-universal if it induces all subgraphs on $k$ vertices.

Another typical property of graphs can be expressed in terms of their edge distribution and edge densities. The edge density of an induced subgraph graph $G[S] \subset G$ is defined as the fraction of edges $|E(S)|/\binom{|S|}{2}$ contained in the induced subgraph $G[S]$. It is expected that a random graph has a uniform distribution of edges throughout the graph with a higher probability. Indeed, it can be shown that for almost every graph $G$, all induced subgraphs of size at least $\mu|G|$ have edge densities closer to 0.5 for some $\mu > 0$.

Both these properties display the generic behaviour of graphs. As almost all graphs contain $H$ as an induced subgraph, it is possible that enforcing $H$-freeness on a graph adds enough structure to skew the edge distributions from typical, thereby linking the two properties above. This idea was concretised and proven by Vojtěch RÖDL [31]. We present his theorem and few other related results in this chapter, for they are relevant to the class of $H$-free graphs and shall be useful in some of the upcoming chapters. We begin by looking at Szemerédi's well-known regularity lemma.

## 3.1 Szemerédi's Regularity Lemma

Given two disjoint set of vertices $A, B \subset V(G)$ of a graph $G$, the edge density $\mu(A, B)$ is defined as:

$$\mu(A, B) \;\coloneqq\; \frac{|E(A, B)|}{|A||B|}$$

where $|E(A, B)|$ equals the number of edges from $A$ to $B$ in the graph $G$. We shall abuse notation slightly by using $\mu(a, B)$ to denote $\mu(\{a\}, B)$. Roughly speaking, the edges between $A$ and $B$ can be said to be evenly distributed if the set of edge densities between various large subparts of $A$ and $B$ do not exhibit much variation. This notion of equidistribution is captured by the following definition of a regular pair.

**Definition 3.1.1.** *Let $\epsilon > 0$. Two disjoint subsets $A, B \subset V(G)$ are said to be an $\epsilon$-regular pair if for all $A' \subset A,\ B' \subset B$ with $|A'| \geq \epsilon|A|,\ |B'| \geq \epsilon|B|$, the following holds:*

$$|\mu(A, B) \;-\; \mu(A', B')| \;\leq \epsilon$$

Note that if $(A, B)$ are an $\epsilon$-regular pair, then they are also $\epsilon'$-regular for $\epsilon' \geq \epsilon$. A useful and direct consequence of the $\epsilon$-regularity of a pair is described in the following lemma.

**Lemma 3.1.1.** *Let $(A, B)$ be an $\epsilon$-regular pair of density $\mu$. Then,*

$$\#\{a \in A \;: (\mu - \epsilon)|B| \leq |E(a, B)| \leq (\mu + \epsilon)|B| \;\} \;\geq\; (1 - 2\epsilon)|A|$$

*Proof.* Let $X \subset A$ be the set of vertices with $|E(a, B)| < (\mu - \epsilon)|B|$, and suppose for contradiction that $|X| \geq \epsilon|A|$. Then, $|E(X, B)| < |X| \cdot (\mu - \epsilon)|B|$, or equivalently $\mu(X, B) <$

$\mu(A,B)-\epsilon$, contradicting the $\epsilon$-regularity of $(A,B)$. Thus, $|X|<\epsilon|A|$. Similarly, if $Y=\{a\in A||E(a,B)|>(\mu+\epsilon)|B|\ \}$, then $|Y|<\epsilon|A|$. The lemma follows by noting that the required set $A\backslash(X\cup Y)$ has cardinality at least $(1-2\epsilon)|A|$. $\square$

Observe that if a pair $(A,B)$ has density $\mu$, then any vertex in $A$ has an expected number of $\mu|B|$ neighbours in $B$. Hence, the above lemma roughly states that for $\epsilon$-regular pair, most vertices of $A$ have nearly the expected number of neighbours in $B$, thereby corroborating the regularity of the pair. The previous lemma extends to the following corollary:

**Corollary 3.1.2.** *Let $A,B_1,B_2,\ldots,B_k\subset V(G)$ be pairwise disjoint vertices of a graph $G$, such that $(A,B_i)$ is an $\epsilon$-regular pair for all $i\leq k$. Then,*

$$\#\left\{a\in A\ :(\mu(A,B_i)-\epsilon)\leq\mu(a,B_i)\leq(\mu(A,B_i)+\epsilon)\quad\forall i\leq k\right\}\ \geq\ (1-2k\epsilon)|A|$$

The proof follows by noting that if we define the sets $X_i,Y_i$, as in the proof of the previous lemma, for each $(A,B_i)$ pair; then the unwanted set of vertices in the corollary above is the union of all $X_i$ and $Y_i$ with size at most $2k\epsilon|A|$. Both these results are useful in finding subgraphs in large graphs. Stated otherwise, every bipartite graph $H$ is a subgraph of all sufficiently large $\epsilon$-regular pair $(A,B)$ for some $\epsilon>0$, where the subgraph $H$ can be found by repeated applications of Lemma 3.1.1. The following lemma by Rödl [31] extends this idea to find induced subgraphs in large graphs with a pairwise $\epsilon$-regular partition. He first defines the following notion:

**Definition 3.1.2.** *A graph $G$ is said to have the $[k,l,\epsilon,\beta]$ property, if $V(G)$ can be partitioned into $k$ pairwise disjoint parts $B_1,B_2,\ldots B_k$, such that each part has size $l$, and every pair of parts is $\epsilon$-regular with density $\mu(B_i,B_j)\in(\beta,1-\beta)$ for $\beta<1/2$.*

Recall that a graph $G$ is $k$-universal if $G$ induces all possible subgraphs on $k$ vertices. It must be noted that if $G$ is $k$-universal, then $G$ is $k'$-universal for $k'\leq k$. Now, the lemma:

**Lemma 3.1.3.** *[31] Given $0<\beta<1/2$ and $k\in\mathbb{N}$, there exist constants $\epsilon_k=\epsilon(k,\beta)$ and $l_k=l(k,\beta)$, such that every graph with the $[k,l,\epsilon_k,\beta]$ property for $l\geq l_k$ is $k$-universal*

*Proof.* We prove the lemma by induction on $k$. Let $k\in\mathbb{N}$ and $\beta<1/2$ be given. The lemma is trivially true for $k=1$, as the base case. Suppose the lemma is true for some integer $k$

and $\beta < 1/2$, and let $\epsilon_k = \epsilon(k, \beta/2)$ and $l_k = l(k, \beta/2)$. We prove the theorem for $k+1$. Define the following:

$$\epsilon_{k+1} = \epsilon(k+1, \beta) \;\coloneqq\; \min\left\{\frac{1}{2(k+1)},\ \frac{\beta\epsilon_k}{2}\right\}$$

$$l_{k+1} = l(k+1, \beta) \;\coloneqq\; \max\left\{\frac{2}{\beta}l_k,\ k+1\right\}$$

Suppose the graph $G$ has the $[k+1, l, \epsilon_{k+1}, \beta]$ property with $l \geq l_{k+1}$. Let $B_1, B_2, \ldots B_{k+1}$ be the partition of $V(G)$, as given by Definition 3.1.2. We need to show that $G$ is $(k+1)$-universal. Let $H$ be any graph on $(k+1)$ vertices with $V(H) = \{v_1, \ldots, v_{k+1}\}$. Pick a vertex $b_{k+1} \in B_{k+1}$, such that the following holds for all $j \leq k$:

$$\mu(B_{k+1}, B_j) - \epsilon_{k+1} \;\leq \mu(b_{k+1}, B_j) \;\leq\; \mu(B_{k+1}, B_j) + \epsilon_{k+1}$$

The existence of such a vertex $b_{k+1}$ is guaranteed by Corollary 3.1.2, and the following inequality, based on the choice of $\epsilon_{k+1}$ and $l_{k+1}$:

$$(1 - 2k\epsilon_{k+1})l_{k+1} \;\geq\; \left(1 - \frac{2k}{2(k+1)}\right)(k+1) \;\geq\; 1$$

Now, for each $j \leq k$, we choose a subset $B'_j \subset B_j$, such that $b_{k+1}$ is complete to $B'_j$ if $(x_j, x_{k+1}) \in E(H)$, and anti-complete otherwise. We investigate the cardinality of these sets $B'_j$. If $(x_j, x_{k+1}) \in E(H)$, then, there are $\mu(b_{k+1}, B_j)|B_j|$ neighbours of $b_{k+1}$ in $B_j$, and by the choice of $b_{k+1}$, we have:

$$|B'_J| \geq (\mu(B_j, B_{k+1}) - \epsilon_{k+1})|B_j| \;\geq\; (\beta - \epsilon_{k+1})|B_j| \;\geq\; (\beta - \frac{\beta}{2}\epsilon_k)|B_j| \;\geq\; \frac{\beta}{2}|B_j|$$

Similarly, one can show that if $(x_j, x_{k+1}) \notin E(H)$, then $|B'_j| \geq \beta|B_j|/2$. In either case, we have that $|B'_j| \;\geq\; \beta|B_j|/2 \;\geq\; \beta l_{k+1}/2 \;\geq\; l_k$ for all $j \leq k$. Now, fix any two such sets $B'_i$ and $B'_j$. We wish to show that their density lies in the interval $(\beta/2, 1-\beta/2)$ and are $\epsilon_k$-regular pairs. For the former, note that, by a calculation similar to the one above, $|B'_j| \geq \frac{\beta}{2}|B_j| \;\geq \frac{\beta\epsilon_k}{2}|B_j| \geq\; \epsilon_{k+1}|B_j|$. Thus, by $\epsilon_{k+1}$-regularity of $(B_i, B_j)$, we have:

$$|\mu(B'_i, B'_j) - \mu(B_i, B_j)| \;\leq\; \epsilon_{k+1} \leq \frac{\beta}{2}\epsilon_k \;\leq\; \frac{\beta}{2}$$

Now, the former claim follows by noting that $\mu(B_i, B_j) \in (\beta, 1-\beta)$. For the $\epsilon_k$-regularity, Let $X_i \subset B'_i$ and $X_j \subset B'_j$ be such that $|X_i| \geq \epsilon_k|B'_i|$ and $|X_j| \geq \epsilon_k|B'_j|$. Then, $|X_i| \geq \epsilon_k|B'_i| \geq \epsilon_k\dfrac{\beta}{2}|B_i| \geq \epsilon_{k+1}|B_i|$. Similarly, $|X_j| \geq \epsilon_{k+1}|B_j|$. Now, by $\epsilon_{k+1}$-regularity of $B_i, B_j$, we have:

$$\begin{aligned} |\mu(X_i, X_j) - \mu(B'_i, B'_j)| &\leq |\mu(X_i, X_j) - \mu(B_i, B_j)| + |\mu(B_i, B_j) - \mu(B'_i, B'_j)| \\ &\leq 2\epsilon_{k+1} < \epsilon_k \end{aligned}$$

Thus, the graph $G'$ induced on $\{B'_i : i \leq k\}$ has the $[k, l_k, \epsilon_k, \beta/2]$ property, and by the induction hypothesis, it is possible to recursively find vertices $b_1, \ldots, b_k$ with $b_i \in B'_i$, such that they induce the graph $H\backslash\{v_{k+1}\}$ in $G'$. Thus, along with the vertex $b_{k+1}$, we find an induced copy of $H$ in the original graph $G$, as was required. □

The above discussion indicates the usefulness of having regular pairs in a graph. Indeed if $V(G)$ can be partitioned in a manner so as to generate a high number of regular pairs, then much can be said about the structure of such sufficiently large graphs $G$. A trivial way is to partition $V(G)$ into singleton sets, to create $\binom{|G|}{2}$ regular pairs in the graph $G$, however, this is not very useful, as each part of the partition is extremely small. Thus, one can ask whether a graph $G$ admits a partition which maximises both the size of the parts and the number of regular pairs in $G$. In a landmark result for extremal graph theory, Szemerédi [32] answers the above question in the affirmative, in the form of the following regularity lemma.

**Definition 3.1.3.** *Given a graph $G$, a partition $C_0, C_1, \ldots, C_k$ of $V(G)$ is said to be an $\epsilon$-regular partition, if the following properties hold:*

- $|C_0| \leq \epsilon|G|$.
- *All $C_i$ have the same size for $i \geq 1$.*
- *All but at most $\epsilon\binom{k}{2}$ many $(C_i, C_j)$ pairs are $\epsilon$-regular for $1 \leq i, j \leq k$.*

**Theorem 3.1.4** (Szemerédi's Regularity Lemma)**.** *For all $\epsilon \in (0, 1)$ and $m \in \mathbb{N}$, there exist constants $M$, $N$ such that every graph on at least $N$ vertices has an $\epsilon$-regular partition $\{C_0, C_1, \ldots C_k\}$ with $m \leq k \leq M$.*

We refer you to [32] or [13] for a proof of the above theorem. The usefulness of the Regularity Lemma stems from the existence of the constant $M(\epsilon, m)$ which does not depend on the graph $G$. Thus, arbitrarily large graphs too can be partitioned into at most $M$ parts, thereby increasing the size of each part as well. The proof of Rödl's theorem in [31] makes use of the Regularity Lemma. See [24] for futher applications of the lemma.

## 3.2 Rödl's Theorem

In 1986, Rödl [31] proved that every sufficiently large graph $G$ is $k$-universal for some $k$, under some conditions of uniform edge distribution. While Rödl proved a more generalised form of the theorem below, we content ourselves with the following version:

**Theorem 3.2.1.** *[31] For all $k \in \mathbb{N}$ and $\sigma < 1/2$, there exists constants $N_0 \in \mathbb{N}$ and $b < 1$, such that for every graph $G$ on at least $N_0$ vertices, the following holds: If every large subset $S \subset V(G), |S| \geq b|G|$ has edge density $\mu(S) \in (\sigma, 1-\sigma)$, then $G$ is $k$-universal.*

The proof of this theorem makes use of the regularity lemma to construct an $\varepsilon$-regular partition of sufficiently large size. The size is chosen large enough, with the help of Ramsey numbers, to ensure the existence of an induced subgraph with the $[k, l, \epsilon, \beta]$ property. The theorem then follows from 3.1.3. We now prove the theorem:

*Proof.* Let $k \in \mathbb{N}$ and $\sigma < 1/2$ be given as above. We can assume without loss of generality that $k \geq 3/\sigma$ (as a $k'$-universal graph is $k$-universal for all $k' \leq k$). Let $m = R_3(k)$ be the smallest number, such that every three colouring of the edges of $K_n$ for $n \geq m$, contains a monochromatic $K_k$. Such a number is known to exist for all $k \in \mathbb{N}$ from Ramsey theory. Further, let $\epsilon_k = \epsilon(k, \sigma/2)$ and $l_k = l(k, \sigma/2)$, as provided by Lemma 3.1.3. Finally, set $\varepsilon = \min\{1/(m-1) - 1/m, \epsilon_k\}$ and let $M$, $N$ be the constants obtained by applying the Regularity lemma with $\varepsilon$ and $m$ as defined above.

We prove the theorem with $N_0 \coloneqq \max\{N, l_k M/(1-\epsilon_k)\}$ and $b \coloneqq k(1-\epsilon_k)/M$. Consider any graph $G$ on at least $N_0$ vertices, such that every large subset $S \subset V(G)$, $|S| \geq b|G|$ has edge density $\mu(S) \in (\sigma, 1-\sigma)$. By the regularity lemma, there exists an $\varepsilon$-regular partition $\{C_0, C_1, \ldots, C_t\}$ with $m \leq t \leq M$. Note that $|C_0| \leq \varepsilon|G|$, and so, for $i \neq 0$, $|C_i| \geq (1-\varepsilon)|G|/M$. Construct the regularity graph $H$, with $V(H) = \{1, 2, \ldots, t\}$ and $(u, v) \in E(H)$ iff $(C_u, C_v)$ are $\varepsilon$-regular pairs. By the regularity lemma, and the choice of $\varepsilon$, we have that:

$$|E(H)| \;\geq\; \binom{t}{2} - \varepsilon\binom{t}{2} \;\geq\; \frac{t^2}{2}\left(1-\frac{1}{t}-\varepsilon\right) \;\geq\; \frac{t^2}{2}\left(1-\frac{1}{m-1}\right)$$

By Turan's theorem, the graph $H$ contains a clique of size $m$, say $H'$ with vertices $\{C_1, \ldots, C_m\}$ taken without loss of generality. Colour the edges of $H'$ in three colours as follows:

- Colour 1: if $\mu(C_i, C_j) \;<\; \sigma/2$.
- Colour 2: if $\mu(C_i, C_j) \;\in\; (\sigma/2, 1-\sigma/2)$.
- Colour 3: if $\mu(C_i, C_j) \;>\; 1-\sigma/2$.

By the choice of $m$, there exists a set $J \subset V(H')$ such that $J$ induces a monochromatic $K_k$ in $H'$. Note that the set $\mathcal{C} = \cup_{i\in J} C_i$ has at least $b|G|$ vertices, and so, the graph $G' = G[\mathcal{C}]$ induced on this set has density $\mu(\mathcal{C}) \in (\sigma, 1-\sigma)$. Now, if the clique $H'[J]$ is monochromatic with colour 1, then, we calculate the edge density of $G'$, by counting the number of edges between two parts, and inside each part $C_i$.

$$\mu(\mathcal{C}) \leq \binom{|C_i|k}{2}^{-1} \cdot \left(\binom{k}{2}\frac{\sigma}{2}|C_i|^2 \;+\; \binom{|C_i|}{2}k\right) \;\leq\; \frac{\sigma}{2} + \frac{1}{k} \;<\; \sigma$$

Here, the final inequality follows from the fact that $k \geq 3/\sigma$. Similarly, if $J$ is monochromatic with colour 3, then, we have:

$$\mu(\mathcal{C}) \geq \binom{|C_i|k}{2}^{-1} \cdot \left(\binom{k}{2}(1-\frac{\sigma}{2})|C_i|^2\right) \;\geq 1 - \frac{\sigma}{2} - \frac{1}{k} \;>\; 1-\sigma$$

Here, the first inequality follows by replacing the denominator with $k^2|C_i|^2/2$. Thus, $J$ cannot be monochromatic with colours 1 or 3, as it contradicts the density requirements on the set $\mathcal{C} \subset V(G)$. Then, $J$ is a monochromatic clique of colour 2, and by the choice of $\varepsilon$ and $N_0$, it follows that the graph $G[\mathcal{C}]$ has the $[k, l_k, \epsilon_k, \sigma/2]$ property. By Lemma 3.1.3, the graph $G[\mathcal{C}]$, and thus, the graph $G$ is $k$-universal. □

This theorem is powerful in the sense that it indicates a non-typical edge distribution for the class of $H$-free graphs. It states that for $|H| = k$, and $\sigma < 1/2$, there exist constants $b$ and $N_0$ such that every $H$-free graph on at least $N_0$ vertices has a subset of size at least

$|S| \geq b|G|$ with a skewed edge density: $\mu(S) < \sigma$ or $\mu(S) > 1 - \sigma$. Later, in [18], Jacob Fox and Benny Sudakov prove Rödl's theorem without using the regularity lemma. They obtained a tighter estimate of the constant $b$, and removed the dependence of $N_0$ from the theorem. Thus, we can rewrite the same theorem as follows:

**Lemma 3.2.2.** *Let $H$ be a graph with $|H| = k$ and let $\sigma < 1/2$, then there exists constant $b < 1$, such that every $H$-free graph $G$ has a subset $S \subset V(G)$ of size $|S| \geq b|G|$ for which either $G[S]$ or $\overline{G}[S]$ has at most $\sigma\binom{|S|}{2}$ edges.*

The authors of [9] show that the above theorem can also be restated to talk about the maximum degree of $G[S]$ instead of the number of edges or edge density. They show that:

**Lemma 3.2.3.** *Let $H$ be a graph with $|H| = k$ and let $\sigma < 1/2$, then there exists constant $b < 1$, such that every $H$-free graph $G$ has a subset $S \subset V(G)$ of size $|S| \geq b|G|$ for which either $G[S]$ or $\overline{G}[S]$ has maximum degree at most $\sigma b|G|$.*

This version of Rödl's theorem turns out to be very useful in various proofs of the EH-property, as we shall see in the next chapter. It allows us to focus on either sparse or dense graphs, if the graph $H$ is self-complementary.

