Asymptotic Symmetries of Spacetime and Memory Effect

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Certificate

This is to certify that this dissertation entitled **Asymptotic Symmetries of Spacetime and Memory Effect** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Kunal Paul at Indian Institute of Science Education and Research, Pune under the supervision of Dr. Sanved Kolekar, Assistant Professor, Indian Institute of Astrophysics, Bangalore, during the academic year 2022-2023.

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Declaration

I hereby declare that the matter embodied in the report entitled **Asymptotic Symmetries of Spacetime and Memory Effect**, are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Sanved Kolekar (IIA, Bangalore), and the same has not been submitted elsewhere for any other degree.

Kunal Paul

Kunal Paul

This thesis is dedicated to my family for their constant support and encouragement

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Abstract

The work presented in this thesis can be broadly divided into two parts: In the first part, we study asymptotic symmetry groups using (i) Gauge-fixing approach and (ii) Geometric approach. The gauge-fixing approach has been studied extensively for Bondi-Metzner-Sachs (BMS) group as well as some extended definitions, including how to construct symmetry group near event horizon. We have also presented a derivation for Barnich-Troessaert group for asymptotically AdS_d space-times and a possible extended BMS group using Geometric approach. Using such symmetry groups as boundary conditions, we present a general recipe to construct perturbative shock waves on a background curved space-time. In the second part, we implant such shock wave on Schwarzschild and Rindler horizon and look at their classical and semi-classical memory effects. For semi-classical case, we have considered a free mass-less scalar quantum field on Rindler space-time. As noted in previous work [1], super-translation produced by shock-waves can produce mode mixing and particle creation. In this work, we find that super-rotation sub-group can only affect mode-mixing and does not contribute to particle creation. Consequently, the entanglement production between positive frequency modes in right wedge of Rindler space-time is not affected by super-rotation.

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Chapter 1

Introduction

Symmetry is a powerful tool to understand the fundamental laws of nature. For instance, the *action* for relativistic free particle in flat space-time can be derived by starting from the definition of *Poincare invariance* of line element in special relativity as well as *locality* [2]. Essentially, the lagrangian has the expression:

$$L(v^2) = \alpha + \beta\sqrt{1 - v^2} \tag{1.1}$$

where α, β are arbitrary constant. For a particle, β is taken to be its mass. Given such a lagrangian, we can derive the relativistic equation of motion using Euler-Lagrange equations. The construction can be generalized to a curved space-time, where we will have

$$S[x^a(\tau), \dot{x}^a(\tau)] = \int \sqrt{\left(g_{ab}(x(\tau))\frac{dx^a(\tau)}{d\tau}\frac{dx^b(\tau)}{d\tau}\right)}\, d\tau \tag{1.2}$$

Similar arguments can be made in classical field theory, where our definition of field elevates from $x(\tau)$ to $\phi_a(x)$ and we have quadratic terms in derivatives of the fields(such as $\partial_\mu\phi_a\partial^\mu\phi_a$ instead of $\frac{dx^a}{d\tau}\frac{dx_a}{d\tau}$). In this thesis, we place our attention to the Einstein-Hilbert action

$$S = \int d^4x\sqrt{|g|}R \tag{1.3}$$

where, metric g_{ab} acts as our dynamical field. In this theory, the condition for *Poincare invariance* of the particle "field" $x(\tau)$ is generalized to *isometry invariance* of our metric field. Isometry transformation $x \to x'$ means that the line-element should be preserved:

$$g_{ab}(x) = g'_{cd}(x')\frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} \tag{1.4}$$

If $x' = x + \epsilon\xi$, then (1.4) implies the Lie-derivative of g_{ab} with respect to ξ is zero (see chapter 4). Such ξ's are called *Killing vector field* Given some matter-field with stress-energy tensor T_{ab}, it is possible to define conserved charges associated to each of these Killing fields. However, as it turns out, a generic solution for ξ vector field need not exist, being severely restricted by the form of Weyl curvature. The non-triviality of (1.4) can exist if we define this transformation only in some appropriate limit of space-time where the geometry (metric) has a very simple, well behaved form. In chapter 3, we have discussed *Asymptotically Simple* space-times using Conformal-Methods and looked extensively at *Asymptotically flat space-times* where space-time becomes Minkowskian in the limit $r \to \infty$. In this limit, it is reasonable to impose (4) and ask how should our ξ vector field look like and what are the associated conserved charges? This is essentially one of the motivation for studying Asymptotic Symmetry group (ASG), which we have discussed extensively in chapter 4. As it will turn out, even if space-time is Minkowskian near infinity, the symmetry group is far larger than the conventional Poincare group, thus it will have a much larger set of conserved charges compared to just 10 Poincare charges in 3+1 dimensional Minkowski space. This is the *Bondi-Metzner-Sachs* (BMS) group.

Historically, the idea of BMS symmetry was developed accidentally by H.Bondi, van der Burg and Metzner [3] while studying gravitational waves in Axis-symmetric space-times. In Part C, section 1 of their article, the authors found certain asymptotic co-ordinate transformation (4.33),(4.34),(4.35) which behaves as Lorentz boost along the axis of symmetry. Later R.K.Sachs extends and give a gauge-fixing approach to construct the symmetry group [4]. At the same time R.Penrose came up with his Conformal method to derive the BMS group [5], which came to be known as the Geometric approach. The flexibility of the gauge fixing approach allows us to define the symmetry transformation for a generic boundary conditions and in the mean time, a number of extended symmetry group has been constructed using this method (see a review of these constructions in [6]) The geometric approach, even though restrictive, gives a very clear visualization of the symmetry transformation. A comparison for each of these methods have been done in chapter 4. We have used the geometric approach to define asymptotic isometry group of space-times which are only locally Minkowskian. Further we have derived the Barnich-Troessaert group for asymptotically AdS_{d+1} space-times using this approach.

Given an ASG, we can ask if there are any physical events, which can induce such symmetry transformation. This is essentially one of the topic of Chapter 5. We can construct space-time models where we stitch two diffeomorphically similar space-times along a hyper-surface N (junction), on which the metric is continuous , while the curvature tensor comes with a delta function. Such models have been considered, example in [7],[8], where it is possible to choose coordinate system where the coordinates on two patch of the space-time are related by *super-translation* map (super-translation is Abelian subgroup of the ASG). Later Hawking,Perry,Strominger [9] considered *shock waves* of the form

$$ds^2 = (g_{ab} + \theta(u - u_0)\mathcal{L}_\xi g_{ab})dx^a dx^b \tag{1.5}$$

and implants it on Schwarzschild horizon. (ξ is the BMS super-translation vector field). One of our aim is to ask whether (1.5) is the most general shock wave that can be implanted on the horizon. Recently, [10],[11] have constructed near horizon symmetry group using gauge fixing approach. We have extended the definition (1.5) to include a general vector field V which simultaneously behaves as generator of BMS group near $r \to \infty$ and near horizon symmetry group as $r \to r_H$.

In final part of chapter 5, we have considered classical and semi-classical Memory effect produced by these shock waves. Memory effect essentially means that there is a permanent change in some configuration of fields as it crosses the shock-waves. In gravitational memory effect, we consider trajectories of test observers which gets permanently displaced on crossing the junction/shock wave (see [12]). The amount of change will depend on the conditions we want to preserve before and after scattering (experimental constraints) as well as the expression of V. We consider the Letaw-Frenet condition [13] to be preserved near Schwarzschild event horizon. It turns out, if an observer is hovering close to the Schwarzschild event horizon maintaining a fixed distance with no transverse motion, then (i) super-rotation of the shock wave can't affect the observer, (ii) the observer will keep on maintaining the fixed distance if super-translation is independent of time. In the semi-classical case, we consider a test free mass-less scalar field ϕ and consider it's scattering with the shock wave in the right wedge of the Rindler space-time. As the scalar field crosses the junction, it will undergo mode-mixing and particle creation. Note, mode-mixing is attributed to *Bogoluibov coefficients* α_{ij} (i.e. a state defined on a given wedge will interfere with rest of the modes in the same wedge). Particle creation is due to the coefficients β_{ij}

$$\langle N \rangle = \sum |\beta|^2 \tag{1.6}$$

We have essentially extended the construction considered in [1]. We find that super-rotation induced by the shock wave doesn't contribute to β_{ij}, only super-translation does. This particular observation can be relevant in the context of information paradox [14].

In chapter 2 , we have discussed the essential mathematical preliminaries needed for later chapters. We have introduced the 2-spinor formalism in GR. The construction is particularly useful in 3+1 dimensional space-times, due to three accidental isometries (see section 2 of [15]). We have used this formalism to derive the Peeling theorem in chapter 3. In few sections, we have also invoked the concept of twistors (for example, in BMS charge construction in chapter 4). By a $n - twistor$, we mean a symmetric spinor field $\omega^{AB\cdots L}$ which satisfy

$$\nabla^{A'(K}\omega^{AB\cdots L)} = 0 \tag{1.7}$$

In general we can have both primed and unprimed indices (both as upper and lower indices). Twistors can be useful for our purpose, since a twistor of the form $\xi^{AA'}$, are essentially the spinor version of conformal Killing field:

$$\nabla^{(A}_{(A'}\xi^{B)}_{B')} = 0 \leftrightarrow \mathcal{L}_\xi g_{ab} = \frac{1}{4}\nabla^c\xi_c g_{ab} \tag{1.8}$$

In this thesis we are simply considering constructions in asymptotically flat space-times. We hope to pursue construction in presence of cosmological constant in later works.

Chapter 2

Mathematical Preliminaries

2.1 2-spinors in General Relativity

2.1.1 Intuitive picture: Stereographic projection of Riemann sphere

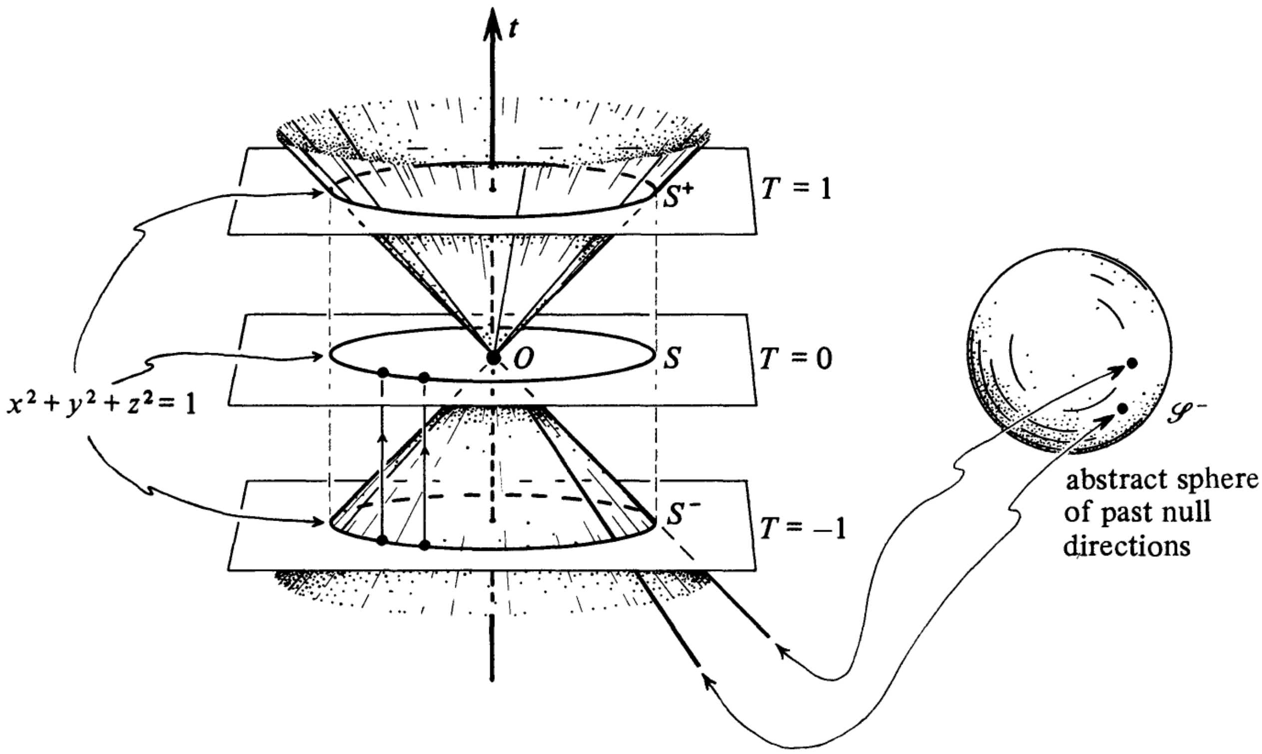


Figure 2.1: Celestial Mapping in $\mathbb{M}$ (diagram obtained from [16])

We begin by considering a scenario as depicted in the diagram above. Imagine there is an observer O in a Minkowski space-time $\mathbb{M}$ (or locally Minkowskian around the neighborhood of O). Consider the two null cones originating from this point which extend all the way upto infinity both in the future and past direction. For the time being, lets assume the observer is stationary, so that the family of his/her plane of simultaneity are given by T =constant surfaces. The observer is situated at $T = 0$ plane. Consider two hyper-surfaces $T = \pm 1$ intersecting the light cones to form corss-sections $S^{\pm}$. These cross-sections are unit sphere, because on the light cone we have

$$T^2 - X^2 - Y^2 - Z^2 = 0 \tag{2.1}$$

with $T^2 = 1$, giving

$$X^2 + Y^2 + Z^2 = 1 \tag{2.2}$$

Let's look at the unit sphere S^+. Consider the diagram (). The unit sphere is centered at the spatial point $C = (0, 0, 0)$. The convention taken here is to take the point $z = 1$ as the North Pole N and $z = -1$ as the south pole S. The Argand Plane Σ is the $z = 0$ plane. To geometrically construct spinors, the first step will be to represent any point $P = (X, Y, Z) \in S^+$ by a complex number ζ. It should be possible as equation (2.2) contains two real degrees of freedom (as it represents a two-dimenional

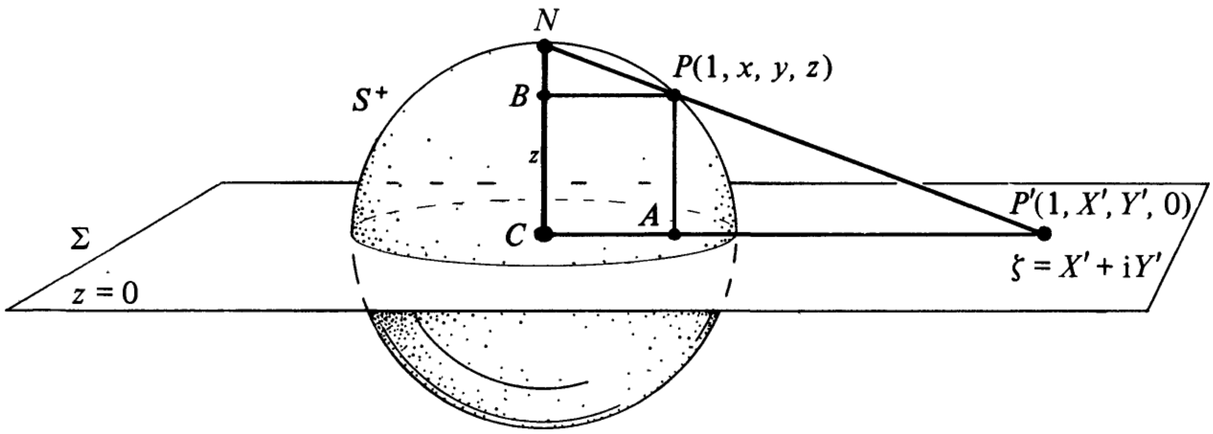


Fig. 1-3. Stereographic projection of S^+ to the Argand plane.

Figure 2.2: Stereographic Projection of Celestial sphere (diagram obtained from [16])

surface) just as a complex number is composed of two real numbers. The idea is to represent each point $P \in S^+$ by an unique complex number ζ on Argand Plane Σ. The procedure is to connect the points P with north pole N and extend the line till it meets Σ. Represent this point of intersection as $P' = (X', Y', Z')$. From the figure 2.2, it can be shown using elementary geometry that

$$\zeta = \frac{X + iY}{1 - Z} \tag{2.3}$$

ζ is called the stereographic coordinate. We can express (X, Y, Z) in terms of ζ and

$\bar{\zeta}$ as

$$X = \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}} \qquad Y = \frac{\zeta - \bar{\zeta}}{i(1 + \zeta\bar{\zeta})} \qquad Z = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1} \tag{2.4}$$

with $T = 1$. However, it is clear from construction that the north pole N can't be assigned a finite ζ. To circumvent this problem, we will need to extend the complex plane Σ by also including the point at infinity $\{\infty\}$. Handling with infinities is not very convenient, so we ask whether we can assign finite number(s) to a point at infinity. For that, let us write

$$\zeta = \frac{\xi}{\eta} \tag{2.5}$$

with $N \equiv (\xi, \eta) = (1, 0)$. We can now express X, Y, Z in terms of ξ, η. Written this way, we see that any rescaling $(\xi, \eta) \to (\lambda\xi, \lambda\eta)$ for any non-zero λ preserves the form (2.5). To break the scale invariance, define a new set of coordinates $K = (T, X, Y, Z)$ as

$$T = \frac{1}{\sqrt{2}}(|\xi|^2 + |\eta|^2) \qquad X = \frac{1}{\sqrt{2}}(\xi\bar{\eta} + \bar{\xi}\eta) \tag{2.6}$$

$$Y = \frac{1}{i\sqrt{2}}(\xi\bar{\eta} - \eta\bar{\xi}) \qquad Z = \frac{1}{\sqrt{2}}(|\xi|^2 - |\eta|^2) \tag{2.7}$$

As we shall see, the pair of complex numbers (ξ, η) is our desired spinors. To prove that, we need to look at their transformation properties. From (2.6),(2.7) we can write each of the combinations $\xi\bar{\xi}$, $\xi\bar{\eta}$, $\eta, \bar{\xi}$, $\eta\bar{\eta}$ as linear combinations of T, X, Y, Z and represent them in the form of a matrix:

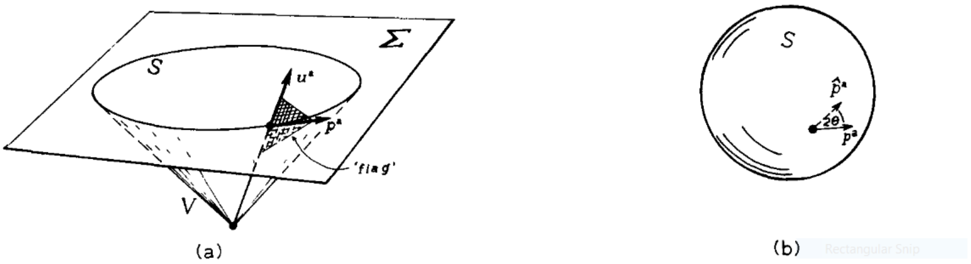
$$\frac{1}{\sqrt{2}}\begin{pmatrix} T + Z & X + iY \\ X - iY & T - Z \end{pmatrix} = \begin{pmatrix} \xi\bar{\xi} & \xi\bar{\eta} \\ \eta, \bar{\xi} & \eta\bar{\eta} \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} \tag{2.8}$$

Thus, any linear transformation of T, X, Y, Z will correspond to a linear transformation of ξ, η. Notice that the determinant of the left matrix is precisely the invariant line element (2.1). Thus, any transformation which will preserve this determinant will be our Lorentz transformation. For a Lorentz transformation T, X, Y, Z to $\tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z}$, let the desired transformation of (ξ, η) be

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \pm A \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{2.9}$$

where A is 2 matrix with complex entries. Substituting in () yields the unitary condition $AA^\dagger = I$. Choosing $\det A = 1$, we obtain that $A \in SL(2, \mathbb{C})$. Because there are two A's in (2.8), for each Lorentz transformation there can be two spinor transformation (2.9), one being the negative of other. This essentially reflects the fact that $SL(2, \mathbb{C})$ forms double cover of $SO^+(1, 3)$, the orthochronous Lorentz group. Indeed, double cover of any $SO(p, q)$ is the spin group $Spin(p, q)$. Thus $SL(2, \mathbb{C})$ is the spin group for $SO^+(1, 3)$.

So far we have only represented the position vector $\vec{K} = (T, X, Y, Z)$ on the null cone (thus 3 real degrees of freedom) with a pair of complex numbers (ξ, η) (which has 4 real degrees of freedom). Thus the spinor representation carries more information than just the coordinate on a null cone. Consider the diagram (), where we look at the tangent vector $\vec{L}$ of S^+ at P. Following the convention in [16], we write $\vec{L}$



One spatial dimension suppressed. Time dimension suppressed.

Fig. 2. (a) The spinor ξ_A defines a "null flag". This may be pictured as a polarisation vector tangent to the "celestial sphere" S. (b) shows how p^b is rotated when the phase of ξ_A is altered.

Figure 2.3: Flag pole and Flag plane of a 2-spinor (diagram obtained from R.Penrose and M.A.H.Callum, Twistor theory: an approach to quantisation of fields and space-time). Here the vector field $\vec{p}$ is essentially our tangent vector $\vec{L}$

in terms of $\zeta, \bar{\zeta}$ and require that they are invariant under spin transformation (2.9). One possible representation of such a vector is

$$\vec{L} = -\frac{1}{\sqrt{2}}\left(\eta^{-2}\frac{\partial}{\partial\zeta} + \bar{\eta}^{-2}\frac{\partial}{\partial\bar{\zeta}}\right) \tag{2.10}$$

An infinitesimally close point $P' \in S^+$ will have the form

$$P' = P + \epsilon\vec{L}(P) = P - \frac{1}{\sqrt{2}}\frac{\epsilon}{\eta^2} \tag{2.11}$$

Notice the non-trivial fact that if we consider transformation $(\xi, \eta) \to (e^{i\theta}\xi, e^{i\theta}\eta)$ for $\theta \in [0, \pi]$, the tip of $\vec{L}$ will make a complete rotation and return to its initial point P'. However, (ξ, η) reverses it's sign. We can represent this pictorially using the idea of *flag planes*. By that we mean the set

$$\Pi(P) = \{a\vec{K} + b\vec{L} | a, b \in \mathbb{R}^+\} \tag{2.12}$$

If we know $\vec{K}$, we can find the pair (ξ, η) upto a phase factor, while knowing $\vec{L}$ gives us information about η^2 and therefore η and ξ upto an overall sign. Thus, apart from the position P on a null cone (3 d.o.f.), the spinor representation also tells us about the *flag plane* orientation $e^{i\theta}$ upto an overall sign (1 d.o.f.). The vector $\vec{K}$ is called the *flag pole*

2.1.2 Constructing tensor algebra from spinors

In § we found the quantity $\zeta = (\xi, \eta) \in \mathbb{C}^2$ which transform as a spinor when null vector $\vec{K}$ undergoes Lorentz transformation. If $V^a = (V^0, V^1, V^2, V^3)$ denotes a generic null vector, then we can re-write relation (2.8) in the form

$$V^{AA'} = \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix} = \frac{1}{\sqrt{2}} V^a \sigma_a \equiv V^a {\sigma_a}^{AA'} \quad (2.13)$$

where the capital indices $A,\ A'$ takes binary value $0, 1$ each. The σ^a are the Pauli-matrices. Defining $v^A = (\xi, \eta)$ and using (2.8), we see that the left most matrix in the above expression can be written as tensor product of v^A and it's complex conjugate:

$$V^{AA'} = v^A \bar{v}^{A'} \quad (2.14)$$

Note that we can only identify v^A upto a phase factor. Given that null vectors V^a form complete basis in $\mathbb{M}$, we should be able to associate any element $V^a \in T_P\mathbb{M} \cong_{iso} \mathbb{M}$ with set of 2-spinors $\{v_i^A\} \subset \mathbb{S}$ (space of all un-primed spinors) identified upto a phase factor. Similar argument also holds for spinors in a curved space-time $\mathcal{M}$. Thus we have the isomorphism

$$T_p\mathcal{M} \cong_{iso} \mathbb{S} \otimes \mathbb{S}' \quad (2.15)$$

The spin space $\mathbb{S}$ can be equipped with three basic operations:

- *scalar multiplication*: $\forall \lambda \in \mathcal{G}$ and $v^A \in \mathbb{S}^A$, $\lambda v^A \in \mathbb{S}^A$
- *addition*: $\forall v^A, w^A \in \mathbb{S}^A$, $v^A + w^A \in \mathbb{S}^A$
- *inner product*: an *anti-symmetric* bi-linear map $\{\cdot, \cdot\} : \mathbb{S} \times \mathbb{S} \to \mathcal{G}$ defined as

$$\{v, w\} = \det[v\ w] \quad (2.16)$$

- *Complex conjugation*: $\forall v^A \in \mathbb{S}^A$ we have $\bar{v^A} = \bar{v}^{A'} \in \mathbb{S}'$.

The anti-symmetric nature of inner-product is clear since on interchange of columns in determinant changes the overall sign. Also notice that when both v and w undergoes the spinor transformation (2.9), the inner-product is preserved, i.e. the value is Lorentz invariant. This motivates us to define the *epsilon spinors*:

$$\epsilon_{AB} v^A w^B = \{v, w\} \quad (2.17)$$

In fact, since RHS is index free (i.e. just an element of $\mathcal{G}$), we can use the above relation to define contraction of spinor indices.

$$\epsilon_{AB} v^A w^B = v_B w^B \leftrightarrow v_B = v^A \epsilon_{AB} \quad (2.18)$$

In general, we have the relation

$$\psi^{\mathcal{C}A} = \epsilon^{AB} {\psi^{\mathcal{C}}}_B = -{\psi^{\mathcal{C}}}_B \epsilon^{BA} \quad (2.19)$$

$\mathcal{C}$ being an arbitrary combination of spinor indices. We have used the fact the underlying module $\mathcal{G}$ is *Grassmann even*, so the relative position of each individual spinor is not relevant. We can also have epsilon spinors with mixed indices:

$$\epsilon_{AB} \epsilon^{CB} = -\epsilon^{CB} \epsilon_{BA} = -{\epsilon^C}_A = {\epsilon_A}^C \quad (2.20)$$

Using above two identities, one can show the general 'see-saw' symmetry of a spinor χ:

$$\chi^{\cdots}{}_{\cdots A}{}^{\cdots}{}_{\cdots}{}^{A} = -\chi^{\cdots}{}_{\cdots}{}^{A\cdots}{}_{\cdots A} \tag{2.21}$$

Another intermediate result which follows from the above definition (and will be relevant in later discussions) is

$$\epsilon_A{}^C\epsilon_B{}^D - \epsilon_B{}^C\epsilon_A{}^D = \epsilon_{AB}\epsilon^{CD} \tag{2.22}$$

Transvecting with a spinor of the form $\phi_{\mathcal{D}CD}$ yields

$$\phi_{\mathcal{D}AB} - \phi_{\mathcal{D}BA} = \phi_{\mathcal{D}C}{}^C\epsilon_{AB} \tag{2.23}$$

Complex conjugation can be used to derive all of the above relations for primed-spinors.

