Near extremal black hole entropy

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Harita P B

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2023

Supervisor: Dr. Suneeta Vardarajan

Certificate

This is to certify that this dissertation entitled Near extremal black hole entropy towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Harita P B at IISER Pune under the supervision of Dr. Suneeta Vardarajan, Associate Professor, Department of Physics, during the academic year 2022-2023.

Harita P B

Dr. Suneeta Vardarajan

Committee:

Dr. Suneeta Vardarajan

Dr. Sachin Jain

This thesis is dedicated to my parents, Kanchana Mala B and Palani Balaji and to my brother, Nishaant Kumar P B

Declaration

I hereby declare that the matter embodied in the report entitled Near extremal black hole entropy are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Suneeta Vardarajan and the same has not been submitted elsewhere for any other degree.

Harita P B

Dr. Suneeta Vardarajan

Acknowledgments

I am grateful to my Supervisor, Dr. Suneeta Vardarajan for mentoring me for the past 1.5 years and for giving me the time and support to learn to do research in theoretical high energy physics. I am grateful for her patience in guiding me through mistakes and setbacks. Her uncompromising demand for perfection helped me bring out the best of my potential and it is one of the many qualities I strive to learn from her example. I also thank Dr. Suneeta Vardarajan for detailed comments on this thesis.

I thank my expert, Dr. Sachin Jain for his timely feedbacks on my presentations. I thank Mohammad Ali for many long and insightful discussions. I am grateful for all the support I received while learning to do independent calculations in this project. The work described in section 3.3 was carried out in collaboration with Mohammad.

I am indebted to my parents, Kanchana Mala and Balaji Palani for their particularly stubborn faith in my potential and to my brother, Nishaant Kumar for his unconditional moral support.

I was partly supported by the INSPIRE (Innovation in Science Pursuit for Inspired Research) scholarship.

Contents

List of Figures

Abstract

Classically, the thermodynamics of near extremal black holes fails below a certain temperature. To avoid this problem, the existence of a "thermodynamic mass gap" between the extremal state and the lightest near extremal state was conjectured. At the same time, for a small Hawking temperature T, temperature dependent quantum corrections to the classical thermodynamic variables can be determined. The effect of these corrections in resolving the existence of a mass gap was studied in recent literature. This thesis is motivated by the problem of understanding the zero temperature limit of the quantum corrected partition function Z and consequently, the entropy S to study the statistical mechanics of extremal black hole states.

We motivate the thermodynamic mass gap problem with relevant background material in chapter 1. We describe how the Jackiw-Teitelboim (JT) theory of $2D$ gravity, and the resulting Schwarzian action describe the dynamics of the dimensionally reduced Einstein-Maxwell theory of near extremal black holes, in the near horizon region. In chapter 2, we discuss the method of coadjoint orbit quantization for quantizing the Schwarzian action. We then discuss how to set up a path integral for Z for a canonical ensemble of non rotating, non supersymmetric, near extremal black holes with a fixed charge Q, and subsequently evaluate the path integral. This is acheived by reducing the theory to an effective $1D$ action at the boundary of the near horizon region, at the level of the path integral. This is then accompanied by a brief account of multi black hole solutions to Einstein-Hilbert and Einstein-Maxwell actions and the method of heat kernel that comes useful in evaluating one loop functional determinants.

In chapter 3, we attempt to look for different effective boundary theories that could possibly rectify the divergence in the entropy in the $T \to 0$ limit. We discuss why the Schwarzian action correctly describes a perturbed JT-boundary theory, where the perturbation takes the extremal Reissner-Nordström (RN) solution to the near extremal RN solution. This is followed by our efforts at resolving the difficulties in determining the contribution of quadratic fluctuations around multi black hole saddles to the path integral. We discuss the applicability of existing methods in the literature in calculating heat kernel coefficients on manifolds with conical singularities, where the metric is conformally related to a coordinate separable one. The possibility of such non perturbative corrections in rectifying the behaviour of Z and S in the zero temperature limit is discussed. We present some conclusions in chapter 4.

Chapter 1

Introduction

That black holes can be described by quantities such as energy, entropy and temperature, characteristic of thermodynamic systems was first considered only to draw an analogy between black hole mechanics and the laws of thermodynamics. It was Hawking's discovery of the thermal behaviour of the radiation from a black hole, at large distances [1] that forged the idea that black holes are indeed thermodynamic systems which can equilibriate by emitting Hawking radiation. Independently, it was discovered that only three independent macroscopic properties can be ascribed to black holes: mass (M), charge (Q) and angular momentum (j). A representative example of static black holes with vanishing angular momentum is the Reissner-Nordström (RN) black hole, a solution to Einstein's field equations sourced by a $U(1)$ Maxwell field, and characterized purely by its mass and charge.

1.1 Near extremal and extremal black holes

A charged black hole is said to be "extremal" if its mass is balanced by its electric and magnetic charges, i.e, $G^2M^2 = Q^2+P^2$[1], where Q and P are its electric and magnetic charges respectively. Consequently, the surface gravity of such black holes which also determines its Hawking temperature can be shown to vanish. Near extremal black holes, which constitute our prime interest, are those whose mass slightly outweighs the combination of electric and magnetic charges in the equality above, and the black hole is said to have a non zero, yet

[1] This condition is valid only for asymptotically flat black holes.

small Hawking temperature.

We will consider the Reissner-Nordström (RN) black hole as an example to demonstrate the properties of these classes of charged black holes. The electrically charged RN black hole in four dimensions given by the metric[2]

$$\begin{aligned} ds^2 &= f(r)d\tau^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2, \\ f(r) &= 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2} \end{aligned} \tag{1.1}$$

constitutes a classical solution to the Euclidean action describing Einstein gravity in an asymptotic AdS_4 spacetime, coupled to a $U(1)$ Maxwell field

$$\begin{aligned} I_{\text{Einstein-Maxwell}} = &-\frac{1}{16\pi G}\left[\int_{\mathcal{M}_4}\sqrt{g}(R+2\Lambda)\,d^4x - 2\int_{\partial\mathcal{M}_4}\sqrt{h}K\,d^3x\right] \\ &+ \frac{1}{4G}\int_{\mathcal{M}_4}\sqrt{g}F_{\mu\nu}F^{\mu\nu}\,d^4x, \end{aligned} \tag{1.2}$$

where M and Q are the mass and charge of the black hole respectively, G is Newton's gravitational constant, L is the AdS_4 curvature radius of the asymptotically AdS black hole, $\Lambda = \frac{3}{L^2}$ denotes the cosmological constant, $g = \det(g_{\mu\nu})$, $h_{\mu\nu}$ is the induced metric along the boundary of the manifold, and K is the extrinsic curvature[3]. The boundary term involving K, referred to as the Gibbons-Hawking-York (GHY) term, is necessary to make the variation of the action 1.2 with respect to the metric $g_{\mu\nu}$ well defined, when dealing with boundary conditions that fix the metric along the boundary of the manifold. For extremal black holes, the metric function becomes

$$f(r)|_{\text{extremal}} = f_0(r) = \frac{(r-r_0)^2}{r^2L^2}(L^2 + r^2 + 2rr_0 + 3r_0^2), \tag{1.3}$$

where r_0 is the radius of the horizon of the extremal black hole. This expression can be obtained from $f(r)$ in 1.1 by using the expressions for the extremal charge and mass of the black hole,

$$Q^2 = r_0^2 + \frac{3r_0^4}{L^2}, \qquad M_0 = \frac{r_0}{G}\left(1 + \frac{2r_0^2}{L^2}\right). \tag{1.4}$$

[2]given in the Euclidean signature, where the time coordinate is Wick rotated to imaginary values, $t \to -i\tau$.

[3]Extrinsic curvature is the curvature of a submanifold which is the boundary here, when viewed as an embedding in the bulk manifold. $K = \nabla_\mu n^\mu$, where n^μ is the normal to the embedded surface.

In the asymptotic flat space limit, $L \to \infty$, 1.3 reduces to

$$f_0(r)|_{L\to\infty} = \frac{(r-r_0)^2}{r^2}. \tag{1.5}$$

1.1.1 Near horizon region

The near horizon region (NHR) of extremal and near extremal black holes can be shown to have the geometry $AdS_2 \times X_{D-2}$, where AdS_2 refers to the Anti-de Sitter geometry in two dimensions, and X_{D-2} denotes a compact space in $D-2$ dimensions, where D is the spacetime dimension. For the extremal RN black hole in $4D$, X_2 becomes S^2, as is evident from the presence of the two sphere metric $d\Omega_2^2$ in 1.1. In the NHR, characterized by $r - r_0 \ll r_0$, the metric function in equation 1.3 becomes

$$f_0(r) = \frac{(r-r_0)^2}{L_2^2}, \tag{1.6}$$

where $L_2 := \frac{Lr_0}{\sqrt{L^2+6r_0^2}}$, and the radius of the S^2 internal space is given by r_0. The universality of the AdS_2 geometry in the NHR offers a remarkable simplification in the study of near extremal and extremal black holes that follows.

1.2 Jackiw-Teitelboim (JT) theory of $2D$ gravity

1.2.1 Dimensional reduction

This subsection is largely based on [2]. Motivated by the $AdS_2 \times S^2$ NHR geometry of near extremal RN black holes, we look for static, spherically symmetric solutions in two dimensions, by integrating out the internal space S^2, thereby reducing the theory to the $r - \tau$ plane. This is achieved by taking the ansatz

$$ds^2 = g_{\alpha\beta}(r,\tau)dx^\alpha dx^\beta + \boldsymbol{\chi}(r,\tau)d\Omega_2^2 \tag{1.7}$$

where $\alpha, \beta = 1, 2$; $x^1 = \tau, x^2 = r$. This resembles the classical solution 1.1 except that now a new field $\boldsymbol{\chi}(r,\tau)$ characterizes the size of the internal space. Substituting this ansatz in the action 1.2, and performing a partial integration over the internal space S^2, followed by the transformation $g_{\alpha\beta} \to \frac{r_0}{\boldsymbol{\chi}^{\frac{1}{2}}} g_{\alpha\beta}$ one obtains[4]

$$
\begin{aligned}
I_{2D} = &-\frac{1}{4G}\left[\int_{\mathcal{M}_2} \sqrt{g}\left(\boldsymbol{\chi} R + \frac{2r_0}{\boldsymbol{\chi}^{1/2}} + 2r_0\boldsymbol{\chi}^{1/2}\Lambda\right) d^2x - \int_{\mathcal{M}_2} \sqrt{g}\frac{\boldsymbol{\chi}^{3/2}}{r_0} F^{\alpha\beta}F_{\alpha\beta}\, d^2x\right] \\
&-\frac{1}{2G}\int_{\partial\mathcal{M}_2} \sqrt{h}\boldsymbol{\chi} K\, dx + \frac{1}{G}\int_{\partial\mathcal{M}_2} \sqrt{h}\frac{\boldsymbol{\chi}^{3/2}}{r_0} n_\alpha F^{\alpha\beta} A_\beta\, dx.
\end{aligned}
\tag{1.8}
$$

With spherical symmetry, the only non zero component of the field strength tensor in the presence of an electric field, corresponds to $E_r = F_{r\tau} = \frac{Q}{r^2}$. If we had chosen magnetically charged black holes instead, one could have simply integrated out the only non zero component of $F_{\mu\nu}$, $F_{\theta\phi}$ during dimensional reduction. We would ultimately like to compute the partition function of a canonical ensemble of such systems. This corresponds to the boundary condition that fixes the charge of the black hole which in turn amounts to fixing the value of the field strength tensor on the boundary of the manifold. Much like the case with the metric tensor, we have added a boundary term for the gauge field in 1.8 where n_α is the outward normal one form along the boundary, to make the variation with respect to the gauge field well defined.

1.8 belongs to the class of "dilaton" gravity models coupled to a gauge field, where the "dilaton" is the field $\boldsymbol{\chi}$, that multiplies the Ricci scalar. This action possesses a solution where the metric is AdS_2, the dilaton takes the constant value, $\chi_0 = r_0^2$ and the field strength tensor is given by

$$
F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta = \frac{Qr_0}{\boldsymbol{\chi}^{3/2}}\sqrt{g}\epsilon_{\alpha\beta}\, dx^\alpha \wedge dx^\beta, \tag{1.9}
$$

with the convention $\epsilon_{r\tau} = 1$, and for the extremal charge in 1.4. This solution clearly describes the NHR of extremal RN black holes. We now introduce a small temperature scale to the problem by slightly breaking the scale invariance of the NHR AdS_2 solution. The near extremal solution in the NHR is achieved by a small shift of the extremal dilaton value, $\boldsymbol{\chi}(r,\tau) \to \chi_0 + \chi$, which gives

[4]The gauge fields have been renormalized at this step, $A_\mu \to \frac{A_\mu}{\sqrt{4\pi}}$.

$$
\begin{aligned}
I_{2D} = &- \frac{\chi_0}{4G}\int_{\mathcal{M}_2^{NHR}} \sqrt{g}R\, d^2x - \frac{\chi_0}{2G}\int_{\partial\mathcal{M}_2^{NHR}} \sqrt{h}K\, du \\
&- \frac{1}{4G}\int_{\mathcal{M}_2^{NHR}} \sqrt{g}\chi\left(R + \frac{2}{L_2^2}\right) d^2x - \frac{1}{2G}\int_{\partial\mathcal{M}_2^{NHR}} \sqrt{h}\chi K\, du + O(\chi^2).
\end{aligned}
\tag{1.10}
$$

If the NHR is a compact region, then the first two terms in the action 1.10 amounts to just a constant multiple of the Euler characteristic, a topological invariant of the manifold, by using the Gauss-Bonet theorem. The next two terms form a special class of dilaton gravity models, the Jackiw-Teitelboim (JT) theory of $2D$ gravity, in the presence of boundary conditions that fix the metric at the boundary. The JT theory possesses the AdS_2 geometry given by $R = -\frac{2}{L_2^2}$ as a classical solution, as is expected from the NHR of extremal RN black holes.

1.2.2 Schwarzian action

The action 1.10 describes the NHR, comprising the AdS_2 geometry, of electrically charged RN black holes. In the Euclidean signature, AdS_2 is just the hyperbolic disk. We will describe the bulk AdS_2 geometry by one of the two coordinate systems

$$
\begin{aligned}
ds^2 &= \frac{L_2^2}{z^2}\left(dt^2 + dz^2\right) \quad \text{(Poincare)} \\
&= L_2^2(d\rho^2 + \sinh^2\rho\, d\tau^2) \quad \text{(Rindler)}.
\end{aligned}
\tag{1.11}
$$

The boundary term that accompanies the JT theory in 1.10 forms the central attention of this section. The presence of a boundary that "cuts" the near horizon AdS_2 geometry, amounts to studying simply connected chunks of the bulk AdS_2 geometry. [3] motivates the interest in such geometries by invoking the "backreaction" problem which describes how the asymptotic geometry of AdS_2 is destroyed by a diverging dilaton field, due to the backreaction from a non zero matter stress tensor. Since JT gravity approximates the full action 1.8 only in the NHR, we could avoid such a divergence in the dilaton by introducing a boundary in the NHR, that cuts out the bulk geometry, before it extends into the asymptotic region. This is the relevance of the boundary term $-\frac{1}{2G}\int_{\partial\mathcal{M}_2}\sqrt{h}\chi K\, du$ in 1.10. Consequently, we will use the renormalized dilaton at the boundary, $\chi_r = \frac{\chi_b}{\epsilon}$ for small ϵ, where $\frac{1}{\epsilon}$ captures the divergence in the dilaton as $\epsilon \to 0$.

Let the boundary curve $(t(u), z(u))$ be parametrized by u, where u is considered to be the proper time along the boundary curve. As long as the boundary cuts out a simply connected chunk of the bulk, the first two terms in the action 1.10 combined, being a topological invariant, is insensitive to the shape of the boundary curve. This is an enormous symmetry which is broken by the $O(\chi)$ boundary term. By moving away from the $O(\chi_0)$ terms, we are explicitly breaking this symmetry. Nevertheless, at this point, it seems plausible to assume that the boundary retains time reparametrization invariance, i.e, invariance under reparametrization of the boundary time, $u \to f(u)$, since the action depends directly only on the metric components (t, z). This full reparametrization symmetry along the boundary, is also the conformal symmetry of the one dimensional boundary theory.

We will impose a Dirichlet boundary condition on the induced metric along the boundary, by assuming that $ds^2|_{\text{boundary}} = g_{uu}du^2 = \frac{L_2^2}{\epsilon^2}du^2$. This implies that

$$z = \epsilon\sqrt{t'^2 + z'^2} = \epsilon t' + O(\epsilon^3), \tag{1.12}$$

where primes denote derivatives with respect to u. This relation recasts the boundary action as a truly one dimensional theory, where the degree of freedom corresponding to the shape of the boundary curve is entirely given by the dynamical variable $t(u)$. Fixing the boundary metric also fixes the proper length of the boundary curve, $l = \int_0^\beta \sqrt{g_{uu}}\, du$, where β is the periodicity of the Euclidean time u. In the presence of such a boundary constraint, reparametrizations of u amounts to more than just relabelling of the boundary time. The shape of the boundary curve changes in order for the metric in the reparametrized coordinates to still obey the Dirichlet boundary constraint, thereby spontaneously breaking the time reparametrization invariance of the system. We emphasize again, that the time reparametrization symmetry is an asymptotic symmetry of AdS_2 spaces and it is broken, the moment we introduce a boundary and impose Dirichlet boundary conditions for the fields on it. Such spaces are referred to as nearly AdS_2 or $\mathscr{N}AdS_2$ spaces. This explicit and spontaneous breaking of the time reparmetrization symmetry of the boundary, crucially determines the thermodynamics of such systems [4].

However, not all reparametrizations map a given cut out of the hyperbolic disc to a new one; rotations and finite translations of a given boundary curve, does not change the bulk we are cutting out. These correspond to the transformations [3]

$$t(u) \to \frac{at(u)+b}{ct(u)+d}; \quad ad-bc=1, \tag{1.13}$$

which constitute the special linear subgroup in two dimensions ($SL(2;\mathbb{R})$) of the group of all reparametrizations. Therefore, in the presence of a Dirichlet boundary constraint, time reparametrization invariance is spontaneously broken and reduced to $SL(2;\mathbb{R})$, and $t(u)$ is the Goldstone mode that emerges as a result.

In the rest of this section, we describe explicitly, how the simplest action that is $SL(2;\mathbb{R})$ invariant, arises from the boundary term under consideration. For a boundary that is one dimensional, the extrinsic curvature is given by

$$K = \frac{h(T, \nabla_T n)}{h(T,T)}{}^{5} \tag{1.14}$$

where T and n refer to the tangent and the unit normal vector fields, along the boundary. Using the leading order boundary constraint 1.12, we get

$$\begin{aligned} K &= \frac{t'(t'^2+z'^2+zz'')-zz't''}{L_2(t'^2+z'^2)^{\frac{3}{2}}} = \frac{1}{L_2}\left(1+\epsilon^2 \mathrm{Sch}(t,u)+O(\epsilon^4)\right) \\ \text{where} \quad \mathrm{Sch}(t,u) &= \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2, \end{aligned} \tag{1.15}$$

and is referred to, as the Schwarzian action. The Schwarzian is indeed $SL(2,\mathbb{R})$ invariant;

$$\mathrm{Sch}(t,u) = \mathrm{Sch}\left(\frac{a\,t+d}{c\,t+d}, u\right), \text{ for } ad-bc=1. \tag{1.16}$$

The Dirichlet boundary conditions on the metric, and renormalization of the dilaton field give $\sqrt{h} = \frac{L_2}{\epsilon}$ and $\chi|_{\text{boundary}} \to \chi_b = \frac{\chi_r}{\epsilon}$. Combining all this, the leading and subleading contributions to the $O(\chi)$ GHY boundary term can be written down as follows,

$$-\frac{1}{2G}\int_{\partial\mathcal{M}_2^{NHR}} \sqrt{h}\chi K\, du = -\frac{1}{2G}\int_{\partial\mathcal{M}_2^{NHR}} \chi_r \left(\frac{1}{\epsilon^2} + \mathrm{Sch}(t,u)\right) du. \tag{1.17}$$

The leading term is divergent in the $\epsilon \to 0$ limit and can be regularized by adding appropriate

[5] $h(X,Y) = h_{ab}X^aY^b$.

counter terms to the action. In section 2.2.2, where setting up a euclidean gravity path integral for the partition function of an effective two dimensional gravity theory is discussed, we show how the path integral over the dilaton effectively fixes the bulk metric to the AdS_2 geometry given by $R = -\frac{2}{L_2^2}$. Therefore, JT gravity in the presence of appropriate boundary conditions, reduces simply to the Schwarzian action, at the boundary.

