Renormalization Group Analysis of Models of Symplectic Fermions

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by

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Certificate

This is to certify that this dissertation entitled **Renormalization Group Analysis of Models of Symplectic Fermions**towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Yuv Agarwal at Indian Institute of Science Education and Research under the supervision of *Dario Benedetti*, Doctor, Department of Physics , during the academic year 2022-2023.

Dario Benedetti

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Dario Benedetti

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To Papa, Mama, & Didi who always encouraged me to go on every adventure, especially this one.

Declaration

I hereby declare that the matter embodied in the report entitled **Renormalization Group Analysis of Models of Symplectic Fermions** are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of *Dario Benedetti* and the same has not been submitted elsewhere for any other degree.

Yuv Agarwal

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Abstract

The framework of the Renormalization Group (RG) forms the foundation of the present understanding of Critical Phenomena, but its fixed points and predictions pertaining to physical quantities, are normally characterized through approximations and truncations. Even so, as demonstrated in the paper [1], the RG fixed points can be given a completely rigorous and non-perturbative characterization. The foundational framework for defining Quantum Field Theories (QFTs), at a non-perturbative level, or beyond perturbation theory, in High Energy Physics, is proffered by the Renormalization Group theory of Critical Phenomena [1] [2–7, 13].

We attempt to comprehend the referenced Rigorous Renormalization Group (RRG) and make connections to Functional Renormalization Group (FRG), [2]

[1] Wilson's Euclidean lattice formulation of gauge theories confers a non-perturbative definition of Quantum Chromo-Dynamics (QCD). [13]

[2] In the absence of small parameters, FRG represents the most structured method for implementing Wilsonian Renormalization Group (WRG) [1].

Contents

Chapter 1

Introduction

Renormalization Group, (RG), is a theoretical framework in physics that allows us to understand the behaviour of complex physical systems by analyzing how their properties change as we zoom in or out on different length scales. It is particularly useful in studying the behaviour of systems that exhibit universal behaviour, meaning that their properties are largely independent of their microscopic details. In this context, the RG is an essential tool for studying the interactions between particles and their properties, allowing us to make precise predictions that can be tested experimentally. The basic idea of the RG, [4,5,7] is to start with a microscopic description of a system and then gradually coarse-grain or integrate out degrees of freedom, effectively describe the system at a larger scale. This process can be repeated iteratively, leading to a flow in the space of theories or models, known as the RG flow. RG has proven to be a highly effective tool for understanding the behaviour of complex systems, including phase transitions, critical phenomena, and the emergence of scaling laws.

1.1 Versions of Renormalization Group Procedure

RRG is a systematic and mathematically rigorous approach to understanding the flow of couplings associated with a QFT, using a rigorous mathematical formulation [1, 8–11]. It establishes the validity and well-posedness of RG flows, providing a foundational framework for the rigorous analysis of the flow and proof of existence and uniqueness of fixed points.

A few key aspects of RRG are:

- The use of functional analysis and Banach spaces to define the RG transformation and the corresponding effective action. Specifically, RRG defines the RG transformation on a Banach space of interactions, which establishes that the RG flow is well-defined and the corresponding effective action is finite. This strategy allows RRG to use a convergent approximation scheme, based on theorems such as the Contraction Mapping Theorem and the Implicit Function Theorem, to rigorously establish the existence of fixed points in a non-perturbative setting.
- Emphasis on the precise mathematical formulation of the RG flow, encompassing a precise definition of the RG transformation and the corresponding effective action. This provides a rigorous and systematic framework for analyzing RG flows, which is significant when developing an understanding of the behavior of systems with strong interactions or long-range order.

FRG is a pragmatic approach to the study of the RG Flow of QFTs and many-body systems, [12, 18, 21, 24]. It is based on the idea of constructing a continuous RG flow of the associated effective action, also termed the Irreducible Vertex (1PI) Generating Functional. FRG is based upon a differential equation referred to as the Exact Flow Equation, which can be further understood as a continuous differential equation in RG time. This flow equation is derived using the Wilsonian approach to the RG, where the effective action is integrated over the high modes of the system and consequently the RG flow is understood in terms of the coarse-grained effective action. The flow equation for the effective action is typically derived using a functional integral representation of the RG transformation. The effective action is expressed as a functional integral over the fields, and the RG transformation is defined, using an infrared (IR) cutoff, which regulates the UV divergences that arise in the RG analysis of QFTs. FRG has applications in a variety of systems, like systems with long-range interactions, theories with long-range order, and systems with strong interactions. It provides a systematic approach for understanding the behaviour of these systems under changes in the length (energy) scale, and for making predictions about their behaviour in the proximity of the critical points.

FRG confers a powerful tool for the study of QFTs. The RG flow of QFTs is an important concept in HEP, as it provides a way to understand the behaviour of these theories under changes in the energy scale. FRG develops a controlled way to study this RG flow, by constructing a continuous RG flow of the corresponding effective action. Applications of FRG,

- Strongly interacting theories [Ex: (QCD)] - In these theories, the strength of the interactions between particles is so strong that perturbative methods, which rely on weak interactions, break down. FRG provides a way to study these strong interactions by constructing an RG flow that incorporates all terms in the associated effective action obeying the symmetries of the system.
- Phase transitions: The behaviour of the system is strongly dependent on the temperature or other external parameters. FRG provides a powerful tool for understanding the behaviour of these systems in the vicinity of critical points, and for making predictions about the nature of the phase transitions.

The FRG is a bridge between the functional methods of QFTs and the RG formalism of successively treating the fluctuations using a scale-dependent framework. Converse to analysing the correlation functions after averaging over all fluctuations, we systematically study the changes induced as infinitesimal momentum shell fluctuations alter the correlation functions. Mathematically, we thus transform the standard field theory's functional integral structure into a functional differential structure. This approach possesses improved analytical and numerical accessibility and stability and increased flexibility to develop approximations customized to a particular physical system. Moreover, using this methodology enables more elegant and efficient structural investigations of field theories from the first principles of renormalizability and proofs. The FRG utilizes a flow equation as its primary tool, which tracks the changes in correlation functions or their generating functional due to fluctuations. This equation establishes a connection between an initial quantity, the correlation functions in a perturbative domain associated with the microscopic description of the theory, and the full correlation functions that result subsequent to accounting for fluctuations. Solutions of the Flow Equation provide information about the complete theory.

This study provides a concise yet comprehensive understanding of the Rigorous RG and Functional RG approaches. By applying these methods to the various models, we demonstrate how each approach independently offers valuable techniques and utilizes a range of mathematical tools to analyze the interaction space of the model.

The implementation of Wilsonian RG in the absence of a small parameter is most systematically carried out by FRG. This study also explores the application of the FRG method to symplectic fermions, (local or long-range), which remains unexplored from an FRG perspective. Our investigation uses the toy model to comprehend the consequences of applying the FRG method rigorously. It is important to note that any Functional RG calculation truncates the interaction space, resulting in only a subset of irrelevant couplings being included, even if it is infinite. Currently, there are no conclusive results on the optimal way to exhaust the interaction space. Our study exhaustively examines the space of interactions using techniques and machinery from FRG and RRG and establishes the existence of a non-trivial fixed point. The analysis of the Long-Range Symplectic Fermion Model put forth in [1] uses a discrete step RG transformation procedure for their analysis, whereas, the formalism of FRG allows us to carry out a continuous RG transformation in order to study the model and its flow. Through this study, we aim to extend the approach put forth by *Guliani et al. 2021* [1], to a continuous RG transformation, thereby establishing the possibility of developing a rigorous approach to FRG.

1.2 Model

The paper [1] focuses on a toy model of Symplectic Fermions, characterized by a *non-local kinetic term*, i.e. long-range, (dependent on the parameter ε) and associated *quartic and quadratic interactions*. It primarily focuses on rigorously defining a Banach Space of Interactions and its respective properties. There exists a fixed point in the Banach space of interactions and the authors determine it by employing a convergent approximation scheme. Furthermore, they rigorously prove the analytic nature of the fixed point with respect to ε. The model, under study, is [1]

$$aMFT(\psi) + \nu_0 \int d^dx\psi^2(x) + \lambda_0 \int d^dx\psi^4(x) \tag{1.1}$$

This model possesses the following characteristics/properties:

- aMFT refers to 'anti-commuting Mean Field Theory of the fermionic field ψ, with an even number **N** of *real Grassmann fields* and $d = 1, 2, 3$.

[1]Fermionic=Anti-Commuting=Grassmann

- The aforementioned aMFT [2] is identified via the two-point function

$$\langle \psi_a(x)\psi_b(y)\rangle = \Omega_{ab} P(x-y) \equiv G_{ab}(x-y) \tag{1.2}$$

where

$$P(x)\ \alpha\ \frac{1}{|x|^{\frac{d}{2}-\varepsilon}} \quad \text{at large distances and } \varepsilon \text{ small and positive}$$

$$\Omega_{ab} = \begin{bmatrix} & 1 & & & & & \\ -1 & & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \\ & & & & & -1 & \end{bmatrix}_{N\times N} \tag{1.3}$$

- The interaction terms (the quadratic term and the quartic term) respect the $Sp(N)$. The fermions (Grassmann fields) are scalar under rotation, and not spinors [3]. Thus the model also preserves $O(d)$ symmetry.
- The model comprises of a non-local (long-range) kinetic term [4]
- The quadratic interaction is $\Omega_{ab}\psi_a\psi_b$ and the quartic interaction is $(\Omega_{ab}\psi_a\psi_b)^2$.

1.3 Structure

The paper is structured as follows:

- Chp:2, lays out the calculations and analysis performed in [1], in a brief yet exhaustive manner. We laid the foundation of the proof of the Existence of the Fixed Point and formally stated the results, presented in Sec(s):[2.2 & 2.3]. Additionally, they involve quick calculation that proves the fixed point equation has an $O(\varepsilon^2)$ solution. The definitions of a Banach Space of Interactions and the corresponding Norms are

[2] aMFT is the reference Gaussian Theory.
[3] Model is not reflection-positive.
[4] Schematically such a term is of the form $\psi(\partial^2)^{\#}\psi$, some non-integer power of the Laplacian.

outlined in the Sec:2.4. In Sec:2.5 we introduce the Renormalization Group Map and rigorously prove that the assumed form of the interaction action is left invariant by the action Renormalization Group Map on it. In this Section, we also state with the aid of Norm Bounds, that the action of the Renormalization Group Map can be expressed as a series that is absolutely convergent in the norm. In conclusion of this Chp:2, the convergence property allowed us to rigorously estimate the action of the Renormalization Group Map, and prove the existence of the Fixed Point, supported by the Key Lemma and Abstract Lemma, details of which are laid out in Sec:2.6.

- Chp:3, lays out the theory and concepts associated with Functional (Exact) Renormalization Group. Sec:3.1 outlines the derivation of the Renormalization Group Flow Equations, with a particular focus on the Polchinski Equation and Wetterich Equation. This derivation comprised, of an additional condition, separation in low modes and high modes. The Flow Equations derived are modelling the Renormalization Group flow of the effective actions after integrating out the high modes. In Sec:3.2, a different approach, involving more mathematical rigour, is used to arrive at the same set of Renormalization Group Flow Equations. In this approach too, the high mode and low mode separation is carried out but in a different method. Subsequently, in Sec:3.3, analogously repeats the aforementioned calculation for an Interacting Fermion Field outlining the differences and similarities with the Scalar Field Renormalization Group Flow Equations. In the concluding remarks of Chp:3, (i.e. Sec:3.4) we discuss in detail the rescaling procedure, in the FRG approach, for a Scalar field.
- Chp:4, lays out the details of the procedure we perform in order to develop a rigorous theory for the FRG Flow Equation - Polchinski Equation for the long-range fermion model. In Sec:4.1, we motivate and derive the Polchinski Equation, for the aforementioned model. Simultaneously, we provide the necessary definitions for the propagator and operators associated with the equation. Subsequently in Sec:4.2, employing the Polchinski Equation, (in its rescaled version) we derive the Fixed Point Equation in terms of the interaction vertices from the Polchinski Equation and attempt to analyze it using the tree expansion procedure. We derive bounds on the derivative, with respect to the scale, on the propagator. We present the algorithm to study the bound on the norm of the re-scaling operator action on interaction kernels and the algorithm to prove the existence of the fixed point.

Chapter 2

Rigorous Renormalization Group

2.1 Algorithm-Rigorous Renormalization Group Approach

The systematic approach to determine the fixed point for the Model, [Ref. Sub-Sec:1.2], can be listed in an ordered manner as follows, (which closely resembles the standard RG procedure):-

- Providing a more detailed and exhaustive view of the model.
- Writing out the main result of the paper [1].
- Determining the Non-trivial Fixed point to the lowest order and proving its existence, in a non-perturbative fashion.

2.2 Model and Results

Ref. Sub-Sec:1.2, we now put forth more characteristics of the model. The propagator[1] [2]

$$P(x) = \int \frac{d^{d}k}{(2\pi)^{d}} \hat{P}(k) \exp(ikx) \quad ; \quad \hat{P}(k) = \frac{\chi(k)}{|k|^{\frac{d}{2}+\varepsilon}} \tag{2.1}$$

Associated with the propagator is the the Gaussian Grassmann Integration measure [3]

$$d\mu_P(\psi) = D\psi \exp(S_2(\psi)) \quad ; \quad S_2(\psi) = \frac{1}{2}\int \frac{d^{d}k}{(2\pi)^{d}} \hat{P}(k)^{-1}\Omega_{ab}\psi_a(k)\psi_b(-k) \tag{2.2}$$

The complete model, comprising of the interacting Gaussian Measure is-

$$\mathscr{Z}^{-1} d\mu_P(\psi) \exp H(\psi) \quad ; \quad \mathscr{Z} = \int d\mu_P(\psi) \exp(H(\psi)) \tag{2.3}$$

The (simplest) interaction, includes the *local* quadratic and quartic terms[4]

$$H_L(\psi) = \nu \int d^{d}x \Omega_{ab}\psi_a\psi_b + \lambda \int d^{d}x (\Omega_{ab}\psi_a\psi_b)^2 \tag{2.4}$$

Further, the following assumption,

$$d \in \{1,2,3\} \quad ; \quad 0 < \varepsilon < min\left(2 - \frac{d}{2}, \frac{d}{6}\right) = \frac{d}{6} \tag{2.5}$$

guarantees that interaction $H_L(\psi)$ is *relevant*. Equivalently, an interaction term consisting of l fields and p derivatives is relevant subject to the constraint,

$$, l[\psi] + p \leq d \quad ; \quad l\left(\frac{d}{4} - \frac{\varepsilon}{2}\right) + p \leq d \tag{2.6}$$

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[1] $\chi(k)$ is the UV-cutoff

[2] The function χ belongs to the Gevery Class G^s (s¿1) of functions, for details Ref Appendix A of [1]

[3] A further rigorous definition of $d\mu_P(\psi)$ entails

$$\langle \psi_{a_1}(x_1)...\psi_{a_{2s}}(x_{2s})\rangle \equiv \int d\mu_p(\psi)\psi_{a_1}(x_1)...\psi_{a_{2s}}(x_{2s}) = \sum_{\pi=Permutation} (-)^{\pi}\Pi_{i=1}^{s} G_{\pi(a_{2i-1})\pi(a_{2i})}(x_{2i-1}, x_{2i})$$

[4] The couplings ν, λ that are considered here are the running couplings, which are different from the previously defined bare couplings ν_0, λ_0. Also it is assumed that $N \geq 4$, to ensure that the quartic term does not vanish identically.

The *Renormalization Transformation* (RGT) operates on a general space of interactions which can be characterized as,

$$H(\psi) = H_L(\psi) + H_{IRR}(\psi) \tag{2.7}$$

, here H_{IRR} denotes the infinitely many numbers of (generally) non-local terms corresponding to Irrelevant Interactions. These interactions, $H_{IRR}(\psi)$ also respect $Sp(N)$ and $O(d)$ symmetries.

The next two main quantities defined are the *re-scaling parameter*, $\gamma \geq 2$ and the *Renormalization Map* which entails two primary operations - integrating-out and dilatation. Analogous to our approach in FRG, (Ref. to Chp:3 for details) the Integrating-Out procedure comprises of separating the Grassmann Field ψ into low mode and high mode fields, as $\psi = \psi_\gamma + \phi$.[5] Our main objective is to eliminate ϕ and consequently define an effective interaction $\exp H_{eff}(\psi_\gamma)$. This is achieved with the help of the following procedure:

- Splitting the Propagator [6]

  $$P(x) = P_\gamma(x) + G(x) \quad ; \quad \tilde{P}_\gamma(k) = \frac{\chi(\gamma k)}{|k|^{\frac{d}{2}+\varepsilon}}; \quad \tilde{G}(k) = \frac{\chi(k) - \chi(\gamma k)}{|k|^{\frac{d}{2}+\varepsilon}} \tag{2.8}$$

  and as a consequence, we can write

  $$d\mu_P(\psi) = d\mu_{P_\gamma}(\psi_\gamma) d\mu_G(\phi) \quad ; \quad \psi = \psi_\gamma + \phi \tag{2.9}$$

- To determine the Correlation functions of ψ_γ capturing the effect of the measure in Eq:2.3, is mathematically equivalent to using the following measure to determine them.[7]

[5] ψ_γ refers to the low mode and ϕ refers to the high mode.

[6] P_γ is scaled version of P.

[7] Relation between P_γ and $P\left(\frac{x}{\gamma}\right)$ is defined as

$$P_\gamma(x) = \gamma^{-2[\psi]} P\left(\frac{x}{\gamma}\right)$$

Normalized Effective Interaction Measure

$$d\mu_{P_\gamma}(\psi_\gamma)\exp\left[H_{eff}(\psi_\gamma)\right] \quad ; \quad \exp\left(H_{eff}(\psi_\gamma)\right) = \int d\mu_G(\phi)\exp\left(H(\psi_\gamma+\phi)\right) \tag{2.10}$$

The dilatation procedure under consideration is governed by the transformation

$$\psi_\gamma(x) = \gamma^{-[\psi]}\psi\left(\frac{x}{\gamma}\right) \tag{2.11}$$

This operation maps Eq:2.10 to a dilated measure

$$d\mu_P(\psi)\exp\left(H'(\psi)\right) \quad ; \quad H'(\psi) = H_{eff}\left[\gamma^{-[\psi]}\psi(./\gamma)\right] \tag{2.12}$$

Thus, the *Renormalization Map* carries out the RGT $\mathfrak{R}(\varepsilon,\gamma) : H \to H'$. And $\mathfrak{R}(\varepsilon,\gamma)$ satisfies the semi-group property [8]

$$\mathfrak{R}(\varepsilon,\gamma_1)\mathfrak{R}(\varepsilon,\gamma_2) = \mathfrak{R}(\varepsilon,\gamma_1\gamma_2) \tag{2.13}$$

Our objective now is to obtain the fixed point of this RGT. [9]
The result of [1] can be summarized with the help of the following theorems,

Fixed Point Theorem
There exists a $\gamma_0 \geq 2$, and a positive continuous function $\varepsilon_0(\gamma)$ defined for $\gamma \geq \gamma_0$ such that for each $\gamma \geq \gamma_0$ and $0 \leq \varepsilon \leq \varepsilon_0(\gamma)$ the Fixed Point Equation Eq:2.61 has a non-trivial solution.

and

[8] $\mathfrak{R}$ depends on d, N, χ also.

[9] While our RGT is obtained by integrating out the degrees of freedom with momenta between $\Lambda \approx 1$ and $\Lambda_{IR} = \frac{\Lambda}{\gamma}$, Λ_{IR} ought not to be regarded as a mass that breaks the fixed point's criticality.

Analytic Fixed Point Theorem
There exists a $\gamma_0 \geq 2$ and a positive continuous function $\varepsilon_0(\gamma)$ defined for $\gamma \geq \gamma_0$ such that for each $\gamma \geq \gamma_0$ and $\varepsilon \in \{z \in \mathbb{C} : |z| \leq \varepsilon_0(\gamma)\} \equiv \mathbb{E}_0$ the Fixed Point Equation Eq:2.61 possesses a solution, analytic in ε. For any $\varepsilon \in \mathbb{E}_0$, such a solution is the unique solution of the Fixed Point Equation in the complex neighbourhood analogue of Eq:2.77.

These aforementioned theorems are based on the *two principle lemmas*

Key Lemma
Choose $d \in \{1, 2, 3\}$, cut-off χ, $N \geq 4$ and an ε satisfying the condition in Eq:2.5. Then

$$\exists \gamma_{key} \geq 2 \tag{2.14}$$

$$\exists\, \delta_0(\gamma),\ A_0(\gamma),\ \{A_k^R(\gamma)\}_{k=0,1,2},\ A(\gamma),\ E_0(\gamma),\ E_1(\gamma) \text{ +ve continuous functs. on } \gamma \geq \gamma_{key} \tag{2.15}$$

with the following property:
Consider any $\gamma \geq \gamma_{key}$, and any $0 < \delta \leq \delta_0(\gamma)$ and any sequence $y = (\nu, \lambda, u)$, satisfying,

$$||y||_{Y(\gamma,\delta)} \leq 1 \tag{2.16}$$

Then the infinite sums defining the functions $e_\nu^{(0)}$, $e_\lambda^{(0)}$ and e_u on the RHS of Eq:2.61 are absolutely convergent and their sums satisfy

$$\begin{aligned} |e_\nu^{(0)}(y)| \leq E_0\delta^2; \quad |e_\lambda^{(0)}(y)| \leq E_1\delta^3 \\ ||e_u(y)||_{B(\gamma,\delta)} \leq \gamma^{-\bar{D}}; \quad \bar{D} = \frac{1}{2} min\,\{D_2 + 2,\ D_4 + 1,\ D_6\} \end{aligned} \tag{2.17}$$

In addition,

$$\begin{aligned} |\partial_i e_\nu^{(0)}(y)| &\leq \frac{E_0\delta^2}{A_0\delta}; \qquad ||\partial_u e_\nu^{(0)}(y)||_{\mathcal{L}(B,\mathbb{R})} \leq E_0\delta^2; \\ |\partial_i e_\lambda^{(0)}(y)| &\leq \frac{E_1\delta^3}{A_0\delta}; \qquad ||\partial_u e_\lambda^{(0)}(y)||_{\mathcal{L}(B,\mathbb{R})} \leq E_1\delta^3; \\ ||\partial_i e_u(y)||_{B(\gamma,\delta)} &\leq \frac{\gamma^{-\bar{D}}}{A_0\delta} \qquad ||\partial_u e_u(y)||_{\mathcal{L}(B,B)} \leq \gamma^{-\bar{D}} \end{aligned} \tag{2.18}$$

where $i = (\nu, \lambda)$ and $\mathcal{L}(B, \mathbb{R})$ is the space of linear operators from $B(\gamma, \delta)$ to $\mathbb{R}$ and similarly for $\mathcal{L}(B, B)$.

and

Abstract Lemma

Suppose that, for a given ε, the constants M_0, $\tilde{\varepsilon}$, α are such that the maps e_j satisfying the bounds

$$|e_\nu(y)| \leq \frac{M_0}{K_1}\varepsilon^2; \quad |e_\lambda(y) \leq \frac{M_0}{K_1}\varepsilon^3; \quad ||e_u(y)||_{B(\gamma,\delta)} \leq 1 \quad (y \in Y_0) \tag{2.19}$$

where

$$Y_0 = \{y : |\nu - \nu_0| \leq M_0\varepsilon^2,\ |\lambda - \lambda_0| \leq M_0\varepsilon^2, ||u||_{B(\gamma,\delta)} \leq 1\} \tag{2.20}$$

and

$$K_1 = K_1(a,b) = max(1 + |a| + |ab|, 1 + |b|) \tag{2.21}$$

also the bounds

$$\begin{aligned} |\partial_i e_\nu| &\leq M_0\varepsilon & ||\partial_u e_\nu||_{\mathcal{L}(B,\mathbb{R})} &\leq M_0\varepsilon\tilde{\varepsilon} \\ |\partial_i e_\lambda| &\leq M_0\varepsilon^2 & ||\partial_u e_\nu||_{\mathcal{L}(B,\mathbb{R})} &\leq M_0\varepsilon^2\tilde{\varepsilon} \\ ||\partial_i e_u||_{B(\gamma,\delta)} &\leq \frac{\alpha}{\tilde{\varepsilon}} & ||\partial_u e_u||_{\mathcal{L}(B,B)} &\leq \alpha \end{aligned} \tag{2.22}$$

where

$$K_2 = K_2(a,b) = max(3 + 3|a| + 2|ab|, 3 + |2b|) \tag{2.23}$$

and lastly the bounds

$$K_1 M_0 \varepsilon \leq 1; \quad K_2 M_0 \varepsilon \leq \frac{1}{2}; \quad K_2 \alpha \leq \frac{1}{2} \tag{2.24}$$

then $F(Y_0) \subset Y_0$ and $||\nabla F(y)|| \leq \frac{1}{2}$ in Y_0, so that F is a contraction in Y_0. and has a unique fixed point there.

Where $F(y) = y - G^{-1}f(y)$, with $y = (\nu, \lambda, u)$, and $f(y)$ as in Eq:2.61, G is some arbitrary invertible (linear) operator and $y_0 = \left(\frac{a}{b}\varepsilon, -\frac{1}{b}\varepsilon, 0\right)\Big|_{b\neq 0}$.

2.3 Fixed Point Equation at the Lowest - Order

While determining the *Fixed Point Equation* (FPE), we consider only the relevant couplings, ν and λ, and their leading order contributions to the beta functions for ν and λ [10]. Apart from these contributions, it can be observed that there exists another order ε^2 contribution to the beta function of λ coupling, due to the self-contraction of the semi-local-sextic term [11] The sextic term inclusion is consistent since - after integrating out the fluctuation field in the local interaction H_L, the resultant, H_{eff}, comprises of a sextic term, (with an order ε^2), tree diagram. Thus in order to determine the FPE at order ε^2, this term is included[12]. This term is denoted by $\mathfrak{X}$. (For the details associated with the inclusion of the sextic term in the FPE, refer to Section 3 of [1].)

Thus,

$$\begin{aligned} \nu &\to \nu + \delta\nu \\ \lambda &\to \lambda + \delta\lambda \\ \mathfrak{X} &\to \mathfrak{X} + \delta\mathfrak{X} \end{aligned} \tag{2.25}$$

after re-scaling, the fields appropriately, Eq:2.25, takes the the form of the FPE, (the $\mathfrak{X}$ term is parametrized as $\mathfrak{X}(x)$, [13]

$$\begin{aligned} \nu &= \gamma^{\frac{d}{2}+\varepsilon}\left[\nu + I_1\lambda + O(\varepsilon^2)\right] \\ I_1 &= 2(N-2)\int \frac{d^dk}{(2\pi)^d}\frac{\chi(k)-\chi(\gamma k)}{|k|^{\frac{d}{2}+\varepsilon}} \end{aligned}$$

[10]The relevant couplings are of order ε, and the beta-functions leading order contributions are of order ε and ε^2 respectively.

[11]Tree Graph contribution, to the sextic interaction, (order ε^2). Even though the sextic term is an irrelevant term, it makes a contribution to the beta function at order ε^2.

[12]It is order ε, so it is included. Only a sharp cutoff would cause this diagram to disappear because its local part would vanish. We had to include it since we employ a smooth cutoff.

[13]The sextic term is called sem-local, since it comprised of 2 groups of ψ^3, interacting via a non-local kernel $\mathfrak{X}(x-y)$. After integrating out the fluctuation field, the contribution to $\mathfrak{X}$ came from the tree-level diagram comprising of two λ-vertices and other contributions are of higher order than ε^2.

$$\delta\mathfrak{X}(x) = -8\lambda^2 g(x)$$

$$\lambda = \gamma^{2\varepsilon}\left[\lambda + I_2\lambda^2 + (N-8)\int d^{d}\mathfrak{X}(x)g(x) + O(\varepsilon^3)\right]$$
$$I_2 = -4(N-8)\int \frac{d^{d}k}{(2\pi)^{d}}\frac{(\chi(k)-\chi(\gamma k))^2}{|k|^{d+2\varepsilon}}$$
$$\mathfrak{X}(x) = \gamma^{2d-6[\psi]}\left[\mathfrak{X}(x\gamma) - 8\lambda^2 g(x\gamma)\right] \tag{2.26}$$

Thus we got the fixed point couplings, ν_* and λ_* at order ε and $\mathfrak{X}_*$ at order ε^2.