Given such a primed and un-primed spin space, equipped with operations considered above, we can now try to construct the tangent vector space, and in general some tensor-space of rank $\begin{pmatrix} p \\ q \end{pmatrix}$. In fact, the identification has been defined in (2.13).

$$V^{AA'} = V^a\sigma_a{}^{AA'} \tag{2.24}$$

The $\sigma_a{}^{AA'}$'s are called *Infeld-van der Waerden symbols* which allow us to go from tensor representation to spinor representation. In fact, these symbols satisfy the 'Clifford identity'. Let's start with two null vectors V, W. The inner-product between these two gives

$$g_{ab}V^aW^b = g_{ab}\sigma^a{}_{AA'}\sigma^b{}_{BB'}V^{AA'}W^{BB'} = \epsilon_{AB}\epsilon_{A'B'}V^{AA'}W^{BB'} \tag{2.25}$$

which allow us to associate

$$g_{ab} = \epsilon_{AB}\epsilon_{A'B'}\sigma_a{}^{AA'}\sigma_b{}^{BB'} \tag{2.26}$$

We can interchange a and b and add it to the original equation. On transvecting with ϵ spinors we can obtain the Clifford relation

$$\sigma_a{}^A{}_{A'}\sigma_{bB}{}^{A'} + \sigma_b{}^A{}_{A'}\sigma_{aB}{}^{A'} = -\epsilon_B{}^A g_{ab} \tag{2.27}$$

Since, we already have the isomorphism for vectors, we can extend it to a generic $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor. It is primarily due to the fact that a tensor can be defined as

$$T_{pq\cdots s}{}^{ab\cdots d} = \sum_i P_p^{(i)}Q_q^{(i)}\cdots S_s^{(i)}A_{(i)}^aB_{(i)}^b\cdots D_{(i)}^d \tag{2.28}$$

for some vectors $P, Q, S, \cdots$. This particular representation is isomorphic to the multi-linear map definition of a tensor, which is true when $\mathcal{G}$ is *totally reflexive* (i.e. finite basis for $\mathbb{S}$ and $\mathbb{S}'$) (see [16])

Since $\mathbb{S}$ is complex 2 dimensional, we can define a spinor basis o^A, ι^A satisfying some condition $f(o^A, \iota^A) = 0$. We can write $v^A = v^0 o^A + v^1 \iota^A$. If $f = 0 \leftrightarrow o_A \iota^A = 1$ then (o_A, ι^A) is called a *spin frame*. Now for any null vector, we can use (2.14) to write

$$V^a = \underbrace{\sigma_{AA'}{}^a o^A o^{A'}}_{l^a} v^0 \bar{v}^{0'} + \underbrace{\sigma_{AA'}{}^a o^A \iota^{A'}}_{m^a} v^0 \bar{v}^{1'} + \underbrace{\sigma_{AA'}{}^a \iota^A o^{A'}}_{\bar{m}^a} v^1 \bar{v}^{0'} + \underbrace{\sigma_{AA'}{}^a \iota^A \iota^{A'}}_{n^a} v^1 \bar{v}^{1'} \tag{2.29}$$

$$= V^{00'} l^a + V^{01'} m^a + V^{10'} \bar{m}^a + V^{11'} n^a \tag{2.30}$$

$$= \frac{1}{\sqrt{2}} \{V^0(l^a + n^a) + V^1(m^a + \bar{m}^a) + V^2 i(m^a - \bar{m}^a) + V^3(l^a - n^a)\} \tag{2.31}$$

$$= V^0 t^a + V^1 x^a + V^2 y^a + V^2 z^a \tag{2.32}$$

We have represented V^a, first in orthonormal null tetrads $(l, n, m, \bar{m})$ (2.29) and then in Minkowski tetrad (t, x, y, z). If $\{\mu_a{}^{AA'}\}$ represents Infeld van der Waerden symbols with respect to (t, x, y, z) then we readily see that $\mu_a{}^{AB'} = \frac{1}{\sqrt{2}}(I, -\vec{\sigma})$:

$$V^a \mu_a{}^{AA'} = (V^0 t^a + V^1 x^a + V^2 y^a + V^2 z^a) \mu_a{}^{AA'} = \frac{1}{\sqrt{2}} V^a \sigma_a \tag{2.33}$$

The reason why we have this relation with Pauli matrices in the first place is due to the definition of ζ we started with. Now, inverting the re-arrangements in (2.31) we can obtain the corresponding symbols for $(l, n, m, \bar{m})$:

$$\sigma_a{}^{CD'} = \delta^C_A \delta^{D'}_{A'} \tag{2.34}$$

For notational convenience, henceforth, we shall avoid writing $\sigma_a{}^{AA'}$ when writing tensors in terms of spinors.

2.1.3 Covariant derivatives and Curvatures

Let $\mathcal{G}^{P \cdots S'}_{B \cdots F'}$ be space of all spinor fields of the form $\chi^{P \cdots S'}_{B \cdots F'}$ with under-lying module $\mathcal{G}$. A *covariant derivative operator* can be defined by the map

$$\nabla_a : \mathcal{G}^{P \cdots S'}_{B \cdots F'} \to \mathcal{G}^{P \cdots S'}_{AA'B \cdots F'} \tag{2.35}$$

satisfying the properties

- *linearity condition* : $\nabla_a(\lambda \chi^{P \cdots S'}_{B \cdots F'} + \mu \xi^{P \cdots S'}_{B \cdots F'}) = \lambda \nabla_a \chi^{P \cdots S'}_{B \cdots F'} + \mu \nabla_a \xi^{P \cdots S'}_{B \cdots F'}$
- *Leibniz property* : $\nabla_a(\chi^{P \cdots S'}_{B \cdots F'} \xi^{P_1 \cdots S'_!}_{B_1 \cdots F'_!}) = \chi^{P \cdots S'}_{B \cdots F'} \nabla_a \xi^{P_1 \cdots S'_1}_{B_! \cdots F'_!} + \xi^{P_1 \cdots S'_!}_{B_! \cdots F'_!} \nabla_a \chi^{P \cdots S'}_{B \cdots F'}$
- *Real operator* : $\overline{\nabla_a \chi^{P \cdots S'}_{B \cdots F'}} = \nabla_a \bar{\chi}^{P' \cdots S}_{B' \cdots F}$

$\forall \chi^{P \cdots S'}_{B \cdots F'}, \xi^{P \cdots S'}_{B \cdots F'} \in \mathcal{G}^{P \cdots S'}_{B \cdots F'}$ and $\forall \lambda, \mu \in \mathbb{C}$.

Given such a definition, we can restrict attention to space of tensor fields $\mathcal{G}^{p \cdots s}_{b \cdots f}$. Let us consider two consecutive action of covariant derivative operators on some element of $\mathcal{G}^{p\cdots s}_{b\cdots f}$. Does the ordering of covariant derivative matters? Define the *commutator* Δ_{ab} as the operator

$$\Delta_{ab} = \nabla_a \nabla_b - \nabla_b \nabla_a \tag{2.36}$$

We observe from the definition of covariant derivative that $\forall A_{\mathcal{C}}, B_{\mathcal{C}} \in \mathcal{G}_{\mathcal{C}}$ and $Q_{\mathcal{D}} \in \mathcal{G}_{\mathcal{D}}$, the conditions

- *linearity condition* : $\Delta_{ab}(A_{\mathcal{C}} + B_{\mathcal{C}}) = \Delta_{ab} A_{\mathcal{C}} + \Delta_{ab} B_{\mathcal{C}}$
- *Leibniz property* : $\Delta_{ab}(A_{\mathcal{C}} Q_{\mathcal{D}}) = A_{\mathcal{C}} \Delta_{ab} Q_{\mathcal{D}} + Q_{\mathcal{D}} \Delta_{ab} A_{\mathcal{C}}$

are automatically satisfied. While the first condition follows since composition of linear operator is linear. The second condition holds because the cross terms essentially cancel out. If we now consider the tensor field $X^{ab} \in \mathcal{G}^{ab}$, then the operator

$$X^{ab} \Delta_{ab} : \mathcal{G} \to \mathcal{G} \tag{2.37}$$

satisfies both linearity condition and Leibniz condition. Same is true if we consider vector field $Y^a \nabla_a : \mathcal{G} \to \mathcal{G}$. The isomorphism is achieved between these two maps by defining the torsion field $T_{ab}{}^c \in \mathcal{G}_{ab}{}^c$, i.e. for each such X, there exists an unique Y such that $X^{ab} T_{ab}{}^c = Y^c$. This amounts to saying

$$\Delta_{ab} f = T_{ab}{}^c \nabla_c f \tag{2.38}$$

$\forall f \in \mathcal{G}$. The anti-symmetry of a, b implies $T_{ab}{}^c = -T_{ba}{}^c$. Thus, if we define a new commutator

$$D_{ab} = \Delta_{ab} - T_{ab}{}^c \nabla_c \tag{2.39}$$

then it follows naturally that $\forall f \in \mathcal{G}$, $D_{ab} f = 0$. Since both Δ_{ab} and ∇_a acting on a vector-field $\mathcal{G}^i$ are linear and satisfies Leibniz property, the same should be true for D_{ab}. Thus, $D_{ab} V^i$ must be linear combinations of V's. The coefficients are called *curvature tensor*:

$$D_{ab} V^i = (\Delta_{ab} - T_{ab}{}^c \nabla_c) V^i = R_{abf}{}^i V^f \tag{2.40}$$

We can extend the above identity for a general element of $H^{p\cdots s}_{c\cdots f} \in \mathcal{G}^{p\cdots s}_{c\cdots f}$ using the definition () and exploiting the linearity condition.

$$D_{ab} H^{p\cdots s}_{c\cdots f} = R_{abp_1}{}^p H^{p_1\cdots s}_{c\cdots f} + \cdots - R_{abc}{}^{c_1} H^{p\cdots s}_{c_1\cdots f} - \cdots \tag{2.41}$$

We can comment about symmetry properties of curvature tensors. Anti-symmetry of a, b means $R_{abf}{}^i = -R_{baf}{}^i$. Further, if $V^f = \nabla^f \alpha$, then for torsion-free case

$$R_{[abf]}{}^i \nabla_i \alpha \propto \nabla_{[a} \nabla_b \nabla_{c]} \alpha = 0 \tag{2.42}$$

thus $R_{[abf]}{}^i = 0$. Similar calculation for tensors of the form $\nabla_a \alpha^h$ gives the Bianchi identity

$$\nabla_{[a} R_{bc]f}{}^h = 0 \tag{2.43}$$

Are covariant derivative operators unique? If we had two such operators $\nabla_a, \tilde{\nabla}_a$ satisfying the conditions (2.35), so should their linear combinations. In fact we can write

$$(\tilde{\nabla}_a - \nabla_a) : \mathcal{G}^{P\cdots S'}_{B\cdots F'} \to \mathcal{G}^{P\cdots S'}_{AA'B\cdots F'} \tag{2.44}$$

We can now try to impose set of conditions and check how these two operators are related to one another:

- $\forall f \in \mathcal{G}$, $\tilde{\nabla}_a f = \nabla_a f$
  If we define
  $$\tilde{\nabla}_a f^C = \nabla_a f^C + \Theta_{aB}{}^C f^B \tag{2.45}$$
  $\forall f^C \in \mathcal{G}^C$ and $\Theta_{aB}{}^C \in \mathcal{G}^C_{aB}$, Then $\forall g_C \in \mathcal{G}_C$, $f^C g_C \in \mathcal{G}$. Thus we will obtain
  $$\tilde{\nabla}_a(f^C g_C) = \nabla_a(f^C g_C) \tag{2.46}$$
  $$\to \tilde{\nabla}_a g_C = \nabla_a g_C - \Theta_{aC}{}^B g_B \tag{2.47}$$
  In general we can have the relation
  $$\begin{aligned}\tilde{\nabla}_a X_{B\cdots F'}{}^{P\cdots S'} &= \nabla_a X_{B\cdots F'}{}^{P\cdots S'} - \Theta_{aB}{}^Y X_{Y\cdots F'}{}^{P\cdots S'} - \cdots - \bar{\Theta}^{Y'}_{aF'} X_{B\cdots Y'}{}^{P\cdots S'} \\ &- \cdots + \Theta_{aY}{}^P X_{B\cdots F'}{}^{Y\cdots S'} + \cdots + \bar{\Theta}^{S'}_{aY'} X_{B\cdots F'}{}^{P\cdots Y'} + \cdots\end{aligned}$$

- Both $\tilde{\nabla}_a, \nabla_a$ are *covariantly constant*, meaning that $\tilde{\nabla}_a = \nabla_a \epsilon_{BC} = 0$, which amounts to
  $$\Theta_{aBC} = \Theta_{aCB} \tag{2.48}$$

- The torsion free condition leads to
  $$\epsilon_{B'C'}\Theta_{A'ABC} + c.c. = \epsilon_{A'C'}\Theta_{B'BAC} + c.c. \tag{2.49}$$
  Symmetrize over A, B, C and contract with $\epsilon^{B'C'}$ to get
  $$\Theta_{A'(ABC)} = 0 \tag{2.50}$$

The last condition allows us to write

$$\Theta_{aBC} = \Lambda_a \epsilon_{BC} + \Upsilon_{A'B}\epsilon_{AC} \tag{2.51}$$

Performing symmetrization over A, C and A', C' allow us to obtain

$$\Lambda_a + \bar{\Lambda}_a = 0 \leftrightarrow \Lambda_a = i\Pi_a \tag{2.52}$$

for real Π_a. While, symmetrization over A, C and B', C' gives

$$\Upsilon_a = \bar{\Upsilon}_a \tag{2.53}$$

Thus, the two covariant derivatives are unique upto the connection term $\Theta_{aB}{}^C = i\Pi_a \epsilon_B{}^C + \Upsilon_{A'B}\epsilon_A{}^C$.

2.1.4 Curvatures in (pseudo-)Riemannian Manifold

The purpose of this section is to come up with spinor version of curvature tensors by considering all of it's symmetries. Consider the expression

$$R_{abcd} = R_{AA'BB'CC'DD'} \tag{2.54}$$

Because Riemann tensor is anti-symmetric in a, b and c, d separately, we can use (2.23) for the two pairs separately to write in terms of symmetric spinors. For simplicity, if we had an anti-symmetric tensor $F_{ab} = F_{ABA'B'}$, then we could write it in the form

$$F_{ab} = \frac{1}{2}(F_{ABA'B'} - F_{ABB'A'}) + \frac{1}{2}(F_{ABA'B'} + F_{ABB'A'}) \tag{2.55}$$

$$= \frac{1}{2}(F_{ABA'B'} - F_{ABB'A'}) + \frac{1}{2}(F_{ABB'A'} - F_{BAB'A'}) \tag{2.56}$$

$$= \underbrace{\frac{1}{2}F_{ABC'}{}^{C'}}_{\phi_{AB}}\epsilon_{A'B'} + \underbrace{\frac{1}{2}F_{A'B'C}{}^{C}}_{\psi_{A'B'}}\epsilon_{AB} \tag{2.57}$$

where in (2.56) we used $F_{ABA'B'} = -F_{BAB'A'}$ and (2.23) in (2.56). ϕ_{AB} and $\psi_{A'B'}$ are symmetric spinors (which follows since F_{ab} and ϵ_{AB} are anti-symmetric). Similar argument for R_{abcd} will yield:

$$R_{abcd} = X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \tag{2.58}$$

$$+ \bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \tag{2.59}$$

where

$$X_{ABCD} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'} \qquad \Phi_{ABC'D'} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{YC'}{}^{Y}{}_{D'} \tag{2.60}$$

with symmetries $X_{ABCD} = X_{(AB)(CD)}$ and $\Phi_{ABC'D'} = \Phi_{(AB)(C'D')}$. The interchange symmetry $R_{abcd} = R_{cdab}$ further leads to $X_{ABCD} = X_{CDAB}$ and $\bar{\Phi}_{ABC'D'} = \Phi_{ABC'D'}$. This essentially puts the reality condition on $\Phi_{ab} = \Phi_{ABA'B'}$. The cyclic property $R_{[abc]d} = 0$ implies (after an long calculation) that

$$X_{AB}{}^{B}{}_{C}\epsilon_{A'C'} = \bar{X}^{B'}_{A'B'C'}\epsilon_{AC} \leftrightarrow \Lambda = \bar{\Lambda} \tag{2.61}$$

where $\Lambda = \frac{1}{6}X_{AB}{}^{AB}$. Contracting with $\epsilon^{A'C'}$ in above equation, we can write

$$X_{ABC}{}^{B} = 3\Lambda\epsilon_{AC} \tag{2.62}$$

It is now possible to express Ricci tensor $R_{ac} = R_{abcd}g^{bd}$ in terms of Λ and Φ_{ac}. The final expression for Ricci tensor will look like

$$R_{ab} = 6\Lambda\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ab} \tag{2.63}$$

Due to symmetry of AB indices, Φ_{ab} is traceless. Thus on contracting with g^{ab}, we get Ricci scalar R in terms of Λ as

$$\Lambda = \frac{R}{24} \qquad \Phi_{ab} = -\frac{1}{2}\left(R_{ab} - \frac{1}{4}g_{ab}R\right) \tag{2.64}$$

Thus Φ_{ab} is related to trace-less part of Ricci tensor, hence $\Phi_{ABA'B'}$ is also called Ricci spinor.

Finally, for X_{ABCD} we may write it in the form

$$X_{ABCD} = \frac{1}{3}(X_{ABCD} + X_{ACDB} + X_{ADCB}) + \frac{1}{3}(X_{ABCD} - X_{ACDB}) + \frac{1}{3}(X_{ABCD} - X_{ADCB}) \tag{2.65}$$

$$= X_{(ABCD)} + \frac{1}{3}\epsilon_{BC}X_{AE}{}^{E}{}_{D} + \frac{1}{3}\epsilon_{BD}X_{AEC}{}^{E} \tag{2.66}$$

$$= \Psi_{ABCD} + \Lambda(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}) \tag{2.67}$$

where in (2.65) we used the symmetries of X_{ABCD} and the definition of Λ. $\Psi_{ABCD} = X_{(ABCD)}$ is called gravitational spinor or Weyl curvature spinor. Substituting (2.63) and (2.67) in (2.54) we obtain

$$R_{abcd} = \Psi_{ABCD}\ \epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\ \epsilon_{AB}\epsilon_{CD} \tag{2.68}$$

$$+ \Phi_{ABC'D'}\ \epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\ \epsilon_{AB}\epsilon_{C'D'} \tag{2.69}$$

$$+ 2\Lambda(\epsilon_{AC}\epsilon_{A'C'}\epsilon_{BD}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{A'D'}\epsilon_{BC}\epsilon_{B'C'}) \tag{2.70}$$

Combination (2.68) is the Weyl tensor C_{abcd}, (2.69) is the contains the traceless part of Ricci tensor , denote this as E_{abcd} while (2.70) is $2\Lambda(g_{ac}g_{bd} - g_{ad}g_{bc})$

2.1.5 Field equations and NP equations

Using equations (2.64), we may write

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = -6\Lambda g_{ab} - 2\Phi_{ab} \tag{2.71}$$

Thus the Einstein's field equation

$$G_{ab} + \lambda g_{ab} = -8\pi G T_{ab} \tag{2.72}$$

can be written in terms of Φ_{ab} and Λ as

$$\Phi_{ab} + (3\Lambda - \frac{1}{2}\lambda)g_{ab} = 4\pi G T_{ab} \tag{2.73}$$

For vacuum space-times, both Φ_{ab} and Λ vanish identically, though the Weyl part Ψ_{ABCD}. Thus it is desirable to express Einstein's field equations in terms of the Weyl curvature. Written this way, we shall see the vacuum field equations closely resembles the free Maxwell's or free spin-1/2 field equations. For that we begin with the Bianchi identity

$$\nabla_{[a}R_{bc]de} = 0 \tag{2.74}$$

Now we need to substitute (2.58), (2.59) in above equation. After much simplification, we are left with

$$\nabla^{A}_{B'}X_{ABCD} = \nabla^{A'}_{B}\Phi_{CDA'B'} \tag{2.75}$$

Next we write use (2.67) and Levi-Civita definition of ∇ to write

$$\nabla^A_{B'}\Psi_{ABCD} = \nabla^{A'}_B\Phi_{CDA'B'} - 2\epsilon_{B(C}\nabla_{D)B'}\Lambda \tag{2.76}$$

If we take symmetrization over B, C, D we get

$$\nabla^A_{B'}\Psi_{ABCD} = \nabla^{A'}_{(B}\Phi_{CD)A'B'} \tag{2.77}$$

Further we could contract both sides with ϵ^{BC} in (2.76). The Weyl part in LHS will vanish identically and we will obtain

$$\nabla^{CA'}\Phi_{CDA'B'} + 3\nabla_{DB'}\Lambda = 0 \tag{2.78}$$

Using (2.64), we see that the above expression is essentially the spinor equivalent of divergence free condition of Einstein tensor. From the field equation (2.72), this is equivalent to conservation equation $\nabla^a T_{ab} = 0$. In (2.77), we can write Φ in terms of stress-energy tensor T to obtain

$$\nabla^A_{B'}\Psi_{ABCD} = 4\pi G\nabla^{A'}_{(B}T_{CD)A'B'} \tag{2.79}$$

In lambda-vacuum scenario, we can set $T_{ab} = 0$

$$\nabla^{AB'}\Psi_{ABCD} = 0 \leftrightarrow \nabla^a C_{abcd} = 0 \tag{2.80}$$

2.1.5.1 Anti-commutation relations in spinor notation

Before going into Newman-Penrose equations, it is necessary to comment about the commutation relations in spinor notations. We will let torsion to be zero, so that $D_{ab} = \Delta_{ab}$. Following (2.55)-(2.57), we can decompose

$$\Delta_{ab} = \epsilon_{A'B'}\Box_{AB} + \epsilon_{AB}\Box_{A'B'} \tag{2.81}$$

where $\Box_{AB} = \nabla_{X'(A}\nabla_{B)}{}^{X'}$. Now consider the commutation relation for a bi-vector k^{ab} in (2.41). If we let $k^{ab} = k^A k^B \epsilon^{A'B'}$, then commutation relation gives

$$k^{(C}\Delta_{ab}k^{D)} = \{\epsilon_{A'B'}X_{ABE}{}^{(C} + \epsilon_{AB}\Phi_{A'B'E}{}^{(C}\}k^{D)}k^E \tag{2.82}$$

which can be simplified to give

$$\Delta_{ab}k^C = \{\epsilon_{A'B'}X_{ABE}{}^C + \epsilon_{AB}\Phi_{A'B'E}{}^C\}k^E \tag{2.83}$$

On comparing the coefficients of ϵ_{AB} and $\epsilon_{A'B'}$ we get

$$\Box_{AB}k^C = X_{ABE}{}^C k^E \qquad \Box_{A'B'}k^C = \Phi_{A'B'E}{}^C k^E \tag{2.84}$$

We can express X in terms of Ψ and Λ. On taking appropriate symmetries we find:

$$\Box_{(AB}k_{C)} = \Psi_{ABCD}k^D \tag{2.85}$$

$$\Box_{AB}k^B = -3\Lambda k_A \tag{2.86}$$

2.1.5.2 Newman-Penrose equations

Newman-Penrose equations are essentially the commutator identities (2.83) expressed in terms of set of first order differential equations involving complex scalar fields. These complex scalar fields are defined with respect to the spin frame $\{o_A, \iota^A\}$. Refer to the section (2.1.2) where we wrote 4-vectors in terms of spin frame. Further, equation (2.28) allows us to extend this relation for arbitrary tensors and in general spinors. Let there be a spinor field $\Xi_{AB\cdots LM'\cdots N'} = \Xi_{(AB\cdots L)(M'\cdots N')}$. We can expand it in polynomials of o_A, ι_A and it's complex conjugates:

$$\Xi_{\underbrace{A\cdots L}_{p}\underbrace{M'\cdots N'}_{q}} = \sum_{0\le p_1\le p}\sum_{0\le q_1\le q}(-1)^{(p+q)-(p_1+q_1)}\Xi_{p_1q_1}\underbrace{o_A\cdots o_P}_{p_1}\underbrace{\iota_Q\cdots\iota_L}_{p-p_1}\underbrace{o_{M'}\cdots o_{S'}}_{q_1}\underbrace{\iota_{T'}\cdots\iota_{N'}}_{q-q_1} \tag{2.87}$$

where the complex scalar field

$$\Xi_{p_1q_1} = \Xi_{\underbrace{1\cdots 1}_{p_1}\underbrace{0\cdots 0}_{p-p_1}\underbrace{1'\cdots 1'}_{q_1}\underbrace{0'\cdots 0'}_{q-q_1}} \tag{2.88}$$

likewise, the covariant derivative operator can be written as

$$\nabla_a = o_A o_{A'} D' - o_A \iota_{A'} \delta' - \iota_A o_{A'} \delta + \iota_A \iota_{A'} D \tag{2.89}$$

where D, D', δ, δ' are directional derivatives along null tetrads $l^a, n^a, m^a, \bar{m}^a$ respectively. Further, if we expand covariant derivative of a spinor , we will come across combinations such as $o^A D o_A, \iota^A \delta o_A, \cdots$ etc. These combinations or scalar functions are called *spin-coefficients*

$$\kappa = o^A D o_A \qquad \varepsilon = \iota^A D o_A \qquad \gamma' = -o^A D\iota_A \qquad \tau' = -\iota^A D\iota_A \tag{2.90}$$
$$\rho = o^A \delta' o_A \qquad \alpha = \iota^A \delta' o_A \qquad \beta' = -o^A \delta'\iota_A \qquad \sigma' = -\iota^A \delta'\iota_A \tag{2.91}$$
$$\sigma = o^A \delta o_A \qquad \beta = \iota^A \delta o_A \qquad \alpha' = -o^A \delta\iota_A \qquad \rho' = -\iota^A \delta\iota_A \tag{2.92}$$
$$\tau = o^A D' o_A \qquad \gamma = \iota^A D' o_A \qquad \varepsilon' = -o^A D'\iota_A \qquad \kappa' = -\iota^A D'\iota_A \tag{2.93}$$

With these definitions, one can express directional derivatives of the spinor dyads in the form

$$Do^A = \varepsilon o^A - \kappa\iota^A \qquad D\iota^A = \gamma'\iota^A - \tau' o^A \tag{2.94}$$
$$\delta' o^A = \alpha o^A - \rho\iota^A \qquad \delta'\iota^A = \beta'\iota^A - \sigma' o^A \tag{2.95}$$
$$\delta o^A = \beta o^A - \sigma\iota^A \qquad \delta\iota^A = \alpha'\iota^A - \rho' o^A \tag{2.96}$$
$$D' o^A = \gamma o^A - \tau\iota^A \qquad D'\iota^A = \varepsilon'\iota^A - \kappa' o^A \tag{2.97}$$

along with their complex conjugates. For completeness, let us also define the curvature scalars. The Ricci scalars (not to be confused with R) are defined by

$$\Phi_{p_1q_1} = \Phi_{\underbrace{1\cdots 1}_{p_1}\underbrace{0\cdots 0}_{2-p_1}\underbrace{1'\cdots 1'}_{q_1}\underbrace{0'\cdots 0'}_{2-q_1}} \tag{2.98}$$

Likewise, the Weyl scalars are given by

$$\Psi_{p_1} = \Psi_{\underbrace{1\cdots1}_{p_1}\underbrace{0\cdots0}_{4-p_1}} \tag{2.99}$$