# Chapter 4

# Known Instances of the EH-Property

In this chapter, we shall look at the various techniques and concepts involved in proving the EH-property known for a few prime graphs, namely $P_4$, $C_5$ and the Bull. The concepts involved mainly include self-complementarity, the perfectness of graphs, substitution operation and combs. The proof for $P_4$ is a well-known result in graph theory, while the others have been proven in [8] and [9]. We present their original results, with occasional modifications to the proofs of some theorems.

## 4.1 The EH-property for $P_4$

We saw that all perfect graphs on $n$ vertices have a homogeneous set of size at least $\sqrt{n}$. The EH-property for $P_4$ follows as a direct consequence of this result, for the class of $P_4$-free graphs form a well-studied subset of perfect graphs. Indeed, Erdős and Hajnal themselves acknowledged this in their 1989 paper while introducing the conjecture. We present a quick proof outlining these pre-established results. Consider first, the well-studied class of co-graphs, defined recursively as follows:

**Definition 4.1.1.** *Co-graphs are defined to be a family of finite graphs $\mathcal{C}$ such that:*

- *The graph $K_1 \in \mathcal{C}$.*
- *If $G_1, G_2 \in \mathcal{C}$, then $G_1 + G_2 \in \mathcal{C}$ and $G_1 \sqcup G_2 \in \mathcal{C}$.*

Stated otherwise, all co-graphs can be obtained from the single vertex graph, $K_1$, through a sequence of join and union operations. Note that the class of co-graphs is closed under taking graph complements, as the complement of a co-graph can be simply obtained by switching joins and unions in the sequence of operations above. Hence, if $G$ is a co-graph, then either $G$ or $\overline{G}$ will be disconnected, depending on whether the final operation is a union or not.

Further, the class of co-graphs also admits a forbidden induced subgraph characterisation in terms of $P_4$, as the join and union operations preserve the $P_4$-freeness of a graph. We shall prove that the class of co-graphs is precisely the set of $P_4$-free graphs.

**Lemma 4.1.1.** *If $G_1$ and $G_2$ are two $P_4$-free graphs, then their join $G_1 + G_2$ and union $G_1 \sqcup G_2$ are $P_4$-free too.*

*Proof.* Let $G_1$ and $G_2$ be $P_4$-free graphs. If $G = G_1 \sqcup G_2$ has an induced $P_4$, then it must lie completely within $G_1$ or $G_2$, as $P_4$ is connected. However, this is not possible, as both $G_1$ and $G_2$ are $P_4$-free. Hence unions preserve $P_4$-freeness. Further, as $P_4$ is self-complementary, a graph $G$ is $P_4$-free if and only if its complement $\overline{G}$ is $P_4$-free. Thus, both $\overline{G_1}$ and $\overline{G_2}$ are $P_4$-free, and the lemma follows by noting that $G_1 + G_2 = \overline{(\overline{G_1} \sqcup \overline{G_2})}$. □

**Theorem 4.1.2.** *A graph $G$ is a co-graph if and only if $G$ is $P_4$-free.*

*Proof.* As $K_1$ is $P_4$-free, the forward implication follows directly from Lemma 4.1.1 and a routine structural induction on the class of co-graphs. It remains to prove that all $P_4$-free graphs are co-graphs. Suppose not! Let $G$ be a counterexample, minimal with respect to the number of vertices in $G$. Then, $G$ is a $P_4$-free graph which is not a co-graph. Further, $G$ has at least 4 vertices, as all graphs with at most three vertices are co-graphs. Finally, note that both $G$ and $\overline{G}$ are connected: for otherwise, the components of $G$ or $\overline{G}$ form smaller sized $P_4$-free graphs, and thus are co-graphs by the minimality assumption above. Taking unions of these components makes $G$ a co-graph, contrary to our assumptions for $G$.

Pick any vertex $v \in V(G)$. Then, $H := G\backslash v$ is $P_4$-free. By the minimality assumption above, $H$ is a co-graph. Hence, either $H$ or $\overline{H}$ is disconnected. Without loss of generality, assume $H = G\backslash v$ is disconnected. As $\overline{G}$ is connected, there exist a vertex $u$ non-adjacent to $v$, and a path $P$ which joins $u$ to $v$ in $G$. As $(u, v) \notin E(G)$, It is possible to find two adjacent vertices $x, y$ on $P$ such that $x$ is adjacent and $y$ is non-adjacent to $v$. Further, let $z$ be any neighbour of $v$ from a component of $H$ not containing $x$ and $y$. Such a $z$ exists by the disconnectedness of $H$. Observe that $z - v - x - y$ induces a $P_4$ in $G$, which is not possible. Thus, by contradiction, all $P_4$-free graphs are co-graphs.

□

Let us now look at how the clique number $\omega(G)$ and the chromatic number $\chi(G)$ interact with the join and union operations on graphs. Consider the union operation. Let $G_1$ and $G_2$ be two graphs and let $G = G_1 \sqcup G_2$. The maximum clique in $G$ lies completely within either $G_1$ or $G_2$ and so the maximum clique size $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$. Similarly, as both components of $G_1$ and $G_2$ can be coloured independently, the chromatic number $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ as well.

The relations above are slightly different for the join operation as every vertex of $G_1$ is adjacent to every vertex of $G_2$. Let $G = G_1 + G_2$. Observe that $\omega(G) = \omega(G_1) + \omega(G_2)$ as the sizes of cliques in $G_1$ and $G_2$ add up to give larger cliques in $G$ by the virtue of $G_1$ being complete to $G_2$. Similarly, no colours assigned to $V(G_1)$ can be repeated in $V(G_2)$ as they are adjacent, thereby making the chromatic number additive as well: $\chi(G) = \chi(G_1) + \chi(G_2)$.

These properties can be summarised as follows:

$$\omega(G_1 \sqcup G_2) = \max\{\omega(G_1), \omega(G_2)\} \quad ; \quad \omega(G_1 + G_2) = \omega(G_1) + \omega(G_2)$$
$$\chi(G_1 \sqcup G_2) = \max\{\chi(G_1), \chi(G_2)\} \quad ; \quad \chi(G_1 + G_2) = \chi(G_1) + \chi(G_2)$$

The mirrored relations for clique number and chromatic number above make the case for the perfectness of co-graphs. As $K_1$ is perfect, the following theorem follows by structural induction on the class of co-graphs.

**Lemma 4.1.3.** *All co-graphs are perfect graphs.*

As any perfect graph $G$ has a homogeneous set of size at least $\sqrt{|G|}$, the EH-property for $P_4$ now follows directly as a consequence of the above theorems with $\gamma(P_4) \geq 1/2$. This bound can be shown to be tight by considering the infinite class of graphs $\{nK_n \ : \ n \in \mathbb{N}\}$. All these graphs are $P_4$-free, as they are disjoint unions of cliques, and have $hom(G) = \sqrt{|G|}$, as seen in Example 2.1 from an earlier chapter.

## 4.2 The EH-property for the Bull

In the previous section, we saw that perfect graphs can be used to find large homogeneous sets in a given class of graphs. Indeed, an equivalent version of the Erdős-Hajnal Conjecture claims that every large enough $H$-free graph $G$ has a perfect induced subgraph of size at least $|G|^c$ for some constant $c$ dependent on $H$. The equivalence follows from the fact that all homogeneous sets are perfect and each perfect graph $G$ has a homogeneous set of size $\sqrt{|G|}$. The notion of $\alpha$-narrowness, first defined in [11], tries to roughly capture this requirement in graphs as follows:

**Definition 4.2.1.** *A function $f : V(G) \to (0,1)$ is said to be* good*, if for every perfect induced subgraph $P \leq G$, we have that $\sum_{v \in V(P)} f(v) \leq 1$. A graph $G$ is said to be $\alpha$-narrow for some $\alpha \geq 1$, if for every good function $f : V(G) \to (0,1)$, the sum $\sum_{v \in V(G)} f(v)^\alpha \leq 1$.*

Observe that all perfect graphs are 1-narrow and an $\alpha$-narrow graph is also $\alpha'$-narrow for $\alpha' \geq \alpha$. Further, it is possible to partition the vertices of any graph in such a way that every part induces a perfect subgraph. For instance, a trivial way of doing so would be to partition $V(G)$ into parts of size at most four, as all graphs on at most four vertices are perfect. Thus, every graph is $\alpha$-narrow for some $\alpha \geq 1$, and this value of $\alpha$ reduces if the graph has larger-sized induced perfect subgraphs, possibly due to a greedy construction of the partition above. Roughly speaking, then, narrowness is a measure of the perfectness of a graph, where graphs with narrowness closer to one have larger induced perfect subgraphs. Following this idea, one can conjecture that if there is a constant $\alpha \geq 1$ such that every $H$-free graph is $\alpha$-narrow, then it has some bearing on the EH-property for the graph $H$; and vice versa. This is indeed true, as proven by Jacob Fox and presented in [10]. We present their proof here:

**Theorem 4.2.1.** *[10] A graph $H$ has the EH-property if and only if there exists a constant $\alpha \geq 1$ such that all $H$-free graphs are $\alpha$-narrow.*

*Proof.* Let $H$ be a given graph. Consider the reverse direction, and suppose that for some $\alpha \geq 1$, every $H$-free graph $G$ is $\alpha$-narrow. Let $K$ be the size of the largest induced perfect subgraph of $G$. Clearly, the constant function $f(v) = 1/K \ \forall v \in V(G)$ is good, as for any perfect induced subgraph $P \leq G$, the sum $\sum_{v \in V(P)} f(v) = \dfrac{|V(P)|}{K} \leq 1$. By $\alpha$-narrowness, we have:

$$\sum_{v \in V(G)} f(v)^\alpha = \frac{|G|}{K^\alpha} \leq 1 \quad \iff \quad K \geq |G|^{\frac{1}{\alpha}}$$

Thus, any $H$-free graph $G$ has a perfect induced subgraph of size at least $|G|^{1/\alpha}$, and so $hom(G) \geq |G|^{1/2\alpha}$. It follows that $H$ has the EH-property with $\gamma(H) \geq \dfrac{1}{2\alpha}$.

Now, consider the forward direction. Let $H$ be a graph having the EH-property, and let $c = 1/\gamma(H)$. We show that $H$-free graphs are $3c$-narrow by induction on the size of the graph. The base case holds trivially as all graphs on at most two vertices are 1-narrow and $3c > 1$. Let $G$ be a large $H$-free graph, and let $f : V(G) \to (0, 1)$ be a good function on $G$. We partition the vertices of $G$ as follows:

$$V_i := \left\{ v \in V(G) \mid \frac{1}{2^i} \leq f(v) < \frac{1}{2^{i-1}} \right\}, \quad \text{and} \quad G_i := G[V_i] \quad \forall i \in \mathbb{N}$$

For any $i \in \mathbb{N}$, we have that $f(v) \geq \frac{1}{2^i}$, $\forall v \in V_i$. As $f$ is a good function, any perfect induced subgraph of $G_i$ has size at most $2^i$. Thus, $hom(G_i) \leq 2^i$, and by the EH-property for $H$, we have:

$$|G_i| \;\leq\; hom(G_i)^{1/\gamma(H)} \;\leq\; 2^{ic} \quad \forall\, i \in \mathbb{N}$$

Now we consider the following two cases:

**Case I: The set** $V_1 = \phi$. Then, we have:

$$\begin{aligned}
\sum_{v \in V(G)} f(v)^{3c} \quad &\leq \sum_{i=2}^{\infty} \sum_{v \in V_i} \left( \frac{1}{2^{i-1}} \right)^{3c} && \text{(By definition of } V_i\text{)} \\
&\leq \sum_{i=2}^{\infty} \frac{2^{ic}}{2^{3c(i-1)}} \;=\; 2^{3c} \sum_{i=2}^{\infty} \frac{1}{2^{2ic}} && \text{(As } |V_i| \leq 2^{ic}\text{)} \\
&= \frac{2^c}{2^{2c} - 1} \;\leq\; 1 && \text{(By Infinite G.P., and as } c > 1\text{)}
\end{aligned}$$

**Case II: The Set** $V_1 \neq \phi$.

Then, let $u \in V_1$ with $f(u) \geq 1/2$. Let $N = N(u)$ and $M = N[u]^c$ be the neighbours and non-neighbours of $u$ in $G$, respectively. Consider the graph $G[N]$ and suppose that $P \subset N$ induces a perfect induced subgraph in $G[N]$. Then, $G[P \cup \{u\}]$ is perfect since joins preserve perfectness, as we saw in the previous section. By the goodness of $f$, we have that

$\sum_{v \in P} f(v) \leq 1 - f(u)$, and so the function $g : N \to (0,1)$ defined by $g(x) = f(x)/(1 - f(u))$ is good on $G[N]$. By the induction hypothesis, $G[N]$ is $3c$-narrow, and we have:

$$\sum_{v \in N} g(v)^{3c} \leq 1 \quad \equiv \quad \sum_{v \in N} f(v)^{3c} \leq (1 - f(u))^{3c}$$

Similarly, repeating the above analysis for $M$, we have:

$$\sum_{v \in M} f(v)^{3c} \leq (1 - f(u))^{3c}$$

Combining both the above inequalities, and by $V(G) = N \cup M \cup \{u\}$,

$$\sum_{v \in V(G)} f(v)^{3c} \leq f(u)^{3c} + 2(1 - f(u))^{3c} \leq 1$$

where, the final inequality follows by noting that $\psi(x) = x^a + 2(1-x)^a \leq 1$ whenever $a \geq 3$ and $x \in (1/2, 1)$. Hence, for every good function $f$, the sum $\sum_{v \in V(G)} f(v)^{3c} \leq 1$, and so the graph $G$ is $3c$-narrow, as required. $\square$

We saw that proving the EH-property for a graph $H$ is equivalent to showing that all $H$-free graphs are $\alpha$-narrow. In 2008, Maria Chudnovsky and Shmuel Safra [8] showed that all Bull-free graphs are 2-narrow, and hence satisfy the EH-property with $\gamma(Bull) \geq 1/4$. The proof is a good example of how forbidding an induced graph helps impose additional structure on the given graph, often altering the properties of the class as a whole. The holistic proof combines the implications of Bull-freeness with self-complementarity, narrowness, perfectness, and the substitution operation to prove the EH-property for the Bull graph. We present their proof in this section. We begin by observing that the substitution operation preserves $\alpha$-narrowness.

**Theorem 4.2.2.** *[8] Suppose $\alpha \geq 1$ and let $G_1, G_2$ be two $\alpha$-narrow graphs. Then, the graph $G = S(G_1, v, G_2)$ is $\alpha$-narrow for all $v \in V(G_1)$.*

*Proof.* Suppose $G_1, G_2$, and $\alpha$ are as given above, and $G = S(G_1, v, G_2)$ for some $v \in V(G_1)$. By Lemma 2.3.1, $S = V(G_2)$ is a non-trivial module of $G$, and let $K, N$ be a partition of $V(G) \backslash S$ such that $S$ is complete to $K$ and anti-complete to $N$. Let $f : V(G) \to (0,1)$ be any good function on $G$. Define the constant $M$ as follows:

$$M := \max_{P \leq G_2} \left\{ \sum_{u \in V(P)} f(u) \right\}$$

where $P$ is a perfect induced subgraph in $G_2$. Let $P^* \subset V(G_2)$ be the perfect graph where the maximum above is attained. Define the function $f_1$ on the graph $G_1$ by the mapping $f_1(v) = M$ and $f_1(x) = f(x)\ \forall x \in V(G_1) \backslash \{v\}$. Then, $f_1$ is a good function on $G_1$, which can be seen as follows: Let $P \subset V(G_1)$ be a perfect graph in $G_1$. If $v \notin P$, then we are done by the goodness of $f$ in $G$. Otherwise, if $v \in P$, consider the graph $G[P']$ obtained by substituting $P^*$ into the vertex $v \in V(P)$. By Lovazs' replacement lemma [27], $G[P']$ is perfect in $G$, and so, we have:

$$\sum_{x \in V(P)} f_1(x) = M + \sum_{x \in V(P) \backslash \{v\}} f(x) = \sum_{x \in V(P^*)} f(x) + \sum_{x \in V(P) \backslash \{v\}} f(x) \ = \sum_{x \in V(P')} f(x) \leq 1$$

Similarly, define a function $f_2$ on $G_2$ as $f_2(x) = f(x)/M \quad \forall x \in V(G_2)$. This is a good function on $G_2$ by the definition of $M$. As $G_1$ and $G_2$ are both $\alpha$-narrow, we have:

$$\sum_{x \in V(G_1)} f_1(x)^\alpha \leq 1 \iff K^\alpha + \sum_{x \in V(G_1 \backslash \{v\})} f(x)^\alpha \leq 1 \tag{4.1}$$

$$\sum_{x \in V(G_2)} f_2(x)^\alpha \leq 1 \iff \sum_{x \in V(G_2)} f(x)^\alpha \leq K^\alpha \tag{4.2}$$

That $G$ is $\alpha$-narrow follows directly by substituting inequality (4.2) into inequality (4.1) above, as:

$$\sum_{x \in V(G)} f(x)^\alpha = \sum_{x \in V(G_2)} f(x)^\alpha + \sum_{x \in V(G_1 \backslash \{v\})} f(x)^\alpha \leq 1$$

□

Recall that an induced $C_k (or \overline{C_k})$ in a graph $G$, for $k \geq 4$ is said to be a *hole (anti-hole)* in $G$. The authors of [8] divide all bull-free graphs into *basic* and *non-basic* graphs as follows:

**Definition 4.2.2.** *A Bull-free graph $G$ is said to be* non-basic *if $G$ has an odd hole or an odd anti-hole $H$, such that there exist vertices $c, a \in V(G \backslash H)$ for which $c$ is complete and $a$ is anti-complete to $H$. All other Bull-free graphs are said to be* basic.