2.4 Banach Space of Interactions

2.4.1 Definition and Representations

Considering the following definition[14]

$$\Psi_P = \begin{cases} \psi_a & when\ P = p \\ \partial_\mu\psi_a & when\ P = (p,\mu) \end{cases} \quad ; \quad \Psi(\mathbf{P},\mathbf{x}) = \Pi_{i=1}^{l}\Psi_{P_i}(x_i) \tag{2.27}$$

we write the generalized definition of an Interaction, with kernels $H(\mathbf{P},\mathbf{x})$ (which are anti-symmetric, and they respect $Sp(N)\times O(d)$ symmetry)

$$H(\psi) = \sum\int d^{d}\mathbf{x}H(\mathbf{P},\mathbf{x})\Psi(\mathbf{x},\mathbf{x}) \tag{2.28}$$

subject to the constraint, the number of legs $|\mathbf{P}| = l$ should be *even* and and $l \geq 2$. This allows for the categorization of the kernels into groups. [15] Since the general representation involves a large amount of redundancy, especially in couplings associated with a small number of legs, the kernels are re-casted in a *Trimmed Representation*, in the following way[16] [17]

[14] $\mathbf{P} = (P_1, ..., P_l$ and $\mathbf{x} = (x_1, ..., x_l)$ are finite sequences. and we considered only zero or one derivative of the fields, the rest are absorbed into the kernel with the help of the Trimming procedure (integration-by-parts).

[15] The ordering in the sequence $\mathbf{P}$, is significant.

[16] $H_{l,d}$ denotes a kernel with l legs and d-derivatives in the fields.

[17] $\mathcal{A}$ - Anti-symmetrization Operation (Operator)

- H_2
  - $H_{2,0}$ - purely local
$$H_{2,0} = \nu\Omega_{ab}\delta(x_1 - x_2) \tag{2.29}$$
  - $H_{2,1} = 0$
  - $H_{2,2} = H_{2R}$
- H_4
  - $H_{4,0}$ - purely local
$$H_{4,0} = \frac{1}{3}\lambda(\Omega_{ab}\Omega_{ce} - \Omega_{ac}\Omega_{be} + \Omega_{a,e}\Omega_{bc})\delta(x_1 - x_2)\delta(the x_1 - x_3)\delta(x_1 - x_4) \tag{2.30}$$
  - $\{H_{4,d}\}_{d\geq 1} = H_{4R}$
- $H_6 = H_{6SL} + H_{6R}$
  - $H_{6,0} = H_{6SL} + H_{6R,0}$
$$H_{6SL} = \mathscr{A}\left[\Omega_{ab}\Omega_{ef}\Omega_{cg}\delta(x_1 - x_2)\delta(x_1 - x_3)\mathfrak{X}(x_1 - x_4)\delta(x_4 - x_5)\delta(x_4 - x_6)\right] \tag{2.31}$$
  - $H_{6,d} = H_{6R,d}$ where $p \geq 1$

Therefore the *Trimmed Representation* maps any general l-external leg sequence to a coupling sequence (H_ℓ), where

$$\ell \in TL = \{2L, 2R, 4L, 4R, 6SL, 6R, 8, 10...\} \tag{2.32}$$

2.4.2 Norms

The norm associated with the Interaction Kernels, is determined by the means of weighted L_1 norm

$$||H(\mathbf{P})||_w \int_{x_1=0} d^d|H(\mathbf{A}, \mathbf{x})|w(\mathbf{x}) \tag{2.33}$$

$w(\mathbf{x})$ - translation-ally invariant weight function. Associated definitions include

$$||H_l||_w = \underset{|\mathbf{P}|=l}{max} ||H(\mathbf{P})||_w \tag{2.34}$$

for an Interaction Kernel with l-legs, norm is defined as the maximum of the weighted norms of all kernels belonging to the corresponding group. and

$$||H_\ell||_w = \underset{|\mathbf{P}|=\ell}{max} ||H(\mathbf{P})||_w \tag{2.35}$$

for an Interaction Kernel characterized by ℓ element of TL, norm is defined as the maximum of the weighted norms of all kernels belonging to the corresponding trimmed coupling.
By choosing the weight function $w(\mathbf{x})$ to grow at infinity appropriately, we are able capture all the relevant information about the decay of the kernels $H(\mathbf{P}, \mathbf{x})$, induced by the fluctuation propagator $g(x)$.

The *norm* of an Interaction H, associated with a trimmed sequence (H_ℓ), $\ell \in TL$, is defined as

$$||H|| = sup_{\ell \in TL} \frac{||H_\ell||_w}{\delta_\ell} \tag{2.36}$$

here, the sequence δ_ℓ, is appropriately defined, (to scale appropriately with ℓ) by estimating the size of H_ℓ at the fixed point. The parametrized form of H, i.e., $H = H(\nu, \lambda, u)$, (in this subspace) is characterized by the norm [18]

$$\begin{aligned} y &= (\nu, \lambda, u) \\ ||y||_Y &= max \left\{ \frac{|\nu|}{A_0 \delta}, \frac{|\lambda|}{A_0 \delta}, \frac{||u_{2R}||_w}{A_0^R \delta^2}, \frac{||u_{4R}||_w}{A_1^R \delta^2}, \frac{||u_{6R}||_w}{A_2^R \delta^2}, sup_{l \geq 8} \frac{||u_l||_w}{A \delta^{\frac{l}{2}-1}} \right\} \\ &\delta - \text{proportional to } \varepsilon \end{aligned} \tag{2.37}$$

The constants $A_0, A_0^R, A_1^R, A_2^R, A$, are fixed (Ref. to Sec:2.6, for initial steps) such that, the action of the RG Map on a sequence y, returns a new sequence $\tilde{y}$ in the same Banach Space.

[18] H is parametrized using $(\nu, \lambda, \mathfrak{X}, u)$, where $u = (H_\ell)_{\ell \neq 2L, 4L, 6SL}$. To determine the fixed point, we fixed $\mathfrak{X} = \mathfrak{X}_*$, in order write it as a function of λ and as a consequence H is now parametrized by (ν, λ, u).

2.5 Renormalization Map

2.5.1 Integration

The equation Eq:2.10 is be re-written in the following form (after taking the natural log of the LHS and RHS)[19]

$$H_{eff}(\psi) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle H(\psi+\phi); H(\psi+\phi); ...; H(\psi+\phi)\rangle_c \tag{2.38}$$

we then express the $H(\psi+\phi)$ term using Eqs[:2.28, 2.29,2.30,2.31, 2.32], where ϕ denotes internal-legs-$\bar{\mathbf{Q}} = \mathbf{P}\backslash\mathbf{Q}$ and ψ denotes external-legs-$\mathbf{Q}$.

$$H(\psi+\phi) = \sum_{\mathbf{P}} \sum_{\mathbf{Q}\subset\mathbf{P}} (-)^{\#} \int d^d\mathbf{x} H(\mathbf{P},\mathbf{x}) \mathbf{\Psi}(\mathbf{Q},\mathbf{x_Q}) \mathbf{\Phi}(\bar{\mathbf{Q}},\mathbf{x}_{\bar{\mathbf{Q}}}) \tag{2.39}$$

substituting Eq:2.39 in Eq:2.38, (and extracting the effective interaction)

$$H_{eff}(\mathbf{Q},\mathbf{x_Q}) = \mathcal{A} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathbf{Q}_1,...,\mathbf{Q}_n, \sum \mathbf{Q}_i = \mathbf{Q}} \sum_{\mathbf{P}_1,...,\mathbf{P}_n, \mathbf{Q}_i\subset\mathbf{P}_i} (-)^{\#} \int d^d\mathbf{x}_{\bar{\mathbf{Q}}} \mathcal{C}(\mathbf{x}_{\bar{\mathbf{Q}}}) \Pi_{i=1}^{n} H(\mathbf{P}_i, \mathbf{x}_{\mathbf{P}_i}) \tag{2.40}$$

where

$$\mathcal{C}(\mathbf{x}_{\bar{\mathbf{Q}}}) = \langle \mathbf{\Phi}(\bar{\mathbf{Q}}_1, \mathbf{x}_{\bar{\mathbf{Q}}_1}); ...; \mathbf{\Phi}(\bar{\mathbf{Q}}_n, \mathbf{x}_{\bar{\mathbf{Q}}_n})\rangle_c \tag{2.41}$$

It is to be noted that $H_{eff}(\mathbf{Q},\mathbf{x_Q})$, generally does not satisfy trimming requirements, (Ref. to Sub-Sec:2.5.2) for the solution).
This map is compactly expressed as

$$(H_{eff})_l = \sum_{(\ell_i)_1^n} \mathcal{S}_l^{\ell_1,...,\ell_n}(H) \quad ; \quad H = (H_\ell)_{\ell\in TL} \tag{2.42}$$

The map, $\sum_{(\ell_i)_1^n} \mathcal{S}_l^{\ell_1,...,\ell_n}(H)$ is the sum of all terms in Eq:2.40 which had $|\mathbf{Q}| = l$ and $H(\mathbf{P}_i, \mathbf{x}_{\mathbf{P}_i}) \in H_{\ell_i}$, where $\mathbf{Q}_i$ is arbitrary.

[19] The γ, is dropped in the argument of LHS, since the momentum range-restriction is not important.

Considering the following definitions and conclusions

$$\begin{aligned} Q_l - &\ \text{Vector Space of coupling } H_l \\ Q_\ell - &\ \text{Vector space of trimmed couplings } H_\ell \\ Q_{TRIM} - &\ \text{Vector Space of trimmed coupling sequences } H = (H_\ell)_{\ell \in TL} = \bigotimes_{\ell \in TL} Q_\ell \\ \mathcal{S}_l^{\ell_1,\ldots,\ell_n} - &\ \text{Homogeneous Map of degree } n \text{ from } B_{TRIM} \text{ to } B_l \end{aligned} \tag{2.43}$$

another map $\mathcal{S}_l^{\ell_1,\ldots,\ell_n}$ is constructed by replacing $\Pi_{i=1}^n H(\mathbf{P}_i, \mathbf{x}_{\mathbf{P}_i})$ by $\Pi_{i=1}^n h_i(\mathbf{P}_i, \mathbf{x}_{\mathbf{P}_i})$ in Eq:2.40, where independent $h_i \in B_{\ell_i}$ [20]

Two key observations are to be made

$$\mathcal{S}_l^{\ell_1,\ldots,\ell_n}(H) = \mathcal{S}_l^{\ell_1,\ldots,\ell_n}(h_1, \ldots, h_n) \quad ; \quad h_i = H_{\ell_i} \tag{2.44}$$

and

$$\text{The maps } \mathcal{S}_l^{\ell_1,\ldots,\ell_n} \text{ and } \mathcal{S}_l^{\ell_1,\ldots,\ell_n} \text{ vanished unless } \sum |\ell_i| \geq l + 2(n-1) \tag{2.45}$$

2.5.2 Trimming Operation and Dilatation

A coupling sequence N_l, is called *Null*, if the corresponding interaction vanishes as a function of the Classical Grassmann Fields $\psi_p(x)$. Two coupling sequences associated to each other via a *null difference* are *equivalent and represent the same interaction*. Equivalent couplings are produced with the help of *Interpolation*. The interpolation identities, expressed $\psi_p(x)$ as a weighted linear combination of $\psi_p(y)$ and $\partial_\mu \psi_p(y)$

$$\begin{aligned} \psi_p(x) &= \int dy \left[f(x,y)\psi_p(y) + f^\mu(x,y)\partial_\mu \psi_p(y) \right] \\ f^\mu(x,y) &= \int_0^1 \frac{ds}{(1-s)^d} \left[(x-z)^\mu f(x,z) \right] \Big|_{z=\frac{y-sx}{1-s}} \end{aligned} \tag{2.46}$$

---

[20] This is a Multi-linear Map.

$$\mathbb{S}_l^{\ell_1,\ldots,\ell_n} \ : \ Q_{\ell_1} \times \ldots \times Q_{\ell_n} \to Q_l$$

This Map is symmetric, i.e., invariant under interchanges of the indices ℓ_i accompanied by simultaneous changes of arguments.

This procedure allows us to write a single interaction term[21] as an equivalent interaction term, made up of a sum of $\not{d}+1$ terms as

$$\int d^{\not{d}}\mathbf{x} H(\mathbf{P},\mathbf{x_P})\mathbf{\Psi}(\mathbf{P},\mathbf{x}) \quad \rightarrow \quad \sum_{\mathbf{Q}} \int d^{\not{d}}\mathbf{x} \tilde{H}(\mathbf{Q},\mathbf{x_Q})\mathbf{\Psi}(\mathbf{Q},\mathbf{x})$$

$$\mathbf{Q}=\mathbf{P} \text{ or is obtained by replacing with corresponding } \mu \text{ based derivative element of sequence.}$$
$$\tilde{H}(\mathbf{Q},\mathbf{x}) \text{ are derived by integrating } H(\mathbf{P},\mathbf{x}) \text{ against } f \text{ and } f^{\mu} \tag{2.47}$$

2.5.3 Trimming Maps

This procedure is performed for $l \leq 6$, since $l \geq 8$ are by definition equivalent to a sequence of trimmed couplings.

This procedure maps a sequence $(H_{eff})_l$ to an equivalent sequence of *trimmed couplings*, $(H_{eff})_{\ell}$.

- $l = 6$

$$\begin{aligned}(H_{eff})_{6SL} &= H_{6SL} + S^{(4L,4L)}_{6SL}(H_{4L},H_{4L}) \\ (H_{eff})_6 &= (H_{eff})_{6SL} + (H_{eff})_{6R}\end{aligned} \tag{2.48}$$

- $l = 4$[22]

$$\begin{aligned}(H_{eff})_{4L} &= \mathcal{T}^{4,0}_{4L}(H_{eff})_{4,0} \\ (H_{eff})_{4R,p} &= \begin{cases} 0 & \text{if } p=0 \\ (H_{eff})_{4,1} + \mathcal{T}^{4,0}_{4R}(H_{eff})_{4,0} & \text{if } p=1 \\ (H_{eff})_{4,p} & \text{if } p>1 \end{cases}\end{aligned} \tag{2.49}$$

[21]The single interaction term considered, corresponding to some sequence $\mathbf{P}$ consisted of atleast one not differentiated field, which is re-substituted with the help of the Interpolation Identity.

[22]For the $l = 4$ and $l = 2$ case, the trimming operation is split into localization and interpolation. Localization extracted the Local Parts of $(H_{eff})_{2,0}$ and $(H_{eff})_{4,0}$. Interpolation re-orders the components setting to zero the parts of $(H_{eff})_{2R,p}$ and $(H_{eff})_{4R,p}$, according to the trimming requirements.

- $l = 2$

$$
\begin{aligned}
(H_{eff})_{2L} &= \mathscr{T}_{2L}^{2,0}(H_{eff})_{2,0} \\
(H_{eff})_{2R,p} &= \begin{cases} 0 & \text{if } p = 0,1 \\ (H_{eff})_{2,2} + \mathscr{T}_{2R}^{2,1}(H_{eff})_{2,1} + \mathscr{T}_{2R}^{2,0}(H_{eff})_{2,0} & \text{if } p = 2 \end{cases}
\end{aligned}
\tag{2.50}
$$

The trimming map, $\mathscr{T}$ is of the form $\mathscr{T}_{\ell}^{l}$, denoting, an input-sequence (interaction) of $(H)_l$-type and output-sequence (interaction) of $(H)_\ell$.

2.5.4 Dilatation

After integrating out the fluctuation field and performing the trimming call, we perform the *re-scaling of fields*, a procedure called *Dilatation*, $\mathscr{D}$. Dilatation preserves the trimmed representation. [23]

$$
\mathscr{D} \quad : \quad H_{\ell,p}(\mathbf{x}) \to \gamma^{-D_l - p}\gamma^{d(l-1)} H_{\ell,p}(\gamma\mathbf{x}) \quad ; \quad D_l = l[\psi] - d = l\left(\frac{d}{4} - \frac{\varepsilon}{2}\right) - d \tag{2.51}
$$

$$
\begin{aligned}
\nu &\to \gamma^{-D_2}\nu \\
\lambda &\to \gamma^{-D_4}\lambda \\
\mathfrak{X}(x) &\to \gamma^{-D_6}\gamma^{d}\mathfrak{X}(\gamma x)
\end{aligned}
\tag{2.52}
$$

2.5.5 Renormalization Map and Fixed Point Equation

Composition of the aforementioned trio of operations generates the *Renormalization map*.[24]

$$
\text{Integration} \circ \text{Trimming} \circ \text{Dilatation} \Rightarrow \mathscr{R} = \mathscr{R}(\varepsilon, \gamma)
$$

$$
\mathscr{R} = \mathscr{R}(\varepsilon, \gamma) \ : \ H \in Q_{TRIM} \to H' \in Q_{TRIM} \tag{2.53}
$$

[23] The condition for irrelevance is $D_l + p > 0$. Since $D_2 + 2, D_4 + 1 > 0$ and $D_l \geq D_6 > 0$ $(\forall l \geq 6)$ hence $H_{2R}, H_{4R}, H_{6SL}, H_{6R}$ *and* H_ℓ $(\forall \ell \geq 8)$ are *irrelevant*. $D_2, D_4 < 0$, and thus ν, λ are *relevant*.

[24] The definitions of $\mathcal{R}$ and $\mathscr{R}$ are analogous to the definitions of $\mathcal{S}$ and $\mathscr{S}$ respectively. They also share the same set of properties and characteristics.

where in

$$H'_\ell = \sum_{(\ell_i)_1^n} \mathcal{R}_\ell^{\ell_1,\dots,\ell_n}(H) \tag{2.54}$$

$$\mathcal{R}_\ell^{\ell_1,\dots,\ell_n} - \text{Homogeneous Map of Degree } n$$

$$\mathcal{R}_\ell^{\ell_1,\dots,\ell_n}(H) = \mathscr{R}_\ell^{\ell_1,\dots,\ell_n}(h_1,...,h_n) \quad ; \quad h_i = H_{\ell_i}$$

Explicitly, the multi-linear map $\mathscr{R}$ is written as, using Eqs:[2.48,2.49,2.50,2.51,2.52]

$$\begin{aligned}
\mathscr{R}_\ell^\ell &= \mathscr{D} \\
\mathscr{R}_\ell^{\ell_1,\dots,\ell_n} &= \mathscr{D}\begin{cases} \mathcal{S}_l^{\ell_1,\dots,\ell_n} & \ell \geq 8 \\ T_\ell^l \mathcal{S}_l^{\ell_1,\dots,\ell_n} & \ell \in \{2L, 4L, 2R, 4R\} \end{cases} \\
\mathscr{R}_{6SL}^{\ell_1,\dots,\ell_n} &= \mathscr{D}\begin{cases} \mathcal{S}_6^{4L,4L} & (\ell_i)_1^n = (4L,4L) \\ 0 & \text{Otherwise} \end{cases} \\
\mathscr{R}_{6R}^{\ell_1,\dots,\ell_n} &= \mathscr{D}\begin{cases} \mathcal{S}_6^{\ell_1,\dots,\ell_n} & (\ell_i)_1^n \neq (6SL);(4L,4L) \\ 0 & \text{Otherwise} \end{cases}
\end{aligned} \tag{2.55}$$

The *Fixed Point Equation* (FPE) is, using Eq:2.54

$$(H'_\ell) = (H_\ell) \tag{2.56}$$

which is explicitly written as [25]

$$\begin{aligned}
\nu &= \gamma^{\frac{d}{2}+\varepsilon}\nu + \mathscr{R}_{2L}^{4L}(\lambda) + \sum_{(\ell_i)_1^n \neq (2L),(4L)} \mathscr{R}_{2L}^{\ell_1,\dots,\ell_n}(H_{\ell_1},...,H_{\ell_n}) \\
\lambda &= \gamma^{2\varepsilon}\lambda + \mathscr{R}_{4L}^{4L,4L}(\lambda,\lambda) + \mathscr{R}_{4L}^{6SL}(\mathfrak{X}) + \sum_{(\ell_i)_1^n \neq (4L),(4L,4L),(6SL)} \mathscr{R}_{4L}^{\ell_1,\dots,\ell_n}(H_{\ell_1},...,H_{\ell_n}) \\
\mathfrak{X}(x) &= \mathscr{R}_{6SL}^{6SL}(\mathfrak{X}) + \mathscr{R}_{6SL}^{4L,4L}(\lambda,\lambda) \; = \gamma^{2d6[\psi]}\left[\mathfrak{X}(x\gamma) - 8\lambda^2 g(x\gamma)\right] \\
u_\ell &= \sum_{(\ell_i)_1^n} \mathscr{R}_\ell^{\ell_1,\dots,\ell_n}(H_{\ell_1},...,H_{\ell_n}) \quad ; \quad \text{if } \ell \neq 2L, 4L, 6SL
\end{aligned} \tag{2.57}$$

[25] A distinction is made for the different elements of the Trimmed List TL, i.e., $\ell = 2L, 4L, 6SL$ and $u = (H_\ell)_{\ell \neq 2L,4L,6SL}$.

It is observed that $\mathfrak{X} = \mathfrak{X}_*$ is the solution for the *FPE*, of $\mathfrak{X}(x)$, where in [26]

$$\mathfrak{X}_*(x) = -8\lambda^2 \sum_{n=1}^{\infty} \gamma^{(2d-6[\psi])n} g(x\gamma^n) \tag{2.58}$$

allows us to re-cast the *FPE*, in the parameter space of $y = (\nu, \lambda, u)$,

$$\boxed{y = \mathscr{R}(y)} \tag{2.59}$$

$$\boxed{\begin{aligned} &\nu = \gamma^{\frac{d}{2}+\varepsilon} (\nu + I_1\lambda) + e_\nu^{(0)}(y) \\ &\lambda = \gamma^{2\varepsilon} \left(\lambda + I_2\lambda^2\right) + e_\lambda^{(0)}(y) \\ &u = e_u(y) \end{aligned}} \tag{2.60}$$

where in (using Eq:2.3)

$$\begin{aligned} \mathscr{R}_{2L}^{4L}(\lambda) &= \gamma^{\frac{d}{2}+\varepsilon} I_1\lambda \\ \mathscr{R}_{4L}^{4L,4L}(\lambda,\lambda) + R_{4L}^{6SL}(\mathfrak{X}_*) &= \gamma^{2\varepsilon} I_2\lambda^2 \end{aligned}$$

and

$$\begin{aligned} e_\nu^{(0)} &= \sum_{(\ell_i)_1^n \neq (2L),(4L)} \mathscr{R}_{2L}^{\ell_1,\ldots,\ell_n}(H_{\ell_1},\ldots,H_{\ell_n}) \\ e_\lambda^{(0)} &= \sum_{(\ell_i)_1^n \neq (4L),(4L,4L),(6SL)} \mathscr{R}_{4L}^{\ell_1,\ldots,\ell_n}(H_{\ell_1},\ldots,H_{\ell_n}) \\ e_u &= \sum_{(\ell_i)_1^n} \mathscr{R}_{\ell}^{\ell_1,\ldots,\ell_n}(H_{\ell_1},\ldots,H_{\ell_n}) \quad ; \quad \text{if } \ell \neq 2L, 4L, 6SL \end{aligned}$$

These expressions are written in their condensed forms as[27]

$$f(y) = 0 \quad ; \quad f(y) = \begin{bmatrix} \nu + a\lambda + e_\nu(y) \\ \varepsilon\lambda + b\lambda^2 + e_\lambda(y) \\ u - e_u(y) \end{bmatrix} \tag{2.61}$$

where in

$$\begin{aligned} (a, e_\nu) &= \frac{1}{1-\gamma^{-\frac{d}{2}-\varepsilon}} \left(I_1, \gamma^{-\frac{d}{2}-\varepsilon} e_\nu^0\right) \\ (b, e_\lambda) &= \frac{\varepsilon}{1-\gamma^{-2\varepsilon}} \left(I_2, \gamma^{-2\varepsilon} e_\lambda^{(0)}\right) \end{aligned}$$

[26]Theoretically, one could have treated $\mathfrak{X}$ in a manner identical to all the other irrelevant couplings. The RG Map would then eventually become contractive in both the $\mathfrak{X}$ direction, and subsequent RG iterations would result in the same solution, $\mathfrak{X} = \mathfrak{X}_*$.

[27]After moving the LHS to the RHS and re-scaling.

2.5.6 Norm Bounds on the Renormalization Map

With the help of the definitions in Sub-Sec:[2.4.1; 2.4.2 & 2.5.5] we write the norm bounds governing the multi-linear operators $\mathscr{R}_{\ell}^{\ell_1,...,\ell_n}$ in the definitions of $e_{\nu}^{(0)}$; $e_{\lambda}^{(0)}$; and e_u as

$$||\mathscr{R}_{\ell}^{\ell}(H_{\ell})||_w \leq \begin{cases} \gamma^{-D_2-2}||H_{2R}||_w & \text{if } \ell = 2R \\ \gamma^{-D_4-1}||H_{4R}||_w & \text{if } \ell = 4R \\ \gamma^{D_l}||H_{\ell}||_w & \text{if } l = |\ell| \geq 6 \end{cases} \tag{2.62}$$
$$|\mathscr{R}_{2L}^{2L}(\nu)| = \gamma^{-D_2}|\nu|$$
$$|\mathscr{R}_{4L}^{4L}(\lambda)| = \gamma^{-D_4}|\lambda|$$

also

$$||\mathscr{R}_{\ell}^{\ell_1,...,\ell_n}(h_1,...,h_n)||_w \leq \gamma^{-D_l}\rho_l(h_1,...,h_n) \quad ; \quad h_i \in Q_{\ell_i}$$
$$\rho_l(h_1,...,h_n) = \begin{cases} C_{\gamma}^{n-1}\Pi_{i=1}^{n}C_0^{|\ell_i|}||h_i||_w & \text{if } \sum_i |\ell_i \geq l + 2(n+1)\Big|_{|\ell|=l} \\ 0 & \text{Otherwise} \end{cases} \tag{2.63}$$

and[28]

$$||\mathscr{D}H_{\ell,p}||_w = \gamma^{-D_l-p}||H_{\ell,p}||_{w(./\gamma)} \leq \gamma^{-D_l-p}||H_{\ell,p}||_w \tag{2.64}$$

C_{γ} and C_0 are constants independent of l, n, ℓ_i. Additionally C_0 is independent of γ also.

Frechet Derivatives of the homogeneous map $\mathcal{R}_{\ell}^{\ell_1,...,\ell_n}$ are also defined, since they are required in the *Analysis* of the FPE (Ref. to Sec:2.6).

$\because \mathcal{R}_{\ell}^{\ell} = \mathscr{R}_{\ell}^{\ell}$ is a linear map, the corresponding Frechet Derivative coincides with it.

$$||\nabla_H \mathcal{R}_{\ell}^{\ell}(H)\delta H||_w \leq \begin{cases} \gamma^{-D_2-2}||H_{2R}||_w & \text{if } \ell = 2R \\ \gamma^{-D_4-1}||H_{4R}||_w & \text{if } \ell = 4R \\ \gamma^{D_l}||H_{\ell}||_w & \text{if } l = |\ell| \geq 6 \end{cases} \tag{2.65}$$

and

$$||\nabla_H \mathcal{R}_{\ell}^{\ell_1,...,\ell_n}(H)\delta H||_w \leq \gamma^{D_l}\sum_{i=1}^{n}\rho_l(H_{\ell_1},...,\delta H_{\ell_i},...,H_{\ell_n}) \tag{2.66}$$

[28]Eq:2.64 justifies the relevant and irrelevant terms conditions.