The Newman-Penrose equations are obtained by setting $k^A = (o^A, \iota^A)$, so that the commutator in (2.83) becomes a first-order relation with respect to the spin-coefficients as defined above. The complete set of equations are

$$D\rho - \delta'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa(\tau' + 2\alpha + \bar{\beta} - \beta') + \rho(\varepsilon + \bar{\varepsilon}) + \Phi_{00} \tag{2.100}$$
$$D'\rho' - \delta\kappa' = \rho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau' - \kappa'(\tau + 2\alpha' + \bar{\beta}' - \beta) + \rho'(\varepsilon' + \bar{\varepsilon}') + \Phi_{22} \tag{2.101}$$

$$D\sigma - \delta\kappa = \sigma(\rho + \bar{\rho} + \bar{\gamma}' - \gamma' + 2\epsilon) - \kappa(\tau + \bar{\tau}' + \alpha - \alpha' + 2\beta) + \Psi_0 \tag{2.102}$$
$$D'\sigma' - \delta'\kappa' = \sigma'(\rho' + \bar{\rho}' + \bar{\gamma} - \gamma + 2\epsilon') - \kappa'(\tau' + \bar{\tau} + \alpha' - \alpha + 2\beta') + \Psi_4 \tag{2.103}$$

$$D\tau - D'\kappa = \rho(\tau - \bar{\tau}') + \sigma(\bar{\tau} - \tau') + \tau(\bar{\gamma}' + \varepsilon) - \kappa(\bar{\gamma} + 2\gamma - \varepsilon') + \Psi_1 + \Phi_{01} \tag{2.104}$$
$$D'\tau' - D\kappa' = \rho'(\tau' - \bar{\tau}) + \sigma'(\bar{\tau}' - \tau) + \tau'(\bar{\gamma} + \varepsilon') - \kappa'(\bar{\gamma}' + 2\gamma' - \varepsilon) + \Psi_3 + \Phi_{21} \tag{2.105}$$

$$\delta\rho - \delta'\sigma = \tau(\rho - \bar{\rho}) + \kappa(\bar{\rho}' - \rho') + \rho(\bar{\alpha} + \beta) - \sigma(\bar{\alpha}' + 2\alpha - \beta') - \Psi_1 + \Phi_{01} \tag{2.106}$$
$$\delta'\rho' - \delta\sigma' = \tau'(\rho' - \bar{\rho}') + \kappa'(\bar{\rho} - \rho) + \rho'(\bar{\alpha}' + \beta') - \sigma'(\bar{\alpha} + 2\alpha' - \beta) - \Psi_2 + \Phi_{21} \tag{2.107}$$

$$\delta\tau - D'\sigma = -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' + \tau(\beta + \bar{\beta}) - \sigma(2\gamma - \varepsilon' + \bar{\varepsilon}') + \Phi_{02} \tag{2.108}$$
$$\delta'\tau' - D\sigma' = -\rho\sigma' - \bar{\sigma}\rho' + \tau'^2 + \kappa'\bar{\kappa} + \tau'(\beta' + \bar{\beta}') - \sigma'(2\gamma' - \varepsilon + \bar{\varepsilon}) + \Phi_{20} \tag{2.109}$$

$$D'\rho - \delta'\tau = \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' + \rho(\gamma + \bar{\gamma}) - \tau(\alpha + \bar{\alpha}') - \Psi_2 - 2\Lambda \tag{2.110}$$
$$D\rho' - \delta\tau' = \rho\bar{\rho} + \sigma'\sigma - \tau'\bar{\tau}' - \kappa'\kappa + \rho'(\gamma' + \bar{\gamma}') - \tau'(\alpha' + \bar{\alpha}) - \Psi_2 - 2\Lambda \tag{2.111}$$

$$D'\beta - \delta\gamma = \tau\rho' - \kappa'\sigma - \bar{\kappa}'\varepsilon + \alpha\bar{\sigma}' + \beta(\rho' + \bar{\varepsilon}' + \gamma) - \gamma(\bar{\beta}' + \alpha' + \tau) - \Phi_{12} \tag{2.112}$$
$$D\beta' - \delta'\gamma' = \tau'\rho - \kappa\sigma' - \bar{\kappa}\varepsilon' + \alpha'\bar{\sigma} + \beta(\rho + \bar{\varepsilon} + \gamma') - \gamma'(\bar{\beta} + \alpha + \tau') - \Phi_{10} \tag{2.113}$$

$$\delta'\varepsilon - D\alpha = \tau'\rho - \kappa\sigma' + \bar{\kappa}\gamma - \beta\bar{\sigma} - \alpha(\rho + \bar{\varepsilon} + \gamma') + \varepsilon(\bar{\beta} + \alpha + \tau') - \Phi_{10} \tag{2.114}$$
$$\delta\varepsilon' - D'\alpha' = \tau\rho' - \kappa'\sigma + \bar{\kappa}'\gamma' - \beta'\bar{\sigma}' - \alpha'(\rho' + \bar{\varepsilon}' + \gamma) + \varepsilon'(\bar{\beta}' + \alpha' + \tau) - \Phi_{12} \tag{2.115}$$

$$D\beta - \delta\varepsilon = \kappa(\rho' - \gamma) - \sigma(\tau' - \alpha) + \beta(\bar{\rho} + \bar{\gamma}') - \varepsilon(\bar{\tau}' + \bar{\alpha}) + \Psi_1 \tag{2.116}$$
$$D'\beta' - \delta'\varepsilon' = \kappa'(\rho - \gamma') - \sigma'(\tau - \alpha') + \beta'(\bar{\rho}' + \bar{\gamma}) - \varepsilon'(\bar{\tau} + \bar{\alpha}') + \Psi_3 \tag{2.117}$$

$$\delta'\gamma - D'\alpha = \kappa'(\rho+\varepsilon) - \sigma'(\tau+\beta) - \alpha(\bar{\rho}'+\bar{\gamma}) + \gamma(\bar{\tau}+\bar{\alpha}) + \gamma\beta' - \varepsilon'\alpha + \Psi_3 \quad (2.118)$$

$$\delta\gamma' - D\alpha' = \kappa(\rho'+\varepsilon') - \sigma(\tau'+\beta') - \alpha'(\bar{\rho}+\bar{\gamma}') + \gamma'(\bar{\tau}'+\bar{\alpha}') + \gamma'\beta - \varepsilon\alpha' + \Psi_1 \quad (2.119)$$

$$D\gamma - D'\varepsilon = \kappa\kappa' - \tau\tau' - \beta(\tau'-\bar{\tau}) - \alpha(\bar{\tau}'-\tau) - \varepsilon(\gamma+\bar{\gamma}) + \gamma(\gamma'+\bar{\gamma}') + \Psi_2 - \Phi_{11} - \Lambda \quad (2.120)$$

$$D'\gamma' - D\varepsilon' = \kappa'\kappa - \tau'\tau - \beta'(\tau-\bar{\tau}') - \alpha'(\bar{\tau}-\tau') - \varepsilon'(\gamma'+\bar{\gamma}') + \gamma'(\gamma+\bar{\gamma}) + \Psi_2 - \Phi_{11} - \Lambda \quad (2.121)$$

$$\delta'\beta - \delta\alpha = \rho\rho' - \sigma\sigma' - \alpha\bar{\alpha} + \beta\bar{\alpha}' + \alpha(\beta-\alpha') + \gamma(\bar{\rho}-\rho) + \varepsilon(\rho'-\bar{\rho}') + \Psi_2 - \Phi_{11} - \Lambda \quad (2.122)$$

$$\delta\beta' - \delta'\alpha' = \rho'\rho - \sigma\sigma' - \alpha'\bar{\alpha}' + \beta'\bar{\alpha} + \alpha'(\beta-\alpha) + \gamma'(\bar{\rho}'-\rho') + \varepsilon'(\rho-\bar{\rho}) + \Psi_2 - \Phi_{11} - \Lambda \quad (2.123)$$

Chapter 3

Conformal Methods in General Relativity

Conformal transformations refers to (local) angle-preserving transformation. The conformal methods in general relativity is originally due to the seminal work by Sir R. Penrose in 1963 [5]. The primary motivation for such a technique was to address the following questions:

1. A co-ordinate free, geometrical definition of Asymptotically flat space-times
2. Covariant definition of incoming and outgoing radiation for spinor fields
3. Conditions for Peeling properties of Riemann tensor and various spinor fields
4. Causality and Cauchy problem in general relativity
5. Understanding the Bondi-Sachs mass loss due to outgoing radiation
6. A geometrical definition of Bondi-Metzner-Sachs group

The basic idea here is that, given a *physical* space-time $\tilde{\mathcal{M}}$, we can perform certain conformal transformations such that the "point at infinity" can be given a "finite description" and is regular. We "squash" the infinity of $\tilde{\mathcal{M}}$, making it an ordinary 3-dimensional boundary (denoted $\mathscr{I}$) of $\mathcal{M}$. The *unphysical* metric g_{ab} is finite on $\mathscr{I}$ and smooth (to some degree). This allows us to look at behaviour of Einstein's field equations in the neighborhood of this boundary. A number of important concepts in general relativity enjoys conformal invariance; such as transformation of a generic free mass-less spinor field (Weyl curvature being a particular example), null congruence, causal structure of space-time etc. We can study these ideas near the boundary using this technique. In general, conformal method serves as an important tool to study asymptotic properties in general relativity.

In this chapter, we are going to look at the conformal treatment to address the questions (1), (2) and (3). In §3.1 and §3.2, we will introduce some basic definitions related to the conformal method and look at transformation properties of curvature spinors. We will also consider the conformal extensions of some simple solutions of Einstein's field equations (i.e. we will construct $\mathcal{M}$ and boundary $\mathscr{I}$ from some exact solutions $\tilde{\mathcal{M}}$).

In §3.3 we will introduce the concept of *asymptotic simplicity* and its consequence. A co-ordinate free definition of asymptotically flat space-time will be introduced (this is the definition used by R. Penrose and by H. Bondi et al. and R. K. Sachs, although there are minute variations in this definition in other literatures*) and the integrability conditions will be considered near the boundary. Finally in §3.4, we will address the Peeling theorem.

3.1 Basics of Conformal Geometry

As preliminaries, it is necessary that we define *conformal transformations.* We will use the Greek alphabets $\alpha,\ \beta,\ \gamma,\cdots$ to represent the co-ordinate indices and lower latin characters $a,\ b,\ c,\cdots$ to represent the *abstract* indices.

3.1.1 Conformal Rescaling and Transformation

Conformal rescaling or **Weyl transformation** refers to the replacement:

$$\tilde{g}_{\alpha\beta} \to g_{\alpha\beta} = \Omega^2 \tilde{g}_{\alpha\beta} \tag{3.1}$$

where Ω is a positive definite smooth function in $\tilde{\mathcal{M}}$. The scalar Ω is called the *conformal factor.* This is not a point-wise transformation (i.e. $\tilde{x}^\alpha = x^\alpha$), so the infinitesimal line element changes by $ds \to \Omega ds$. The generators of this transformation forms an infinite-dimensional Abelian Lie group.

We can define **conformal class** of a metric $[\tilde{g}]$ to be the set of all metrices g, which are related to $\tilde{g}$, following equation (3.1). Note that (3.1) is required to preserve the identity relation:

$$\tilde{g}_{\alpha\beta}\tilde{g}^{\beta\gamma} = \tilde{g}_\alpha^\gamma \qquad\qquad g_{\alpha\beta}g^{\beta\gamma} = g_\alpha{}^\gamma \tag{3.2}$$

Thus, the inverse metric should satisfy

$$\tilde{g}^{\alpha\beta} \to g_{\alpha\beta} = \Omega^{-2}\tilde{g}^{\alpha\beta} \tag{3.3}$$

In contrast, a **conformal transformation** or *conformal mapping* refers to the diffeomorphism transformation $\mu : \tilde{\mathcal{M}} \to \mathcal{M}$ such that the pull-back of g is conformally related to $\tilde{g}$ for some scalar $\Omega > 0$:

$$\mu^* g_{\alpha\beta} = \Omega^2 \tilde{g}_{\alpha\beta} \tag{3.4}$$

Note that definitions (3.1) and (3.4) can be written compactly using the *abstract* indices as

$$g_{ab} = \Omega^2 \tilde{g}_{ab} \tag{3.5}$$

For that, we use the relation between *abstract* and *coordinate* indices [16] The *epsilon* spinors will therefore transform as

$$\epsilon_{AB} = \Omega \tilde{\epsilon}_{AB} \qquad \epsilon^{AB} = \Omega^{-1} \tilde{\epsilon}^{AB} \tag{3.6}$$

As an illustration, let us look at conformal mapping of Minkowski space-time $\mathbb{M}$ to itself. We note that the general rule of transformation of co-ordinates in this case to be given by (see [17])

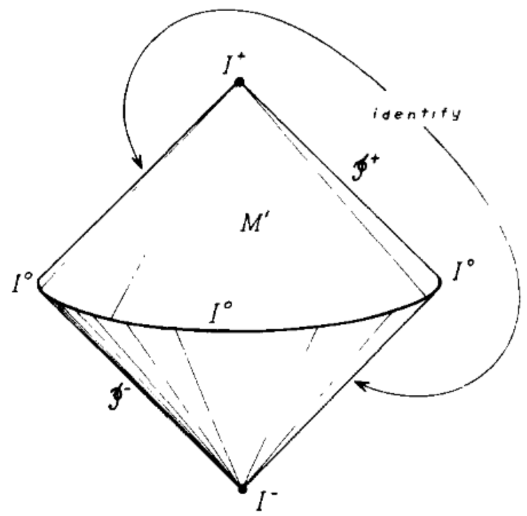


Figure 3.1: Compactified Minkowski space-time where we identify $\mathcal{I}^+$ with $\mathcal{I}^-$

$$X_c = P_c + M_{bc}x^b + Ax_c + (2B_b x^b x_c - B_c x^b x_b) \tag{3.7}$$

defined by 15-parameter Lie group: P_a and M_{ab} constitutes the usual 10-parameter Poincare group; A denotes the dilation and B_a corresponds to special conformal transformation. Note that, even if we just consider the integral curves for non-zero B_a, the transformation is not well-defined at all points:

$$X^a = x^a(s) = \frac{x^a(0) - sB^a x^b(0) x_b(0)}{1 - 2sB^b x_b(0) + s^2 B^b B_b x^c(0) x_c(0)} \tag{3.8}$$

X^a reaches infinity for finite s and it is not even a point of $\mathbb{M}$. Thus, with regards to the mapping μ considered above, we see that $\tilde{\mathcal{M}} = \mathbb{M}$ but $\mu(\mathbb{M}) \subset \mathcal{M}$. In this case, we say that $\mathcal{M}$ is a **conformal extension** of $\mathbb{M}$. To make sense of transformation (3.8), we will somehow need to incorporate the point at infinity by giving it a finite co-ordinate description. The details will be discussed in the later sections, but the solution for the above problem is to define $\mathcal{M} = \mathbb{M}^\#$ (the **compactified** Minkowski space-time). We can visualize $\mathbb{M}^\#$ consisting of two null cones joined base to base.

The interior is the standard Minkowski space-time. The null cones themselves define the boundary $\mathscr{I}$, while the top and bottom vertices along with the equator representing the three points at infinity (denoted by i^+, i^-, i^0 respectively). **Compactification**, here, is done by identifying the bottom cone ($\mathscr{I}^-$) with the top cone ($\mathscr{I}^+$) and replace the three points $i^{\pm}, i^0$ by a single interior point I (see the diagrams below). The result is a compact conformal manifold. In general, a **conformal compactification** of $(\tilde{\mathcal{M}}, \tilde{g})$ refers to the conformal transformation $\mu : \tilde{\mathcal{M}} \to \mathcal{U}$, where $\mathcal{U}$ is relatively compact (closure of $\mathcal{U}$ is compact), connected, open set of a manifold $\mathcal{M}$ such that

$$\mu^* g = \Omega^2 \tilde{g} \tag{3.9}$$

where (1) $\Omega|_{\mathcal{U}} > 0$ and (2) $\Omega|_{\partial\mathcal{U} > 0}$ ($\partial\mathcal{U}$ is the conformal boundary of $\tilde{\mathcal{M}}$)

3.2 Asymptotic Simplicty

Having looked at some very special cases of solutions, we now turn to a more wider class of space-times which can allow a smooth conformal boundary. Let us begin with the *Schwarzschild* solution. The line element in Eddington-Finkelstein outgoing co-ordinate system is given as

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.10}$$

Choose conformal factor $\Omega = r^{-1} = w$, so that the *unphysical* line-element has the form

$$ds^2 = (w^2 - 2mw^3)du^2 - 2dudw - (d\theta^2 + \sin^2\theta d\phi^2) \tag{3.11}$$

Note that the metric and its determinant are finite and non-zero on $w = 0$. In this case, the surface $w = 0$ defines $\mathscr{I}^+$ and the physical space-time is defined by $w > 0$.

If we had started with the incoming coordinate system, the physical line element would look like

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\nu^2 - 2d\nu dr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.12}$$

while the *unphysical* line element has the form

$$ds^2 = (w^2 - 2mw^3)d\nu^2 + d\nu dw - (d\theta^2 + \sin^2\theta d\phi^2) \tag{3.13}$$

In this case, $w = 0$ defines $\mathscr{I}^-$. Notice that, if we replace $w \to -w$ and $m \to -m$ in the advanced line element (3.13), we readily obtain the retarded line element (3.11). Since the sign of mass flips as we go from $w > 0$ to $w < 0$, we can not canonically identify $\mathscr{I}^+$ with $\mathscr{I}^-$. This is related to the fact that the *mass aspect* depends on the unphysical Weyl curvature at infinity. However, the Weyl curvature is divergent at $i^{\pm}, i^0$. Secondly, following Penrose's argument (Twistor Newsletter (TN30-02)) the Strong Null Convergence Condition doesn't seem to allow a well-defined negative mass at i^0. Therefore, for our discussion we will exclude the points $i^0, i^{\pm}$.

Now consider a general line element of the form

$$ds^2 = r^{-2}Adr^2 + 2B_i dx^i dr + r^2 C_{ij} dx^i dx^j \tag{3.14}$$

where $\tilde{x}^\alpha = (r, x^1, x^2, x^3) = (r, x^i)$. x^1 could represent either advanced or retarded time co-ordinate while x^2, x^2 could be the angular coordinates. For $\Omega = r^{-1}$, the unphysical line element will look like

$$ds^2 = Adx^o dx^0 - 2B_i dx^i dx^0 + C_{ij} dx^i dx^j \tag{3.15}$$

where $x^0 = r^{-1}$. Note that the metric and its determinant is well-defined and finite on $x^0 = 0$ if the functions A, B_i, C_{ij} are Taylor expandable about $x^0 = 0$. This is true for Schwarzschild space-time, Bondi-Sachs formalism, Robinson-Trautman space-times etc. For all of these cases, the surface $x^0 = 0$ can be treated as the "conformal infinity".

We now state the definition of **asymptotic simplicity**. The following definition is adapted from R. Penrose and W. Rindler (1984):

(**Asymptotic Simplicity**) A space-time $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ is called k-asymptotically simple if a $\mathcal{C}^{k+1}$ manifold $(\mathcal{M}, g_{ab})$ exists with the conformal boundary $\mathcal{I} = \partial\mathcal{M}$ and the scalar field Ω such that

(a) $\tilde{\mathcal{M}} = \mathrm{int}\mathcal{M}$

(b) $g_{ab} = \Omega^2 \tilde{g}_{ab}$ in $\tilde{\mathcal{M}}$

(c) Ω and g_{ab} are $\mathcal{C}^k$ smooth throughout $\tilde{\mathcal{M}}$

(d) $\Omega > 0$ in $\tilde{\mathcal{M}}$ and $\Omega = 0$, $\nabla_a\Omega = 0$ on $\mathcal{I}$

(e) Every null geodesics in $\tilde{\mathcal{M}}$ acquires a past and future end-points on $\mathcal{I}$

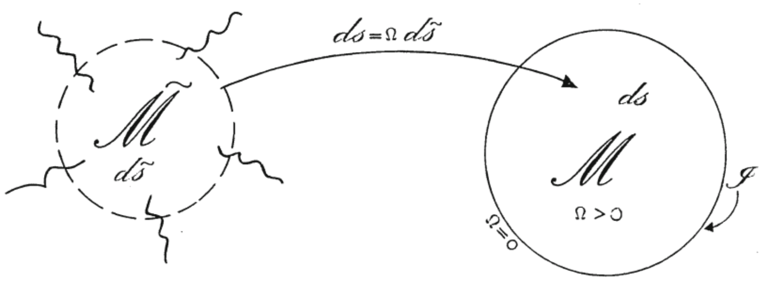


Figure 3.2: Caption

In addition to the above five conditions, if $\tilde{R}_{ab} = \mathcal{O}(\Omega^2)$ in some neighborhood of $\mathcal{I}$ in $\tilde{\mathcal{M}}$, then $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ is said to be **asymptotically flat**. If $\tilde{R}_{ab} = 0$, then space-time is said to be **asymptotically empty**.

insert image

The question stands as of now is how generic is the above definition. It is a delicate and open problem as of now*. However, it should be pointed out that the conditions (a)-(d) can be satisfied trivially by any manifold: simply choose $\mathcal{M} = \tilde{\mathcal{M}}$, $\Omega = 1$ and $\mathcal{I} = \emptyset$. However, there are subtleties with condition (e): it is very restrictive, in fact many of the known standard examples need not satisfy this condition. For example, helical null trajectories at $r = 3m$ in Schwarzschild space-time need not reach $\mathcal{I}$. Moreover, there are null rays which falls into the black-hole. The reason for this is that the Cauchy surface for astro-physical schwarzschild black-holes is $\mathcal{I} \cup \mathcal{H}^+$. We need a weaker version of the above *asymptotic simplicity*, by incorporating other possible end points. As of now, all examples of asymptotically simple space-times turns out to be time independent and it is not known whether there exists any time dependent solution of vacuum Einstein's equations which will satisfy the criterion (a)-(e). Some notable dynamical solutions exists, albeit with some pathology : e.g. Bicak and Schmidt (1989), Bicak (2000), Griffiths and Podolsky (1981), Ashtekar and Dray (1981). (**weakly asymptotically simple space-times**:) A space-time $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ is said to be **weakly asymptotically simple** if there exists an asymptotically simple space-time $(\tilde{\mathcal{M}}', \tilde{g}'_{ab})$ and a neighborhood $\mathcal{U}'$ of $\mathcal{I}' = \partial\tilde{\mathcal{M}}'$ such that $\mu^{-1}(\mathcal{U}') \cap \tilde{\mathcal{M}}'$ is isometric to the open subspace $\tilde{\mathcal{U}}$ of $\tilde{\mathcal{M}}$ Although this is just a mathematical definition, the extent to which this is physically reasonable remains to be verified. Geroch and Horowitz argues that no physical constraints have been imposed on $\tilde{\mathcal{M}}'$. Other approaches have been proposed *, however, as of now there is no universally applicable prescription known for this. We will assume Einstein's field equations to be satisfied in the neighborhood of $\mathcal{I}$ in $\tilde{\mathcal{M}}$.

3.2.1 Geometry of $\mathcal{I}$

The purpose of this section is to look at certain geometrical properties of the boundary assuming some weaker version of asymptotic simplicity to be defined for the manifold and that Einstein's field equations is satisfied near the boundary. To begin with, consider some conformally invariant mass-less fields in the neighborhood of $\mathcal{I}$. Conformal invariance mandates $\tilde{T}^a_a = 0$ so that the Ricci scalar is given by

$$\tilde{R} = 4\tilde{\lambda} \tag{3.16}$$

Now following the calculations in [18], we can express the physical Schouten tensor in terms of the unphysical one as

$$\tilde{P}_{ab} = P_{ab} + \Omega^{-1}\nabla_a\nabla_b\Omega - \frac{1}{2}\Omega^{-2}g_{ab}\nabla^c\Omega\nabla_c\Omega \tag{3.17}$$

Taking the trace and using the definition of Schouten tensor, we obtain

$$-\frac{2}{3}\tilde{\lambda} = \Omega^2 P_a{}^a + \Omega\nabla_a\nabla^a\Omega - 2\nabla_a\Omega\nabla^a\Omega \tag{3.18}$$

When restricted on $\mathscr{I}$, we have $\Omega = 0$. Thus the vector field

$$N_a = -\nabla_a \Omega \tag{3.19}$$

is orthogonal to the boundary $\mathscr{I}$ and satisfies

$$N^a N_a \approx \frac{1}{3}\tilde{\lambda} \tag{3.20}$$

The symbol $\approx$ refers to equality on $\mathscr{I}$. Thus the conformal boundary is space-like, time-like or null provided $\tilde{\lambda} > 0$, < 0 or $= 0$. Asymptotic flatness mandates $\tilde{\lambda} = 0$, making $\mathscr{I}$ a null surface. Similar to the discussion for Schwarzschild space-time, we will ignore the points $i^{\pm}, i^0$. Thus, pictorially we can visualize our space-time as composed of two inverted cones ($\mathscr{I}^{\pm}$) without the vertices and *almost* touching base to base.

insert diagram

Following (3.20), we have for the case of null $\mathscr{I}$ **(Topology of Conformal Infinity)** : If $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ is null everywhere, then

$$\mathscr{I}^{\pm} \cong_{iso} S^2 \times \mathbb{R} \tag{3.21}$$

Let us look at further consequences of Einstein's field equations near $\mathscr{I}$. Recall that the *trace-free* part of Ricci curvature given by equation([18]). Apart from the cosmological constant, if we allow $\tilde{T}_{ab} = \mathcal{O}(\Omega^2)$ (like free Maxwell's equations), then we notice that $\tilde{\Phi}_{ab} = \mathcal{O}(\Omega^2)$. Now, we have the relation

$$\tilde{\Phi}_{ab} = \Phi_{ab} + \Omega^{-1}\nabla_{A'(A}\nabla_{B)B'}\Omega \tag{3.22}$$

For $\mathcal{C}^{\geq 3}$ space-times, we get Φ_{ab} to be at least continuous, therefore, finite when $\Omega = 0$. Thus the second term $\Omega^{-1}\nabla_{A'(A}\nabla_{B)B'}\Omega$ must be well defined, which is true only when $\nabla_{A'(A}\nabla_{B)B'}\Omega = \mathcal{O}(\Omega)$. The **weak aymptotic Einstein condition** refers to the restriction

$$\nabla_{A'(A}\nabla_{B)B'}\Omega \approx 0 \tag{3.23}$$

In tensor indices, this corresponds to ()

$$\nabla_a \nabla_b \Omega \approx \frac{1}{4} g_{ab} \nabla^c \nabla_c \Omega \tag{3.24}$$

which in terms of vector field N_a reads

$$\nabla_a N_b \approx \frac{1}{4} g_{ab} \nabla^c N_c \tag{3.25}$$

Thus the vector field N_a on $\mathscr{I}$ is (1) twist free (since it is hyper-surface orthogonal from definition) and (2) shear free from the weak asymptotic Einstein condition.

For the asymptotically flat case, using the relation

$$\tilde{R}_{ab} = R_{ab} - 2\Omega^{-1}\nabla_a\nabla_b\Omega - g_{ab}(\Omega^{-1}\nabla_c\nabla^c\Omega - 3\Omega^{-2}\nabla_c\nabla^c\Omega) \tag{3.26}$$

and null condition for N_a on $\mathcal{I}$ we get

$$2\nabla_a\nabla_b\Omega + g_{ab}\nabla_c\nabla^c\Omega \approx 0 \tag{3.27}$$

On taking the trace, it yields

$$\nabla_c\nabla^c\Omega \approx 0 \tag{3.28}$$

or, from asymptotic condition, this reads

$$\nabla_a\nabla_b\Omega \approx 0 \tag{3.29}$$

Thus for asymptotic flat case, N_a is also divergence free on $\mathcal{I}$. Say, if we are considering only null $\mathcal{I}^+$, we can write

$$N^b \approx A\iota^B\iota^{B'} \tag{3.30}$$

where A is some positive function on $\mathcal{I}^+$. Using (3.29) we can show that

$$\sigma' \approx 0 \qquad \kappa' \approx 0 \qquad \rho' \approx \bar{\rho}' \tag{3.31}$$

3.2.2 Condition on Weyl curvature

Consider *asymptotically empty* space-times. This means the vacuum Einstein's equations is satisfied near $\mathcal{I}$:

$$\tilde{\nabla}^{AA'}\tilde{\Psi}_{ABCD} = 0 \tag{3.32}$$

Now, recall the conformal invariance property of free mass-less field equations: $\tilde{\nabla}^{AA'}\phi_{AB\cdots L} = 0 \leftrightarrow \nabla^{AA'}\phi_{AB\cdots L} = 0$ which is true provided $\tilde{\phi}_{AB\ldots} \to \phi_{AB\ldots} = \Omega^{-1}\tilde{\phi}_{AB\ldots}$. This means we also have the relation

$$\nabla^{AA'}(\Omega^{-1}\tilde{\Psi}_{ABCD}) = 0 \tag{3.33}$$

However, the Weyl curvature spinor is invariant under conformal transformations. Thus:

$$\Psi_{ABCD}\underbrace{\nabla^{AA'}\Omega}_{-N^{AA'}} = \underbrace{\Omega\nabla^{AA'}\Psi_{ABCD}}_{\approx 0} \tag{3.34}$$

If cosmological constant $\tilde{\lambda} \neq 0$, then the Hermitian matrix $N^{AA'}$ is obtainable from condition (3.20) and (3.19). For arbitrary $N^{AA'}$, this would imply $\Psi_{ABCD} \approx 0$.

If $\tilde{\lambda} = 0$, we had $N^{AA'} \approx A\iota^A\iota^{A'}$. Thus the Weyl curvature is of Petrov Type-N:

$$\Psi_{ABCD} \approx \psi\iota_A\iota_B\iota_C\iota_D \tag{3.35}$$

where $\psi = \Psi_0 = \Psi_{ABCD}o^Ao^Bo^Co^D$ has a spin weight 2. Following the directions in [18], first we differentiate (3.33) to obtain:

$$N_{EE'}\nabla^{AA'}\Psi_{ABCD} \approx N^{AA'}\nabla_{EE'}\Psi_{ABCD} + \Psi_{ABCD}\nabla_{EE'}N^{AA'} \tag{3.36}$$

Next we lower A' and symmetrize with E' and invoke the weak asymptotic Einstein condition (3.29):

$$N_{E(E'}\nabla^A{}_{A')}\Psi_{ABCD} - N^A{}_{(A'}\nabla_{E')E}\Psi_{ABCD} \approx 0 \tag{3.37}$$

Contracting with $\epsilon_A{}^E$, and using the expression for $N^{AA'}$ we obtain:

$$\iota^A \nabla_{AE'} \Psi_{EBCD} \approx 0 \tag{3.38}$$

Contracting with $o^{E'} o^E o^B o^C o^D$ we obtain

$$\eth' \psi \approx 0 \tag{3.39}$$

which implies $\psi = 0$. Thus all *asymptotically empty* space-times admit $\Psi_{ABCD} \approx 0$ or $C_{abcd} \approx 0$. This is also known as **strong asymptotic Einstein's condition**.