Lagrangians containing higher order time derivatives like the Schwarzian aren't usually encountered in physics. Using the standard Hamiltonian description for higher order Lagrangians, one can see that the Hamiltonian is again given by the Schwarzian. The presence of higher order derivatives raises concerns about the Ostrogradsky instability which concerns the boundedness of the Hamiltonian function. However, the Schwarzian is bounded below as pointed out by Witten in [5], and argued in [6]. The idea is to determine critical points of the Hamiltonian which obey some physical constraints, which turns out to be uniquely determined. Finding the Hessian of the Hamiltonian for fluctuations about this critical point to be positive, and realizing that the phase-space is connected, suggests that the Hamiltonian function is bounded below.

1.3 Thermodynamic mass gap problem

Let M_0 and S_0 be the mass and entropy of the extremal RN black hole. We are interested in the thermodynamics of near extremal black holes, with temperatures only slightly above zero. A purely classical analysis can reveal that the internal energy and entropy of near extremal black holes of a fixed charge Q, scale as [7]

$$E(\beta, Q) = M_0 + \frac{2\pi^2}{M_{SL(2)}} T^2 + ..., \quad S(\beta, Q) = S_0 + \frac{4\pi^2}{M_{SL(2)}} T + ... \tag{1.18}$$

where following [7], we define a energy scale $M_{SL(2)} = \frac{G}{r_0 L_2^2}$. The linear in T dependence of the entropy comes purely from the contribution of the unique physically relevant classical solution to the Schwarzian action. Details of this solution and the physical constraints imposed to arrive at it are discussed in section 2.2.3. We are interested in understanding whether near extremal black holes with non zero Hawking temperature can thermalize, or come to equilibrium by emitting Hawking radiation, because only then can we proceed to ask how thermodynamic variables such as the entropy, internal energy, etc. behave as a

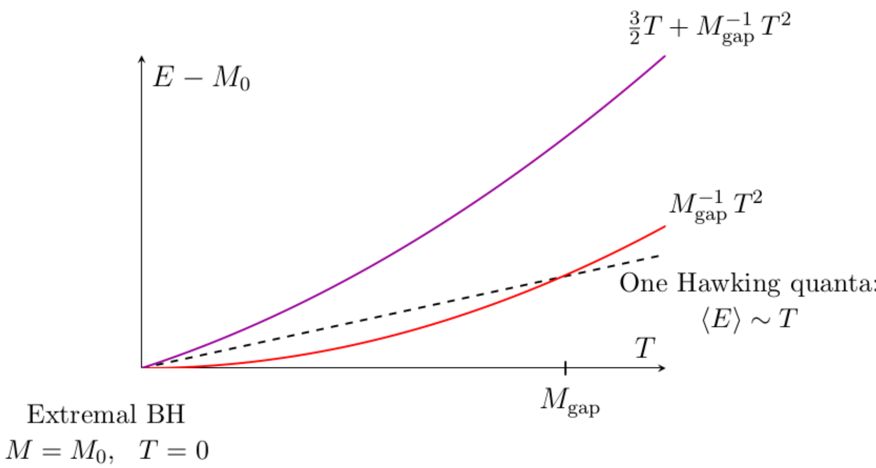

Figure 1.3.1: Classical contribution to energy above extremality at fixed charge (red curve), quantum corrections to the classical energy (purple curve), and energy of typical Hawking quanta (dashed line) as a function of Hawking temperature, from [7]. M_{gap} and $M_{SL(2)}$ refer to the same variable.

function of temperature. The energy of a typical Hawking quanta emitted by a black hole of temperature T, scales roughly as T itself. Combining this with the energy scale in the expression 1.18 gives us Figure 1.3, borrowed from [7].

Focussing on only the red and dashed curves, we infer that for $T < M_{SL(2)}$, the black hole lacks sufficient excitation energy to emit even a single Hawking quanta of average energy, T. This seems to imply that thermodynamics fails for near extremal black holes in the temperature range $0 < T < M_{SL(2)}$. The thermodynamic mass gap conjecture avoids this problem by stating that the region in the black hole spectrum corresponding to $T < M_{SL(2)}$ be interpreted as a literal mass gap, i.e, there are no near extremal black hole states in this range.

While this conjecture has been partly supported for supersymmetric black holes[6], the resolution for non supersymmetric near extremal black holes has been unsatisfacory to this date. Iliesiu and Turiaci [7] propose to solve the mass gap problem by finding the quantum corrections at small T, using the euclidean gravity path integral formalism.

The purple curve corresponding to the expression for $E_{[RN]}$ in 1.19, displayed in Fig. 1.3

[6]For more details, the reader is referred to recent works such as [8]

suggests that after accounting for their one-loop quantum correction, the near extremal black hole's temperature is always above the excitation energy needed to emit a Hawking quanta even for arbitrarily small temperatures, thereby seemingly resolving the mass gap problem. Moreover, [7] also shows that the consequent density of states is non zero for $E < M_{SL(2)}$; it smoothly goes to zero as $E - M_0 \to 0$. However, this comes at the cost of the partition function and the entropy being ill defined in the $T \to 0$ limit. For non rotating black holes of fixed charge, at finite inverse temperature β, their calculations give

$$\begin{aligned} Z_{[RN]}[\beta, Q] &= \left(\frac{\Phi_b}{\beta}\right)^{\frac{3}{2}} e^{\pi\Phi_0 - \beta M_0 + \frac{2\pi^2}{\beta}\Phi_b}, \\ S_{[RN]}[\beta, Q] &= (1 - \beta\partial_\beta)\ln Z = S_0 + \frac{3}{2} + \frac{4\pi^2\Phi_b}{\beta} - \frac{3}{2}\ln\frac{\beta}{e\Phi_b}, \\ E_{[RN]}[\beta, Q] &= M_0 + \frac{2\pi^2\Phi_b}{\beta^2} + \frac{3}{2\beta} \end{aligned} \tag{1.19}$$

where $S_0 = \pi\Phi_0$, $\Phi = \frac{\chi}{G}$, $\Phi_b = M_{gap}^{-1}$, and the internal energy, E is obtained from Helmholtz free energy, F by using $F = -\frac{1}{\beta}\ln Z$, and $E = F + TS$. The $\ln T$ correction to S and the $O(T)$ term in E come from quantum corrections at one loop order (quadratic order in fluctuations) of the path integral for the partition function, in the semiclassical approximation. The $T \to 0$ limit crucially determines the statistical mechanics of extremal black holes. However from expressions 1.19, we infer that as $T \to 0$, $Z_{[RN]} \to 0$, indicating that there are no extremal states and $S_{[RN]} \to -\infty$ which cannot be ascribed to any physical explanation. In this thesis, we look at several appproaches aimed at resolving this problem.

Chapter 2

Methods

2.1 Coadjoint Orbit Quantization

In the path integral approach to quantum field theory, the ambiguity in the measure of the path integral is an unsolved problem. In most applications, this ambiguity is taken care of by normalizing physical quantities appropriately. However, in gravity, the space of fields over which we perform the path integral can contribute non trivially to the measure. Contributions from the measure can involve parameters that characterize the system such as the temperature, and hence become important in studying leading quantum corrections to the classical temperature dependence of thermodynamic quantities such as entropy, internal energy, etc. The mathematical tool of coadjoint oribit quantization becomes very useful in deriving the structure of such contributions. In this section, we develop this concept with tools from group theory. This section is largely based on [9], [6], and [5].

2.1.1 Definitions

Let G be a Lie group with an associated Lie algebra $\mathscr{G}$. Let $\mathscr{G}^*$ be the dual space of $\mathscr{G}$, i.e, the space of linear forms $p : \mathscr{G} \to \mathbb{R}$. Similarly, G^* is defined as the dual space of G. The elements of $\mathscr{G}$ and $\mathscr{G}^*$ are referred to as adjoint and coadjoint vectors respectively. Let $X, Y \in \mathscr{G}$, $f, g \in G$, $u, v \in G^*$, and $a, b \in \mathscr{G}^*$.

Definition 2.1.1. *The adjoint representation of a group G is the homomorphism*

$$Ad : G \to GL(\mathscr{G}) : g \to Ad_g \tag{2.1}$$

where Ad_g is the linear operator that acts on $\mathscr{G}$ according to

$$Ad_g(X) = \frac{d}{dt}(ge^{tX}g^{-1})|_{t=0}. \tag{2.2}$$

By $GL(\mathscr{G})$, we denote the general linear group of representations of $\mathscr{G}$. For matrix groups $Ad_g(X) = gXg^{-1}$. The adjoint representation of the algebra $\mathscr{G}$ is defined as the differential of 2.1 at identity

$$ad : \mathscr{G} \to GL(\mathscr{G}) : X \to ad_X \tag{2.3}$$

where ad_X is the linear operator that acts on $\mathscr{G}$ according to

$$ad_X(Y) \equiv \frac{d}{dt}(Ad_{e^{tX}}Y)|_{t=0} = \frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0} = [X,Y]. \tag{2.4}$$

where $[X,Y] = XY - YX$ is the Lie bracket of X and Y.

Definition 2.1.2. *The coadjoint representation of a group G is the homomorphism*

$$Ad^* : G \to GL(\mathscr{G}^*) : f \to Ad_{f^*} \tag{2.5}$$

which is the dual to the adjoint representation

$$Ad_{f^*}(b) = b \circ (Ad_f)^{-1}, \tag{2.6}$$

i.e, $\langle Ad_{f^*}(b), X\rangle \equiv \langle b, Ad_{f^{-1}}(X)\rangle \ \forall\, b \in \mathscr{G}^*$ and any $X \in \mathscr{G}$. The set

$$W_b \equiv \{Ad^*_g(b) |\, \forall g \in G\} \tag{2.7}$$

is called the *coadjoint orbit* of b. The space $\mathscr{G}^*$ can be foliated into disjoint coadjoint orbits. In other words, W_b consists of coadjoint vectors that can be obtained from b by the G action. However, not all adjoint vectors generate a distinct coadjoint vector. We will explore this redundancy in section 2.1.3. The coadjoint representation of the algebra $\mathscr{G}$, is the dual of the infinitesimal adjoint representation 2.4, i.e, the differential of 2.5 at identity.

$$ad^*_X(b) \equiv \frac{d}{dt}(Ad^*_{e^{tX}}(b))|_{t=0} = -b \circ ad_X = -b[X, \cdot]. \tag{2.8}$$

We will also need the definition of a symplectic manifold for what follows.

Definition 2.1.3. *A symplectic manifold is the pair $(\mathcal{M}, \omega)$ of a smooth, even dimensional manifold $\mathcal{M}$, and a symplectic form, ω. A symplectic form on $\mathcal{M}$, is a non degenerate, closed two form ω on $\mathcal{M}$.*

A differential form is said to be closed if its exterior derivative is zero, i,e, $d\omega = 0$. It is non degenerate if $\forall\, p \in \mathcal{M}$, a vector $v \in T_p\mathcal{M}$ such that $\omega_p(v, w) = 0\ \forall\, w \in T_p\mathcal{M}$ necessarily vanishes. In local coordinates, non degeneracy means that the components ω_{ij} constitute an invertible, antisymmetric matrix.

2.1.2 Symplectic structure

We begin by understading how adjoint vectors generate translations in the space of coadjoint vectors. In other words, any tangent vector X to the coadjoint orbit can be associated with an adjoint vector that generates translation in the X-direction. The coadjoint vector obtained as a result of an adjoint vector, U acting on a coadjoint vector, a is described by its action on another adjoint vector, V

$$U(a)(V) = -a([U, V]). \tag{2.9}$$

This definition ensures that the pairing of V and a, given by $V(a)$ is invariant under infinitesimal transformations of V and a generated by another adjoint vector U, i.e,

$$(U(a))(V) + a(U(V)) = 0, \tag{2.10}$$

where the action of one adjoint vector on the other generates a new adjoint vector given by their Lie bracket, $U(V) = [U, V]$. One immediately recognizes 2.9 with the coadjoint representation of U. The above properties motivate the existence of a natural G-invariant symplectic structure ω on the tangent space to W_b at b, where $b \in \mathscr{G}^*$, given by[1].

$$\omega(a, a') = -b([U, U']) \quad \text{where } U(b) = a, U'(b) = a'; \quad a, a', b \in \mathscr{G}^* \text{ and } U, U' \in \mathscr{G}. \tag{2.11}$$

[1]For an elaborate proof, the reader is referred to [5].

ω is referred to as the Kirillov-Kostant symplectic form on the symplectic manifold, (W_b, ω).

2.1.3 Coadjoint orbits of the Virasoro group

Let us start by describing the group of diffeomorphisms of the circle, denoted by $\text{Diff}(\mathbb{S}^1)$. Elements of $\text{Diff}(\mathbb{S}^1)$ consist of monotonous, single valued functions $\Phi : [0, 2\pi) \to [0, 2\pi)$ such that $\Phi(\theta + 2\pi) = \Phi(\theta) + 2\pi$[2], and the Lie algebra consists of vector fields $f(\theta)\frac{d}{d\theta}$ with the associated Lie bracket given by

$$[f, g] = (f(\theta)g'(\theta) - g(\theta)f'(\theta))\frac{d}{d\theta}. \tag{2.12}$$

Coadjoint vectors are given by a quadratic differential $b(\theta)d\theta^2$, and the pairing of coadjoint and adjoint vectors gives the real number

$$\langle b, f \rangle = \frac{1}{2\pi}\int_0^{2\pi} b(\theta)f(\theta)d\theta. \tag{2.13}$$

The Virasoro group, a Lie group denoted by $\widehat{\text{Diff}(\mathbb{S}^1)}$ is the central extension of $\text{Diff}(\mathbb{S}^1)$. This forms a part of the asymptotic symmetry group of many gravitational systems. In order to develop the notion of a central extension, we will develop some concepts for the associated Lie algebra since we will mostly be working with elements of the algebra. For parallel definitions associated to the Lie group, we refer the reader to [9].

A representation of the Lie algebra $\mathscr{G}$ in a vector space $\mathbb{V}$ is a linear map $\mathscr{T} : \mathscr{G} \to \text{End}(\mathbb{V})$ such that $\mathscr{T}[X] \circ \mathscr{T}[Y] - \mathscr{T}[Y] \circ \mathscr{T}[X] = \mathscr{T}[[X, Y]], \forall\, X, Y \in \mathscr{G}$.

Definition 2.1.4. *Let k be a non negative integer and $\mathscr{T}$ a representation of $\mathscr{G}$ in $\mathbb{V}$. A $\mathbb{V}$-valued k-cochain on $\mathscr{G}$ is a continuous, multilinear, completely antisymmetric map*

$$c : \underbrace{\mathscr{G} \times \mathscr{G} \times ... \times \mathscr{G}}_{k\ times} \to \mathbb{V} : (X_1, X_2, ..., X_k) \to c(X_1, X_2, ..., X_k), \tag{2.14}$$

where $0 \leq k \leq dim(\mathscr{G})$.

A *k-cocycle* is a k-cochain c such that $d_k c = 0$, where d is the Chevalley-Eilenberg

[2]This condition says that such functions have a winding number 1, around the circle.

differential [9]. Specifically, a two cochain is a cocyle when

$$\begin{aligned} & c([X,Y],Z) + c([Y,Z],X) + c([Z,X],Y) = \\ & = \mathscr{T}[X] \cdot c[Y,Z] + \mathscr{T}[Y] \cdot c[Z,X] + \mathscr{T}[Z] \cdot c[X,Y]. \end{aligned} \tag{2.15}$$

The central extension of a Lie algebra is defined as follows.

Definition 2.1.5. *Let $\mathscr{G}$ be a real Lie algebra and let c be a real two-cocycle on $\mathscr{G}$. Then c defines a central extension $\widehat{\mathscr{G}}$ of $\mathscr{G}$, which is a Lie algebra whose underlying vector space*

$$\widehat{\mathscr{G}} = \mathscr{G} \oplus \mathbb{R} \tag{2.16}$$

is endowed with the centrally extended Lie bracket

$$[(X,\lambda),(Y,\mu)] = ([X,Y], c(X,Y)). \tag{2.17}$$

In particular, elements of $\widehat{\mathscr{G}}$ are pairs (X,λ) where $X \in \mathscr{G}$ and $\lambda \in \mathbb{R}$, and by definition $\mathbb{R}$ is an abelian subalgebra of $\widehat{\mathscr{G}}$. Under the trivial representation of the algebra $\mathscr{G}$, the RHS of 2.15 becomes zero and consequently the Lie bracket defined as in 2.17 obeys the Jacobi identity for any two cocyle. We now extend these considerations to the specific case of the Virasoro group and its corresponding algebra.

Definition 2.1.6. *The Virasoro group denoted by $\widehat{Diff(\mathbb{S}^1)}$, and diffeomorphic to $Diff(\mathbb{S}^1) \times \mathbb{R}$, consists as elements, pairs (f,λ) where $f \in Diff(S^1)$, and $\lambda \in \mathbb{R}$ with the group operation given by*

$$(f,\lambda) \circ (g,\mu) = (f \circ g, \lambda + \mu + \mathscr{C}(f,g)) \tag{2.18}$$

where $\mathscr{C}$ is the Bott-Thurston cocyle.

The corresponding Virasoro algebra is defined according to definition 2.1.5, where the cocyle c that defines the Lie bracket is related to the differential of the Souriau cocycle associated with $\mathscr{C}$ [9]. Vectors of the algebra $\widehat{\mathrm{Diff}(\mathbb{S}^1)}$ are pairs $(f(\theta),a)$, of a vector field $f(\theta)\frac{\partial}{\partial\theta}$, and a real number a which multiples the central element, which is also a number z[3]. Following [6], we also denote $(f(\theta),a)$ by $f(\theta)\frac{\partial}{\partial\theta} - iaz$. The Lie bracket is given explicitly by

$$\left[f_1(\theta)\frac{\partial}{\partial\theta} - ia_1 z, f_2(\theta)\frac{\partial}{\partial\theta} - ia_2 z\right] = (f_1 f_2' - f_2 f_1')\frac{\partial}{\partial\theta} + \frac{iz}{48\pi}\int_0^{2\pi} (f_1 f_2''' - f_2 f_1''')\, d\theta. \tag{2.19}$$

[3]A central element is that element of the group which commutes with all other elements of the group.