2.6 Fixed Point - Construction

2.6.1 Key Lemma and Abstract Lemma

Following the set-up and the definitions presented in Sec(s):[2.2-2.5] we constructed the fixed point. It is assumed that γ is sufficiently large and the norm of $y = (\nu, \lambda, u)$ is bounded to a value smaller than 1;[29] The constraint placed on ε, is that it allows for the irrelevant directions, u_ℓ to be irrelevant. [30]

For the sake of convenience a new definition is introduced,

$$||u||_{Q(\gamma,\delta)} = max\left\{\frac{||u_{2R}||_w}{A_0^R(\gamma)\delta^2}, \frac{||u_{4R}||_w}{A_1^R(\gamma)\delta^2}, \frac{||u_{6R}||_w}{A_2^R(\gamma)\delta^2}, sup_{l\geq 8}\frac{||u_l||_w}{A(\gamma)\delta^{\frac{l}{2}-1}}\right\} \tag{2.67}$$

which allows us to write Eq:2.37, as

$$||y||_{Y(\gamma,\delta)} = max\left\{\frac{|\nu|}{A_0\delta}, \frac{|\lambda|}{A_0\delta}, ||u||_{Q(\gamma,\delta)}\right\} \tag{2.68}$$

We then use the *Key Lemma*, 2.2 the bounds from which imply that the *FPE*, (Eq:2.61) possesses a solution, $f(y) = 0$. The solution present in a *suitable neighbourhood of the Banach Space Y* and is *unique*.

With concerns to the re-scaling in Eq:2.61, it can be shown through a short calculation that,

$$\frac{\varepsilon}{1-\gamma^{-2\varepsilon}} = \frac{1}{2log\gamma}\left(1 + O(\varepsilon\log\gamma)\right) \tag{2.69}$$

and consequently[31]

$$a = 2(N-2)\left[\int \frac{d^d}{(2\pi)^d}\frac{\chi(k)}{|k|^{\frac{d}{2}}} + O(\varepsilon\log\gamma)\right] \quad ; \quad b = -2(N-8)\left[\frac{S_d}{(2\pi)^d} + O(\varepsilon\log\gamma)\right] \tag{2.70}$$

[29] That is the constants $A_0, A_0^R, A_1^R, A_2^R, A$ are fixed in a suitable γ dependent way, and δ (also dependent on γ is chosen to be sufficiently small.

[30] In the future, it is assumed that δ can be identified as ε up to a constant, but presently it is assumed that γ and ε are independent.

[31] S_d is the area of the unit sphere in $\mathbb{R}^d$.

We also assume $\varepsilon \leq \frac{c}{\log \gamma}$. Under these assumptions ($N \neq 8$ and $b \neq 0$), $a, b, \frac{1}{b}$ are all $O(1)$.

We now calculate an *approximate solution*

$$f_0(y) = 0 \quad ; \quad f_0(y) = \begin{bmatrix} ,\nu + a\lambda \\ \varepsilon\lambda + b\lambda^2 \\ u \end{bmatrix} \tag{2.71}$$

$$y_0 = (\nu_0, \lambda_0, u_0) = \left(\frac{a}{b}\varepsilon, -\frac{1}{b}\varepsilon, 0\right) \tag{2.72}$$

Since we are aiming to apply a *Contraction Argument*, we re-write the *FPE*, (Eq:2.61) as

$$f(y) = 0 \quad \Leftrightarrow \quad y = F(y); \quad F(y) = y - G^{-1}f(y) \tag{2.73}$$

where G, is an arbitrary invertible operator. Further on a G is to be chosen such that $F(y)$ is a contraction in a small neighbourhood of y_0.[32]

For simplicity

$$G = \nabla f_0(y_0) \tag{2.74}$$

gives us

$$G = \begin{bmatrix} 1 & a & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \quad ; \quad G^{-1} = \begin{bmatrix} 1 & \frac{a}{\varepsilon} & 0 \\ 0 & -\frac{1}{\varepsilon} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \tag{2.75}$$

thus the map $F(y)$ takes the form,

$$F(y) = \begin{bmatrix} F^{\nu}(y) \\ F^{\lambda}(y) \\ F^{u}(y) \end{bmatrix} = \begin{bmatrix} -2a\lambda - ab\frac{\lambda^2}{\varepsilon} - e_\nu - a\frac{e_\lambda}{\varepsilon} \\ 2\lambda + b\frac{\lambda^2}{\varepsilon} + \frac{e_\lambda}{\varepsilon} \\ e_u \end{bmatrix} \tag{2.76}$$

The *contraction argument* is applied to $F(y)$, in a neighbourhood Y_0, of y_0[33]

$$Y_0 = \{y : |\nu - \nu_0| \leq M_0\varepsilon^2;\ |\lambda_0 - \lambda_0| \leq M_0\varepsilon^2, ||u||_Q \leq 1\} \tag{2.77}$$

[32] The reason for defining a new map $F(y)$ and not using the renormalization map itself, is that contraction cannot be applied directly to R. R is not fully contracting.

[33] Size of the neighbourhood depends on ε and a parameter M_0.

Writing $\lambda = \lambda_0 + \delta\lambda$, allows us to write $F(y)$ as

$$F(y) = \begin{bmatrix} \nu_0 - \frac{ab}{\varepsilon}(\delta\lambda)^2 - e_\nu - \frac{a}{\varepsilon}e_\lambda \\ \lambda_0 + \frac{b}{\varepsilon}(\delta\lambda)^2 + \frac{1}{\varepsilon}e_\lambda \\ e_u \end{bmatrix} \tag{2.78}$$

as a consequence of which for any $y \in Y_0$ satisfying

$$K_1 max\left(M_0^2\varepsilon^3, |e_\nu(y)|, \frac{|e_\lambda(y)|}{\varepsilon}\right) \leq M_0\varepsilon^2 \quad ; \quad ||e_u(y)||_Q \leq 1 \tag{2.79}$$

where

$$K_1 = K_1(a,b) = max(1 + |a|a + |ab|, 1 + |b|) \tag{2.80}$$

we conclude $F(Y_0) \subset Y_0$ is constrained by

$$|e_\nu(y)| \leq \frac{M_0}{K_1}\varepsilon^2; \quad |e_\lambda(y)| \leq \frac{M_0}{K_1}\varepsilon^3; \quad ||e_u(y)||_Q \leq 1; \quad y \in Y_0 \tag{2.81}$$

These relations and equations (Eq:2.76-Eq:2.81) allow us to ensure that

F maps Y_0 to itself.

Using the norm[34]

$$||y||_Y = max\left\{\frac{|\nu|}{\tilde{\varepsilon}}, \ \frac{|\lambda|}{\tilde{\varepsilon}}, ||u||_Q\right\} \tag{2.82}$$

we thus conclude that F is a contraction in Y_0.
We also analyze the ∇F (Frechet Derivative) and arrange for its operator norm to be less than 1. [35] Using Eq:2.78

$$\frac{\partial(F^\nu, F^\lambda, F^u)}{\partial(\nu, \lambda, u)} = \begin{bmatrix} -\partial_\nu e_\nu - \frac{a}{\varepsilon}\partial_\nu e_\lambda & -\frac{2ab}{\varepsilon}(\lambda - \lambda_0 - \partial_\lambda)e_\nu - \frac{a}{\varepsilon}\partial_\lambda e_\lambda & -\partial_u e_\nu - \frac{a}{\varepsilon}\partial_\nu e_\lambda \\ \frac{1}{\varepsilon}\partial_\nu e_\lambda & \frac{2b}{\varepsilon}(\lambda - \lambda_0) + \frac{1}{\varepsilon}\partial_\lambda e_\lambda & \frac{1}{\varepsilon}\partial_u e_\lambda \\ \partial_\nu e_u & \partial_\lambda e_u & \partial_u e_u \end{bmatrix} \tag{2.83}$$

[34] A new parameter $\tilde{\varepsilon}$ is introduced. This is done to ensure that all term in Eq:2.82, are of the same order, (ν, λ are of $O(\varepsilon)$ and $||u||_B$ is of $O(1)$). In the future this is fixed to be $A_0\delta$ to make all equations and definitions consistent.

[35] The gradient is discussed briefly, but the approach and techniques involves are analogous to the ones used prior to this calculation.

Consequently we also study the norm of $\nabla F(y) \in \mathcal{L}Y, Y$ where $y \in Y_0$. Assuming

$$\delta y \in Y; \quad ||\delta y||_Y \leq 1; \quad \Leftrightarrow \quad |\delta \nu| \leq \tilde{\varepsilon}; \; |\delta \lambda| \leq \tilde{\varepsilon}; \; ||\delta u||_Q \leq 1 \tag{2.84}$$

we get

$$\nabla F(y)\delta = \begin{bmatrix} \partial_\nu F^\nu \delta\nu + \partial_\lambda F^\nu \delta\lambda + \partial_u F^\nu \delta u \\ \partial_\nu F^\lambda \delta\nu + \partial_\lambda F^\lambda \delta\lambda + \partial_u F^\lambda \delta u \\ \partial_\nu F^u \delta\nu + \partial_\lambda F^u \delta\lambda + \partial_u F^u \delta u \end{bmatrix}\Bigg|_y \tag{2.85}$$

After analysing the norm of $\nabla F(y)$, we get

$$\begin{aligned} ||\nabla F(y)||_{\mathcal{L}(Y,Y)} &= sup_{||\delta y||_Y \leq} ||\nabla F(y)(\delta y)||_Y \\ &= sup_{||\delta y||_Y \leq 1} ||\nabla F(y)(\delta y)||_Y \\ &= sup_{||\delta y||_Y \leq 1} max \begin{bmatrix} \tilde{\varepsilon}^{-1} |\partial_\nu F^\nu \delta\nu + \partial_\lambda F^\nu \delta\lambda + \partial_u F^\nu \delta u| \\ \tilde{\varepsilon}^{-1} |\partial_\nu F^\lambda \delta\nu + \partial_\lambda F^\lambda \delta\lambda + \partial_u F^\lambda \delta u| \\ ||\partial_\nu F^u \delta\nu + \partial_\lambda F^u \delta\lambda + \partial_u F^u \delta u||_Q \end{bmatrix} \\ &\leq \begin{bmatrix} |\partial_\nu F^\nu| + |\partial_\lambda F^\nu| + \varepsilon^{-1} ||\partial_u F^\nu||_{\mathcal{L}(Q,\mathbb{R}} \\ |\partial_\nu F^\lambda| + |\partial_\lambda F^\lambda| + \varepsilon^{-1} ||\partial_u F^\lambda||_{\mathcal{L}(Q,\mathbb{R}} \\ \tilde{\varepsilon} ||\partial_\nu F^u||_Q + \varepsilon ||\partial_\lambda F^u||_Q + ||\partial_u F^u||_{\mathcal{L}(Q,Q)} \end{bmatrix} \end{aligned} \tag{2.86}$$

Subsequently, using Eq:2.83 and Eq:2.86, for $y \in Y_0$ (We also use, $|\lambda - \lambda_0| \leq M_0 \varepsilon^2$ in Y_0.) we get

$$\begin{aligned} ||\nabla F(y)||_y \leq K_2 max \Big[&|\partial_\nu e_\nu|, \; \frac{1}{\varepsilon} |\partial_\nu e_\lambda, \; M_0 \varepsilon, |\partial_\lambda e_\nu|, \; \frac{1}{\varepsilon} |\partial_\lambda e_\lambda|, \; \frac{1}{\tilde{\varepsilon}} ||\partial_u e_\nu||_{\mathcal{L}(Q,\mathbb{R})}, \\ &\frac{1}{\tilde{\varepsilon}\varepsilon} ||\partial_u e_\lambda||_{\mathcal{L}(Q,\mathbb{R})}, \; \tilde{\varepsilon} ||\partial_\nu e_u||_Q, \; \tilde{\varepsilon} ||\partial_\lambda e_u||_Q, ||\partial_u e_u||_{\mathcal{L}(Q,Q)} \Big] \end{aligned} \tag{2.87}$$

where

$$K_2(a,b) = max(3 + 3|a| + 2|ab|, 3 + 2|b|) \tag{2.88}$$

Following this discussion we demanded, that the following condition holds uniformly for $y \in Y_0$.

$$\begin{aligned} &|\partial_i e_\nu| \leq M_0 \varepsilon && ||\partial_u e_\nu||_{\mathcal{L}(B,\mathbb{R})} \leq M_0 \varepsilon \tilde{\varepsilon} \\ &|\partial_i e_\lambda| \leq M_0 \varepsilon^2 && ||\partial_u e_\nu||_{\mathcal{L}(B,\mathbb{R})} \leq M_0 \varepsilon^2 \tilde{\varepsilon} \\ &||\partial_i e_u||_{B(\gamma,\delta)} \leq \frac{\alpha}{\tilde{\varepsilon}} && ||\partial_u e_u||_{\mathcal{L}(B,B)} \leq \alpha \end{aligned} \tag{2.89}$$

where α is another parameter. Subject to the conditions in Eq:2.89, Eq:2.87 implies

$$||\nabla F(y)||_{\mathcal{L}(Y,Y)} \leq max(K_2 M_0 \varepsilon, K_2 \alpha); \quad y \in Y_0 \tag{2.90}$$

Summarizing and collecting the results from the above relations, (Eq:2.67-Eq:2.90), we wrote the *Abstract Lemma*, 2.2.

2.6.2 Fixed Point Theorem

In conclusion, we now put together the results of Key Lemma and Abstract Lemma, to prove the existence of a unique solution to the *FPE*, (Eq:2.61). This is formally stated in the form of the *Fixed Point Theorem*, 2.2. The strategy to prove the Fixed Point Theorem is simple: With the help of the Key Lemma, for some $\gamma \geq \gamma_0$ and $0 < \varepsilon \leq \varepsilon_0(\gamma)$, conditions of Abstract Lemma can be satisfied.
To achieve this we identified, the Banach Space Q, with the Banach Space $Q(\gamma, \delta)$ and identified the Banach space Y, with the Banach Space $Y(\gamma, \delta)$. This is followed up by setting

$$\tilde{\varepsilon} = A_0 \delta \tag{2.91}$$

The parameter δ in the Key Lemma is now chosen proportional to ε.[36]

$$\delta = h\varepsilon \tag{2.92}$$

Examining the neighbourhood of Y_0, as required by the Abstract Lemma, calls for setting $K_1 M_0 \varepsilon \leq 1$, allowing the points of Y_0 to satisfy

$$\begin{gathered} |\nu|, |\lambda| \leq K_3 \varepsilon \\ K_3 = K_3(a,b) = max\left(\frac{|a|}{|b|} + \frac{1}{K_1}, \frac{1}{|b|} + \frac{1}{K_1}\right) \end{gathered} \tag{2.93}$$

By choosing $h = \frac{K_3}{A_0}$ and using Eq:2.92, we get

$$Y_0 \subset \{y : ||y||_Y \leq 1\} \tag{2.94}$$

[36] h is fixed momentarily.

Therefore the basic assumption Eq:2.16 holds in Y_0, which allowed us to use the Key Lemma to estimate e_j and their corresponding derivatives in Y_0.
Also, considering

$$\alpha = \gamma^{-\bar{D}} \tag{2.95}$$

and using Eq:2.91 and Eq:2.95, the bounds on the derivatives in Eq:2.89 coincide with the corresponding bounds for the derivatives in Eq:2.21 (from the Key Lemma).[37]
Next we chose γ_0 as,

$$\gamma_0 = max\left(\gamma_{key}, (2K_2)^{\frac{1}{\bar{D}}}\right) \tag{2.96}$$

As a consequence of which we have $\gamma \geq \gamma_0, \gamma_{key}$ and we could use the Key Lemma and simultaneously we satisfy, the third condition in Eq:2.24.

FPE, Eq:2.61, displays that e_ν, e_λ equals $e_\nu^{(0)}$, $e_\lambda^{(0)}$ times a γ dependent factor, f_γ.[38]
Therefore, (using the Lemma, Abstract Lemma and Eq:2.92), $(i = \nu, \lambda)$ we get,

$$\begin{aligned}
&|e_\nu| \leq E'_0 h^2 \varepsilon^3 && |e_\lambda| \leq E'_1 h^3 \varepsilon^3 \\
&|\partial_i e_\nu \leq \frac{E'_0}{A_0} h\varepsilon && |\partial_i e_\lambda| \leq \frac{E'_1}{A_0} h^2 \varepsilon^2 \\
&||\partial_u e_\nu||_{\mathcal{L}(Q,\mathbb{R}} \leq E'_0 h^2 \varepsilon^2 && ||\partial_u e_\lambda||_{\mathcal{L}(Q,\mathbb{R}} \leq E'_1 h^3 \varepsilon^3
\end{aligned} \tag{2.97}$$

since these have the same scaling in ε, as their corresponding estimates in Eq:2.81 and Eq:2.89. Setting $\frac{\tilde{\varepsilon}}{\varepsilon} = A_0 h$, and choosing M_0 sufficiently large to satisfy the conditions in Eq:2.81 and Eq:2.89

$$M_0 = max\left(K_1 E'_0 h^2, K_1 E'_1 h^3, \frac{E'_0}{A_0} h, \frac{E'_1}{A_0} h^2\right) \tag{2.98}$$

and lastly choosing,

$$\varepsilon_0(\gamma) = min\left(\frac{\delta_0}{h}, \frac{1}{K_1 M_0}, \frac{1}{2K_2 M_0}\right) \tag{2.99}$$

allows us to satisfy the first two conditions in Eq:2.24 and $\delta = h\varepsilon \leq \delta_0$.
Thus, for any $0 < \varepsilon \leq \varepsilon_0(\gamma)$, conditions of the Abstract Lemma are satisfied, and hence a unique Fixed Point exists.[39]

[37] The requirement of $||e_u||_B \leq 1$ in Y_0, as need by Eq:2.81, is also satisfied by the bound on $||e_u||_Q$ in Eq:2.17 and Eq:2.18

[38] Key Lemma gave us estimates for $e_\nu^{(0)}$, e_λ^0, and by scaling with f_γ we can write the estimates for e_ν, e_λ, denoted by the *p*rimed notation.

[39] For the detailed proof of the Key Lemma, refer to Section 7 of [1].

Chapter 3

Functional Renormalization Group

3.1 FRG for a Scalar Field

We consider a Euclidean Version of the (one component) Scalar Field [14,15]. The propagator in this theory is denoted by $\mathscr{P}$ [1],[2],[3] The *partition function* (regulated by an overall momentum cut-off Λ_0) is

$$Z[J] = \int \mathscr{D}\phi \exp\left(-\frac{1}{2}(\phi, \mathscr{P}^{-1}\phi) - S_{\Lambda_0}[\phi] + (J,\phi)\right) \tag{3.1}$$

Equation Eq:3.1 is re-written as

$$Z[J] = \int d\mu_{\mathscr{P}}[\phi] \exp\left(-S_{\Lambda_0}[\phi] + (J,\phi)\right) \tag{3.2}$$

We proceed with writing this partition function, by splitting it into high and low modes, with the help of the following definitions [4]

$$\mathscr{P} = \mathscr{P}_{>} + \mathscr{P}_{<} \quad ; \quad \phi = \phi_{<} + \phi_{>} \tag{3.3}$$

[1]The *n-point Green's Functions* are defined when $\mathbf{p}_1 + \mathbf{p}_2 + ... + \mathbf{p}_n = 0$. And the propagator $\mathscr{P}(p)$ only depends on the magnitude of the two momenta involved,i.e. for $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p} + (-\mathbf{p}) = 0$ and $\mathscr{P} = \mathscr{P}(|\mathbf{p}| = p)$

[2]Notation, $(\phi, J) \equiv \phi_x J_x \equiv \int d^dx \phi(\mathbf{x}) J(\mathbf{x})$.

[3]Notation,

$$(\phi, \mathscr{P}^{-1}\phi) \equiv \int \frac{d^dp}{(2\pi)^d} \phi(\mathbf{p})[\mathscr{P}^{-1}]\phi(-\mathbf{p})$$

[4]We used, like in the paper, $\theta_\varepsilon(p,\Lambda)$, a smooth cut-off function, bounded between 0 *and* 1. This cut-off function satisfied, $\theta_\varepsilon(p,\Lambda) \approx 0$ for $p < \Lambda - \varepsilon$ and $\theta_\varepsilon(p,\Lambda) \approx 1$ for $p > \Lambda + \varepsilon$.

This gives us,

$$Z[J] = \int \mathcal{D}\phi_> \mathcal{D}\phi_< \exp\Bigg(-\frac{1}{2}(\phi_>, \mathcal{P}_>^{-1}\phi_>) - \frac{1}{2} \\ (\phi_<, \mathcal{P}_<^{-1}\phi_<) - S_{\Lambda_0}[\phi_> + \phi_<] + (J, \{\phi_> + \phi_<\})\Bigg) \tag{3.4}$$
$$= \int d\mu_{\mathcal{P}_<}[\phi_<] d\mu_{\mathcal{P}_>}[\phi_>] \exp\left(-S_{\Lambda_0}[\phi_> + \phi_<] + (J, \phi_< + \phi_>)\right)$$

where

$$\mathcal{P}_>(p) = (\theta_\varepsilon(p,\Lambda) - \theta_\varepsilon(p,\Lambda_0)\mathcal{P}(p) \quad and \quad \mathcal{P}_<(p) = (1 - \theta_\varepsilon(p,\Lambda))\mathcal{P}(p) \tag{3.5}$$

Ref. Eq:3.5, $\mathcal{P}_>(p)$ is cut-off from below by Λ and above Λ_0 and $\mathcal{P}_<(p)$ cut-off from above by Λ. Integrating out the high-modes, $\phi_>$ [5]

$$Z[J] = \int \mathcal{D}\phi_< \exp\left(-\frac{1}{2}(\phi_<, \mathcal{P}_<^{-1}\phi_<) - S_\Lambda[(P, J) + \phi_<] + (J, \phi_<) + \frac{1}{2}(J, \mathcal{P}_> J)\right)$$
$$= \int d\mu_{\mathcal{P}_<}[\phi_<] \exp\left(-S_\Lambda[(\mathcal{P}_>, J) + \phi_<] + (J, \phi_<) + \frac{1}{2}(J, \mathcal{P}_> J)\right) \tag{3.6}$$
$$= \int d\mu_{\mathcal{P}_<}[\phi_<] \mathcal{Z}_\Lambda[\phi_<, J]$$

where we define

$$\mathcal{Z}_\Lambda[\phi_<, J] = \exp\left(W_\Lambda[\phi_<, J]\right.$$
$$W_\Lambda[\phi_<, J] = -S_\Lambda[(\mathcal{P}_>, J) + \phi_<] + (J, \phi_<) + \frac{1}{2}(J, \mathcal{P}_> J) \tag{3.7}$$

and

$$\exp\left(-S_\Lambda[\phi_<]\right) = \int d\mu_{\mathcal{P}_>}[\phi_>] \exp\left(-S_{\Lambda_0}[\phi_< + \phi_>]\right) \tag{3.8}$$

Differentiating, with respect to $\xi = \ln \frac{\Lambda}{\mu_0}$, we get

$$\frac{\partial}{\partial \xi}\mathcal{Z}_\Lambda[\phi_<, J] = \frac{1}{2}\left(\frac{\delta}{\delta\phi_<}\right) \cdot \left(\frac{\partial}{\partial \xi}\mathcal{P}_>\right) \cdot \left(\frac{\delta}{\delta\phi_<}\right) \mathcal{Z}_\Lambda[\phi_<, J] \tag{3.9}$$

[5]The algorithm, followed is, isolating the integral over the high modes $\phi_>$ dependent terms, and substituting $\phi_> = \phi - \phi_<$

and consequently, we arrive at another important quantity,

$$\frac{\partial}{\partial \Lambda} W_\Lambda[\phi_<, J] = \frac{1}{2}\left(\frac{\delta W_\Lambda[\phi_<, J]}{\delta \phi_<}.\frac{\partial \mathscr{P}_>}{\partial \xi}\frac{\delta W_\Lambda[\phi_<, J]}{\delta \phi_<} + Tr\left[\frac{\partial \mathscr{P}}{\partial \xi}.\frac{\delta^2 W_\Lambda[\phi_<, J]}{\delta \phi_< \delta_<}\right]\right) \tag{3.10}$$

evaluating Eq:3.10, for $J = 0$, we get the *Polchinski's Equation* [6]

$$\boxed{\frac{\partial S_\Lambda[\phi_<]}{\partial \xi} = -\frac{1}{2}\frac{\delta S_\Lambda}{\phi_<}.\frac{\partial \mathscr{P}_>}{\partial \xi}.\frac{\delta S_\Lambda}{\phi_<} + \frac{1}{2}Tr\left[\frac{\partial \mathscr{P}}{\partial \xi}.\frac{\delta^2 S_\Lambda[\phi_<]}{\delta \phi_< \delta \phi_<}\right]} \tag{3.11}$$

Note: A few useful relations used in the calculations ahead

$$\begin{aligned} \frac{\delta W_\Lambda[\phi_<, J]}{\delta \phi_<} &= -\mathscr{P}_>^{-1}\frac{\delta S_\Lambda[\phi_<, J]}{\delta J} + J = \mathscr{P}_>^{-1}\left(\frac{\delta W_\Lambda[\phi_<, J]}{\delta J} - \phi_<\right) \\ \frac{\delta^2 W_\Lambda[\phi_<, J]}{\delta \phi_< \delta \phi_<} &= \mathscr{P}_>^{-2}\frac{\delta^2 S_\Lambda[\phi_<, J]}{\delta J \delta J} + J = \mathscr{P}_>^{-2}\left(\frac{\delta^2 W_\Lambda[\phi_<, J]}{\delta J \delta J} - \mathscr{P}_>\right) \end{aligned} \tag{3.12}$$

We now define,

$$\varphi = \frac{\delta W_\Lambda[\phi_<, J]}{\delta J} \tag{3.13}$$

and using Eq:3.13, we then re-write Eq:3.10

$$\frac{\partial}{\partial \xi} W_\Lambda[\phi_<, J] = -\frac{1}{2}\left((\varphi - \phi_<).\frac{\partial \mathscr{P}_>^{-1}}{\partial \xi}.(\varphi - \phi_<) + Tr\left[\frac{\partial \mathscr{P}_>^{-1}}{\partial \xi}.\left(\frac{\delta^2 W_\Lambda[\phi_<, J]}{\delta J \delta J} - \mathscr{P}_>\right)\right]\right) \tag{3.14}$$

This gives us the *Effective Average Action*, using the Legendre Transform relation,

$$\tilde{\Gamma}_\Lambda[\varphi, \phi_<] = ((J, \varphi) - W_\Lambda[\phi_<, J])\Big|_{\frac{\delta W_\Lambda}{\delta J} = \varphi} \tag{3.15}$$

the *Interacting Part*, of the Effective Average Action defined in Eq:3.15 is defined as[7]

$$\Gamma[\varphi] = \tilde{\Gamma}_\Lambda[\varphi, \phi_<] - \frac{1}{2}(\varphi - \phi_<)\mathscr{P}_>^{-1}(\varphi - \phi_<) \tag{3.16}$$

[6]These set of Equations Eq:3.10, Eq:3.11, can also be written in terms of $\partial_\xi \mathscr{P}_<$ instead of $\partial_\xi \mathscr{P}_>$ subject to the conditions

- $-\partial_\xi \mathscr{P}_< = \partial_\xi \mathscr{P}_>$
- Carefully tending to non-differentiated $\mathscr{P}_>$ in the equations.