3.3 Peeling theorem

Intuitively, the **Peeling theorem** states that a spinor field will appear more and more algebraically special as we go further away from its source. Mathematically, we shall see shortly that the **weak asymptotic Einstein's condition** will guarantee such spinor fields $\phi_{AB\cdots L}$ in physical space-time to be represented as polynomials of $1/\tilde{r}$:

$$\phi_{AB\cdots L} = \sum_{i=1}^{n} \phi^{(i)}_{AB\cdots L} \tilde{r}^{-i} + o(\tilde{r}^{-n}) \tag{3.40}$$

Here $\phi^{(i)}_{A\cdots L}$ has at least $n - i + 1$ PNDs, and is constant along null rays γ. The radial coordinate $\tilde{r}$ is an affine parameter on these trajectories. On the right hand side, the symbol $o(r^{-n})$ has the definition that $\lim_{|r|\to\infty} r^n o(r^{-n}) = 0$. In contrast, the big $\mathcal{O}$ means that $\lim_{|r|\to\infty} |r^n \mathcal{O}(r^{-n})| < C$ for some number C. The aim of this section is to derive this result and look at the peeling property of Weyl curvature (**Sachs peeling effect**) (see [19]). We outline the proof as given in [18] and [20]:

Lets consider a geodetic null congruence $\mathcal{C}$ in $(\tilde{\mathcal{M}}, \tilde{g})$ whose tangent vector field is $\tilde{\boldsymbol{l}} = \frac{\partial}{\partial \tilde{r}}$. Following the definition of asymptotic simplicity, we have that the end points of the integral curves γ will (almost) touch the boundary $\mathcal{I}$. Since $\tilde{l}$ is null, define $\tilde{l}^a = \tilde{o}^A \tilde{o}^{A'}$. Likewise, in the unphysical space-time $(\mathcal{M}, g)$, we have $\boldsymbol{l} = \frac{\partial}{\partial r}$. Let

$$o_A = \tilde{o}_A \qquad\qquad o^A = \Omega^{-1} \tilde{o}^A \tag{3.41}$$

The normalisation condition

$$l^a \nabla_a r = Dr = 1 \qquad\qquad \tilde{l}^a \tilde{\nabla}_a \tilde{r} = 1 \tag{3.42}$$

First we note how asymptotic simplicity condition imposes restriction on the form of conformal factor Ω. Using definition $N_a \approx A \iota_A \iota_{A'}$, we can write

$$-l^a N_a \approx -A o^A o^{A'} \iota_A \iota_{A'} = -A \tag{3.43}$$

Further, we have the definition $N_a = -\nabla_a \Omega$, so that

$$-l^a N_a = l^a \nabla_a \Omega = \frac{\partial \Omega}{\partial r} \tag{3.44}$$

Thus $\frac{\partial \Omega}{\partial r} \approx -A$. According to the definition of asymptotic simplicity, Ω is a $\mathcal{C}^k$ smooth function. Thus we may write

$$\Omega = -Ar - A_2 r^2 - A_3 r^3 - \cdots - A_k r^k + o(r^k) \tag{3.45}$$

For asymptotically flat space-times, recall that we obtained $\nabla_a \nabla_b \Omega \approx 0$. So the coefficient $-A_2$ can be written as

$$-A_2 \approx D^2 \Omega = l^a \nabla_a (l^b \nabla_b \Omega) = l^a l^b \nabla_a \nabla_b \Omega \approx 0 \tag{3.46}$$

where we used the geodetic and affine property of l^a vector field: $l^b \nabla_b l^a = 0$. Thus the weak asymptotic Einstein condition in an asymptotically flat space-time means $A_2 = 0$. What is its significance? Note that

$$D\tilde{r} = l^a \nabla_a \tilde{r} = \Omega^{-2} \tilde{l}^a \tilde{\nabla}_a \tilde{r} = \Omega^{-2} \tag{3.47}$$

This can be integrated to express r in terms of $\tilde{r}$, and therefore Ω in terms of $\tilde{r}$:

$$\tilde{r} = \int A^{-2} r^{-2} \left(1 + \frac{A_2}{A} r + \frac{A_3}{A} r^2 + \cdots + \frac{A_k}{A} r^{k-1} + o(r^{k-1})\right)^{-2} dr \tag{3.48}$$

where we have used the expression for Ω. The aim now is to look at this integral in the limit where r is small, which means we can do a binomial expansion of the terms in the brackets, only upto $o(r^{k-1})$:

$$\tilde{r} = \int A^{-2} r^{-2} \left(1 + C_1 r + C_2 r^2 + \cdots + C_{k-1} r^{k-1} + o(r^{k-1})\right)^{-2} dr \tag{3.49}$$

The new coefficients C_i's are obtained by expanding the integral (3.49) and collecting all terms order by order in powers of r. It can be verified that the C_1 term can be contributed by A_2 terms only. This is important: if we allow $A_2 \neq 0$, then we will get

$$\tilde{r} = -A^{-2} r^{-1} + \frac{C_1}{A^2} \log r + \frac{C_2}{A^2} r + \cdots + o(r^{k-2}) \tag{3.50}$$

Note that the logarithmic terms are associated with C_1 or A_2 factors only. Let $A_2 = 0$, then we can make an ansatz that

$$r = \sum_i B_i \tilde{r}^{-i} \tag{3.51}$$

and substitute in the (3.52) to obtain:

$$r = -A^2 \tilde{r}^{-1} + B_2 \tilde{r}^{-2} + B_3 \tilde{r}^{-3} + \cdots \tag{3.52}$$

where B_i's are some combination of C_i's. Finally, we substitute this in (3.45) to obtain the asymptotic form of the scalar field Ω:

$$\Omega = A^{-1} \tilde{r}^{-1} + E_2 \tilde{r}^{-2} + E_3 \tilde{r}^{-3} + \cdots + \cdots o(\tilde{r}^{-k}) \tag{3.53}$$

To complete the derivation of Peeling expression, it is necessary to comment about the fall-off behavior of the physical spin frame $(\tilde{o}_A, \tilde{\iota}^A)$. We already had the transformation between o_A and $\tilde{o}_A$ (). Now define:

$$\iota^A = \tilde{\iota}^A - \nu \tilde{o}^A \tag{3.54}$$

Note that it satisfies the normality condition

$$o_A \iota^A = \tilde{o}_A \tilde{\iota}^A = 1 \tag{3.55}$$

The transformation and spin frames are defined such that we have

$$Do^A = 0 \qquad D\iota^A = 0 \qquad \tilde{D}\tilde{o}^A = 0 \qquad \tilde{D}\tilde{\iota}^A = 0 \tag{3.56}$$

Now if we start with the condition $\tilde{D}\tilde{o}_A = 0$, then

$$Do_A = \Omega^{-2}\tilde{l}^b\tilde{\nabla}_b\tilde{o}_A = \Omega^{-2}\tilde{l}^b(\tilde{\nabla}_b\tilde{o}_A - \tilde{\Upsilon}_{AB'}\tilde{o}_B) = \Omega^{-2}\tilde{D}\tilde{o}_A = 0 \tag{3.57}$$

Further, if we impose $D\iota^A = 0$ and $\tilde{D}\tilde{\iota}^A = 0$, we get:

$$D\nu = \Omega^{-2}\delta'\Omega \tag{3.58}$$

On $\mathcal{I}$, we want the natural identification $o_A \approx \tilde{o}_A$ and $\iota^A \approx \tilde{\iota}^A$, so from definition of ι^A, we have $\Omega\nu \to 0$. From (3,58), we have

$$\Omega^2 D\nu = \delta'\Omega \tag{3.59}$$

Further, using (3.47), we have

$$\nu D\Omega^2 = 2\Omega\nu D\Omega \approx -2A\Omega\nu = 0 \tag{3.60}$$

Adding the two relations above, we get

$$\delta'\Omega \approx D(\Omega^2\nu) = D\Omega.\underbrace{\Omega\nu}_{\to 0} + \underbrace{\Omega}_{\approx 0} D(\Omega\nu) \tag{3.61}$$

LHS is simply $-o^A\iota^{A'}N_{AA'}$. Thus, if one express N_a as linear combinations of null-tetrads on $\mathcal{I}$, we find that N_a should only be a linear combination of $l_a = o_A o_{A'}$ and $n_a = \iota_A\iota_{A'}$. Recall the inner-product (3.20) and the expression for N_a when $\lambda = 0$. Keeping these special cases in mind, we can write

$$N^b \approx An^b + \frac{1}{6}\tilde{\lambda}A^{-1}l^b \tag{3.62}$$

Thus, in the neighborhood of $\mathcal{I}$, we may write $N^b - An^b - \frac{1}{6}\tilde{\lambda}A^{-1}l^b = \mathcal{O}(\Omega)$. Recalling that Ω is $\mathcal{C}^k$ smooth, let Q_b be a $\mathcal{C}^{k-2}$ smooth co-vector field defined by

$$\Omega Q_b = N_b - An_b - \frac{1}{6}\tilde{\lambda}A^{-1}l_b \tag{3.63}$$

Take derivative ∇_c on both sides and contract with $o^{B'}l^c$ to get

$$AQ_b o^{B'} \approx 0 \tag{3.64}$$

where we have invoked the asymptotic Einstein condition. Thus, if we define a $\mathcal{C}^{k-3}$ scalar μ such that

$$-Q_b \bar{m}^b = \Omega\mu \tag{3.65}$$

then it follows that

$$D\nu = \mu \tag{3.66}$$

So, we can now express $\tilde{\iota}^A$ as

$$\tilde{\iota}^A = \iota^A + (\nu_1 \tilde{r}^{-1} + \cdots + \nu_{k-1}\tilde{r}^{-(k-1)} + o(\tilde{r}^{-(k-1)}))o^A \tag{3.67}$$

For the derivation of the Peeling relation, let us assume that there is a symmetric spinor field $\theta_{A\cdots MN'\cdots L'}$ which is $\mathcal{C}^h$ smooth in $\mathcal{M}$. Here the smoothness applies to each component of the spinor field separately. Further, assume that it is a conformal density of weight $-w$, i.e. $\tilde{\theta} = \Omega^w\theta$. To begin with, define

$$\theta_q = \theta_{\underbrace{A\cdots D}_{q_1}\underbrace{E\cdots M}_{p_1}\underbrace{N'\cdots P'}_{q_2}\underbrace{Q'\cdots L'}_{p_2}} o^A \cdots o^D \iota^E \cdots \iota^M o^{M'} \cdots o^{P'} \iota^{Q'} \cdots \iota^{L'} \tag{3.68}$$

where $q_1 + q_2 = q$. The smoothness condition here means that

$$\theta_q = \theta_q^{(0)} + \theta_q^{(1)} r + \theta_q^{(2)} r^2 + \cdots + \theta_q^{(h)} r^h + o(r^h) \tag{3.69}$$

As a conformal density, we have

$$\tilde{\theta} = \Omega^w \theta_{\underbrace{A\cdots D}_{q_1}\underbrace{E\cdots M}_{p_1}\underbrace{N'\cdots P'}_{q_2}\underbrace{Q'\cdots L'}_{p_2}} \tilde{o}^A \cdots \tilde{o}^D \tilde{\iota}^E \cdots \tilde{\iota}^M \tilde{o}^{M'} \cdots \tilde{o}^{P'} \tilde{\iota}^{Q'} \cdots \tilde{\iota}^{L'} \tag{3.70}$$

Expressing $(\tilde{o}_A, \tilde{\iota}^A)$ in terms of (o_A, ι^A) along with (3.69) and (3.70) yields expression of the form

$$\tilde{\theta} = \sum_{i=w+q}^{w+q+h} \theta^{(i)} \tilde{r}^{-i} + o(\tilde{r}^{-(w+q+h)}) \tag{3.71}$$

For Weyl curvature, it is the special case where $w = 1$ and number of indices $n = 4$. In this case we assume the $\mathcal{C}^h$ condition on the unphysical Weyl curvature $\psi_{ABCD} =$

$\Omega^{-1}\Psi_{ABCD}$. Thus we have the **Sachs peeling theorem**:

$$\tilde{\Psi}_0 = \sum_{i=5}^{h+5} \tilde{\Psi}_0^{(i-5)}\tilde{r}^{-i} + o(\tilde{r}^{-h-5}) \tag{3.72}$$

$$\tilde{\Psi}_1 = \sum_{i=4}^{h+4} \tilde{\Psi}_1^{(i-4)}\tilde{r}^{-i} + o(\tilde{r}^{-h-4}) \tag{3.73}$$

$$\tilde{\Psi}_2 = \sum_{i=3}^{h+3} \tilde{\Psi}_2^{(i-3)}\tilde{r}^{-i} + o(\tilde{r}^{-h-3}) \tag{3.74}$$

$$\tilde{\Psi}_3 = \sum_{i=2}^{h+2} \tilde{\Psi}_3^{(i-2)}\tilde{r}^{-i} + o(\tilde{r}^{-h-2}) \tag{3.75}$$

$$\tilde{\Psi}_4 = \sum_{i=1}^{h+1} \tilde{\Psi}_4^{(i-1)}\tilde{r}^{-i} + o(\tilde{r}^{-h-1}) \tag{3.76}$$

If we set $A = 1$, then one can verify that $\psi_r \approx \tilde{\Psi}_r^0$. Also note that $\tilde{\Psi}_r \approx 0$ which, again, follows from the strong asymptotic Einstein condition. However, there exists sharp k_* value, which is a positive integer, such that even if Ψ_{ABCD} is $\mathcal{C}^{k_*}$ smooth at $\mathcal{I}$, the Peeling condition will still be satisfied [20]

Chapter 4

Asymptotic Symmetries of Spacetime

4.1 Introduction

Highly symmetric space-times such as Minkowski space-time $\mathbb{M}$ or Friedmann-Walker space-times admit class of co-ordinate transformations which preserves line element exactly. These transformations have a special name, they are called the isometry transformation. Given a coordinate system $\{x^\alpha\}$, we want to move to a new coordinate system $\{y^\alpha\}$ such that

$$g_{\alpha\beta}(x)dx^\alpha dx^\beta = g_{\alpha\beta}(y)dy^\alpha dy^\beta \tag{4.1}$$

If we look at only the infinitesimal transformations, i.e. of the form

$$y^\alpha = x^\alpha + K^\alpha(x) + \mathcal{O}(K^2) \tag{4.2}$$

we can substitute this expression in the line element and expand till linear order in K. The RHS of (4.1) is

$$(g_{\alpha\beta}(x) + \underbrace{K^\gamma \partial_\gamma g_{\alpha\beta} + g_{\alpha\gamma}\partial_\beta K^\gamma + g_{\beta\gamma}\partial_\alpha K^\gamma)}_{\mathcal{L}_K g_{\alpha\beta}} dx^\alpha dx^\beta + \mathcal{O}(K^2) \tag{4.3}$$

which is easy to check, since

$$dy^\alpha = dx^\alpha + \partial_\beta K^\alpha dx^\beta + \mathcal{O}(K^2) \tag{4.4}$$

$$g_{\alpha\beta}(y) = g_{\alpha\beta}(x) + \partial_\gamma g_{\alpha\beta} dx^\gamma + \mathcal{O}(K^2) \tag{4.5}$$

The vector field K is called the **Killing Vector-field** and it constitutes the **isometry group**. Thus the isometry condition implies that

$$\mathcal{L}_K g_{\alpha\beta} = 0 \tag{4.6}$$

This is called the **Killing field equation**.

We can also consider coordinate transformations which preserve the line element upto a scale factor. This is precisely the *conformal transformations*, we considered in the previous chapter. The generators of such transformation satisfy

$$\mathcal{L}_K g_{\alpha\beta} \propto g_{\alpha\beta} \tag{4.7}$$

The scale factor k is given by

$$k = \frac{1}{4} g^{\alpha\beta} \mathcal{L}_K g_{\alpha\beta} = \frac{1}{4} \nabla_\alpha K^\alpha \tag{4.8}$$

In the RHS, we have written the Lie derivative expression as covariant derivative of co-vector components K_α

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0 \tag{4.9}$$

The *isometry transformation* is just that special case where $k = 0$. As an example, let us consider the explicit solution for Minkowski space-time $\mathbb{M}$. In this case, we will consider the procedure outlined in [17].

To begin with let us take covariant derivative of (4.7)

$$\nabla_\gamma \nabla_\alpha K_\beta + \nabla_\gamma \nabla_\beta K_\alpha = \nabla_\gamma k g_{\alpha\beta} \tag{4.10}$$

For $\mathbb{M}$, the Riemann curvature tensor is trivial. This means

$$\nabla_\alpha \nabla_\beta K_\gamma - \nabla_\beta \nabla_\alpha K_\gamma = 0 \tag{4.11}$$

This simplification allows us to write

$$\nabla_\alpha \nabla_\beta K_\gamma = \frac{1}{2}(\nabla_\alpha k g_{\beta\gamma} + \nabla_\beta k g_{\gamma\alpha} - \nabla_\gamma k g_{\alpha\beta}) \tag{4.12}$$

Let us take one more derivative of the above equation

$$\nabla_\rho \nabla_\alpha \nabla_\beta K_\gamma = \frac{1}{2}(\nabla_\rho \nabla_\alpha k g_{\beta\gamma} + \nabla_\rho \nabla_\beta k g_{\gamma\alpha} - \nabla_\rho \nabla_\gamma k g_{\alpha\beta}) \tag{4.13}$$

Anti-symmetrizing over ρ and α gives

$$\nabla_\beta \nabla_{[\rho} k g_{\alpha]\gamma} - \nabla_\gamma \nabla_{[\rho} k g_{\alpha]\beta} = 0 \tag{4.14}$$

Next we take trace over α and β. After some manipulation, we get

$$g_{\rho\gamma} \Box k + 2 \nabla_\rho \nabla_\gamma k = 0 \tag{4.15}$$

On taking trace over ρ and γ, we get $\Box k = 0$. Thus

$$\nabla_\rho \nabla_\gamma k = 0 \tag{4.16}$$

Since $\nabla_a = \partial_a$, the solution for the scale factor is simply

$$k = 2A + 4B_\sigma x^\sigma \tag{4.17}$$

Substitute this expression in (4.12) to get

$$\nabla_\alpha \nabla_\beta K_\gamma = 2B_\alpha g_{\beta\gamma} + 2B_\beta g_{\gamma\alpha} - 2B_\gamma g_{\alpha\beta} \tag{4.18}$$

Integrating twice w.r.t. x yields

$$K_\gamma(x) = P_\gamma + M_{\beta\gamma}x^\beta + Ax_\gamma + (2B_\beta x^\beta x_\gamma - B_\gamma x^\beta x_\beta) = K(x_\gamma) \tag{4.19}$$

This is the Conformal Killing vector for $\mathbb{M}$. If $k = 0$, then $A = 0$ and $B_\alpha = 0$ so that the expression simplifies to give just the Poincare vector.

Note that the above expression was dependent on the fact that the Riemann curvature tensor is trivial, Just by looking at the isometry transformation, we see that there are four translations (generated by P_a) and six Lorentz rotation generated by M_{ab}. Thus the group is essentially a ten parameter Lie group. We can think of it as following: given any point $p \in \mathbb{M}$, we can define four axis along which one can do translation. This gives us four translation parameters P_a. We can also do rotation of any axis about any other axis, which can be done in ${}^4C_2 = 6$ ways. Thus there are $4 + 6 = 10$ parameter group. Can space-time with non-trivial curvature still admit ten parameter Lie group? The answer is yes and such space-times are said to be **Maximally symmetric**. The Riemann curvature tensor for such space-times have the form

$$R_{\rho\sigma\mu\nu} = \frac{R}{12}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{4.20}$$

To get the intuition why this should be the form of curvature, let us define coordinates in a small neighborhood U centered around point $p \in \mathcal{M}$ such that the metric evaluated at point p gives the standard Minkowski metric. The 10 parameter Lie group as defined above will preserve the form of $g_{\alpha\beta}|_p = \eta_{\alpha\beta}$, but not its derivative. By maximal symmetry, we mean that the Riemann tensor should also be invariant under the action of this group. This implies that the curvature can be written as sum of terms containing only the metric and none of its derivative. Considering the symmetry of Riemann tensor and its relation to Ricci tensor and Ricci scalar, the only possible expression is

$$R_{\rho\sigma\mu\nu}|_p = \frac{R}{12}(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\rho\nu}\eta_{\sigma\mu}) \tag{4.21}$$

For Maximal space-times, this should be true for all $p \in \mathcal{M}$, which leads us to the form (4.20). Comparing with the Ricci decomposition form, we see that

$$C_{\rho\sigma\mu\nu} = E_{\rho\sigma\mu\nu} = 0 \tag{4.22}$$

Thus, while Ricci tensor need not be zero, the Weyl curvature should be zero. The Weyl curvature carries the gravitational degrees of freedom, and a generic vacuum Einstein equation is given by

$$\nabla^\rho C_{\rho\sigma\mu\nu} = 0 \tag{4.23}$$

so one could ask whether it is possible to define Killing fields in presence of non-zero Weyl curvature. It turns out that the Killing fields, or more generically the Conformal Killing fields puts severe restriction on possible solution of Weyl curvature. Note that the CKVFs can be considered as the primary part of Hermitian twistor (). In general, for a symmetric $n-$twistor $\omega^{AB\cdots L}$ we have the consistency condition:

$$\Psi_F{}^{(ABC}\omega^{DE\cdots L)F} = 0 \tag{4.24}$$

The Weyl curvature spinor Ψ_{ABCD} shares same PNDs with that of twistor ω, in fact we have the following result due to B. Jeffryes: If $\omega^{AB\cdots L}$ is a symmetric $n(\leq 4)$ twistor, then Ψ_{ABCD} is proportional to $(\omega_{AB\cdots L})^{4/n}$ It should be noted that the Killing fields have great importance with regards to the conservation laws in curved space-time. Consider a free falling particle travelling with velocity field u^α. For affine parameter, the equation of motion is simply

$$u^\beta \nabla_\beta u^\alpha = 0 \tag{4.25}$$

Let K^α be a KVF. Let us define the quantity $Q_K = K^\alpha u_\alpha$. Then it is easy to check that Q_K is conserved along the trajectories of the particles:

$$\frac{DQ_K}{d\tau} = \nabla_u Q_K = \underbrace{K_a u^b \nabla_b u^a}_{=0} + \underbrace{u^a u^b \nabla_b K_a}_{=0} = 0 \tag{4.26}$$

For mass-less particles, it suffices that K be just the CKVF, since the second term $u^a u^b \nabla_b K_a$ will vanish as u is a null vector. In $\mathbb{M}$, the KVFs P_a generates translation motion, thus Q_P's are the associated conserved momentum of the system. Likewise there are six conserved angular momentums Q_M.

For fields, we have the conservation equation

$$\nabla^{\alpha\beta} T_{\alpha\beta} = 0 \tag{4.27}$$

Similar to the point particle case, we find that the quantity

$$J^K_\alpha = T_{\alpha\beta} K^\beta \tag{4.28}$$

behaves as a conserved current

$$\nabla^a (J_K)_a = \underbrace{K^b \nabla^a T_{ab}}_{=0} + \underbrace{T_{ab} \nabla^a K^b}_{=0} = 0 \tag{4.29}$$

For conformally invariant fields, we have $T = 0$, thus it suffices that K is a CKVF. Given such a conserved current, we can define conserved charges by integrating over arbitrary 3-surface Σ:

$$Q[k] = \int_\Sigma T_{\alpha\beta} K^\alpha d\sigma^\beta \tag{4.30}$$

The above definition of conserved quantities in a curved space-time faces some ambiguities, and it primarily has to do with nature of Weyl curvature and equivalence principle. It should be noted that gravitational field has no canonical stress-energy tensor and this is due to the fact that there always exist a local inertial frame where space-time looks like a patch of $\mathbb{M}$. Consequently, a frame independent definition of gravitational stress energy tensor will mean that if the field is zero in one frame , it should be true for all frames. Thus the very definition of conserved momentum and conserved angular-momentum for an isolated gravitating system becomes *quasi-local* rather than local (as considered above). Apart from that, the condition () makes life difficult to define KVFs on $\mathcal{M}$, at least in the bulk. However, life becomes simpler when we look at the asymptotics of space-time. Take for instance, the Sachs peeling effect we discussed in the previous chapter. For asymptotically simple space-times (and some weaker definition of that), the peeling theorem implies that the Weyl curvature should vanish asymptotically. To what extent can the consistency conditions be satisfied in this limit?

4.2 Isometry near infinity

Take the simpler case of $\mathbb{M}$. So far we have considered isometry of the full space-time and we have noted that $\mathbb{M}$ admits a non-trivial ten parameter lie group. We can ask, how the isometry group will look like if we carry out our discussion only near infinity. Let us consider the line element in Eddington-Finkelstein coordinates (see chapter 3 for Schwarzschild case):

$$ds^2 = du^2 + 2dudr - r^2 d\Omega^2 \tag{4.31}$$

We now choose new co-ordinates $(\bar{u}, \bar{r}, \bar{\theta}, \bar{\phi})$ such that isometry is satisfied asymptotically:

$$ds^2 = d\bar{u}^2 + 2d\bar{u}d\bar{r} - (\bar{r}^2 d\bar{\Omega}^2 + \mathcal{O}(\bar{r})) + \mathcal{O}(\bar{r}^{-1}) \tag{4.32}$$

For that, we make the following ansatz:

$$u = a^0(\bar{u}, \bar{x}^A) + a^1(\bar{u}, \bar{x}^A)\bar{r}^{-1} + \mathcal{O}(\bar{r}^{-2}) \tag{4.33}$$

$$r = R(\bar{u}, \bar{x}^A)\bar{r} + \rho^0(\bar{u}, \bar{x}^A) + \mathcal{O}(\bar{r}^{-1}) \tag{4.34}$$

$$x^A = g^{0A}(\bar{u}, \bar{x}^A) + g^{1A}(\bar{u}, \bar{x}^A)\bar{r}^{-1} + \mathcal{O}(\bar{r}^{-2}) \tag{4.35}$$

Note that the asymptotic Einstein condition allows us to define functions as polynomial of $1/\bar{r}$. To find the coefficients, we can begin with the equation (4.1) and equate the coefficients order by order. The isometry condition

$$\bar{g}_{ab} = g_{cd}\frac{\partial x^c}{\partial \bar{x}^a}\frac{\partial x^d}{\partial \bar{x}^b} \tag{4.36}$$

is evaluated for all values of a, b. However, for asymptotic behavior, it suffices to look at the leading order expansions only (i.e. $a^0,\ R,\ g^{0A}$). We have the following results:

For $a = b = \bar{u}$, the coefficient of $\bar{r}^2$ is

$$R^2 q_{AB}\frac{\partial g^{0A}}{\partial \bar{u}}\frac{\partial g^{0B}}{\partial \bar{u}} = 0 \tag{4.37}$$

and the coefficient of $\bar{r}$ is

$$2\frac{\partial R}{\partial \bar{u}}\frac{\partial a^0}{\partial \bar{u}} - q_{AB}R^2\left(\frac{\partial {g^0}^A}{\partial \bar{u}}\frac{\partial {g^1}^B}{\partial \bar{u}} + (A \leftrightarrow B)\right) = 0 \tag{4.38}$$

For $a = \bar{u}$, $b = \bar{r}$, the order $\mathcal{O}(1)$ term in the expansion is

$$R\frac{\partial a^0}{\partial \bar{u}} + R^2 q_{AB}\frac{\partial {g^0}^A}{\partial \bar{u}}{g^1}^A = 1 \tag{4.39}$$

For $a = \bar{u}$ and $b = B$, the leading coefficient of $\bar{r}^2$ is

$$R^2 q_{CD}\frac{\partial {g^0}^C}{\partial \bar{u}}\frac{\partial {g^0}^D}{\partial \bar{x}^B} = 0 \tag{4.40}$$

The above condition, along with (4.37) suggests that $\frac{\partial {g^0}^C}{\partial \bar{u}} = 0$. Substitute this relation in (4.41) to obtain

$$\frac{\partial R}{\partial \bar{u}}\frac{\partial a^0}{\partial \bar{u}} = 0 \tag{4.41}$$

We can't take $\frac{\partial a^0}{\partial \bar{u}}$ to be zero, because line element () tells that $\bar{u}$ mimics u near infinity, therefore should be linearly dependent on one another in this limit. Thus we have

$$\frac{\partial R}{\partial \bar{u}} = 0 \leftrightarrow R = R(\bar{x}^A) \tag{4.42}$$

Given only the angular dependence of R, we can now use (4.42) in (4.39) to get a^0:

$$a^0(\bar{u}, \bar{x}^A) = R^{-1}(\bar{u} + \beta(\bar{x}^A)) \tag{4.43}$$

The unknown function R is essentially the scale-factor associated with conformal motion of unit sphere. For that , assume $a = A$ and $b = B$. The $\bar{r}$ term in this expansion is

$$R^2 q_{CD}\frac{\partial {g^0}^C}{\partial \bar{x}^A}\frac{\partial {g^0}^D}{\partial \bar{x}^B} = q_{AB} \tag{4.44}$$

Thus, in the limit $\bar{r} \to \infty$, we have

$$x^A = {g^0}^A \approx \bar{x}^A + \bar{f}^A \tag{4.45}$$

where in the RHS, $\bar{f}^A$ is the conformal Killing vector of 2-sphere. The above equation can be subtituted in (4.44) and calculated till linear order in $\bar{f}$. It may be shown that

$$R = e^{\bar{D}_A \bar{f}^A} \tag{4.46}$$

Thus in the asymptotic limit, we have the following infinitesimal transformations:

$$u = e^{-\bar{D}_A \bar{f}^A}[\bar{u} + \alpha(\bar{x}^A)] \qquad\qquad x^A = \bar{x}^A + \bar{f}^A \tag{4.47}$$

where $\bar{f}^A$ is a CKVF on 2-sphere. Inverting this relation we get

$$\bar{u} = e^{-D_A f^A}[u + \alpha(x^A)] \qquad \bar{x}^A = x^A + f^A \tag{4.48}$$

It is known that Lorentz transformation is isomorphic to Mobius transformation of a Riemann sphere. Thus we can regard f^A as generator of Lorentz transformation at infinity. If we take $\alpha = 0$, the transformation

$$\bar{u} = e^{-D_A f^A} u \tag{4.49}$$

also has a special geometric interpretation. As we shall see later, the transformation (4.48) corresponds to conformal motion of infinity $\mathcal{I}$. The condition (4.49) ensures that the retarded time co-ordinate always remain as the *affine* parameter for the orthogonal vector N_a at infinity. This is corollary to the **strong conformal geometry** on $\mathcal{I}$. Another consequence of this condition is that it preserves **null angle** at infinity. In general, transformation (4.48) where $\alpha = 0$ is called **super-rotation**.