A coadjoint vector of the Virasoro group is the pair $(b(\theta), t)$ of a quadratic differential $b(\theta)d\theta^2$, and a real number t which multiples $\tilde{z}$, dual of the central element z i.e, $\tilde{z}(z) = 1$. The Virasoro invariant pairing of (b, t) with (f, a) is

$$\langle (b,t),(f,a)\rangle = \int_0^{2\pi} bf\, d\theta + ta. \tag{2.20}$$

Classification of orbits

One can study the transformation generated by an adjoint vector $(\phi(\theta), a)$ on the space of coadjoint orbit of a constant coadjoint vector (b_0, t_0) of the Virasoro group. By demanding that the variation of the pairing 2.20 under a transformation generated by another adjoint vector vanishes[4], one arrives at the finite form of the transformation generated by $(\phi(\theta), a)$

$$(b_0, t_0) \to (b(\phi), t_0) = \left(b_0 \phi'^2 - \frac{t_0\, \mathrm{Sch}(\phi, \theta)}{24\pi}, t_0 \right), \tag{2.21}$$

where now $\phi(\theta)$ acts as the inverse $\mathrm{Diff}(\mathbb{S}^1)$ element, and $\mathrm{Sch}(\phi, \theta)$ is the Schwarzian derivative of ϕ with respect to θ defined by

$$\mathrm{Sch}(\phi, \theta) = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2. \tag{2.22}$$

In particular, t_0 is invariant under transformations generated by adjoint vectors. This implies that t_0 is the central charge associated with an orbit. Therefore, we will hereafter identify the Virasoro central charge z with t_0. Points along the coadjoint orbit of (b_0, t_0) can be parametrized by the adjoint vectors $(\phi(\theta), a)$ modulo those adjoint vectors that do not generate a distinct coadjoint vector and hence "stabilize" the orbit. Let us denote by S, the set of stabilizers of the coadjoint orbit. The coadjoint orbit is therefore isomorphic to $\mathrm{Diff}(\mathbb{S}^1)/S$. The stabilizer set S, depends on the choice of coadjoint vector, and can be classified into the following three categories.

1. $b_0 \neq -\frac{t_0 n^2}{48\pi}$, where n is an integer. These are "ordinary" coadjoint orbits of the Virasoro group. The only stabilizers of this orbit are constant shifts of $\phi(\theta)$, $\phi(\theta) \to \phi(\theta) + a$ for any constant a. In other words, $\phi(\theta) + a$ generates the same coadjoint vector as

[4]This condition makes the pairing 2.20 G-invariant.

$\phi(\theta)$, evident from 2.21. Constant shifts of elements of $\text{Diff}(\mathbb{S}^1)$ is the $U(1)$ symmetry group, and hence, the stabilizer set is $U(1)$.

2. $b_0 = -\frac{t_0}{48\pi}$. This is referred to as the "first exceptional orbit" of the Virasoro group. The transformation 2.21 becomes

$$b(\phi) = -\frac{t_0}{24\pi}\text{Sch}\left(\tan\left(\frac{\phi}{2}\right), \theta\right). \tag{2.23}$$

   The stabilizer consists of fractional linear transformations of $\tan\left(\frac{\phi}{2}\right)$

$$\tan\left(\frac{\phi}{2}\right) \to \frac{a\tan\left(\frac{\phi}{2}\right)+b}{c\tan\left(\frac{\phi}{2}\right)+d}, \quad \text{such that} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2;\mathbb{R}). \tag{2.24}$$

   where $PSL(2;\mathbb{R})$ is the projective special linear group given by $PSL(2;\mathbb{R}) = SL(2;\mathbb{R})/\{\pm I\}$, where I is the 2×2 identity matrix, and $SL(2;\mathbb{R})$ is the special linear group defined as the set of 2×2 real matrices with determinant one. The resulting coadjoint orbit is isomorphic to $\text{Diff}(\mathbb{S}^1)/PSL(2;\mathbb{R})$.

3. $b_0 = -\frac{t_0 n^2}{48\pi}; n > 1$. These are referred to as "higher exceptional oribits" of the Virasoro group. The transformation 2.21 becomes

$$b(\phi) = -\frac{t_0}{24\pi}\text{Sch}\left(\tan\left(\frac{n\phi}{2}\right), \theta\right). \tag{2.25}$$

   which is invariant under the stabilizer $PSL^{(n)}(2;\mathbb{R})$ that corresponds to the transformations

$$\tan\left(\frac{n\phi}{2}\right) \to \frac{a\tan\left(\frac{n\phi}{2}\right)+b}{c\tan\left(\frac{n\phi}{2}\right)+d}. \tag{2.26}$$

   For further discussion of this category, the reader is referred to [5].

We will find the categories 1 and 2 most useful in our study of asymptotic symmetries of some special class of $2D$ gravitational systems.

2.1.4 Quantization of coadjoint orbits

The phase space of a classical mechanical system leads to the notion of a symplectic manifold. Since the generalized momentum transforms like a covector under coordinate transformations, the phase space of a classical system is decribed by the cotangent bundle of a manifold. It can be shown that every cotangent bundle has a canonical symplectic form.

Quantizing a classical mechanical system amounts to mapping the classical phase space to a Hilbert space, and classical observables which are smooth functions on the symplectic manifold to quantum observables which are self adjoint operators on the Hilbert space, in such as way so that the Poisson bracket goes to a commutator under this map. For a more precise notion of quantization, we refer the reader to [10].

The idea of quantization with respect to our current setting is nicely summarized in [6], "Quantizing these (coadjoint) orbits leads to elegant quantum mechanical models with a Hilbert space that is typically a single irreducible representation of G. Correlation functions and partition functions are simply group-theoretic functions (like characters) of this representation." The Quantization of coadjoint orbits can be performed using two different methods: geometric quantization, and phase space path integral quantization.

A Hamiltonian that enacts the transformation 2.9 can be naturally associated to the phase-space of the coadjoint orbit W_b. In path integral quantization, one can quantize this Hamiltonian system defined on the phase-space W_b by performing the path integral

$$\int [dx_i]\mathrm{Pf}(\omega)e^{iS} \tag{2.27}$$

where the Pfaffian of the symplectic form, $\mathrm{Pf}(\omega)$ defined as the square root of the determinant of ω, provides a natural measure on the symplectic manifold, W_b. When the Hamiltonian for the system is as described above, one can employ the Duistermaat-Heckman formula to show that the path integral 2.27 is one-loop exact, i.e, quantization upto the quadratic order in fluctuations provides the exact answer. Later on, we will also be interested in more general Hamiltonians that do not necessarily generate the transformations 2.9 on W_b. Albeit, in such cases the space over which we perform the path integral will turn out to be a symplectic manifold and consequently, the measure $\mathrm{Pf}(\omega)$ will become the most important ingredient we derive from this procedure.

Computing the symplectic form

Combining 2.19, 2.20 and 2.21, the symplectic form ω acting on a pair of coadjoint vectors X_1 and X_2, generated by the action of adjoint vectors $F_1 = (f_1(\theta), a_1)$ and $F_2 = (f_2(\theta), a_2)$ respectively on the constant coadjoit vector (b_0, t_0), can be explicitly written down as

$$\begin{aligned}\omega(X_1, X_2) &= -\langle (b_0, t_0), [F_1, F_2]\rangle \\ &= -\int_0^{2\pi} \left\{ \frac{t_0}{48\pi}(f_1' f_2'' - f_2' f_1'') + \left(b_0 \phi'^2 - \frac{t_0}{24\pi}\mathrm{Sch}(\phi, \theta) \right)(f_1 f_2' - f_2 f_1') \right\} d\theta,\end{aligned} \tag{2.28}$$

where we have performed an integration by parts to arrive at the first term inside the flower bracket. By noting that an infinitesimal transformation by $F = (f, a)$ transforms ϕ according to

$$\partial_F \phi = f(\theta)\frac{\partial}{\partial\theta}\phi = f\phi', \tag{2.29}$$

we can rewrite the symplectic form in terms of a one form $d\phi$, such that $d\phi(F) = \partial_F\phi = f\phi'$, with the understanding that $\phi(\theta)$ labels points on the coadjoint orbit according to 2.21. In terms of this one form, ω becomes

$$\begin{aligned}\omega &= -\int_0^{2\pi} \left\{ \frac{t_0}{48\pi}\left(\frac{d\phi}{\phi'}\right)' \wedge \left(\frac{d\phi}{\phi'}\right)'' + \left(b_0 - \frac{t_0}{24\pi\phi'^2}\mathrm{Sch}(\phi, \theta)\right) d\phi \wedge d\phi' \right\} d\theta \\ &= -\int_0^{2\pi} \left\{ \frac{t_0}{48\pi}\frac{d\phi' \wedge d\phi''}{\phi'^2} + b_0 d\phi \wedge d\phi' \right\} d\theta.\end{aligned} \tag{2.30}$$

d is the exterior derivative that acts only on the field, ϕ used to label point on the phase-space, i.e, the coadjoint orbit. It does not act on the spatial coordinate θ. Therefore, we have used identities such as $(d\phi)' = d\phi'$ and $d\phi' \wedge d\phi' = 0$, in going from the first equality to the second, in 2.30. We expand upon how to compute the symplectic form explicitly using 2.30, in section 2.2.3.

2.2 One loop contribution to the partition function

2.2.1 Effective $2D$ action

In this section, we roughly follow [7] and [2], to outline the steps involved in setting up a path integral for the partition function of near extremal black holes.

In order to make use of the simplification offered by the universality of the near horizon geometry of near extremal black holes, we partition the spacetime into regions, the NHR obeying $r - r_0 \ll r_0$ and the far away region (FAR) obeying $r - r_0 \gg \delta r_h$, where δr_h denotes the difference in horizon radius of the near extremal (r_h) and extremal (r_0) black holes. The temperature of near extremal RN black holes depends on their geometry by

$$\beta = \frac{4\pi}{|f'(r_h)|} \tag{2.31}$$

where $f(r)$ is the function in 1.1. The temperature is related to δr_h as

$$\delta r_h = \frac{2\pi}{\beta} L_2^2 + ... \implies T \sim \frac{\delta r_h}{L_2^2}, \tag{2.32}$$

where the dots denote subleading terms in the large β limit. For near extremal black holes that only differ slightly from extremality such that $\delta r_h \ll r_0$, it is easy to see that the NHR and FAR regions have a non trivial overlap. These regions are depicted in a cartoon borrowed from [7] in Figure 2.2.1.

The relevant degrees of freedom in the dimensionally reduced theory 1.8 are the dilaton, the metric components, and the $U(1)$ Maxwell gauge field. In addition to these, [7] introduces a $SO(3)$ Yang-Mills field that slightly breaks the spherical symmetry in the full $4D$ theory. While this field is necessary to extend the analysis to rotating black holes with a small angular momentum labelled by the quantum number j, we will restrict our attention to non rotating black holes with $j = 0$. The path integral for the partition function becomes

$$Z = \int Dg_{\mu\nu} D\boldsymbol{\chi} DA_\mu e^{-I_{2D}}, \tag{2.33}$$

where I_{2D} is the action in 1.8. Using results from [11], [7] shows how to exactly integrate out the gauge field coupled to a dilaton, by taking advantage of working in $2D$. This gives them

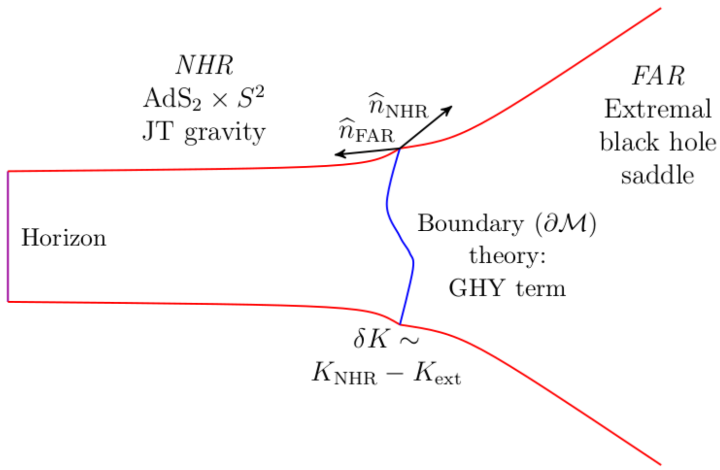

Figure 2.2.1: Figure from [7] depicting the NHR and FAR regions of a near extremal RN black hole. The GHY boundary term captures the effective action in the dimensionally reduced theory.

a sum of partition functions, over all possible values of the quantized charge Q weighted by factors dependent on Q. Fixing the charge of the black hole amounts to choosing a fixed Q sector from the infinite sum. Here, following [2], we will choose the more familiar route of retaining the gauge terms explicitly. Both these methods give rise to the same effective boundary action, as elaborated below. Performing the path integral directly, for the full action I_Q^{2D}[5] is out of reach, due to non trivial coupling of the dilaton with the metric components. To make use of the simplification provided by the near horizon geometry, we split the limits of integration of the action I_Q^{2D} that runs over the full spacetime, into

$$I_Q^{2D} = \underbrace{\int_{r_h}^{r_{\partial\mathcal{M}_{NHR}}} ()_{\text{bulk}}}_{I_{NHR}^{2D}} + \underbrace{\int_{r_{\partial\mathcal{M}_{NHR}}}^{r_{\partial\mathcal{M}_2}} ()_{\text{bulk}} + \int_{r_{\partial\mathcal{M}_2}} ()_{\text{boundary}}}_{I_{FAR}^{2D}}, \tag{2.34}$$

where $r_{\partial\mathcal{M}_2}$ denotes the asymptotic infinity of the $2D$ manifold, and $r_{\partial\mathcal{M}_{NHR}}$ is a boundary introduced in the overlap region of the NHR and FAR regions, $\delta r_h \ll r - r_0|_{r=r_{\partial\mathcal{M}_{NHR}}} \ll r_0$ such that the induced metric and the dilaton at this boundary obey the following Dirichlet

[5]The subscript Q denotes that we have chosen boundary conditions that fix the charge of the black hole.

boundary conditions.

$$\boldsymbol{\chi}|_{\partial\mathcal{M}_{NHR}} = \boldsymbol{\chi}_b, \quad h_{uu} = \frac{L_2^2}{\epsilon^2} \implies l = \int_0^\beta \sqrt{h}\, du. \tag{2.35}$$

The significance of the choice of these boundary constraints in defining the thermodynamics of the near horizon region have been explored in section 1.2.2. We have

$$\begin{aligned}
I_{NHR}^{2D} =& -\frac{1}{4G}\int_{r_h}^{r_{\partial\mathcal{M}_{NHR}}} \sqrt{g}\left(\boldsymbol{\chi} R + \frac{2r_0}{\boldsymbol{\chi}^{1/2}} + 2r_0\boldsymbol{\chi}^{1/2}\Lambda\right) d^2x + \frac{1}{4G}\int_{r_h}^{r_{\partial\mathcal{M}_{NHR}}} \sqrt{g}\frac{\boldsymbol{\chi}^{3/2}}{r_0}F^{\alpha\beta}F_{\alpha\beta}\, d^2x \\
&+ \frac{1}{G}\int_{r_{\partial\mathcal{M}_{NHR}}} \sqrt{h}\frac{\boldsymbol{\chi}^{3/2}}{r_0} n_\alpha F^{\alpha\beta}A_\beta\, dx, \\
I_{FAR}^{2D} =& -\frac{1}{4G}\int_{r_{\partial\mathcal{M}_{NHR}}}^{r_{\partial\mathcal{M}_2}} \sqrt{g}\left(\boldsymbol{\chi} R + \frac{2r_0}{\boldsymbol{\chi}^{1/2}} + 2r_0\boldsymbol{\chi}^{1/2}\Lambda\right) d^2x + \frac{1}{4G}\int_{r_{\partial\mathcal{M}_{NHR}}}^{r_{\partial\mathcal{M}_2}} \sqrt{g}\frac{\boldsymbol{\chi}^{3/2}}{r_0}F^{\alpha\beta}F_{\alpha\beta}\, d^2x \\
&- \frac{1}{2G}\int_{r_{\partial\mathcal{M}_2}} \sqrt{h}\boldsymbol{\chi} K\, dx + \frac{1}{G}\int_{r_{\partial\mathcal{M}_2}} \sqrt{h}\frac{\boldsymbol{\chi}^{3/2}}{r_0} n_\alpha F^{\alpha\beta}A_\beta\, dx + \underbrace{\frac{1}{2GL_2}\int_{r_{\partial\mathcal{M}_2}} \sqrt{h}\boldsymbol{\chi}\, dx}_{I_{CT}^{\partial\mathcal{M}_2}} \\
&- \frac{1}{G}\int_{r_{\partial\mathcal{M}_{NHR}}} \sqrt{h}\frac{\boldsymbol{\chi}^{3/2}}{r_0} n_\alpha F^{\alpha\beta}A_\beta\, dx,
\end{aligned} \tag{2.36}$$

where $I_{CT}^{\partial\mathcal{M}_2}$ is a counter term added at asymptotic infinity, to cancel the divergence that arises from the on shell evaluation of the GHY boundary term. We have also added a charge-fixing boundary term for the gauge field at $\partial\mathcal{M}_{NHR}$ to I_{NHR}^{2D}, and subtracted the same from I_{FAR}^{2D}. The presence of this term in I_{NHR}^{2D} enables us to fix the charge Q, which we fix to that of the classical extremal solution 1.4 while approximating the near extremal dilaton field as a perturbation around its extremal value as elaborated in section 1.2. After a relabelling of fields $\frac{\boldsymbol{\chi}}{G} \to \Phi$, and performing an expansion of I_{NHR}, in $\frac{1}{\Phi_0}$, where $\Phi_0 = \frac{r_0^2(Q)}{G}$, we get the perturbed bulk JT action, $\delta I_{JT}^{\text{bulk}}$

$$I_{NHR} = \frac{1}{4}\int_{r_h}^{r_{\partial\mathcal{M}_{NHR}}} \sqrt{g}\left[-\Phi_0 R - \Phi\left(R + \frac{2}{L_2^2}\right) + O\left(\frac{\Phi^2}{\Phi_0}\right)\right] d^2x. \tag{2.37}$$

In the FAR region, [11] treats the near extremal classical solution of the fields $g_{\mu\nu}$ and $\boldsymbol{\chi}$ as a first order perturbation from the classical extremal solution

$$g_{\mu\nu}|_{\text{classical}} = g_{\mu\nu}^{\text{ext}} + \delta g_{\mu\nu}^{\text{near ext}}, \quad \boldsymbol{\chi}|_{\text{classical}} = \boldsymbol{\chi}^{\text{ext}} + \delta\boldsymbol{\chi}^{\text{near ext}}. \tag{2.38}$$

Unlike the semiclassical expansion of the fields, in quantum fluctuations about their classical solutions, both the terms in each expansion above are classical contributions. Expanding I_{FAR} in the space of classical solutions gives

$$I_{FAR}[g_{\mu\nu}, \boldsymbol{\chi}] = I_{FAR}[g_{\mu\nu}^{\text{ext}}, \boldsymbol{\chi}^{\text{ext}}] + \frac{\delta I_{FAR}}{\delta g_{\mu\nu}}\bigg|_{\text{ext}} \delta g_{\mu\nu}^{\text{near ext}} + \frac{\delta I_{FAR}}{\delta \boldsymbol{\chi}}\bigg|_{\text{ext}} \delta \boldsymbol{\chi}^{\text{near ext}}. \tag{2.39}$$

From Appendix A of [2], we learn how the last two terms in 2.39 reduce to an effective action at $\partial\mathcal{M}_{NHR}$. Since I_{FAR} is a sum of bulk and boundary components, each of these terms will be a sum of $\delta I_{FAR}^{\text{bulk}}$ and $\delta I_{FAR}^{\text{boundary}}$. Variation of I_{FAR}^{bulk} will generate some boundary terms after an integration by parts. Since, there are two boundaries to I_{FAR}, namely the inner boundary at $r_{\partial\mathcal{M}_{NHR}}$, and the outer boundary at $r_{\partial\mathcal{M}_2}$, two sets of additional boundary terms will be generated from $\delta I_{FAR}^{\text{bulk}}$, one at each boundary. Now, the GHY and charge fixing boundary terms were added at asymptotic infinity, precisely to make the variational principle well defined. In other words, the variation of the boundary terms from $\delta I_{FAR}^{\text{boundary}}$ exactly cancel the terms at asymptotic infinity that get generated from $\delta I_{FAR}^{\text{bulk}}$.