[7]$\Gamma_\Lambda[\varphi]$ as defined, *does not* depend on $\phi_<$

Differentiating Eq:3.16 with ξ we find,

$$\begin{aligned}\frac{\partial}{\partial\xi}\Gamma_\Lambda[\varphi] &= -\frac{\partial}{\partial\xi}W_\Lambda[\phi_<, J] - \frac{1}{2}\left(\varphi-\phi_<\right).\frac{\partial\mathscr{P}_>^{-1}}{\partial\xi}.\left(\varphi-\phi_<\right) \\ &= \frac{1}{2}Tr\left[\frac{\partial\mathscr{P}_>^{-1}}{\partial\xi}.\frac{\delta^2 W_\Lambda[\phi_<, J]}{\delta J\delta J}\right] - \frac{1}{2}Tr\left[\mathscr{P}_>.\frac{\partial\mathscr{P}_>^{-1}}{\partial\xi}\right]\end{aligned} \tag{3.17}$$

Using the property, we termed as Identity Relation[8],

$$\int d^d z \frac{\delta^2\tilde{\Gamma}_\Lambda[\varphi,\phi_<]}{\delta\varphi(x)\delta\varphi(z)}\frac{\delta^2 W_\Lambda[\phi_<, J]}{\delta J(z)\delta J(y)} = \delta^{(d)}(x-y) \tag{3.18}$$

we get the equation,

$$\boxed{\begin{aligned}\frac{\partial}{\partial\xi}\Gamma_\Lambda[\varphi] &= \frac{1}{2}Tr\left[\left(\frac{\delta^2\Gamma_\Lambda[\varphi]}{\delta\varphi\delta\varphi}+\mathscr{P}^{-1}\right)^{-1}.\frac{\partial\mathscr{P}_>^{-1}}{\partial\xi}\right] - \frac{1}{2}Tr\left[\mathscr{P}_>.\frac{\partial\mathscr{P}_>^{-1}}{\partial\xi}\right] \\ &= \frac{1}{2}Tr\left[\frac{\delta^2\Gamma_\Lambda[\varphi]}{\delta\varphi\delta\varphi}.\left(\mathscr{P}.\frac{\delta^2\Gamma_\Lambda[\varphi]}{\delta\varphi\delta\varphi}+1\right)^{-1}+\frac{\partial\mathscr{P}_>}{\partial\xi}\right]\end{aligned}} \tag{3.19}$$

This is the *Wetterich Equation.*[9]

[8] Implicitly the $\tilde{\Gamma}_\Lambda$ and W_Λ, post the operation of functional derivatives are expressed using the same parameter space, using Eq:3.13.

[9] In case of a Free Theory,

$$W_\Lambda[\phi_<, J] = \frac{1}{2}J.\mathscr{P}_>.J + J.\phi_<$$

followed from

$$\Gamma_\Lambda[\varphi] = 0$$

and vacuum term in Eq:3.19, establishes that for a Free Theory

$$\frac{\partial}{\partial\xi}\Gamma_\Lambda[\varphi] = 0$$

3.2 FRG for a Scalar Field - A Different Approach

We again consider a Euclidean Version of a Scalar Field Theory, but this time in d dimensions [16]. The field is denoted by ϕ, and the corresponding partition function is,[10]

$$Z[J] = \int d\mu_{C_\Lambda}(\phi) \exp\left(- \int V(\phi) + \int J\phi \right) \tag{3.20}$$

$$d\mu_{C_\Lambda} = \mathcal{D}\phi \exp\left(-\frac{1}{2} \int_{x,y} \phi(x) C_\Lambda^{-1}(x-y)\phi(y) \right)$$
$$C_\Lambda = (1 - \theta_\epsilon(p, \Lambda))C(p)$$
$$C(p) = \frac{1}{p^2 + m^2}$$

Separation of $\phi(p)$ into low modes and high modes, is done with regards to the momentum scale p. Thus, working in Fourier Space,

$$\phi_p = \phi_{p,<} + \phi_{p,>} \tag{3.21}$$

$$\begin{aligned} \phi_p &\to C_\Lambda(p) \\ \phi_{p,<} &\to C_p(p) \\ \phi_{p,>} &\to C_\Lambda - C_p(p) \end{aligned} \tag{3.22}$$

Additionally, we have the identity-

$$d\mu_{C_\Lambda}(\phi) = d\mu_{C_p}(\phi_<) d\mu_{C_\Lambda - C_p}(\phi_>) \tag{3.23}$$

After performing an integration over the high modes $\phi_>$, we define a running potential V_p (associated scale is p)-

$$\exp\left(- \int V_p(\phi_<) \right) = \int d\mu_{C_\Lambda - C_p}(\phi_>) \exp\left(V(\phi_> + \phi_<\right) \tag{3.24}$$

and the Partition Function can be re-written as-

$$Z = \int d\mu_{C_p}(\phi_<) \exp\left(\int V_p(\phi_<) \right) \tag{3.25}$$

With these definitions, we can derive the Polchinski Equation, a differential equation that captures the evolution of V_p as p changes. Another mathematically equivalent formulation *Effective Average Action Method*, is considered to more convenient. The idea proposed by Kadanoff and Wilson is

[10] θ_ϵ is the cut-off function - a step function in momentum space, beginning at Λ and is a smooth function in an interval ϵ around Λ

mapping Hamiltonians to Hamiltonians at larger scales. The Hamiltonians obtained in this manner are the Hamiltonians of the modes that have not yet been integrated out in the partition function, i.e. $\phi_<$. So rather than computing this sequence of Hamiltonians, we can compute the Gibbs free energy $\Gamma[M]$ of the already integrated out rapid modes, i.e. $\phi_>$. So we construct a one-parameter family of models, characterized by a scale p such that-

- $p = \Lambda \rightarrow$ the case where in no fluctuation had been integrated out, the Gibbs free energy $\Gamma_p[M]$ is the microscopic Hamiltonian.

$$\Gamma_{p=\Lambda}[\varphi] = H[\phi = \varphi] \tag{3.26}$$

- $p = 0 \rightarrow$ the case where all the fluctuations had been integrated out, $\Gamma_{p=0}$, is the Gibbs Free Energy of the original model.

$$\Gamma_{p=0}[\varphi] = \Gamma[\varphi] \tag{3.27}$$

Consequently, integrating out more and more fluctuations decreases p. This allows us to make two pivotal observations:-

- p denotes the UV-cut-off for the low modes $\phi_<$ in the Wilson-Polchinski formulation. But it plays the role of IR-cut-off in the *Effective Average Action method*; Γ_p is the free energy of high modes.
- Low modes are a crucial part of the Wilson-Polchinski Formulation, but they are absent from the *Effective Average Action method*, which is based on the free energy of high modes. Fortunately, the object $\Gamma_p[\varphi]$ comprises all the information about the model, which is a major plus point over the Wilson-Polchinski Formulation.

The explicit construction of the one-parameter family of Γ_p, is achieved by giving the low modes a large mass, which decouples them in the partition function[11]
Thus, a one-parameter family of models is built, with the help of a momentum-dependent mass

[11] With regards to Particle Physics, a large mass translates to a small Compton Wavelength and, consequently to a small range of distances where quantum fluctuations are significant. Since a very heavy particle is decoupled from low energy (relative to its mass) physics because it can only play a role at energies below its mass threshold via virtual processes. Inverse powers of the mass of the heavy particle coming from its propagator suppress these processes.

term infused in the original Hamiltonian,

$$\mathscr{Z}_p[J] = \int \mathscr{D}\phi(x) \exp\left[-H[\phi] - \Delta H_p[\phi] + \int J\phi\right] \quad ; \quad \Delta H_p[\phi] = \frac{1}{2}\int R_p(q)\phi_q\phi_{-q} \tag{3.28}$$

here $R_p(q)$ is the cut-off function[12], chosen with the following properties-

- $p = 0$, $R_{p=0}(q) = 0$ identically $\forall q \rightarrow \mathscr{Z}_{p=0}[J] = \mathscr{Z}[J]$ - This guarantees that the original model is recovered once all fluctuations have been accounted for [13]
- $p = \Lambda$ - Fluctuations are frozen[14] Tying the modes $q \in [0, \Lambda]$ to an infinite mass, freezes their propagation $\Rightarrow R_{p=\Lambda}(q) = \infty, \ \forall q$.
- $0 \leq p \leq \Lambda$ - High modes ($|q| > p$) are unaffected by $R_p(q)$. Therefore, $R_p(|q| > p) \approx 0$ for those modes. The low modes had a mass that decoupled them from long-range Physics.

We now define,

$$W_p[J] = \log Z_p[J] \tag{3.29}$$

and the Legendre Transform of $W_p[J]$ is defined as,

$$\tilde{\Gamma}_p[\varphi] + W_p[J] = \int J\phi \tag{3.30}$$

$$\varphi(x) = \frac{\delta W_p[J]}{\delta J(x)} \tag{3.31}$$

On careful, observation, it can be seen that even though,

$$p \rightarrow 0 \Rightarrow \{R_p \rightarrow 0, \ W_p \rightarrow W \text{ and } \tilde{\Gamma}_p \rightarrow \Gamma\}$$

but still[15]

$$\tilde{\Gamma}_\Lambda[\varphi] \neq H[\varphi]$$

A convenient re-definition (with the aide of R_p) allows us to modify the equations in order to get $\Gamma_{p=\Lambda}[\varphi] = H[\varphi]$, as, [16]

$$\Gamma_p[\varphi] + W_p[J] = \int J\varphi - \frac{1}{2}\int_q R_p(q)\varphi_q\varphi_{-q} \tag{3.32}$$

[12] A slight abuse of nomenclature has been exercised by the author.
[13] Refer to Eq:3.27, and the corresponding point for details.
[14] Ref to Eq:3.26 and corresponding point for details.
[15] Consequence of large $\Delta H_{p=\Lambda}$ term.
[16] R_p term vanishes in the limit $p \rightarrow 0 \ \forall q$. Thus the limiting case relations remained consistent.

Using Eq:3.28 and Eq:3.32, we can write,

$$J(x) = \frac{\delta\Gamma_p(x)}{\delta\varphi(x)} \int_y R_p(x-y)\varphi(y) \tag{3.33}$$

and on substituting of Eq:[3.32, 3.33] in Eq:3.29, we observe that,

$$\begin{aligned} \exp\left(-\Gamma_p[\varphi]\right) = \int \mathcal{D}\phi \exp\left(-H[\phi] + \int_x \frac{\Gamma_p[\varphi]}{\varphi(x)}(\phi(x)-\varphi(x))\right) \\ \times \exp\left(-\frac{1}{2}\int_{x,y}(\phi(x)-\varphi(x))R_p(x-y)(\phi(y)-\varphi(y))\right) \end{aligned} \tag{3.34}$$

Setting the condition $R_p(q)$ diverges $\forall q$ when $p \to \Lambda$, gives us the relation[17]

$$\exp\left(-\frac{1}{2}\int_{x,y}(\phi(x)-\varphi(x))R_p(x-y)(\phi(y)-\varphi(y))\right) \approx \delta(\phi-\varphi)$$

Thus we get the relation[18]

$$\Gamma_p[\varphi] \to H[\phi=\varphi] \; as \; p \to \Lambda$$

Proceeding on to deriving the *Exact RG Equations*.

$$\begin{aligned} \partial_p \exp\left(W_p\right) &= -\frac{1}{2}\int \mathcal{D}\phi \left(\int_{x,y}\phi(x)R_p(x-y)\phi(y)\right)\exp\left(-H[\phi] - \frac{1}{2}\int_q R_p(q)\phi(q)\phi_{-q} + \int J\phi\right) \\ &= \left(-\frac{1}{2}\int_{x,y}\partial_p R_p(x-y)\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}\right)\exp\left(W_p[J]\right) \end{aligned} \tag{3.35}$$

or equivalently, we get the *Polchinski Equation*

$$\boxed{\partial_p W_p[J] = -\frac{1}{2}\int_{x,y}\partial_p R_p(x-y)\left(\frac{\delta^2 W_p[J]}{\delta J[x]\delta J[y]} + \frac{\delta W_p[J]}{\delta J[x]}\frac{\delta W_p[J]}{\delta J[y]}\right)} \tag{3.36}$$

Using Eq:3.31, we can write[19]

$$\frac{\delta\Gamma_p[\varphi]}{\varphi(x)} = J(x) - \int_x R_p(x-y)\varphi(y) \tag{3.37}$$

---

[17] A functional Dirac-Delta.

[18] Subject to the condition R_k diverges in the limiting case.

[19] A useful equation, used in the calculation hence,

$$\partial_{p_{|J}} = \partial_{p_{|\varphi}} + \int_x \partial_p\varphi(x)_{|J}\frac{\delta}{\delta\varphi(x)}$$

and

$$\partial_p\Gamma_k[\varphi]_{|J}+\partial_p W_k[J]_{|J} = \int_x J\partial_p\varphi_{|J} - \frac{1}{2}\int_{x,y}\partial_p R_k(x-y)\varphi(x)\varphi(y) - \int_{x,y} R_p(x-y)\varphi(x)\partial_p\varphi(y)_{|J} \quad (3.38)$$

On substituting, Eqs:[3.36,3.37] in Eq:3.38, we obtain,

$$\partial_p\Gamma_p[\varphi] = \frac{1}{2}\int_{x,y}\partial_p R_p(x-y)\frac{\delta^2 W_p[J]}{\delta J(x)\delta J(y)} \quad (3.39)$$

Considering Eq:3.37, we observe that,

$$\begin{aligned}\delta(x-z) &= \int_y \frac{\delta^2 W_p[J]}{\delta J(x)\delta\varphi(z)} \\ &= \int_y \frac{\delta^2 W_p[J]}{\delta J(x)\delta J(y)}\frac{\delta B_y}{\delta\varphi(z)} \\ &= \int_y \frac{\delta^2 W_p[J]}{\delta J(x)\delta J(y)}\left(\frac{\delta^2\Gamma_p[\varphi]}{\delta\varphi(y)\delta\varphi(z)} + R_p(y-z)\right)\end{aligned} \quad (3.40)$$

gives us an *identity-equation*, which allows us to define *inverse-operators*,

$$\left[\frac{\delta^2 W_p[J]}{\delta J(x)\delta J(y)}\right]^{-1} = \frac{\delta^2\Gamma_p[\varphi]}{\delta\varphi(y)\delta\varphi(z)} + R_p(y-z)$$

Therefore, Eq:3.39, can be re-written as,

$$\boxed{\partial_p\Gamma_k[\varphi] = \frac{1}{2}\int_{x,y}\partial_p R_p(x-y)\left(\frac{\delta^2\Gamma_p[\varphi]}{\delta\varphi(y)\delta\varphi(z)} + R_p(y-z)\right)^{-1}} \quad (3.41)$$

which is the *Wetterich Equation*.

3.3 FRG for a Fermion Field

Next, we consider a system of interacting fermions [17, 18, 22, 23], characterized with the help of Grassmann Fields, $\psi, \bar{\psi}$, with the action[20][21],

$$S[\psi,\bar{\psi}] = -(\bar{\psi}, \mathscr{P}^{-1}\psi) + V[\psi,\bar{\psi}] \quad (3.42)$$

[20] $\mathscr{P}$ is the Propagator of the non-interacting system. The (., .), denotes $\sum_x \bar{\psi}(x)(\mathscr{P}^{-1}\psi)(x)$, associated with the definition, $(\mathscr{P}^{-1}\psi)(x) = \sum_{x'}\mathscr{P}^{-1}(x,x')\psi(x')$

[21] $V[\psi,\bar{\psi}]$ - Arbitrary many-body Interaction

The Generating Functional is expressed with the help of the following path integral,

$$\mathcal{Z}[\eta,\bar{\eta}] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left\{-S[\psi,\bar{\psi}] + (\bar{\eta},\psi) + (\bar{\psi},\eta)\right\} \tag{3.43}$$

the associated Connected Green's Functions are obtained, [18], with the help of the Generating Functional defined as,

$$\mathcal{G}[\eta,\bar{\eta}] = -\ln\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-S[\psi,\bar{\psi}] + (\bar{\eta},\psi) + (\bar{\psi},\eta)\right) \tag{3.44}$$

this is equivalent to writing, [22]

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-S[\psi,\bar{\psi}] + (\bar{\eta},\psi) + (\bar{\psi},\eta)\right) = Z \exp\left[-\left(\bar{\eta},\mathcal{P}\eta\right)\right] \tag{3.45}$$

for vanishing source field [23],

$$\begin{aligned} \mathcal{G}[0,0] &= -\ln \mathcal{Z} \\ \mathcal{Z} &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-S[\psi,\bar{\psi}]\right) \end{aligned} \tag{3.46}$$

The *Connected-n-Particle Greens Function* is given by [24]

$$\begin{aligned} G^{(2n)}(x_1,x2,...,x_n;y_1,y_2,...,y_n) &= -\langle\psi(x_1\psi(x_2)...\psi(x_n)\bar{\psi}(y_n)....\bar{\psi}(y_2)\bar{\psi}(y_1)\rangle_c \\ &= (-1)^n \frac{\delta^{2m}\mathcal{G}[\eta,\bar{\eta}]}{\delta\bar{\eta}(x_1)...\delta\bar{\eta}(x_n)\delta\eta(y_m)...\delta\eta(y_1)}\bigg|_{\eta,\bar{\eta}=0} \end{aligned} \tag{3.47}$$

[22] In the non-interacting case, i.e., $V[\psi,\bar{\psi}] = 0$ -

$$\mathcal{G}[\eta,\bar{\eta}] = -\ln Z + (\bar{\eta},\mathcal{P}\eta)$$

[23] Partition Function of the Interacting System.

[24] Carrying out the expansion of $\mathcal{G}[\eta,\bar{\eta}]$, gave us a power series, with the Connected Green's Functions as co-efficients-

$$\mathcal{G}[\eta,\bar{\eta}] = -\ln \mathcal{Z}(\bar{\eta},G^{(2)}\eta) + \frac{1}{(2!)^2}\sum_{x_1,x_2,y_1,y_2} G^{(4)}(x_1,x_2;y_1,y_2)\bar{\eta}(x_1)\bar{\eta}(x_2)\eta(y_1\eta(y_2)$$

To get the *Effective Average Action*, we consider the Legendre Transform of $\mathcal{G}[\eta,\bar{\eta}]$

$$\begin{aligned}\Gamma[\psi,\bar{\psi}] &= (\bar{\eta},\psi)+(\bar{\psi},\eta)+\mathcal{G}[\eta,\bar{\eta}]\\ \psi &= -\frac{\delta\mathcal{G}[\eta,\bar{\eta}]}{\delta\bar{\eta}}\\ \bar{\psi} &= \frac{\delta\mathcal{G}[\eta,\bar{\eta}]}{\delta\eta}\end{aligned} \tag{3.48}$$

which generates the *one-particle irreducible vertex functions*

$$\mathit{\Gamma}^{(2n)}(y_1,y_2,...,y_n;x_1,x2,...,x_m) = \frac{\delta^{2n}\Gamma}{\delta\bar{\psi}(y_1)...\delta\bar{\psi}(y_n)\delta\psi(x_n)...\delta\psi(x_1)}\bigg|_{\psi,\bar{\psi}=0} \tag{3.49}$$

using Eq:3.47 and Eq:3.49 and their corresponding functionals ($\mathcal{G}$ and Γ)-

$$\begin{aligned}\mathit{\Gamma}^{(2)}(x_1';x1) &= (G^{(2)}(x_1;x_1'))^{-1}\\ G^{(4)}(x_1,x_2;x_1',x_2') &= \sum_{y_1,y_2,y_1',z_2'} G^{(2)}(x_1,y_1')G^{(2)}(x_2,y_2')\mathit{\Gamma}^{(4)}(y_1',y_2';y_1,y_2)G^{(2)}(y_1,x_1')G^{(2)}(y_2,x_2')\end{aligned} \tag{3.50}$$

The Effective Average Action, $\Gamma[\psi,\bar{\psi}]$, obey the *reciprocity relations*

$$\frac{\delta\Gamma[\psi,\bar{\psi}]}{\delta\psi} = -\bar{\eta} \quad ; \quad \frac{\delta\Gamma[\psi,\bar{\psi}]}{\delta\bar{\psi}} = \eta \tag{3.51}$$

We can also prove that the second functional derivatives of $\mathcal{G}$ and Γ, with respect to the fields, are inverses/reciprocal in a certain sense. For which, the following definitions of the second-order derivatives of $\mathcal{G}$ and Γ, are considered

$$\begin{aligned}\mathcal{G}^{(2)}[\eta,\bar{\eta}] &= -\begin{bmatrix}\frac{\partial^2\mathcal{G}}{\partial\bar{\eta}(x)\partial\eta(x')} & -\frac{\partial^2\mathcal{G}}{\partial\bar{\eta}(x)\partial\bar{\eta}(x')}\\ \frac{\partial^2\mathcal{G}}{\partial\eta(x)\partial\eta(x')} & -\frac{\partial^2\mathcal{G}}{\partial\eta(x)\partial\bar{\eta}(x')}\end{bmatrix} = \begin{bmatrix}\langle\psi(x)\bar{\psi}(x')\rangle & \langle\psi(x)\psi(x')\rangle\\ \langle\bar{\psi}(\bar{\psi}(x)\bar{\psi}(x')\rangle & \langle\bar{\psi}(x)\psi(x')\rangle\end{bmatrix}\\ \mathbf{\Gamma}^{(2)}[\psi,\bar{\psi}] &= \begin{bmatrix}\frac{\partial^2\Gamma}{\partial\bar{\psi}(x')\partial\psi(x)} & \frac{\partial^2\Gamma}{\partial\bar{\psi}(x')\partial\bar{\psi}(x)}\\ \frac{\partial^2\Gamma}{\partial\psi(x')\partial\psi(x)} & \frac{\partial^2\Gamma}{\partial\psi(x')\partial\bar{\psi}(x)}\end{bmatrix} = \begin{bmatrix}\bar{\partial}\partial\Gamma[\psi,\bar{\psi}](x',x) & \bar{\partial}\bar{\partial}\Gamma[\psi,\bar{\psi}](x',x)\\ \partial\partial\Gamma[\psi,\bar{\psi}](x',x) & \partial\bar{\partial}\Gamma[\psi,\bar{\psi}](x',x)\end{bmatrix}\end{aligned} \tag{3.52}$$

these satisfy the relation[25]

$$\mathbf{\Gamma}^{(2)}[\psi,\bar{\psi}] = (\mathcal{G}[\eta,\bar{\eta}])^{-1} \tag{3.53}$$

[25]Only when $\eta=\bar{\eta}=0$ and $\psi=\bar{\psi}=0$ (and in the absence of a $U(1)$ charge symmetry, does the relation

$$\Gamma^{(2)}[\psi,\bar{\psi}] = (\mathcal{G}[\eta,\bar{\eta}])^{-1}$$

hold.

We also define *Effective Interaction* in case of the Fermions[26],

$$\mathcal{V}[\chi,\bar{\chi}] = -\ln\left(\frac{1}{Z}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(\bar{\psi},\mathcal{P}^{-1}\psi\right)\exp -V[\psi+\chi,\bar{\psi}+\bar{\chi}]\right) \tag{3.54}$$

which is equivalently written as

$$\mathcal{V}[\chi,\bar{\chi}] = \mathcal{G}[\eta,\bar{\eta}] + \ln(Z) - (\bar{\eta},\mathcal{P}\eta) \quad ; \quad \chi = \mathcal{P}\eta; \quad \bar{\chi} = \mathcal{P}^T\bar{\eta}; \quad \mathcal{P}^T(x,x') = \mathcal{P}(x',x) \tag{3.55}$$

Thus, with the help of these definitions and calculations, we can determine that the functional derivatives of $\mathcal{V}[\chi,\bar{\chi}]$ with respect to, χ and $\bar{\chi}$ generate the Connected Green's functions, with bare propagators amputated from external legs, in their respective Feynman Diagrams. This is due to the $(\ln(Z) - (\bar{\eta},\mathcal{P}\eta))$ term, which allows us to eliminate, the non-interacting part of $\mathcal{G}[\bar{\eta},\eta]$ [27].

A short calculation, allows us to re-write the Effective Interaction, in-terms of functional-derivatives[28],

$$\begin{aligned}
\exp -\mathcal{V}[\chi,\bar{\chi}] &= \frac{1}{Z}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp(\bar{\psi},\mathcal{P}^{-1}\psi)\exp(-V[\psi+\chi,\bar{\psi}+\bar{\chi}] \\
&= \frac{1}{Z}\exp\left(-V\left[\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta\eta}\right]\right)\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(\bar{\psi},\mathcal{P}^{-1}\psi\right)\exp\left(\bar{\eta},\psi+\chi\right) + \left(\eta,\bar{\psi}+\bar{\chi}\right)\Big|_{\eta,\bar{\eta}=0} \\
&= \exp\left(-V\left[\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta\eta}\right]\right)\exp(\bar{\eta},\mathcal{P}\eta)\exp\left[(\bar{\eta},\chi)+(\eta,\bar{\chi})\right]\Big|_{\eta,\bar{\eta}=0} \\
&= \exp\left(-V\left[\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta\eta}\right]\right)\exp\left(\frac{\delta}{\delta\chi},\mathcal{P}\frac{\delta}{\delta\bar{\chi}}\right)\exp\left[(\bar{\eta},\chi)+(\eta,\bar{\chi})\right]\Big|_{\eta,\bar{\eta}=0} \\
&= \exp\left(\frac{\delta}{\delta\chi},\mathcal{P}\frac{\delta}{\delta\bar{\chi}}\right)\exp(-V[\chi,\bar{\chi}])
\end{aligned} \tag{3.56}$$

We consider a scale Λ, (analogous to our prior calculations and definitions and $\Lambda < \Lambda_0$), and construct a theory. This theory is characterized by the propagator $\mathcal{P}^{\Lambda}_{>}$ and the bare action of this theory (based on the propagator) is denoted as $S^{\Lambda}_{>}[\psi,\bar{\psi}]$. In the following calculations, this notation is extended in order to denote the Generating Functionals defined above as in the case of fermions, in this version of the Λ-scaled theory; that is we will consider the functionals - $\mathcal{G}^{\Lambda}_{>}[\eta,\bar{\eta}]$, $\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]$

[26]
$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp(\bar{\psi},\mathcal{P}^{-1}\psi)$$

[27] That is $\mathcal{V}[\chi,\bar{\chi}] = 0$ when, $V[\psi,\bar{\psi}] = 0$.

[28]
$$\left(\frac{\delta}{\delta\chi},\mathcal{P}\frac{\delta}{\delta\bar{\chi}}\right) = \sum_{x,x'}\frac{\delta}{\delta\chi(x)}\mathcal{P}(x,x')\frac{\delta}{\delta\bar{\chi}(x')}$$

and $\Gamma^{\Lambda}_{>}[\psi,\bar{\psi}]$ [29].