They show that all basic graphs are 2-narrow and all non-basic graphs are non-primes. The EH-property for the bull graph then follows from Theorems 4.2.2 and 4.2.1. We present the proofs of both these statements in the remaining section. Note that as the bull-graph is self-complementary, the class of bull-free graphs is closed under complements. Consequently, by the definition above, both non-basic and basic graphs are closed under complements.

### Basic graphs are 2-narrow

For simplicity, we shall call a graph *narrow* if it is 2-narrow. If a basic graph $G$ has an odd hole $H$, then it must have a vertex $c$ complete to $H$ or a vertex $a$ anti-complete to $H$, but not both. Note that either the vertex $c$ or $a$ described above should exist; for otherwise, the complement $\overline{G}$, and hence $G$, would be non-basic, contrary to our assumptions. We begin by looking at the structure of basic graphs for each of the two possibilities mentioned above.

**Lemma 4.2.3.** *[8] Let $G$ be basic with an odd hole $H$ and a vertex $c \notin V(H)$ complete to $H$. If $u \in V(G \backslash H)$ is non-adjacent to $c$, then $u$ has at least $|H|-2$ neighbours in $H$.*

*Proof.* Let $G, H, c$, and $u$ be as above, and let $|H| = k$ with $H = \{h_1, h_2, \dots h_k\}$. As $G$ is basic, $u$ is not anti-complete to $H$. If $u$ is complete to $H$, then we are done, and so we assume that $u$ is neither complete nor anti-complete to $H$. Then, there exist adjacent vertices, say $h_1, h_2 \in V(H)$ such that $u$ is adjacent to $h_1$ but non-adjacent to $h_2$. If $h_i$ is a vertex of $H$ non-adjacent to both $h_1$ and $h_2$, then $h_i$ must be adjacent to $u$, for otherwise the vertices $\{h_1, h_2, h_i, c, u\}$ would induce a Bull graph in $G$. Thus, $u$ is complete to the set $V(H)\backslash\{h_2, h_3, h_k\}$. It suffices to show that $u$ is also adjacent to $h_3$ or $h_k$. This follows as, if $u$ is non-adjacent to $h_k$, then $u$ must be adjacent to $h_3$, for otherwise the set $\{h_k, h_1, h_3, c, u\}$ would induce a Bull graph in $G$. Thus, $u$ has at least $|H|-2$ neighbours in $H$. □

Similarly, we have the following lemma for the second possibility:

**Lemma 4.2.4.** *[8] Let $G$ be basic with an odd hole $H$ and a vertex $a \notin V(H)$ anti-complete to $H$. If $u \in V(G \backslash H)$ is adjacent to $a$, then $u$ has at least $\frac{|H|+1}{2}$ non-neighbours in $H$.*

*Proof.* Let $G, H, a$, and $u$ be given as above, and let $|H| = k$ with $H = \{h_1, h_2, \dots h_k\}$. Suppose for contradiction that $u$ has two consecutive neighbours, say $h_1, h_2 \in V(H)$. Let

$i \leq k$ be the smallest index so that $u$ is non-adjacent to $h_i$, whose existence is guaranteed as $G$ is basic. Then, $S = \{h_{i-2}, h_{i-1}, h_i, u, a\}$ induces a Bull graph in G, as $a$ is anti-complete to $H$ and adjacent to $u$, giving a contradiction. Thus, $u$ does not have consecutive neighbours in $H$, and by the pigeon-hole principle, has at least $\lceil \frac{|H|}{2} \rceil$ non-neighbours in $H$. As $H$ is an odd hole, $u$ has at least $\frac{|H|+1}{2}$ non-neighbours in $H$ □

Using the previous lemmata, we can show that for a basic graph $G$, and any vertex $v \in V(G)$, either the neighbours or anti-neighbours of $v$ induce a perfect graph in $G$, which would be useful in setting up induction while proving that basic graphs are narrow.

**Lemma 4.2.5.** *Let $G$ be a basic graph with any vertex $x \in V(G)$. Let $N = N(x)$ and $M = N[x]^c$ be the neighbours and non-neighbours of $x$ respectively. Then, either $G[N]$ or $G[M]$ is perfect.*

*Proof.* Let $G, N, M$, and $x$ be defined as above. Suppose for contradiction that neither $G[N]$ nor $G[M]$ is perfect. Then, the following observations can be made:

**A. At most one of $G[N]$ and $G[M]$ contains an odd hole**

Suppose both $G[N]$ and $G[M]$ have odd holes $H_N, H_M$ of size $n, m \geq 5$ respectively. Note that $x$ is complete to $H_N$ and anticomplete to $H_M$. Then, by Lemma 4.2.3, every vertex of $H_M$ has at least $|H_N| - 2$ neighbours in $H_N$. Thus, there are at least $m(n-2)$ edges from $H_M$ to $H_N$. Similarly, by Lemma 4.2.4, every vertex of $H_N$ has at least $(|H_M|+1)/2$ non-neighbours in $H_M$. Thus, there are at least $n(m+1/2) \geq n(m/2)$ non-edges between $H_N$ and $H_M$. As the sum of edges and non-edges between $H_N$ and $H_M$ equals $mn$, we have:

$$mn \geq m(n-2) + n\left(\frac{m}{2}\right)$$

or equivalently, $n \leq 4$, a contradiction. Thus at most one of $G[N]$ and $G[M]$ has an odd hole. further, as basic graphs are closed under complements, we have that at most one of $G[N]$ and $G[M]$ has an odd anti-hole. By the *strong perfect graph theorem*, both $G[N]$ and $G[M]$ must have an odd hole or anti-hole. Thus, we have the following two cases:

**Case I: $G[N]$ has an odd hole and $G[M]$ has an odd anti-hole.**

Suppose that $G[N]$ has an odd hole $H_N$ on $n$ vertices and $G[M]$ has an odd antihole $AH_M$ on $m$ vertices. By Lemma 4.2.3, every vertex in $AH_M$ has at least $|H_N| - 2$ neighbours in $H_N$, so there are at least $m(n-2)$ edges between $AH_M$ and $H_N$. Further, $AH_M$ and $H_N$ switch their roles in the complement graph $G$, and so by the same analysis, there are at least $n(m-2)$ edges between $AH_M$ and $H_N$ in the graph $\overline{G}$. Thus, there are at least $n(m-2)$ non-edges between $AH_M$ and $H_N$ in $G$. Once again, as the sum of edges and non-edges between $H_N$ and $AH_M$ equal $mn$, we have:

$$mn \;\geq\; m(n-2) + n(m-2)$$

which is not true for $m, n \geq 5$.

**Case II:** $G[N]$ **has an odd anti-hole and** $G[M]$ **has an odd hole.**

Let $H_M$ be an odd hole in $G[M]$ of size $m$, and let $AH_N$ be an odd anti-hole of size $n$. We proceed in a manner similar to Case I, by using Lemma 4.2.4 instead of 4.2.3, to obtain the following impossibility:

$$mn \;\geq\; m\left(\frac{n+1}{2}\right) + n\left(\frac{m+1}{2}\right) \quad\equiv\quad m+n \leq 2$$

Thus, by contradiction, either $G[N]$ or $G[M]$ is a perfect subgraph of $G$. □

Observe that the above proof and its use of the strong perfect graph theorem helps motivate the definition of basic and non-basic graphs as defined above. We now use these lemmata to prove that all basic graphs are narrow.

**Theorem 4.2.6.** *If $G$ is a basic graph, then $G$ is 2-narrow.*

*Proof.* We prove this by induction on the size of the graph. For the base case, note that all graphs on at most four vertices are perfect, and thus are 2-narrow as well. Let $G$ be a basic graph, and $f : V(G) \to (0,1)$ be a good function. Pick the vertex $u \in V(G)$ for which $g(u)$ is maximum and let $N, M$ be the neighbours and non-neighbours of $u$ in $G$. By the previous lemma, either $G[N]$ or $G[M]$ is perfect, and by switching to the complement if necessary, assume that $G[N]$ is perfect. Note that the switching is possible as basic graphs are closed under complements. As $G[N]$ is perfect and joins preserve perfectness, we have that $G[N \cup \{u\}]$ is perfect too, and so $f(u) + \sum_{v \in N} f(v) \leq 1$. Hence, along with the maximality of $f(u)$, we have:

$$\sum_{v \in N} f(v)^2 \ \leq \ f(u) \sum_{v \in N} f(v) \ \leq \ f(u)(1 - f(u))$$

Now, for the subset $M \subset V(G)$, define $g : M \to (0,1)$ by the rule $g(v) = \dfrac{f(v)}{1 - f(u)}$. If $P \subset M$ induces a perfect graph in $G[M]$, then $G[P \cup \{u\}]$ is also perfect as the union operation preserves perfectness. By goodness of $f$ on $G$, we have $f(u) + \sum_{v \in P} f(v) \leq 1$ and so, $\sum_{v \in P} g(v) \leq 1$ for all perfect subgraphs $G[P] \leq G[M]$. Thus $g$ is a good function on $G[M]$, and as $G[M]$ is narrow by induction, we have: $\sum_{v \in M} f(v)^2 \leq (1 - f(v))^2$.

Combining all inequalities, we see that for all good functions $f$:

$$\begin{aligned} \sum_{v \in V(G)} f(v)^2 &= \ f(u)^2 \ + \ \sum_{v \in N} f(v)^2 \ + \ \sum_{v \in M} f(v)^2 \\ &\leq \ f(u)^2 \ + \ f(u)(1 - f(u)) \ + \ (1 - f(u))^2 \\ &\leq \ (f(u) + 1 - f(u))^2 \ = \ 1 \end{aligned}$$

Thus, G is narrow, as was required. □

## Non-basic graphs are not Substitution Primes

Let us begin by defining the following notion of a *split set* in a graph $G$.

**Definition 4.2.3.** *Given a graph $G$, a subset $S \subset V(G)$ of vertices is said to be a split set if for every $x \in S$, neither complete nor anti-complete to $S$, there exist vertices $u, v, w$ such that either of the following holds:*

- *$u, v, x$ form a $C_3$; and $w$ is adjacent to $v$ and non-adjacent to $x, u$*
- *$u, v, x$ form an $I_3$; and $w$ is adjacent to $x, u$ but non-adjacent to $v$*

In essence, the vertices $\{u, v, w, x\}$ induce either of the two labelled graphs shown in the figure above. Note that both these graphs are complements of each other, and hence, a set $S \subset V(G)$ is a split set in $G$ if and only if $S$ is a split set in $\overline{G}$ as well. The authors of [8] showed that if a Bull-free graph $G$ has a split set $S$ and there exist vertices $c, a$ respectively

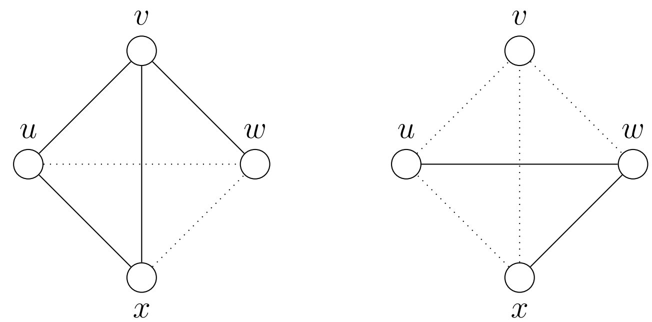


Figure 4.1: Graphs induced by $\{x, u, v, w\}$ for $x \in S$

complete and anti-complete to $S$, then $G$ is a non-prime graph. Further, the holes and anti-holes of a Bull-free graph are split sets, and hence, it follows that all non-basic graphs are non-prime. Let's begin by proving the first lemma:

**Lemma 4.2.7.** *Let $G$ be a Bull-free graph and $S \subset V(G)$ be a split set in $G$ of size at least two. If there exist vertices $c, a \in V(G) \backslash S$, such that $c$ is complete and $a$ is anti-complete to $S$, then $G$ is a non-prime graph.*

*Proof.* Let $G$ be a Bull-free graph with split set $S$. Let $C$ and $A$ be the set of vertices in $V(G)\backslash S$ which are complete and anti-complete to $S$, respectively; and let $X = V(G)\backslash(S \cup C \cup A)$. We try to construct a non-trivial module in $G$. It is given that $A, C \neq \phi$. Choose any $a \in A$ and $c \in C$, We make the following claim:

**(I)** *If $(a, c) \in E(G)$, then $X$ is complete to $C$; and*
*If $(a, c) \notin E(G)$, then $X$ is anti-complete to $A$.*

Suppose, for contradiction, that (I) is false, and let $x \in X$ be a counter-example. As $S$ is a split set, there exist vertices $u, v, w$ such that one of the following holds:

`Case I: The vertices` $\{x, u, v, w\}$ `form the first graph of Figure 4.1.`
Here, $x$ is not adjacent to $a$, for otherwise, $\{a, x, u, v, w\}$ induces a bull in $G$. As $x$ is a counter-example, $a$ is adjacent to $C$, and hence $(x, c) \notin E(G)$. Then, {a, c, v, w, x} induces a bull in $G$, giving a contradiction.

`Case II: The vertices` $\{x, u, v, w\}$ `form the second graph of Figure 4.1.`
Here, $x$ is adjacent to $c$, for otherwise, $\{c, x, u, v, w\}$ induces a bull in $G$. As $x$ is a counter-example, $a$ is not adjacent to $C$, and hence $(x, a) \in E(G)$. Then, {a, c, v, w, x} induces a

bull in $G$, again giving a contradiction.

Thus, claim (I) holds true. Note that if there exists $c \in C$ with no neighbour in $A$, then every vertex of $A$ has $c \in C$ as a non-neighbour. Thus, either every $c \in C$ has a neighbour in $A$ or every $a \in A$ has a non-neighbour in $C$. We assume the former without loss of generality, by switching to $\overline{G}$ if necessary. This is possible as non-primes are closed under complements and $C, A$ switch roles on taking complements. Now, for every $c \in C$, there exists $a \in A$, such that $(a, c) \in E(G)$, and so by (I), $C$ is complete to $X$.

Finally, fix some $x \in X$ and define the set $A' \subset A$ to be the set vertices $a \in A$ which have an $x - a$ path going through $A \cup \{x\}$. Let $a \in A'$ and suppose $P = (x = a_0), a_1, a_2, \ldots, (a_k = a)$ is an $x - a$ path through $A \cup \{x\}$. Then, each $a_i \in A'$, $\; i \leq k$. If $c \in C$, then $(c, x) \in E(G)$ as $C$ is complete to $X$. Also, $(c, a_1) \in E(G)$ by claim (I) and as $(x, a_1) \in E(G)$. Further, If $a_{i-1}, a_{i-2}$ are adjacent to $c$, then $a_i$ is adjacent to $c$, for otherwise, $\{c, s, a_{i-1}, a_{i-2}, a_i\}$ induce a Bull in $G$ for any $s \in S$. Thus, by induction, $C$ is complete to $A'$. Further, $A'$ is anticomplete to $A \backslash A'$, for otherwise, there would be an $x - a$ path extending to $A \backslash A'$.

Define $Z = S \cup A' \cup X$, then $Z$ is a non-trivial module of size at least two, complete to $C$ and anticomplete to $A \backslash A'$. Thus $G$ is non-prime, by Lemma 2.3.1. □

Finally, we prove that non-basic Bull-graphs are non-prime.

**Theorem 4.2.8.** *All non-basic graphs are not substitution prime.*

*Proof.* Let $G$ be a non-basic graph, and by switching to $\overline{G}$ if necessary, assume that $H$ is a hole in $G$ with $|H| = k$. Let $c, a \in V(G) \backslash H$ be complete and anti-complete to $H$ respectively. Pick any $x \in V(G) \backslash H$ neither complete nor anti-complete to $H$. We need to find $u, v, w \in H$, such that $\{x, u, v, w\}$ induces one of the labelled graphs in Figure 4.1.

Assume without loss of generality that $(x, h_1) \in E(G)$ and $(x, h_2) \notin E(G)$ for some $h_1, h_2 \in H$. If $(x, h_k) \in E(G)$, then set $(u, v, w) = (h_k, h_1, h_2)$ to get the first graph in Figure 4.1. Otherwise, $(x, h_k) \notin E(G)$. In this case, setting $(u, v, w) = (h_k, h_2, h_{k-1})$ if $(x, h_{k-1}) \in E(G)$, and $(u, v, w) = (h_2, h_{k-1}, h_1)$ otherwise, induces the second graph from Figure 4.1 in $G$. Thus, $H$ is a split set, and so by the previous lemma, G is not a substitution prime graph. □

Thus, the theorems in this section imply that the bull graph has the EH-property with $\gamma(Bull) \geq 1/4$. The tightness of this bound can be shown by the following example from [6]: Consider, for any $m \in \mathbb{N}$, the $K_3$-free graph $T$, constructed by Kim [23] with $|T| = m$, as we saw in Section 2.1. Then, $T$ is bull-free as well. Construct the graph $G$ by substituting $\overline{T}$ in every vertex of $T$. Then, $G$ is bull-free by a proof similar to that of Theorem 2.4.3 as the bull graph is prime and as both $T, \overline{T}$ are bull-free. Further, $hom(G) = \max\{\ \omega(T)\omega(\overline{T}),\ \alpha(T)\alpha(\overline{T})\ \} = \omega(T)\alpha(T) \leq 2hom(T)$, as $T$ is triangle free. So, $hom(G) \leq 18\sqrt{|T| \log |T|} = 18(|G| \log |G|)^{1/4}$, where $|G| = m^2\ \forall m \in \mathbb{N}$. Now, similar to the triangle-free case in Section 2.1, it follows that $\gamma(Bull) \leq 1/4$.

## 4.3 The EH-property for $C_5$

The Erdős-Hajnal Property for $C_5$, the cycle on five vertices, was proved recently by Maria Chudnovsky, Alex Scott, Paul Seymour, and Sophie Spirkl in their paper - "Erdős-Hajnal for graphs with no 5-hole" [9]. It was an important result, as for a long time it was felt that the EH-property might not hold true for the case of $C_5$. However, in an interesting proof by contradiction, it was shown that $C_5$ has the EH-property. More importantly, $C_5$ is the first non-perfect graph which has the EH-property. By a theorem of Lovazs [27], the substitution operation preserves the perfectness of graphs, and hence all previously known cases of $P_4$, bull, etc only proved the EH-property for some perfect graph $H$.

Indeed, the nagging case of $C_5$ was pointed out by Gryarfas as well, in [20] where he mentions the following discussion with Erdős:

> "We have the following problem with Hajnal. If $G(n)$ has $n$ points and does not contain induced $C_4$, is it true that it has either a clique or an independent set with $n^\varepsilon$ points?...": E.P.
>
> ... I realized soon that $\frac{1}{3}$ is a good $\varepsilon$ ... About a month later Paul arrived and said he meant $C_5$ for $C_4$. And this minor change of subscript gave a problem still unsolved...