The bulk components from $\delta I_{FAR}^{\text{bulk}}$, when evaluated at the extremal solution as mandated by 2.39 vanish since the FAR action is extremized at classical solutions. After some simplifications as elaborated in [2], it is easy to see that δI_{FAR} reduces to a boundary term at $r_{\partial\mathcal{M}_{NHR}}$[6]. The structure of the boundary term can be greatly simplified by rewriting the near extremal solution as a small perturbation about the extremal solution, as motivated by 2.39. In order to do this, we first observe that the resulting boundary term at $r_{\partial\mathcal{M}_{NHR}}$ arose from a variation of I_{FAR}, and $r_{\partial\mathcal{M}_{NHR}}$ was chosen to lie at the overlap of the NHR and FAR regions. This was what that enabled us to split the limits of integration of the full action, in the first place. To account for this condition, we need to tell this effective boundary term, that $r_{\partial\mathcal{M}_{NHR}}$ also lies in NHR. This is done by taking the extremal solution to be that of the near horizon AdS_2 metric.

In the near horizon limit, f_0 given by 1.6 is taken to be the extremal solution, while the

[6]There are no GHY and charge fixing boundary terms at $r_{\partial\mathcal{M}_{NHR}}$ to begin with, to cancel the terms generated from $\delta I_{FAR}^{\text{bulk}}$.

near extremal solution is obtained by varying the extremal solution by

$$\begin{aligned}
\delta g_{\tau\tau} &= \delta f_0 = -\frac{2G\delta M}{r}, \\
\delta g_{rr} &= \delta \frac{1}{f_0} = -\frac{1}{f_0^2}\delta f_0 = \frac{2G\delta M}{f_0^2 r}, \\
ds^2|_{\text{near ext}} &= (f_0(r) + \delta g_{\tau\tau})d\tau^2 + \left(\frac{1}{f_0(r)} + \delta g_{rr}\right) dr^2,
\end{aligned} \tag{2.40}$$

where δM is the difference in mass between the extremal and near extremal black hole. δM is related to the Hawking temperature T of the near extremal black hole by $\delta M = \frac{\pi^2 T^2 L^2 r_0}{3G}$ [12].

In the NHR, $r \to r_0 + \delta r$ and the leading order contribution to δf_0 is a constant, $\delta f_0 = -\frac{2G\delta M}{r_0}$ which we refer to by "$-A$" in subsequent discussions. While A is a constant in spacetime coordinates, it is important to note that A is temperature dependent through δM. We will return to this temperature dependence later, when calculating the entropy.

Using the explicit form for the extremal and near extremal solutions 2.40, and assuming that the NHR boundary in the extremal solution is present at a constant value of the coordinate $r_{\partial\mathcal{M}_{NHR}}$, simplifies the effective boundary term at $\partial\mathcal{M}_{NHR}$ to the following simple structure

$$\delta I_{FAR} = \delta I_{GHY} + \frac{1}{2GL_2}\int_{r_{\partial\mathcal{M}_{NHR}}} \delta(\sqrt{h})\boldsymbol{\chi}\, dx = \delta I_{JT}^{\text{boundary}}. \tag{2.41}$$

where the second term in the middle expression is the variation of the counter term associated with the GHY term at $\partial\mathcal{M}_{NHR}$. Together with δI_{GHY}, this gives $\delta I_{JT}^{\text{boundary}}$. Combining this result with 2.37, we display the significance of the JT theory in describing the dimensionally reduced theory of near extremal black holes.

We now simplify the boundary term in 2.41 further. δI_{GHY} contains terms proportional to $\delta\boldsymbol{\chi}$, $\delta\sqrt{h}$, and δK. Strict Dirichlet boundary conditions on the induced metric and dilaton at $\partial\mathcal{M}_{NHR}$ set $\delta\sqrt{h}$ and $\delta\boldsymbol{\chi}$ to 0. This leaves us with

$$I_{FAR}[g_{\mu\nu}, \boldsymbol{\chi}] = I_{FAR}[g_{\mu\nu}^{\text{ext}}, \boldsymbol{\chi}^{\text{ext}}] - \frac{1}{2G}\int_{r_{\partial\mathcal{M}_{NHR}}} \sqrt{h}\boldsymbol{\chi}\delta K\, du. \tag{2.42}$$

We proceed to understand the meaning of δK, in the expression above. We emphasize that "δ" in this calculation refers to a very specific variation; from the extremal solution, to the near extremal solution. Therefore, $\delta K = K_{\text{near ext}} - K_{\text{ext}}$. Although both the terms in δK originally came from a variation of the FAR action, these terms are treated as though they are induced by the NHR metric, along the boundary ∂M_{NHR} since the FAR action is now completely reduced to an action at this boundary.

$K_{\text{ext}} = \frac{1}{L_2}$, is the extrinsic curvature of the extremal AdS_2 metric with the metric function given in 1.6. This is more correctly the on shell part of extrinsic curvature, where the boundary is that of the hyperbolic disc located at $z = 0$ in the Poincare description, or equivalently at $r = \infty$ in spherical coordinates[7]. This is because, the NHR action associated with the extremal solution which comprises the $O(\chi_0)$ terms in 1.10 is a topological invariant, and hence, is degenerate under asymptotic time reparametrizations. This is an asymptotic symmetry of AdS_2. This degeneracy is slightly broken in the presence of near extremal deviations corresponding to the $O(\chi)$ terms.

Now, δK in [7] is essentially $K_{\mathcal{N}AdS_2} - K_{AdS_2}$ which is known to give the Schwarzian action in the presence of appropriate boundary conditions (see section 1.2.2). While we understand how $I_{\text{JT}}^{\text{bdy}}$ gives the Schwarzian from section 1.2.2, it is not immediately clear why $\delta I_{\text{JT}}^{\text{bdy}}$ (2.41 and 2.42) which now includes temperature dependent corrections to the NHR metric, should also give the Schwarzian action. We describe our understanding of this step in great detail in the next chapter.

In [7], Iliesiu and Turiaci use the simplification provided by the NHR geometry to write the near extremal dilaton as a perturbation over the extremal dilaton solution $\boldsymbol{\chi} = \chi_0 + \chi$, and go on to calculate the contribution from I_{FAR}^{bulk}, appropriately regularize it by including counter terms to the $I_{FAR}^{\text{boundary}}$ terms at asymptotic infinity, and arrive at the following action

$$\begin{aligned} I_{\text{eff}}^{2D} = \beta M_0(Q) &- \frac{1}{4}\int_{r_h}^{r_{\partial \mathcal{M}_{NHR}}} \sqrt{g}\left[\Phi_0 R + \Phi\left(R + \frac{2}{L_2^2}\right) + O\left(\frac{\Phi^2}{\Phi_0}\right)\right] d^2x \\ &- \frac{1}{2}\int_{r_{\partial \mathcal{M}_{NHR}}} \sqrt{h}\left[\Phi_0 K_{\mathcal{N}AdS_2} + \frac{\Phi_b}{\epsilon}\left(K_{\mathcal{N}AdS_2} - \frac{1}{L_2}\right)\right] du, \end{aligned} \tag{2.43}$$

where $\beta M_0(Q)$ which comes from the on shell evaluation of the FAR action, shifts the ground state energy of the system. We immediately note that contribution from the FAR terms do not explicitly appear in the effective $2D$ action 2.43. This is an enormous simplification,

[7]For a comment on the relation between these different coordinates, we refer the reader to [2].

which was however achieved at the expense of classically expanding the FAR action about the extremal solution. Therefore, quantum effects in the FAR region due to fluctuations in the metric were ignored in this calculation. This seems to be a common assumption in the study of statistical mechanics of near extremal black holes; quantum corrections from the near horizon region seem to give the leading temperature dependent correction to the classical thermodynamic variables. We expand on this observation in the next chapter.

2.2.2 Partition function and Entropy

Due to boundary conditions that fix the charge of the black hole to that of its extremal value 1.4, the only degrees of freedom in our effective theory are the dilaton along with the bulk and boundary metric components. The partition function set up in 2.33 now becomes

$$\begin{aligned} Z &= \int Dg_{\mu\nu} D\Phi e^{-I_{\text{eff}}^{2D}} \\ &= e^{\pi\Phi_0(Q)-\beta M_0} \int Dg_{\mu\nu} e^{\frac{1}{2}\frac{\Phi_b}{\epsilon}\int_{\partial\mathcal{M}_{NHR}}\sqrt{h}\left(K_{\mathscr{N}AdS_2}-\frac{1}{L_2}\right)du} \int D\Phi e^{\frac{1}{4}\int_{\mathcal{M}_{NHR}}\sqrt{g}\Phi\left(R+\frac{2}{L_2^2}\right)d^2x}, \end{aligned} \tag{2.44}$$

where $e^{\pi\Phi_0(Q)}$ comes from evaluating the $O(\Phi_0)$ terms in I_{eff}^{2D} using the Gauss-Bonet theorem, $Dg_{\mu\nu}$ above counts both the bulk and boundary metric components, and Φ_b being a constant comes out of the $D\Phi$ integral. Wick rotating the dilaton to the complex plane $\Phi \to i\Phi$, enables us to perform the Φ integral,

$$\begin{aligned} &\int D\Phi e^{\frac{i}{4}\int_{\mathcal{M}_{NHR}}\sqrt{g}\Phi\left(R+\frac{2}{L_2^2}\right)d^2x} = \delta\left(R+\frac{2}{L_2^2}\right); \\ &Z = e^{\pi\Phi_0(Q)-\beta M_0}\int Dg_{\mu\nu}\delta\left(R+\frac{2}{L_2^2}\right)e^{\frac{1}{2}\frac{\Phi_b}{\epsilon}\int_{\partial\mathcal{M}_{NHR}}\sqrt{h}\left(K_{\mathscr{N}AdS_2}-\frac{1}{L_2}\right)du}. \end{aligned} \tag{2.45}$$

This implies that each NHR patch contributing to the path integral is a cut out of AdS_2 with constant negative curvature $R = -\frac{2}{L_2^2}$, cut along a curve with a fixed induced metric and proper length. Now, as elaborated in section 1.2.2, in the presence of such boundary conditions,

$$K_{\mathscr{N}AdS_2} - \frac{1}{L_2} = \frac{\epsilon^2}{L_2}\text{Sch}(t,u), \tag{2.46}$$

and we are left to quantizing the Schwarzian action.

For later use, we mention here that the equation of motion of the Schwarzian action is given by

$$\frac{(\mathrm{Sch}(t,u))'}{t'(u)} = 0. \tag{2.47}$$

2.2.3 Quantizing the Schwarzian action

We now have all the necessary tools at our disposal to quantize the Schwarzian action. This subsection largely follows [13], and the methods developed in section 2.1. However, unlike [13], we retain and track explicit factors of β, that primarily enters the problem as the periodicity of Euclidean time. We will begin by reviewing a theorem that offers a remarkable simplicity in quantizing this action.

Duistermaat-Heckman (DH) theorem

Theorem 2.2.1. *The integral over a symplectic manifold of e^{H/g^2}, where H generates a $U(1)$ symmetry of the manifold, is one-loop exact.*

The only effective degree of freedom in the path integral for the partition function, 2.45 is the Goldstone mode $t(u)$ that emerged as a result of spontaneous breaking of the asymptotic time reparametrization symmetry in the NHR, where u is the proper time along the boundary, $\partial\mathcal{M}_{NHR}$ with periodicity β. For the purpose of this section, we switch to a new coordinate, τ which is related to the Poincare time, t by $t(u) = \tan\left(\frac{B\tau(u)}{2}\right)$, where $B = \frac{2\pi}{\beta}$. The functions $\tau(u)$ are restricted to those that are monotone increasing and that wind once around the boundary circle, ensured by the condition $\tau(u+\beta) = \tau(u) + \beta$. This ensures that they are elements of $\mathrm{Diff}(\mathbb{S}^1)$. However, since the Schwarzian action is stabilized by $PSL(2,\mathbb{R})$ transformations of $\tan(\frac{\pi\tau}{\beta})$, we are essentially integrating over the space of functions $\mathrm{Diff}(\mathbb{S}^1)/PSL(2,\mathbb{R})$, which is the first exceptional orbit of a constant coadjoint vector of the Virasoro group. Since coadjoint orbits are symplectic manifolds, the measure should now rightly include the Pfaffian of the symplectic form.

Before going into the measure, let us study the applicability of the DH theorem to the problem at hand. Clearly, $U(1)$ time translations form a subgroup of $SL(2,\mathbb{R})$, and hence is also a symmetry of the effective action, the Schwarzian. We can fully apply the DH theorem to show that the integral over the Schwarzian action is one loop exact if we can show that the Hamiltonian generates this very $U(1)$ symmetry.

The Schwarzian action with a $1/g^2$ coupling can be rewritten in the new coordinate,

$$\begin{aligned} I &= -\frac{1}{g^2}\int \mathrm{Sch}\left(\tan\frac{B\tau}{2}, u\right) du = -\frac{1}{g^2}\int \left[\mathrm{Sch}(\tau,u) + \frac{B^2\tau'^2}{2}\right] du \\ &= \frac{1}{2g^2}\int \left[\frac{\tau''^2}{\tau'^2} - B^2\tau'^2\right] du. \end{aligned} \tag{2.48}$$

In going from the first line to the second, we have used

$$\frac{\tau'''}{\tau'} = \frac{d}{du}\left(\frac{\tau''}{\tau'}\right) + \frac{\tau''^2}{\tau'^2} \tag{2.49}$$

to render the third derivative Schwarzian action as an effective second derivative action, upto a total derivative term. When $\tau(u) = u$, $\mathrm{Sch}(t, u)$ is a constant, and 2.47 implies that we have a classical solution. Indeed, this is the simplest classical solution that obeys the physical constraints described earlier, necessary to be a valid element of $\mathrm{Diff}(\mathbb{S}^1)$. Let us evaluate the action at this classical solution.

$$I_{\mathrm{cl}} = -\frac{1}{2g^2}\int_0^\beta B^2\, du = -\frac{2\pi^2}{\beta g^2}. \tag{2.50}$$

In order to compute the contribution of the one loop term, we need $\mathrm{Pf}(\omega)$. We need to slightly modify the results of section 2.1, to account for the change in periodicity of the angular coordinate (which corresponds to the boundary time) from 2π to β. An adjoint vector of the Virasoro group of periodicity β, τ is related to the corresponding adjoint vector of periodicity 2π, ϕ by $\tau(u) = \frac{\beta}{2\pi}\phi(\theta)$. Accounting for this scaling recasts 2.21 as,

$$(b_0, t_0) \to (b(\tau), t_0) = \left(b_0 B^2\tau'^2 - \frac{t_0\,\mathrm{Sch}(\tau,u)}{24\pi}, t_0\right). \tag{2.51}$$

Because of how the Schwarzian transforms as shown in the first line of equation 2.48, it is easy to see that the conditions for classification of coadjoint orbits of a constant coadjoint vector (b_0, t_0) remains the same as in section 2.1. Using the same scaling argument, one can see how the symplectic form in 2.30 gets modified to,

$$\omega = -\int_0^\beta \left\{\frac{t_0}{48\pi}\frac{d\tau'\wedge d\tau''}{\tau'^2} + b_0 B^2 d\tau\wedge d\tau'\right\} du. \tag{2.52}$$

For the first exceptional orbit of the Virasoro group, we recall here that $b_0 = -\frac{t_0}{48\pi}$. With this brief detour, we go back to realizing the full applicability of the DH theorem. The $U(1)$ time translation symmetry acts on $\tau(u)$ by $\delta\tau = \tau'(u)$. Let this transformation be generated by a vector V. The condition that a Hamiltonian H generates this symmetry on a symplectic manifold with symplectic form ω, is equivalent to $V^i\omega_{ij} = \partial_j H \implies V^i = (\omega^{-1})^{ij}\partial_j H$. The expression on the left of the above statement, can be rewritten as $\imath_V\omega = dH$, where $\imath_V\omega$ denotes the operation of contracting ω with V. For a closed two form such as the symplectic form, this implies that $(d\imath_V + \imath_V d)\omega = 0$, which is the condition that V leaves ω fixed. We show that this Hamiltonian function H is the Schwarzian upto an overall factor. In fact, we describe the proof for a general Diff($\mathbb{S}^1$) vector V that generates $\delta\tau = \alpha(u)\tau'(u)$, which leaves the symplectic form (Diff($\mathbb{S}^1$)) invariant. Time translation corresponds to the special case when $\alpha(u) = 1$.

In dealing with ω, one has to be careful with the extra "$-$" signs that arise because of the wedge product, which antisymmetrizes the resulting two form. Since we are dealing with the first exceptional orbit of a constant coadjoint vector of the Virasoro group,

$$\begin{aligned} &\omega = -\frac{t_0}{48\pi}(\omega_1 + \omega_2) \quad \text{where} \\ &\omega_1 = \int_0^\beta \left[\frac{d\tau' \wedge d\tau''}{\tau'^2}\right] du, \quad \omega_2 = \int_0^\beta -B^2 d\tau \wedge d\tau'\, du. \end{aligned} \tag{2.53}$$

We first contract V with ω_2, by contracting with each component one-form at a time, and taking into account the "$-$" sign introduced by the wedge product. Then, after an integration by parts, we get

$$\begin{aligned} \imath_V\omega_2 &= -2B^2 \int_0^\beta (\alpha\tau'\partial_u d\tau)\, du \\ &= -2B^2 \int_0^\beta (\alpha\tau' d\tau')\, du. \end{aligned} \tag{2.54}$$

But this is nothing but dH_2, where

$$H_2 = -B^2 \int_0^\beta \alpha\tau'^2\, du. \tag{2.55}$$

Similarly, we compute $\imath_V\omega_1$, and write it in terms of a differential of a function H_1.

$$\begin{aligned}
\imath_V\omega_1 &= 2\int_0^\beta \left(\frac{\partial_u(\alpha\tau')}{\tau'}\partial_u\frac{d\tau'}{\tau'}\right) du \\
&= 2\int_0^\beta \left(\left(\alpha' + \alpha\frac{\tau''}{\tau'}\right) d\left(\frac{\tau''}{\tau'}\right)\right) du = dH_1, \text{ where} \\
H_1 &= \int_0^\beta \left(2\alpha'\frac{\tau''}{\tau'} + \alpha\left(\frac{\tau''}{\tau'}\right)^2\right) du.
\end{aligned} \tag{2.56}$$

Combining H_1 and H_2, we get,

$$H = -\frac{t_0}{48\pi}\int_0^\beta \left[\alpha\left(-B^2\tau'^2 + \left(\frac{\tau''}{\tau'}\right)^2\right) + 2\alpha'\frac{\tau''}{\tau'}\right] du. \tag{2.57}$$

Hence, $\imath_V\omega = dH$. Using 2.48, when $\alpha(u) = 1$, we see that the Hamiltonian function above is nothing but the Schwarzian upto an overall constant. Since the phase space of functions $t(u)$, $\text{Diff}(\mathbb{S}^1)/SL(2,\mathbb{R})$ is a symplectic manifold, and the Schwarzian action generates the $U(1)$ time translation symmetry, the DH theorem implies that the path integral of the action 2.48, over the symplectic manifold with the appropriate measure, is one loop exact. With this enormous simplification at hand, we proceed to evaluate the one loop term.

We start by explicitly calcultating the symplectic form and then its Pfaffian.

Computing the Pfaffian

Let us expand $\tau(u)$ about its classical solution, in terms of fourier modes where each mode comes with a coupling g.

$$\tau(u) = u + g\sum_{n\in\mathbb{Z}} u_n e^{inu\frac{2\pi}{\beta}}, \quad d\tau(u) = g\sum_{n\in\mathbb{Z}} du_n e^{inu\frac{2\pi}{\beta}}. \tag{2.58}$$

While the u_n are complex numbers in general, since the fluctuation about $\tau(u) = u$ is real we get $u_{-n} = u_n^*$.