Referring to Eq:3.56, in the scaled-theory, we observe that it takes the form

$$\exp\left(-\mathcal{V}^{\Lambda}_{>}\right) = \exp\left(\frac{\delta}{\delta\chi},\mathcal{P}^{\Lambda}_{>}\frac{\delta}{\delta\bar{\chi}}\right)\exp\left(-V[\chi,\bar{\chi}]\right) \tag{3.57}$$

Next, we define the separation between high modes and low modes as,

$$\mathcal{P} = \mathcal{P}^{\Lambda}_{>} + \mathcal{P}^{\Lambda}_{<} \tag{3.58}$$

and using Eq:3.58, the following relation is written[30]

$$\begin{aligned}
\exp\left(-\mathcal{V}[\chi,\bar{\chi}]\right) &= \exp\left(\frac{\delta}{\delta\chi},\mathcal{P}\frac{\delta}{\delta\bar{\chi}}\right)\exp\left(-V[\chi,\bar{\chi}]\right) \\
&= \exp\left(\frac{\delta}{\delta\chi},\mathcal{P}^{\Lambda}_{>}\frac{\delta}{\delta\bar{\chi}}\right) + \left(\frac{\delta}{\delta\chi},\mathcal{P}^{\Lambda}_{<}\frac{\delta}{\delta\bar{\chi}}\right)\exp\left(-V[\chi,\bar{\chi}]\right) \\
&= \exp\left(\frac{\delta}{\delta\chi},\mathcal{P}^{\Lambda}_{<}\frac{\delta}{\delta\bar{\chi}}\right)\exp\left(-V^{\Lambda}_{>}[\chi,\bar{\chi}]\right)
\end{aligned} \tag{3.59}$$

Once again, a short calculation allows us to determine the *Functional Renormalization Group Equation* that governs the Effective Interaction. Thus using Eq:3.57, we get,

$$\begin{aligned}
\frac{\partial}{\partial\Lambda}\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}] &= -\exp\left(\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]\right)\frac{\partial}{\partial\Lambda}\exp\left(-\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]\right) \\
&= -\exp\left(\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]\right)\frac{\partial}{\partial\Lambda}\left[\exp\left(\frac{\delta}{\delta\chi},\mathcal{P}^{\Lambda}_{>}\frac{\delta}{\delta\bar{\chi}}\right)\exp\left(-V[\chi,\bar{\chi}]\right)\right] \\
&= -\exp\left(\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]\right)\left(\frac{\delta}{\delta\chi},\frac{\partial\mathcal{P}^{\Lambda}_{>}}{\partial\Lambda}\frac{\delta}{\delta\bar{\chi}}\right)\exp\left(-\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]\right) \\
&= -\left(\frac{\delta\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]}{\delta\chi},\frac{\partial\mathcal{P}^{\Lambda}_{>}}{\partial\Lambda}\frac{\delta\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]}{\delta\bar{\chi}}\right) - Tr\left(\frac{\partial\mathcal{P}^{\Lambda}_{>}}{\partial\Lambda}\frac{\delta^{2}\mathcal{V}^{\Lambda}_{>}[\chi,\bar{\chi}]}{\delta\bar{\chi}\chi}\right)
\end{aligned} \tag{3.60}$$

This is the *Renormalization Group Equation* for the functional $\mathcal{V}^{\Lambda}[\chi,\bar{\chi}]$. The expansion of this functional in powers of χ and $\bar{\chi}$ develops the *Fermionic Analog of Polchinski's Flow Equation*, for *amputated Connected n-particle Green's Functions*, ${V^{(2n)}}^{\Lambda}$.

[29] The *original* Functionals $\mathcal{G},\mathcal{V} and \Gamma$, are defined, in the limit $\Lambda \to 0$.

[30] $V^{\Lambda}_{>}[\chi,\bar{\chi}]$ performs two important tasks

- Acts as the Generating Functional for Amputated Green's functions of a (scaled) Theory with the scale cut-off Λ.
- Acts as the Interaction for the leftover low modes.

Using Eq:3.55 and Eq:3.60, we obtain the Renormalization Group Flow Equation for the Functional, $\mathcal{G}^{\Lambda}[\eta, \bar{\eta}]$.

$$\frac{\partial}{\partial \Lambda}\mathcal{G}^{\Lambda}[\eta,\bar{\eta}] = \left(\frac{\partial \mathcal{G}^{\Lambda}[\eta,\bar{\eta}]}{\partial \eta}, \frac{\partial (\mathcal{P}^{\Lambda}_{>})^{-1}}{\partial \Lambda}\frac{\partial \mathcal{G}^{\Lambda}[\eta,\bar{\eta}]}{\partial \bar{\eta}}\right) + Tr\left(\frac{\partial (\mathcal{P}^{\Lambda}_{>})^{-1}}{\partial \Lambda}\frac{\partial^2 \mathcal{G}^{\Lambda}}{\partial \bar{\eta}\partial \eta}\right) \tag{3.61}$$

Analogously the expansion of the functional $\mathcal{G}^{\Lambda}[\eta, \bar{\eta}]$, in the Renormalization Group Equation, in powers of η and $\bar{\eta}$ provides us with *Fermionic Analog of the Renormalization Group Flow Equation* for *Connected n-particle Green's Functions*, $G^{(2n)^{\Lambda}}$.

For the purpose of determining the *Renormalization Group Flow Equations* for the *one-particle irreducible vertex function* $\Gamma^{(2n)^{\Lambda}}$, we consider the scale-dependent Effective Action[31].

$$\Gamma^{\Lambda}[\psi,\bar{\psi}] = \left(\bar{\eta}^{\Lambda},\psi\right) + \left(\bar{\psi},\eta^{\Lambda}\right) + \mathcal{G}^{\Lambda}[\eta^{\Lambda},\bar{\eta}^{\Lambda}] \quad ; \quad \psi = -\frac{\partial \mathcal{G}^{\Lambda}[\eta,\bar{\eta}]}{\partial \bar{\eta}}; \quad \bar{\psi} = \frac{\partial \mathcal{G}^{\Lambda}[\eta,\bar{\eta}]}{\partial \eta} \tag{3.62}$$

Since the Λ-dependence has no effect on the structure of the action as a function of the fields, all of the standard relations between the connected Green's Functions and the Vertex Functions are preserved.

Considering Eq:3.61 and Eq:3.62[32],

$$\begin{aligned}\frac{\partial}{\partial \Lambda}\Gamma^{\Lambda}[\psi,\bar{\psi}] &= \left(\frac{\partial \bar{\eta}^{\Lambda}}{\partial \Lambda},\psi\right) + \left(\bar{\psi},\frac{\partial \eta^{\Lambda}}{\partial \Lambda}\right) + \frac{\partial}{\partial \Lambda}\mathcal{G}^{\Lambda}[\eta^{\Lambda},\bar{\eta}^{\Lambda}] \\ &= \frac{\partial}{\partial \Lambda}\mathcal{G}^{\Lambda}[\eta^{\Lambda},\bar{\eta}^{\Lambda}]\Big|_{\eta^{\Lambda},\bar{\eta}^{\Lambda}-fixed}\end{aligned} \tag{3.63}$$

and using Eqs:[3.51,3.53,3.61], the above equation Eq:3.63, can be re-written as

$$\frac{\partial}{\partial \Lambda}\Gamma^{\Lambda}[\psi,\bar{\psi}] = -\left(\bar{\psi},\frac{\partial (\mathcal{P}^{\Lambda}_{>})^{-1})}{\psi}\right) - \frac{1}{2}Tr\left[\frac{\partial}{\partial \Lambda}\left\{Diag\left(\mathcal{P}^{\Lambda}_{>}, -(\mathcal{P}^{\Lambda}_{>})^{T}\right)\right\}.\left\{\mathbf{\Gamma}^{2^{\Lambda}}[\psi,\bar{\psi}]\right\}^{-1}\right] \tag{3.64}$$

$$\text{Diag}\left(\mathcal{P}^{\Lambda}_{>}, (\mathcal{P}^{\Lambda}_{>})^{T}\right) = \begin{bmatrix}\mathcal{P}^{\Lambda}_{>} & 0 \\ 0 & -(\mathcal{P}^{\Lambda}_{>})^{T}\end{bmatrix} \quad ; \quad (\mathcal{P}^{\Lambda}_{>})^{T}(x,x') = \mathcal{P}^{\Lambda}_{>}(x',x) \tag{3.65}$$

[31] η^{Λ} and $\bar{\eta}^{\Lambda}$ are the scale-dependent, i.e., Λ-dependent equations, [Ref. to Eq:3.62]

[32] The derivative operator, which acted on $\mathcal{G}^{\Lambda}[\eta, \bar{\eta}]$ also captured the Λ-dependence of η^{Λ} and $\bar{\eta}^{\Lambda}$. These observations helped cancel most of the terms.

3.4 Rescaling in FRG

Any *Wilsonian RG* procedure, consists of two main steps, (with reference to the theory considered in Sec:3.1)

- *Integration* - An integration over the fluctuations (generally in the interval $e^{-\xi} < |\mathbf{p}| \leq 1$, which are known as the high modes.)[33]
- *Rescaling* - The procedure entails, re-scaling all linear dimensions to the original scale of the system/model, (with the help of a factor of $e^{-\xi}$).

For convenience, in this section, we consider the action $S[\phi]$, in the momentum-space, [34]
Notation,[35]

$$\int_p \equiv \int \frac{d^d p}{(2\pi)^d} \quad ; \quad \phi(\mathbf{x}) = \int_p \phi_p e^{i\mathbf{p}.\mathbf{x}}$$

Therefore the action, $S[\phi]$ is defined as [36]

$$S[\phi] = \sum_n^{\infty} \int_{p_1,\dots p_n} u_n(\mathbf{p}_1, ..., \mathbf{p}_n)\phi_{p_1}, ..., \phi_{p_n}\hat{\delta}(\mathbf{p}_1 + ... + \mathbf{p}_n) \tag{3.66}$$

and the corresponding functional derivative is defined as

$$\frac{\delta}{\delta\phi_p} = \int d^d x e^{i\mathbf{p}.\mathbf{x}} \frac{\delta}{\delta\phi(\mathbf{x})} \tag{3.67}$$

[33]This leaves the Partition Function, Eq:3.1 Invariant.

[34]The action, $S[\phi]$, is considered to be a semi-local functional of ϕ, which effectively meant that, we could construct $S[\phi]$, with te help of various powers of ϕ and of its derivatives, only.

[35]Here d is some general dimensions. And $p = |\mathbf{p}|$

[36]Here

$$\hat{\delta}(\mathbf{p} \equiv (2\pi)^d \delta^d(\mathbf{p})$$

where

$$\delta^d = \int_p e^{i\mathbf{p}.\mathbf{x}}$$

3.4.1 Field Variable Transformation

We need to consider a general transformation of the field that leaves the *Partition Function* invariant.[37]

$$\tilde{\phi}_p = \phi_p + \sigma\psi_p[\phi] \tag{3.68}$$

then

$$S[\tilde{\phi}] = S[\phi] + \sigma \int_p \psi_p[\phi] \frac{\delta S[\phi]}{\delta \phi_p} \tag{3.69}$$

thus [38] imposing the aforementioned transformation to the constraint of leaving the Partition Function invariant, we get

$$\begin{aligned} Z[\phi] &= \int \mathcal{D}\tilde{\phi} \exp\left(-S[\tilde{\phi}]\right) \\ &= \int \mathcal{D}\phi \exp\left(-S[\phi] - \sigma\Box\{\psi\}S[\phi]\right) \end{aligned} \tag{3.70}$$

where

$$\Box\{\psi\}S[\phi] = \int_p \left(\psi_p \frac{\delta S}{\delta \phi_p} - \frac{\delta \psi_p}{\delta \phi_p}\right) \tag{3.71}$$

or equivalently

$$\frac{\partial S[\phi]}{\partial \sigma} = \Box\{\psi\}S[\phi] \tag{3.72}$$

3.4.2 Rescaling

For the purpose of rescaling, details in [19], we consider an infinitesimal change of the momentum scale.[39]

$$p \to \hat{p} = (1+\sigma)p \tag{3.73}$$

which induces the following change in the action $S[\phi]$

$$S \to \hat{S} = S + \sigma \triangle S \tag{3.74}$$

[37] σ is an infinitesimal quantity.

[38] Notation

$$\int \mathcal{D}\tilde{\phi} = \int \mathcal{D}\phi \frac{\partial\{\tilde{\phi}\}}{\partial\{\phi\}} = \int \mathcal{D}\phi \left(1 + \sigma \int_p \frac{\delta\psi_p[\phi]}{\delta\phi_p}\right)$$

[39] The hat on the terms, denotes the rescaling changes that are made.

equivalently expressed in the form,

$$\frac{\partial S}{\partial \sigma} = \triangle S \tag{3.75}$$

Using the definition of $S[\phi]$ in Eq:3.66, we get the following list of changes induced by the rescaling Eq:3.73, ($Change \equiv \sigma \triangle S$)

- Change induced by Differential Volume ($\triangle S_1$)

$$\Pi_{i=1}^{n} d^{\not{d}} p_i = (1+\sigma)^{-n\not{d}} \Pi_{i=1}^{n} d^{\not{d}} \hat{p}_i \tag{3.76}$$

induces the change

$$\tilde{\triangle} S_1 = -\sigma \left[\not{d} \int_p \phi_p \frac{\delta}{\delta \phi_p} \right] S \tag{3.77}$$

- Change induced by Couplings ($\triangle S_2$)

$$u_n(\mathbf{p}_1, ..., \mathbf{p}_n) = u_n((1+\sigma)^{-1}\hat{\mathbf{p}}_1, ..., (1+\sigma)^{-1}\hat{\mathbf{p}}_n) \tag{3.78}$$

induces the change [40]

$$\tilde{\triangle} S_2 = -\sigma \left[\int_p \phi_p \mathbf{p} \cdot \partial'_p \frac{\delta}{\delta \phi_p} \right] S \tag{3.79}$$

- Change induced by the Delta-Function ($\triangle S_3$)

$$\hat{\delta}(\mathbf{p}_1 + ... + \mathbf{p}_n) = \hat{\delta}((1+\sigma)^{-1}\hat{\mathbf{p}}_1 + ... + (1+\sigma)^{-1}\hat{\mathbf{p}}_n) \tag{3.80}$$

induces the change

$$\tilde{\triangle} S_3 = \sigma(\not{d} S) \tag{3.81}$$

- Change induced by the Field ($\triangle S_4$)

$$\phi(\mathbf{p}) = \phi((1+\sigma)^{-1}\hat{\mathbf{p}}) \tag{3.82}$$

induces the change [41]

$$\phi((1+\sigma)^{-1}\hat{\mathbf{p}}) = (1+\sigma)^{\not{d}-\not{d}_\phi} \tag{3.83}$$

[40] ∂'_p, denotes the collection of momentum derivatives that do not act on the delta-functions.

[41] $\not{d}_\phi$ is the adjustable dimension of the field,

$$\phi(a\mathbf{x}) = a^{\not{d}_\phi}\phi(\mathbf{x}) \quad ; \quad \phi(a\mathbf{p}) = a^{\not{d}_\phi - \not{d}}\phi(\mathbf{p})$$

which to first order in σ induces the change

$$\tilde{\triangle} S_4 = \sigma \left[(d - d_\phi) \int_p \phi_p \frac{\delta}{\delta \phi_p} \right] S \tag{3.84}$$

To get the *total change* induced we sum up all the four contributions and arrive at

$$\begin{aligned} \tilde{\triangle} S &= \sum_{i=1}^{4} \tilde{\triangle} S_i \\ &= \sigma \left[d S - \int_p \phi_p \mathbf{p} \cdot \partial'_p \frac{\delta}{\delta \phi_p} - d_\phi \int_p \phi_p \frac{\delta}{\delta \phi_p} \right] S \end{aligned} \tag{3.85}$$

Further simplifications can be performed using the action of the momentum derivative ∂_p to act also on the $\delta - functions$ and integration by parts, giving

$$\triangle S = \left[\int_p (\mathbf{p} \cdot \partial_p \phi_p) \cdot \frac{\delta}{\delta \phi_p} + (d - d_\phi) \int_p \phi_p \frac{\delta}{\delta \phi_p} \right] S \tag{3.86}$$

The rescaling can also be expressed as [20]

$$\triangle \equiv d - \triangle_\partial - d_\phi \triangle_\phi \tag{3.87}$$

where in

$$\begin{aligned} &\triangle_\phi \equiv \phi \cdot \frac{\delta}{\delta \phi} \quad \text{phi-ness counting operator : \#(occurrences of field } \phi \text{ in a vertex)} \\ &\triangle_\partial \equiv d + \int_p \phi_p \mathbf{p} \cdot \partial_p \frac{\delta}{\delta \phi_p} \quad \text{momentum scale counting operator } + d \end{aligned} \tag{3.88}$$

Thus the *Re-scaled-Polchinski Equation* is written as

$$\frac{\partial S_\Lambda[\phi_<]}{\partial \xi} = -\frac{1}{2} \frac{\delta S_\Lambda[\phi_<]}{\phi_<} . \frac{\partial \mathscr{P}_>}{\partial \xi} . \frac{\delta S_\Lambda[\phi_<]}{\phi_<} + \frac{1}{2} Tr \left[\frac{\partial \mathscr{P}_>}{\partial \xi} . \frac{\delta^2 S_\Lambda[\phi_<]}{\delta \phi_< \delta \phi_<} \right] + \triangle S_\Lambda[\phi_<] \tag{3.89}$$

and the *Fixed Point Equation* is written as

$$\begin{aligned} \frac{\partial S_\Lambda[\phi_<]}{\partial \xi} &= 0 \\ \left[\triangle_\partial + d_\phi \triangle_\phi - d \right] S_\Lambda[\phi_<] &= -\frac{1}{2} \frac{\delta S_\Lambda[\phi_<]}{\phi_<} . \frac{\partial \mathscr{P}_>}{\partial \xi} . \frac{\delta S_\Lambda[\phi_<]}{\phi_<} + \frac{1}{2} Tr \left[\frac{\partial \mathscr{P}_>}{\partial \xi} . \frac{\delta^2 S_\Lambda[\phi_<]}{\delta \phi_< \delta \phi_<} \right] \end{aligned} \tag{3.90}$$

From [21], we analogously derived the position space equivalent of Eq:3.89 employing the definition in Eq:3.87

$$\frac{\partial S_\Lambda[\phi_<]}{\partial \xi} = -\frac{1}{2}\frac{\delta S_\Lambda[\phi_<]}{\phi_<}.\frac{\partial \mathscr{P}_>}{\partial \xi}.\frac{\delta S_\Lambda[\phi_<]}{\phi_<} + \frac{1}{2}Tr\left[\frac{\partial \mathscr{P}_>}{\partial \xi}.\frac{\delta^2 S_\Lambda[\phi_<]}{\delta\phi_<\delta\phi_<}\right] + \bar{\triangle}S_\Lambda[\phi_<] \tag{3.91}$$

where in

$$\bar{\triangle} \equiv -\int_x \left[(d\!\!\!/_\phi + \mathbf{x}\cdot\partial_x)\phi_x\right]\frac{\delta}{\delta\phi_x} \tag{3.92}$$

Chapter 4

Attempting to Make the Connection

The FRG Equations in Sec(s):[3.1 & 3.3] are determined for a general model. For our model, we derive the *Polchinski Equation*, i.e. an equation of the form of Eq:3.11. This equation Eq:4.5, is then used to analyse and determine the fixed point, using the formalism provided by [1]. The derivation and subsequent analysis rely on the machinery developed in Chp:2 and Chp:3.

4.1 The FRG Flow Equation

On careful observation, using Eq:2.40 and Eq:2.41, we can derive,[1]

$$H_{eff}(\mathbf{Q}, \mathbf{x_Q}) = \mathcal{A} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathbf{Q}_1,...,\mathbf{Q}_n, \sum \mathbf{Q}_i = \mathbf{Q}} \sum_{\mathbf{P}_1,...,\mathbf{P}_n, \mathbf{Q}_i \subset \mathbf{P}_i} (-)^{\#} \int d^d \mathbf{x}_{\bar{\mathbf{Q}}} \mathcal{C}(\mathbf{x}_{\bar{\mathbf{Q}}}) \Pi_{i=1}^{n} H(\mathbf{P}_i, \mathbf{x}_{\mathbf{P}_i}) \quad (2.40)$$

$$\begin{aligned} H_{eff}(\psi) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathbf{P}_1,...,\mathbf{P}_n} \sum_{Q_1,...,Q_n \ ; \ \mathbf{Q}_i \subset P_i} \Big\{ (-)^{\#} \int d^d \mathbf{x} \Psi(\mathbf{Q}_1, \mathbf{x}_{\mathbf{Q}_1}) ... \Psi(\mathbf{Q}_n, \mathbf{x}_{\mathbf{Q}_n}) \times \\ \times \Pi_{i=1}^{n} H(\mathbf{P}_i, x_{\mathbf{P}_i}) \langle \Phi(\bar{\mathbf{Q}}_1, \mathbf{x}_{\bar{\mathbf{Q}}_1}); ...; \Phi(\bar{\mathbf{Q}}_n, \mathbf{x}_{\bar{\mathbf{Q}}_n}) \rangle_c \Big\} \end{aligned} \quad (4.1)$$

$$H_{eff}(\psi) = \Big\{ \int d^d \mathbf{x_Q} \Psi(\mathbf{Q}_1, \mathbf{x}_{\mathbf{Q}_1}) ... \Psi(\mathbf{Q}_n, \mathbf{x}_{\mathbf{Q}_n}) \times H_{eff}(\mathbf{Q}, \mathbf{x_Q}) \Big\} \quad (4.2)$$

---

[1]

$$\int d^d \mathbf{x} = \int d^d \mathbf{x_Q} \times \int d^d \mathbf{x}_{\bar{\mathbf{Q}}}$$

where $\mathbf{Q} = \sum_{i=1}^{n} \mathbf{Q}_i$ and $\bar{\mathbf{Q}} = \sum_{i=1}^{n} \bar{\mathbf{Q}}_i$, a *concatenation.*

These Equations, (Eq:4.1 and Eq:4.2) are used to derive the corresponding Polchinski Equation, (after taking the derivative with respect to a rescaling parameter). But such a procedure requires a re-labelling of symbols for systematic book-keeping.

4.1.1 Book-Keeping Notation

We primarily need a redefinition of the propagator. This redefinition allows us to recast Eq:4.1, as a Polchinski Equation.
With the definition of $P(x)$ as in Eq:2.17,[2]

$$\mathbb{G}^{xy}_{AB} = \Omega_{AB}\mathbb{P}_{xy} = \begin{cases} \langle\psi_a(x)\psi_b(y)\rangle & = \Omega_{ab}P_{xy} \quad ; \quad A = (a),\ B = (b) \\ \langle[\partial_\mu\psi_a(x)]\psi_b(y)\rangle & = \Omega_{ab}\partial_\mu P_{xy} \quad ; \quad A = (a,\mu),\ B = (b) \\ \langle\psi_a(x)[\partial_\nu\psi_b(y)]\rangle & = \Omega_{ab}\partial_\nu P_{xy} \quad ; \quad A = (a),\ B = (b,\nu) \\ \langle[\partial_\mu\psi_a(x)][\partial_\nu\psi_b(y)]\rangle & = \Omega_{ab}\partial_\mu\partial_\nu P_{xy} \quad ; \quad A = (a,\mu), B = (b,\nu) \end{cases} \tag{4.3}$$

or using the definitions in Eq:2.27,

$$\mathbb{G}^{xy}_{AB} = \Omega_{AB}\mathbb{P}_{xy} = \langle\Psi_A(x)\Psi_B(y)\rangle \tag{4.4}$$

4.1.2 Polchinski Equation and Vertex-Polchinski Equation

Employing a procedure similar to one presented in Chp:3, and using Eq:2.40 the corresponding Polchinski Equation for our model is written for $H_{eff} = \mathscr{H}$ [3] [4].

$$\boxed{\frac{\partial}{\partial\xi}\mathscr{H} = \frac{1}{2}\left\{\frac{\delta}{\delta\Psi^x_A}\mathscr{H}\right\}\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left\{\frac{\delta}{\delta\Psi^y_B}\mathscr{H}\right\} + \frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left\{\frac{\delta^2}{\delta\Psi^x_A\delta\Psi^y_B}\mathscr{H}\right\}\right)} \tag{4.5}$$

which is diagrammatically expressed as

[2] $P_{xy} = P(x-y)$ and $\Omega_{ab} \equiv$ Symplectic Matrix

[3] The scale parameter in FRG is Λ, and in RRG is γ, and their respective derivatives are related by an overall $-ve\ sign$.

$$-\gamma\frac{\partial}{\partial\gamma}\mathscr{F} = \Lambda\frac{\partial}{\partial\Lambda}\mathscr{F} = \frac{\partial}{\partial\xi}\mathscr{F}$$

[4] Analogously $-S^I$ in Sec:3.1 is replaced by $\mathscr{H}$.

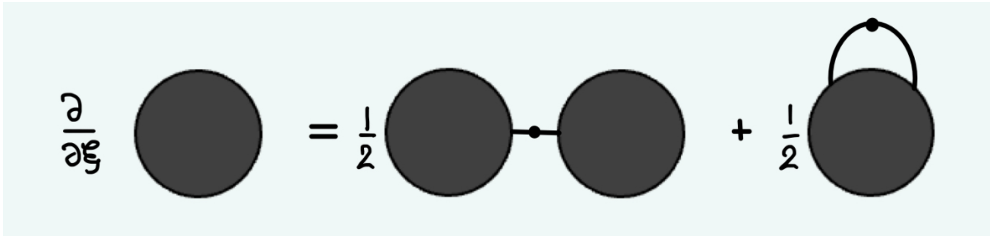

Figure 4.1: Diag. Representation of the Polchinski Eqn

A diagrammatic representation of the analogues of the Polchinski Equation Eq:3.11. The dotted line is representative of the derivative of the propagator, i.e. $\frac{\partial \mathbb{G}^{xy}_{AB}}{\partial \xi}$. This representation comprises of two types, referred to as Straight Road and Loop Road, respectively.

and we systematically derived the analogue of the Polchinski Equation for the vertices as

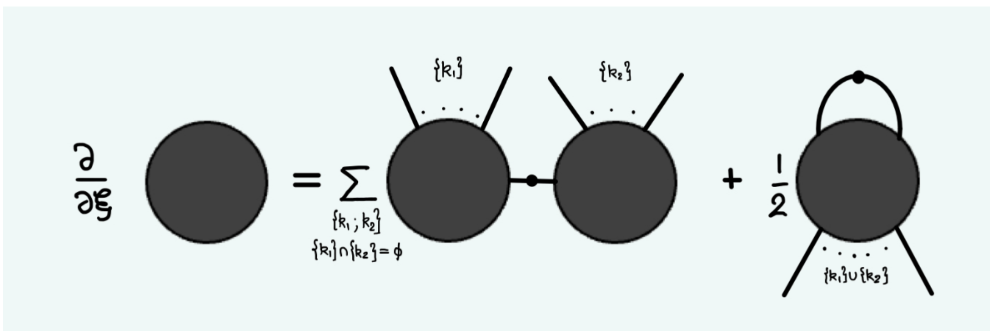

Figure 4.2: Diag. Representation of the Polchinski Eqn

A diagrammatic representation of the analogues of the Polchinski Equation Eq:3.11 for the vertices. The dotted line is representative of the derivative of the propagator, i.e. $\frac{\partial \mathbb{G}^{xy}_{AB}}{\partial \xi}$. This representation comprises of two types, referred to as Straight Road and Loop Road, respectively. The open-ended lines are representative of the n-vertex accordingly.