In this section, we present the proof by Chudnovsky et al. The proof goes essentially via contradiction, and makes use of the structure of a comb in a graph, as we shall see. These theorems shall be useful to make some headway on working towards the EH-property for $P_5$ as well, as we shall show in the upcoming chapter. Let us begin by defining a comb.

**Definition 4.3.1.** *Given constants $t, k \geq 0$, the set $\{(a_i, B_i) : 1 \leq i \leq t\}$ is said to be a $(t,k)$-comb in $G$ if the following properties hold:*

- *All $a_i$ are distinct vertices in $G$.*
- *All $B_i$ are pairwise disjoint subsets of $V(G) \backslash \{a_1, \ldots a_t\}$, such that $|B_i| \geq k$.*
- *$B_i \subset N(a_i)$ and $B_j \cap N(a_i) = \phi, \quad \forall i \neq j$.*

We shall refer to the sets $B_1, B_2, \ldots B_t$ as the *teeth* of the comb and the set $\mathcal{A} \coloneqq \{a_1, a_2, \ldots\}$ as the *head* of the comb. Note that the definition adds no restriction on edges lying inside the sets $B_1, \ldots, B_t$ or $\mathcal{A}$. Futher, given two disjoint subsets $X, Y \subset V(G)$ in a graph $G$, we say that $G$ has a $(t,k)$-*comb in* $(X, Y)$ if $\mathcal{A} \subset X$ and $B_i \subset Y \quad \forall i \leq t$. The following lemma talks about the existence of combs in a given graph.

**Lemma 4.3.1.** *[9] Let $\Delta, c, d > 0$ be fixed constants with $d < 1$. Further, let $G$ be a graph with two disjoint subsets of vertices, $A$ and $B$, such that every vertex in $B$ has a neighbour in $A$ and every vertex in $A$ has at most $\Delta$ neighbours in $B$. Then, at least one of the following holds:*

- *$G$ has a $(t,\ c\ t^{-1/d})$-comb in $(A, B)$ for some $t > 0$.*
- *$|B| \leq \dfrac{3^{d+1}}{3/2 - (3/2)^d}\ c^d\ \Delta^{1-d}$.*

*Proof.* Suppose we are given a graph $G$, with disjoint subsets of vertices $A, B$ as described above. We begin by partitioning $B$ into $k$-parts, $C_1, C_2, \ldots C_k$, such that every vertex in $A$ has at most $(2/3)^i\Delta$ neighbours in $D_i \coloneqq B \backslash (C_1 \cup \ldots \cup C_i)$ for all $i \leq k$. This partition is constructed as follows:

Let $C_0 = \phi$ and $D_0 = B$. Now suppose $C_0, C_1, C_2, \ldots, C_{s-1}$ have already been chosen, and the set $C_s$ has to be chosen from $D_{s-1}$ (the remaining vertices of $B$). Note that every vertex in $A$ has at most $(2/3)^{s-1}\Delta$ neighbours in $D_{s-1}$. Order all vertices of $A$ in a manner, such that for some $m \leq |A|$, The sequence $A_s \coloneqq \{a_1, a_2, \ldots, a_m\} \subset A$ is of maximal length with the following property: For each $i \leq m$, the vertex $a_i$ has at least $(2/3)^s\Delta$ adjacent vertices in $D_{s-1}$ which are not adjacent to any of $a_1, a_2, \ldots, a_{i-1}$. Define $C_s \coloneqq D_{s-1} \cap N(A_s) \subset B$.

By construction, every vertex in $A$ has at most $(2/3)^s\Delta$ neighbours in $D_s$: for otherwise, some vertex $a \notin A_s$ would have at least $(2/3)^s\Delta$ neighbours in $D_s$, which are not adjacent to any vertex of $A_s$, thereby contradicting the maximality of $m$ above. Further, as $|A_s| \geq 1$ whenever $D_{s-1} \neq \phi$, and as every vertex of $B$ has a neighbour in $A$, the process terminates in finitely many steps once $B$ is covered, thereby returning a partition $C_1, C_2, \ldots, C_k$ of $B$.

Consider any part $C_s$ from the partition above, and let $A_s \subset A$ be defined as in the construction above, with $|A_s| = m$. For all $i \leq m$, Let $P_i \subset D_{s-1}$ be the set of vertices adjacent to $a_i$ but non-adjacent to $a_j$ for $j < i$. Similarly, let $Q_i \subset D_{s-1}$ be the neighbours of $a_i$ in $D_{s-1} \backslash P_i$. By definition of $A_s$, we have $|P_i| \geq (2/3)^s\Delta$, and as every vertex of $A$ has at most $(2/3)^{s-1}\Delta$ neighbours in $D_{s-1}$, the cardinality $|Q_i| \leq (2/3)^s\Delta/2$.

Now, going backwards from $a_m$, we shall call $a_i \in A$ to be *good* if at most $|P_i|/2$ vertices of $P_i$ are adjacent to some good vertices of $\{a_{i+1}, \ldots, a_m\}$. Let $I \subset [m]$ be the indexing set for good vertices, and let $Q = \cup_{i \in I} Q_i$. Note that if $j \notin I$, then at least $|P_j|/2$ elements of $P_j$ are adjacent to some $a_k$, for $k > j, k \in I$. Thus, these elements lie in $Q_k$, and hence in $Q$. We get the following estimate for $|Q|$:

$$(m - |I|) \left(\frac{2}{3}\right)^s \frac{\Delta}{2} \;\leq\; |Q| \;\leq\; |I| \left(\frac{2}{3}\right)^s \frac{\Delta}{2}$$

Thus, we have $|I| \geq m/2$. Now, let $B_i = P_i \backslash Q$ for $i \in I$. Then, by definition of good vertices, we have $|B_i| \geq (2/3)^s\Delta/2$. Clearly, the good elements $a_i$, along with their sets $B_i$, form a $(|I|, (2/3)^s\Delta/2)$-comb in $G$. Set $t = |I|$. If $(2/3)^s\Delta/2 \geq ct^{-1/d}$, then we are done. otherwise, we have $t = |I| \leq [(3/2)^s 2c/\Delta]^d$. As $m \leq 2|I|$ and $C_s = D_{s-1} \cap N(A_s)$, we have:

$$|C|_s \;\leq\; 2^{d+1} \left(\frac{2}{3}\right)^{s-sd-1} c^d \Delta^{1-d}$$

If no $C_s$ contains a $(t, ct^{-1/d})$ comb, then, the latter conclusion of the theorem follows by summing the above inequality over all values of $s$. □

The previous lemma instructed us on the existence of combs in a given graph $G$. The proof for the EH-property for $C_5$, essentially goes via contradiction, wherein the properties of a minimal counter-example are used to reach an impossibility. These minimal counter-examples satisfy the property of $\tau$-criticality which can be defined as follows:

**Definition 4.3.2.** *Given a constant $\tau > 0$, we say that a graph $G$ is $\tau-critical$ if $\alpha(G)\omega(G) < |G|^\tau$ and $\alpha(G')\omega(G') \geq |G'|^\tau$ for all proper induced subgraphs $G' \leq G$.*

Note that $G$ is $\tau$-critical if and only if $\overline{G}$ is $\tau$-critical. It is also possible to refine the structure of a comb in a given graph. We can define the notion of a tied comb as follows:

**Definition 4.3.3.** *A $(t,k)$-comb $\{(a_i, B_i)\}$ in a graph $G$ is said to be a* tied $(t,k)$-comb, *if the vertices $a_i$ are all pairwise independent, and there exists $v \in V(G)\backslash \cup_i (\{a_i\} \cup \{B_i\})$ such that $v$ is adjacent to all $a_i$ and anti-complete to all $B_i$.*

*We shall call the vertex $v$, the* knot *of the tied comb.*

The following theorem discusses the existence of tied combs in linear-sized induced subgraphs of $\tau$-critical graphs under certain conditions, as dictated by Rödl's theorem.

**Theorem 4.3.2.** *[9] Given constants $\sigma, b > 0$ with $\sigma < 0.05$, there exists $\tau \in (0,1)$ with the following property: If $G$ is a $\tau$-critical graph, $X$ is a subset of vertices of $G$ such that $|X| > b|G|$ and $G[X]$ has a maximum degree no more than $\sigma b|G|$, then $G[X]$ has a tied $(t, \dfrac{b|G|}{400\sigma t^2})$-comb with $t \geq 1/400\sigma$*

*Proof.* Let $\sigma, b > 0$ be given as above. Pick a $\tau > 0$, such that the following holds:

$$\frac{2^{1-1/\tau}}{b} + \left(\sigma + \frac{19}{20}\right)(\sigma b)^{-\tau} < 1 \tag{4.3}$$

Such a $\tau > 0$ exists, as the left-hand side of the inequality tends to $\sigma + \frac{19}{20}$ as $\tau$ tends to zero, and $\sigma + \frac{19}{20} < 1$ by the restriction on $\sigma$. Now suppose that $G$ is a $\tau$-critical graph, and let $X \subset V(G)$ be such that $|X| > b|G|$ and $G[X]$ has minimum degree at least $\sigma b|G|$. We partition the set $X$ as follows:

Set $X_0 = X$. Given $X_{i-1}$, pick a vertex $v_i \in X_{i-1}$ with the largest degree in $G[X_{i-1}]$. Let $A_i = N(v_i) \cap X_{i-1}$ and $C_i$ be the largest independent set in $A_i$. Finally, let $D_i = N(C_i)\backslash(A_i \cup \{v_i\})$ and set $X_i = X_{i-1}\backslash(\{v_i\} \cup A_i \cup D_i)$. Repeat this until $X_i = \phi$. Note that the sets $Y_i := \{v_i\} \cup A_i \cup D_i$ partition the set $X$. Let the number of parts be $s$.

Set $\lambda = \dfrac{b}{400\sigma}$. Now, consider any $Y_i$. By $\tau$-criticality, we have:

$$|C_i| \;=\; \alpha(A_i) \;\geq\; \frac{|A_i|^\tau}{\omega(A_i)} \;\geq\; \frac{|A_i|^\tau}{\omega(G)}$$

We now make use of Lemma 4.3.1. Set $c = \lambda|X|/b$, $\Delta = |A_i|$, and $d = 0.5$. Note that every vertex in $C_i$ has at most $d(v_i) = |A_i|$ neighbours in $D_i$. If there is a $(t, ct^{-2})$-comb in $(C_i, D_i)$, then it is a tied-comb with the knot $v_i$, of required size, and the condition that all the teeth of the comb lie in $X$ implies that $t \cdot ct^{-2} \leq |X|$ or, $t \geq 1/400\sigma$. If such a comb does not exist, then, by Lemma 4.3.1, we have:

$$|D_i| \leq \frac{3^{3/2}}{3/2 - (3/2)^{1/2}} \left(\lambda|X|\frac{|A_i|}{b}\right)^{\frac{1}{2}} \leq 19 \left(\lambda|X|\frac{|A_i|}{b}\right)^{\frac{1}{2}} \tag{4.4}$$

suppose for contradiction, none of the $Y_i$s have a tied-comb in $(C_i, D_i)$ as mentioned above, making the previous bounds on $D_i$ applicable $\forall i \leq s$. As the family of sets $\{Y_i : i \leq s\}$ partitions $X$, and each $Y_i = \{v_i\} \cup A_i \cup D_i$, we have:

$$\sum_{i=1}^{s} \frac{|Y_i|}{|X|} \;=\; \sum_{i=1}^{s} \left(\frac{1}{|X|} \;+\; \frac{|A_i|}{|X|} \;+\; \frac{|D_i|}{|X|}\right) \;=\; 1 \tag{4.5}$$

Let us calculate each of the three terms above. For the first term, note that the vertices $v_i$ form an independent set in $G$, and so $s \leq \alpha(G) \leq |G|^\tau/\omega(G) \leq |G|^\tau$. Further, as $\alpha(G)\omega(G) \geq 2$, it follows from $\tau$-criticality, that $|G| \geq 2^{1/\tau}$. Combining these inequalities, we have:

$$\sum_{i=1}^{s} \frac{1}{|X|} \;\leq\; \frac{|G|^\tau}{b|G|} \;\leq\; \frac{2^{1-1/\tau}}{b} \tag{4.6}$$

Next, set $a_i = \frac{|A_i|}{|X|}$. Observe that $|C_i|\omega(G) \;\geq\; (x_i b|G|)^\tau \;\geq\; (x_i b)^\tau \alpha(G)\omega(G)$, by $\tau$-criticality of $G$. Further, the union of all $C_i$s forms an independent set by construction, and thus $\sum\limits_{i=1}^{s} |C_i| \;\leq\; \alpha(G)$. Combining both these inequalities, we have: $\sum\limits_{i=1}^{s} x_i^\tau \;\leq\; b^{-\tau}$. On the contrary, as $G[X]$ has maximum degree $\sigma b|G|$, we have, $|A_i| = d(v_i) \leq \sigma b|G|$. This gives an upper bound on $x_i$ as follows: $x_i = \dfrac{|A_i|}{|X|} \;\leq\; \dfrac{\sigma b|G|}{b|G|} = \sigma$. we have the following:

$$\sum_{i=1}^{s} \frac{|A_i|}{|X|} \;=\; \sum_{i=1}^{s} (x_i)^\tau (x_i)^{1-\tau} \;\leq\; \sigma^{1-\tau} \sum_{i=1}^{s} x_i^\tau \;=\; \sigma(\sigma b)^{-\tau} \tag{4.7}$$

Finally, using 4.4, and a technique similar to the inequality above, we have the following bound for the third term:

$$\sum_{i=1}^{s} \frac{|D_i|}{|X|} \leq 19 \left(\frac{\lambda}{b}\right)^{\frac{1}{2}} \sum_{i=1}^{2} \sqrt{x_i} \;\leq\; \frac{19}{20}(\sigma b)^{-\tau} \tag{4.8}$$

Substituting inequalities 4.6, 4.7, and 4.8, into Equation 4.5, we obtain the following contradiction to our choice of $\tau$ in 4.3:

$$\frac{2^{1-1/\tau}}{b} \;+\; \left(\sigma \;+\; \frac{19}{20}\right)(\sigma b)^{-\tau} \;\geq\; 1$$

$\square$

Note that if the above theorem holds for some $\tau = \tau_0$, then it holds for all constants $\tau \in (0, \tau_0)$. Hence, it is possible to choose small enough non-zero values of $\tau$ for which the theorem holds. We shall use this fact while proving the EH-property for $C_5$ below. The above theorem is essentially used in the following proof to find a large homogeneous set in a $\tau$-critical $C_5$-free graph $G$, thereby leading to a contradiction.

We end this chapter with the proof of the EH-property for $C_5$.

**Theorem 4.3.3.** *[9] The cycle on five vertices, $C_5$, has the EH-property.*

*Proof.* Fix some $\sigma < 1/400$ and $b > 0$ such that Theorem 3.2.3 is satisfied for $\sigma$ and $k = 5$. Now choose a constant $0 < \tau < 1/2$, small enough so that Theorem 4.3.2 holds and the following inequality is satisfied:

$$\frac{b}{400\sigma} \;\geq\; (400\sigma)^{\frac{1}{\tau}-2} \tag{4.9}$$

Choosing such a small enough $\tau$ is made possible by the remark above, and due to the fact that $400\sigma < 1$ makes the right side of the inequality strictly increasing in $\tau$ and tending to zero at $\tau = 0$.

**Claim:** $C_5$ has the EH-property with $\gamma(C_5) \geq \tau/2$.

Suppose Not! Then, there exist $C_5$-free graphs $G$ with $hom(G) < |G|^{\tau/2}$, and hence $\alpha(G)\omega(G) < |G|^{\tau}$. Let $G$ be a graph satisfying the latter inequality with a minimal number of vertices. Then, $G$ is $\tau$-critical, as for all proper induced subgraphs $G' < G$, we have that $\alpha(G')\omega(G') \geq |G'|^\tau$. By Theorem 3.2.3, there exists $X \subset V(G)$ such that $|X| \geq b|G|$ and either $G[X]$ or $\overline{G}[X]$ has maximum degree no more than $\sigma b|G|$. As $\tau$-criticality and $C_5$-freeness are preserved under taking complements, we can choose our $\tau$-critical graph $G$ such that $G[X]$ has maximum degree $\sigma b|G|$, by switching $G$ and $\overline{G}$ if necessary.

Applying Theorem 4.3.2 to this graph, we see that $G$ has a tied $(t, \dfrac{b\ |G|}{400\sigma\ t^2})$-comb in G[X] with $t \geq 1/400\sigma$. Consider any tooth $B_i$ of the tied-comb, then we have:

$$
\begin{aligned}
\alpha(B_i)\omega(B_i) \ &\geq |B_i|^\tau \ \geq \left(\frac{b}{400\sigma\ t^2}\right)^\tau \cdot |G|^\tau && \text{(By $\tau$-criticality of graph G)} \\
&\geq (400\sigma)^{\tau(\frac{1}{\tau}-2)} \cdot \frac{|G|^\tau}{t^{2\tau}} && \text{(From Equation (1))} \\
&\geq t^{2\tau-1} \cdot \frac{|G|^\tau}{t^{2\tau}} \ = \ \frac{1}{t}|G|^\tau && \text{(As $t \geq \dfrac{1}{400\sigma}$ and $\tau \leq 1/2$)}
\end{aligned}
$$

Further, let $v \in V(G)$ be the knot of the tied comb, and pick any two vertices $b_i \in B_i$ and $b_j \in B_j$ from two distinct teeth. If $b_i, b_j$ are adjacent, then the vertices $\{v, a_i, a_j, b_i, b_j\}$ induce a $C_5$ in $G$, by the structure of a tied-comb. As $G$ is $C_5$-free, all $B_i$ are pairwise anti-complete and thus, $\alpha(G) \geq \sum_{i=1}^t \alpha(B_i)$. Consequently, we have:

$$
\alpha(G)\omega(G) \ \geq \ \sum_{i=1}^{t} \alpha(B_i)\omega(G) \ \geq \ \sum_{i=1}^{t} \alpha(B_i)\omega(B_i) \ \geq \ \sum_{i=1}^{t} \frac{1}{t}|G|^\tau \ = \ |G|^\tau
$$

Thus, $\alpha(G)\omega(G) \geq |G|^\tau$, thereby contradicting the $\tau$-criticality of $G$. Thus, by contradiction, $C_5$ has the EH-property with $\gamma(C_5) \geq \tau/2$. $\square$

# Chapter 5

# Towards the EH-Property for $P_5$

Presently, $P_5$, the path on five vertices, and its complement, are the only graphs with at most five vertices for which the EH-property is not known. In this chapter, we attempt to make some progress towards investigating the EH-property for these graphs, specifically, for $P_5$. This chapter comprises entirely original work.