Since we showed that ω is $\text{Diff}(\mathbb{S}^1)$ invariant, we can evaluate ω at the identity element of $\text{Diff}(\mathbb{S}^1)$, namely at $\tau(u) = u$. This sets τ'^2 in ω_1 to 1. Now, differentiating 2.58 with respect to u to get $d\tau'$ and $d\tau''$, and substituting in 2.53, we get

$$\omega = -\frac{t_0}{48\pi}\int_0^\beta -ig^2\left(\frac{2\pi}{\beta}\right)^3 \sum_{nn'} e^{iu\frac{2\pi}{\beta}(n+n')}[(nn'^2+n')du_n \wedge du_{n'}]\, du. \tag{2.59}$$

Since $\text{Pf}(\omega)$ eventually contributes to the measure, from here onwards we drop numerical constants and only keep track of factors of physical constants such as g and β. After a change of variables $u \to \tilde{u} = u\frac{2\pi}{\beta}$, and performing the integral over $\tilde{u}$, we get

$$\begin{aligned}\omega &\sim iB^2g^2\sum_n (n^3-n)du_n \wedge du_{-n} = iB^2g^2\sum_n (n^3-n)du_n \wedge du_n^* \\ &= B^2g^2\sum_n (n^3-n)du_n^{\text{Re}} \wedge du_n^{\text{Im}}\end{aligned} \tag{2.60}$$

where $u_n = u_n^{\text{Re}} + iu_n^{\text{Im}}$ and we have used $du_n \wedge du_{-n} = -i(du_n^{\text{Re}} \wedge du_n^{\text{Im}})$. This ω is expressed purely in terms of real modes, and is degenerate because of the zero modes at $n = -1, 0, +1$ which is reminiscent of the degeneracy caused by the stabilizer set, $SL(2,\mathbb{R})$. In order to arrive at the symplectic form which is non degenrate, we remove these zero modes from the sum over n in 2.60. Since ω is a two-form, it can be recast as an antisymmetric matrix,

$$\omega = \begin{array}{c c} & \begin{array}{cccc} du_2^{\text{Re}} & du_2^{\text{Im}} & du_3^{\text{Re}} & \end{array} \\ \begin{array}{c} du_2^{\text{Re}} \\ du_2^{\text{Im}} \\ du_3^{\text{Re}} \\ \\ \end{array} & \begin{pmatrix} 0 & B^2g^2(2^3-2) & 0 & \dots \\ -B^2g^2(2^3-2) & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

Clearly, this is a block diagonal matrix whose determinant is the product of determinants of the smaller 2×2 blocks. We get

$$\det(\omega) = \prod_{n=2}^{\infty} CB^4g^4(n^3-n)^2 \implies \det(\omega)^{1/2} = \text{Pf}(\omega) = \prod_{n=2}^{\infty} C^{1/2}B^2g^2(n^3-n), \tag{2.61}$$

where C is a positive constant containing numerical factors. The Pfaffian can be rendered in a form suitable for zeta function regularization.

$$
\begin{aligned}
\mathrm{Pf}(\omega) &= \exp\left\{\log\left(\prod_{n=2}^{\infty} C^{1/2}B^2g^2(n^3-n)\right)\right\} \\
&= \exp\left\{\sum_{n=2}^{\infty}\left[\log(C^{1/2}B^2g^2)+\log(n)+\log(n+1)+\log(n-1)\right]\right\}.
\end{aligned}
\tag{2.62}
$$

All the individual terms in the sum above can be regularized with the Riemann zeta function,

$$
\begin{aligned}
\sum_{n=1}^{\infty} 1 &= \zeta(0) = -\frac{1}{2}, \\
\sum_{n=1}^{\infty} \log(n) &= -\zeta'(0) = \frac{1}{2}\log(2\pi).
\end{aligned}
\tag{2.63}
$$

Tracking only powers of physical constants, we get[8]

$$
\mathrm{Pf}(\omega) \sim (B^2g^2)^{-3/2} \sim \left(\frac{\beta}{g}\right)^3. \tag{2.65}
$$

The final regularized result would have been independent of g if all the fourier modes u_n were included. Every fourier mode that was removed amounted to giving a factor of $1/g$ in the final result, since each of these modes came with a factor of g.

Evaluating the one loop term

To arrive at the one loop action $I^{(2)}$, we expand the field τ around the classical solution $\tau = u + g\epsilon$, to quadratic order in ϵ in the last equality in 2.48.

$$
I^{(2)} = \frac{1}{2}\int_0^{\beta} [\epsilon''^2 - B^2\epsilon'^2]\, du. \tag{2.66}
$$

[8]For instance, the first term in the sum can be regularized by

$$
\begin{aligned}
& e^{\sum_{n=2}^{\infty}\log(C^{1/2}B^2g^2)} = \frac{1}{C^{1/2}B^2g^2} e^{\sum_{n=1}^{\infty}\log(C^{1/2}B^2g^2)} \\
&= \frac{1}{C^{1/2}B^2g^2} e^{-\frac{1}{2}\log(C^{1/2}B^2g^2)} = (C^{1/2}B^2g^2)^{-3/2}.
\end{aligned}
\tag{2.64}
$$

The quadratic term contains all those terms of order $O(g^0)$. Expanding ϵ in fourier modes as in 2.58, the path integral upto one loop order can be written as

$$e^{-S_{\text{cl}}}\int\prod_{n\in\mathbb{Z}^+\setminus\{1\}} Du_n Du_n^* \,\text{Pf}(\omega)\exp\left\{\left[-\frac{1}{2}\left(\frac{2\pi}{\beta}\right)^3\sum_{n=2}^{\infty}u_n(n^4-n^2)u_n^*\right]\right\}=$$
$$e^{-S_{\text{cl}}}\int\prod_{n\in\mathbb{Z}^+\setminus\{1\}} Du_n^{\text{Re}} Du_n^{\text{Im}} \,\text{Pf}(\omega)\exp\left\{\left[-\frac{1}{2}\left(\frac{2\pi}{\beta}\right)^3\sum_{n=2}^{\infty}\left(u_n^{\text{Re}}(n^4-n^2)u_n^{\text{Re}}+u_n^{\text{Im}}(n^4-n^2)u_n^{\text{Im}}\right)\right]\right\},\tag{2.67}$$

where we accounted for the degeneracy in the modes, $u_n^* = u_{-n}$ by restricting n to $\mathbb{Z}^+$. The path integral consists of integrals over two independent real scalar modes. The respective quadratic operators are already expressed in the diagonalized form and the eigenvalues (λ_n^{Re} and λ_n^{Im}) can be read off directly. The result of the path integral is

$$e^{-S_{\text{cl}}}\,\text{Pf}(\omega)\left(\prod_{n=2}^{\infty}\frac{\lambda_n^{\text{Re}}}{2\pi}\right)^{-1/2}\cdot\left(\prod_{n=2}^{\infty}\frac{\lambda_n^{\text{Im}}}{2\pi}\right)^{-1/2}\sim e^{-S_{\text{cl}}}\,\text{Pf}(\omega)\prod_{n=2}^{\infty}\frac{\beta^3}{n^4-n^2},\tag{2.68}$$

where we have retained only factors of β. Performing zeta function regularization for the β dependent term and assuming that the n dependent term can be appropriately regularized, gives the factor $(\beta^3)^{-3/2}$. Combining 2.45 and 2.67, Z becomes

$$\begin{aligned}Z &= \frac{1}{g^3\beta^{3/2}}e^{\pi\Phi_0-\beta M_0}e^{\frac{2\pi^2}{\beta g^2}}\\ &= \left(\frac{\Phi_b}{\beta}\right)^{3/2}e^{\pi\Phi_0-\beta M_0}e^{\frac{2\pi^2\Phi_b}{\beta}},\end{aligned}\tag{2.69}$$

where in the second line we have replaced $1/g^2$ with our original coupling constant Φ_b which then matches with Iliesiu's results given in equation 1.19.

At first glance, it can be surprising to realize that the degrees of freedom of the full spacetime including the dilatonic mode was completely reduced to a one dimensional mode at a boundary that was introduced artificially into the theory, based on the symmetries of the NHR and FAR regions. The partition function seems to bear reminiscences of this addition, by containing explicit dependencies on the information at the boundary, such as Φ_b. However, we expect the final result to be independent of information of fields at the boundary, since the boundary was not there in the original theory to begin with, and the path integral if performed correctly, should have integrated over all possible boundary information. We now clarify this point by noting that Φ_b can be written entirely in terms of physical constants of the theory, and thus is not an unphysical, leftover degree of freedom.

We start by reconsidering the Dirichlet boundary conditions for the fields, at this boundary.

$$\boldsymbol{\chi}_b = G\left(\Phi_0 + \frac{\Phi_b}{\epsilon}\right), \text{ where } \frac{\Phi_b}{\epsilon} = \frac{2r_0\delta r_{\text{bdy}}}{G} \tag{2.70}$$

which is obtained by evaluating the classical solution for $\boldsymbol{\chi} = r^2$ at $r_{\partial\mathcal{M}_{NHR}} = r_0 + \delta r_{\text{bdy}}$, and expanding to linear order in δr_{bdy}. Now, the Dirichlet boundary condition on the metric gives

$$\begin{aligned}\left(\frac{(r-r_0)^2}{L_2^2}dt^2 + \frac{L_2^2}{(r-r_0)^2}dr^2\right)\Big|_{r_{\partial\mathcal{M}_{NHR}}} &= \frac{(\delta r_{\text{bdy}})^2}{L_2^2}dt^2 \\ &= \frac{L_2^2}{\epsilon^2}du^2.\end{aligned} \tag{2.71}$$

While the second equality above seems to imply that t can be any constant times u, in order to incorporate the classical solution of the boundary mode, $t(u)$ has to be u. Therefore, equating the coefficients, we get

$$\epsilon = \frac{L_2^2}{\delta r_{\text{bdy}}} \implies \Phi_b = \frac{2r_0L_2^2}{G}, \tag{2.72}$$

and hence, Φ_b is indeed composed of physical constants of the theory, and the resulting partition function does not depend on any remaining degrees of freedom.

2.3 Multi black hole solutions

From a talk by Luca Iliesiu [14], we were motivated to look for non perturbative corrections to the partition function that could possibly fix its behaviour in the $T \to 0$ limit. Suppose the partition function receives a $O(1)$ correction denoted by Z_0. This would not only give a non zero value to the partition function in the zero temperature limit, but also renders the zero temperture entropy as a finite positive quantity, ln Z_0. Such non perturbative corrections could possibly come from quantum corrections around other highly non trivial classical solutions such as collinear, charged black holes in Einstein-Maxwell gravity.We eventually plan to employ the method of heat kernel discussed in section 2.4 to compute one loop contributions of fluctuations around these saddles. But first, we attempt to approach a simpler problem by choosing the classical solution comprising two Schwarzschild black holes. This is a simpler case because such solutions belong to the class of static, axisymmetric, "vacuum" spacetime.

2.3.1 Two Schwarzschild Black Holes

This solution is reviewed from [15]. The metric for any static axisymmetric vacuum spacetime with two commuting killing vectors $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial t}$ can be written in the form,

$$\begin{aligned} ds^2 &= -V(\rho,z)dt^2 + V^{-1}(\rho,z)\gamma_{ij}dx^i dx^j, \\ \gamma_{ij}dx^i dx^j &= e^{2K(\rho,z)}(d\rho^2+dz^2)+\rho^2 d\phi^2. \end{aligned} \tag{2.73}$$

It was first shown by Weyl [16] that the problem of finding such solutions can be reduced to a problem in Newtonian gravity. This association is made explicit by defining a function $U(\rho,z)$ by $V=e^{2U}$, where U is the Newtonian potential outside two rods of lengths $\mu_i = 2M_iG$ and masses M_i, where M_i refers to the mass of the black holes for $i=1,2$. Each rod is the locus of the holes' horizon. Interacting black holes in thermal equilibrium can be shown to have $M_1 = M_2 = M$. The generalization to multi black hole solutions can be easily done [15].

We are interested in inspecting the metric in the region connecting both rods. Along this section,

$$\begin{aligned} ds^2 &\sim (-V(z)dt^2 + V^{-1}(z)e^{2K}dz^2) + V^{-1}(z)(e^{2K}d\rho^2+\rho^2 d\phi^2), \\ V(z) &= \frac{z-(z_1-\mu/2)}{z-(z_1+\mu/2)}\frac{z-(z_2+\mu/2)}{z-(z_2-\mu/2)}, \\ e^K &= \frac{(\Delta z+2\mu)\Delta z}{(\Delta z+\mu)^2}, \end{aligned} \tag{2.74}$$

where Δz is the Newtonian distance between the rods, $\Delta z = z_1 - z_2 - \mu$, and z_1 and z_2 refer to the z coordinates of the midpoint of these rods. The solution is sketched in our Figure 2.3.1. We require Δz to be positive, and hence, $z_1 - z_2 > \mu$. The spatial part of the metric is not conformally flat as it seems to be in 2.74, but can be shown to host a conical singularity

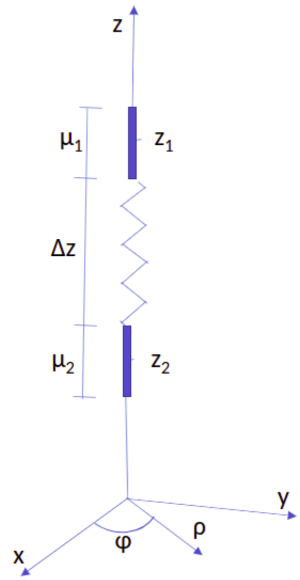

Figure 2.3.1: The two Schwarzschild black holes solution from [15].

along $\rho = 0$. Along the section connecting both rods, the deficit angle along the ϕ-direction,

$$\delta = 2\pi(1 - e^{-K}) = -2\pi \frac{\mu^2}{(\Delta z + 2\mu)\Delta z} \tag{2.75}$$

is negative, suggesting that the polar angle is excess of 2π. The presence of conical singularities hints towards an instability; two Schwarzschild black holes cannot be in equilibrium in flat space. An effective, non zero energy momentum tensor can be associated to the conical singularity, and the tension along this cosmic string like solution is actually negative as a consequence of the excess angle. Therefore this solution is referred to as a “strut”, and it provides the correct outward pressure to counter the gravitational attraction of the black holes, hence keeping the two Schwarzschild black holes in equilibrium. In other words, the conical singularity is the cost for keeping the separation between the black holes, Δz fixed. For large Δz, this interpretation is consistent with Newtonian gravity. This is the significance of the presence of conical singularities in the study of multi black holes in equilibrium. Despite conical singularities, this solution has a well defined gravitational action, and hence the standard tools for studying gravitational thermodynamics can be applied to this system.

From now on, we refer to the two dimensional subspace (ρ, ϕ) of the metric in 2.74, that has a conical singularity at $\rho = 0$ as C_2. Let the (z, t) subspace be referred to as

Σ_2. The metric along the section between the rods, 2.74 tells us that the spacetime in this region is topologically separable; for every fixed value of the coordinate z, there is an internal space C_2 characterized by (ρ, ϕ). In addition to this, it is easy to see from 2.74 that the metric is coordinate-separable after a conformal transformation. We denote this by $ds^2 = \Omega^2(z)(C_2 \times \Sigma_2)$, where $\Omega^2(z) = V^{-1}(z)$ is the conformal factor.

The Euclidean time periodicity is given by,

$$\frac{1}{T} = \beta = 4\pi\mu \frac{\Delta z + 2\mu}{\Delta z + \mu}. \tag{2.76}$$

2.3.2 Two non extremal RN black holes

Here, we briefly describe the case of two, non extremal RN black holes in equilibrium which belongs to the class of asymptotically flat, static solutions to Einstein-Maxwell theory. For a more complete discussion of this solution, we refer the reader to [17]. The metric and the electromagnetic potential can be written in the form,

$$\begin{aligned} ds^2 &= -f(\rho,z)dt^2 + f^{-1}(\rho,z)[h^2(\rho,z)(d\rho^2 + dz^2) + \rho^2 d\phi^2], \\ A_t &= -\Phi(\rho,z),\ A_\rho = A_z = A_\Phi = 0. \end{aligned} \tag{2.77}$$

The metric solution takes the same form as 2.73 with the identification $V \to f$, $e^K \to h$. It is symmetric about the z axis, and as before the horizons are coordinate singular rods placed on this axis at $(z_H - \Sigma, z_H + \Sigma)$ and $(z_h - \sigma, z_h + \sigma)$ where the half lengths Σ and σ are functions of the masses and charges of the black holes and the separation parameter, $R = |z_H - z_h|$. Like before, we require $R > \sigma + \Sigma$.

For an explicit solution to the metric functions f, h, and Φ, we refer the reader to [17]. We would eventually like to study the one loop contribution of quantum fluctuations about this classical solution.

The extremal multi black hole solution given by Majumdar and Papapetrou is the unique static and regular multi black hole solution in Einstein-Maxwell theory. Here, the gravitational attraction between the two near extremal black holes slightly outweighs their electrostatic repulsion. Hence, a strut comprising conical singularities and providing positive pressure in the section between the black holes is necessary to achieve equilibrium.

2.4 The method of heat kernel

The method of heat kernel for evaluating functional determinants becomes very useful in computing one loop contributions from quadratic actions that involve off diagonal mixing among the different quantum fluctuation fields such as 3.15. It provides a prescription to calculate necessary quantities in terms of just a few geometric invariants and comes particularly useful in working with complicated goemetries such as manifolds with boundaries or singularities. However, it is important to bear in mind that the method of heat kernel expansion does not apply beyond the one loop approximation. This section is largely based on [18]. We will continue to work in the Euclidean signature.

Let D denote the quadratic fluctuation operator of a theory, obtained by expanding the associated lagrangian around a classical solution. Then, the heat kernel is a function parametrized by an auxiliary coordinate, t and defined as

$$K(t;x,y;D) = \langle x| \exp(-tD) |y\rangle \,, \tag{2.78}$$

where x and y are two spacetime points. For a self adjoint operator D on a smooth compact Riemannian manifold, using its complete set of orthonormal eigenfunctions $\{\psi_n\}$ with corresponding eigenvalues $\{\lambda_n\}$, the heat kernel can be expressed as

$$K(t;x,y;D) = \sum_n e^{-t\lambda_n}\psi_n^*(x)\psi_n(y). \tag{2.79}$$

The heat kernel is a solution of the heat conduction equation,

$$(\partial_t + D|_x)K(t;x,y;D) = 0 \tag{2.80}$$

with the initial condition, $K(0,x,y;D) = \delta(x,y)$.

When t is positive, the operator e^{-tD} is trace class on the space of square integrable functions, $L^2(V)$. This enables us to define the trace of the heat kernel as follows[9].

$$K(t,D) = \mathrm{Tr}_{L^2(V)}(e^{-tD}) = \int \mathrm{tr}_V K(t;x,x;D)\sqrt{g}\, d^n x. \tag{2.81}$$

[9] We have set the auxiliary function used in [18] to identity.

The operator D, and consequently $K(t;x,x;D)$ are matrices on the internal space, V of quantum fields. The set $\{\psi_n\}$ provides a basis on $L^2(V)$, with the index n running over the internal space, V. tr_V denotes the trace on this internal space[10].

On manifolds without boundaries or on manifolds with boundaries with the fields subject to local boundary conditions, $K(t,D)$ supports an asymptotic expansion as $t \downarrow 0$.

$$K(t,D) \approx \sum_{k\geq 0} t^{(k-n)/2} \underbrace{\int a_k(x,x;D)\sqrt{g}\, d^n x}_{a_k(D)}, \tag{2.82}$$

where the $a_k(x,x;D)$ are known as Seeley-DeWitt coefficients or heat kernel coefficients, and are given in terms of local invariants constructed from the background fields and their derivatives.