The form of the *Polchinski Equation for the Vertices*, is derived mathematically as follows: Considering the following definition for a vertex-interaction kernel[5]

$$\frac{\delta^{|\mathbf{C}|}\mathscr{H}}{\delta\boldsymbol{\Psi}(\mathbf{C},\mathbf{z_C})} = \mathscr{H}(\mathbf{C},\mathbf{z_c}) \tag{4.6}$$

defined with the help of the notation

$$\frac{\delta^{|\mathbf{C}|}}{\delta\boldsymbol{\Psi}(\mathbf{C},\mathbf{z_C})} = \frac{\delta^{|\mathbf{C}|}}{\delta\Psi(C_1,z_{C_1})...(C_l,z_{C_l})} \quad ; \quad \mathbf{C} = \{C_1,...,C_l\}\ ,\ l = |\mathbf{C}| \tag{4.7}$$

[5]The definition of the sequences are the same as that in Sec:2.4

The Vertex-Polchinski Equation is obtained by the action of $\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}$ on the Polchinski Equation, Eq:4.5 and setting $\mathbf{\Psi}=0$

$$\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\left\{\frac{\partial}{\partial\xi}\mathcal{H}=\frac{1}{2}\left[\frac{\delta}{\delta\Psi_A^x}\mathcal{H}\right]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta}{\delta\Psi_B^y}\mathcal{H}\right]+\frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta^2}{\delta\Psi_A^x\delta\Psi_B^y}\mathcal{H}\right]\right)\right\}\Bigg|_{\Psi=0} \tag{4.8}$$

which after the relevant rescaling procedure Sec:3.4, takes the form

$$\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\left\{\frac{\partial}{\partial\xi}\mathcal{H}=\frac{1}{2}\left[\frac{\delta}{\delta\Psi_A^x}\mathcal{H}\right]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta}{\delta\Psi_B^y}\mathcal{H}\right]+\frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta^2}{\delta\Psi_A^x\delta\Psi_B^y}\mathcal{H}\right]\right)+\bar{\triangle}\mathcal{H}\right\}\Bigg|_{\Psi=0} \tag{4.9}$$

Considering the following calculation it is further simplified to

$$\frac{\partial}{\partial\xi}\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\mathcal{H}\Bigg|_{\Psi=0}=\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\left\{\frac{1}{2}\left[\frac{\delta}{\delta\Psi_A^x}\mathcal{H}\right]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta}{\delta\Psi_B^y}\mathcal{H}\right]+\frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta^2}{\delta\Psi_A^x\delta\Psi_B^y}\mathcal{H}\right]\right)+\bar{\triangle}\mathcal{H}\right\}\Bigg|_{\Psi=0} \tag{4.10}$$

The Equation Eq:4.9, after the action of $\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}$ represents the diagrammatic representation in Fig:4.2. But in our analysis of the Polchinski Equation for Vertices, we re-write the equation using the notation developed in Sub-Sec:2.5.2. [6]

We use the *Trimming Map* defined in Sub-Sec:2.5.2, to determine the Polchinski Equation for a given Interaction-Kernel, which is characterized by the Eqs:[2.48, 2.49, 2.50]

$$\mathcal{T}_{\ell}^{l}\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\frac{\partial}{\partial\xi}\mathcal{H}\Bigg|_{\Psi=0}=\mathcal{T}_{\ell}^{l}\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\left\{\frac{1}{2}\left[\frac{\delta}{\delta\Psi_A^x}\mathcal{H}\right]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta}{\delta\Psi_B^y}\mathcal{H}\right]+\frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta^2}{\delta\Psi_A^x\delta\Psi_B^y}\mathcal{H}\right]\right)+\bar{\triangle}\mathcal{H}\right\}\Bigg|_{\Psi=0} \tag{4.11}$$

$$\frac{\partial}{\partial\xi}\mathcal{T}_{\ell}^{l}\left\{\mathcal{H}(\mathbf{C},\mathbf{z_C})\right\}_l=\mathcal{T}_{\ell}^{l}\frac{\delta^{|\mathbf{C}|}}{\delta\mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\left\{\frac{1}{2}\left[\frac{\delta}{\delta\Psi_A^x}\mathcal{H}\right]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta}{\delta\Psi_B^y}\mathcal{H}\right]+\frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}_{AB}^{xy}\right]\cdot\left[\frac{\delta^2}{\delta\Psi_A^x\delta\Psi_B^y}\mathcal{H}\right]\right)+\bar{\triangle}\mathcal{H}\right\}\Bigg|_{\Psi=0} \tag{4.12}$$

[6] The subsequent procedure is motivated by the reasoning of employing the mathematical structure and bounds presented in [1] to study the Vertex-Polchinski Equation.

$$\frac{\partial}{\partial \xi}\left\{\mathcal{H}(\mathbf{C},\mathbf{z_C})\right\}_{\ell} = \mathcal{T}^{l}_{\ell}\frac{\delta^{|\mathbf{C}|}}{\delta \mathbf{\Psi}(\mathbf{C},\mathbf{z_C})}\Bigg\{\frac{1}{2}\left[\frac{\delta}{\delta \Psi^x_A}\mathcal{H}\right]\cdot\left[\frac{\partial}{\partial \xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left[\frac{\delta}{\delta \Psi^y_B}\mathcal{H}\right] + \frac{1}{2}Tr\left(\left[\frac{\partial}{\partial \xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left[\frac{\delta^2}{\delta \Psi^x_A \delta \Psi^y_B}\mathcal{H}\right]\right) + \bar{\triangle}\mathcal{H}\Bigg\}\Bigg|_{\Psi=0} \tag{4.13}$$

and following the approach in [14], and using Eq:3.65 and Eq:4.6. We also re-define the action of the Re-scaling operator, $\bar{\Delta}$, appropriately for the Polchinski Equation for the Vertices and details as present in Eq:4.18 and Eq:4.19.

$$\frac{\partial}{\partial \xi}\left\{\mathcal{H}(\mathbf{C},\mathbf{z_C})\right\}_{\ell} = \mathcal{T}^{l}_{\ell}\Bigg\{\sum_{\{(\mathbf{C}_1,\mathbf{z}_{\mathbf{C}_1}),(\mathbf{C}_2,\mathbf{z}_{\mathbf{C}_2})\}}\left[\mathcal{H}(A,\mathbf{C}_1;x,\mathbf{z}_{\mathbf{C}_1})\right]\cdot\left[\frac{\partial}{\partial \xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left[\mathcal{H}(B,\mathbf{C}_2;x,\mathbf{z}_{\mathbf{C}_2})\right] + \frac{1}{2}Tr\left(\left[\frac{\partial}{\partial \xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left[\mathcal{H}(A,B,\mathbf{C};x,y,\mathbf{z_C})\right]\right) + \Box\mathcal{H}(\mathbf{C},\mathbf{z_C})\Bigg\}_{l} \tag{4.14}$$

subject to the following constraints

$$\begin{aligned}\mathbf{C}_1 \cup \mathbf{C}_2 &= \mathbf{C}\\ \mathbf{C}_1 \cap \mathbf{C}_2 &= \phi\end{aligned} \tag{4.15}$$

$\sum_{\{(\mathbf{C}_1,\mathbf{z}_{\mathbf{C}_1}),(\mathbf{C}_2,\mathbf{z}_{\mathbf{C}_2})\}} \equiv$ Sum over all the disjoint subsets with each pair being counted only once.

Example, let us consider writing the Polchinski Equation for the 2-point-local Vertex Polchinski Equation

$$\frac{\partial}{\partial \xi}\left\{\mathcal{H}(\mathbf{C},\mathbf{z_C})\right\}_{2L} = \mathcal{T}^{2}_{2L}\Bigg\{\sum_{\{(C_1,z_{C_1}),(C_2,z_{C_2})\}}\left[\mathcal{H}(A,C_1;x,z_{C_1})\right]\cdot\left[\frac{\partial}{\partial \xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left[\mathcal{H}(B,C_2;x,z_{C_2})\right] + \frac{1}{2}Tr\left(\left[\frac{\partial}{\partial \xi}\mathbb{G}^{xy}_{AB}\right]\cdot\left[\mathcal{H}(A,B,\mathbf{C};x,y,\mathbf{z_C})\right]\right) + \Box\mathcal{H}(\mathbf{C},\mathbf{z_C})\Bigg\}_{2} \tag{4.16}$$

In Equation Eq:4.16, the following definitions and properties of sets are considered.

$$\begin{aligned}|C_1| &= |C_2| = 1\\ \mathbf{C} &= C_1 \cup C_2\\ \mathbf{z_C} &= z_{C_1} \cup z_{C_2}\end{aligned} \tag{4.17}$$

The action of the Rescaling operator on the Polchinski Equation for the Vertices is defined as

$$\begin{aligned}
\mathcal{T}_\ell^l \frac{\delta^{|\mathbf{C}|}}{\delta \boldsymbol{\Psi}(\mathbf{C}, \mathbf{z_C})} \bar{\triangle} \mathcal{H} &= -\mathcal{T}_\ell^l \frac{\delta^{|\mathbf{C}|}}{\delta \boldsymbol{\Psi}(\mathbf{C}, \mathbf{z_C})} \int_w [(d_\Psi + w \cdot \partial_w) \Psi_D^w] \frac{\delta}{\delta \Psi_D^w} \mathcal{H} \\
&= -\sum_{i=1}^{l} \mathcal{T}_\ell^l \int_w \left[(d_\Psi + w \cdot \partial_w) \frac{\delta}{\delta \Psi_{(\mathbf{C}_i)}^{z(\mathbf{C}_i)}} \Psi_D^w \right] \Pi_{j \neq i} \frac{\delta}{\delta \Psi_{(\mathbf{C}_j)}^{z(\mathbf{C}_j)}} \cdot \frac{\delta}{\delta \Psi_D^w} \mathcal{H} \\
&= -\sum_{i=1}^{l} \mathcal{T}_\ell^l \int_w [(d_\Psi + w \cdot \partial_w) \{\delta(w - z_{\mathbf{C}_i}) \delta_{\mathbf{C}_i D}\}] \Pi_{j \neq i} \frac{\delta}{\delta \Psi_{(\mathbf{C}_j)}^{z(\mathbf{C}_j)}} \cdot \frac{\delta}{\delta \Psi_D^w} \mathcal{H} \\
&= -\sum_{i=1}^{l} \mathcal{T}_\ell^l \int_w \{\delta(w - z_{\mathbf{C}_i}) \delta_{\mathbf{C}_i D}\} \Big[(d_\Psi - d) \mathcal{H}(\{\mathbf{C}_{i \neq j}\}, D; \mathbf{z}_{\{\mathbf{C}_{i \neq j}\}}, w) \\
&\quad + w \cdot \partial_w \mathcal{H}(\{\mathbf{C}_{i \neq j}\}, D; \mathbf{z}_{\{\mathbf{C}_{i \neq j}\}}, w) \Big] \\
&= -\mathcal{T}_\ell^l \{ l (d_\Psi - d) \mathcal{H}(\mathbf{C}, \mathbf{z_C}) \} + \mathcal{T}_\ell^l \sum_{i=1}^{l} z_{\mathbf{C}_i} \cdot \partial_{z_{\mathbf{C}_i}} \mathcal{H}(\mathbf{C}, \mathbf{z_C}) \\
&= -\mathcal{T}_\ell^l \left[l (d_\Psi - d) - \sum_{i=1}^{l} z_{\mathbf{C}_i} \cdot \partial_{z_{\mathbf{C}_i}} \right] \mathcal{H}(\mathbf{C}, \mathbf{z_C})
\end{aligned} \tag{4.18}$$

To keep our expressions compact we will denote the operator derived above using $\square$,

$$\begin{aligned}
\square &= -\left[l (d_\Psi - d) - \sum_{i=1}^{l} z_{\mathbf{C}_i} \cdot \partial_{z_{\mathbf{C}_i}} \right] \\
\bar{\square} &= -\mathcal{T}_\ell^l \left[l (d_\Psi - d) - \sum_{i=1}^{l} z_{\mathbf{C}_i} \cdot \partial_{z_{\mathbf{C}_i}} \right]
\end{aligned} \tag{4.19}$$

4.2 Fixed Point Equations

4.2.1 Construction of the Fixed Point Equation

The next step comprises of constructing the Fixed Point Equations for the *Polchinski Equation* (in this case the Polchinski Equation for the Vertices). The *Fixed Point Equation* is derived by setting the LHS of Equation Eq:4.14 as zero [7] [8]

$$\frac{\partial}{\partial \xi}\left\{\mathscr{H}(\mathbf{C},\mathbf{z_C})\right\}_\ell = 0 \tag{4.20}$$

$$\begin{aligned}
&\mathcal{T}^l_\ell \Bigg\{ \sum_{\{(\mathbf{C}_1,\mathbf{z}_{\mathbf{C}_1}),(\mathbf{C}_2,\mathbf{z}_{\mathbf{C}_2})\}} [\mathscr{H}(A,\mathbf{C}_1;x,\mathbf{z}_{\mathbf{C}_1})]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot[\mathscr{H}(B,\mathbf{C}_2;x,\mathbf{z}_{\mathbf{C}_2})] \\
&\qquad + \frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot[\mathscr{H}(A,B,\mathbf{C};x,y,\mathbf{z_C})]\right) + \Box\mathscr{H}(\mathbf{C},\mathbf{z_C})\Bigg\}_l = 0 \\
&-\bar{\Box}\{\mathscr{H}(\mathbf{C},\mathbf{z_C})\}_\ell = \mathcal{T}^l_\ell \Bigg\{ \sum_{\{(\mathbf{C}_1,\mathbf{z}_{\mathbf{C}_1}),(\mathbf{C}_2,\mathbf{z}_{\mathbf{C}_2})\}} [\mathscr{H}(A,\mathbf{C}_1;x,\mathbf{z}_{\mathbf{C}_1})]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot[\mathscr{H}(B,\mathbf{C}_2;x,\mathbf{z}_{\mathbf{C}_2})] \\
&\qquad + \frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot[\mathscr{H}(A,B,\mathbf{C};x,y,\mathbf{z_C})]\right)\Bigg\}_l
\end{aligned} \tag{4.21}$$

For convenience and simplicity, the RHS of Eq:4.21 is notationally denoted as, where in we refer to the newly defined operator as *Polchinski Operators*

$$-\bar{\Box}\{\mathscr{H}(\mathbf{C},\mathbf{z_C})\}_\ell = \sum_{(\ell_1,\ell_2)}\left[\{\mathscr{P}_S\}^{\ell_1,\ell_2}_\ell(\mathscr{H}_{\ell_1},\mathscr{H}_{\ell_2})\right] + \sum_{(\ell')}\left[\{\mathscr{P}_L\}^{\ell'}_\ell(\mathscr{H}_{\ell'})\right] \tag{4.22}$$

where in

$$\begin{aligned}
&\sum_{(\ell_1,\ell_2)}\left[\{\mathscr{P}_S\}^{\ell_1,\ell_2}_\ell(\mathscr{H}_{\ell_1},\mathscr{H}_{\ell_2})\right] \equiv \\
&\qquad \mathcal{T}^l_\ell \Bigg\{ \sum_{\{(\mathbf{C}_1,\mathbf{z}_{\mathbf{C}_1}),(\mathbf{C}_2,\mathbf{z}_{\mathbf{C}_2})\}} [\mathscr{H}(A,\mathbf{C}_1;x,\mathbf{z}_{\mathbf{C}_1})]\cdot\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot[\mathscr{H}(B,\mathbf{C}_2;x,\mathbf{z}_{\mathbf{C}_2})]\Bigg\}_l
\end{aligned} \tag{4.23}$$

and

$$\left[\{\mathscr{P}_L\}^{\ell'}_\ell(\mathscr{H}_{\ell'})\right] \equiv \mathcal{T}^l_\ell\Bigg\{\frac{1}{2}Tr\left(\left[\frac{\partial}{\partial\xi}\mathbb{G}^{xy}_{AB}\right]\cdot[\mathscr{H}(A,B,\mathbf{C};x,y,\mathbf{z_C})]\right)\Bigg\}_l \tag{4.24}$$

[7] This is the definition of the *Fixed Point Equation* for FRG.

[8] The action of the Trimming map $\mathcal{T}^l_\ell$ on an Interaction kernel in the trimmed representation is trivial and $\mathcal{T}^l_\ell\{\mathscr{H}\}_\ell = \{\mathscr{H}\}_\ell$ and $l = |\ell|$. Also the relation

$$-\mathcal{T}^l_\ell\Box\{\mathscr{H}(\mathbf{C},\mathbf{z_C})\}_\ell = -\bar{\Box}\{\mathscr{H}(\mathbf{C},\mathbf{z_C})\}_\ell$$

is assumed in the expressions that follow.

Considering the example Eq:4.16 the *Fixed Point Equation* is expressible as

$$
\begin{aligned}
0 = \mathcal{T}_{2L}^2 \Bigg\{ \sum_{\{(C_1,z_{C_1}),(C_2,z_{C_2})\}} [\mathcal{H}(A,C_1;x,z_{C_1})] \cdot \left[\frac{\partial}{\partial \xi}\mathbb{G}_{AB}^{xy}\right] \cdot [\mathcal{H}(B,C_2;x,z_{C_2})] \\
+ \frac{1}{2} Tr\left(\left[\frac{\partial}{\partial \xi}\mathbb{G}_{AB}^{xy}\right] \cdot [\mathcal{H}(A,B,\mathbf{C};x,y,\mathbf{z_C})]\right) + \square\mathcal{H}(\mathbf{C},\mathbf{z_C}) \Bigg\}_2 \\
-\bar{\square}\{\mathcal{H}(\mathbf{C},\mathbf{z_C})\}_{2L} = \mathcal{T}_{2L}^2 \Bigg\{ \sum_{\{(C_1,z_{C_1}),(C_2,z_{C_2})\}} [\mathcal{H}(A,C_1;x,z_{C_1})] \cdot \left[\frac{\partial}{\partial \xi}\mathbb{G}_{AB}^{xy}\right] \cdot [\mathcal{H}(B,C_2;x,z_{C_2})] \\
+ \frac{1}{2} Tr\left(\left[\frac{\partial}{\partial \xi}\mathbb{G}_{AB}^{xy}\right] \cdot [\mathcal{H}(A,B,\mathbf{C};x,y,\mathbf{z_C})]\right) \Bigg\}_2
\end{aligned}
\tag{4.25}
$$

or simply, using the definition in Eq:4.24 and Eq:4.23, we can write,

$$
-\bar{\square}\{\mathcal{H}(\mathbf{C},\mathbf{z_C})\}_{2L} = \sum_{(\ell_1,\ell_2)} \left[\{\mathcal{P}_S\}_{2L}^{\ell_1,\ell_2} (\mathcal{H}_{\ell_1},\mathcal{H}_{\ell_2}) \right] + \sum_{(\ell')} \left[\{\mathcal{P}_L\}_{2L}^{\ell'} (\mathcal{H}_{\ell'}) \right] \tag{4.26}
$$

We now attempt to employ the formalism of *tree expansion* present in App:B, and place bounds on the various aforementioned operators, thereby developing a rigorous analysis of the Functional Renormalization Group approach. To cast the equation in form, as present in App:B in Eq:B.1, we add a $\{\mathcal{H}\}$ from the *LHS* and *RHS* of Eq:4.22,

$$
\begin{aligned}
\{\mathcal{H}\}_\ell - \bar{\square}\{\mathcal{H}\}_\ell = \sum_{(\ell_1,\ell_2)} \left[\{\mathcal{P}_S\}_{\ell}^{\ell_1,\ell_2} (\mathcal{H}_{\ell_1},\mathcal{H}_{\ell_2}) \right] + \sum_{(\ell')} \left[\{\mathcal{P}_L\}_{\ell}^{\ell'} (\mathcal{H}_{\ell'}) \right] + \{\mathcal{H}\}_\ell \\
\left(\mathbf{1} - \bar{\square}\right) \{\mathcal{H}\}_\ell = \sum_{(\ell_1,\ell_2)} \left[\{\mathcal{P}_S\}_{\ell}^{\ell_1,\ell_2} (\mathcal{H}_{\ell_1},\mathcal{H}_{\ell_2}) \right] + \sum_{(\ell')} \left[\{\mathcal{P}_L\}_{\ell}^{\ell'} (\mathcal{H}_{\ell'}) \right] + \{\mathcal{H}\}_\ell
\end{aligned}
\tag{4.27}
$$

Therefore, the final functional form of the *Fixed Point Equation* for the Polchinski Equation is,

$$
\{\mathcal{H}\}_\ell = \bar{\blacksquare} \left[\sum_{(\ell_1,\ell_2)} \left[\{\mathcal{P}_S\}_{\ell}^{\ell_1,\ell_2} (\mathcal{H}_{\ell_1},\mathcal{H}_{\ell_2}) \right] + \sum_{(\ell')} \left[\{\mathcal{P}_L\}_{\ell}^{\ell'} (\mathcal{H}_{\ell'}) \right] + \{\mathcal{H}\}_\ell \right] \tag{4.28}
$$

where in,

$$
\bar{\blacksquare} = (1 - \bar{\square})^{-1} = 1 + \bar{\square} + \bar{\square}^2 + \bar{\square}^3 + \cdots \tag{4.29}
$$

Using the power series expansion for $\mathcal{H}_\ell$, we derive the tree expansion corresponding to the *Polchin-*

ski Equation for our model.

$$\mathcal{H}_\ell = \sum_{u,v \geq 0} \mathcal{H}^{(u,v)} \nu^u \lambda^v \tag{4.30}$$

is substituted in Eq:4.28 and simplified.

$$\begin{aligned} \mathcal{H}_\ell^{(u,v)} \nu^u \lambda^v = \bar{\blacksquare} \Bigg[\sum_{(\ell_1,\ell_2)} \Bigg[\{\mathscr{P}_S\}_\ell^{\ell_1,\ell_2} \Bigg(\sum_{u_1,v_1} \mathcal{H}_{\ell_1}^{(u_1,v_1)} \nu^{u_1} \lambda^{v_1}, \sum_{u_2,v_2} \mathcal{H}_{\ell_2}^{(u_2,v_2)} \nu^{u_2} \lambda^{v_2} \Bigg) \Bigg] \\ + \sum_{(\ell')} \Bigg[\{\mathscr{P}_L\}_\ell^{\ell'} \Bigg(\sum_{u_3,v_3} \mathcal{H}_{\ell'}^{(u_3,v_3)} \nu^{u_3} \lambda^{v_3} \Bigg) \Bigg] + \sum_{u_4,v_4} \mathcal{H}_\ell^{(u_4,v_4)} \nu^{u_4} \lambda^{v_4} \Bigg] \end{aligned} \tag{4.31}$$

We then study Eq:4.31, line by line, understanding and analysing its structure and developing an analogous tree expansion algorithm to determine the fixed point. Considering, the first term on the *RHS* of Eq:4.31,

$$\begin{aligned} &\bar{\blacksquare} \Bigg\{ \sum_{(\ell_1,\ell_2)} \Bigg[\{\mathscr{P}_S\}_\ell^{\ell_1,\ell_2} \Bigg(\sum_{u_1,v_1} \mathcal{H}_{\ell_1}^{(u_1,v_1)} \nu^{u_1} \lambda^{v_1}, \sum_{u_2,v_2} \mathcal{H}_{\ell_2}^{(u_2,v_2)} \nu^{u_2} \lambda^{v_2} \Bigg) \Bigg] \Bigg\} \\ &= \bar{\blacksquare} \mathcal{T}_\ell^l \Bigg\{ \sum_{u_1,v_1} \sum_{u_2,v_2} \mathcal{H}_{\ell_1}^{(u_1,v_1)} \nu^{u_1} \lambda^{v_1} \cdot [\partial_\xi \mathbb{G}]_l^{\ell_1,\ell_2} \cdot \mathcal{H}_{\ell_2}^{(u_2,v_2)} \nu^{u_2} \lambda^{v_2} \Bigg\} \\ &= \bar{\blacksquare} \mathcal{T}_\ell^l \Bigg\{ \sum_{u_1,v_1} \sum_{u_2,v_2} \mathcal{H}_{\ell_1}^{(u_1,v_1)} \cdot [\partial_\xi \mathbb{G}]_l^{\ell_1,\ell_2} \cdot \mathcal{H}_{\ell_2}^{(u_2,v_2)} \nu^{u_1+u_2} \lambda^{v_1+v_2} \Bigg\} \end{aligned} \tag{4.32}$$

Eq:4.32 is constrained by, $|\ell_1 + \ell_2| = |\ell + 2|$ and $|\ell| = l$. Isolating the coefficient of $\{\nu^u \lambda^v\}$, gives the contribution of the *straight road Polchinski Operator* to the Fixed Point Equation. The implicit summation on the *RHS*, of Eq:4.32, due to the presence of the propagator, establishes the constraints, thereby ensuring that the system of equations is consistent.

Similarly, the second term on the *RHS* of Eq:4.31,

$$\begin{aligned} &\bar{\blacksquare} \Bigg\{ \sum_{(\ell')} \Bigg[\{\mathscr{P}_L\}_\ell^{\ell'} \Bigg(\sum_{u_3,v_3} \mathcal{H}_{\ell'}^{(u_3,v_3)} \nu^{u_3} \lambda^{v_3} \Bigg) \Bigg] \Bigg\} \\ &= \bar{\blacksquare} \mathcal{T}_\ell^l \Bigg\{ \frac{1}{2} Tr \Bigg[\sum_{u_3,v_3} [\partial_\xi \mathbb{G}]_l^{\ell'} \cdot \mathcal{H}_{\ell'}^{(u_3,v_3)} \nu^{u_3} \lambda^{v_3} \Bigg] \Bigg\} \end{aligned} \tag{4.33}$$

is associated with $|\ell'| = |\ell + 2|$ and $|\ell| = l$. Identically, isolating the coefficient of $\{\nu^u \lambda^v\}$, gives the contribution of the *loop road Polchinski Operator* to the Fixed Point Equation. And similar to the reasoning above, about the implicit summation present in Eq:4.33, the system of equations is consistent with the constraints.

Lastly, the third term on the RHS of Eq:4.31,

$$\bar{\blacksquare}\left(\sum_{u_4,v_4}\mathscr{H}_{\ell}^{(u_4,v_4)}\nu^{u_4}\lambda^{v_4}\right) \equiv \text{ is a trivial expression adjusted by the } \bar{\blacksquare} \text{ operator.} \tag{4.34}$$

and is associated with itself trivially, with the same value of ℓ (from the trimming map list) as in the LHS.

Therefore, the term $\mathscr{H}_{\ell}^{(u,v)}$, is expressible in the form

$$\begin{aligned}\mathscr{H}_{\ell}^{(u,v)} &= \sum_{(\ell_1,\ell_2)}\sum_{\{u_i,v_i\}_{i=1,2}}\bar{\blacksquare}\{\mathscr{P}\}_{\ell}^{\ell_1,\ell_2}\left(\mathscr{H}_{\ell_1}^{(u_1,v_1)},\mathscr{H}_{\ell_2}^{(u_2,v_2)}\right)\Bigg|_{|\ell_1+\ell_2|=|\ell|+2=l+2} \\ &+\sum_{(\ell')}\sum_{\{u_3,v_3\}}\bar{\blacksquare}\{\mathscr{P}\}_{\ell}^{\ell'}\left(\mathscr{H}_{\ell}^{(u_3,v_3)}\right)\Bigg|_{|\ell'|-2=|\ell|=l} \\ &+\bar{\blacksquare}\left(\mathscr{H}_{\ell}^{(u_4,v_4)}\right)\end{aligned} \tag{4.35}$$

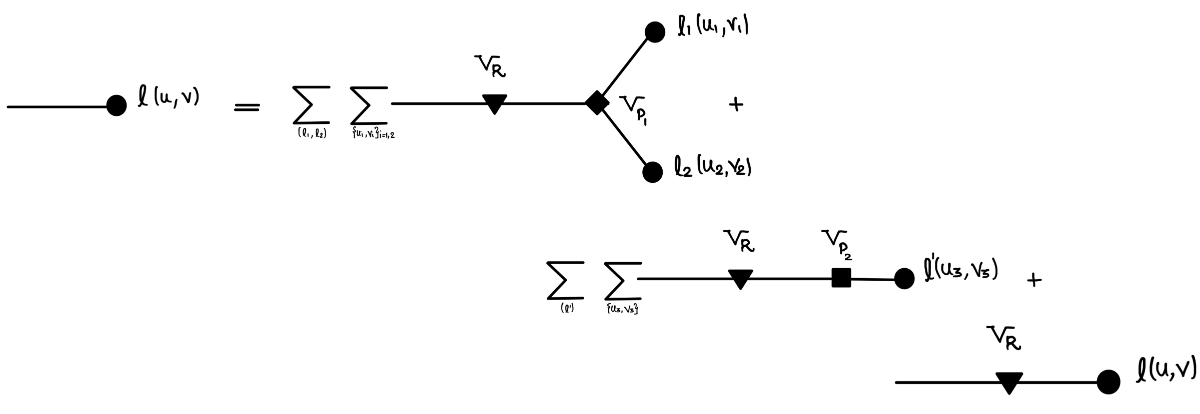

Figure 4.3: Graphical Representation of the Eq:4.35

A graphical representation of Eq:4.35. This representation allowed us to write the interaction kernel arguments in the form of a tree-expansion. The vertex labelled v_R on the RHS denotes the operator $(1-\bar{\triangle})^{-1} = \bar{\blacksquare}$; the vertex labelled v_{P_1} on the RHS denotes the operator $\{\mathscr{P}\}_{\ell}^{\ell_1,\ell_2}$; and the vertex labelled v_{P_2} on the RHS denotes the operator $\{\mathscr{P}\}_{\ell}^{\ell'}$.