$\alpha(x^A)$ is an arbitrary function on 2-sphere which appeared as an "integration constant". Any such functions on a sphere can be expanded in terms of spherical harmonics :

$$\alpha(x^A) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(x^A) \tag{4.50}$$

Lets take $f^A = 0$, then $l \leq 2$ harmonics gives us the familiar *translation* motion:

$$\alpha = \epsilon_o + \epsilon_1 \sin\theta\cos\phi + \epsilon_2 \sin\theta\sin\phi + \epsilon_3 \cos\theta \tag{4.51}$$

For a general l, this is known as **super-translation motion**. Thus unlike in the Poincare group, which had just four translation parameter, the super-translation α is an arbitrary function and is defined by infinitely many parameters $\{\alpha_{lm}\}$. Thus isometry condition defined only for a subspace of Minkowksi space can be considerably larger than the usual transformation (4.19). The transformations (4.48) generates the symmetry group called **Bondi-Mentzner-Sachs group**. The derivation we have considered in this section is just a simplified version of the analysis done by H.Bondi et al [3] for axis-symmetric space-times. The original analysis by H. Bondi [3] and R.K.Sachs [21] involved a number of assumptions on the behavior of metric components.

1. The Sommerfeld condition on metric components γ:

$$\lim_{r\to\infty} \frac{\partial(r\gamma)}{\partial r}|_{u=const} = 0 \tag{4.52}$$

   which is an ad-hoc condition implying that the functions can be written as polynomials of $1/r$ and that one could go to the limit $r \to \infty$ for a fixed value of u.

2. Boundary conditions, which essentially amounts to space-time being Minkowskian at infinity.

Using the Conformal methods discussed in previous chapter we obtained the Peeling theorem, which gives geometric justification for the ad-hoc assumption (i). We will assume the boundary conditions for asymptotically flat space-times while deriving the BMS group using gauge-fixing approach. Justification for such a condition will be given using the conformal methods in section (4.4).

4.3 Gauge-Fixing Approach

In our previous section we saw that the isometry group at infinity is much larger than the conventional Poincare group, primarily due to the arbitrary function α. The condition we imposed was that the transformation should preserve line element in the limit $r \to \infty$. The boundary conditions we want to preserve can be more abitrary, and need not necessarily correspond to isometry transformation. In this section we are going to discuss the BMS group in the framework of gauge-fixing approach. More generically, if some boundary conditions are specified near infinity, the *weak* version of definition of asymptotic symmetry group is defined as

$$G_{weak} = [\textit{Residual gauge diffeomorphism preserving boundary conditions}] \tag{4.53}$$

The *stronger* definition further requires the associated charge to be non-trivial. The symmetry group can be defined in the following steps. We are essentially going to follow the procedure outlined in [6]. However, before we dig into the procedure of the gauge-fixing approach, it is necessary to define the framework and the underlying assumptions.

We begin by considering the *jet-bundle* $(J, \pi, \mathcal{M})$. The base space $\mathcal{M}$ is our usual Lorentzian manifold. On each point $p \in \mathcal{M}$, we can define the tangent space $T_p\mathcal{M} \cong_{iso} \mathbb{M}$. And this vector space admits a natural Hamel's basis $\{\partial_\mu\}$. Likewise, the dual space $T_p^*\mathcal{M}$ will admit basis $\{dx^\mu\}$ defined such that $dx^a\left(\frac{\partial}{\partial x^b}\right) = \delta^a_b$. For sake of completeness, let us also define the space of $k-$forms as $\Omega^k(\mathcal{M})$. $k-$forms are essentially $(0,k)$ alternating tensor fields defined on $\mathcal{M}$. The d operator takes elements of the space Ω^k to Ω^{k+1}. Note that $d^2 = 0$. This is due to the fact that for any $k-$ form L, $d^2L = \frac{\partial^2 L}{\partial x^a \partial x^b} \wedge dx^a \wedge dx^b = 0$. In this construction, the exterior derivatives are more important as relevant quantities of interest that we are going to consider are these $k-$forms.

Each point of *total space* J corresponds to collections of abstract quantities $\{\Phi^i, \Phi^i_\mu, \Phi^i_{\mu\nu}, \cdots\}$. The indices $\mu, \nu, \cdots$ are all symmetrized. We can as well define co-tangent space at a given point in J, and for a given such space, let us denote the co-tangent basis by the collection $\{\delta\Phi^i, \delta\Phi^i_\mu, \delta\Phi^i_{\mu\nu}, \cdots\}$. Thus the exterior derivative which can act on $0-$forms are given by the *variational operator*

$$\delta = \delta\Phi^i \frac{\partial}{\partial \Phi^i} + \delta\Phi^i_\mu \frac{\partial}{\partial \Phi^i_\mu} + \delta\Phi^i_{\mu\nu} \frac{\partial}{\partial \Phi^i_{\mu\nu}} + \cdots \tag{4.54}$$

The $\delta^2 = 0$ condition mandates that the abstract variations $\delta\Phi^i_{\mu\nu\cdots}$ are Grassmann odd numbers.

The map $\pi : J \to \mathcal{M}$ is called the *projection map.* The fibres at a given point $p \in \mathcal{M}$ is the space $\pi^{-1}(\{p\}) = \{\Phi^i(p), \Phi^i_\mu(p), \Phi^i_{\mu\nu}(p), \cdots\}$. The fibres represent the collection of fields $\Phi^i(x)$ and its symmetrized derivatives evaluated at $x = p$. Note that exterior derivative d acts on the base manifold only, while δ acts along a given fibre.

Definition [Gauge transformation and Gauge symmetry] Let us now define the *gauge transformation*, by which we mean transformations of the fields Φ^i induced by parameters $F = \{f^\alpha\}$, where f^α's are some arbitrary functions. We write

$$\delta_F \Phi^i = R_\alpha f^\alpha + R^\mu_\alpha \partial_\mu f^\alpha + R^{(\mu\nu)}_\alpha \partial_\mu \partial_\nu f^\alpha + \cdots \tag{4.55}$$

The coefficients $R^{\mu\nu\cdots}_\alpha$ are functions of x, Φ^i and its derivatives. *Gauge symmetry* is defined with respect to a given lagrangian. We know that the classical physics of fields are all encoded in the lagrangian L, which is a 4-form. The gauge transformation (4.55) is a symmetry of our theory, if it preserves L upto a surface term (a 3-form):

$$\delta_F L = dB_F \tag{4.56}$$

The surface terms are important as we shall see later that they contribute to the definition of *charge*.
As an example, consider the free *Maxwell field* $F_{ab} = 2\partial_{[a}\Phi_{b]}$. This can be considered as components of the 2-form $F = d\Phi$, for some smooth 1-form Φ. Since F is *exact*, transformation $\Phi \to \Phi + d\alpha$ keeps F invariant: $F = d\Phi \to d\Phi + \underbrace{d^2\alpha}_{=0}$. The last term is zero due to the identity $d^2 = 0$. Since F is preserved, the Lagrangian $L \sim F^2$ is preserved as well. Thus $\Phi \to \Phi + d\alpha$ is a *gauge transformation* and is also a symmetry of the free *Maxwell* theory.

Definition [Gauge fixing] The symmetry of our theory is generated by the set of parameters $\{f^\alpha\}$, and we may want to study this theory for some particular choice of these parameters. This is called *gauge fixing.* More precisely, we have a set of algebraic or differential constraints on the field.

$$G[\Phi] = 0 \tag{4.57}$$

These constraints should be obtainable by some gauge transformation. Further the transformation should use all the freedom of arbitrary parameters f^α. Thus the number of gauge fixing conditions should be equal to number of arbitrary parameters defining the gauge transformation.

Definition [Residual gauge transformation] The gauge fixing conditions (4.57) may itself possess some symmetry, i.e.

$$\delta_F G[\Phi] = 0 \tag{4.58}$$

The transformations (4.58) are called residual gauge transformations.

Definition [Background conditions] The explicit solution of the field Φ for a given gauge fixing condition and residual condition will depend on the choice of boundary conditions The choice of boundary conditions will determine the asymptotic symmetry group. Too severe conditions can make the symmetry group trivial. The exact choice will depend on the particular problem we want to consider.

To demonstrate examples for each of these definitions, let us consider the case for electro-magnetic field. Begin with the *gauge transformation* $\Phi_a \to \Phi'_a = \Phi_a + \partial_a \alpha$. If our *gauge fixing* condition is $A_2 = 0$, then it suffices that $\frac{\partial \alpha}{\partial x^2} = 0$. In other words, $\alpha(x) = \alpha(x^1, x^A)$. However, this function is still arbitrary. We can fix α by specifying a *boundary condition* on some surface $\mathcal{S} = \mathcal{S}(x^1, x^A)$: let's say $A_1(x)|_{\mathcal{S}} = 0$. Now, even if α is completely specified, there still exists the *residual freedom* $A_{3,4} \to A_{3,4} + \partial_{3,4}\beta(x^3, x^4)$. Note that the choice of β, even though arbitrary at this point, does not affect the *gauge* $A_2 = 0$, nor the *boundary condition.* This function β can be fixed by additional *boundary conditions* such that the solution for the Maxwell equations is well-posed.

In the next section we are going to look at a very similar setting, but for gravitational field. The metric g_{ab} will be our relevant field of interest. For the construction of BMS group in a more general setting, we will first introduce the Bondi-Sachs formalism briefly (though we have already used a simplified version of it in the previous section). We will also briefly comment about the symmetry group for other boundary conditions.

4.3.1 The Bondi-Mentzner-Sachs Group

4.3.1.1 The Bondi Sachs co-ordinate system and conventions

The basic consideration here is to start with a scalar field u (or ν) such that the u =constant hypersurfaces are outgoing of future light cones (similarly ν =constant represents incoming or past light cones). We further assume that the generator (represented by the null tetrad $\boldsymbol{l} = l^a \partial_a$) of null surfaces u =constant is geodetic, i.e.

$$l^\alpha \nabla_\alpha l^\beta \propto l^\beta \tag{4.59}$$

Define $l^\alpha = \frac{dx^\alpha}{dr} = \delta^\alpha_r$. Thus, $l^\alpha \nabla_\alpha = D = \frac{\partial}{\partial r}$ when acting on scalars. Choose r to be an affine parameter on the integral curves of l^α. Thus we have

$$Dl^\alpha = 0 \tag{4.60}$$

The angular co-ordinates $x^A = (\theta, \phi)$ are defined such that they are constant along the flow of vector field l^α:

$$Dx^A = 0 \tag{4.61}$$

The Bondi-Sachs coordinate $x^\alpha = (u, r, x^A)$ if we are considering outgoing congruence, or (ν, r, x^A) for incoming. The outgoing coordinate system is relevant when we are studying radiations from isolated system, which will be considered here as well. The incoming coordinate is important in the case of cosmology, since all astro-physical observables are essentially defined on observer's past light cone. In this case, the r

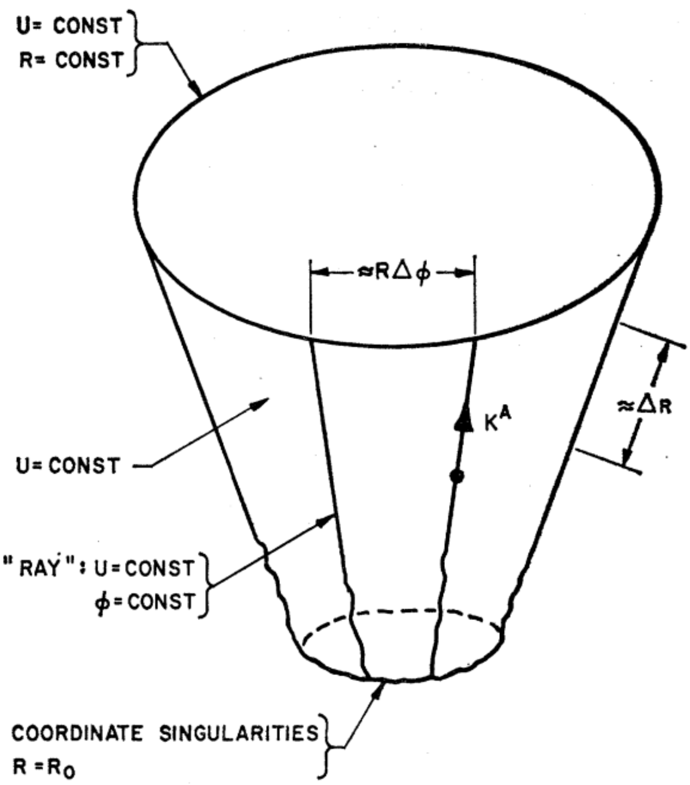


Figure 4.1: Outgoing null cone in Bondi-Sachs coordinates (figure obtained from [4])

coordinate could represent the correxted luminosity distance, or the observed redshift parameter as the relevant problem may be.

Given such a co-ordinate system we can comment about the general form of metric g_{ab}: We can observe that the null condition on l^α implies that

$$l^\alpha l_\alpha = 0 = g_{\alpha\beta}\delta^\alpha_r \delta^\beta_r = g_{rr} \tag{4.62}$$

Likewise, the angular relations (4.61) means

$$l^\alpha \partial_\alpha x^A = g_{\alpha\beta}\delta^\alpha_r \partial^\beta x^A = 0 \tag{4.63}$$

which would imply $g_{rA} = 0$. Thus the generic form of metric can be written as

$$g_{ab} = \begin{pmatrix} \alpha & \beta & U_A \\ \beta & 0 & \mathbf{0} \\ U_A & \mathbf{0} & g_{AB} \end{pmatrix} \tag{4.64}$$

for some functions $\alpha,\ \beta,\ U^A,\ g_{AB}$.

Following the convention used in [21], we write the line-element in the form:

$$ds^2 = \frac{V}{r}e^{2\beta}du^2 + 2e^{2\beta}dudr - r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du) \tag{4.65}$$

The angular part of the line element are written in terms of two functions $\gamma(u, r, x^A)$ and $\delta(u, r, x^A)$ as

$$h_{AB}dx^A dx^B = (e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2)\cosh(2\delta) + 2\sin\theta\sinh(2\delta)d\theta d\phi \tag{4.66}$$

The two functions reflects the fact that gravitational field carries two degrees of freedom, or in perturbative gravitational wave analysis, as the + and × polarizations [22].

It's worth noting that the determinant of h_{AB} is simply $\sin^2\theta$. Thus, the set of conditions

$$g_{rr} = 0 \qquad g_{rA} = 0 \qquad \det[h_{AB}] = \sin^2\theta \tag{4.67}$$

represents our *gauge fixing* conditions. Thus, the *residual gauge symmetry* can be given by transformatons which will preserve (4.67) :

$$\delta g_{rr} = 0 \qquad \delta g_{rA} = 0 \qquad \delta\det[g_{AB}] = 0 \tag{4.68}$$

The functions in line element are all polynomials in $1/r$, as also stated in the previous section. To obtain the expression for the metric functions, we will need to solve the vacuum Einstein's field equations $R_{ab} = 0$. Similar to the discussion in §, we make ansatz that

$$V = r - 2M(u, x^A) + \mathcal{O}(r^{-1}) \tag{4.69}$$
$$h_{AB} = q_{AB} + c_{AB}(u, x^A)r^{-1} + \mathcal{O}(r^{-2}) \tag{4.70}$$
$$\beta = \beta_0(u, x^A)r^{-1} + \mathcal{O}(r^{-2}) \tag{4.71}$$
$$U^A = U_0^A(u, x^A)r^{-1} + \mathcal{O}(r^{-2}) \tag{4.72}$$

To solve the field equations, we impose the *boundary conditions* [21],[4]:

$$\lim V/r = 1 \qquad \lim(rU^A) = \lim\beta = \lim h_{AB} = 0 \tag{4.73}$$

where lim is the limit at $r \to \infty$. The asymptotic forms satisfying field equations and respecting the boundary conditions are given by:

$$V = r - 2M(u, x^A) + \mathcal{O}(r^{-1}) \tag{4.74}$$
$$h_{AB} = q_{AB} + \mathcal{O}(r^{-1}) \tag{4.75}$$
$$\beta = \frac{1}{4}|c(u, x^A)|^2 r^{-2} + \mathcal{O}(r^{-3}) \tag{4.76}$$
$$U^A = \mathcal{O}(r^{-2}) \tag{4.77}$$

4.3.1.2 Explicit construction due to Sachs

Let us first solve the *residual gauge symmetries* (4.68).

The expression (4.68)(i) implies that

$$\delta g_{rr} = 0 \leftrightarrow \nabla_r \xi_r = 0 \tag{4.78}$$

All the required Christoffel symbols for this calculation are mentioned towards the end of this chapter. The RHS of the above expression can be wriiten as

$$\partial_r \xi_r = \Gamma^a_{rr}\xi_a = 2\partial_r\beta\xi_r \tag{4.79}$$

$$\xi_r = e^{2\beta} f(u, x^A) \tag{4.80}$$

where f is an arbitrary function. Likewise (4.68)(ii) implies

$$\begin{aligned}
\delta g_{rA} = 0 \implies \partial_r \xi_A + \partial_A \xi_r &= 2\Gamma^a_{Ar}\xi_a \\
\partial_r \xi_A + 2\partial_A \beta \xi_r + e^{2\beta}\partial_A f &= 2\Gamma^a_{Ar}\xi_a
\end{aligned}$$

On substituting the expression for Γ^a_{Ar}, we obtain:

$$\begin{aligned}
\partial_r \xi_A + 2\partial_A \beta \xi_r + e^{2\beta}\partial_A f &= 2\Gamma^r_{Ar}\xi_r + 2\Gamma^B_{Ar}\xi_B \\
&= 2\partial_A \beta \xi_r + f r^2 h_{AB}\partial_r U^B + 2\frac{\xi_A}{r} + (\partial_r h_{AC})h^{BC}\xi_B
\end{aligned}$$

On RHS, we do some re-arranging of derivatives followed by transvection with g^{AD}:

$$\begin{aligned}
[-g_{AB} f \partial_r U^B - \frac{2\xi_A}{r} + h_{AC}\partial_r(h^{BC}\xi_B) &= -e^{2\beta}\partial_A f] \times g^{AD} \\
f\partial_r U^D + \frac{2\xi_A h^{AD}}{r^3} - \frac{\partial_r(\xi_A h^{AD})}{r^2} &= -(\partial_A f)e^{2\beta} g^{AD} \\
\partial_r(fU^D) - \partial_r\left(\frac{\xi_A h^{AD}}{r^2}\right) &= -(\partial_A f)e^{2\beta} g^{AD} \\
\partial_r(fU^D + \xi_A g^{AD}) &= -(\partial_A f)e^{2\beta} g^{AD} \\
\xi_B g^{BD} &= f^D - fU^D - \int e^{2\beta} g^{AD}\partial_A f dr'
\end{aligned}$$

Thus, the angular part of diffeomorphism vector field ξ is given by

$$\xi_C = f_C - fU_C + r^2 h_{DC}\partial_A f \int_r^\infty dr' \frac{e^{2\beta}h^{AD}}{r'^2} \tag{4.81}$$

In this case, we obtain an arbitrary vector field f^A. Finally for (4.68)(iii) we shall get the ξ_u component:

$$\begin{aligned}
\delta \det g = 0 \implies g^{AB}\mathcal{L}_\xi g_{AB} &= 0 \\
g^{AB} \times [\nabla_A \xi_B + \nabla_B \xi_A] &= 0 \\
g^{AB} \times [\partial_A \xi_B + \partial_B \xi_A - 2\Gamma^c_{AB}\xi_c] &= 0
\end{aligned}$$

On RHS, let us just consider the expression for Γ^u_{AB}, so that we can express ξ_u in terms of ξ_r and ξ_A as follows:

$$g^{AB}(2\xi_{A,B}) + \frac{4e^{-2\beta}}{r}\xi_u + (2\Gamma^r_{AB}\xi_r - 2\Gamma^C_{AB}\xi_C)g^{AB} = 0$$

Thus, we have the following final expression

$$\xi_u = \frac{1}{2} r e^{2\beta}(-\xi_{A,B} + \Gamma^r_{AB}\xi_r + \Gamma^C_{AB}\xi_C)g^{AB} \tag{4.82}$$

$$= -\frac{1}{2}\frac{e^{2\beta}}{r}(-\xi_{A,B} + \Gamma^r_{AB}\xi_r + \Gamma^C_{AB}\xi_C)h^{AB} \tag{4.83}$$

Note that (f, f^A) obtained as integration constants, are arbitrary. In contrast, the calculation for asymptotic Minkowski case had a specific expression for f in terms of f^A, and we also had f^A as CKVF on unit 2-sphere. To obtain this particular form, Sachs considers a set of additional conditions:

$$\delta g_{uu} = \mathcal{O}(r^{-1}) \qquad \delta g_{uA} = \mathcal{O}(1) \tag{4.84}$$

$$\delta g_{ur} = \mathcal{O}(r^{-2}) \qquad \delta g_{AB} = \mathcal{O}(r) \tag{4.85}$$

Lets begin with (4.85)(ii). The aim will be to expand the metric coefficients following the asymptotic forms () and compare order-by-order in powers of r. For this condition we obtain

$$\partial_A \xi_B + \partial_B \xi_A - 2\Gamma^u_{AB}\xi_u - 2\Gamma^r_{AB}\xi_r - 2\Gamma^C_{AB}\xi_C = \mathcal{O}(r) \tag{4.86}$$

After some thorough inspection, we find that the r^2 term in the LHS of (4.86) has the form

$$-2f_{(A,B)} + q_{AB}\partial_C f^C + \gamma^C_{EF} q_{AB} q^{EF} f_C + 2\gamma^C_{AB} f_C = 0 \tag{4.87}$$

where γ^A_{BC} is the Christoffel symbol w.r.t. the unit sphere metric q_{AB}. After some manipulation it can be shown that

$$D_A f_B + D_B f_A = q_{AB} D_C f^C \tag{4.88}$$

which agrees with (4.44).

Lets repeat the same procedure for ()(ii). We have

$$\partial_u \xi_A + \partial_A \xi_u - 2\Gamma^c_{uA}\xi_c = \mathcal{O}(1) \tag{4.89}$$

The only r^2 term in LHS is

$$\partial_u f_A = 0 \tag{4.90}$$

Likewise, the r terms can be collectively written as

$$\partial_A \left[\partial_u f - \frac{1}{4}\partial_D (q_{EF} f^D) q^{EF} \right] = 0 \tag{4.91}$$

For (4.85)(i), we note that the order of 1 term is essentially

$$\partial_u f - \frac{1}{4}\partial_D (q_{EF} f^D) q^{EF} = 0 \tag{4.92}$$

This we can integrate to obtain

$$f(u, x^A) = \alpha(x^A) + \frac{u}{2} D_A f^A \tag{4.93}$$

Note that this particular form of f and f^A will automatically satisfy the remaining conditions.

We can try to find the expression for $\xi^a = g^{ab}\xi_b$. Here, is the following asymptotic forms of each component:

$$\xi^u = f(u, x^A) \tag{4.94}$$

$$\xi^r = -r\partial_u f + \frac{1}{2}D_C D^C f + \mathcal{O}(r^{-1}) \tag{4.95}$$

$$\xi^A = f^A - \frac{q^{AB}\partial_B f}{r} + \mathcal{O}(r^{-2}) \tag{4.96}$$

This completes the derivation of the *Bondi-Metzner-Sachs* group using the *gauge fixing approach.*

4.3.1.3 The group structure

The infinitesimal transformations of (u, x^A) under the action of f, f^A near $r \to \infty$ has the form

$$u \to u + \xi(u) = u + \xi^u = u + \left[\alpha + \frac{u}{2}D_A f^A\right] \tag{4.97}$$

$$x^A \to x^A + \xi(x^A) = x^A + f^A \tag{4.98}$$

Let us define vector fields

$$\xi_T = \alpha\partial_u \tag{4.99}$$

$$\xi_R = f^A\partial_A + \frac{u}{2}D_C f^C\partial_u \tag{4.100}$$

Then, any element of *BMS* group $\mathcal{B}$ can be written as $\xi = \xi_T + \xi_R$. Note that the expression for ξ_T automatically suggest that

$$[\xi_{T_1}, \xi_{T_2}] = 0 \tag{4.101}$$

for all ξ_{T_1} and ξ_{T_2}. We say that ξ_T generates *super-translation* motion. We shall denote $\mathcal{S}$ as the *super-translation* group.

The commutation relation for ξ_R's is a bit more involved:

$$[\xi_{R1}, \xi_{R2}] = [f_1^A\partial_A + \frac{u}{2}D_C f_1^C\partial_u, f_2^A\partial_A + \frac{u}{2}D_C f_2^C\partial_u] \tag{4.102}$$

$$= (f_1^A\partial_A f_2^C - (1 \leftrightarrow 2))\partial_C + \frac{u}{2}(f_1^A\partial_A D_C f_2^C - (1 \leftrightarrow 2))\partial_u \tag{4.103}$$

We first manipulate the ∂_u term. Note that on expanding the D_A operator, we can re-arrange it in the form

$$\frac{u}{2}D_C((f_1^A\partial_A f_2^C - (1 \leftrightarrow 2)) + \frac{u}{2}(\partial_A\gamma^C_{CB})(f_1^A f_2^B - (1 \leftrightarrow 2)) \tag{4.104}$$

where $\gamma^C_{CB} = \frac{\log\sqrt{\det q}}{\partial x^B} = \cot\theta\delta^\theta_B$. However, that makes $\partial_A\gamma^C_{CB} = -\csc^2\theta\delta^\theta_A\delta^\theta_B$, i.e. symmetric in A and B. Since the term in bracket is anti-symmetric in A and B, this makes the entire second term to be zero. Denoting

$$\hat{f}^C = (f_1^A\partial_A f_2^C - (1 \leftrightarrow 2) \tag{4.105}$$

we see that the commutator relation simplifies to just

$$[\xi_{R1}, \xi_{R2}] = \xi_{\hat{R}} \tag{4.106}$$

We say that ξ_R is the generator of *super-rotation* group $\mathcal{R}$.

Finally, we may look at the commutation relation between ξ_R and ξ_T:

$$[\xi_R, \xi_T] = [f^A\partial_A + \frac{u}{2}D_C f^C\partial_u, \alpha\partial_u] = \left(f^A\partial_A\alpha - \frac{\alpha}{2}D_C f^C\right)\partial_u = \xi_{\hat{T}} \tag{4.107}$$

This, relation along with () suggests that for any $\xi \in \mathcal{B}$ and for any $\xi_T \in \mathcal{S}$, we have

$$[\xi, \xi_T] = \xi_{\hat{T}} \in \mathcal{S} \tag{4.108}$$

Thus, we have the following theorem:

[**Theorem** :] The super-translation group $\mathcal{S}$ forms an Abelian Normal subgroup of the total BMS group $\mathcal{B}$

Let us take $r \in \mathcal{R}$ and $s \in \mathcal{S}$. Let $x^a = (u, x^A)$ (note that these are co-ordinates on the null infinity $\mathcal{I}^+$). We may write

$$rs(u, x^A) = r(u + \alpha, x^A) = (K^{-1}(u + \alpha), r(x^A)) = b(u, x^A) \tag{4.109}$$

where $b \in \mathcal{B}$. This particular identification implies that we can write $\mathcal{B}$ as the semi-direct product of $\mathcal{S}$ and $\mathcal{R}$:

$$\mathcal{B} = \mathcal{R}\mathcal{S} \tag{4.110}$$

Note that, (4.107) suggests that for any $b \in \mathcal{B}$

$$b^{-1}\mathcal{S}b \cong_{iso} \mathcal{S} \tag{4.111}$$

So, if we just restrict to the case of $\mathcal{R}$ followed by some re-arranging, we will have

$$s^{-1}\mathcal{R}s \cong_{iso} \mathcal{R} \tag{4.112}$$

This is important as it suggests that there is no unique rotation subgroup of the total BMS group.