While the heat kernel expansion can be shown to explain features such as the short distance behaviour of the propagator and the large mass expansion of the effective action, we will be interested in analyzing the one loop effective action and its regularization. The one loop effective action is given by

$$W = \frac{1}{2} \ln \det(\mathrm{D}). \tag{2.83}$$

We want to relate W to the heat kernel. In order to achieve this, we realize that every positive eigenvalue, λ of D obeys the identity

$$\ln \lambda = -\int_0^\infty \frac{dt}{t} e^{-\lambda t}, \tag{2.84}$$

which can be shown by differentiating both sides with respect to λ. Now, using $\ln \det(D) = \mathrm{Tr} \ln D$ and extending 2.84 to the whole operator D, we get

$$W = -\frac{1}{2} \int_0^\infty \frac{dt}{t} K(t,D). \tag{2.85}$$

The integral in 2.85 can be divergent at both limits. The divergence at $t=\infty$ comes from zero or negative eigenvalues of D and hence, this is an IR divergence. We assume that the inherent energy scale in the problem such as the masses of the fields are large enough such

[10] The trace of any operator defined on this internal space is, $\mathrm{tr}_V O^m_n = O^m_n \delta^n_m$.

that the IR divergence can be avoided. The UV divergence at $t = 0$ needs a more careful treatment. We introduce a UV cutoff at $t = \Lambda^{-2}$.

$$W_\Lambda = -\frac{1}{2}\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} K(t, D). \tag{2.86}$$

It can be shown that the divergent part of W_Λ is characterized by the coefficients $a_k(x, x; D)$ with $k \le n$ [18].

2.4.1 Zeta function regularization of the one loop effective action

For a positive operator D[11], we define the associated zeta function by

$$\zeta(s, D) = \mathrm{Tr}_{L^2(V)}(D^{-s}). \tag{2.87}$$

In terms of the heat kernel, this becomes

$$\zeta(s, D) = \Gamma(s)^{-1}\int_0^\infty t^{s-1} K(t, D)\, dt. \tag{2.88}$$

The relation above can be inverted to show that

$$K(t, D) = \frac{1}{2\pi i}\oint t^{-s}\Gamma(s)\zeta(s, D)\, ds, \tag{2.89}$$

where the contour of integration encircles all poles of the integrand. The integrated form of the Seeley-DeWitt coefficients are related to the residues at the poles as follows.

$$a_k(D) = \mathrm{Res}_{s=(n-k)/2}(\Gamma(s)\zeta(s, D)). \tag{2.90}$$

Since $\Gamma(s)$ has a simple pole at $s = 0$, $a_n = \zeta(0, D)$. As we will see, in n dimensions a_n plays a special role in the regularization of W.

W is explicitly divergent[12] in the UV limit due to its dependence on t^{-1}. For integer powers greater than -1, W is not manifestly UV divergent. Therefore, it seems that W can be regularized by shifting the power of t in the integrand for W to a value slightly greater

[11]For an operator with negative modes, the zeta function can be defined as, $\zeta(s) = \sum_{\lambda\neq 0} |\lambda|^{-s}$.

[12]It is also implicitly divergent due to the overall $t^{-n/2}$ normalization of the heat kernel coefficients.

than -1. However, in order to keep the one loop effective action dimensionless in units where $\hbar = 1$, we introduce a constant, $\tilde{\mu}$ with the dimension of mass.

$$W_s = -\frac{1}{2}\tilde{\mu}^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(t, D). \tag{2.91}$$

Note that since the term $-tD$ is exponentiated in the definition of the heat kernel, it should be dimensionless implying that t has a dimension of length squared. In terms of the zeta function, 2.91 becomes

$$W_s = -\frac{1}{2}\tilde{\mu}^{2s}\Gamma(s)\zeta(s, D). \tag{2.92}$$

For a small s near $s = 0$, the gamma function looks like the following.

$$\Gamma(s) = \frac{1}{s} - \gamma_E + O(s), \tag{2.93}$$

where $\gamma_E \sim 0.577$ is the Euler-Mascheroni constant. Taylor expanding W_s around $s = 0$,

$$W_s = -\frac{1}{2}\left(\frac{1}{s} - \gamma_E + \ln \tilde{\mu}^2\right)\zeta(0, D) - \frac{1}{2}\zeta'(0, D), \tag{2.94}$$

we see that the regularized effective action also has a pole at $s = 0$, and $a_n = \zeta(0, D)$ characterizes the divergent contribution. The pole term has to be removed by appropriate renormalization. The renormalized one loop effective action becomes,

$$W^{\text{ren}} = -\frac{1}{2}\zeta'(0, D) - \frac{1}{2}\ln(\mu^2)\zeta(0, D), \tag{2.95}$$

where we have introduced a new parameter, $\mu^2 = e^{-\gamma_E}\tilde{\mu}^2$. The ambiguity in the finite part of the effective action due to the freedom in the choice of the parameter μ^2, is reminiscent of the apparent ambiguity in the choice of the renormalization scheme.

A note on the classical, background manifold

Since the quadratic fluctuation operator, D is evaluated at a classical solution, the nature of the spectrum of D depends on the classical solution. When this solution is a compact manifold such as S^2, D supports a discrete spectrum which can be found exactly for simple enough operators. For non compact manifolds, the one loop effective action receives a divergent contribution from the inifnite volume of the manifold. This divergence can be regularized by confining the solution to a box and imposing appropriate boundary conditions on all fields at the boundary of the box. Then, subtracting a reference heat kernel removes dependencies on the size of the box.[13]

2.4.2 Evaluating Seeley DeWitt coefficients

Here, we briefly summarize results from [18] on explicit expressions for evaluating the Seeley DeWitt coefficients. In this section, we summarize results on smooth compact Riemannian manifolds for operators of the Laplace type. These are minimal, partial differential operators of second order. Minimal operators are those that have a scalar principal part i.e, in this context the second derivatives are contracted with the metric and the internal index structure of the second derivative term is trivial. Such operators can be written in the form,

$$D = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\sigma\partial_\sigma + b), \tag{2.96}$$

where a^σ and b are matrix valued functions on the manifold. The matrix indices run over the internal space of fields, V. D can be expressed in the form

$$D = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E), \tag{2.97}$$

for a unique connection ω on V, and unique endomorphism E of V where the covariant derivative ∇ is a sum of the Riemann ∇^R, and gauge ω parts; $\nabla = \nabla^R + \omega$. We express

$$\begin{aligned} \omega_\delta &= \frac{1}{2}g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma^\nu_{\mu\sigma}), \\ E &= b - g^{\nu\mu}(\partial_\mu\omega_\nu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma^\sigma_{\nu\mu}). \end{aligned} \tag{2.98}$$

Let $\Omega_{\mu\nu}$ be the field strength of the connection ω.

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu. \tag{2.99}$$

On a smooth compact Riemannian manifold without boundary the Seeley DeWitt coefficients with odd index, a_{2j+1} vanish and those with an even index, a_{2j} are locally computable in terms of geometric invariants constructed from $E, \Omega, R_{\mu\nu\rho\sigma}$, and their derivatives. The

[13] For an explicit application of this regularization procedure, we refer the reader to [19].

odd index coefficients must vanish since the geometric invariants in odd dimension must necessarily be written as total derivatives of the even dimensional quantities E, Ω, and $R_{\mu\nu\rho\sigma}$. The total derivatives can be integrated out to give an effective even dimensional theory. Hence, one cannot construct odd dimensional invariants on manifolds without boundaries.

We quote the expressions for the first three non zero coefficients from [18].

$$\begin{aligned}
a_0(D) &= (4\pi)^{-n/2} \int \dim(V) \sqrt{g}\, d^n x, \\
a_2(D) &= (4\pi)^{-n/2} \frac{1}{6} \int \mathrm{tr}_V \{6E + R\} \sqrt{g}\, d^n x, \\
a_4(D) &= (4\pi)^{-n/2} \frac{1}{360} \int \mathrm{tr}_V \{60(\nabla^R)^2 E + 60RE + 180E^2 + 12(\nabla^R)^2 R \\
&\qquad + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 30\Omega_{\mu\nu}\Omega^{\mu\nu}\} \sqrt{g}\, d^n x.
\end{aligned} \tag{2.100}$$

Literature on the method of heat kernel for manifolds with conical singularities

On manifolds with conical singularities, components of the Riemann tensor can be given only in terms of distributions. It is important to get a prescription to arrive at these distributions, since geometric invariants of the manifold are essential ingredients in calculating heat kernel coefficients. Let Σ be the conical-singular subspace of the manifold. By treating the conical-singular manifold as a limit of a converging sequence of smooth manifolds, [20] gives us a precription for arriving at expressions for components of the Riemann curvature tensor near Σ. Computing logarithm of the scalar field determinant on manifolds with conical singularities appeared in works such as [21] and [22]. Generalization to higher spin fields has been achieved in [23]. However, their analysis of spin 2 fields was greatly simplified by the choice of a compact manifold with its smooth part having a constant curvature.

All these works rely crucially on the separability of the background manifold into $C_2 \times \Sigma$, in the neighborhood of Σ. This facilitates isolating and separately calculating the contributions of the two $2D$ subspaces, C_2 and Σ to the heat kernel on the full background manifold. However, as demonstrated in section 2.3.1, the solutions that interests us are of the form $ds^2 = \Omega^2(z)(C_2 \times \Sigma_2)$. In section 3.3, we discuss our attempts at understanding the applicability of the methods developed in the literature for manifolds that are related to "coordinate-separable" cases by a conformal transformation.

Chapter 3

Results and Discussion

3.1 Understanding δK at $\partial\mathcal{M}_{NHR}$

The aim of this section is to understand how a boundary action given by $\delta I_{\mathrm{JT}}^{\mathrm{bdy}}$ where the variation takes the extremal RN solution to the near extremal RN solution is described by the Schwarzian action, in the presence of appropriate Dirichlet boundary conditions for the quantum fields at $\partial\mathcal{M}_{NHR}$. We were motivated to check whether $\delta I_{\mathrm{JT}}^{\mathrm{bdy}}$ would give way to a different effective boundary theory that could possibly fix the divergence in the entropy in the zero temperature limit. We also come to understand why metric fluctuations in the NHR crucially determine the leading temperature dependent correction to the partition function of near extremal black holes.

3.1.1 Attempts at generalization

We try to generalize the evaluation of the $\delta I_{\mathrm{JT}}^{\mathrm{bdy}}$ boundary term by employing the full extremal solution 1.3 as against the near horizon approximated solution 1.6 that went into reducing the FAR action to an effective boundary theory. We also try to extend to the case of asymptotically flat black holes through the $L \to \infty$ limit.

We calculate extrinsic curvature 1.14, for our extremal metric characterized by the metric function f_0, for the most general boundary curve parametrized by $(\tau(u), r(u))$,

$$K_{\text{ext}} = \frac{r'\tau'' - \tau' r'' + \frac{f_0 \tau'^3 \partial_r f_0}{2} + \frac{3\tau' r'^2 \partial_r f_0}{2 f_0}}{\left(\frac{r'^2}{f_0} + f_0 \tau'^2\right)^{3/2}}. \tag{3.1}$$

On fixing the "r" coordinate at the boundary of the extremal metric, K_{ext} reduces to $1/L_2$. For the near extremal metric given in 2.40, with δf_0 taken to be some constant "$-A$" as described earlier, K becomes

$$K_{\text{near ext}} = \frac{f_0}{(f_0^2 - A^2)^{1/2}} \frac{\left(\frac{f_0^2 - A^2}{f_0^2}\right)(r'\tau'' - \tau' r'') + \frac{(f_0 - A)\tau'^3 \partial_r f_0}{2} + \tau' r'^2 \left[\frac{3(f_0 + A)\partial_r f_0}{2 f_0^2} - \frac{A^2 \partial_r f_0}{f_0^3}\right]}{\left[\frac{r'^2 (f_0 + A)}{f_0^2} + (f_0 - A)\tau'^2\right]^{3/2}}, \tag{3.2}$$

where primes denote derivatives with respect to the boundary coordinate u. For $A = 0$, 3.2 reduces to 3.1 as expected.

On imposing the Dirichlet boundary condition for the boundary curve parametrized by $(\tau(u), r(u))$, we get

$$g_{\tau\tau}\tau'^2 + g_{rr} r'^2 = \frac{L_2^2}{\epsilon^2}, \tag{3.3}$$

using which we can express the boundary action δI_{GHY} only in terms of the degree of freedom corresponding to reparametrizations of the boundary time, $\tau(u)$. For the extremal metric described by 1.6, the boundary is the asymptotic boundary of the hyperbolic disc. This is nothing but a circle at asymptotic infinity. Since $r'(u) = 0$ for a circle, imposing this in the constraint 3.3, and solving for $\tau(u)$ renders it independent of the radius of this circle, and we get $\tau(u) = u$.

We take advantage of the generalizations listed above, to first work out corrections for flat black holes, whose full extremal solution, f_0 takes a very simple form 1.5. For the near extremal metric 2.40, naively imposing 3.3 involves solving a higher than quadratic order polynomial. To avoid this complication, we use the extremal metric, but allow for fluctuations in the boundary curve $(\tau(u), r(u))$ and use 3.3 to derive the boundary constraint $f_0(r) \approx \frac{L_2^2}{\epsilon^2 \tau'^2}$ upto leading order in ϵ. Since $\partial \mathcal{M}_{NHR}$ is positioned in the overlap of the NHR and FAR regions, using the NHR limit, $r - r_0 \ll r_0$, one gets the leading order relation,

$$
\begin{aligned}
f_0 &\approx \frac{(r-r_0)^2}{r_0^2} \approx \frac{L_2^2}{\epsilon^2 \tau'^2} \\
\Rightarrow r(u) &= r_0 + \frac{L_2 r_0}{\epsilon \tau'(u)}.
\end{aligned}
\tag{3.4}
$$

Using 3.4 in 3.2, we get the leading term in $K_{\text{near ext}}$ to be of $O(\epsilon^2)$.

$$
K^{FAR}_{\text{near ext}} = \frac{\epsilon^2}{r_0^4}\left(1 - \frac{A^2}{2} - \frac{A^4}{2}\right)\left(\frac{\tau'^2}{\tau''}\right)^3\left(\frac{\tau'''}{\tau'} - 3\frac{\tau''^2}{\tau'^2}\right) \tag{3.5}
$$

where we have used $L_2 \approx r_0$ for asymptotically flat black holes. The boundary action now becomes,

$$
\begin{aligned}
\delta I_{FAR} &= -\frac{1}{2G}\int_{\partial\mathcal{M}_{NHR}} \sqrt{h}\chi(K^{FAR}_{\text{near ext}} - K^{FAR}_{\text{ext}})\, du \\
&= -\frac{1}{2G}\int_0^{\beta} \frac{r_0}{\epsilon}\frac{\chi_r}{\epsilon}\left[\frac{\epsilon^2}{r_0^4}\left(1 - \frac{A^2}{2} - \frac{A^4}{2}\right)\left(\frac{\tau'^2}{\tau''}\right)^3\left(\frac{\tau'''}{\tau'} - 3\frac{\tau''^2}{\tau'^2}\right) - \frac{1}{L_2}\right] du,
\end{aligned}
\tag{3.6}
$$

where the term involving $K_{\text{ext}} = 1/L_2$ is divergent in the small ϵ limit, and the counter term that would normally be employed to cancel such divergences cannot be invoked here since this boundary was not present in the full $2D$ spacetime to begin with. However, with the understanding that we cutoff the AdS_2 space, and ϵ given in 2.72 is a small quantity, the $\epsilon \to 0$ limit is avoided as long we deal with cutout versions where the boundary is at a finite coordinate distance.

Before proceeding, we should also check whether our Hamiltonian is bounded below and whether the simplest reparametrization of the boundary time u obeying $\tau(u+\beta) = \tau(u) + \beta$, namely $\tau(u) = u$ is a classical solution of 3.6. The equation of motion for a single variable, third order Lagrangian L is given by

$$
\frac{\partial L}{\partial \tau} - \frac{d}{du}\frac{\partial L}{\partial \tau'} + \frac{d^2}{du^2}\frac{\partial L}{\partial \tau''} - \frac{d^3}{du^3}\frac{\partial L}{\partial \tau'''} = 0. \tag{3.7}
$$

For the action in 3.6, the equation of motion reduces to

$$
12\tau'^2 - \frac{8\tau'^3\tau'''}{\tau''^2} + \frac{3\tau'^4\tau'''^2}{\tau''^4} - \frac{\tau'^4\tau''''}{\tau''^3} = 0, \tag{3.8}
$$

which does not possess $\tau(u) = u$ as a solution. Hence, on physical grounds, we drop further analysis of this action.

On failing to identify simple classical solutions to 3.6, we revisit the boundary constraint that was employed in arriving at δI_{FAR}. If instead of using the near horizon approximation to obtain the boundary relation as in 3.4, we use the exact extremal solution for f_0 given by 1.5, we get a much simpler boundary constraint at leading order in ϵ,

$$r(u) = \epsilon\tau'(u). \tag{3.9}$$

This gives $K^{FAR}_{\text{near ext}}$ upto $O(\epsilon^2)$ as

$$K^{FAR}_{\text{near ext}} = -\frac{r_0}{\tau'^2\epsilon^2} + \left(\frac{5A}{2r_0} - \frac{3}{2r_0}\right) + \frac{\epsilon\tau'}{r_0^2}\left(\frac{13A}{2} - \frac{73}{2}\right) + \frac{\epsilon^2\tau'^2}{r_0^3}\left(\frac{85}{2} - 12A + A^2\right) + O(\epsilon^3). \tag{3.10}$$

As before, $K^{FAR}_{\text{ext}} = 1/L_2$.

Although the resulting action, $\delta I_{FAR} \sim \delta K$ does possess $\tau(u) = u$ as a classical solution, observing that it is stabilized by $U(1)$ time translation, and quantizing it by realizing that the integration space is the normal coadjoint orbit of a constant coadjoint vector of the Virasoro group, continues to be divergent in entropy, in the $T \to 0$ limit. In fact, in addition to the logarithmic divergence in β, additional factors of β brought by A also seem to be ill behaved in the $T \to 0$ limit of the entropy.

3.1.2 Unique boundary action

We tried to work with the full extremal solution of asymptotically flat black holes which is not simplified by the NHR geometry, to include the contribution of metric solution in the FAR. However, this solution in the presence of different boundary constraints either gave us an over simplified effective action 3.10, or an action too complicated to possess the $\tau(u) = u$ solution 3.6.

Through these attempts, we come to understand why it is enough to work with the NHR region to derive leading temperature dependent corrections to the classical thermodynamic variables. This can be seen by recasting the near extremal solution 2.40 in coordinates where

$\rho = r - r_0$ as

$$ds^2|_{\text{near ext}} = \left(\frac{\rho^2 - \delta r_h^2}{L_2^2}\right) d\tau^2 + \left(\frac{L_2^2}{\rho^2 - \delta r_h^2}\right) d\rho^2. \tag{3.11}$$

The leading temperature dependent corrections from 3.11 in the limit of large AdS black holes i.e, $r_0 \gg L \implies L_2 \approx L/\sqrt{6}$ gives the metric 2.40. The geometry described by 3.11 is still AdS_2 with $R = -2/L_2^2$, which now describes the NHR of near extremal black holes. It is now easy to see that the leading correction to the extremal metric in the presence of a small temperature is of the order of $(\delta r_h/\rho)^2$ which dies down in the FAR region described by $\rho \gg \delta r_h$.