## 5.1 Reduction to Dense Graphs

Let us revisit the proof of Theorem 4.3.3. It can be observed that there are precisely two instances where individual properties of the graph $C_5$ get used while proving the EH-property:

1. To show that the teeth of the tied comb are all pairwise disjoint, for otherwise, we could find an induced $C_5$ in the given $C_5$-free graph; and

2. To make Theorem 4.3.2 applicable in both the cases generated by Rödl's theorem, by allowing to jump between $G$ and $\overline{G}$, while preserving the $C_5$-freeness. This is possible as $C_5$ is self-complementary.

The former of these instances was useful in constructing a large independent set in the chosen $C_5$-free $\tau$-critical graph in order to reach a contradiction. The construction was primarily facilitated by the structure of the tied-comb and the $C_5$-freeness of the graph as follows: If $v$ is the knot of the tied-comb, and $b_i, b_j$ are vertices from two distinct teeth, Then consider the set of vertices $S = \{v, a_i, a_j, b_i, b_j\}$ in the graph $G$. By the definition of a comb, $(a_i, b_i), (a_j, b_j) \in E(G)$ and $(a_i, b_j), (a_j, b_i) \notin E(G)$. further, $v$ is adjacent to $a_i, a_j$, but is non-adjacent to $b_i, b_j$; and $(a_i, a_j) \notin E(G)$ as they form an independent set. As a result, $S$ would induce a $C_5$ in $G$ if $b_i, b_j$ are adjacent vertices. It follows from the $C_5$-freeness of $G$, that all $B_i$ are pairwise anti-complete, and so each of their independent sets, of size $\alpha(B_i)$, add up to return a large independent set in $G$.

We make the following observation. Note that other than $(b_i, b_j)$, all other edges and non-edges in $S$ have been fixed by the structure of the tied comb, and we saw that $S$ induces a $C_5$ if $b_i, b_j$ are adjacent. However, if $b_i, b_j$ are non-adjacent, then the same set of vertices $S$ induces a $P_5$ in the graph $G$! Thus now, the teeth of a tied comb in any $P_5$-free graph must be pairwise complete, and so the cliques in $B_i$ can be merged, instead of independent sets, to obtain a sufficiently large clique in the $P_5$-free graph $G$.

However, the proof for $C_5$ does not entirely follow for $P_5$, simply because $P_5$ is not self-complementary, and thus does not satisfy the second instance above. Stated otherwise, the case from the proof of Theorem 4.3.3, where $\overline{G}[X]$ has a bounded maximum degree, does not get resolved without the self-complementarity of $C_5$, and hence, stays unresolved while extending the proof to $P_5$. Despite several attempts, we found it difficult to find a similar version of Lemma 4.3.2, which works for the case above. Nevertheless, we obtain an equivalent reduction of the EH-property for $P_5$ in terms of 'dense' graphs, as we shall prove next.

Let us begin by making precise the notion of *dense graphs* used by us. Classically, they contain the set of all graphs with $\Theta(|G|^2)$ many edges. However, we shall restrict ourselves to the subclass of such graphs with a minimum degree at least $\beta \cdot |G|$ for some constant $\beta > 1/2$. More formally,

**Definition 5.1.1.** *A Graph $G$ is defined to be $\beta$-dense for some constant $\beta > \frac{1}{2}$, if it has minimum degree at least $\beta \cdot |G|$.*

We shall occasionally use the term dense graphs to denote $\beta$-dense graphs when the choice of $\beta$ is unclear or immaterial. Consider the following version of the EH-property restricted to dense graphs:

**Definition 5.1.2.** *A Graph $H$ is said to have the* dense EH-property*, if there exist constants*

$\beta > 1/2, c > 0$, *and* $N_0 \in \mathbb{N}$, *such that every* $\beta$*-dense* $H$*-free graph on at least* $N_0$ *vertices has* $hom(G) \geq |G|^c$.

We prove the following theorem for $P_5$, thereby giving an equivalent, but reduced version of the $EH$-property for $P_5$.

**Theorem 5.1.1.** *$P_5$ has the dense EH-property if and only if it has the EH-property.*

*Proof.* If $P_5$ has the EH-property, then it also has the dense EH-property, since every $P_5$-free graph $G$, including dense graphs, has $hom(G) \geq |G|^{\gamma(P_5)}$. It then suffices to show that the dense EH-property for $P_5$ implies the EH-property as well.

So, suppose $P_5$ has the dense EH-property. Then, there exist constants $\beta > 1/2, c > 0$ and $N_0 \in \mathbb{N}$ such that, every large enough $P_5$-free $\beta$-dense graph $G$ with $|G| \geq N_0$ has $hom(G) \geq |G|^c$. Fix some constant $0 < \sigma < \min\{1/400, 1-\beta\}$ and choose $b > 0$ such that Theorem 3.2.3 holds for the chosen value of $\sigma$ and $k = 5$. Then, for every $P_5$-free graph G, there exists $X_G \subset V(G)$ such that $|X_G| \geq b|G|$ and either $G[X_G]$ or $\overline{G}[X_G]$ has maximum degree no more than $\sigma b|G|$. We choose one of these possibilities for each such graph $G$ and partition the class of $P_5$-free graphs into two disjoint sets $\mathcal{G}_1$ and $\mathcal{G}_2$, as follows:

$$\mathcal{G}_1 = \{G \in \mathcal{G},\ P_5\text{-free } : G[X_G] \text{ has maximum degree at most } \sigma b|G|\ \}$$

$$\mathcal{G}_2 = \{G \in \mathcal{G},\ P_5\text{-free } : \overline{G}[X_G] \text{ has maximum degree at most } \sigma b|G|\ \}$$

We look at graphs in each of these parts separately.

**Case I:** $G$ is a $P_5$-free graph with $G \in \mathcal{G}_2$.

Let $0 < \tau_1 < c$. Assume $G$ is large enough so that the following hold:

$$b|G| - 1 \approx b|G| \text{ and } |G| \geq \max\left\{\frac{N_0}{b},\ \left(\frac{1}{b}\right)^{\dfrac{c}{c-\tau_1}}\right\}.$$

Let $N$ be the size of the smallest graph in $\mathcal{G}_2$ that satisfies these conditions. Note that the conditions ensure that for all graphs $G \in \mathcal{G}_2$ on at least $N$ vertices, we have $|X_G| \geq N_0$ and $(b|G|)^c \geq |G|^{\tau_1}$. Now, as $\overline{G}[X_G]$ has maximum degree at most $\sigma b|G|$ and $|X_G| \geq b|G|$, we see that the $P_5$-free graph $G[X_G]$ has minimum degree given by:

$$\delta(G[X_G]) \;=\; |X_G| - 1 - \Delta(\overline{G}[X_G]) \;\geq\; |X_G| - \sigma b|G| \;\geq\; (1-\sigma)|X_G| \;\geq\; \beta|X_G|$$

Thus, $G[X_G]$ is $\beta$-dense with $|X_G| \geq N_0$. By the dense EH-property, we see that for all graphs $G \in \mathcal{G}_2$ with $|G| \geq N$, we have:

$$hom(G) \;\geq\; hom(G[X_G]) \;\geq\; |X_G|^c \;\geq\; (b|G|)^c \;\geq\; |G|^{\tau_1}$$

Now, as we have done earlier, we shift from this asymptotic statement to its non-asymptotic version, which is crucial for the use of $\tau$-criticality in the next case. Recall that in Section 2.4, we defined for every graph $G$, the constant $\delta_G = \log(hom(G))/\log(|G|)$ such that $hom(G) = |G|^{\delta_G}$. Then define $\tau^*$ as follows:

$$\tau^* \;\coloneqq\; \min\{\tau_1,\; \min\{\delta_G \;:\; G \in \mathcal{G}_2 \text{ and } |G| < N\}\}$$

As there are finitely many graphs on at most $N$ vertices and as each $\delta_G > 0$, the last term is non-zero, thereby making $\tau^* \neq 0$. Hence, for all graphs $G \in \mathcal{G}_2$, we have $hom(G) \geq |G|^{\tau^*}$ for some constant $\tau^* > 0$.

**Case II:** $G$ is a $P_5$-free graph with $G \in \mathcal{G}_1$.

Now choose a constant $0 < \tau < \min\{\frac{1}{2}, \tau^*\}$, small enough so that Theorem 4.3.2 holds for the chosen $\sigma, b$; and the following inequality is satisfied:

$$\frac{b}{400\sigma} \;\geq\; (400\sigma)^{\frac{1}{\tau}-2} \tag{5.1}$$

We have seen in the proof of Theorem 4.3.3 that choosing such a $\tau$ is indeed possible.

Suppose, for contradiction, that there exist $P_5$-free graphs $G \in \mathcal{G}_1$ with $hom(G) < |G|^{\tau/2}$, and hence $\alpha(G)\omega(G) < |G|^{\tau}$. Then, there is a graph $G \in \mathcal{G}_1$ satisfying the latter inequality with a minimal number of vertices. Every proper induced subgraph $G' < G$ is such that either $G \in \mathcal{G}_1$ and $\alpha(G')\omega(G') \geq |G'|^{\tau}$ by minimality; or $G' \in \mathcal{G}_2$. In the latter case, we have $\alpha(G')\omega(G') \;\geq\; hom(G') \;\geq\; |G'|^{\tau^*} \;\geq |G'|^{\tau}$. Thus, by definition, $G$ is $\tau$-critical and satisfies the requirements of Theorem 4.3.2 since $G \in \mathcal{G}_1$.

Applying Theorem 4.3.2 to this graph, we see that $G$ has a tied $(t, \dfrac{b\;|G|}{400\sigma\; t^2})$-comb in

$G[X_G]$ with $t \geq 1/400\sigma$. Consider any tooth $B_i$ of the tied-comb, then we have:

$$\begin{aligned}
\alpha(B_i)\omega(B_i) \geq |B_i|^\tau \ &\geq \left(\frac{b}{400\sigma\ t^2}\right)^\tau \cdot |G|^\tau && \text{(By } \tau\text{-criticality of graph G)} \\
&\geq (400\sigma)^{\tau(\frac{1}{\tau}-2)} \cdot \frac{|G|^\tau}{t^{2\tau}} && \text{(From Equation (1))} \\
&\geq t^{2\tau-1} \cdot \frac{|G|^\tau}{t^{2\tau}} \ = \ \frac{1}{t}|G|^\tau && \text{(As } t \geq \frac{1}{400\sigma} \text{ and } \tau \leq 1/2)
\end{aligned}$$

Further, let $v \in V(G)$ be the knot of the tied comb, and pick any two vertices $b_i \in B_i$ and $b_j \in B_j$ from two distinct teeth. If $b_i, b_j$ are non-adjacent, then the vertices $\{v, a_i, a_j, b_i, b_j\}$ induce a $P_5$ in $G$, by the structure of the tied-comb. As $G$ is $P_5$-free, all $B_i$ are pairwise complete and thus, $\omega(G) \geq \sum_{i=1}^{t} \omega(B_i)$. Consequently, we have:

$$\alpha(G)\omega(G) \ \geq \ \sum_{i=1}^{t} \alpha(G)\omega(B_i) \ \geq \ \sum_{i=1}^{t} \alpha(B_i)\omega(B_i) \ \geq \ \sum_{i=1}^{t} \frac{1}{t}|G|^\tau \ = \ |G|^\tau$$

Thus, $\alpha(G)\omega(G) \geq |G|^\tau$, thereby contradicting the $\tau$-criticality of $G$. Thus, by contradiction, all graphs $G \in \mathcal{G}_1$ have $hom(G) \geq |G|^{\tau/2}$. Further, as $\tau \leq \tau^*$, it follows from Case I and Case II, that all $P_5$-free graphs have $hom(G) \geq |G|^{\tau/2}$ and consequently, $P_5$ has the EH-property with $\gamma(P_5) \geq \tau/2 > 0$. □

Thus, we can conclude that in order to prove the EH-property for $P_5$, we can restrict ourselves to working with dense $P_5$-free graphs. However, note that while there could be more ways to prove the EH-property for a graph $H$; having a class of graphs closed under complements is often more helpful in finding a proof, simply because homogeneous sets are preserved under complements as well. Take for instance, the class of $\mathcal{H}$-free graphs with $\mathcal{H} = \{P_5, \overline{P_5}\}$. This is a subclass of $P_5$-free graphs, with the added benefit of being closed under complements. Thus, using a method similar to the proof of Theorems 4.3.3 and 5.1.1, it can be shown that $\mathcal{H}$ has the EH-property as well. Stated otherwise, It can be shown that $\mathcal{H} = \{P_5, \overline{P_5}\}$ had the EH-property as well:

**Theorem 5.1.2.** *There exists a constant $c > 0, N \in \mathbb{N}$, such that every $\{P_5, \overline{P_5}\}$-free graph $G$ on at least $N$ vertices has $hom(G) \geq |G|^c$.*

The above result was first proven by Gyárfás in [20], where he used a generalised notion of *strong perfectness* to show that $\chi(G) \leq \omega(G)^2$ for all $\{P_5, \overline{P_5}\}$-free graphs $G$. The *reflections* by Gyárfás also motivated the search for polynomial-sized homogeneous sets in $\{H, \overline{H}\}$-free graphs, for any given graph $H$, and presently, the EH-property is known to be true for many such sets of the form $\mathcal{H} = \{H, \overline{H}\}$. For instance, in 2014, Maria Chudnovsky and Paul Seymour [10] proved that $\mathcal{H} = \{P_6, \overline{P_6}\}$ had the EH-property by showing that $\mathcal{H}$-free graphs are $\alpha$-narrow. Around the same time, N. Bousquet, A. Lagoutte and S. Thomassé, [4] independently showed that the set $\{P_k, \overline{P_k}\}$ has the EH-property for all $k \in \mathbb{N}$. They used the notion of *bicliques* (also known as *pure pairs*) and the Rödl's Theorem (Theorem 3.2.1) to find polynomial-sized perfect induced subgraphs in $\{P_k, \overline{P_k}\}$-free graphs. However, the ability to switch between a graph and its complement was an essential component in the proofs of all the results mentioned above.

## 5.2 Dense Graphs and Distance to Co-graphs

The previous section, specifically Theorem 5.1.1, suggests that we focus our attention on large homogeneous sets in dense $P_5$-free graphs. One method to do so, as we saw in the case of $P_4$ and the Bull, is to search for large induced perfect subgraphs in the given graph. In this section, we explore whether it is possible to find large co-graphs in dense $P_5$-free graphs.

Recall that co-graphs, a subset of perfect graphs, are precisely the set of $P_4$-free graphs, and any co-graph $G$ has a homogeneous set of size at least $\sqrt{|G|}$. Thus, if a graph can be made $P_4$-free by deleting a few vertices, then the above facts can be used to find large homogeneous sets in the graph. Keeping this in mind, we shall call a subset of vertices, $S \subset V(G)$, an *induced $P_4$ vertex cover* if it intersects the vertex set of every induced $P_4$ in the graph $G$. The minimum cardinality of such a set is also called the **distance to cograph** number of the graph $G$, denoted here by $IPVC_4(G)$.

Note that as $P_4$ is a self-complementary graph, every subset $S \subset V(G)$ induces a $P_4$ in $G$ if and only if it induces a $P_4$ in $\overline{G}$ as well. Hence, for all graphs $G$, we have $IPVC_4(G) = IPVC_4(\overline{G})$. Further, mapping all induced $P_4$s in the graph $G$ to its outer pair of edges gives an injective function from the set of induced $P_4$s in $G$ to $E(G) \times E(G)$, and thus every graph $G$ with $m$ edges contains at most $m^2$ many induced $P_4$s. Both these facts roughly suggest that the class of sparse, and hence dense graphs too, may have a small, perhaps linearly bounded, distance to co-graph number. This explains our motivation to look for large co-graphs in dense $P_5$-free graphs. More precisely, if there exists a constant $c > 0$

such that for all $\beta$-dense $P_5$-free graphs $G$, we have $IPVC_4(G) \leq c|G|$, then removing these few vertices gives us a $P_4$-free induced subgraph of size at least $(1-c)|G|$. Consequently, $hom(G) \geq \sqrt{(1-c)|G|} \geq |G|^{\tau}$ for some $\tau < 1/2$ and $|G|$ sufficiently large, thereby proving that $P_5$ has the dense EH-property, and hence the EH-property, as was required.

Let us first just focus on dense graphs to get a flavour of the problem. We shall look at $P_5$-free dense graphs later in the section. Recall that by *dense graphs*, we simply refer to $\beta$-dense graphs for some $\beta > 1/2$. We wish to determine if there exist constants $0 < c < 1$ and $\beta > 1/2$, such that for every $\beta$-dense graph $G$, we have $IPVC_4(G) \leq c|G|$. Unfortunately, we answer this question in the negative as follows:

**Lemma 5.2.1.** *For every constant $c \in (0,1)$, there is some graph $G$ for which the distance to co-graph number is greater than $c|G|$.*

*Proof.* Fix some $c \in (0,1)$. Suppose, for contradiction, that every graph $G$ has an induced $P_4$ vertex cover of size at most $c|G|$. Then, $G$ has an induced $P_4$-free subgraph of size at least $(1-c)|G|$, and thus has a homogeneous set of size at least $k = \sqrt{(1-c)|G|}$. Consequently, any graph on $\dfrac{k^2}{1-c}$ vertices contains a $K_k$ or an $I_k$, and hence the Ramsey number $R(k,k) \leq \dfrac{k^2}{1-c}$ $\forall\ k \in \mathbb{N}$. This contradicts Erdős' lower bound given by $R(k,k) > 2^{\frac{k}{2}}$ [15]. Thus, by contradiction, there exists some graph with $IPVC_4(G) \leq c|G|$. $\square$

Now that we have shown the existence of a graph with arbitrarily large distance to co-graph, we use it to construct $\beta$-dense graphs $G$ with $IPVC_4(G) > c|G|$, for any $\beta > 1/2$ and $c > 0$, as we had claimed earlier.

**Theorem 5.2.2.** *For all constants $c \in (0,1)$ and $\beta > 1/2$, there exists a $\beta$-dense graph $G$ for which $IPVC_4(G) > c|G|$.*

*Proof.* Fix constants $c, \beta$ as above. By the previous lemma, there exists a graph, say $H$, for which $IPVC_4(H) > c|H|$. Choose $m \in \mathbb{N}$ such that the following holds:

$$m > \frac{\Delta(H) + 1}{(1-\beta) \cdot |H|}$$

where $\Delta(H)$ refers to the maximum degree of $H$. Now, we construct a graph $G^*$ by taking $m$ disjoint copies of the graph $H$ and let the graph $G = \overline{G^*}$ be its complement. We prove that $G$ is the required graph. Note that $\Delta(G^*) = \Delta(H)$. Further,

$$IPVC_4(G) = IPVC_4(G^*) = \sum_m IPVC_4(H)$$

where the summation is taken over all $m$ disjoint copies of the graph $H$. Thus, $IPVC_4(G) > m \cdot c|H| = c|G|$, as was required. The theorem now follows by the following calculation of $\delta(G)$:

$$\delta(G) \;=\; |G| - 1 - \Delta(G^*) \;=\; |G| - (1 + \Delta(H)) \;>\; |G| - (1-\beta)m|H| \;=\; \beta|G|$$

$\square$

Thus, our question gets answered negatively when focusing on just dense graphs. Let us now enforce the additional restriction of $P_5$-freeness, as was originally required, and ask whether there exist constants $0 < c < 1$ and $\beta > 1/2$, such that every $\beta$-dense $P_5$-free graph $G$ has $IPVC_4(G) \le c|G|$. It is expected that both, the high density of edges and $P_5$-freeness, should act together to constrain the number of induced $P_4$s in the graph $G$, and provide an answer in the affirmative. However, once again, we show that our question takes on a negative answer.