Recall that the $l \leq 2$ modes of the super-translation parameter α corresponds to the translation group $\mathcal{T}$. In fact, following from above Theorem, we see that $\mathcal{T}$ form the unique normal 4-dimensional subgroup of the total super-translation group. Thus, if we wish to find the *Poincare* subgroup $\mathcal{P}$ from $\mathcal{B}$, then there will be arbitrariness in the choice of Lorentz group, due to the isomorphism (4.112).

4.3.1.4 Few comments about Poincare group at infinity

Although, there is no unique way one can select out a Poincare group from $\mathcal{B}$, in practice, we can choose a particular rotation group $\mathcal{R}$ near infinity using the idea of *good cuts*. A *good cut* is the *cross-section* on $\mathcal{I}$ formed by its intersection with a null cone originating from an interior point $p \in \tilde{\mathcal{M}}$. If there is no such interior point, then it is called a *bad cut*. The convention is that we choose the rotation group $\mathcal{R}$ as the generator of conformal motion of these *good cuts*. Let us consider the example for Minkowski space-time.

Consider the flat space line element:

$$ds^2 = du^2 + 2dudr - r^2 q_{AB} dx^A dx^B \tag{4.113}$$

To define the Lorentz group at infinity, we need to choose an origin, similar to the consideration in §2.1. We let $r = 0$ to be our *axis*. The null surfaces originating from each point p on the axis defines our outgoing surfaces u =const, and these surfaces will intersect the boundary forming a 2-sphere. Thus $r = 0$ forms the set of all interior points for the *good cuts*. While this is very convenient in a flat space-time, such a construction need not hold for a general asymptotically flat space-tmes. The intricate structure of the bulk can modify these definitions of *good cuts*. One way to circumvent this problem is to look at the optical scalars, or any of it's combinations associated with the outgoing null congruence, which do not change under the influence of curvature. The asymptotic shear σ of the outgoing null surfaces is a convenient choice. The peeling theorem mandates that

$$\sigma = \sum_{i=0} \sigma_i r^{-i} \tag{4.114}$$

For *good cuts*, we have $\sigma_0 = 0$. Thus, the standard choice is to look at all those cross-sections on $\mathfrak{I}$, which have $\sigma \approx 0$.

Note that σ is a complex number, while the identification of *good cuts* involves only one real condition. Thus, in general it is not true that good cuts defined this way will always exist. In fact, existence of gravitational radiation need not preserve the *good cuts*. Details of this calculation will not pursued here, but has been explained in [18]. To summarize, for a Bondi system $\tau \approx 0$, the evolution of shear σ follows this particular relation

$$\partial_u^2 \sigma \approx -A\bar{\psi}_4 \tag{4.115}$$

We can model shock waves, in general, which can induce a generic BMS transformation near null infinity (this will be discussed in the next chapter). If we simply consider the super-translation effect, we will see that the asymptotic shear will change following the relation

$$\sigma_2 \approx \sigma_1 + \frac{1}{2}\eth^2\alpha \tag{4.116}$$

Note that solution for $\eth^2\alpha = 0$ is true if and only if $\alpha \in \mathcal{T}$. Thus, only the translation group is invariant. Since σ changes, these cross-sections are now *bad cuts* and we will now need to choose a new set of *good cuts*. This essentially reflects the fact that the rotation group $\mathcal{R}$ is not preserved under the conjugate action of the super-translation group. Since, the concept of rotation shifts in this process, we say that super-translation changes the angular-momentum of the system. In other words, the charges associated to (higher modes of) super-translation are essentially the angular-momentum.

4.3.2 Other asymptotic definitions

The arbitrariness in the choice of gauge fixing condition makes this particular approach highly generic. Let us consider a couple of other symmetry groups which has been constructed using this approach:

4.3.2.1 Barnich-Troessaert group

Let us consider the case for null infinity: Note that in the BMS derivation, we had defined the determinant of the angular part of the metric as $r^4 \sin^2\theta = \det g_{AB}$. In general, let us write $\det g_{AB} = r^4 b$, where $b = b(u, x^A) = \frac{1}{4}e^{4\tilde{\varphi}}$ is some arbitrary function. The consequence is that the leading term of the angular part of line element is only conformal to a unit sphere line element. On top of that, let us modify the fall-off conditions on metric coefficients:

$$\beta = \mathcal{O}(r^{-2}) \qquad U^A = \mathcal{O}(r^{-2}) \qquad V/r = -2r\partial_u\tilde{\varphi} + \tilde{\Delta}\tilde{\varphi} + \mathcal{O}(r^{-1}) \tag{4.117}$$

The associated residual symmetries should satisfy

$$\delta g_{rr} = 0 \qquad \delta g_{rA} = 0 \qquad \delta \det[g_{AB}] = 4\tilde{\omega} \tag{4.118}$$

The determinant condition shows that we are only preserving the angular part upto a scale factor, where $\omega(u, x^A)$ is an arbitrary function. The additional conditions on other metric components are

$$\delta g_{ur} = \mathcal{O}(r^{-2}) \qquad \delta g_{rA} = \mathcal{O}(1) \qquad \delta g_{AB} = \mathcal{O}(r) \tag{4.119}$$

$$\delta g_{uu} = -2r\partial_u\omega - 2\omega\tilde{\Delta}\tilde{\varphi} + \tilde{\Delta}\omega + \mathcal{O}(r^{-1}) \tag{4.120}$$

The exact solution for such a diffeomorphism vector field is given by

$$\xi^u = f \tag{4.121}$$

$$\xi^r = f^A - f_{,B}\int_r^\infty dr'(e^{2\beta}g^{AB}) \tag{4.122}$$

$$\xi^r = -\frac{1}{2}r(\bar{D}_A\xi^A - f_{,B}U^B + 2f\partial_u\tilde{\varphi} - 2\omega) \tag{4.123}$$

where $f = f(u, x^A)$ satisfies

$$\partial_u f = f\partial_u\varphi + \frac{1}{2}\bar{D}_A f^A - \omega \tag{4.124}$$

Here $\bar{D}_A$ and f^A are the covariant derivatives and conformal Killing field for the metric $e^{2\tilde{\varphi}}q_{AB}$. The arbitrariness of ω also makes f^A to be an element of infinite-dimensional Virasoro algebra rather than 6-dimensional Lorentz algebra as was in the case of BMS. Therefore, the transformation as defined above generates a group which is isomorphic to (Vir×Vir) $\ltimes$ $\mathcal{S}$.

For asymptotically AdS space-times, we could start with line element written in Fefferman-Graham form:

$$ds^2 = \frac{l^2}{r^2}dr^2 + g_{AB}(r, x^C)dx^A, dx^B \tag{4.125}$$

where $g_{AB} = r^2 e^{2\tilde{\varphi}}\gamma_{AB} + \mathcal{O}(1)$. $\gamma_{AB}dx^A dx^B$ is the flat space-time line element.

The residual conditions are

$$\delta g_{rr} = 0 \qquad \delta g_{rA} = 0 \qquad \delta g_{AB} = \mathcal{O}(1) \tag{4.126}$$

The solution for such a vector field is given as

$$\xi^r = -\frac{1}{2}r\bar{D}_A f^A \tag{4.127}$$

$$\xi^A = f^A - \frac{l^2}{2}\partial_B(\bar{D}_C f^C)\int_r^\infty \frac{dr'}{r'}g^{AB} \tag{4.128}$$

Similar arguments hold for asymptotically dS space-times as well.

4.3.2.2 Campiglia-Laddha group

For a line element of the form

$$ds^2 = \mathcal{O}(1)du^2 + (2 + \mathcal{O}(r^{-1}))dudr - (r^2 q_{AB} + \mathcal{O}(r))dx^A dx^B + \mathcal{O}(1)dudx^A \tag{4.129}$$

we simply look at the condition

$$\lim_{r\to\infty} \nabla_{(a}\xi_{b)} \to 0 \tag{4.130}$$

The residual symmetries in this case are:

$$\delta g_{ur} = \mathcal{O}(r^{-1}) \qquad \delta g_{uA} = \mathcal{O}(r) \qquad \delta g_{uu} = \mathcal{O}(1) \tag{4.131}$$

Upon solving, we obtain the following expression for ξ:

$$\xi^u = \underbrace{\alpha(x^A) + \frac{u}{2}D_A f^A}_{f} + \mathcal{O}(r^{-1}) \tag{4.132}$$

$$\xi^r = -r\partial_u f + \mathcal{O}(1) \tag{4.133}$$

$$\xi^A = f^A + \mathcal{O}(r^{-1}) \tag{4.134}$$

Here f^A and f are arbitrary. The symmetry group in this case is Diff(S^2) $\ltimes$ $\mathcal{S}$

4.4 Geometric Approach

The *Geometric approach* was originally proposed by Penrose in his seminal paper [5]. The lengthy calculation for the BMS group can be elegantly and intuitively derived as conformal motion of the boundary $\mathscr{I}$. In fact, this is what we are going to discuss in this section and also comment about its relation to include Barnich-Troessaert group (both asymptotically flat or (A)dS) and beyond.

4.4.0.1 BMS group

In the previous chapter, we talked about properties of infinity: the divergence free and shear free condition on N_a vector field. Let us now choose $\Omega = l = \frac{1}{r}$ as our conformal factor. It is easy to see for standard examples that N_a vector field does satisfy the necessary conditions for this particular choice. For our physical line element, let us choose the Bondi-Sachs form. The unphysical line element will therefore look like

$$ds^2 = \Omega^2 d\tilde{s}^2 = l^3 V e^{2\beta} du^2 - 2e^{2\beta} du dl - h_{AB}(dx^A - U^A du)(dx^B - U^B du) \quad (4.135)$$

However, for now let's just assume that each of the metric functions are Taylor expandable about $l = 0$. Now, let us consider the solutions for Einstein's equations on the unphysical line-element [22]:

$$h_{AB} = H_{AB}(u, x^C) + \mathcal{O}(l) \quad (4.136)$$

$$\beta = H(u, x^C) + \mathcal{O}(l^2) \quad (4.137)$$

$$U^A = H^A(u, x^C) + \mathcal{O}(l) \quad (4.138)$$

$$l^2 V = D_A H^A + \mathcal{O}(l) \quad (4.139)$$

ehre D_A is covariant derivative w.r.t. H_{AB}. The N^a vector field will look like

$$N^a = (-e^{-2H}, 0, e^{-2H} H^A) \quad (4.140)$$

If we impose that x^A are constants along N^a, we get $H^A = 0$:

$$N^a \partial_a x^A \approx 0 \implies H^A = 0 \quad (4.141)$$

Furthermore, the *strong conformal geometry* , implies $H = 0$:

$$\hat{N}^a \partial_a u \approx 1 \implies H = 0 \quad (4.142)$$

Finally, $\nabla_a \nabla_b \Omega \approx 0$ implies $\partial_u H_{AB} = 0$. If we consider H_{AB} as metric on the cross-section of $\mathscr{I}^+$, then it can only be conformal to a unit sphere metric. Then $\partial_u H_{AB} = 0$ implies that $H_{AB} = \omega^2(x^C) q_{AB}$. On suitable rescaling and redefinition we finally have

$$g_{ab} \approx \begin{pmatrix} 0 & -1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -q_{AB} \end{pmatrix} \quad (4.143)$$

BMS group is precisely the diffeomorphism transformation which will preserve the above form of metric upto a scale-factor. The scale factor is such that the transformation is well defined under the conformal map. To prove this, let us start with the physical line element If $\tilde{x} \to \tilde{x}'$ is the required transformation, we have

$$\tilde{g}_{ab}(\tilde{x}') = \tilde{g}_{cd}(\tilde{x}) \frac{\partial \tilde{x}^c}{\partial \tilde{x}'^a} \frac{\partial \tilde{x}^d}{\partial \tilde{x}'^b} \tag{4.144}$$

Say $x \to x'$ is the induced transformation in $\mathcal{M}$, then it preserves the space-time interval in $\mathcal{M}$ upto a scale factor:

$$g_{ab}(x') = \frac{\Omega^2(x)}{\Omega^2(x')} g_{cd}(x) \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} \tag{4.145}$$

Since (4.145) holds at $r \to \infty$, we should regard the above relation to hold on $\mathcal{I}$. If $x' = x + \epsilon\xi$, then

$$dx'^a = (\delta^a_b + \epsilon \partial_b \xi^a) dx^b \tag{4.146}$$

$$g_{ab}(x') = g_{ab}(x) + \epsilon \xi^c \partial_c g_{ab}(x) \tag{4.147}$$

$$\Omega^{-2}(x') = \Omega^{-2}(x)(1 - 2\epsilon \Omega^{-1} \xi^c \partial_c \Omega) \tag{4.148}$$

We see that condition (4.145) is true upto $\mathcal{O}(\epsilon^2)$ if

$$(\mathcal{L}_\xi g_{ab} - 2\Omega^{-1}\xi^c \partial_c \Omega g_{ab})|_{\mathcal{I}} = 0 \tag{4.149}$$

Since $\Omega|_{\mathcal{I}} = 0$, we need $\xi^c \partial_c \Omega = \mathcal{O}(\Omega)$ so that the scale factor is finite. Let's Taylor expand ξ about $\mathcal{I}$ w.r.t. the coordinate system $x = (u, l, x^A)$:

$$\xi^a = \xi^a_0 + \xi^a_1 l + \mathcal{O}(l^2)$$

Choosing $\Omega = l$, $\xi^c \partial_c l = \xi^l = \mathcal{O}(l)$, or $l^{-1}\xi^l|_{\mathcal{I}} = \xi^l_1 = \partial_l \xi^l|_{\mathcal{I}}$. Lets denote each expression in () as $Q_{ab} = 0$. Upon substituting the expression for g_{ab} we obtain the following form (all equalities are defined on $l = 0$)

$$Q_{uu} = 0 \implies \partial_u \xi^l = 0 \tag{4.150}$$

$$Q_{AB} = 0 \implies D_A \xi_B + D_B \xi_A = 2\partial_l \xi^l q_{AB} \implies \partial_l \xi^l = \frac{1}{2} D_A \xi^A \tag{4.151}$$

$$Q_{ul} = 0 \implies \partial_l \xi^l + \partial_u \xi^u - 2\partial_l \xi^l = 0 \implies \partial_u \xi^u = \partial_l \xi^l \tag{4.152}$$

$$Q_{uA} = 0 \implies \partial_A \xi^l + q_{AB} \partial_u \xi^B = 0 \tag{4.153}$$

$$Q_{ll} = 0 \implies \partial_l \xi^u = 0 \tag{4.154}$$

$$Q_{lA} = 0 \implies \partial_A \xi^u + q_{AB} \partial_l \xi^B = 0 \tag{4.155}$$

While (4.151) implies $\xi^l_1 = \frac{1}{2} D_A \xi^A_0$, the condition (4.152) can be integrated to obtain $\xi^u_0 = \alpha + \frac{u}{2} D_A \xi^A_0$. (4.154) implies $\xi^u_1 = 0$, while (4.153) leads to $\partial_u \xi^B_0 = 0$. Equation (4.155) gives $\xi^A_1 = -q^{AB} \partial_B \xi^u_0$. Thus, we obtain the following asymptotic expression for ξ:

$$\xi^u = \underbrace{\alpha(x^A) + \frac{u}{2} D_A f^A}_{f} + \mathcal{O}(l^2) \tag{4.156}$$

$$\xi^l = \frac{1}{2} D_A f^A l + \mathcal{O}(l^2) \tag{4.157}$$

$$\xi^A = f^A - q^{AB} \partial_B f l + \mathcal{O}(l^2) \tag{4.158}$$

where α is arbitrary and f^A is CKVF on unit sphere. This proves our claim.

4.4.0.2 Extended BMS group

The derivation of BMS group required a very particular form of g_{ab} at infinity. Although, it is always possible to arrange for such a simplistic form in Bondi-Sachs formalism, we see that the BMS transformation only preserve the metric upto a scale factor. Now, there exists other exact solutions for Einstein's field equations, which do not satisfy the above mentioned boundary conditions. Consider the Robinson-Trautman space-times, which have the form:

$$ds^2 = \left(\Delta \log P - 2r \partial_u \log P - \frac{2m(u)}{r} \right) du^2 + 2dudr - \frac{2r^2}{P^2} d\zeta d\bar{\zeta} \tag{4.159}$$

where $P = P(u, \zeta, \bar{\zeta})$ satisfies

$$\Delta\Delta \log P + 12m \partial_u \log P - 4\partial_u m = 0 \tag{4.160}$$

The operator $\Delta = 2P^2 \partial_\zeta \partial_{\bar{\zeta}}$. This is a vacuum solution and involves outgoing gravitational radiation, signified by the non-zero Weyl curvature Ψ_4:

$$\Psi_4 = r^{-2} \partial_{\bar{\zeta}} [P^2 \partial_{\bar{\zeta}} \{ \frac{1}{2} \Delta \log P - r \partial_u \log P \}] \tag{4.161}$$

In fact it can be shown using Lyapunov-functional argument that for any general initial data on $u =$const , the solution will emit gravitational radiation and asymptotically reach Schwarzschild solution. If we now take $\Omega = l = 1/r$ and look at unphysical metric, then at $l = 0$, it takes the form:

$$g_{ab}|_{l=0} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P^{-2} \\ 0 & 0 & -P^{-2} & 0 \end{pmatrix} \tag{4.162}$$

Since we have outgoing gravitational radiation, $\partial_u P \neq 0$, hence we can't allow any transformation so that the angular part reduces to a unit-sphere metric. Thus, while the usual BMS symmetry group can't be used in this example, we can still use the definition (4.149) to construct diffeomorphism which will preserve the line element upto a scale factor. For the time being, let's assume $g_{ul} = -W(u, x^A)$ (some arbitrary function)

$$\xi^u = f + \mathcal{O}(l^2) \tag{4.163}$$

$$\xi^l = \left(\frac{1}{2}\mathcal{D}_A f^A + \frac{1}{2} f \partial_u \log\sqrt{|H|}\right) l + \mathcal{O}(l^2) \tag{4.164}$$

$$\xi^A = f^A - W H^{AB}\mathcal{D}_B f\, l + \mathcal{O}(l^2) \tag{4.165}$$

where f and f^A satisfies:

$$\partial_u f = \frac{1}{2}\mathcal{D}_A f^A + \frac{1}{2} f \partial_u \log\sqrt{|H|} - f\partial_u \log|W| - f^A \partial_A \log|W| \tag{4.166}$$

$$\mathcal{D}_{(A} f_{B)} - \frac{1}{2}\mathcal{D}_C f^C H_{AB} + \frac{1}{2} f(\partial_u H_{AB} - H_{AB}\partial_u \log\sqrt{|H|}) = 0 \tag{4.167}$$

Note that the solution space is much larger than usual BMS group, as both (f, f^A) are infinite-dimensional.For Robinson-Trautman metric, $W = -1$ and $H_{AB} = -P^{-2} q_{AB}$ is the angular part. Since the angular part of metric is conformal to unit sphere, it may be shown that

$$\partial_u H_{AB} = H_{AB}\partial_u \log\sqrt{|H|} \tag{4.168}$$

which makes f^A as the CKVF on the cross-sections. If we let $\partial_u f^A = 0$, then with the condition $\omega = 0$, it is possible to show that (4.166) agrees with the Barnich-Troessaert form. Thus the ξ diffeomorphism field we have constructed generates asymptotic isometry of space-times which are asymptotically flat locally.

4.4.0.3 For asymptotically AdS space-times

To end this section, let us also comment about symmetry group of asymptotically (A)dS$_{3+1}$ space-times. Consider the line element in Fefferman-Graham form:

$$ds^2 = -\frac{l^2}{r^2}dr^2 + \underbrace{g_{AB}(r, x^C)}_{r^2 e^{2\varphi}\eta_{AB}} dx^A dx^B \tag{4.169}$$

where $\eta_{AB} dx^A dx^B = dt^2 - \cdots$. Once again, let $\Omega = 1/r$ so that the unphysical line element takes the form:

$$ds^2 \to \frac{l^2}{r^2} d\left(\frac{1}{r}\right)^2 + e^{2\varphi}\eta_{AB} dx^A dx^B \tag{4.170}$$

The metric on the boundary is given by

$$g_{ab}|_{\mathcal{I}} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{\gamma}_{AB} \end{pmatrix} \tag{4.171}$$

where $\gamma_{\bar{A}B} = e^{2\varphi}\eta_{AB}$. It may be verified that the solution for asymptotic relation by taking the above form of metric is given by

$$\xi^l = \frac{1}{2}\bar{D}_C Y^C l + \mathcal{O}(l^2) \tag{4.172}$$

$$\xi^A = Y^A + \mathcal{O}(l^2) \tag{4.173}$$

In physical co-ordinates we get

$$\xi^r = -\frac{1}{2} r \bar{D}_C Y^C + \mathcal{O}(1) \tag{4.174}$$

$$\xi^A = Y^A + \mathcal{O}(r^{-2}) \tag{4.175}$$

where Y^A is a CKVF w.r.t. 2-metric $\bar{\gamma}$ with $\bar{D}$ as the corresponding covariant derivative. The leading terms exactly matches with the asymptotic expression discussed in [6], thereby validating the usefull-ness of geometric approach. It should be noted that the definition (4.149) which followed from isometry condition of physical space-time only determines the leading two terms of the asymptotic expansion, while the gauge fixing approach provided an *unique* vector field with all higher order terms pre-determined from metric coefficients. Thus, in this sense the vector-field obtained from the geometric approach is more generic. It is therefore, more meaningful to consider the conformal motion of boundary $\mathscr{I}$ if we want to look at isometry near infinity in physical space-time. However, such transformation need not extend into the bulk in any meaningful way.

In general, an asymptotic symmetry group in geometric approach is defined as the quotient group:

$$G = \frac{\textit{Gauge transformations preserving boundary conditions}}{\textit{Trivial gauge transformations}} \tag{4.176}$$

The trivial transformations are the one which has a vanishing surface charge. The weak point of this construction is that the boundary conditions are defined in a particular way, and it is not trivial to fiddle with them. Thus the construction is much more rigid compared to the flexible nature of gauge fixing approach

4.5 Asymptotic Symmetry group near Null surfaces in the Bulk

4.5.0.1 Coordinate conventions

As of now, all of our constructions were defined near infinity, where for a particular choice of coordinates, the metric assumed a very simple form for which we are able to construct the symmetry transformations. We now ask, whether similar construction can also be defined in the bulk, near some family of hypersurfaces. In fact, it is always possible to find co-ordinates $x^a = (u, \rho, x^A)$ such that line element near any hyper-surface takes the simplified form:

$$ds^2 = 2C_\rho du^2 + 2dud\rho + 2\rho\theta_A dx^A du - (\Omega_{AB} + \rho\lambda_{AB})dx^A dx^B + \mathcal{O}(\rho^2) \qquad (4.177)$$

The resemblance with the asymptotic line element (4.135) makes the analysis for asymptotic symmetry group convenient. To begin with, let us consider family of 2-dimensional hypersurfaces $\{S\}$ with co-ordinates $Z^a(x^A)$. If g_{ab} is the total space-time metric , the induced metric on the 2-surface S will be given by

$$q_{AB} = e^a_A e^b_B g_{ab} \qquad (4.178)$$

where $e^a_A = \frac{\partial Z^a}{\partial x^A}$. Also note that the 2-surface S will admit two orthogonal transverse null directions, denoted by l and n. Transverse condition implies that $l \cdot n = 1$. There is a gauge freedom associated to the choice of these null directions: i.e. the rescaling

$$l \to l' = \lambda l \qquad (4.179)$$

$$n \to n' = \lambda^{-1} n \qquad (4.180)$$

preserves the normality condition. The total space-time metric g_{ab} can now be written as

$$g_{ab} = l_a n_b + l_b n_a - q_{ab} \qquad (4.181)$$

In Newman-Penrose formalism we may write $q_{ab} = 2m_{(a}\bar{m}_{b)}$.

Let us assume that the null surface H is foliated by these family of 2-surfaces $\{S_u\}_u$ where we choose scalar field u to parameterize each member of this family. Now, it may be shown that for any H, there exists an unique vector-field $\mathcal{V} \in TH$ such that

1. $\mathcal{V}$ is normal to S_u
2. $\mathcal{V}$ is tangent to H
3. the Lie derivative $\mathcal{L}_\mathcal{V} u = 1$

The vector field $\mathcal{V} = \frac{\partial}{\partial u}$ defines the evolution from one cross-section S_u to other. Just like in ADM formalism we may write

$$\frac{\partial}{\partial u} = \mathcal{V} + \tilde{\mathcal{V}} \qquad (4.182)$$

where $\tilde{\mathcal{V}}$ is tangent to S_u (analogue to "shift vector" in 3+1 case). $\mathcal{V}$ can be thought of as the null vector field, whose integral curves pass through all the cross-sections having the same x^A co-ordinates. This means

$$\mathcal{L}_\mathcal{V} x^A = 0 \qquad (4.183)$$

Thus we have $\tilde{\mathcal{V}}^A = -\mathcal{L}_\mathcal{V} x^A$. Since, for a given null surface we already have an unique vector $\mathcal{V}$ satisfying the above properties, let us choose

$$l = \mathcal{V} = \frac{\partial}{\partial u} - \tilde{\mathcal{V}} \qquad (4.184)$$

This of course fixes the gauge freedom in the choice of transverse null vectors discussed earlier. To completely fix the gauge freedom in NP tetrad, let us also impose the condition

$$\tilde{\mathcal{V}}^A|_H = 0 \tag{4.185}$$

Now, define the other null vector

$$n = \frac{\partial}{\partial \rho} \tag{4.186}$$

where ρ is an affine parameter on the null rays generated by n^a. So, if we work with co-ordinate system $x^a = (u, \rho, x^A)$, its easy to guess the form of null tetrads on H: $l^a|_H = (1, B(u, x^A), \mathbf{0})$, $n^a|_H = (0, 1, \mathbf{0})$. We can compute g^{ab} (using ()) and therefore the line element has the form:

$$ds^2 = 2\kappa\rho du^2 + 2du d\rho + 2\rho\theta_A dx^A du - (\Omega_{AB} + \lambda_{AB}\rho)dx^A dx^B + \mathcal{O}(\rho^2) \tag{4.187}$$

for some functions $\kappa, \theta_A, \lambda_{AB}$. Similar construction can be carried out near space-like or time-like surfaces, and is a generic result (see [23])

4.5.0.2 Extended symmetry near null surface H

Let us consider a particular construction of a symmetry group which is defined near a null surface H. The extended symmetry group by L. Donnay et al essentially corresponds to diffeomorphism transformation which preserves the co-ordinate convention as defined above. That is, the residual symmetry should satisfy:

$$\delta g_{\rho\rho} = \delta g_{\rho A} = \delta g_{u\rho} = 0 \tag{4.188}$$

Lets denote the diffeomorphism vector field near H as χ. The first condition implies

$$\mathcal{L}_\chi g_{\rho\rho} = 0 \leftrightarrow 2g_{\rho a}\partial_\rho \chi^a = 0 \tag{4.189}$$

or $\chi^u = f(u, x^A)$. The second condition can be written as:

$$\mathcal{L}_\chi g_{\rho A} = 0 \leftrightarrow \partial_A \chi^u + g_{Au}\partial_\rho \chi^u + g_{AB}\partial_\rho \chi^B = 0 \tag{4.190}$$

The second term is zero due to (4.189). Thus

$$\chi^B = Y^B - \int_0^\rho d\rho' g^{AB}\partial_A f \tag{4.191}$$

Finally, the third condition yields

$$\mathcal{L}_\chi g_{u\rho} = 0 \leftrightarrow \partial_\rho \chi^\rho + \partial_u \chi^u + g_{uA}\partial_\rho \chi^A = 0 \tag{4.192}$$

Substituting the expression for χ^A, we get

$$\chi^\rho = Z - \rho\partial_u f + \int_0^\rho d\rho' g_{uA} g^{AB}\partial_B f \tag{4.193}$$

The asymptotic form of this vector field is therefore:

$$\chi^u = f(u, x^A) \tag{4.194}$$

$$\chi^\rho = Z(u, x^A) - \partial_u f \rho + \mathcal{O}(\rho^2) \tag{4.195}$$

$$\chi^A = Y^A + \Omega_{AB}\partial_B f \rho + \mathcal{O}(\rho^2) \tag{4.196}$$

The higher order terms in $\mathcal{O}(\rho^2)$ can be determined exactly, if the complete asymptotic expansion of g_{ab} is known. Let us now compute the Lie derivative of rest of the metric components:

$$\mathcal{L}_\chi g_{AB} = \chi^c\partial_c g_{AB} + g_{Ac}\partial_B\chi^c + g_{Bc}\partial_A\chi^c \tag{4.197}$$

The first term in the RHS can be expanded as: $\chi^a\partial_a g_{AB} = -(f\partial_u\Omega_{AB} + Z\lambda_{AB} + Y^C\partial_C\Omega_{AB}) + \mathcal{O}(\rho)$. Likewise, $g_{Ac}\partial_B\chi^c = -\Omega_{AC}\partial_B Y^C + \mathcal{O}(\rho)$. Thus

$$\mathcal{L}_\chi g_{AB} = \underbrace{-(f\partial_u\Omega_{AB} + Z\lambda_{AB} + \mathcal{L}_Y\Omega_{AB})}_{\delta\Omega_{AB}} + \mathcal{O}(\rho) \tag{4.198}$$

Similarly for δg_{uu} , we have

$$\mathcal{L}_\chi g_{uu} = \chi^c\partial_c g_{uu} + 2g_{uc}\partial_u\chi^c \tag{4.199}$$

The first term is $\chi^c\partial_c g_{uu} = 2\kappa Z + \mathcal{O}(\rho)$, while the second term $2g_{uc}\partial_u\chi^c = 2\partial_u Z + \mathcal{O}(\rho)$. Thus

$$\mathcal{L}_\chi g_{uu} = \underbrace{2\partial_u Z + 2\kappa Z}_{=0} + \mathcal{O}(\rho) \tag{4.200}$$

Similar calculations for $\mathcal{L}_\chi g_{uA}$ yields

$$\mathcal{L}_\chi g_{uA} = \underbrace{(Z\theta_A + \partial_A Z - \Omega_{AB}\partial_u Y^B)}_{=0} + \mathcal{O}(\rho) \tag{4.201}$$

We will assume that $Z = 0$ (this will be important later for definition of charge) and $\partial_u Y^B = 0$. Note that

$$\mathcal{L}_\chi g_{u\rho}|_H = 0 \qquad\qquad \mathcal{L}_\chi g_{AB}|_H = -(f\partial_u\Omega_{AB} + \mathcal{L}_Y\Omega) \neq 0 \tag{4.202}$$

4.6 Charge construction

For flat space-time, we noted that for a given matter field T_{ab}, we can construct ten conserved charges associated to each Killing field. The situation in presence of gravitational field is ambiguous for two reasons - non-local nature of gravitational energy and non-existent of a generic Killing field. If we look at the expression for charge in (4.30), we note that its an integral over a 3-dimensional surface. If we can express it as an integral on the boundary of this surface, then the Killing field on the boundary can be naively identified with an element of the asymptotic symmetry group. However, there is still an issue with the T_{ab}: how do we take into account gravitational energy? In this section, we are going to briefly outline the original Penrose's quasi-local charge construction and its expression on cross-section $\mathcal{S} \subset \mathcal{I}$. Using L.Mason's argument, we will show how this construction will agree with the one obtained from Covariant Phase space formalism. Finally we will look at charges for extended symmetries.