The *relevance condition* for the interaction vertices remains the same, that is $u + v \geq \frac{|\ell|}{2} - 1$

for $\{\mathscr{H}\}_\ell$ to be non-trivial. These conditions are stated in App:B, and are,

$$\begin{aligned} \{H\}_\ell^{(u,v)} &\neq 0 \quad \text{if } u+v \geq \frac{\ell}{2}-1 \\ \ell = 2R,\ 4R &\quad \text{for } u+v \geq 2 \\ \ell = 6R &\quad \text{for } u+v \geq 3 \end{aligned} \tag{4.36}$$

The explicit expressions for the Fixed Point Equations of the are

$$\begin{aligned} \{\mathscr{H}\}_{2L}^{(u,v)} &= \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}_{2L}^{2L,2L}\left(\mathscr{H}_{2L}^{(u_1,v_1)}, \mathscr{H}_{2L}^{(u_2,v_2)}\right) + \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}_{2L}^{2L,2R}\left(\mathscr{H}_{2L}^{(u_1,v_1)}, \mathscr{H}_{2R}^{(u_2,v_2)}\right) \\ &+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}_{2L}^{2R,2L}\left(\mathscr{H}_{2R}^{(u_1,v_1)}, \mathscr{H}_{2L}^{(u_2,v_2)}\right) \\ &+ \sum_{u_3,v_3} \bar{\blacksquare}\{\mathscr{P}_L\}_{2L}^{4L}\left(\mathscr{H}_{4L}^{(u_3,v_3)}\right) + \sum_{u_3,v_3} \bar{\blacksquare}\{\mathscr{P}_L\}_{2L}^{4R}\left(\mathscr{H}_{4R}^{(u_3,v_3)}\right) + \bar{\blacksquare}\{\mathscr{H}\}_{2L}^{(u_4,v_4)} \\ &= \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}_{2L}^2\left\{\mathscr{H}_{2L}^{(u_1,v_1)} \cdot [\partial_\xi \mathbb{G}]_2^{2L,2L} \cdot \mathscr{H}_{2L}^{(u_2,v_2)}\right\} \\ &+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}_{2L}^2\left\{\mathscr{H}_{2L}^{(u_1,v_1)} \cdot [\partial_\xi \mathbb{G}]_2^{2L,2R} \cdot \mathscr{H}_{2R}^{(u_2,v_2)}\right\} \\ &+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}_{2L}^2\left\{\mathscr{H}_{2R}^{(u_1,v_1)} \cdot [\partial_\xi \mathbb{G}]_2^{2R,2R} \cdot \mathscr{H}_{2R}^{(u_2,v_2)}\right\} \\ &+ \frac{1}{2}\bar{\blacksquare}\mathcal{T}_{2L}^2\left\{Tr\left[\sum_{u_3,v_3}[\partial_\xi \mathbb{G}]_{2L}^{4L} \cdot \mathscr{H}_{4L}^{(u_3,v_3)}\right]\right\} \\ &+ \frac{1}{2}\bar{\blacksquare}\mathcal{T}_{2L}^2\left\{Tr\left[\sum_{u_3,v_3}[\partial_\xi \mathbb{G}]_{2L}^{4R} \cdot \mathscr{H}_{4R}^{(u_3,v_3)}\right]\right\} + \bar{\blacksquare}\{\mathscr{H}\}_{2L}^{(u_4,v_4)} \end{aligned} \tag{4.37}$$

$$\begin{aligned} \{\mathscr{H}\}_{4L}^{(u,v)} &= \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}_{4L}^{2L,4L}\left(\mathscr{H}_{2L}^{(u_1,v_1)}, \mathscr{H}_{4L}^{(u_2,v_2)}\right) + \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}_{4L}^{2R,4L}\left(\mathscr{H}_{2R}^{(u_1,v_1)}, \mathscr{H}_{4L}^{(u_2,v_2)}\right) \\ &+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}_{4L}^{2L,4R}\left(\mathscr{H}_{2L}^{(u_1,v_1)}, \mathscr{H}_{4R}^{(u_2,v_2)}\right) + \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}_{4L}^{2R,4R}\left(\mathscr{H}_{2R}^{(u_1,v_1)}, \mathscr{H}_{4R}^{(u_2,v_2)}\right) \\ &+ \sum_{u_3,v_3} \bar{\blacksquare}\{\mathscr{P}_L\}_{4L}^{6SL}\left(\mathscr{H}_{6SL}^{(u_3,v_3)}\right) + \sum_{u_3,v_3} \bar{\blacksquare}\{\mathscr{P}_L\}_{4L}^{6R}\left(\mathscr{H}_{6R}^{(u_3,v_3)}\right) \\ &+ \bar{\blacksquare}\{\mathscr{H}\}_{4L}^{(u_4,v_4)} \\ &= \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}_{4L}^4\left\{\mathscr{H}_{2L}^{(u_1,v_1)} \cdot [\partial_\xi \mathbb{G}]_4^{2L,4L} \cdot \mathscr{H}_{4L}^{(u_2,v_2)}\right\} \\ &+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}_{4L}^4\left\{\mathscr{H}_{2R}^{(u_1,v_1)} \cdot [\partial_\xi \mathbb{G}]_4^{2R,4L} \cdot \mathscr{H}_{4L}^{(u_2,v_2)}\right\} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{4}_{4L}\Big\{ \mathscr{H}^{(u_1,v_1)}_{2L} \cdot [\partial_\xi \mathbb{G}]^{2L,4R}_{4} \cdot \mathscr{H}^{(u_2,v_2)}_{4R} \Big\} \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{4}_{4L}\Big\{ \mathscr{H}^{(u_1,v_1)}_{2R} \cdot [\partial_\xi \mathbb{G}]^{2R,4R}_{4} \cdot \mathscr{H}^{(u_2,v_2)}_{4R} \Big\} \\
&+ \frac{1}{2}\bar{\blacksquare}\mathcal{T}^{4}_{4L}\left\{ Tr\left[\sum_{u_3,v_3} [\partial_\xi \mathbb{G}]^{6SL}_{4} \cdot \mathscr{H}^{(u_3,v_3)}_{6SL} \right] \right\} + \frac{1}{2}\bar{\blacksquare}\mathcal{T}^{6R}_{4L}\left\{ Tr\left[\sum_{u_3,v_3} [\partial_\xi \mathbb{G}]^{6R}_{4} \cdot \mathscr{H}^{(u_3,v_3)}_{6R} \right] \right\} \\
&+ \bar{\blacksquare}\{\mathscr{H}\}^{(u_4,v_4)}_{4L} \qquad (4.38)
\end{aligned}
$$

$$
\begin{aligned}
\{\mathscr{H}\}^{(u,v)}_{6SL} &= \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{2L,6SL}_{6SL}\left(\mathscr{H}^{(u_1,v_1)}_{2L}, \mathscr{H}^{(u_2,v_2)}_{6SL} \right) + \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{2R,6SL}_{6SL}\left(\mathscr{H}^{(u_1,v_1)}_{2R}, \mathscr{H}^{(u_2,v_2)}_{6SL} \right) \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{2L,6R}_{6SL}\left(\mathscr{H}^{(u_1,v_1)}_{2L}, \mathscr{H}^{(u_2,v_2)}_{6R} \right) + \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{2R,6R}_{6SL}\left(\mathscr{H}^{(u_1,v_1)}_{2R}, \mathscr{H}^{(u_2,v_2)}_{6R} \right) \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{4L,4L}_{6SL}\left(\mathscr{H}^{(u_1,v_1)}_{4L}, \mathscr{H}^{(u_2,v_2)}_{4L} \right) + \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{4R,4R}_{6SL}\left(\mathscr{H}^{(u_1,v_1)}_{4R}, \mathscr{H}^{(u_2,v_2)}_{4R} \right) \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{4L,4R}_{6SL}\left(\mathscr{H}^{(u_1,v_1)}_{4L}, \mathscr{H}^{(u_2,v_2)}_{4R} \right) \\
&+ \sum_{u_3,v_3} \bar{\blacksquare}\{\mathscr{P}_L\}^{8}_{6SL}\left(\mathscr{H}^{(u_3,v_3)}_{8} \right) + \bar{\blacksquare}\{\mathscr{H}\}^{(u_4,v_4)}_{6SL} \\
&= \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{6}_{6SL}\Big\{ \mathscr{H}^{(u_1,v_1)}_{2L} \cdot [\partial_\xi \mathbb{G}]^{2L,6SL}_{6} \cdot \mathscr{H}^{(u_2,v_2)}_{6SL} \Big\} \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{6}_{6SL}\Big\{ \mathscr{H}^{(u_1,v_1)}_{2R} \cdot [\partial_\xi \mathbb{G}]^{2R,6SL}_{6} \cdot \mathscr{H}^{(u_2,v_2)}_{6SL} \Big\} \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{6}_{6SL}\Big\{ \mathscr{H}^{(u_1,v_1)}_{2L} \cdot [\partial_\xi \mathbb{G}]^{2L,6R}_{6} \cdot \mathscr{H}^{(u_2,v_2)}_{6R} \Big\} \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{6}_{6SL}\Big\{ \mathscr{H}^{(u_1,v_1)}_{2R} \cdot [\partial_\xi \mathbb{G}]^{2R,6R}_{6} \cdot \mathscr{H}^{(u_2,v_2)}_{6R} \Big\} \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{6}_{6SL}\Big\{ \mathscr{H}^{(u_1,v_1)}_{4L} \cdot [\partial_\xi \mathbb{G}]^{4L,4L}_{6} \cdot \mathscr{H}^{(u_2,v_2)}_{4L} \Big\} \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{6}_{6SL}\Big\{ \mathscr{H}^{(u_1,v_1)}_{4R} \cdot [\partial_\xi \mathbb{G}]^{4R,4R}_{6} \cdot \mathscr{H}^{(u_2,v_2)}_{4R} \Big\} \\
&+ \sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathcal{T}^{6}_{6SL}\Big\{ \mathscr{H}^{(u_1,v_1)}_{4L} \cdot [\partial_\xi \mathbb{G}]^{4L,4R}_{6} \cdot \mathscr{H}^{(u_2,v_2)}_{4R} \Big\} \\
&+ \frac{1}{2}\bar{\blacksquare}\mathcal{T}^{6}_{6SL}\left\{ Tr\left[\sum_{u_3,v_3} [\partial_\xi \mathbb{G}]^{8}_{6} \cdot \mathscr{H}^{(u_3,v_3)}_{8} \right] \right\} + \bar{\blacksquare}\{\mathscr{H}\}^{(u_4,v_4)}_{6SL} \qquad (4.39)
\end{aligned}
$$

and, $\sum^*$ denotes that the sum is carried out over all possible even partitions of $|\ell| + 2 = l + 2$ and

$\sum^{\#}$ denotes that the sum is carried out over all possible values of $|\ell'| = |\ell| + 2$.

$$\{\mathscr{H}\}^{(u,v)}_{\ell\geq 8} = \sum^{*}\sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\{\mathscr{P}_S\}^{\ell_1,\ell_2}_{\ell}\left(\mathscr{H}^{(u_1,v_1)}_{\ell_1}, \mathscr{H}^{(u_2,v_2)}_{\ell_2}\right) + \sum^{\#}\sum_{u_3,v_3} \bar{\blacksquare}\{\mathscr{P}_L\}^{\ell'}_{\ell}\left(\mathscr{H}^{(u_3,v_3)}_{\ell'}\right)$$
$$+ \bar{\blacksquare}\{\mathscr{H}\}^{(u_4,v_4)}_{\ell} \tag{4.40}$$
$$= \sum^{*}\sum_{u_1,v_1}\sum_{u_2,v_2} \bar{\blacksquare}\mathscr{T}^{l=|\ell|}_{\ell}\left\{\mathscr{H}^{(u_1,v_1)}_{\ell_1} \cdot [\partial_\xi \mathbb{G}]^{\ell_1,\ell_2}_{l=|\ell|} \cdot \mathscr{H}^{(u_2,v_2)}_{\ell_2}\right\}$$
$$+ \sum^{\#}\frac{1}{2}\bar{\blacksquare}\mathscr{T}^{l=|\ell|}_{\ell}\left\{Tr\left[\sum_{u_3,v_3}[\partial_\xi \mathbb{G}]^{\ell'}_{l=|\ell|} \cdot \mathscr{H}^{(u_3,v_3)}_{\ell'}\right]\right\} + \bar{\blacksquare}\{\mathscr{H}\}^{(u_4,v_4)}_{\ell} \tag{4.41}$$

These are the formal definitions of the relevant Fixed Point Equations for the interaction couplings of the model.

4.2.2 Analysis of the Fixed Point Equation

Taking motivation from the definition of vector spaces, as defined in Eq:2.43, we define the following vector spaces to define the Polchinski Operators' action explicitly,

$$\begin{aligned}
Q_l -&\ \text{Vector Space of coupling } \{\mathscr{H}\}_l \\
Q_\ell -&\ \text{Vector space of trimmed couplings } \{\mathscr{H}\}_\ell \\
Q_S -&\ \text{Vector Space of trimmed coupling sequences } H = (H_\ell)_{\ell\in TL} = Q_{\ell_1}\otimes Q_{\ell_2} \quad \text{for } \{\mathscr{P}_S\}^{\ell_1,\ell_2}_{\ell} \\
Q_L -&\ \text{Vector Space of trimmed coupling sequences } H = (H_\ell)_{\ell\in TL} = Q_{\ell} \quad \text{for } \{\mathscr{P}_L\}^{\ell'}_{\ell}
\end{aligned} \tag{4.42}$$

Using Eq:4.42, it is evident, the following definitions of the Polchinski Operators hold true,

$$\{\mathscr{P}_S\}^{\ell_1,\ell_2}_{\ell} \ : \ Q_S \to Q_\ell \ \text{ or more explicitly } \ Q_S \to Q_l \to Q_\ell\Big|_{|\ell_1|+|\ell_2|=|\ell|+2=l+2} \tag{4.43}$$

and

$$\{\mathscr{P}_L\}^{\ell'}_{\ell} \ : \ Q_L \to Q_\ell \ \text{ or more explicitly } \ Q_L \to Q_l \to Q_\ell\Big|_{|\ell'|=|\ell|+2=l+2} \tag{4.44}$$

Similar to our approach in App:B, we now will determine the bounds on the different parts of the Fixed Point Equations.

First, we determine the bound on $[\partial_\xi \mathbb{G}]^{\ell_1,\ell_2}_{l=|\ell|}$, that is the propagator of the theory. This propagator, $\mathbb{G}]^{\ell_1,\ell_2}_{l=|\ell|}$ can take two forms in the FPEs above. (In this situation using the Norm Bounds characterized in App:C, we can write[9].)

- One term with two Fields - $n = 1$.

$$|\mathbb{G}|^{\ell_1,\ell_2}_{l=|\ell|}| = |\langle \Psi_{\ell_1,\ell_2} \rangle_c| \leq CM_0 \tag{4.45}$$

  This kind of propagator is used to characterize the Loop Road Polchinski Operator's bound.

- Two terms with one Field each - $n = 2$.

$$|\mathbb{G}|^{\ell_1,\ell_2}_{l=|\ell|}| = |\langle \Psi_{\ell_1} \Psi_{\ell_2} \rangle_c| \leq M_{\ell_1,\ell_2} \tag{4.46}$$

From the bound on the propagator, we will use the arguments presented in Section [2,4,5] and Appendix [A] of [1]. The paper clearly presents the result,

$$|G(x)|, |\partial_\mu G(x)|, |\partial_\nu \partial_\mu G(x)| \leq M(x) \equiv C_{\chi_1} e^{-C_{\chi_2}|x/\gamma|^\sigma} \quad ; \quad x \in \mathbb{R}^d; \; \sigma = 1/s < 1 \tag{4.47}$$

here the definitions of C_{χ_1}, C_{χ_2} are dependent on χ, which is defined in Eq:2.8. The details of the proof of this result are presented in Appendix [A] of [1] and we will use an analogous approach to put a bound on $\partial_\xi \mathbb{G}$. It is to be noted that the symplectic matrix does not affect the norm of the propagator or its scale-based derivative in any way.
Using the definition of the propagator independent of the symplectic matrix acted upon by the derivative with respect to γ (and by extension with respect to ξ), we can see that,

$$\gamma \frac{\partial}{\partial \gamma} \mathbb{G} = -\frac{\partial}{\partial \gamma} \frac{\chi(\gamma k)}{|k|^{d/2+\varepsilon}} \tag{4.48}$$

and the action of ∂_γ can be expressed as,

$$\gamma \partial_\gamma \chi(\gamma k) = \sum_{\mu=1}^{d} k^\mu \partial_{k^\mu} \chi(\gamma k) \tag{4.49}$$

Similar to the approach in [1], we now bound the value of the term in Eq:4.49, wherein we have

[9]Definition of s,

$$s = \frac{1}{2} \left(\sum |\ell_i| - 2(n-1) \right)$$

demanded that the function $\chi(k)$ belong to the *Gevrey Class* G^s $(s>1)$ [10]

$$\max_{k\in\mathbb{R}^{d}} |\partial_{k^\mu}\chi(\gamma k)| \le \gamma\mathscr{C} \quad \Rightarrow \quad |\partial_{k^\mu}\chi(k)| \le \gamma\mathscr{C} \tag{4.50}$$

and

$$\begin{aligned}
|\partial_\gamma\chi(\gamma k)| &= |\sum_{\mu=1}^{d} k^\mu\partial_\mu\chi(k)| \\
&\le \sum_{\mu=1}^{d} |k^\mu\partial_{k^\mu}\chi(k)| \\
&\le \sum_{\mu=1}^{d} |k^\mu|\gamma\mathscr{C} \quad \text{using } |k^\mu| \le |k| \text{ by definition, we get} \\
&\le d|k|\gamma\mathscr{C}
\end{aligned} \tag{4.51}$$

Thus,

$$|\partial_\gamma\hat{G}(k)| = \frac{|\partial_\gamma(\chi(k)-\chi(\gamma k))|}{|k|^{d/2+\varepsilon}} \le \frac{\gamma d\mathscr{C}}{|k|^{d/2+\varepsilon-1}} \tag{4.52}$$

and

$$|\partial^a(\partial_\gamma\hat{G}(k))| \le \gamma d\mathscr{C}\frac{(C\gamma)^n n^{ns}}{|k|^{d/2+\varepsilon-1}} \quad ; \quad \text{any } k\in\mathbb{R}^{d} \text{ and } n=|a|\ge 0 \tag{4.53}$$

Using the definition of the Fourier Transform [11] we can bound the scale-based derivative of the propagator by [12]

$$\sup_x |x^a\gamma\partial_\gamma G(x)| \le \gamma\tilde{\mathscr{C}}(C\gamma)^n n^{ns} \tag{4.54}$$

where C is an appropriate constant. Considering, $\mathcal{C} = max\{\gamma\tilde{\mathscr{C}}, C\}$, we can write the bound as,

$$|\partial_\gamma G(x)| \le \mathcal{C}M_{\mathcal{C}}(x) \quad ; \quad \text{where } M_{\mathcal{C}}(x) \text{ is appropriately defined using } \mathcal{C} \tag{4.55}$$

---

[10] The Gevrey condition states, the derivatives of $\chi(k)$ of arbitrary order a are uniformly bounded by

$$\max_{k\in\mathbb{R}^{d}} |\partial^a\chi(k)| \le C^{|a|}|a|^{|a|s}$$

[11] The Fourier Transform of $(ix)^aF(x)$ is $\partial^a\hat{F}(k)$ and suitable bounds can be defined as,

$$\sup_x |x^aF(x)| \le \frac{1}{(2\pi)^{d}}||\partial^a\hat{F}||_{L^1}$$

[12] Also it is assumed that since $\hat{G}(k)$ has compact support and

$$\int_{|k|\le 1}\frac{d^{d}k}{|k|^{d/2+\varepsilon}} \le \infty \quad ; \quad \int_{|k|\le 1}\frac{d^{d}k}{|k|^{d/2+\varepsilon-1}} \le \infty$$

Therefore, we have a bound on the $[\partial_\xi \mathbb{G}]_l^{\ell_1,\ell_2}$, term in our prior expressions.

4.3 Further Work

We are currently attempting to place a bound on the norm of the Re-scaling operator, $\bar{\square}$. A suitable bound on the norm of the map is necessary to prove $\bar{\blacksquare}$ can be expressed as an expansion in the form presented Eq:4.29. To study the bounds on the norm of $\bar{\square}$, we will consider the following approach:

- Test the action of $\bar{\square}$ on the *local elements* of the Trimmed List - TL, which include $\ell = 2L$ and $\ell = 4L$. - For the interaction kernels associated with the *local terms* in TL, the action of the rescaling operator, $\bar{\square}$ will be trivial and can be expressed purely as a number $(n \neq 0)$. This is because the functional form of these respective interaction kernels comprises combinations δ-functions and the symplectic matrices. These are expected to take the form $\square = -l[d_\Psi - (\#)d]$ or equivalently, $\bar{\square} = \mathcal{T}_\ell^l \cdot (-l[d_\Psi - (\#)d]) = (-l[d_\Psi - (\#)d]) \cdot \mathcal{T}_\ell^l$, where in $\#$ denotes a (integer) number.
- Test the action of $\bar{\square}$ on the non-local and non-trivial elements of the Trimmed List - TL which include $\ell = 2R, 4R, 6SL, 6R, 8, \ldots$. These elements have non-trivial contributions which arise due to the presence of derivatives and non-local pieces of functions that make up the respective interaction kernels (interaction vertices) multiplied to the $\boldsymbol{\Psi}$ field when acted upon by $\square$. The bound on the norm of $\bar{\square}$ for these interaction kernels (interaction vertices) is however expected to be consistent with the bounds on the norm that can be defined analogously to the bounds in the previous point. Particularly, the bound the norm of the action of $\bar{\square}$ above referred elements of the trimmed list are expected to be consistent with the bound on the norm of the action of $\bar{\square}$ on a purely local interaction kernel with l external legs.

Subsequent to placing bounds on the norms on the propagator, the rescaling operator and the trimming map, we can develop bounds on the norms of $\{\mathscr{P}_S\}_\ell^{\ell_1,\ell_2}$, $\{\mathscr{P}_L\}_\ell^{\ell'}$ and $\bar{\blacksquare}$, using properties of the product of norms and Triangle Inequalities. Following the approach presented in App:B, considering $|\nu|, \|\lambda| \leq \delta$, and constructing the completely reduced version of each tree diagram, i.e., completely reduced version corresponds to the tree diagram with only ν, λ and $\mathfrak{X}_*$ vertices as endpoints. Considering the bounds on the norm of the irrelevant interaction vertices, $\{\mathscr{H}\}_\ell$,

$\ell = 2R, 4R, 6R, 8...$ presented in we can [1], we write the bound on the tree as the product of the norms of its endpoints and the corresponding bounds on the vertices that are not endpoints as the norms of the action of the Polchinski Operators, $\{\mathscr{P}_S\}_{\ell}^{\ell_1,\ell_2}$, $\{\mathscr{P}_L\}_{\ell}^{\ell'}$. In addition, subsequent steps involve determining the number of possible trees and a consistent overall bound on $\bar{\blacksquare}$. Consequently, we can write the FRG-defined Fixed Point equations for the irrelevant vertices, $\{\mathscr{H}\}_{\ell}$, $\ell = 2R, 4R, 6R, 8, ...$ as the using the relevant interaction vertices, ν and λ. This would simultaneously establish the analyticity of $\{\mathscr{H}\}_{\ell}$ in ν and λ, in the corresponding neighbourhood of the origin in $\mathbb{C}^2$.

The beta functions equations for the relevant couplings can also be derived using the tree expansion technique. Employing an equivalent strategy, we can express the fixed point equations for ν and λ using the Polchinski Operators $\{\mathscr{P}_S\}_{\ell}^{\ell_1,\ell_2}$ and, $\{\mathscr{P}_L\}_{\ell}^{\ell'}$ and the inverse of their respective (trivial) re-scaling operator.Therefore at the Fixed Point, the Fixed Point equations of ν, λ (and $\{\mathscr{H}\}_{\ell=2R,4R,6R,8,..}$) can be expressed as analytic functions of the ν, λ and ε, possessing suitable norm bounds. Consequently, by the Implicit Function Theorem, these equations will thus possess a unique solution present analytically close to the lowest order approximated solution of the Fixed Point Equations.

4.4 Discussion

We have established a suitable framework to develop the Functional Renormalization Group analysis of the Symplectic Long-Range Fermion Model and focused on study the characteristics of the non-Gaussian Fixed Point of the fermion model. To facilitate our understanding of these topics, we conducted a comprehensive literature review of the Wilsonian RG, the Rigorous RG (Constructive RG), and the Functional RG procedures. We consulted various sources listed in the bibliography to gather the necessary background information. After laying the essential foundation, we established the connection between the techniques employed in Rigorous RG and Functional RG procedures for the aforementioned fermion model. This connection provided a structured mathematical approach to perform calculations in Functional RG, devoid of arbitrary truncations while simultaneously establishing the groundwork for the Functional RG studies of the Symplectic fermions.

We employed our understanding of the Wilsonian RG Procedure and Fixed Points to perform a detailed review of the non-Gaussian Fixed Point for a Long-range Fermion model with a relevant quartic interaction. This was motivated by the research conducted by the authors of [1] on the aforementioned toy model. Through this review, we learned about the Constructive RG procedure, a method to construct the non-Gaussian Fixed points in a mathematically rigorous manner. In our review, we referred to this procedure and method as Rigorous RG. Through this review study, we algorithmically categorized the Rigorous RG procedure for the Fermion model as follows:

- Constructing a Banach space which was invariant under the action of the respective RG Map. The Banach space was not restricted to the space of relevant couplings and local irrelevant interactions, since the action of the RG Map generated both local and non-local irrelevant couplings irrespective of the input action.
- Defining a non-perturbative RG Map, by utilizing the fermionic nature of the fields to simplify its definition and organizing the associated perturbative expansion as a series of determinants. Instead of expressing the perturbative expansion as a series of Feynman diagrams, using a series of determinants conferred two significant advantages. First, it allowed for a more structured reorganization of terms while simultaneously performing partial summation, and second, unlike the Feynman diagram expansion, the determinant expansion was absolutely convergent.
- Demonstrating the contractive property of the RG Map in the neighbourhood of the lowest-order fixed point which was trivially determined. This evidence, therefore, supported the existence of a unique non-Gaussian Fixed point in the neighbourhood of the trivially determined fixed point. We simultaneously studied the analyticity of the fixed point with respect to ε.

After reviewing the Constructive/Rigorous RG procedure, we presented the foundations and formalism associated with the Functional RG. The effect of integrating the high modes, step one of the RG procedure, was expressed using a hierarchy of functional differential equations for a class of Greens Functions or Vertex Functions. The Polchinski Equation and the Wetterich Equation for a single Scalar Field action were derived in two different methods, namely using the covariance split technique or the cut-off function technique. The corresponding Polchinski and Wetterich Equations for the Fermion field were also explicitly worked out, and the difference between the scalar field and the fermion field case was studied. The theory associated with the rescaling of fields in the Functional RG procedure, step two of the RG procedure, was also presented, and the re-scaling operator was derived explicitly for the scalar field.