We look at the following construction suggested by Dr Alex Scott to show that the constants $c, \beta$ above, do not exist. Let $G = C_5$, the cycle graph on five vertices, and let $S_G = \{G_n\}$ be the sequence of graphs obtained after successive substitutions, as defined in Section 2.4. As $C_5$ is $\overline{P_5}$-free and $\overline{P_5}$ is substitution prime, it follows from Lemma 2.4.3 that all $G_n \in S_G$ are $\overline{P_5}$-free as well. Further, we have $|G_n| = 5^{n+1}$, with $G_0 = C_5$.

Let us calculate the distance to co-graph $IPVC(G_i)$ for each $G_i \in S_G$ recursively. Recall that the graph $G_{i+1} \in S_G$ can be thought of as the graph $G$, each of whose vertices (called bags) contain a copy of the graph $G_i$. Let $X, X_i$, and $X_{i+1}$ be the smallest induced $P_4$ vertex covers of $G, G_i$, and $G_{i+1}$, respectively. Then, any induced $P_4$ vertex cover of $G_{i+1}$ must contain entire bags in $X \subset V(G)$ (to cover all induced $P_4$s in the skeleton graph $G$) and must also contain the vertices $X_i$ from each of the remaining bags in $V(G)\backslash X$ (to cover all induced $P_4$s in $G_i$s of the remaining bags). It can be checked that this construction gives the smallest induced $P_4$ vertex cover of $G_{i+1}$. Thus, we have the following recurrence relation:

$$IPVC_4(G_{i+1}) \;=\; IPVC_4(G) \cdot |G_i| \;+\; (|G| - IPVC_4(G)) \cdot IPVC_4(G_i)$$

By substituting $|G| = 5$ and $IPVC_4(G) = IPVC_4(C_5) = 2$, we obtain:

$$IPVC_4(G_{i+1}) \ = \ 2|G_i| \ + \ 3 \cdot IPVC_4(G_i)$$

Solving the recurrence relation using generating functions, with the initial condition $IPVC_4(G) = 2$, we obtain the following solution:

$$IPVC_4(G_i) \ = \ \left[1 - \left(\frac{3}{5}\right)^{i+1}\right] |G_i|$$

As the term $1 - (3/5)^{i+1} \to 1$ as $i \to \infty$, given any constant $c < 1$, we can find $i \in mathbbN$ such that $IPVC_4(G_i) \geq c|G_i|$. Thus, we have the following analogue of Lemma 5.2.1:

**Lemma 5.2.3.** *For every constant $c \in (0, 1)$, there is some $\overline{P_5}$-free graph $G$ for which the distance to co-graph number is greater than $c|G|$.*

Now, we proceed as in the proof of Theorem 5.2.2. Let $0 < c < 1, \beta > 1/2$ be given. By the previous theorem, there is a $\overline{P_5}$-free graph $G$, with $IPVC_4(G) \geq c|G|$. Construct the graph $\overline{G^*}$ by taking $m$ many disjoint copies of $G$, and consider the graph $G^*$. Observe that $\overline{G^*}$ is $\overline{P_5}$-free, and thus, the graph $G^*$ is $P_5$-free. The following theorem now follows by an appropriate choice of $m$:

**Theorem 5.2.4.** *For all constants $c \in (0, 1)$ and $\beta > 1/2$, there exists a $\beta$-dense $P_5$-free graph $G$ for which $IPVC_4(G) > c|G|$.*

Thus, contradictory to our intuitions, it is not possible to make all dense $P_5$-free graphs $G$ $P_4$-free, by deleting at most linearly many vertices. The above theorems also shed some light on the size of the largest induced cographs in a graph. Define $CoG(G)$ to be the size of the largest induced subgraph in $G$ which is a cograph. Note that $S \subset V(G)$ is a cograph in $G$ if and only if $V(G) \backslash S$ is an induced $P_4$ vertex cover in $G$. Thus, we have:

$$IPVC_4(G) > c|G| \iff CoG(G) < (1 - c)|G|$$

The previous theorem indicates that $IPVC_4(G)$ is not bounded above by $c|G|$ when restricted to the class of $\beta$-dense $P_5$-free graphs. Consequently, the $CoG(G)$ too is not bounded below by some linear function $(1 - c)|G|$ over the class of $\beta$-dense $P_5$-free graphs for any $\beta > 1/2$. Stated otherwise, we obtain the following theorem:

**Theorem 5.2.5.** *For all constants $c, \beta \in (0,1)$, there exists a $\beta$-dense $P_5$-free graph $G$ which contains induced cographs of size at most $c|G|$.*

Therefore, in this section, we attempted at trying to find linear-sized co-graphs in all dense $P_5$-free graphs. This is a much stronger requirement than what is claimed by the EH-property for $P_5$. Thus, finding an efficient vertex deletion algorithm to obtain a $P_4$-free subgraph might still be potentially useful. While we looked at linear-sized induced $P_4$-vertex covers, showing that $IPVC_4(G) \leq \left(1 - \frac{1}{|G|^c}\right)|G|$ for any $0 < c < 1$ is sufficient to prove the EH-property as well. We hope that the assumption of high density might be instrumental in proving the bounds above.

# Chapter 6

# Paley Graphs and Self-Complementarity

Over the past few chapters, we saw that the self-complementary nature of $H$ was highly useful in most of the proofs seen till now. This is expected, as the Erdős-Hajnal conjecture primarily deals with $hom(G)$, which is preserved under complements, and with cliques and independent sets, which switch roles on taking the complement. Thus, some freedom of switching from $G$ to $\overline{G}$ is highly desirable and can be achieved if and only if $H$ is self-complementary. In this section, we show that it is indeed possible to reduce the conjecture by focusing our attention on a class of self-complementary prime graphs. We shall define the *Paley* graphs, which are of interest to us, and use them to equivalently reduce the Erdős-Hajnal Conjecture in the last section. Proofs in the last section constitute original work. We first begin with a short background in quadratic residues of a finite field and their properties.

## 6.1 Quadratic Residues

Let $F$ be a field and let $F^\times$ be the set of non-zero elements in $F$. We are interested in studying the properties of the set of non-zero squares in this field, which are usually known as the quadratic residues of $F$.

**Definition 6.1.1.** *An element $x \in F^\times$ is defined to be a quadratic residue of $F$, if there exists an element $a \in F^\times$ such that $x = a^2$ in the field $F$. Else, $x$ is said to be a quadratic non-residue of $F$.*

The set of quadratic residues and the set of quadratic non-residues of a field $F$ shall be denoted as $Res(F)$ and $NRes(F)$, respectively. Further, the term *quadratic* shall be dropped whenever it does not call for any confusion. Note that by definition, the additive identity $0_F$ is neither a residue nor a non-residue of $F$.

We shall restrict ourselves to exploring residues and non-residues of finite fields with odd orders. Recall that the *characteristic* of a field, denoted by $char(F)$, is the smallest integer $m$ for which the sum of $m$ multiplicative identities $1_F + 1_F + \ldots + 1_F$ equals the additive identity $0_F$ in the field $F$. Then, the following properties of a finite field are well-known (See Chapter 2 of [25]):

**Theorem 6.1.1.** *Let $\mathbb{F}_q$ be a finite field of order $q$ and characteristic $p$, then we have:*

*(a) All finite fields on $q$ elements are isomorphic. Thus, $\mathbb{F}_q$ is unique up to isomorphism.*

*(b) $char(\mathbb{F}_q) = p$ is prime, and there exists an integer $r$ such that $q = p^r$.*

*(c) The map $x \mapsto x^p$ is an automorphism on $\mathbb{F}_q$ (known as the Frobenius map).*

*(d) The multiplicative group $\mathbb{F}_q^\times$ is cyclic with order $q-1$.*

Observe that if $\mathbb{F}_q$ is a finite field of even order, then it has characteristic 2. Consequently, the surjective property of the Frobenius map from the previous theorem implies that $Res(\mathbb{F}_q) = \mathbb{F}_q^\times$. Things become more interesting as non-residues emerge in finite fields of odd order.

Let us look at some examples. Consider $q = 5$. Then $char(\mathbb{F}_q) = 5$, and $\mathbb{F}_q^\times$ has four elements, namely $1, 2, 3$, and $4$. Squaring each of these gives the set of quadratic residues of $\mathbb{F}_5$. So, $1_F^2 = 1_F$, $2_F^2 = 4_F$, $3_F^2 = 9 = 4_F$, and $4_F^2 = 16 = 1_F$, as multiples of five equal zero in fields of characteristic 5. Thus, $Res(\mathbb{F}_5) = \{1, 4\}$ and $NRes(\mathbb{F}_5) = \{2, 3\}$. Similarly, residues and non-residues can be calculated for other primes. For instance, $Res(\mathbb{F}_{11}) = \{1, 3, 4, 5, 9\}$ and $Res(\mathbb{F}_{13}) = \{1, 3, 4, 9, 10, 12\}$.

Note that finite fields with non-prime order $q$ enjoy a slightly different structure as compared to finite fields of prime order, for they are not a subset of integers and hence, are not isomorphic to $\mathbb{Z}_q$, the ring of integers modulo $q$. See [25] for more details. For instance, let us look at the finite field of order 9. As $9 = 3^2$, the finite field $\mathbb{F}_9$ is said to be an *extension* of

$\mathbb{F}_3$ of order 2, and is defined as a two-dimensional vector space over $\mathbb{F}_3$ generated by the basis elements $\{1_F, x\}$. Here $x$ is simply a root of the irreducible polynomial $X^2 + 1$ over $\mathbb{F}_3$, and thus satisfies the equation $x^2+1 = 0$. This equation, along with the fact that $3_F = 0_F$, can be used to generate the multiplication table for $\mathbb{F}_9 = \{0,\ 1,\ 2,\ x,\ x{+}1,\ x{+}2,\ 2x,\ 2x{+}1,\ 2x{+}2\}$. For example, $(x+1)^2 = x^2+2x+1 = 2x$, or $(2x)^2 = 4x^2 = 3x^2 + x^2 = x^2 = x^2+1+2 = 2$. Now, the residues can be calculated simply by squaring each non-zero element, as before. We obtain $Res(\mathbb{F}_9) = \{1, 2, x, 2x\}$ and $NRes(\mathbb{F}_9) = \{x+1,\ 2x+1,\ x+2,\ 2x+2\}$.

However, the residues of finite fields of odd orders share many common properties which we shall now explore.

**Theorem 6.1.2.** *If $q$ is an odd prime power, then $|Res(\mathbb{F}_q)| = |NRes(\mathbb{F}_q)| = \dfrac{q-1}{2}$.*

*Proof.* Let $q = p^r$ be an odd prime power, and let $\mathbb{F}_q$ be the finite field on $q$ elements. Define the surjective function $f : \mathbb{F}_q^\times \to Res(\mathbb{F}_q)$ with the map $x \mapsto x^2$. Let $w \in Res(\mathbb{F}_q)$, then there exists $x \in \mathbb{F}_q^\times$ such that $x^2 = w$. It follows that $\{x, -x\} \subset\ f^{-1}(w)$. further, for any $y \in f^{-1}(w)$, we have $y^2 = x^2$, and hence $y = \pm x$. Consequently, $f^{-1}(w) = \{x, -x\}\ \forall w \in Res(\mathbb{F}_q)$.

As $char(\mathbb{F}_q) = p > 2$, we see that for every non-zero element $x$, the difference $x - (-x) = x(1_F + 1_F) \neq 0$. Thus, the elements $x$ and $-x$ are distinct whenever $x \neq 0$, and so every element of $Res(\mathbb{F}_q)$ has a pre-image under $f$ with exactly two elements. It follows that

$$|Res(\mathbb{F}_q)| = \frac{|\mathbb{F}_q^\times|}{2} = \frac{q-1}{2} \quad \text{and} \quad |NRes(\mathbb{F}_q)| = |\mathbb{F}_q^\times| - |Res(\mathbb{F}_q)| = \frac{q-1}{2}$$

$\square$

Henceforth, we shall only consider finite fields of odd order in our explorations. Let us begin by looking at the multiplicative properties of residues and non-residues over such fields.

**Theorem 6.1.3.** *Let $\mathbb{F}_q$ be a finite field of odd order, then:*

- *If $x, y \in Res(\mathbb{F}_q)$, then $x \cdot y \in Res(\mathbb{F}_q)$.*
- *If $x, y \in NRes(\mathbb{F}_q)$, then $x \cdot y \in Res(\mathbb{F}_q)$.*
- *If $x \in Res(\mathbb{F}_q)$ and $y \in NRes(\mathbb{F}_q)$ , then $x \cdot y \in NRes(\mathbb{F}_q)$.*

In order to prove this theorem, we shall define and study scaling functions.

**Definition 6.1.2.** *Given a field $\mathbb{F}_q$ and $a \in \mathbb{F}_q^\times$, we define the function $SC_a : \mathbb{F}_q^\times \to \mathbb{F}_q^\times$ mapping $x \mapsto a \cdot x$ as the* scaling map *at $a$.*

Note that the scaling maps are bijective with inverses given by $SC_a^{-1} = SC_{a^{-1}}$. Further, the product $a \cdot b \in \mathbb{F}_q^\times$ can be represented as both $SC_a(b)$ and $SC_b(a)$, which would be useful in our proof. Let us look at scaling maps at residues and scaling maps at non-residues separately:

**Lemma 6.1.4.** *Let $\mathbb{F}_q$ be a finite field of odd order $q$ and let $a \in Res(\mathbb{F}_q)$. Then, the scaling map $SC_a$ maps residues to residues, and non-residues to non-residues.*

*Proof.* As $a \in Res(\mathbb{F}_q)$, there exists a non-zero element $b$ such that $b^2 = a$. If $x \in Res(\mathbb{F}_q)$, then $x = y^2$ for some $y \in \mathbb{F}_q^\times$, and thus $SC_a(x) = b^2y^2$ is a residue of $\mathbb{F}_q$. Now consider the case when $x$ is a non-residue of $\mathbb{F}_q$. If $SC_a(x) \in Res(\mathbb{F}_q)$, then there is a non-zero element $y$ for which $SC_a(x) = a \cdot x = y^2$. As $a \neq 0$, we have that $x = a^{-1} \cdot y^2 = (b^{-1}y)^2$, thereby contradicting the fact that $x$ is a non-residue. Thus $SC_a$ maps non-residues to non-residues, as was required. $\square$

**Lemma 6.1.5.** *Let $\mathbb{F}_q$ be a finite field of odd order $q$ and let $a \in NRes(\mathbb{F}_q)$. Then, the scaling map $SC_a$ maps residues to non-residues, and non-residues to residues.*

*Proof.* Let $a$ be a non-residue of $\mathbb{F}_q$. Suppose $x \in Res(\mathbb{F}_q)$, Then Lemma 6.1.4 implies that $SC_a(x) = SC_x(a)$ is a non-residue of $\mathbb{F}_q$. Thus, $SC_a$ maps residues to non-residues. Further, since the scaling maps are bijective and $|Res(\mathbb{F}_q)| = |NRes(\mathbb{F}_q)|$, we conclude that each non-residue has a residue as its pre-image under $SC_a$. Consequently, the scaling function $SC_a$ will map each non-residue to a residue, as was required. $\square$

Theorem 6.1.3 now follows directly from Lemmata 6.1.4 and 6.1.5. This multiplicative property will be crucial for our proofs in the upcoming sections. In order to manifest this property in a more applicable manner, the following function can be defined:

**Definition 6.1.3.** *The Quadratic Character on the finite field $\mathbb{F}_q$ of odd order $q$, is defined as the function $\chi_q : \mathbb{F}_q \to \{-1, 0, 1\}$ with the mapping:*

$$\chi_q(a) \coloneqq \begin{cases} 1, & \text{if } a \in Res(\mathbb{F}_q) \\ -1, & \text{if } a \in NRes(\mathbb{F}_q) \\ 0, & \text{if } a = 0 \end{cases}$$

Note that a *character* on a field $F$ is a group homomorphism from $F^\times$ to $\mathbb{C}^\times$. The domain of a character can be extended to $F$ by mapping $0_F$ to itself, thereby preserving its multiplicative nature. Theorem 6.1.3 justifies the use of the term character in the above definition.

Next, we are interested in looking at the conditions for which the set of residues of $\mathbb{F}_q$ are closed under additive inverses. Such a property would be useful to ensure that the defined *Paley* graphs are undirected, as shall be seen in the upcoming sections. By the multiplicative properties discussed above, it suffices to look for conditions when $-1_F$ is a residue of the field, for $\chi(-a) = \chi(a)$ iff $\chi(-1_F) = 1$. The following theorem lays out this condition:

**Theorem 6.1.6.** *Given the finite field $\mathbb{F}_q$ of odd order $q$, The element $-1_{\mathbb{F}_q}$ is a residue of $\mathbb{F}_q$ if and only if $q \equiv 1 \mod 4$*

*Proof.* Let $\mathbb{F}_q$ be the finite field of odd order $q$. From Theorem 6.1.1, we know that $\mathbb{F}_q^\times$ is a cyclic group of order $q-1$. Let $a \in \mathbb{F}_q^\times$ be the generator of this group. Then $q-1$ is the smallest positive integer $n$ for which $a^n = 1$. Further, any non-zero element $x$ of the field is of the form $a^k$ for some integer $k \in [q-1]$ and satisfies the equality $x^{q-1} = 1_F$.