We now incorporate these insights carefully by dealing with asymptotically AdS black holes, and working with the contribution of the NHR goemetry 1.6 and its fluctuations. With the understanding that the NHR of the near extremal solution also possesses the AdS geometry, we proceed to verify whether including the contribution of the temperature dependent corrections to the extremal geometry, to the extrinsic curvature correctly reproduces the Schwarzian action, as in [7]. The boundary constraint 3.4 now gets modified to

$$\begin{aligned} f_0 = \frac{(r - r_0)^2}{L_2^2} &\approx \frac{L_2^2}{\epsilon^2 \tau'^2} \\ \implies r(u) = r_0 &+ \frac{L_2^2}{\epsilon \tau'(u)}. \end{aligned} \tag{3.12}$$

With this, the extrinsic curvature of the near extremal metric, 3.2 becomes

$$K_{\text{near ext}} = \frac{1}{L_2} + \frac{\epsilon^2}{L_2}\left[\text{Sch}(\tau, u) + \frac{A}{2L_2^2}\tau'^2\right]. \tag{3.13}$$

In the limit of large AdS black holes $A/(2L_2^2)$ becomes $B^2/2$, and thereby the action $-\int \sqrt{h}\frac{\Phi_b}{\epsilon}\delta K\, d^2x$ where $\delta K = K_{\text{near ext}} - K_{\text{ext}} = K_{\text{near ext}} - 1/L_2$ correctly gives the Schwarzian 2.48.

The explains why the $\delta I_{JT}^{\text{bdy}}$ term comprising the difference in the extrinsic curvature induced at $\partial\mathcal{M}_{NHR}$, between the extremal and the near extremal NHR solutions is correctly described by the Schwarzian action. An explanation for the consequent divergence in the entropy in the zero temperature limit, is an open problem for the future.

3.2 The full $2D$ path integral

We summarize some insights derived from the path integral calculations of the partition function. The calculations described in section 2.2 relied on the following perturbative approximations.

1. $\boldsymbol{\chi} = \chi_0 + \chi$: Truncating at $O(1)$ perturbation to the extremal dilaton value paved the way to tap into the simplification offered by JT theory where the dilaton appears only as a Lagrange multiplier.

2. Leading order, $O(\epsilon^2)$ in boundary data: A non zero but small ϵ reminds us that we are dealing with cutout versions of AdS_2 with a boundary that is close enough to the asymptotic AdS_2 boundary. Truncating at second order rendered the effective $2D$ boundary action independent of boundary values of the quantum fields.

3. The FAR action classically perturbed about the extremal NHR metric solution, to first order in near extremal perturbation: This rendered the entire theory as an effective boundary action at $\partial\mathcal{M}_{NHR}$.

It is uncommon in the path integral formalism to modify the action by classically perturbing about a solution, before arriving at the quadratic (quantum) fluctuation term, as was done to the FAR action (section 2.2.1). In other words, the FAR action was put on shell since as discussed in section 3.1.2, quantum fluctuations due to a small temperature are supressed in the FAR. Here, we analyze the raw quadratically expanded action of the complete $2D$ theory without reducing it to an effective action at $\partial\mathcal{M}_{NHR}$. For this, we work with the dilaton gravity action as presented in [7] in their equation (2.25), and focus only on the bulk terms. This has the form

$$I_{2D} = -\frac{1}{4G}\int_{\mathcal{M}_2} [\boldsymbol{\chi} R + V(\boldsymbol{\chi})]\sqrt{g}\, d^2x, \tag{3.14}$$

where the potential term, $V(\boldsymbol{\chi})$ is parametrized by the charge of the black hole which we have supressed in the notation. The bulk part of the quadratic fluctutation action turns out to be

$$\begin{aligned}\delta I_{2D}^{(2)} =& -\frac{1}{4G}\int \Bigg[\frac{h^2}{4}(\boldsymbol{\chi} R + V(\boldsymbol{\chi})) - \frac{1}{2}h_{\mu\nu}h^{\mu\nu}(\boldsymbol{\chi} R + V(\boldsymbol{\chi})) - R^{\alpha\beta}h_{\alpha\beta}h\boldsymbol{\chi} + h\boldsymbol{\chi}\nabla_\alpha\nabla_\mu h^{\alpha\mu} \\ &- \frac{1}{2}\boldsymbol{\chi}\nabla^\mu h\nabla_\mu h - \boldsymbol{\chi} h^{\alpha\beta}\nabla_\mu\nabla_\alpha h^\mu_\beta + \nabla_\alpha\boldsymbol{\chi}(2h^\alpha_\kappa\nabla_\mu h^{\kappa\mu} + h^{\kappa\mu}\nabla_\mu h^\alpha_\kappa) + \frac{\boldsymbol{\chi}}{2}h^{\alpha\beta}\Box h_{\alpha\beta} \\ &- \frac{3}{2}h^{\lambda\kappa}\nabla^\nu h_{\lambda\kappa}\nabla_\nu\boldsymbol{\chi} - 2h^{\alpha\beta}\nabla_\beta\boldsymbol{\chi}\nabla_\alpha h + 2\boldsymbol{\chi} h^{\alpha\kappa}h^\beta_\kappa R_{\alpha\beta} + \delta\boldsymbol{\chi} h(R + V'(\boldsymbol{\chi})) \\ &- 2\delta\boldsymbol{\chi} R^{\alpha\beta}h_{\alpha\beta} - 2\delta\boldsymbol{\chi}\Box h + 2h^{\alpha\mu}\nabla_\mu\nabla_\alpha\delta\boldsymbol{\chi} + V''(\boldsymbol{\chi})(\delta\boldsymbol{\chi})^2\Bigg]\sqrt{g}\, d^2x,\end{aligned} \tag{3.15}$$

where $h_{\mu\nu}$ denotes metric fluctuation, $g_{\mu\nu}$ refers to the near extremal background metric, $h = g^{\mu\nu}h_{\mu\nu}$, and $\delta\boldsymbol{\chi}$ denotes fluctuation in the dilaton. We observe several features of this quadratic action.

1. The coefficients of the quadratic fluctuation terms when evaluated at the background solution 2.40, can be written in the form $\widehat{C} + T^2\widehat{D}$ by perturbatively expanding in δM where $\hat{C}$ is the contribution to the quadratic operator from the extremal classical solution. We can possibly expand the $e^{-T^2\widehat{D}}$ factor in the path integral and study only the leading order temperature dependence, provided that we can prove the analyticity of the partition function. However, if Z were analytic in T, one could not explain the leading $T^{3/2}$ contribution from near extremal RN black holes.

2. The path integral is not separable into separate integrals over the dilaton and components of the metric fluctuation field.

3. Even among quadratic terms in the metric fluctuation components, there are several off diagonal terms that cannot be dealt with without invoking techniques such as the heat kernel method for calculating one loop determinants.

3.3 $O(1)$ correction to Z?

In this section, we describe our efforts toward quantizing fluctuations around multi black hole saddles (see section 2.3) which gives a non perturbative correction to the path integral. If this correction turns out to be of $O(1)$, then the zero temperature limit of the partition function and the entropy would become physically meaningful, as described towards the beginning of section 2.3. In order to derive the one loop effective action, we rely on the heat kernel method discussed in section 2.4.

We begin by attempting to arrive at one loop contribution to the partition function of Einstein-Hilbert action in $4D$, for a semiclassical expansion around the two Schwarzschild black hole solution. We obtain the quadratic fluctuation operator after the addition of gauge fixing terms to the action from [24] which works with Einstein-Maxwell theory. For fluctuations of the form, $g_{\mu\nu} + \sqrt{2}h_{\mu\nu}$ and for the quadratic operator of Einstein-Hilbert theory we isolate those components that depend only on the metric to obtain,

$$\begin{aligned} \delta I^{(2)} &= -\frac{1}{32\pi G}\int (h^{\mu\nu}(\Delta h)_{\mu\nu} + \text{diffeomorphism-ghost term})\sqrt{g}\, d^4x, \\ (\Delta h)_{\mu\nu} &= -\Box h_{\mu\nu} - R^{\tau}_{\mu} h_{\tau\nu} - R^{\tau}_{\nu} h_{\tau\mu} - 2R_{\mu}{}^{\rho}{}_{\nu}{}^{\tau} h_{\rho\tau} + \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}\Box h_{\rho\sigma} \\ &\quad + R h_{\mu\nu} + (g_{\mu\nu}R^{\rho\sigma} + R_{\mu\nu}g^{\rho\sigma})h_{\rho\sigma} - \frac{1}{2} R g_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma}. \end{aligned} \tag{3.16}$$

Seeley DeWitt coefficients on backgrounds with conical singularities

We wish to evaluate the quadratic fluctuation operator $\delta I^{(2)}$ above on a background with two Schwarzschild black holes in equilibrium (see section 2.3.1). This solution supports a strut; a $2D$ surface of conical singularities in the section between the black holes. The full $4D$ manifold, $\mathcal{M}$ can be decomposed into a small neighborhood of the singular surface Σ_2 where the spacetime is topologically separable as $C_2 \times \Sigma_2$ and the rest of the manifold which is smooth, $\mathcal{M}\backslash C_2 \times \Sigma_2$. The trace of the heat kernel $K(t; x, x; D)$ can be written in the form

$$\begin{aligned} K(t; D) &= \int_{\mathcal{M}} \mathrm{tr}_V K(t; x, x; D)\sqrt{g}\, d^4x \\ &= \int_{C_2\times\Sigma_2} \mathrm{tr}_V K(t; x, x; D|_{C_2\times\Sigma_2})\sqrt{g}\, d^4x + \int_{\mathcal{M}\backslash C_2\times\Sigma_2} \mathrm{tr}_V K(t; x, x; D_{\text{smooth}})\sqrt{g}\, d^4x. \end{aligned} \tag{3.17}$$

The second integral above goes over the smooth part of $\mathcal{M}$ where the standard treatment for obtaining Seeley DeWitt coefficients as discussed in section 2.4.2 holds. These coefficents receive corrections from the singular region, which require a special treatment. This is because, on such singular backgrounds the Riemann curvature tensor, Ricci tensor and Ricci scalar all of which appear in $\delta I^{(2)}$ can be given only in terms of delta function distributions (see for example, [20]). The general structure of the Seeley DeWitt coefficients is thus,

$$a_k = a_k|_{\mathcal{M}\setminus C_2\times\Sigma_2} + a_k|_{C_2\times\Sigma_2}. \tag{3.18}$$

3.3.1 Heat kernel in $2D$

In $2D$, the quadratic operator in 3.16 can be greatly simplified by employing the following identities.

$$\begin{aligned} R_{\mu\alpha\nu\beta} &= \frac{R}{2}(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\nu\alpha}), \\ R^{\nu}_{\mu} &= \frac{1}{2}R\delta^{\nu}_{\mu} \end{aligned} \tag{3.19}$$

While the second identity is obtained from vacuum Einstein equation, to obtain the first identity we first note that the Riemann tensor has only one independent component in two dimensions. The RHS of the first line spans the vector space of tensors having the correct symmetries of the Riemann tensor in two dimensions, and gives the correct expressions for Ricci tensor and Ricci scalar on contracting with the background metric. Hence, the potential terms in the quadratic operator for the metric fluctuation field in 3.16 can be expressed completely in terms of the background metric and its Ricci scalar.

For coordinate separable manifolds that have $ds^2 = C_2 \times \Sigma_2$, [23] claims the following relation for the Lichnerowicz operator.

$$\mathrm{Tr}\, K^{(2)}|_{C_2\times\Sigma_2} = \mathrm{Tr}\, K^{(2)}|_{C_2}\mathrm{Tr}\, K^{(0)}|_{\Sigma_2} + \mathrm{Tr}\, K^{(0)}|_{C_2}\mathrm{Tr}\, K^{(2)}|_{\Sigma_2} + \mathrm{Tr}\, K^{(1)}|_{C_2}\mathrm{Tr}\, K^{(1)}|_{\Sigma_2}. \tag{3.20}$$

The Lichnerowicz operator appears in the quantization of gravitational field in the harmonic gauge. The relation above presents an enormous simplification by recasting a heat kernel calculation in $4D$ as a sum of products of $2D$ heat kernel traces. We set out to verify the validity of this result for coordinate separable manifolds for our quadratic operator 3.16 that was obtained on averaging over all possible gauges, with the aim of understanding how far it generalizes to manifolds with metrics that are conformally related to $ds^2 = C_2 \times \Sigma_2$. The meaning of the indices that appear on each of the terms in 3.20 is explained later in this section.

We begin by recasting the operator in 3.16 in coordinate-separable form by decomposing the components of the metric fluctuation field into trace and traceless parts,

$$h_{\mu\nu} = f_{\mu\nu}{}^{1} + \frac{1}{4} g_{\mu\nu} h. \tag{3.21}$$

This gives us,

$$\begin{aligned} h^{\mu\nu}(\Delta h)_{\mu\nu} \to & - f^{\mu\nu}\Box f_{\mu\nu} - f^{\mu\nu} R^{\tau}_{\mu} f_{\tau\nu} - f^{\mu\nu} R^{\tau}_{\nu} f_{\tau\mu} + f^{\mu\nu} R f_{\mu\nu} \\ & - 2 f^{\mu\nu} R_{\mu}{}^{\rho}{}_{\nu}{}^{\tau} f_{\rho\tau} + \frac{1}{4} h \Box h. \end{aligned} \tag{3.22}$$

Since $f_{\mu\nu}$ is the traceless part of $h_{\mu\nu}$, $g^{\mu\nu} f_{\mu\nu} = 0$ and the above decomposition is orthogonal. This means that the one loop path integral can be performed independently over $f_{\mu\nu}$ and the massless scalar field, h. However, since $\delta I^{(2)}$ appears in the Euclidean gravity path integral as $e^{-\delta I^{(2)}}$, it is easy to recognize that the kinetic term for h has a wrong sign. This is the conformal mode problem in quantum gravity. One can deal with this problem by analytically continuing the field to imaginary values, $h \to ih$ [25].

Let us now restrict the background manifold to a small neighborhood of Σ_2, where the metric is separable $ds^2 = C_2 \times \Sigma_2$. Let $\{i, j, k, l\}$, and $\{a, b, c, d\}$ denote indices for metric components of C_2 and Σ_2 respectively. The quadratic terms involving $f_{\mu\nu}$ can be written in the form,

$$\begin{aligned} & f^{ij}(\underbrace{(-\Box_{C_2} - \Box_{\Sigma_2})\delta_i^k \delta_j^l + O_{ij}^{kl}}_{\widetilde{O}_{ij}^{kl}}) f_{kl} + f^{ab}(\underbrace{(-\Box_{C_2} - \Box_{\Sigma_2})\delta_a^c \delta_b^d + O_{ab}^{cd}}_{\widetilde{O}_{ab}^{cd}}) f_{cd} \\ & f^{ia}\underbrace{(-\Box_{C_2} - \Box_{\Sigma_2} + R)\delta_i^j \delta_a^b}_{\widetilde{O}_{ia}^{jb}} f_{jb}, \end{aligned} \tag{3.23}$$

where we have decomposed the box operator into the two $2D$ subspaces, facilitated by coordinate separability of the full manifold into $C_2 \times \Sigma_2$. In the $\{i, j, k, l\}$ subspace, f_{kl} is a covariant rank 2 tensor with respect to $\Box_{C_2}$, while a scalar with respect to $\Box_{\Sigma_2}$. Similarly, f_{cd} is a rank two tensor with respect to the action of $\Box_{\Sigma_2}$, and a scalar with respect to $\Box_{C_2}$. On the other hand, for the third term in 3.23 which contains mixed indices, f_{ia} is a

[1] $f_{\mu\nu}$ can be further decomposed into longitudinal and transverse traceless parts but this step is not required in our current analysis.

covariant rank 1 tensor with respect to both $\Box_{C_2}$ and $\Box_{\Sigma_2}$. The net contribution to the one loop effective action can be written as,

$$W = \frac{1}{2}\left(\ln\,\det\, O_h + \ln\,\det\,\widetilde{O}_{ij}^{kl} + \ln\,\det\,\widetilde{O}_{ab}^{cd} + \ln\,\det\,\widetilde{O}_{ia}^{jb}\right) \tag{3.24}$$

where $O_h = \frac{1}{4}\Box$. The decomposition that we have performed did not completely diagonalize the operators in 3.24. Moreover, for backgrounds with conical singularities one has to deal with delta function distributions of the Riemann curvature tensor, Ricci tensor and Ricci scalar in the quadratic fluctuation operator. This does not render it feasible to directly compute the eigen spectrum and consequently the determinant and hence, it is best to deal with such one loop integrals using the method of heat kernel. Let us first pick the operator $\widetilde{O}_{ij}^{kl}$, and understand how its corresponding heat kernel decomposes to heat kernels in two dimensions.

Using the representation 2.79 for the heat kernel on a compact manifold and by observing that each of the quadratic operators in 3.24 can be written as a sum of operators on the two $2D$ subspaces, $O = O_{\Sigma_2} + O_{C_2}$ we infer that the respective eigenvalue equation supports separable solutions and obtain for the operator $\widetilde{O}_{ij}^{kl}$,

$$\begin{aligned}\mathrm{Tr}\,K|_{C_2\times\Sigma_2} &= \int \sqrt{g}\,d^4x \sum_{n,m} e^{-t\left(\lambda_n^{C_2}+\lambda_m^{\Sigma_2}\right)}\psi_n^{*\,C_2}(\rho,\phi)\psi_m^{*\,\Sigma_2}(t,z)\psi_n^{C_2}(\rho,\phi)\psi_m^{\Sigma_2}(t,z)\\ &= \int \sqrt{h_{C_2}}\,d^2x \underbrace{\sum_n e^{-t\lambda_n^{C_2}}\psi_n^{*\,C_2}\psi_n^{C_2}}_{K_{C_2}^{(2)}} \int \sqrt{h_{\Sigma_2}}\,d^2y \underbrace{\sum_m e^{-t\lambda_m^{\Sigma_2}}\psi_m^{*\,\Sigma_2}\psi_m^{\Sigma_2}}_{K_{\Sigma_2}^{(0)}}.\end{aligned} \tag{3.25}$$

where the labels in superscript for the respective $2D$ heat kernels is the rank of the covariant tensor of the eigenfunctions determined with respect to the action of the kinetic operator, as discussed earlier. With a similar analysis of the other operators in 3.24, we obtain the expression 3.20 for the heat kernel trace of the full quadratic operator for the traceless part of the metric fluctuation field, $f_{\mu\nu}$. Putting everything together, the effective one loop action is,

$$W = \frac{1}{2}\ln\,\det\,O_h - \frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}\,K^{(2)}|_{C_2\times\Sigma_2}. \tag{3.26}$$

where the second term captures the contribution of the traceless components of the metric fluctuation field. Going to two dimensions greatly simplifies the quadratic operator as described towards the beginning of this section. For a compact background manifold one could readily extend the approach of [23] to obtain $a_{k=2}$ by analyzing how the spectrum of the quadratic operator changes on introducing a conical singularity to the background manifold which can then be mapped to the symmetries that are broken on introducing the conical singularity.

However, we need upto a_4 from the $2D$ heat kernel coefficients to determine the A_4 coefficient of the $4D$ asymptotic expansion. Each of the Seeley-DeWitt coefficients receive contributions from the smooth and singular parts of the background manifold separately as noted earlier. While the form of these contributions has been explicitly detailed out in [21] for a scalar field on a compact background, higher spin generalizations are not available as far as we know. We proceed to analyze the applicability of the considertions in this section to our conformally trasformed background manifolds in section 3.3.2.