4.6.0.1 Penrose's Quasi-local charge motivation

The motivation stems from the fact that the total Riemann curvature in the weak field limit on a background $\mathbb{M}$ space-time has the structure of Weyl curvature:

$$K_{abcd} = \phi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + c.c. \tag{4.203}$$

which is also very similar to the expression for Maxwell tensor:

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + c.c. \tag{4.204}$$

The charge in Maxwell theory can be written in terms of Hodge dual $*F_{ab}$ as:

$$q = \frac{1}{4\pi}\Im \oint_{\mathcal{S}} *F \tag{4.205}$$

The question is whether we can write F_{ab} in terms of K_{abcd} and what will the corresponding q physically represent. So, for a given symmetric 2-twistor σ^{AB}, it follows that the spinor

$$\chi_{AB} = \phi_{ABCD}\sigma^{AB} \tag{4.206}$$

will satisfy free Maxwell's equations if ϕ_{ABCD} satisfies the vacuum Einstein's equation. We can also write the above equation in tensor notation: for that, we define skew tensor $Q_{ab} = i\sigma^{AB}\epsilon^{A'B'} + c.c.$. Then it may be shown that

$$^{*}P_{ab} = -\frac{1}{2}K_{abcd}Q^{cd} \tag{4.207}$$

where P_{ab} is the Maxwell tensor associated to χ_{AB}. Also note that $\xi^a = \frac{1}{3}\nabla_b Q^{ba}$ is a Killing field. Now, since we started with the vacuum solution for K_{abcd}, it may be shown that $K_{abcd} = K^*_{abcd}$ [16]. Using this alteration and Einstein's equations, we can write (4.205) in the form

$$q = -\frac{1}{32\pi G}\oint_{\partial\mathcal{V}} K^*_{abcd}Q^{cd}dx^a \wedge dx^b = -\frac{1}{6}\int_{\mathcal{V}} e_{abc}{}^{d}T_{df}\xi^f dx^a \wedge dx^b \wedge dx^c \tag{4.208}$$

For the asymptotic case, we take $\mathcal{S} \subset \mathcal{I}$, where space-time is vacuum. So we may simply write $K_{abcd} = C_{abcd}$. Also, since we are considering calculations on $\mathcal{I}$, we should regard the curvatures here to be unphysical. With some manipulation, the integral in LHS of (4.208) can be written as

$$q = -\frac{1}{4\pi G}Im\oint_{\mathcal{S}}\{\sigma^{00}\psi_1 + 2\sigma^{01}\psi_2 + \sigma^{11}\psi_3\}\mathcal{S} \tag{4.209}$$

σ^{AB} can be written as symmetrized product of 1-twistor ω_i^A. Let's take $\sigma^{AB} = \omega^{(A}\omega^{B)}$. Thus, the expression for q is

$$q = -\frac{1}{4\pi G}Im\oint_{\mathcal{S}}\{\psi_1\omega_1^0\omega_2^0 + (\omega_1^0\omega_2^1 + \omega_2^0\omega_1^1)\psi_2 + \psi_3\omega_1^1\omega_2^1\}\mathcal{S} \tag{4.210}$$

Details of this calculation can be found in [18].

4.6.0.2 BMS charge using Penrose's definition

According to Penrose, it suffices that ω_i^A satisfy twistor relation only on $\mathcal{S}$, though even in this case the solution space for twistors can be highly restrictive. Therefore, we only consider the subset of those equations:

$$\eth'\omega^0 = 0 \qquad\qquad \eth\omega^1 = \sigma\omega^0 \tag{4.211}$$

For a Bondi system, a number of spin coefficient vanishes:

$$\kappa' = \sigma' = \tau' = \rho' = \tau = Im\rho = 0 \tag{4.212}$$

For such a simplified system, it may be shown that for any two twistors $\omega_{1,2}^A$ satisfying (4.211), the vector-field

$$\xi^a = gn^a + \eta_c^a + \bar{\eta}_{c'}^a \tag{4.213}$$

will behave as a BMS vector field on $\mathcal{I}$, where $\eta_c^a = \frac{1}{2}u\eth cn^a + cm^a$. Here $g = i(\omega_1^0\omega_2^1 + \omega_2^0\omega_1^1)$ and $c = 2i\omega_1^0\omega_2^0$. The integrand in (4.210) can also be manipulated such that we only have g and c terms (procedure outlined in [**TDray**]) The final expression looks like

$$q = Q(g, c, c') = -\frac{i}{8\pi G}\oint \omega_1^0\omega_2^0(2\psi_1 + 2\sigma\eth\bar{\sigma} + \eth(\sigma\bar{\sigma}))\mathcal{S} \tag{4.214}$$

$$- \frac{i}{4\pi G}\oint(\omega_1^0\omega_2^1 + \omega_2^0\omega_1^1)\left(\psi_2 - \sigma N + \frac{1}{2}\eth^2\bar{\sigma} - \frac{1}{2}\eth'^2\sigma\right)\mathcal{S} \tag{4.215}$$

$$= Q_c + Q_g \tag{4.216}$$

Note that the combination which appears in the integral (4.215) with g is the mass-aspect, while the combination in integral (4.214) is the angular-momentum aspect.

4.6.0.3 Relation with Phase space definition

In covariant phase space formalism, one can define *surface charge* from the integral

$$H_F[\Phi] = \int_\Sigma \mathbf{j}_F = \int_\Sigma \mathbf{S}_F + \oint_{\partial\Sigma}\mathbf{K}_F \approx \oint_{\partial\Sigma}\mathbf{K}_F \tag{4.217}$$

The surface term $\mathbf{K}_F$ is identified by defining exact reducibility parameters [6]. Note that we can write

$$\int_\Sigma \mathbf{S}_F = \int_\Sigma n^a\left(\frac{1}{8\pi G}G_{ab} + T_{ab}\right)d^3x^b \approx 0 \tag{4.218}$$

where component of n^a orthogonal to Σ is lapse function and the one parallel to it is called shift vector. If we naively take n to be null, we could write it in terms of spinor fields, say $n^a = \lambda^A\bar{\lambda}^{A'}$. It follows [24], that we can have the identity

$$d\Lambda = \Gamma - \frac{1}{2}n^aG_{ab}d^3x^b \tag{4.219}$$

where $\Lambda = -i\bar{\lambda}_{A'} d\lambda_A \wedge dx^a$ and $\Gamma = -id\bar{\lambda}_{A'} \wedge d\lambda_A \wedge dx^a$. The d operator is defined as $d\lambda^A = \nabla_b \lambda^A dx^b$. Substituting (4.218) and (4.219) in (4.217) yields the surface 2-form $\mathbf{K}_F$ to be $\frac{1}{4\pi G}\Lambda$. The identification with Penrose quasi-local charge definition follows from the identity

$$k^{AA'} = -iK^{\alpha\beta}\omega^A_\alpha \bar{\pi}^{A'}_\beta \tag{4.220}$$

where k^a is the 10-parameter Killing field, and $K^{\alpha\beta} = K^{(\alpha\beta)}$ are ten numbers for $\alpha, \beta = 0, 1, 2, 3$. $(\omega^A_\alpha, \bar{\pi}_{A'\alpha})$ is the twistor field. Thus, we replace n^a with k^a in () and make the identification $\lambda^A \to \omega^A$ and $\bar{\lambda}^{A'} \to \bar{\pi}^{A'}$. Then, it may be shown that

$$H_F(k^a) = A_{\alpha\beta}K^{\alpha\beta} \tag{4.221}$$

$A_{\alpha\beta}$ being the momentum angular-momentum twistor.

While the above identification works for the original Penrose's definition, the modified charge expression has no Hamiltonian analogue. In literature, there are other ways one can choose the surface form $\mathbf{K}_F$. It turns out that if we simply use these definition of charge for extended symmetries, then it can yield divergent results, which can be removed by some renormalization scheme (e.g. see the renormalization of charges for Campliglia-Laddha group in Robinson-Trautman space-time [25]).

4.7 List of Christoffel symbols

Here are list of Christoffel symbols which were required to derive the BMS group:

$$\Gamma^r_{rr} = 2\partial_r\beta \tag{4.222}$$

$$\Gamma^r_{Ar} = \partial_A\beta + \frac{1}{2}r^2 e^{-2\beta} h_{AB}\partial_r U^B \tag{4.223}$$

$$\Gamma^B_{Ar} = \frac{1}{r}\delta^B_A + \frac{1}{2}h^{BC}\partial_r h_{AC} \tag{4.224}$$

$$\Gamma^u_{AB} = e^{-2\beta} r h_{AB} + \frac{1}{2}r^2 e^{-2\beta}\partial_r h_{AB} \tag{4.225}$$

$$\Gamma^r_{AB} = r^2 e^{-2\beta}\partial_{(A}U_{B)} + \frac{1}{2}r^2 e^{-2\beta}\partial_u h_{AB} - Ve^{-2\beta}h_{AB} - \frac{1}{2}rVe^{-2\beta}\partial_r h_{AB} - r^2 e^{-2\beta}U^C\gamma_{ACB} \tag{4.226}$$

$$\Gamma^C_{AB} = \gamma^C_{AB} + re^{-2\beta}U^C h_{AB} + \frac{1}{2}r^2 e^{-2\beta}U^C\partial_r h_{AB} \tag{4.227}$$

$$\Gamma^u_{Au} = \partial_A\beta - re^{-2\beta}h_{AB}U^B - \frac{1}{2}r^2 e^{-2\beta}\partial_r(h_{AB}U^B) \tag{4.228}$$

$$\Gamma^r_{Au} = \frac{\partial_A V}{2r} - \frac{1}{2}r^2 e^{-2\beta}(U_B\partial_A U^B + U^B\partial_B U_A) + Ve^{-2\beta}(U_A + r/2\partial_r U_A) - \frac{1}{2}r^2 e^{-2\beta}U^B\partial_u h_{AB} \tag{4.229}$$

$$\Gamma^B_{Au} = U^B\partial_A\beta - re^{-2\beta}U^B(\partial_r U_A) - h^{BC}\partial_{[A}U_{B]} + \frac{1}{2}h^{BC}\partial_u h_{AC} \tag{4.230}$$

Chapter 5

Shock Waves and Memory Effect

5.1 Shock Waves

Shock waves that we are going to consider in this chapter are the matter-field or gravitational radiations, which are confined on some hyper-surface (time-like or null) such that they discontinuously deform the geometry around its neighborhood. Just to set the convention, let us start with a line element

$$ds^2 = g_{ab} dx^a dx^b \tag{5.1}$$

Assume $f(x)$ is some smooth function and V is some vector field, such that for $f(x) > 0$ we have the discontinuous transformation of coordinates

$$x^a \to x^a + V^a \tag{5.2}$$

We take the differential on both sides and substitute the above expression in (5.1). On Taylor expanding, we will get

$$ds^2 = (g_{ab} + \mathcal{L}_V g_{ab}) dx^a dx^b + \mathcal{O}(V^2) \tag{5.3}$$

We define the shock wave line element to have the particular form

$$ds^2 = (g_{ab} + \theta(f(x)) \mathcal{L}_V g_{ab}) dx^a dx^b + \mathcal{O}(V^2) \tag{5.4}$$

Note that the coordinates we have used are discontinuous across $f(x) = 0$. However, we may choose new coordinates $\bar{x}^a$ such that the metric components are continuous. We can also choose coordinates where $\delta-$function appears on the line element, so that patches $f(x) > 0$ and $f(x) < 0$ will look the same. The metric, therefore is $\mathcal{C}^0$ on $f(x) = 0$. It should be pointed out that in literature such as [26], considers line element of the above form as defining *impulsive waves*. For impulsive waves, the curvature is proportional to $\delta-$function on $f(x) = 0$, while it is just discontinuous for a shock wave. For example

$$g_{ab} = (1 - \theta) g^{-}_{ab} + \theta g^{+}_{ab} \tag{5.5}$$

is manifestly discontinuous where the curvature terms contain step functions. The aim of our analysis is to extend the work by Hawking, Perry and Strominger [27],[9] , so to avoid confusion, we will stick to their definition of shock waves.

5.1.1 Earlier works

5.1.1.1 Shock wave for zero rest mass particle

Pirani [28] and later Aichelburg and Sexl [7] found that space-time around a fast moving particle has the geometry of a plane gravitational wave, much in the same way how a fast moving charged particle produce plane electro-magnetic wave. T.Dray and t'Hooft pointed out that such an energetic particle can induce super-translation and refraction on any transverse null rays which will cross its path [8].

We begin with the Schwarzschild space-time for a massive particle of mass m. Since we can write this in the Kerr-Schild form, we perform Lorentz boost to describe the particle in a moving frame. In the limit where $v \to 1$, we let $m \to 0$, keeping momentum of the particle fixed. This involves taking appropriate limit followed by coordinate transformations.

Consider the line element

$$ds^2 = \frac{(1-A)^2}{(1+A)^2}dt^2 - (1+A)^4(dx^2+dy^2+dz^2) \tag{5.6}$$

where $A = \frac{m}{2r}$. Now perform a Lorentz boost along x direction. Denote $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ as our new co-ordinates. The line-element takes the form

$$ds^2 = (1+A)^2\eta_{ab}d\bar{x}^a d\bar{x}^b - (8A - 2A^2 + \cdots)\frac{(d\bar{t} - vd\bar{x})^2}{1-v^2} \tag{5.7}$$

The expression for A is

$$A = \frac{p(1-v^2)}{2\sqrt{(\bar{x}-v\bar{t})^2 + (1-v^2)\rho^2}} \tag{5.8}$$

which goes to zero as $v \to 1$. Here $\rho^2 = \bar{y}^2 + \bar{z}^2$. However, notice that in this limit, there will be a non-zero contribution from the term

$$\lim_{v\to 1}\frac{8A}{1-v^2} = \frac{4p}{|\bar{x}-\bar{t}|} \tag{5.9}$$

in line element (5.7). However, the above expression blows up when $\bar{x} \to \bar{t}$. To remove the divergence, one performs a new coordinate transformation: $x' - vt' = \bar{x} - v\bar{t}$ and $x' + vt' = \bar{x} + v\bar{t} - 4p\log[\sqrt{(\bar{x}-v\bar{t})^2 + (1-v^2)} - (\bar{x}-\bar{t})]$. In this coordinate system, the line element looks like

$$ds^2 = \eta_{ab}dx'^a dx'^b - 4p\left\{\frac{1}{\sqrt{(\bar{x}-v\bar{t})^2+(1-v^2)\rho^2}} - \frac{1}{\sqrt{(\bar{x}-v\bar{t})^2+(1-v^2)}}\right\}(dt'-dx')^2 \tag{5.10}$$

On taking the limit (see appendix of [7]) we finally obtain

$$ds^2 = \eta_{ab}dx'^a dx'^b + 8p\log|\rho|\delta(t'-x')(dt'-dx')^2 \tag{5.11}$$

Using advanced and retarded co-ordinates, we may re-write this as

$$ds^2 = du(d\nu + 8p\log|\rho|\delta(u)du) - q_{AB}dx^A dx^B \tag{5.12}$$

where $x^A = (y', z')$. Notice the delta type singularity at $u = 0$. T. Dray and t'Hooft showed that we can obtain the same line element as above if we stitch two portions of Minkowski space-time whose coordinates are related to one another by a super-translation motion. To prove this, say for $u < 0$, the line element is given by

$$ds^2 = dud\nu - q_{AB}dx^A dx^B \tag{5.13}$$

while for $u > 0$ we write

$$ds^2 = du(d\nu - \theta f_{,i}dx^i) - q_{AB}dx^A dx^B \tag{5.14}$$

where $f = 8p\log|\rho|$. Under transformation $u = \hat{u}$, $\hat{\nu} = \nu - \theta f$ and $\hat{x}^A = x^A$ we get the required metric (5.11).

5.1.1.2 Shock wave construction in HPS

In [27],[9] the authors consider shock wave construction along the line considered in section 5.1. They consider Schwarzschild space-time as the background metric and $f(x) = \nu - \nu_0$. The vector field V is chosen as the BMS super-translation vector field:

$$V = \zeta = \alpha\partial_\nu + \frac{1}{r}D^A\alpha\partial_A - \frac{1}{2}D^2\alpha\partial_r + \cdots \tag{5.15}$$

The perturbed metric components are

$$\mathcal{L}_\zeta g_{\nu\nu} = -\frac{mD^2\alpha}{r^2} \tag{5.16}$$

$$\mathcal{L}_\zeta g_{\nu A} = D_A(V\alpha + \frac{1}{2}D^2\alpha) \tag{5.17}$$

$$\mathcal{L}_\zeta g_{AB} = rq_{AB}D^2\alpha - 2rD_AD_B\alpha \tag{5.18}$$

The line element therefore looks like

$$\begin{aligned} ds^2 = &\left(V - \theta\frac{mD^2\alpha}{r^2}\right)d\nu^2 - 2d\nu dr + \theta D_A(V\alpha + \frac{1}{2}D^2\alpha)d\nu dx^A \\ &- (r^2 q_{AB} + \theta(2rD_AD_B\alpha - rq_{AB}D^2\alpha))dx^A dx^B \end{aligned}$$

The shock wave shifts the position of event horizon by amount $r|_{2M} \to \hat{r} = 2M - \frac{1}{2}D^2\alpha$. One also computes the Einstein tensor for the above metric and obtain

$$T_{\nu\nu} = \left(\frac{\hat{T}}{4\pi r^2} + \frac{\hat{T}^{(1)}}{4\pi r^3}\right)\delta(\nu - \nu_0) \qquad T_{\nu A} = \frac{\hat{T}_A}{4\pi r^2}\delta(\nu - \nu_0) \tag{5.19}$$

where the $\hat{T}$'s are some combination of ζ and its derivatives so that they satisfy the conservation equation. Essentially, we have $\hat{T}^{(1)} = D_A\hat{T}^A$ and will satisfy $(D^2 + 2)D_A\hat{T}^A = -6M\hat{T}$, all of which follows from the field equations.

[29] introduces the new variable $\rho = r - \hat{r}$ as a new variable to write the line element in the form

$$ds^2 = \frac{\rho}{\hat{r}}d\nu^2 - 2d\nu d\rho + \frac{2}{\hat{r}}\rho D_A f d\nu dx^A - (\hat{r}^2 q_{AB} + 2\hat{r}D_A D_B f)dx^A dx^B + \cdots \tag{5.20}$$

which is of the form (4.177). We can read of the coefficients θ_A, κ, λ_{AB} etc. Setting the super-translation $f = 0$ near horizon yields the identificaton such as $Y^A = \frac{1}{\hat{r}}D_A\alpha$ and so on.

5.1.2 Implanting shock waves on BH horizon

The original work by HPS considered super-translation BMS vector field ζ which they have extended all the way from infinity $\mathcal{I}^-$ to the future event horizon $\mathcal{H}^+$. However, the question remains whether this is the only possible shock wave which can be implanted on a BH and simultaneously induce BMS super-translation at infinity. Notably, definitions like [10], [11] near event horizon or any null surfaces in general implies that there could be other ways to extend a diffeomorphism vector field. To address this issue, let us first recall the geometric definition of BMS group we discussed in previous chapter. The BMS vector field was defined as the one which generates conformal motion of $\mathcal{I}$. Indeed, the isometry condition only determined the first two correction terms in asymptotic expansion of ξ vector field about $1/r = 0$. The higher order terms are completely arbitrary and becomes relevant when we consider dynamics in the bulk. Moreover, introducing new coordinate ρ near event horizon as considered by L. Donnay will involve non-trivial intermixing of all of the components when we try to express our field variables in terms of ρ. We can exploit these arbitrariness to define independent boundary conditions for the diffeomorphism vector field V near event horizon $\mathcal{H}$. The boundary conditions we are going to choose is that V should behave as the χ vector field near $\mathcal{H}$ (as defined in §) and simultaneously behave as BMS vecor field near infinity. Such conditions can put constraints on the correction terms. Moreover, we will require such shock waves to be physically well defined, which would amount to satisfying some suitable energy conditions.

To analyze shock waves near both event horizon and infinity requires two set of coordinates which can be defined near these hyper-surfaces. We already introduced the Bondi-Sachs coordinates in chapter 4 which will represent our coordinates near infinity. Likewise, the discussion in (4.5) shows that it is equally possible to construct the Gaussian null coordinates (u, ρ, x^A) near event horizon. There is freedom in how we choose the near horizon coordinate ρ. Consider the following scenario:

We start by considering family of hyper-surfaces defined by $\Sigma - r = 0$ where $\Sigma = \Sigma(u, x^A)$.Let us define a new distance coordinate $\rho = r - \Sigma$, so that the Bondi-Sachs line element when restricted on $\rho = 0$ has the form:

$$g_{ab}|_{\rho=0} = \begin{pmatrix} 0 & W & 0 \\ W & 0 & 0 \\ 0 & 0 & H_{AB} \end{pmatrix} \tag{5.21}$$

This is a null hyper-surface, since the hyper-surface orthogonal $n_a = \nabla_a \rho$ is null w.r.t. g_{ab} on $\rho = 0$. To have this particular form of metric, Σ should satisfy certain conditions. The metric functions can now be expanded about $\rho = 0$ surface. Assuming small $|\rho/\Sigma|$, we expand $V/r = V_0 + \mathcal{O}(\rho)$, $e^{2\beta} = W + \mathcal{O}(\rho)$, $h_{AB} = h^0_{AB} + \mathcal{O}(\rho)$ and so on. To have the particular off diagonal form of metric , it may be verified that Σ should satisfy:

$$\Sigma^2 h^0_{AB} U^{0B} + W \frac{\partial \Sigma}{\partial x^A} = 0 \tag{5.22}$$

$$V^0 W - \Sigma^2 h^0_{AB} U^{0A} U^{0B} + 2 \frac{\partial \Sigma}{\partial u} W = 0 \tag{5.23}$$

As a particular example, consider Schwarzschild space-time, for which $W = 1$, $H_{AB} = -\Sigma^2 q_{AB}$, $U^A = 0$ and $V_0 = 1 - \frac{2M}{\Sigma}$. Thus (5.22) implies $\frac{\partial \Sigma}{\partial x^A} = 0$ while integrating equation (5.23) yields $u + 2\Sigma^* =$const , where $\Sigma^* = \Sigma + 2M \log|\Sigma/2M - 1|$. The null surface $\rho = 0$ will then correspond to $u + 2r^* =$const or $\nu =$const. Thus $\rho = 0$ for Schwarzschild space-time expressed in outgoing Eddington-Finkelstein co-ordinate corresponds to family of incoming null surfaces $\nu =$const. In particular, for $r = 2M$, $\frac{\partial \Sigma}{\partial u} = -\frac{1}{2} V_0|_{r=2M} = 0$. Thus we can safely interpret Σ as constant on $r = 2M$. Thus, for the particular case of Schwarzschild space-time, we define $\rho = r - 2M$. Note that it is almost same as the definition considered in ([29]), except the authors assumed super-translated event horizon in place of just $2M$.