We presented our formalism for studying the Long-range Symplectic Fermion model using the techniques of Constructive/Rigorous RG and Functional RG. Using the covariance splitting technique and the definition of the Effective interaction kernel presented in the Rigorous RG Chapter, we derived the form of the Functional RG Flow Equation for the Effective Interaction (Effective Hamiltonian since we considered Euclidean space-time) with a suitable re-definition of the field and the propagator. This calculation presented us with the Polchinski Equation for the toy model under consideration. Furthermore, we explicitly derived the Vertex-Polchinski Equation or Polchinski Equation for the Vertices (vertices defined as the interaction kernels associated with the trimmed list elements) using the action of a suitable l-order variation of the Polchinski Equation and the Trimming operator (as defined in the Rigorous RG Formalism). Through this derivation, we explicitly derived the Polchinski Operators $\{\mathscr{P}_S\}_{\ell}^{\ell 1,\ell 2}$ and $\{\mathscr{P}_L\}_{\ell}^{\ell'}$, and the variation re-defined re-scaling operator $\bar{\square}$, and defined their respective input interaction kernels, the parameters of the input interaction kernels, and actions on interaction kernels while writing the Fixed Point Equations from the Flow Equations. Carefully observing and using the aforementioned definitions, we expressed this system of equations in the form of a tree expansion that we expect to play a pivotal role in proving the existence of a fixed point using the Implicit Function Theorem. Next, we defined the required vector spaces associated with the inputs of the Polchinski Operators, explicitly stating the Polchinski Operators as maps between the corresponding multi-dimensional vector spaces. In addition to this, we formally proved the existence and explicitly determined a bound on the norm of the derivative with respect to the scale of the propagator of the toy model in Functional RG formalism. Further work on determining the bounds on the norm of the action of the re-scaling operator $\bar{\square}$ on the interactions is under investigation, and we presented our hypothesis on the same in the previous section. Formal completion of the proof of the existence of the fixed point through the algorithm and steps presented in Sec:4.3 will allow us to present a rigorous understanding of the Functional RG approach towards the analysis of systems and serve as a basis for a similar formalism to be developed for the Wetterich Equation, which in theoretical studies is deemed to be a more local quantity.

Appendix A

Useful Theorems

The Banach fixed-point theorem and Implicit Function Theorems hold significant relevance in the domain of metric spaces. These theorems establish the existence and uniqueness of fixed points for a class of self-maps of metric spaces and also provide a constructive approach to determining such fixed points [25, 26].

A.1 Theorem 1 : Contraction Mapping Theorem

The *mathematically* precise statement of the *Contraction Mapping Theorem*, [27], used in the *Construction-based proof* of the Fixed Point in Chp:2.

Let (X, d) be a *complete* metric space, and $\mathscr{F}$ be a map defined from the X, onto itself, i.e., $\mathscr{F} : X \to X$. Consider that $\exists$ a constant α, belonging to the interval $(0, 1)$, i.e., $\alpha \in (0, 1)$, and

$$d[\mathscr{F}(x), \mathscr{F}(y)] \leq \alpha d[x, y] \quad ; \quad \forall x, y \in X \tag{A.1}$$

Thus, $\mathscr{F} : X \to X$ is a contraction mapping. The *contraction maps* are responsible for bringing the points *closer*. For every $x \in X$ and an arbitrary $r > 0$, all points in the ball $\mathscr{B}_r(x)$ are mapped to a ball $B_{r'}(\mathscr{F}(x))$, subject to the constraint, $r' < r$.[1]

Then $\mathscr{F}$ possess a unique fixed point $\tilde{x} \in X$, satisfying,

$$\mathscr{F}(\tilde{x}) = \tilde{x} \tag{A.2}$$

[1] $\alpha < 1$ is referred to as a strict contraction.

and for any $x \in X$ and $n \geq 0$,

$$d[\mathscr{F}^n(x), \tilde{x}] \leq \frac{\alpha^n}{1-\alpha} d[\mathscr{F}(x), x] \tag{A.3}$$

The *Contraction Mapping Theorem* states that a strict contraction, on a complete metric space has a unique fixed point. Contraction mapping theorems are a *subset* of sorts of the more general *Fixed Point Theorems*[2]. The *Schauder fixed point Theorem* states that a continuous map on a convex compact subset of a Banach Space, posses a *fixed point*.

A.2 The Implicit Function Theorem I & II

The Implicit Function Theorem I

The *mathematically* precise statement of the *Implicit Function Theorem I* , [27], used in the *proof* of the Fixed Point in App:B.

Let B_1, B_2, B_3, be Banach spaces, with B_1 and B_2 comprising of the open sets X_1 and X_2, respectively.Consider

$$\mathscr{F} : S_1 \times S_2 \to B_3 \tag{A.4}$$

is differentiable in $s_1 \in S_1$, and $\mathscr{F}$ & $\nabla\mathscr{F}$ are continuous in $(s_1, s_2) \in S_1 \times S_2$. If $\exists$ a point $(s_1^*, s_2^*) \in X_1 \times X_2$, such that

$$\mathscr{F}(s_1^*, s_2^*) = 0 \tag{A.5}$$

and $\nabla_{s_1}\mathscr{F}(s_1^*, s_2^*)$ possess a bounded inverse, then $\exists$ a neighbourhood $\tilde{S}_1 \times \tilde{S}_2 \subset S_1 \times S_2$ of (s_1^*, s_2^*) and a continuous function $f : \tilde{S}_2 \to \tilde{S}_1$ with $f(s_2^*) = s_1^*$ such that $\mathscr{F}(\tilde{s}_1, \tilde{s}_2)$ for $(\tilde{s}_1, \tilde{s}_2) \in \tilde{S}_1 \times \tilde{S}_2$ if and only if $\tilde{s}_1 = f(\tilde{s}_2)$.

The Implicit Function Theorem II

The *mathematically* precise statement of the *Implicit Function Theorem II*, [27], used in the *proof* of the Fixed Point in App:B.

[2]The Fixed Point Theorem in Chp:2 is also a particular theorem belonging to this broader class of Fixed Point Theorems

Let B_1, B_2, B_3, be Banach spaces, with B_1 and B_2 comprising of the open sets X_1 and X_2, respectively.Consider

$$\mathcal{F} : S_1 \times S_2 \to B_3 \tag{A.6}$$

is a continuously differentiable in $s_1 \in S_1$ and $s_2 \in S_2$. If $\exists$ a point $(s_1^*, s_2^*) \in X_1 \times X_2$, such that

$$\mathcal{F}(s_1^*, s_2^*) = 0 \tag{A.7}$$

and $\nabla_{s_1}\mathcal{F}(s_1^*, s_2^*)$ possess a bounded inverse, then $\exists$ a neighbourhood $\tilde{S}_1 \times \tilde{S}_2 \subset S_1 \times S_2$ of (s_1^*, s_2^*) and a continuously differentiable function $f : \tilde{S}_2 \to \tilde{S}_1$ with $f(s_2^*) = s_1^*$ such that $\mathcal{F}(\tilde{s}_1, \tilde{s}_2)$ possessing $f(s_2^*) = s_1^*$ such that

$$\mathcal{F}(\tilde{s}_1, \tilde{s}_2) = 0 \quad \text{for } (\tilde{s}_1, \tilde{s}_2) \in \tilde{S}_1 \times \tilde{S}_2 \text{ if and only if } \tilde{s}_1 = f(\tilde{s}_2) \tag{A.8}$$

and

$$\nabla f(s_2) = -\frac{1}{\nabla_{s_1}\mathcal{F}(f(s_2, s_2)}\nabla_{s_2}\mathcal{F}(f(s_2, s_2) \tag{A.9}$$

Additionally, if $\mathcal{F}$ is analytic in a neighbourhood of (s_1^*, s_2^*) then f is also analytic in a neighbourhood of s_2^*.

Appendix B

Gallavotti-Nicol‘o trees

Another method for constructing the Fixed Point is employed by us. The general framework of the construction remains the same: trimmed interaction kernels, weighted norm characterizing the bounds on the couplings, norm bounds of the maps, etc. This approach is presented in [1] and the corresponding theory of employing tree expansions to study the field-theoretic structures of models is explored in detail in [28–31].

We consider the Fixed Point Equation for the irrelevant interaction kernels u_ℓ, where $\ell = 2R, 4R, 6R, 8, ...$ [1]

$$u_\ell = \mathscr{D}u_\ell + \sum_{n\geq 1}\sum_{(\ell_i)_1^n}^{*} \mathscr{R}_\ell^{\ell_1,...,\ell_n}(H_{\ell_1}, ..., H_{\ell_n}) \tag{B.1}$$

Next, we consider,

$$u_\ell = \sum_{k_1,k_2\geq 0} u_\ell^{(k_1,k_2)}\nu^{k_1}\lambda^{k_2} \tag{B.2}$$

as our ansatz [2] and substituting Eq:B.2 in Eq:B.1, we arrive at

$$u_\ell^{(k_1,k_2)} = \sum_{n\geq 1}\sum_{(\ell)_1^n}^{*}\sum_{\{k_{1,i},k_{2,i}\}_{i=1,2,...,n}}^{(k_1,k_2)} (1-\mathscr{D})^{-1}\mathscr{R}_\ell^{\ell_1,...,\ell_n}(H_{\ell_1}^{(k_{1,1},k_{2,1})}, ..., H_{\ell_n}^{(k_{1,n},k_{2,n})}) \tag{B.3}$$

subject to the constraints,

$$\begin{aligned} k_{1,1} + k_{1,2} + ... + k_{1,n} &= k_1 \\ k_{2,1} + k_{2,2} + ... + k_{2,n} &= k_2 \end{aligned} \tag{B.4}$$

[1] The $*$ denotes the constraint, if $n = 1$ then $\ell_1 \neq \ell$.

[2] Subject to the constraint $u^{(k_1,k_2)} = 0$ if $k_1+k_2 < |\ell|/2-1$ and $k_1+k_2 \geq 2$ for $\ell = 2R, 4R$ and $k_1+k_2 \geq 3$ for $\ell = 6R$.

,and (the interaction kernel arguments $H_{\ell_i}^{(k_{1,i},k_{2,i})}$ are denoted by,

$$H_{\ell_i}^{(k_{1,i},k_{2,i})} = \begin{cases} u_{\ell_i}^{(k_{1,i},k_{2,i})} & \text{if } \ell_i \neq 2L, 4L, 6SL \\ \nu & \text{if } \ell_i = 2L \ ; \ (k_{1,i}, k_{2,i}) = (1,0) \\ \lambda & \text{if } \ell_i = 4L \ ; \ (k_{1,i}, k_{2,i}) = (0,1) \\ \mathfrak{X}_* & \text{if } \ell_i = 6SL \ ; \ (k_{1,i}, k_{2,i}) = (0,2) \end{cases} \tag{B.5}$$

Operator $(1 - \mathscr{D})$ is invertible, since $(1 - \mathscr{D})^{-1} = 1 + \mathscr{D} + \mathscr{D}^2 + ...$ and the subsequent series is *convergent* in $L_1 - Norm\ Space$ because $\gamma^{-D_l - p} < 1$ which is the condition for the irrelevance of the interaction kernels. The equation Eq:B.3, can be graphically written as

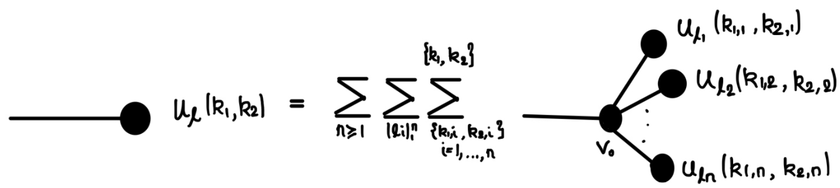

Figure B.1: Graphical Representation of the Power Series Expansion

A graphical representation of Eq:B.3. This representation allows us to write the interaction kernel arguments in the form of a tree-expansion. The vertex labelled v_0 on the RHS denotes the action of $(1 - \mathscr{D})^{-1}\mathscr{R}_{\ell}^{\ell_1,\ldots\ell_n}$ operator and the n branches '*enter*' the vertex v_0 and the branch to the left of v_0 '*exits*' from the vertex v_0 and it carries the label ℓ and its left end-point is termed as the *root*.

and

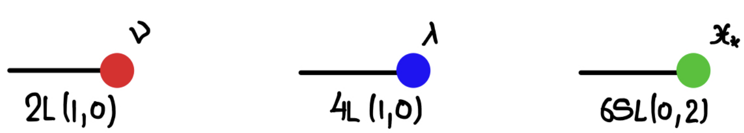

Figure B.2: Graphical Representation of the End Points of Tree Expansion

A graphical representation of the *end points* of Eq:B.3. This representation allows us to write the end points of the trees of interaction kernel arguments.

using the graphical notation presented in Fig:B.1 and Fig:B.2, we now characterize each tree diagram with a specific contribution. The following example of a tree, presents this notion concisely.

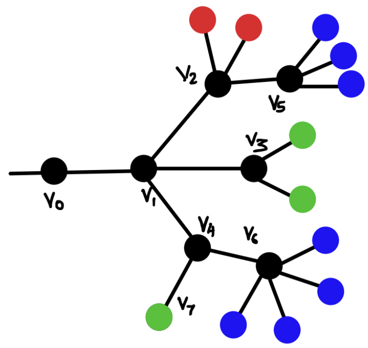

Figure B.3: Example : Tree Form Representation of a Tree Diagram from an Arbitrary Expansion

An example of a tree diagram of the graphical representation of an arbitrary contribution to Eq:B.3.

In Fig:B.3 the branch labels are implicitly defined : any vertex v_j denotes the action of the operator $(1-\mathscr{D})^{-1}\mathscr{R}_{\ell}^{\ell_1,\ldots,\ell_n}$, with a branch labelled ℓ *exiting* from v_j and n branches *entering* into the vertex, each with a label ℓ_i. Denoting the number of end-points of each type by n_ν, n_λ, $n_{\mathfrak{X}_*}$ respectively, and ℓ_0 to denote the branch *exiting* v_0 (in this example $n_\nu = 2$, $n_\lambda = 7$ and $n_{\mathfrak{X}_*} = 3$) we therefore establish that the tree has a contribution of $u^{(n_\nu, n_\lambda + 2n_{\mathfrak{X}_*})}$.

Assuming $|\nu|, |\lambda| \leq \delta$, we now use the $||\cdot||_w$ norm bounds from Sec:2.5 of $\mathscr{D}$ and $\mathscr{R}_{\ell}^{\ell_1,\ldots,\ell_n}$ to determine a bound on any arbitrary tree. $\mathscr{D}$ is characterized by a norm bound and thus $(1-\mathscr{D})^{-1}$ is also bound by some d_γ, where in $d_\gamma > 1$. Employing the norm bounds from Eq:2.63 and Eq:2.64 with the norm bound of $(1-\mathscr{D})^{-1}$ we can characterize the bound on the value of a tree by the product of the norms of its endpoints, which themselves are bound by

$$\delta^{n_\nu + n_\lambda}(B_\gamma \delta^2)^{n_{\mathfrak{X}_*}} \leq (B_\gamma \delta)^{n_\nu + n_\lambda + 2n_{\mathfrak{X}_*}} \tag{B.6}$$

multiplied with the product over vertices v_j (vertices that are not characterized as end-points) [3]

$$\tilde{\Pi}_v \left[d_\gamma C_\gamma^{n_v - 1} \gamma^{-D_{l_v}} C_0^{\sum_{i=1}^{n_v} |\ell_i(v)|} \right] \tag{B.7}$$

where in

[3]Product is considered over vertices that are not end points and this is denoted by $\tilde{\Pi}$ and the product that is considered over vertices that are end points, is denoted by $\bar{\Pi}$. Analogous notation is considered for the case of sum over vertices that are not end points and are end points, respectively.

- $l_v = |\ell_v|$, and ℓ_v characterizes the line exiting the vertex labelled v.
- $\ell_i(v)$ label the $i-th$ line entering the vertex v.

For some certain values of i for a given value of v, the vertex, can be an end-point and therefore Eq:B.7, is written as,

$$C_0^{-|\ell_0|} \times \tilde{\Pi} d_\gamma C_\gamma^{n_v-1} \gamma^{-D_{l_v}} C_0^{l_v} \times \bar{\Pi} C_0^{l_v} \tag{B.8}$$

Using the inequalities,

$$C_0^{-|\ell_{v0}|} \leq 1 \tag{B.9}$$

and

$$\bar{\sum} l_v = 2n_v + 4n_\lambda + 6m_{\mathfrak{X}_*} \leq 4(n_\nu + n_\lambda + 2n_{\mathfrak{X}_*}) = 4\kappa \tag{B.10}$$

along with Eq:B.6, the contribution of a given tree is constrained by (bounded),

$$(B_\gamma C_0^4 \delta)^\kappa \tilde{\Pi} d_\gamma C_\gamma^{n_v-1} \gamma^{-D_{l_v}} C_0^{l_v} \tag{B.11}$$

Following this, a summation over all possible values of ℓ_i, labelling the tree branches, is carried out. Assuming, γ to be sufficiently large, to be consistent with the inequality, (using Eq:2.51 for the definition of D_l)

$$\gamma^{-\frac{d}{4}-\frac{\varepsilon}{2}} C_0 < 1 \tag{B.12}$$

gives us

$$\sum_{l=2}^{\infty} \gamma^{-D_l} (C_0)^l < \infty \tag{B.13}$$

Considering the constraints and bounds in Eq:B.13, Eq:B.11, the sum over all tree branches, denotes by $\sum_{\ell_v}(Eq:B.11)$

$$\sum_{\ell_v}(Eq:B.11) \leq (B_\gamma C_0^4 \delta)^\kappa \tilde{\Pi} \tilde{d}_\gamma C_\gamma^{n_v-1} \tag{B.14}$$

where in, $\tilde{d}_\gamma$, is a re-adjusted constant, capturing d_γ and the finite constant defined in Eq:B.13.

Using,

$$\tilde{\sum}(n_v - 1) = n_v + n_\lambda + n_{\mathfrak{X}_*} - 1 \equiv \text{(Total \#(end points))-1} \tag{B.15}$$

and

$$\tilde{\sum} 1 \leq \frac{1}{2} \bar{\sum} l_v \quad \Leftarrow \quad \left\{ |\ell| \leq \sum_i |\ell_i| - 2 \right\}_{\forall\ v} \tag{B.16}$$

gives, the bound in Eq:B.14, the form,

$$\sum_{\ell_v} (Eq : B.11) \leq (B_\gamma C_0^4 C_\gamma \tilde{d}_\gamma^2 \delta)^\kappa \tag{B.17}$$

Lastly, we sum over all trees subject to the constraint, that the $\#(\text{end points}) = n_\nu + n_\lambda + 2n_{\mathfrak{X}_*} = \kappa$ for the tree being a part of the summation. Since, from []

$$\#(trees)\Big|_{\#(\text{end points})=\kappa} \leq 4^\kappa \tag{B.18}$$

the contribution of order κ to u_ℓ is bounded in the $||\cdot||w$ norm by $(A\gamma\delta)^\kappa$, where A_γ is a constant dependent on γ. Thus the tree expansion for u_ℓ is convergent for $\delta \leq \delta_0(\gamma)$ being sufficiently small.

Therefore the *irrelevant couplings* can be constructed employing the *relevant couplings* ν and λ. This also is proof of the fact that u_ℓ are analytical in ν and λ in the neighbourhood, $|\nu|,\ |\lambda| \leq \delta_0(\gamma)$ of the *origin* in $\mathbb{C}^2$ and analyticity in ε iff u_ℓ belongs to the complex half-plane $Re[\varepsilon] < \frac{d}{6}$, where in all the couplings u_ℓ are irrelevant.

For the $\beta-$functions of the relevant couplings, we consider a similar approach and observe that they are given by the first two equations of Eq:2.60, wherein we write $e_\nu^{(0)}$ and $e_\lambda^{(0)}$ as tree expansions which are convergent for sufficiently small δ. Thus the at *Fixed Point* satisfies,

$$\begin{aligned} \nu &= \gamma^{\frac{d}{2}+\varepsilon}(\nu + I_1\lambda) + e_\nu^{(0)}(\nu, \lambda, \varepsilon) \\ \lambda &= \gamma^{2\varepsilon}(\lambda + I_2\lambda^2) + e_\lambda^{(0)}(\nu, \lambda, \varepsilon) \end{aligned} \tag{B.19}$$

where, $e_\nu^{(0)}$ and $e_\lambda^{(0)}$ are analytical functions of $\nu, \lambda, \varepsilon$ for $|\nu| \leq \delta_0(\gamma)$, $|\lambda| \leq \delta_0(\gamma)$ and $Re[\varepsilon] < \frac{d}{6}$. Also the conditions, $\mathcal{O}_{[\delta]}(e_\nu^{(0)}) = 2$ and $\mathcal{O}_{[\delta]}(e_\lambda^{(0)}) = 3$, where $\delta = max\{|\nu|, |\lambda|\}$ and for $\varepsilon \in$ Compact Subset of $Re[\varepsilon] < \frac{d}{6}$. Simultaneously I_1 and I_2 are analytic in ε.

Therefore, by the *Implicit Function Theorem*, Eq:B.19 have a unique solution $\{\nu_*(\varepsilon), \lambda_*(\varepsilon)\}$, in the disk $|\varepsilon| \leq \varepsilon_0(\gamma)$ and is *analytically close* to the lowest order solution, from Sub-Sec:2.3, $\{\nu_0, \lambda_0\} = \left\{ \frac{I_1\lambda_0}{1-\gamma^{\frac{d}{2}+\varepsilon}},\ \frac{1-\gamma^{2\varepsilon}}{I_2} \right\}$.

Appendix C

Norm Bounds on Maps

Here we provide details of the bounds for the Renormalization Map, expressed in Eqs:[2.62 & 2.63. To understand these bounds, we first start from, the bounds on the $\mathcal{S}$ and $\mathcal{S}$ map. The bound on $\mathcal{S}$ map can be expressed as,

$$||\mathcal{S}_l^{\ell_1,\ldots,\ell_n}(h_1,...,h_n)||_w \leq \rho_l(h_1,...,h_n) \quad ; \quad h_i \in Q_{\ell_i} \tag{C.1}$$

which can be explicitly expressed using Eq:2.40. Considering a fixed sequence $\mathbf{Q}$ and fixed lengths $|\mathbf{P}_i| = l_i$, the number of sequences can be estimated and subsequently bounded by,

$$\#(\text{ terms in Eq:2.40 }) = \binom{\sum l_i}{l} [N d + N]^{\sum l_i - l} \leq 2^{\sum l_i} \times [N d + N]^{\sum l_i} \tag{C.2}$$

Additionally for some fixed $\mathbf{Q}_i$ and $\mathbf{P}_i$, the bound on the Integration kernel, can be stated using the GKL bound [],

$$|\mathcal{C}(\mathbf{x}_{\bar{\mathbf{Q}}})| = |\langle \Phi(\bar{\mathbf{Q}}_1, \mathbf{x}_{\bar{\mathbf{Q}}}); ...; \bar{\mathbf{Q}}_n, \mathbf{x}_{\bar{\mathbf{Q}}_n})\rangle_c| \leq (C_{GH})^s \sum_{\mathcal{T}} \Pi_{(xx')\in\mathcal{T}} M(x - x') \tag{C.3}$$

where in $s = \frac{1}{2}\sum|\mathbf{B}_i| - (n-1) \leq \frac{1}{2}\sum l_i$. We now define the following quantity for the ease of calculations,

$$\mathcal{M}_{(\mathbf{Q}_i,\mathbf{P}_i)_1^n}(\mathbf{x}_{\mathbf{Q}}) = \int d^d\mathbf{x}_{\bar{\mathbf{Q}}} \mathcal{C}(\mathbf{x}_{\bar{\mathbf{Q}}}) \Pi_{i=1}^n h_i(\mathbf{P}_i, \mathbf{x}_{\mathbf{P}_i}) \tag{C.4}$$

and the associated norm of $\mathcal{M}(\mathbf{x}_{\mathbf{Q}})$ is defined, using the weighted-norm procedure defined in Sec:2, as,

$$||\mathcal{M}||_w \int_{x_1=0} d^d\mathbf{x}_{\mathbf{Q}} |K(\mathbf{x}_{\mathbf{Q}})| w(\mathbf{x}_{\mathbf{Q}}) \tag{C.5}$$

Using the theory presented in App:[E & D] of [1], we can write the bound on the norm of $\mathcal{M}(\mathbf{x_Q})$ as,

$$\begin{aligned} ||K||_w &\leq (C_{GH})^s \sum_{\mathcal{T}} \int_{x_1=0} d^d \mathbf{x}_{\mathbf{Q}\cup\bar{\mathbf{Q}}} \Pi_{(xx')\in\mathcal{T}} M(x-x') w(\{x,x'\}) \Pi_{i=1}^n h_i(\mathbf{P}_i, \mathbf{x}_{\mathbf{P}_i} w(\mathbf{x}_{\mathbf{P}_i}) \\ &= (C_{GH})^s N_{\mathcal{T}} ||M||_w^{n-1} \Pi_{i=1}^n ||h_i||_w \end{aligned} \tag{C.6}$$

and the number of anchored trees,

$$N_{\mathcal{T}} \leq n! 4^{\sum |\mathbf{x}_{\bar{\mathbf{B}}_i}|} = n! 4^{\sum l_i - l} \leq n! 4^{\sum l_i} \tag{C.7}$$

Collecting all the bounds together, $\#(terms)$ in Eq:2.40, from Eq:C.2; norm bound of each individual term in Eq:C.6; number of anchored trees in Eq:C.7 and finally, s is bounded by $\frac{1}{2}\sum l_i$ gives us the bound

$$||\mathcal{S}^{\ell_1,\ldots,\ell_n}||_w \leq C_\gamma^{n-1} \Pi_{i=1}^n C_0^{l_i} ||h_i||_w \quad \text{if } \sum l_i \leq l + 2(n+1) \tag{C.8}$$

where in $C_\gamma = ||M||_w$ and $C_0 = 8(Nd + N)\sqrt{C_{GH}}$, which is rewritten in the form of,

$$||\mathcal{S}^{\ell_1,\ldots,\ell_n}||_w \leq \rho_l(h_1, \ldots, h_n) \quad ; \quad h_i \in Q_{\ell_i} \tag{C.9}$$

where in this is the same $\rho_l(h_1, \ldots, h_n)$ defined and used in Eq:2.63. This completes the norm-bound definitions for the map $\mathcal{S}_l^{\ell_1,\ldots,\ell_n}$.

For the case of the Trimming map defined in Sub-Sec:2.5.3, the bounds are defined as[1],

- The localization maps follow the respective norm bounds,

$$\begin{aligned} ||(H_{eff})_{2L}||_w &\leq ||(H_{eff})_{2,0}||_w \\ ||(H_{eff})_{4L}||_w &\leq ||(H_{eff})_{4,0}||_w \end{aligned} \tag{C.10}$$

- The interpolation maps follow the respective norm bounds,

$$\begin{aligned} ||T_{4R}^{4,0}(H_{eff})_{4,0}||_{w(\cdot/\gamma)} &\leq C_R \gamma ||(H_{eff})_{4,0}||_w \\ ||T_{2R}^{2,1}(H_{eff})_{2,1}||_{w(\cdot/\gamma)} &\leq C_R \gamma ||(H_{eff})_{2,1}||_w \\ ||T_{2R}^{2,0}(H_{eff})_{2,0}||_{w(\cdot/\gamma)} &\leq C_R \gamma^2 ||(H_{eff})_{2,0}||_w \end{aligned} \tag{C.11}$$

Lastly, the case of the norm bound on the Dilatation map is stated in Eq:2.64. Collecting all the norm bounds it is evident that the norm bounds described in Eqs:[2.62 & 2.63] are true.

[1] C_R depends on σ and C_w, defined in Sub-Section 4.2 of [1] in the definition of the Steiner Map.

- $\ell \geq 8$ and $\ell \in \{6R, 6SL\}$: These can be bounded using Eqs:[C.8 & 2.64 with the least ideal case of $p = 0$.
- $\ell \in \{2R, 4R\}$: These can be bounded using Eqs:[C.11 & 2.64]. The factor of γ that is lost in the case of the norm bound on the Trimming Map (Interpolation Map in Eq:C.11) is restored due to the occurrence of the derivatives in the respective interaction couplings. [2]
- $\ell \in \{2L, 4L\}$: These can be bounded using Eqs:[C.8 & 2.64] with the least ideal case of $p = 0$ along with Eq:C.10.

[2]The summation of the interpolation maps present in the definitions of the Trimming Maps, in Eqs:[2.49 & 2.50], introduces an additional factor for each of the terms. The C_0 factor in Eq:2.63 can be redefined using a larger value, to compensate for the additional factors allowing the inequalities to remain consistent.

List of Important Equations and Theorems

List of Figures

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