3.3.2 Considerations for conformally transformed manifolds

On smooth manifolds, the result of conformal transformations on components of the Riemann tensor are well known. By approximating manifolds with conical singularities as limits of converging sequences of smooth manifolds, [20] is able to derive the distributional form for the Riemann tensor components. This is done by using the conformal transformation relations on the regularized smooth spaces and then taking the limit where the regularization parameter is taken to zero after crucially employing special asymptotic properties of the conformal factor which gives a coordinate separable geometry, $C_2 \times \Sigma_2$ in the neighborhood of the singular hypersurface, Σ_2. While their conformal factor is e^{σ}, in the vicinity of conical singularities i.e, in a small neighborhood of the $\rho = 0$ hypersurface (Σ_2), [20] assumes

$$\sigma = \sigma_1 \rho^2 + \sigma_2 \rho^4 + ... \tag{3.27}$$

where σ_1 and σ_2 can be functions of ϕ. As $\rho \to 0$, the conformal factor becomes unity giving the coordinate separable geometry, $C_2 \times \Sigma_2$. On going to the regularized smooth spaces we get for the action of a conformal transformation on the Ricci scalar

$$\begin{aligned} R[g] &= e^{-\sigma}\left(R[\tilde{g}] + \frac{1}{2}(d-1)\sigma^{\alpha}_{\alpha}\right), \\ \sigma_{\mu\nu} &= -2\widetilde{\nabla}_{\mu}\widetilde{\nabla}_{\nu}\sigma + \widetilde{\nabla}_{\mu}\sigma\widetilde{\nabla}_{\nu}\sigma - \frac{1}{2}\tilde{g}_{\mu\nu}(\widetilde{\nabla}\sigma)^2 \end{aligned} \tag{3.28}$$

where $\tilde{g}_{\mu\nu}$ is the regularized smooth metric and $g_{\mu\nu} = e^{\sigma}\tilde{g}_{\mu\nu}$.

Here, we do not assume such asymptotic properties for our conformal factor. We generalize the expression for the scalar curvature in [20] for a $2D$ conical metric labelled by (ρ, ϕ) and where Σ is just the point $\rho = 0$, to the case where $\sigma(\rho = 0, \phi) \neq 0$ and obtain

$$R = R_{\text{smooth}} + e^{-\sigma}\frac{2(1-\alpha)}{\alpha}\delta(\rho), \tag{3.29}$$

where e^{σ} is used interchangeably with Ω^2, and the delta function is normalized as $\int_0^{\infty} \rho\delta(\rho)\, d\rho = 1$. The first term is the standard Ricci scalar calculated on the smooth portion of the background manifold, which comes from simplifying the $\frac{3}{2}e^{-\sigma}\sigma^{\alpha}_{\alpha}$ term in 3.28 in the limit where the regularization is removed. $2\pi\alpha$ is the periodicity of the polar angle, ϕ. 3.29 differs from the expression in [20] in the second term where the contribution from the conical singularity is modulated by the reciprocal of the conformal factor. This term comes from integrating the first term in 3.28 containing the scalar curvature on the regularized manifold, and then reducing to a local picture on the original unregularized manifold. In $2D$, invoking the Gauss-Bonet theorem renders the integral of the scalar curvature as a topological invariant (Euler characteristic) and hence the result 3.29, especially the term containing the contribution of Σ_2 is independent of the choice of the regularized manifold.

Their results for curvature tensors in higher dimensions are more non trivially affected on lifting the assumptions on the conformal factor. Since the multi black hole solution of our interest 2.74 possesses a conformal factor that preserves axisymmetry, we will study those conformal factors that are independent of the coordinate ϕ. The contribution from the hypersurface of conical singularities is now proportional to δ_{Σ}. In D dimensions when σ is any function of ρ, the integration of the first term in equation 3.28 now depends on the choice of the regularization function. Moreover, for $D > 2$, this term contains explicit powers of the conformal factor and the integral over the regularized manifold cannot be performed without assuming the special asymptotic property of the conformal factor.

The $\frac{3}{2}e^{-\sigma}\sigma^{\alpha}_{\alpha}$ term from 3.28 now gives the leading terms in R_{smooth} in the $\rho \to 0$ limit. However, when σ is taken to be a function of z, a coordinate on the hypersurface Σ_{D-2} which is applicable to our solution of interest 2.74, $\frac{3}{2}e^{-\sigma}\sigma^{\alpha}_{\alpha}$ does not give R_{smooth} even in the vicinity of Σ_{D-2} i.e, in the neighborhood of $\rho = 0$. R_{smooth} now contains derivatives of metric functions of the Σ_{D-2} subspace that cannot be ignored in the $\rho \to 0$ limit. This restricts the applicability of the results in [20] to our choice of multi black hole solution in $4D$.

The background metric is continuous and therefore can be differentiated once. However, first derivatives are discontinuous and consequently second derivatives are not defined on the hypersurface of conical singularity, Σ_2. Since Christoffel symbols are linear in derivatives of the metric, its transformation for a conformal transformation of the metric is valid on the full background solution. For $\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$,

$$\widetilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + C^{\rho}_{\mu\nu}, \tag{3.30}$$

where $C^{\rho}_{\mu\nu}$ is a tensor given by [26],

$$C^{\rho}_{\mu\nu} = \Omega^{-1}(\delta^{\rho}_{\mu}\nabla_{\nu}\Omega + \delta^{\rho}_{\nu}\nabla_{\mu}\Omega - g_{\mu\nu}g^{\rho\lambda}\nabla_{\lambda}\Omega). \tag{3.31}$$

We use 3.30 to determine how the Laplacian in D spacetime dimensions acting on a covariant rank 2 tensor transforms on the smooth part of the background manifold,

$$\begin{aligned}(\widetilde{\nabla}^2 h_{\mu\nu})h^{\mu\nu} = \Omega^{-2}\{&(\nabla^2 h_{\mu\nu})h^{\mu\nu} + 2\Omega^{-2}(4-D)(\nabla\Omega)^2 h_{\mu\nu}h^{\mu\nu} - 2\Omega^{-1}h_{\mu\nu}h^{\mu\nu}\nabla^2\Omega \\ &+ 4\Omega^{-2}hh^{\mu\nu}\nabla_{\nu}\Omega\nabla_{\mu}\Omega + 2\Omega^{-2}(4-D)h_{\lambda\alpha}h^{\mu\lambda}g^{\alpha\rho}\nabla_{\rho}\Omega\nabla_{\mu}\Omega \\ &+ \Omega^{-1}g^{\beta\alpha}h^{\mu\nu}((D-6)\nabla_{\alpha}\Omega\nabla_{\beta}h_{\mu\nu} - \underline{4\nabla_{\mu}\Omega\nabla_{\alpha}h_{\beta\nu}} + 4\nabla_{\alpha}\Omega\nabla_{\mu}h_{\beta\nu})\}.\end{aligned} \tag{3.32}$$

All the metric functions and its derivatives on the RHS of 3.32 now correspond to that of the coordinate separable metric on $C_2 \times \Sigma_2$. Inside the curly brackets in the RHS of 3.32, the coefficients of the metric fluctuation components in all the terms can be separated (or decomposed) into $2D$ operators that are a function of either (ρ, ϕ) or (t, z) with the exception of the underlined term. This term mixes coordinates from the two $2D$ subspaces and prevents us from reducing the problem to $2D$ and arriving at a result analogous to 3.20. In addition to the tranformation of individual terms in $\delta I^{(2)}$ under a conformal transformation of the metric, one needs to account for the coordinate dependent factors coming from $\sqrt{g} = e^{2K}V^{-1}(z)\rho$ in the measure. For the metric in 2.74, $\Omega^2 = V^{-1}$. V^{-1} from the measure cancels the overall Ω^{-2} factor on the RHS of 3.32, and we retain an overall coordinate dependent term, ρ from the measure. This ρ can be absorbed into the measure of the conformally transformed $2D$ metric on the cone.

Since the singular hypersurface Σ_2 is two dimensional, reducing the problem to $2D$ greatly helps in separating the contributions of the smooth and singular parts of the solution which can then be treated independently using techniques that work only for the respective parts. We proceed to see if this situation concerning separability of the terms in the conformally transformed laplacian and also the applicability of the method of [20] can be improved by applying some approximations to the conformal factor.

Approximations to the conformal factor

$V(z)$ in 2.74, for $z_2 + \mu/2 \leq z \leq z_1 - \mu/2$ is a smoothly varying function that vanishes at points on the horizon i.e, at $z = z_2 + \mu/2$ and $z = z_1 - \mu/2$. From the attempts described in the previous sections, it is clear that the conformal factor $\Omega^2(x) = V^{-1}(z)$ becoming unity near Σ_2 is crucial to achieve separability into two $2D$ subspaces. Here, we explore whether by carefully choosing special cases of the multi black hole solution 2.74 parametrized by z_1, z_2, μ, and Δz, $V(z)$ and consequently $\Omega^2(x)$ can be expressed as a small correction to 1. Then, the metric would be coordinate separable at leading order and how the subleading corrections break coordinate separability can be possibly studied using the methods discussed so far.

Indeed, such special choice of parameters exist, bound to the constraint

$$\Delta z = z_1 - z_2 - \mu > 0, \tag{3.33}$$

where Δz is the length of the strut. One such choice is where the coordinate positions z_1 and z_2 are infinitesimally close i.e, $V(z) \approx 1$ when $z_1 - z_2 = \epsilon$ for small ϵ. The constraint 3.33 implies that μ be small enough such that $\epsilon > \mu$. In this case, the conformal factor becomes

$$V^{-1}(z) = \left(1 - \frac{\epsilon}{z - z_2 + \frac{\mu}{2}}\right)^{-1} \left(1 - \frac{\epsilon}{z - z_2 - \frac{\mu}{2}}\right). \tag{3.34}$$

However, in the section between the rods

$$1 < \frac{\epsilon}{z - z_2 + \frac{\mu}{2}} < \frac{\epsilon}{\mu}, \tag{3.35}$$

indicating that the first term in the expression for the conformal factor 3.34 cannot be approximated by terminating the Taylor expansion at any finite order in ϵ. A second choice of parameter values for which the conformal factor can be approximated to 1 is a large value for μ i.e, $\mu = 1/\epsilon$ for small ϵ. In this case, the constraint 3.33 gives $z_1 - z_2 > 1/\epsilon$. However through a similar analysis, we again find the conformal factor to not permit a finite order approximation in ϵ.

We also realize that for a semi classical expansion of the path integral for the partition function to be valid, the general relativity description should hold for the multi Schwarzschild classical solution as highlighted in [15]. This happens for a small deficit angle, $|\delta|$ or effectively for a strut tension well below the Planck scale, ensured by $\frac{\mu}{\Delta z} \ll 1$ which is achieved for a large proper distance between the black holes. We can then define the scale $\epsilon = \frac{\mu}{\Delta z} \ll 1$.

By dividing the numerator and denominator in the expression for $V(z)$ 2.74 by Δz, one can readily employ the small ϵ limit to obtain

$$V^{-1}(z) = \left(1 + \frac{\epsilon \Delta z}{2(z - z_1)}\right)^{-1} \left(1 - \frac{\epsilon \Delta z}{2(z - z_1)}\right) \left(1 - \frac{\epsilon \Delta z}{2(z - z_2)}\right)^{-1} \left(1 + \frac{\epsilon \Delta z}{2(z - z_2)}\right). \tag{3.36}$$

Using $z_2 + \mu/2 \leq z \leq z_1 - \mu/2$ and 3.33 we deduce

$$\frac{\mu}{2\Delta z + \mu} \leq \frac{\epsilon \Delta z}{2(z - z_2)} \leq 1, \quad \text{and} \quad \frac{\mu}{2\Delta z + \mu} \leq -\frac{\epsilon \Delta z}{2(z - z_1)} \leq 1, \tag{3.37}$$

which naively seems to imply that the terms in 3.36 can now be approximated by a Taylor expansion to finite order in ϵ. However, at points on the horizon given by $z = z_2 + \mu/2$ and $z = z_1 - \mu/2$, the inequalities above achieve their upper limits and $V^{-1}(z)$ blows up at these points. Therefore, at points infinitesimally close to the horizon, $V^{-1}(z)$ cannot be approximated as a small correction to 1.

Chapter 4

Conclusions and Outlook

The entropy of black holes can be computed purely from a low energy prescription. The classical Bekenstein-Hawking result relating entropy to the area of the black hole receives quantum corrections which can be obtained from the one loop contribution from massless modes to the gravitational path integral for the partition function. The significance of these corrections lies in the fact that they can be completely extracted from the IR data of the theory, and hence, can be used to constrain their UV completions.

We discuss the effective action approach of [7] in which the dimensionally reduced Einstein-Maxwell theory is reduced to an effective $1D$ Schwarzian action at the level of the path integral. On taking the zero temperature limit of the quantum corrections, the partition function vanishes suggesting an absence of extremal black hole states, and the entropy diverges to $-\infty$.

[27] get the Schwarzian mode through a slightly different approach, by analyzing the asymptotic behaviour of tensor zero modes which are large diffeomorphisms of the extremal, AdS_2 metric. A very recent work [28] displays similar corrections at leading order in T, to the logarithm of the partition function from a direct computation of the $4D$ path integral for near extremal black holes. Thus, the temperature dependence of these thermodynamic variables remains to be well understood in the $T \to 0$ limit. At leading order in T, [28] hint towards the vanishing density of states in this limit, suggesting that the entropy loses a statistical description in the zero temperature limit.

Previous studies in the literature such as [2] have established the significance of a perturbed $2D$ JT theory of gravity in describing the dynamics of near extremal RN black holes. The perturbation under consideration is a very specific one, that takes the extremal RN solution to the near extremal RN solution. The work of Iliesiu and Turiaci [7] has shown how the path integral for the partition function can be reduced to an integral over an effective boundary mode described by the perturbed JT boundary term, and why the Schwarzian action correctly describes this boundary term.

Through explicit attempts at evaluating the $\delta I_{JT}^{\text{bdy}}$ term, we could not arrive at a different effective boundary theory that could possibly rectify the zero temperature limit of the entropy. While the effect of introducing a small temperature dies off in the metric of the FAR, the possibility of including the contribution of metric fluctuations in the FAR was considered. However, it was difficult to solve for the boundary constraint for the complete extremal solution of asymptotically AdS black holes 1.3 which now involved finding roots of a quartic polynomial.

We performed a quadratic fluctuation of the full $2D$ dilaton gravity action with a potential term, devoid of any approximations and obtained 3.15 for the bulk quadratic action. Isolating the temperature dependence, we identified that the quadratic fluctuation operators can be decomposed in the form, $\widehat{C} + T^2\widehat{D}$. However, we could not proceed to determine the leading order temperature dependence from the $e^{-T^2\widehat{D}}$ term in the path integral due to concerns regarding the analyticity of the partition function.

We then explored the idea of non perturbative contributions to the path integral from non trivial saddle points such as multi black hole solutions, which could make the partition function and the entropy well defined in the zero temperature limit provided that such solutions contribute to a $O(1)$ correction to the partition function. Multi black hole solutions in equilibrium required the presence of a strut with conical singularities in the spacetime. We reviewed past works in the literature on employing the method of heat kernel to manifolds with conical singularities. Almost all previous works that we are aware of rely on the separability of the metric into a $2D$ conical metric and a $D-2$ dimensional singular hypersurface Σ, in the vicinity of Σ to obtain the Seeley DeWitt coefficients. However, the multi black hole solutions that constitute our interest (section 2.3) are not coordinate separable; instead they are conformally related to such coordinate separable geometries. This posed serious limitations on trying to extend the considerations of works such as [20, 21, 22, 23].

Reducing the problem to two dimensions seems crucial to isolate the contribution of the singular and smooth parts of the background manifold to the Seeley DeWitt coefficients. The presence of a z-dependent conformal factor prevents this approach. It also renders the regularization procedure of [20] which gives the components of the Riemann tensor as a sum of the smooth part and a distrubutional contribution from the singular parts, invalid. Arriving at a suitable regularization procedure for conformally trasformed separable metrics could be a first step in attempting to deal with such geometries.

Bibliography

[1] S. W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. **43**, 199-220 (1975).

[2] U. Moitra, S. K. Sake, S. P. Trivedi and V. Vishal, *Jackiw-Teitelboim Gravity and Rotating Black Holes*, JHEP **11**, 047 (2019).

[3] G. Sárosi, *AdS_2 holography and the SYK model*, PoS **Modave2017**, 001 (2018).

[4] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, PTEP **2016**, no.12, 12C104 (2016).

[5] E. Witten, *Coadjoint Orbits of the Virasoro Group*, Commun. Math. Phys. **114**, 1 (1988).

[6] J. Cotler and K. Jensen, *A theory of reparameterizations for AdS_3 gravity*, JHEP **02**, 079 (2019).

[7] L. V. Iliesiu and G. J. Turiaci, *The statistical mechanics of near-extremal black holes*, JHEP **05**, 145 (2021).

[8] M. Heydeman, L. V. Iliesiu, G. J. Turiaci and W. Zhao, *The statistical mechanics of near-BPS black holes*, J. Phys. A **55**, no.1, 014004 (2022).

[9] B. Oblak, *BMS Particles in Three Dimensions*, arXiv:1610.08526 [hep-th].

[10] N. Moshayedi, *Notes on Geometric Quantization*, arXiv:2010.15419 [math-ph].

[11] L. V. Iliesiu, *On 2D gauge theories in Jackiw-Teitelboim gravity*, arXiv:1909.05253 [hep-th].

[12] P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi and V. Vishal, *On the Dynamics of Near-Extremal Black Holes*, JHEP **09**, 048 (2018).

[13] D. Stanford and E. Witten, *Fermionic Localization of the Schwarzian Theory*, JHEP **10**, 008 (2017).

[14] L. V. Iliesiu, *Black hole microstate counting from the gravitational path integral*, Quantum Spacetime seminars, TIFR (2022).

[15] M. S. Costa and M. J. Perry, *Interacting black holes*, Nucl. Phys. B **591**, 469-487 (2000).

[16] H. Weyl, *The theory of gravitation*, Annalen Phys. **54**, 117-145 (1917).

[17] P. Krtouš and A. Zelnikov, *Thermodynamics of two black holes*, JHEP **02**, 164 (2020).

[18] D. V. Vassilevich, *Heat kernel expansion: User's manual*, Phys. Rept. **388**, 279-360 (2003).

[19] A. Sen, *Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole Entropy in Different Dimensions*, JHEP **04**, 156 (2013).

[20] D. V. Fursaev and S. N. Solodukhin, *On the description of the Riemannian geometry in the presence of conical defects*, Phys. Rev. D **52**, 2133-2143 (1995).

[21] D. V. Fursaev, *Spectral geometry and one-loop divergences on manifolds with conical singularities*, Phys. Lett. B **334**, 53-60 (1994).

[22] G. Cognola, K. Kirsten and L. Vanzo, *Free and self-interacting scalar fields in the presence of conical singularities*, Phys. Rev. D **49**, 1029-1038 (1994).

[23] D. V. Fursaev and G. Miele, *Cones, spins and heat kernels*, Nucl. Phys. B **484**, 697-723 (1997).

[24] S. Bhattacharyya, B. Panda and A. Sen, *Heat Kernel Expansion and Extremal Kerr-Newmann Black Hole Entropy in Einstein-Maxwell Theory*, JHEP **08**, 084 (2012).

[25] G. W. Gibbons, S. W. Hawking and M. J. Perry, *Path Integrals and the Indefiniteness of the Gravitational Action*, Nucl. Phys. B **138**, 141-150 (1978).

[26] S. M. Carroll, *Spacetime and Geometry, An Introduction to General Relativity*, Cambridge University Press, (2019).

[27] L. V. Iliesiu, S. Murthy and G. J. Turiaci, *Revisiting the Logarithmic Corrections to the Black Hole Entropy*, arXiv:2209.13608 [hep-th].

[28] N. Banerjee and M. Saha, *Revisiting Leading Quantum Corrections to Near Extremal Black Hole Thermodynamics*, arXiv:2303.12415 [hep-th].