Now $-1_F = a^m$ for some $m \in [q-1], m \neq q-1$. Then $a^{2m} = 1$ and so $q-1$ divides $2m$. As $2m < 2(q-1)$, it follows that $2m = q-1$ or $m = \dfrac{q-1}{2}$. Finally, an element $x = a^k$ is a residue if there exists some element $y = a^j$ for which $x = y^2$. Thus, $k = 2j \mod (q-1)$. It follows that $k$ is even, as $q-1$ is even too. Consequently, $-1_F$ is a residue if and only if the integer $m = \dfrac{q-1}{2}$ is even, or when $q \equiv 1 \mod 4$. □

Recall that for primes $p$, the finite field $\mathbb{F}_p$ is isomorphic to $\mathbb{Z}_p$, the ring of integers modulo $p$, and so, addition and multiplication in $\mathbb{F}_p$ simply overlap with the usual operations on $\mathbb{Z}$ taken modulo $p$. Thus, it is possible to define the quadratic residues (non-residues) of a prime $p$ by identifying them with the set $Res(\mathbb{F}_p)$ ($NRes(\mathbb{F}_p)$). Perhaps this also explains the origin of the term 'quadratic residues', for in $\mathbb{F}_p$, they are simply the remainders of perfect squares which remain after division by a prime $p$. The domain of the quadratic character $\chi_p$ too can be extended to all integers by equating $\chi_p(a) = \chi_p(b)$ whenever $a \equiv b \mod p$. As a result, it is possible to study the residues and non-residues of primes using number theoretic concepts (see [33] for a detailed overview). The following theorems are examples of some of the known results on the residues and non-residues of primes (See Sections 6.11 - 6.13 of [21] for proofs and other discussions):

**Theorem 6.1.7.** *Let $p$ be an odd prime $p$, then 2 is a residue of $p$ if and only if $p \equiv 1 \mod 8$ or $p \equiv 7 \mod 8$. It is a non-residue otherwise.*

**Theorem 6.1.8** (Law of Quadratic Reciprocity)**.** *Let $p, q$ be two odd primes. Then,*

$$\chi_p(q) \cdot \chi_q(p) = (-1)^{\frac{1}{4}(p-1)(q-1)}$$

*Consequently, $\chi_q(p) = \chi_p(q)$ if and only if $p \equiv 1 \mod 4$ or $q \equiv 1 \mod 4$.*

Finally, we are interested in looking at a theorem on the distribution of integers into residues and non-residues for the proof of Theorem 6.3.2. Given a bi-partition $(A, B)$ of a finite set of primes, Lemma 4.4 from [33] states that there exist infinitely many primes $p$ for which $A \subset Res(\mathbb{F}_p)$ and $B \subset NRes(\mathbb{F}_p)$. We further observed that all these primes are congruent to $1 \mod 4$, which shall be crucially used in our results from the upcoming section. We include a proof of this lemma to emphasise this fact:

**Lemma 6.1.9.** *[33] If $\Pi = \{p_1, p_2, \ldots, p_k\}$ is a non-empty finite set of primes and if $\varepsilon : \Pi \to \{-1, 1\}$ is a fixed function, then there are infinitely many primes $p \equiv 1 \mod 4$ for which*

$$\chi_p(p_i) \;=\; \varepsilon(p_i) \quad \forall\, i \in [k]$$

In order to prove Lemma 6.1.9, we require the following well-known theorems from number theory:

**Theorem 6.1.10** (The Chinese Remainder Theorem)**.** *Let $m_1, m_2, \ldots, m_k$ be pairwise co-prime integers greater than 1, and let $a_1, a_2, \ldots, a_k$ be integers, then there is a unique integer modulo $M = m_1 \cdot m_2 \cdot \ldots \cdot m_k$ which satisfies the following system of congruences:*

$$X \equiv a_i \mod m_i \;\; \forall\, i \in [k]$$

**Theorem 6.1.11** (Dirichlet's Theorem on Primes in Arithmetic Progressions)**.** *If $a, b$ are two co-prime positive integers, then the arithmetic progression $AP(a, b)$, with first term $a$ and common difference $b$, contains infinitely many primes.*

See Section 8.1 in [21] and Section 4.4 in [33] for a proof of Theorem 6.1.10 and a discussion on Theorem 6.1.11, respectively. Note that $AP(a, b)$ refers to the arithmetic progression with first term $a$ and common difference $b$. We now end this section with a proof of Lemma 6.1.9:

*Proof.* Let the set $\Pi$ and the function $\varepsilon$ be given as above. Without loss of generality, assume that the prime $2 \in \Pi$, for otherwise, it can be added to the set $\Pi$ and the function $\varepsilon$ can be extended to get a new equivalent instance of the theorem.

Arrange the primes in $\Pi$ in ascending order, say, $2 = p_1 < p_2 < \ldots < p_k$, and consider the case when $\varepsilon(2) = 1$. Define a map $p_i \mapsto w_i$, for $i \geq 2$, such that: If $\varepsilon(p_i) = -1$, then $w_i$ is an odd number co-prime to $p_i$ for which $\chi_{p_i}(w_i) = -1$; and $w_i = 1$ otherwise. Note that it is always possible to choose $w_i$ in the former case, as $p_i$ has a non-residue, say $x < p_i$. If $x$ is odd, then set $w_i = x_i$, else set $w_i = x_i + p_i$. Clearly, $p_i$ does not divide both $x_i$ and $x_i + p_i$, and so $w_i$ is co-prime to $p_i$. Observe that $w_i$ has been chosen in such a way that $\chi_{p_i}(w_i) = \varepsilon(p_i)$.

Now consider the set:

$$S = AP(1, 8) \cap \left( \bigcap_{i=2}^{k} AP(w_i, 2p_i) \right)$$

Let $p \in S$ be prime. Then $p \equiv 1 \mod 8$, and thus $p \equiv 1 \mod 4$. Further, from Theorem 6.1.7, $\chi_p(2) = 1 = \varepsilon(2)$. Moreover, for $i \geq 2$, we have:

$$\begin{aligned}
\chi_p(p_i) &= \chi_{p_i}(p) && \text{(From Theorem 6.1.8 as } p \equiv 1 \mod 4) \\
&= \chi_{p_i}(w_i + 2p_i n) && \text{(As } p \in AP(w_i, 2p_i)) \\
&= \chi_{p_i}(w_i) \;=\; \varepsilon(p_i) && \text{(By definition of } \chi_{p_i} \text{ and the choice of } w_i)
\end{aligned}$$

Thus, any $p \in S$ satisfies the requirements of the theorem, and hence, it suffices to show that there are infinitely many primes in the set $S$. By Theorem 6.1.10, there is a positive integer $M$ which satisfies the following system of congruences (These are well defined as all $w_i$ are odd):

$$M \equiv \frac{w_i - w_2}{2} \mod p_i \ \ \forall\, i \geq 2\ ; \quad M \equiv \frac{1 - w_2}{2} \mod 4$$

We claim that the sequence $A = AP(w_2 + 2M,\ 8p_2p_3 \dots p_k) \subset S$. Let $x \in A$, then $x = w_2 + 2M + n \cdot 8P^1$ where, $P^1 = p_2p_3 \dots p_k$. First, note that using the congruences above, $2M = 8q + 1 - w_2$ for some integer $q$, and so $x = 1 + 8(q + nP^1)$. Hence, $x \in AP(1, 8)$. Next, fix any $i \geq 2$. Again, using the congruences above, $2M = 2p_iq + w_i - w_2$ for some integer $q$, and so $x = w_i + 2p_i \cdot (q + 4n\frac{P^1}{p_i})$. Hence, $x \in AP(w_i, 2p_i)$. It follows that the progression $A$ is a subset of $S$.

Finally, we show that $GCD(8P^1,\ w_2 + 2M) = 1$. Suppose not. then, they have a common factor greater than 1. As $w_2 + 2M$ is odd, by choice of $w_2$, the common factor must be divisible by $p_i$ for some $i \geq 2$. But this is not possible, as $w_2 + 2M \equiv w_i \mod p_i \not\equiv 0 \mod p_i$ (Since $w_i$ and $p_i$ are co-prime). By contradiction, $GCD(w_2 + 2M,\ 8P^1) = 1$.

It follows from Theorem 6.1.11, that the set $A \subset S$ contains infinitely many primes. Further, all these primes are of the form $1 \mod 8$ and hence are congruent to $1 \mod 4$. Finally, for the case $\varepsilon(2) = -1$, replace $AP(1, 8)$ with $AP(5, 8)$ above, and the same proof follows. Note that any number of the form $5 \mod 8$ is congruent to $1 \mod 4$ as well. $\square$

## 6.2 Paley Graphs

Let $q$ be a prime power such that $q \equiv 1 \mod 4$, and consider the finite field $\mathbb{F}_q$ with $q$ elements. The class of *Paley* graphs is defined as follows:

**Definition 6.2.1.** *The $Paley_q$ graph is the unique graph with vertex set $\mathbb{F}_q$ and edge set*

$$E = \{(u, v) : u - v \ \in \ Res(\mathbb{F}_q^{\times})\}$$

Note that choosing $q \equiv 1 \mod 4$ ensures that the $Paley_q$ graph is undirected by the virtue of Theorem 6.1.6. Further, the uniqueness of the $Paley_q$ graph follows from the uniqueness of $\mathbb{F}_q$ as stated in Theorem 6.1.1. Refer to [22] for a more detailed history and introduction to *Paley* graphs.

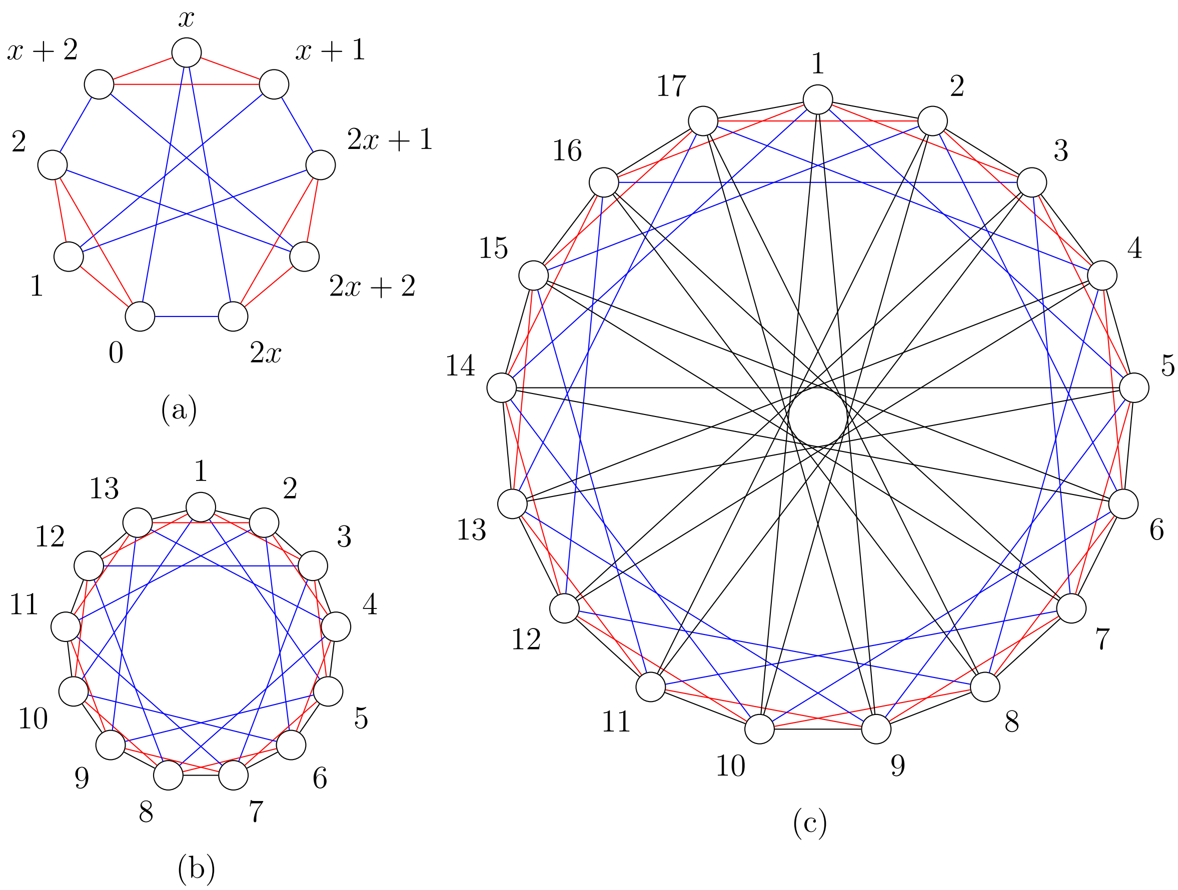


Figure 6.1: Examples of *Paley* graphs: (a) $Paley_9$, (b) $Paley_{13}$, and (c) $Paley_{17}$.

Let us look at some examples of *Paley* graphs. The smallest *Paley* graph is $C_5$, the cycle on 5 vertices, with vertex set $\{1, 2, 3, 4, 5\}$ and two nodes are adjacent if and only if their difference is 1 or 4 modulo 5. Observe that the residues $x$ and $-x$ contribute to the same edge. See Figure 6.1 for more examples of *Paley* graphs (their residues were calculated in the previous section). Colours have been used to distinguish the set of edges formed by different residues. All these *Paley* graphs have some surprising common properties. Let us begin by proving the property most important to us, namely self-complementarity.

**Theorem 6.2.1.** *The $Paley_q$ graph is self-complementary.*

*Proof.* Let $G$ be the $Paley_q$ graph, with vertex set $\mathbb{F}_q$. Let $a$ be any non-residue of $\mathbb{F}_q^{\times}$, and define the function $f : V(G) \to V(\overline{G})$ by the mapping $f(x) = a \cdot x$. We prove that $f$ is an isomorphism from $G$ to $\overline{G}$.

That $f$ is bijective follows from the existence of its inverse $g : V(\overline{G}) \to V(G)$ defined by the mapping $g(x) = a^{-1} \cdot x$. Note that $a^{-1}$ exists as $a \neq 0$. Next, observe that $(x, y)$ is an edge in $G$ if and only if the quadratic character $\chi_q(y - x) = 1$. Similarly, $(x, y)$ is an edge in $\overline{G}$ if and only if it is not an edge in $G$ and so $\chi_q(y - x) = -1$. Further, by the multiplicative property of the quadratic character, $\chi_q(f(y) - f(x)) = \chi_q(a)\chi_q(y - x) = -1 \cdot \chi_q(y - x)$. Hence, $(f(x), f(y))$ is an edge of $\overline{G}$ if and only if $(x, y)$ is an edge of $G$, as was required. □

As a corollary, we observe that *Paley* graphs are connected as they are self-complementary and the complement of a disconnected graph is always connected.

*Paley* graphs are rich in properties. For instance, they are self-complementary, strongly regular, connected graphs, with a well-known spectrum of eigenvalues. Roughly speaking, most of these properties arise as a natural consequence of the high degree of 'symmetry' held by these graphs. Here, the notion of symmetry can be made precise by looking at the automorphism group of *Paley* graphs and focusing on the properties of vertex and arc transitivity satisfied by this class of graphs (See [19] for the notion of transitivity). Note that arcs are simply edges in a graph $G$ with an assigned direction.

**Definition 6.2.2.** *A graph $G$ is said to be* ***vertex transitive****, if for every pair of vertices $x, y$, there is an automorphism of $G$ which maps $x$ to $y$.*

**Definition 6.2.3.** *A graph $G$ is said to be* ***arc transitive****, if for every pair of directed edges $e, f$, there is an automorphism of $G$ which maps $e$ to $f$ while preserving their direction.*

We first prove that *Paley* graphs are vertex and arc transitive. The proofs follow from [14].

**Lemma 6.2.2.** *[14] Let $G$ be the $Paley_q$ graph. Let $a \in Res(\mathbb{F}_q^\times)$ and $b \in \mathbb{F}_q$. Then the function $f : V(G) \to V(G)$ defined by $f(x) = ax + b$ is an automorphism on $G$.*

*Proof.* Let G be the $Paley_q$ graph, and let $a, b$, and $f : V(G) \to V(G)$ be defined as above. It suffices to prove that $f$ admits an inverse and preserves adjacencies. As $a \neq 0$, the multiplicative inverse $a^{-1}$ exists in $\mathbb{F}_q$. Let $g : V(G) \to V(G)$ be a function defined by $g(x) = a^{-1}(x - b)$. It is easy to check that $g$ is the desired inverse:

$$f(g(x)) = f(a^{-1}(x - b)) = a(a^{-1}(x - b)) + b = (x - b) + b = x$$

$$g(f(x)) = g(ax + b) = a^{-1}((ax + b) - b) = a^{-1}(ax) = x$$

To show that $f$ preserves adjacency, consider any two vertices $x, y \in V(G)$. By definition of the $Paley_q$ graph, $(x, y) \in E(G)$ if and only if the quadratic character $\chi_q(x-y) = 1$. Further, as $a$ is a residue of $(F)_q^\times$, we have that $\chi_q(a) = 1$. By the multiplicative property of the quadratic character, we have: $\chi_q(f(x) - f(y)) = \chi_q(a(x-y)) = \chi_q(x-y)$. Thus, $(x, y) \in E(G)$ if and only if $(f(x), f(y)) \in E(G)$, as was to be shown. □

**Theorem 6.2.3.** *[14] Paley graphs are vertex-transitive and arc-transitive.*

*Proof.* Let $G$ be the $Paley_q$ graph. We begin by proving that $G$ is vertex-transitive. Let $u, v$ be any two vertices in G. Set $w = v - u$ and define the function $f : V(G) \to V(G)$ using the rule $f(x) = x + w$. By the previous lemma, $f$ is an automorphism as $1 \in Res(\mathbb{F}_q^\times)$. Further, $f(u) = v$, as was required.

For arc-transitivity, let $(u_1, v_1)$ and $(u_2, v_2)$ be any two edges in $G$. Define the function $g : V(G) \to V(G)$ as follows:

$$g(x) = \frac{u_2 - v_2}{u_1 - v_1} \cdot x \ + \ \frac{u_1 v_2 - v_1 u_2}{u_1 - v_1}$$

Note that this function is well-defined as $G$ contains no self-loops. It is easy to see that $g(u_1) = u_2$ and $g(v_1) = v_2$. Further, as $(u_1, v_1)$ and $(u_2, v_2)$ are edges in G, we have: $\chi_q(v_2 - u_2) = \chi_q(v_1 - u_1) = 1$. By the multiplicative property of $X_q$ and the fact that $v_1 - u_1 \neq 0$, we see that $\chi_q((v_1 - u_1)^{-1}) = \dfrac{\chi_q(1)}{\chi_q(v_1 - u_1)} = 1$, and consequently, $\chi_q(a) = \chi_q((u_2 - v_2)(u_1 - v_1)^{-1}) = 1$. It thus follows from the previous lemma that $g$ is an automorphism on $G$. □

The transitivity properties ensure that all vertices, and likewise all edges, of a *Paley* graph, share many common properties. This ensures that these graphs are Regular:

**Theorem 6.2.4.** *The $Paley_q$ graph is $\dfrac{q-1}{2}$ regular.*

*Proof.* Let $G$ be the $Paley_q$ graph with vertex set $\mathbb{F}_q$. Let $x$ represent the vertex in $G$ corresponding to the additive identity $0_{\mathbb{F}_q} \in \mathbb{F}_q$. Then, any vertex $y \in V(G)$ is adjacent to $x$ if and only if $y$ is a quadratic residue in $\mathbb{F}_q^\times$. It follows from Theorem 6.1.2 that $d(x) = \dfrac{q-1}{2}$. We conclude by noting that any vertex $v \in V(G)$ has the same degree as $x$, for there exists an automorphism mapping the vertex $v$ to $x$ by Theorem 6.2.3. □

In fact, *Paley* graphs satisfy a much stronger form of regularity which emerges when any two vertices are considered simultaneously. We shall define and prove this property next, for it plays a crucial role in proving that *Paley* graphs are substitution prime.

**Definition 6.2.4.** *A graph $G$ is said to be* ***strongly regular*** *with parameters $(n, k, \lambda, \mu)$ if it is a $k$-regular graph on $n$-vertices such that:*

- *Every pair of adjacent vertices have $\lambda$ common neighbours.*
- *Every pair of non-adjacent vertices have $\mu$ common neighbours.*

Let us look at the number of common neighbours for adjacent and non-adjacent vertices in the $Paley_q$ graph.

**Lemma 6.2.5.** *[14] Every pair of adjacent vertices of the $Paley_q$ graph has exactly $\dfrac{q-5}{4}$ common neighbours.*

*Proof.* Let $G$ be the $Paley_q$ graph and let $x, y$ be any two adjacent vertices in $G$. Define $N = N(x)$ and $M = N[x]^c$ to be the set of neighbours and non-neighbours of $x$ respectively. Suppose that $y$ has $k$ neighbours in $M$. Pick any vertex $v \in N$. By Theorem 6.2.3, there is an automorphism $f : V(G) \to V(G)$ that maps the directed-edge $(x, y)$ to the directed-edge $(x, v)$ (making $x$ a fixed point). As this map preserves adjacencies, it bijectively maps elements of $M$ onto $M$, and also maps all neighbours of $y$ in $M$ injectively into $M$: for if $w \in M$ was adjacent to $y$, then $f(w)$ is adjacent to $f(y) = v$ and gets mapped to $M$ as it is non-adjacent to $f(v) = v$. Thus, every vertex in $N$ has exactly $k$ neighbours in $M$.

Next, as $G$ is self-complementary, the sets $M$ and $N$ simply switch roles in the complement graph $\overline{G}$. Consequently, in the graph $G$, every vertex in $M$ has exactly $k$ non-neighbours in $N$. Adding the number of edges and non-edges going across $M$ and $N$, we have: $k|N| + k|M| = |M| \cdot |N|$. It follows from $|M| = |N| = \dfrac{q-1}{2}$, that $k = \dfrac{q-1}{4}$.

To conclude, observe that $y$ has $d(y) - k - 1 = \dfrac{q-5}{4}$ neighbours in $N$ (by subtracting the $k$ neighbours in $M$, and the neighbour $x$). Consequently, every pair of adjacent vertices has exactly $\dfrac{q-5}{4}$ common neighbours in $G$.

□

**Lemma 6.2.6.** *[14] Every pair of non-adjacent vertices of the $Paley_q$ graph has exactly $\frac{q-1}{4}$ common neighbours.*

*Proof.* Let $G$ be the $Paley_q$ graph and let $x, y$ be any two non-adjacent vertices in $G$. Define the sets $A = N(x)$ and $B = N(y)$, each of cardinality $\frac{q-1}{2}$. As $G$ is self-complementary and $x, y$ are adjacent in $\overline{G}$, they have $\frac{q-5}{4}$ common neighbours in $\overline{G}$ by Lemma 6.2.5. In other words, there are $\frac{q-5}{4}$ vertices in $G$ which are neighbours with neither $x$ nor $y$ in $G$, and so $|A \cup B| = q - \frac{q-5}{4}$. The lemma follows from the following equation:

$$|A \cap B| = |A| + |B| - |A \cup B| = \frac{q-1}{2} + \frac{q-1}{2} - q + \frac{q-5}{4} \;=\; \frac{q-1}{4}$$

□

The following theorem now follows from Theorem 6.2.4, and Lemmata 6.2.5 and 6.2.6.

**Theorem 6.2.7.** *[14] The $Paley_q$ graph is strongly regular with parameters* $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$

## 6.3 Reducing the Erdős-Hajnal Conjecture

Recall that, by Theorem 2.2.1, proving all substitution prime graphs satisfy the EH-property is sufficient to prove the Erdős-Hajnal Conjecture. We first show that *Paley* graphs, as introduced in the previous section, are substitution prime as well, thereby forming an important class of graphs with respect to the conjecture.

**Theorem 6.3.1.** *All Paley graphs are substitution prime graphs.*

*Proof.* We shall prove by contradiction. Let $G$ be the $Paley_q$ graph, and suppose that $G$ is not a substitution prime graph. Then, by Theorem 2.3.1, the vertices of $G$ can be partitioned into three sets: $S$, $K$, and $N$ where $S$ is a non-trivial module and $K$ (respectively $N$) is the set of vertices complete (respectively anti-complete) to $S$. We shall count the elements of $S$ in two distinct ways:

**Method 1:** By definition of a non-trivial module, $|S| \geq 2$. Let $x$ and $y$ be two elements in $S$. Note that all elements of $K$ are common neighbours of $x$ and $y$. By Theorem 6.2.7, if $x, y$ are adjacent, then they have exactly $\frac{q-5}{4}$ common neighbours, which lie in $S$ or $K$. Thus, $|K| \leq \frac{q-5}{4}$. As $x$ has exactly $\frac{q-1}{2}$ neighbours, including $y$ and elements of $K$, we have:

$$|S| \geq |S \cap N[x]| = \frac{q-1}{2} - |K| + |\{x\}| \geq \frac{q+7}{4}$$

Similarly, If $x, y$ were not adjacent, then they have exactly $\frac{q-1}{4}$ common neighbours, which lie in $S$ and $K$. Hence, $|K| \leq \frac{q-1}{4}$. Counting as above, we have:

$$|S| \geq |S \cap (N[x] \cup \{y\})| = \frac{q-1}{2} - |K| + |\{x, y\}| \geq \frac{q+7}{4}$$

In either cases, we have that $|S| \geq \frac{q+7}{4}$.

**Method 2:** Let us focus on the set $K$. Note that $K$ is non-empty, else either $N$ would be empty or a disconnected component of $G$. Both these conditions are not viable, as the former violates the definition of a non-trivial module, and the latter contradicts the fact that Paley graphs are connected. Thus, $|K| \geq 1$.

Further, for any $v \in K$, we note that $S \subset N(v)$. Therefore, as $d(v) = \frac{q-1}{2}$, we conclude that $|S| \leq \frac{q-1}{2}$. Similarly, for any $w \in S$, the set $N$ is a subset of non neighbours of $w$, and so, $|N| \leq \frac{q-1}{2}$. If $|K| = 1$ (say, $K = \{v\}$), then $|S \sqcup N| = q - 1$, which is only possible if $|S| = |N| = \frac{q-1}{2}$. As a result, $K = \{v\}$ has all its neighbours in $S$, and so $N$ becomes a disconnected component in $G$. As Paley graphs can are connected, we conclude by contradiction, that $|K| \geq 2$. Let $x, y$ be two vertices in $K$. Then, all elements of $S$ are common neighbours of $x$ and $y$. Thus, irrespective of the adjacency of $x, y$, $|S| \leq \frac{q-1}{4}$.

Comparing both methods, we observe that:

$$|S| \leq \frac{q-1}{4} < \frac{q+7}{4} \leq |S|$$

Thereby contradicting the existence of $S$. Hence, $G$ is a substitution prime graph. $\square$

We wish to show that proving the EH-property for Paley Graphs is sufficient to prove the conjecture. For this purpose, we make the following observation: If $H_1$ is an induced subgraph of $H_2$, then, every $H_1$-free graph is $H_2$-free as well. This follows from the fact that any graph containing an induced $H_2$ must contain an induced $H_1$ too, by the transitivity of the induced subgraph relation. Thus every induced subgraph of a graph $H$ satisfies the EH-property if $H$ satisfies the property itself, and consequently, it suffices for us to prove that every graph is an induced subgraph of some *Paley* graph. We prove that next.

Note that it becomes easier to work with $Paley_q$ graphs of prime order as the finite field $\mathbb{F}_p$ coincides with $\mathbb{Z}_p$, the ring of integers modulo a prime $p$. This allows for the vertices to be labelled using integers and makes way for the use of various number theoretic arguments to study these graphs. We shall define such *Paley* graphs to be primitive.

**Definition 6.3.1.** *The $Paley_q$ graph is defined to be a primitive Paley graph if $q$ is an odd prime congruent to $1 \mod 4$.*

Further, a graph $G$ is an induced subgraph of a primitive $Paley_q$ graph if and only if we can label each vertex using distinct integers in such a way that the difference in the labels of adjacent (non-adjacent) vertices are residues (non-residues) of $\mathbb{F}_q$ for every pair of vertices in $G$. Our aim would be to find such labellings for every graph $G$ by inducting on the number of vertices in $G$.

**Definition 6.3.2.** *We shall say that a graph $G$ admits a residue numbering if there exists an odd prime $q \equiv 1 \mod 4$, and an injective label map $a : V(G) \to \mathbb{Z}_q$, such that for every pair of vertices $x, y$, the difference $a(x) - a(y) \in Res(\mathbb{F}_q)$ whenever $(x, y)$ is an edge and is a non-residue otherwise.*

Note that every induced subgraph of a primitive *Paley* graph admits a residue numbering with the label map as the inclusion function obtained by the virtue of the induced subgraph relation.

We now prove that every graph is an induced subgraph of some primitive *Paley* graph. It was brought to our attention that a similar theorem was proven by Béla Bollobás and Andrew Thomason in [2] using "Weil's theorem proving the Riemann hypothesis for algebraic curves over finite fields". However, we provide an independent proof using Lemma 6.1.9:

**Theorem 6.3.2.** *Any graph is an induced subgraph of infinitely many primitive Paley graphs.*

*Proof.* We induct on the number of vertices in the graph $G$.

The base case follows from the fact that $K_1, K_2$, and $I_2$ are induced subgraphs of all primitive *Paley* Graphs on at least five vertices. Now, let $G$ be a graph on $n$ vertices, and let $v \in V(G)$ be an arbitrary vertex of $G$. Then, by the induction hypothesis, there is an odd prime $z \equiv 1 \mod 4$ such that $G \backslash \{v\}$ is an induced subgraph of the $Paley_z$ graph.

Let $\{a_1, a_2, \ldots, a_{n-1}\}$ be the numbers assigned to vertices of $G \backslash \{v\}$ in its embedding as an induced subgraph of $Paley_z$, taken in ascending order and let $X$ be the number to be assigned to the vertex $v$ in order to obtain a residue numbering for $G$. Define $T$ to be the set of prime factors for the elements of the set: $\{\ a_j \ - \ a_i \ : \ i < j, \ i, j \in [n-1]\}$.

As $T$ is a finite set, we can pick a set of primes $P = \{p_1, p_2, \ldots, p_{n-1}\}$ not in $T$. Define the following quantities:

$$M = \prod_{i=1}^{n-1} p_i, \qquad M_i = \prod_{j \neq i} p_j$$

As $M_i$ is co-prime to $p_i$, $\exists\ y_i$ such that $y_i M_i \ \equiv \ 1 \bmod p_i$, and let $M_{i,k} = \dfrac{M_i}{p_k}$. For each $i \in [n-1]$, define the terms $b_i$ as follows:

$$b_i = a_i \cdot \frac{M_i y_i - 1}{p_i} \quad + \quad \sum_{k \neq i} a_k \cdot M_{i,k} \cdot y_k$$

Note that $b_i$ is an integer, and by the Chinese remainder theorem, there is an integer $c$ such that $c \equiv y_i \ - \ b_i \cdot y_i \bmod p_i \ \ \forall\ i \in [n-1]$. Define:

$$X \ = \ \sum_{k=1}^{n-1} a_k \cdot M_k \cdot y_k \quad + \quad c \cdot M; \qquad X_i = X - a_i \ \ \forall\ i \in [n-1]$$

**Claim: 1.** $p_i | X_i$, **2.** $p_j \nmid X_i \ \forall\ i \neq j$, and **3.** $p_i^2 \ \nmid \ X_i$.

Observe that $X_i \ \equiv \ a_i \cdot (y_i \cdot M_i - 1) \bmod p_i \ \equiv \ 0 \bmod p_i$, thereby proving 1. Similarly, for $i \neq j$, $X_i \ \equiv a_j - a_i \bmod p_j$. As $p_j \ \notin \ T$, we see that $a_j - a_i \ \not\equiv \ 0 \bmod p_j$, and hence $p_j$ does not divide $X_i$. Finally, 3. follows from the following equations:

$$\frac{X_i}{p_i} = b_i + c \cdot M_i \ \equiv \ b_i + (y_i - b_i y_i) \cdot M_i \bmod p_i \ \equiv \ 1 \bmod p_i$$

We have shown that in the set $P$, only $p_i$ divides $X_i$ and does so exactly. Let $S$ be the set of prime factors of $X_i$ which are not in $P \cup T$ for all $i \in [n-1]$ and let $W = P \cup T \cup S$. We then define a multiplicative function $\varepsilon : P \to \{-1, 1\}$ as follows:

$$\varepsilon(p) = \chi_z(p) \; \forall \; p \in T, \quad \varepsilon(p) = 1 \; \forall \; p \in S$$

And $\varepsilon(p_i)$ are chosen such that $\varepsilon(X_i) = \varepsilon(p_i) \cdot \varepsilon \left( \dfrac{X_i}{p_i} \right) = 1$ iff $(v, v_i)$ are adjacent, and $-1$ otherwise.

By Lemma 6.1.9, there is a prime $q$ congruent to 1 mod 4, such that the quadratic character $\chi_q$, when restricted to $W$, equals the mapping given by $\varepsilon$. It follows that $G$ admits a residue numbering: for $\chi_q(a_i - a_j) = \chi_z(a_i - a_j)$ equals 1 whenever $(a_i, a_j) \in E(G)$ by the induction hypothesis, and by the choice of $\varepsilon(p_i)$, the difference $X_i \in Res(\mathbb{F}_q)$ if and only if $v$ is adjacent to $a_i$. Thus, $G$ is an induced subgraph for some primitive $Paley_q$ graph with $q \equiv 1 \bmod 4$. The infinitude of primes in Lemma 6.1.9 ensures that the assigned label map $V(G) \mapsto \{a_1, \ldots, a_{n-1}, X\} \subset \mathbb{Z}_q$ is an injective function for large enough primes $q$. □

Essentially, in the proof of the previous theorem, we tried to extend a given residue numbering of $G \backslash \{v\}$ to the graph $G$. In order to do so, we used the primes in $P$ as control switches to ensure that the differences $X_i$ were residues if and only if $(a_i, v) \in E(G)$. This control was enabled by ensuring that each $p_i \in P$ divides exactly one difference $X_i$, and none other. Finally, the condition $p_i^2 \nmid X_i$ was necessary to establish the control, for otherwise, $\chi_q(X_i) = \chi_q(p_i)^2 \cdot \chi_q(X_i/p_i)$, and all control over $\chi_q(X_i)$ would be lost as $\chi_q(p_i)^2 = 1$ irrespective of the value of $\chi_q(p_i)$. As a consequence of Theorems 6.3.1 and 6.3.2, we obtain the following reduced conjecture equivalent to the Erdős-Hajnal conjecture:

**Conjecture 6.3.3.** *Every Paley graph of prime order $q$ has the EH-property.*

Thus, it is possible to just focus on the self-complementary and substitution prime family of primitive *Paley* graphs. This has several potential advantages, as the properties of these graphs sit at a confluence of finite fields, graph theory, spectral graph theory, and number theory. Further, the self-complementarity enables the use of various theorems, such as Rödl's Theorem and its versions (Theorem 3.2.3), which for instance, allow us to restrict to the class of *dense* or *sparse* $H$-free graphs, as is best suited. Finally, there are also possibilities of attempting some form of inductive proof for Conjecture 6.3.3, with the EH-property for $C_5$ as the base case!

# Chapter 7

# Conclusion and Future Directions

The Erdős-Hajnal Conjecture has been an exciting problem of interest over the past 25 years. The present work done towards the problem has opened doors to many useful properties of $H$-free graphs, for a given graph $H$. The conjecture also comes with some algorithmic implications, as shown in [3]. For instance, the improved bounds claimed by the conjecture provide for better fixed-parameter tractable algorithms of the Minimum Independent Set problem, when restricted to certain classes of $H$-free graphs. The improvement generally comes in the form of a kernel of lower size.

In this thesis, we took a look at some of the known results pertaining to this conjecture and tried to incorporate them under a uniform framework of notations. We saw how the substitution operation preserves the EH-property, and looked at the known proofs of the property for $P_4$, $C_5$, and the bull. By taking inspiration from the case for $C_5$, we were able to reduce the EH-property for $P_5$ to finding large homogeneous sets in dense $P_5$-free graphs. We further noted the significance of self-complementary graphs in relation to the Erdős-Hajnal conjecture and saw how it facilitates the various proofs by allowing to switch between $G$ and its complement. To harness this desirable property of self-complementarity, we looked at the class of *Paley* graphs. We showed that all *Paley* graphs are substitution prime and that every graph is an induced subgraph of some *Paley* graph of prime order. Consequently, we further reduced the Erdős-Hajnal conjecture to the equivalent problem of focusing on the EH-property for the substitution prime self-complementary class of primitive *Paley* graphs. As discussed towards the end of Section 6.3, we hope that this reduction proves useful in solving the conjecture.

While we explored a classical approach to the conjecture, there have been many other approaches towards making progress on the problem. For instance, the authors of [26] take a probabilistic approach to show that for any given graph $H$, almost all $H$-free graphs have polynomial-sized homogeneous sets. The problem was also mapped to an equivalent version in terms of *tournaments*, a type of directed graph, and much work has been done in this version (See [6] for an overview). The authors of [5] tried to improve Erdős and Hajnal's original $2^{c\sqrt{\log|G|}}$ bound to show that all large enough $H$-free graphs $G$ have $hom(G) \geq 2^{c\sqrt{\log|G|\log\log|G|}}$, which, as explained in [5], acts as a mid-point between Erdős-Hajnal's bound and the conjecture. Finally, the analogues of the conjecture for hypergraphs was also studied in [12]. They showed that there is a version of the conjecture for uniform 3-hypergraphs which holds true; however, no analogue of the conjecture can be true in $k$-uniform hypergraphs for $k \geq 4$.

Searching for more graph operations which preserve the EH-property, and studying the properties of $\tau$-critical graphs, as done in the proof for $C_5$ [9], are some possible future directions of work on the conjecture. Further, it is also possible to look at a restricted version of the conjecture as follows: If $\mathcal{A}$ is a proper sub-class of graphs, then we can ask for all graphs $H$, whether every $H$-free graph in $\mathcal{A}$ has polynomial-sized homogeneous sets, instead of looking at all possible $H$-free graphs. For instance, the conjecture readily holds when restricted to perfect graphs or planar graphs. However, both these restrictions are trivial, as they have polynomially large homogeneous sets even without the constraint of $H$-freeness. The case of non-trivial restrictions is more interesting, and to the best of my knowledge, the conjecture is not known to be true under any non-trivial restrictions. The family of Ramsey graphs are a potentially good candidate for such a non-trivial restriction, and we believe that they could hold some key insights into the conjecture.

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