The boundary conditions we need to look at are as follows:

$$V^u = f + A_2 r^{-2} + A_3 r^{-3} + \cdots \tag{5.24}$$

$$V^r = -r\partial_u f - B_2 - B_3 r^{-1} \cdots \tag{5.25}$$

$$V^A = f^A - q^{AB} \partial_B f r^{-1} + \Gamma^A_2 r^{-3} + \Gamma^A_3 r^{-4} + \cdots \tag{5.26}$$

are essentially the conditions near infinity. Near horizon $\rho = 0$, we let

$$V^u = f_H + \tilde{A}_2 \rho^2 + \cdots \tag{5.27}$$

$$V^\rho = -\partial_u f_H \rho + \tilde{B}_2 \rho^2 + \cdots \tag{5.28}$$

$$V^A = Y^A + \frac{q^{AB} \partial_B f_H}{(2M)^2} \rho + \tilde{\Gamma}^A_2 \rho^2 + \cdots \tag{5.29}$$

To express f_H, Y^A in terms of $f, f^A, \{A_i\}, \{B_i\}$, we first define coordinate transformation $V^{\hat{a}} = V^b \frac{\partial x^{\hat{a}}}{\partial x^b}$ to go from (V^u, V^r, V^A) to (V^u, V^ρ, V^A) followed by Taylor expansion about $\rho = 0$. We skip the details of this calculation. The final result is as follows:

$$V_0^u = f_H = f + \sum_{k=2} \frac{A_k}{(2M)^k} \tag{5.30}$$

$$V_1^u = -\sum_{k=2} \frac{kA_k}{(2M)^{k+1}} = 0 \tag{5.31}$$

$$V_0^\rho = -2M\partial_u f - \sum_{k=2} \frac{B_k}{(2M)^{k-2}} = 0 \tag{5.32}$$

$$V_1^\rho = -\partial_u f_H = -\partial_u f + \sum_{k=2} \frac{(k-2)B_k}{(2M)^{k-1}} \tag{5.33}$$

$$V_0^A = Y^A = f^A - \frac{q^{AB}\partial_B f}{2M} + \sum_{k=2} \frac{\Gamma_k^A}{(2M)^k} \tag{5.34}$$

$$V_1^A = \frac{q^{AB}\partial_B f_H}{(2M)^2} = \frac{q^{AB}\partial_B f}{(2M)^2} - \sum_{k=2} \frac{k\Gamma_k^A}{(2M)^{k+1}} \tag{5.35}$$

If we compare everything in orders of $2M$, we see that $B_k = -\frac{\partial_u A_{k-1}}{k-2}$ while $\Gamma_k^A = -\frac{q^{AB}\partial_B A_{k-1}}{k}$, all for $k > 2$. B_2 can be obtained from the condition (5.32) but Γ_2^A terms are indeterminate from these restrictions unless Y^A are specified. Its interesting that although $\{A_i\}$'s are all arbitrary at this stage, the expression for $\{B_i\}$'s and $\{\Gamma_i^A\}$'s are completely different from that of the standard BMS definition

5.1.3 Field equations

We will analyze the field equations near event horizon. The calculations are not *proper*, in the sense that from a more analytical point of view we can't have differential equations in $\mathcal{C}^0$ smooth metric field, rather we should strictly think in terms of differential inclusion. Nevertheless, we will carry forward the procedures considered in the previous articles for our present discussion. Thus, for all calculations, we take $\theta\mathcal{L}_V g_{ab}$ as perturbation on back-ground metric g_{ab}. We will consider all calculations till first order in V. First we note that if our total metric $g' = g + h$, then the inverse of this metric will be

$$g'_{ab} = g_{ab} + h_{ab} \longleftrightarrow g'^{ab} = g^{ab} - h^{ab} + \mathcal{O}(V^2) \tag{5.36}$$

The fact that it is true can be verified by taking the contraction

$$g'^{ac}g'_{cb} = (g^{ac} - h^{ac})(g_{cb} + h_{cb}) = \underbrace{g^{ac}g_{cb}}_{\delta^a_b} - \mathcal{O}(h^2) \tag{5.37}$$

The Levi-Civita connection upto linear order in h can be written as

$$\Gamma^a_{bc} = \frac{1}{2}g'^{as}(\partial_b g'_{cs} + \partial_c g'_{sb} - \partial_s g'_{bc}) \tag{5.38}$$

$$= \bar{\Gamma}^a_{bc} + \frac{1}{2}g^{as}(\partial_b h_{cs} + \partial_c h_{sb} - \partial_s h_{bc}) - \frac{1}{2}h^{as}(\partial_b g_{cs} + \partial_c g_{sb} - \partial_s g_{bc}) \tag{5.39}$$

$$= \bar{\Gamma}^a_{bc} + \delta\Gamma^a_{bc} \tag{5.40}$$

Expression (5.39) can be written in a much more compact notation. First note that covariant derivative of the perturbation has the form

$$P_{bcs} = \nabla_b h_{cs} = \partial_b h_{cs} - \bar{\Gamma}^m_{bc} h_{ms} - \bar{\Gamma}^m_{bs} h_{mc} \tag{5.41}$$

We can do cyclic permutation of the indices and consider the sum $P_{bcs} + P_{csb} - P_{sbc}$ to obtain $\delta\Gamma_{bcs}$. Contracting with g^{as} we can re-write the perturbed connection as

$$\delta\Gamma^a_{bc} = \frac{1}{2}g^{as}(\nabla_b h_{cs} + \nabla_c h_{sb} - \nabla_s h_{bc}) \tag{5.42}$$

We can use the expression for perturbed Christoffel symbols to get the first order expression for Curvature tensors [30]

$$R_{abcd} = \bar{R}_{abcd} + 2\nabla_{[c|}\nabla_{(a} h_{b)|d]} - h_{[a}{}^e \bar{R}_{b]ecd} \tag{5.43}$$

Contracting with g'^{ac} we can obtain the first order Ricci tensor:

$$R^{(1)}_{ab} = \nabla^c \nabla_{(a} h_{b)c} - \frac{1}{2}(\Box h_{ab} + \nabla_a \nabla_b h) \tag{5.44}$$

We shall consider construction on background Schwarzschild metric, so that $\bar{R}_{ab} = 0$ and we estimate the 1st order Einstein tensor G_{ab}. Near $r = 2M$, the metric has the form

$$ds^2 = \left(\frac{\rho}{2M} - \cdots\right) du^2 + 2du d\rho - (4M^2 + \cdots)(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.45}$$

The perturbations for the vector field V has the form (excluding the overall step function factor)

$$h_{uu} = 2\left[\frac{\partial_u f_H}{4M} - \partial_u^2 f_H\right]\rho + \mathcal{O}(\rho^2) \tag{5.46}$$

$$h_{u\rho} = 2\tilde{B}_2(u,\theta,\phi)\rho + \mathcal{O}(\rho^2) \tag{5.47}$$

$$h_{u\theta} = \left[-4M^2\partial^2_{u\theta} f_H + \frac{\partial_\theta f_H}{2M} - \partial^2_{u\theta} f_H\right]\rho + \mathcal{O}(\rho^2) \tag{5.48}$$

$$h_{u\phi} = \left[-4M^2\partial^2_{u\phi} f_H + \frac{\partial_\phi f_H}{2M} - \partial^2_{u\phi} f_H\right]\rho + \mathcal{O}(\rho^2) \tag{5.49}$$

$$h_{\rho\rho} = 4\tilde{A}_2(u,\theta,\phi)\rho + \mathcal{O}(\rho^2) \tag{5.50}$$

$$h_{\rho\theta} = [\partial_\theta f_H - 4M^2\partial_\theta f_H] + [-4M\partial_\theta f_H - 8M^2\tilde{\Gamma}^\theta_2(u,\theta,\phi)]\rho + \mathcal{O}(\rho^2) \tag{5.51}$$

$$h_{\rho\phi} = [\partial_\phi f_H - 4M^2\partial_\phi f_H] + [-4M\partial_\phi f_H - 8M^2\sin^2\theta\tilde{\Gamma}^\phi_2(u,\theta,\phi)]\rho + \mathcal{O}(\rho^2) \tag{5.52}$$

$$h_{\theta\theta} = -8M^2\partial_\theta\Upsilon^\theta - [4M\left(-2M\partial^2_{\theta\theta}f_H + \partial_u f_H - \partial_\theta\Upsilon^\theta\right) - 4M\partial_\theta\Upsilon^\theta]\rho + \mathcal{O}(\rho^2) \tag{5.53}$$

$$h_{\theta\phi} = -4M^2\left(\partial_\phi\Upsilon^\theta + \sin^2\theta\partial_\phi\Upsilon^\phi\right) + \mathcal{O}(\rho) \tag{5.54}$$

$$h_{\phi\phi} = 4M\sin\theta\left(-2M\cos\theta\Upsilon^\theta - 2M\sin\theta\partial_\phi\Upsilon^\phi\right) + \mathcal{O}(\rho) \tag{5.55}$$

The components of perturbed Einstein tensor are as follows:

$$G_{uu} = -\left(\delta' + \frac{1}{4M}\delta\right)D_A\Upsilon^A + \mathcal{O}(\rho) \tag{5.56}$$

$$G_{u\rho} = \delta\left(\frac{1}{M}D_AY^A + D^2f_H - \frac{1}{M}\partial_u f_H + \frac{1}{2}D^2f_H\right) + \mathcal{O}(\rho) \tag{5.57}$$

$$G_{u\theta} = -\frac{\delta\left(8M^3\partial^2_{u\theta}f_H - 6M\partial^2_{u\theta}f + \partial_\theta f_H - 2M\csc^2\theta\partial^2_\phi\Upsilon^\theta + 2M\partial^2_{\theta\phi}\Upsilon^\phi + 4M\cot\theta\partial_\phi\Upsilon^\phi\right)}{4M} \tag{5.58}$$

$$+\frac{1}{2}\left(1-4M^2\right)\delta'\partial_\theta f_H + \delta\Upsilon^\theta + \mathcal{O}(\rho) \tag{5.59}$$

$$G_{u\phi} = \frac{\delta\left(-8M^3\partial^2_{u\phi}f_H + 6M\partial^2_{u\phi}f_H - \partial_\phi f_H - 2M\partial^2_{\theta\phi}\Upsilon^\theta + 2M\cot\theta\partial_\phi\Upsilon^\theta\right)}{4M} \tag{5.60}$$

$$+\frac{\left(2M\sin^2\theta\partial^2_\theta\Upsilon^\phi + 3M\sin 2\theta\partial_\theta\Upsilon^\phi\right) + 2M\left(1-4M^2\right)\delta'\partial_\phi f_H}{4M} \tag{5.61}$$

$$G_{\rho\rho} = -\delta\frac{2\tilde{A}_2}{M}\rho + \mathcal{O}(\rho^2) \tag{5.62}$$

$$G_{\rho\theta} = \frac{\delta\left(\left(8M^2-1\right)\partial_\theta f + 8M^3\tilde{\Gamma}^\theta_2\right)}{2M} + \mathcal{O}(\rho) \tag{5.63}$$

$$G_{\rho\phi} = \frac{\delta\left(\left(8M^2-1\right)\partial_\phi f + 8M^3\sin^2\theta\tilde{\Gamma}^\phi_2\right)}{2M} + \mathcal{O}(\rho) \tag{5.64}$$

$$G_{\theta\theta} = -\delta(8M_2^2 + 4M^2\cot\theta\partial_\theta f_H + 4M^2\csc^2\theta\partial^2_\phi f_H - 4M\partial_u f_H + \cot\theta\partial_\theta f_H \tag{5.65}$$

$$+\csc^2\theta\partial^2_\phi f + 4M\cot\theta\Upsilon^\theta + 4M\partial_\phi\Upsilon^\phi) + \mathcal{O}(\rho) \tag{5.66}$$

$$G_{\theta\phi} = -\delta\left(\left(4M^2+1\right)\left(\cot\theta\partial_\phi f_H - \partial^2_{\theta\phi}f_H\right) - 2M\partial_\phi\Upsilon^\theta - 2M\sin^2\theta\partial_\theta\Upsilon^\phi\right) + \mathcal{O}(\rho) \tag{5.67}$$

$$G_{\phi\phi} = -\sin^2\theta\delta\left(8M^2\tilde{B}_2 + 4M^2\partial^2_\theta f_H - 4M\partial_u f_H + \partial^2_\theta f_H + 4M\partial_\theta\Upsilon^\theta\right) + \mathcal{O}(\rho) \tag{5.68}$$

Note that here $\delta = \delta(u - u_0)$ and $\delta' = \delta'(u - u_0)$. Thus, the curvature is concentrated on $u = u_0$ null surface. Intuitively, we can think of the δ term as a monopole contribution while δ' as the dipolar contribution from the matter-field T_{ab}. Note that the field equations, imply that $T_{ab} = -\frac{1}{8\pi G} G_{ab}$. However, the matter-field as written above need not be physically well defined. We can further look into the Energy conditions satisfied by this shock wave.

5.1.4 Energy conditions

We are going to use the Hawking-Ellis classification of Stress energy tensor to comment about some generic properties about our shock waves and the conditions needed to be satisfied for that. The essential idea is to write our stress-energy tensor in terms of orthonormal tetrads and find the eigen vectors of the system

$$T_{(a)}{}^{(b)} = \lambda \eta_{(a)}{}^{(b)} \tag{5.69}$$

If there are 1 timelike and 3 space-like eigenvectors, then it is called a Type-I stress energy tensor (some examples include perfect fluids, massive scalar field, non-null EM field etc). If (5.69) admits one double null-eigen vector, then it is Type-II (e.g. null dust solution or classical radiation). If there are one triple null eigen vectors, then it is Type-III (there are no classical examples). At last, if there are no causal eigen vectors, then it is said to be Type-IV. If we consider higher order corrections in V near event horizon to be negligible, the Eigen vectors for Einstein tensor. In this case there is a double null eigen vector $k^a = (0, 1, 0, 0)$ with eigen value $-\mu = -\frac{1}{8\pi G} \times \delta(u - u_0)\nabla_A \nabla^A f_H$. Thus, our stress-energy tensor is Type-II. Likewise, let $p_2.p_3$ be other two non-degenerate eigen values, (essentially, the eigen vectors for these cases are complicated radicals containing f_H and Y^A). Then Null energy condition is satisfied if $\mu + p_i \geq 0$. Weak energy condition will further require $\mu \geq 0$, while Strong energy condition will require $\sum p_i \geq 0$. Dominant energy condition means that $|p_i| \leq \mu$ with $\mu \geq 0$

5.2 Memory effects

In this section we are going to comment about observational significance of such shock wave construction. Memory effect, by itself plays an essential role in the infrared dynamics of mass-less particles. The basic idea is to look at permanent distortion of some configuration as it crosses such shock waves. Now the shock wave construction as considered in previous sections have associated conserved charges. The conserved charge expression have terms which appear with the *mass aspects* and *angular-momentum aspects*. Classically, the memory effect which are sourced by the mass aspect is called *gravitational/displacement memory effect* while the one sourced by the *angular-momentum aspect* is called *Spin memory effect*. For the classical limit, we will restrict attention to the gravitational case only. We will also consider a particular example for Quantum Memory effect (strictly speaking semi-classical) by considering scattering of mass-less minimally coupled scalar field in background Rindler space-time and comment about change in entanglement of subsystems.

5.2.0.1 Gravitational/displacement memory effect

The idea is to consider some family of trajectories, which passes through the shock wave. Practically, such trajectories could represent (test) observers undergoing some kind of motion (it could either be a free-fall motion or accelerated). Each of these observers will carry clock, which are synchronized : the time taken for two way transfer of signal between adjacent observers can give a measure of the length of displacement vector between these trajectories. Now, imagine these observers bump into the gust of radiation. Even if they try to maintain their flow of trajectory after the impact, their clocks will be out of synchronization: there will be a change in time taken for two way transfer of signal. This change will primarily depend on the super-translation and super-rotation effect (provided we are analyzing the motion near horizon or infinity). The choice of trajectories can be arbitrary, depending on the experimental constraints. For our purpose, we are going to look at Letaw-Frenet equations which gives a covariant formulation of uniform linearly accelerated observers. One motivation for such choice of trajectories is related to Hawking-Unruh type effect in quantum field theory : the Minkowski vacuum appears to be a thermal bath to an uniform linearly accelerated observer, while it is not so for other type of trajectories.

The Letaw-Frenet condition mandates that $w^c = -a^2 u^b \nabla_b u^c$, where $w^c = u^b \nabla_b a^c$ is the jerk and $a^c = u^b \nabla_b u^c$ is the acceleration associated to the family of trajectories. Let $u^a = (u^u, u^\rho, u^A)$ be the velocity vector field. After scattering with the shock wave, let the new velocity field be $u'^a = u^a + \delta u^a$. The variation δu^a here is considered to be just linear in V. If we want the inner-product $u^c u_c$ to be preserved till linear order in V, then we must have

$$(u^a + \delta u^a)(g_{ab} + h_{ab})(u^b + \delta u^b) = u^a u_a + \mathcal{O}(V^2) \tag{5.70}$$

For special case, if we assume $u^a = \left(\frac{1}{\sqrt{g_{uu}}}, 0, 0, 0\right)$, then we must have

$$\delta u^\rho = -u^u h_{uu} = -2\theta(u-u_0)\left[\frac{\partial_u f_H}{2\sqrt{2M}} - \sqrt{2M}\partial_u^2 f_H\right]\sqrt{\rho} + \cdots \tag{5.71}$$

It may be verified that the Letaw-Frenet condition is satisfied for the choice of $u^a = (1/\sqrt{g_{uu}}, 0, 0, 0)$. Assume $\delta u^u = 0$. On top of this, if we substitute the perturbation δu^a and demand preservation of Frenet condition, we obtain equation for δu^A:

$$\sqrt{2}\kappa^2 \partial_u^2 \delta u^A - 2\kappa^4 \delta u^A + 2\sqrt{\kappa\rho}\,\delta'(u-u_0)\kappa\partial^A f_H = 0 \tag{5.72}$$

where $\kappa = \frac{1}{4M}$. Solving above equation yields

$$\delta u^A = -\theta(u-u_0)\cosh\kappa(u-u_0)\frac{\sqrt{2\kappa\rho}}{\kappa}\partial^A f_H \tag{5.73}$$

Notice that the sign of δu^ρ is negative when the terms in the square bracket is positive. In this scenario, the observer after impact with the shock wave will divert towards the Black hole event horizon. Also notice that there is no contribution from the super-rotation term which is primarily due to the $u^A = 0$ before scattering.

We can repeat these calculations near infinity as well (the super-translation case has been considered here [13]). We could as well consider inertial observers near infinity and demand conservation of free fall condition [12], which essentially leads to a change in geodesic deviation δq being proportional to the asymptotic shear σ. However, σ changes under the influence of both super-translation and super-rotation, so both of these contributions will be reflected in the δq.

5.2.0.2 Quantum memory effect

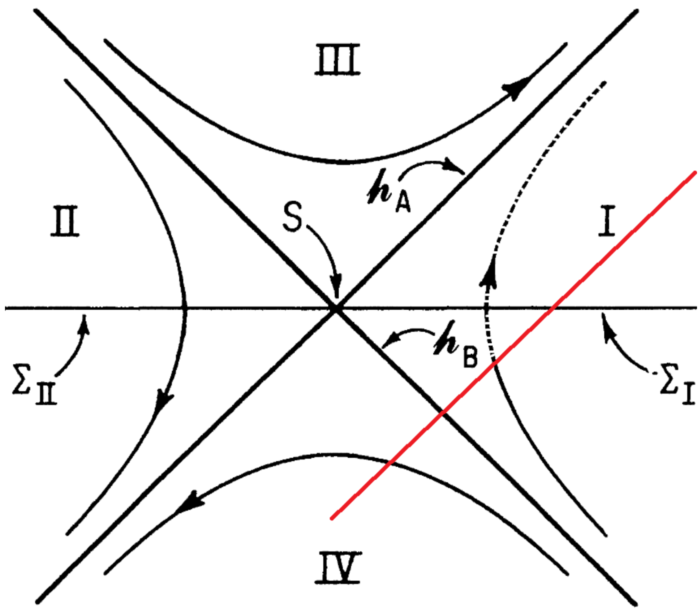


Figure 5.1: Rindler space-time with Cauchy surface $\Sigma_I \cup \Sigma_{II}$. A shock-wave has been implanted on the right wedge I on horizon h_B. Picture borrowed from [31]

For the semi-classical memory effect, consider the particular case of Rindler space-time, whose line element has the form

$$ds^2 = 2\kappa\rho du^2 + 2du d\rho - \delta_{AB} dx^A dx^B \tag{5.74}$$

where $\kappa = a$. Let us now consider a test scalar field ϕ which satisfies the mass-less free field equation:

$$\Box\phi = g^{ab}\nabla_a\nabla_b\phi = 0 \tag{5.75}$$

Now, the d'Alembert operator for Rindler space-time is essentially $\partial^2 - 2\kappa\partial_\rho$, thus the equation of motion is

$$(2\kappa\rho\partial_\rho^2 - 2\partial_u\partial_\rho + \delta^{AB}\partial_A\partial_B + 2\kappa\partial_\rho)\phi = 0 \tag{5.76}$$

Positive frequency solution means that

$$\mathcal{L}_K \phi = -i\omega\phi \tag{5.77}$$

for $\omega > 0$. Here $K = \partial_u$ defines the trajectory of the observer. Thus we can do separation of variables to write

$$\phi = e^{-i\omega u}\phi_\rho(\rho)e^{ik_A x^A} \tag{5.78}$$

Substituting this expression in (5.76) yields

$$2\kappa\rho\partial_\rho^2\phi_\rho + 2(\kappa + i\omega)\partial_\rho\phi_\rho - k^2\phi_\rho = 0 \tag{5.79}$$

The solution for ϕ_ρ is given in terms of Modified Bessel's function [1] :

$$\phi_\rho = N \times K_{-i\omega/\kappa}\left(\frac{2k\sqrt{\rho}}{\sqrt{2\kappa}}\right) \tag{5.80}$$

Given such a space of positive frequency solution, we can equip an inner-product

$$\langle\phi_1, \phi_2\rangle = -i\int_o^\infty d\rho \int dx^1 dx^2 (\phi_1\partial_\rho\phi_2^* - \phi_2^*\partial_\rho\phi_1) \tag{5.81}$$

The normalisation factor N in (5.80) can be chosen such that the solutions form an orthonormal basis with respect to the above inner-product. For the definition above, we may choose

$$\phi_{\omega,\vec{k}} = \sqrt{\frac{|\sinh\pi\omega/\kappa|}{4\pi^4\kappa}}e^{-i\omega u}\left(\frac{2k\sqrt{\rho}}{\sqrt{2\kappa}}\right)^{-\omega/\kappa} K_{-i\omega/\kappa}\left(\frac{2k\sqrt{\rho}}{\sqrt{2\kappa}}\right)e^{ik_A x^A} \tag{5.82}$$

and it will satisfy the orthonormal condition

$$\langle\phi_{\omega,\vec{k}}, \phi_{\omega',\vec{k}'}\rangle = \delta(\omega - \omega')\delta^2(\vec{k} - \vec{k}') \tag{5.83}$$

$$\langle\phi^*_{\omega,\vec{k}}, \phi^*_{\omega',\vec{k}'}\rangle = -\delta(\omega - \omega')\delta^2(\vec{k} - \vec{k}') \tag{5.84}$$

$$\langle\phi_{\omega,\vec{k}}, \phi^*_{\omega',\vec{k}'}\rangle = 0 \tag{5.85}$$

Now, let us consider the situation when this scalar field crosses the shock-wave. For $u > u_0$, we have

$$\phi^1_{\omega,\vec{k}}(x) = \phi_{\omega,\vec{k}}(x + V) = \phi_{\omega,\vec{k}}(x) + \underbrace{V^a\partial_a\phi_{\omega,\vec{k}}}_{\delta\phi} + \mathcal{O}(V^2) \tag{5.86}$$

If we demand, that $\phi^1_{\omega,\vec{k}}$ should satisfy mass-less free field equation with respect to the metric g'_{ab}, then we can find constraint where $\delta\phi$ is related to base solution ϕ. If we integrate the expression around $u = u_0$, then only terms with delta function (and it's derivatives) will survive. For simplicity we assume V to be exact till linear order in ρ. Then we shall obtain

$$\lim_{\epsilon\to 0}\int_{u_0-\epsilon}^{u_0+\epsilon} du(2\partial_u\partial_\rho\delta\phi + \delta(u - u_0)(\partial_A Y^A + \rho\delta^{AB}\partial_A\partial_B f_H)\partial_\rho\phi) = \mathcal{O}(V^2) \tag{5.87}$$

On integrating w.r.t. u followed be ρ the $\delta\phi$ takes the form

$$2\delta\phi^{+} - 2\delta\phi^{-} = 2\delta\phi|_{u=u_0} = \int_{\rho}^{\infty} (\partial_A Y^A + \rho' \delta^{AB}\partial_A\partial_B f_H)\partial_{\rho'}\phi d\rho'|_{u=u_0} \tag{5.88}$$

Further we can assume that the states $\{\phi^{(1)}_{\omega,\vec{k}}\}$ to be O.N. w.r.t. the inner-product (5.81). Thus, we may write for $\{\phi_{\omega,\vec{k}}\}$ or equivalently the $\{\delta\phi\}$'s as linear superposition of $\{\phi^{(1)}\}$'s

$$\delta\phi_{\omega,\vec{k}} = \int_0^{\infty} d\omega' \int d^2k' (\alpha_{\omega,\vec{k};\omega',\vec{k}'}\phi^{(1)}_{\omega',\vec{k}'} + \beta^{(1)}_{\omega,\vec{k};\omega',\vec{k}'}\phi*^{(1)}_{\omega',\vec{k}'}) \tag{5.89}$$

where α, β's are Bogoluibov coefficients:

$$\alpha_{\omega,\vec{k};\omega',\vec{k}'} = \delta(\omega-\omega')\delta^2(\vec{k}-\vec{k}') + \alpha^{(1)}_{\omega,\vec{k};\omega',\vec{k}'} + \cdots \tag{5.90}$$

$$\beta_{\omega,\vec{k};\omega',\vec{k}'} = \beta^{(1)}_{\omega,\vec{k};\omega',\vec{k}'} + \cdots \tag{5.91}$$

Note that the $\alpha^{(1)}, \beta^{(1)}$ are non-trivial terms arising from the scattering , so they are taken to be linear order correction in V. Now we substitute (5.89) in (5.88) to obtain

$$2\int d\omega' d^2k' (\alpha^{(1)}_{\omega,\vec{k};\omega',\vec{k}'}\phi_{\omega',\vec{k}'} + \beta^{(1)}_{\omega,\vec{k};\omega',\vec{k}'}\phi^{*}_{\omega',\vec{k}'}) = -2V^a\partial_a\phi_{\omega,\vec{k}}|_{u=u_0} \tag{5.92}$$

$$+ \int_{\rho}^{\infty} (\partial_A Y^A + \rho'\delta^{AB}\partial_A\partial_B f_H)\partial_{\rho'}\phi d\rho'|_{u=u_0} \tag{5.93}$$

Using (5.83)-(5.85) we may write

$$\alpha^{(1)}_{\omega,\vec{k};\omega',\vec{k}'} = -\langle V^a\partial_a\phi_{\omega,\vec{k}}, \phi_{\omega',\vec{k}'}\rangle + \frac{1}{2}\langle \int_{\rho}^{\infty} (\partial_A Y^A + \rho'\delta^{AB}\partial_A\partial_B f_H)\partial_{\rho'}\phi_{\omega,\vec{k}} d\rho', \phi_{\omega',\vec{k}'}\rangle \tag{5.94}$$

$$\beta^{(1)}_{\omega,\vec{k};\omega',\vec{k}'} = \langle V^a\partial_a\phi_{\omega,\vec{k}}, \phi^{*}_{\omega',\vec{k}'}\rangle - \frac{1}{2}\langle \int_{\rho}^{\infty} (\partial_A Y^A + \rho'\delta^{AB}\partial_A\partial_B f_H)\partial_{\rho'}\phi_{\omega,\vec{k}} d\rho', \phi^{*}_{\omega',\vec{k}'}\rangle \tag{5.95}$$

Compare the expressions obtained above with [1]. Apart from the sign convention, we find that there no super-rotation contribution in $\beta^{(1)}$ term

$$\frac{1}{2}\langle \int_{\rho}^{\infty} (\partial_A Y^A)\partial_{\rho'}\phi_{\omega,\vec{k}} d\rho', \phi^{*}_{\omega',\vec{k}'}\rangle = -\frac{1}{2}\langle (\partial_A Y^A)\phi_{\omega,\vec{k}}, \phi^{*}_{\omega',\vec{k}'}\rangle = 0 \tag{5.96}$$

However, there is a non-trivial contribution in $\alpha^{(1)}$:

$$\frac{1}{2}\langle \int_{\rho}^{\infty} (\partial_A Y^A)\partial_{\rho'}\phi_{\omega,\vec{k}} d\rho', \phi_{\omega',\vec{k}'}\rangle = -\frac{i}{2}\tilde{k}^{-}_{A}\tilde{Y}^A(\tilde{k}^{-})\delta(\omega-\omega') \tag{5.97}$$

$$-\langle Y^A\partial_A\phi_{\omega,\vec{k}}, \phi_{\omega',\vec{k}'}\rangle = -\frac{i}{2}(\tilde{k}^{+} + \tilde{k}^{-})_A\tilde{Y}^A(\tilde{k}^{-})\delta(\omega-\omega') \tag{5.98}$$

where $\tilde{k}^{\pm} = \tilde{k} \pm \tilde{k}'$. Here, the tilde ove Y refers to the Fourier transform of $Y(x)$. Eventually, our result can be compactly written as

$$\alpha^{(1)}_{\omega,\vec{k};\omega',\vec{k}'} = \alpha^{(I)}_{\omega,\vec{k};\omega',\vec{k}'} - i\left(\tilde{k}^{-}_{A} + \frac{1}{2}\tilde{k}^{+}_{A}\right)\tilde{Y}^{A}(\tilde{k}^{-})\delta(\omega-\omega') \tag{5.99}$$

$$\beta^{(1)}_{\omega,\vec{k};\omega',\vec{k}'} = \beta^{(I)}_{\omega,\vec{k};\omega',\vec{k}'} \tag{5.100}$$

$\alpha^{(I)}_{\omega,\vec{k};\omega',\vec{k}'}, \beta^{(I)}_{\omega,\vec{k};\omega',\vec{k}'}$ being the result obtained from [1]. Thus, till first order in V, the super-rotation doesn't contribute to particle creation at all. It would also be true if we had started with non-trivial higher order corrections in V.

Note, that the basis (5.82) allows us to define positive frequency element in the right wedge of Rindler space-time (the positive energy condition (5.77)). We can as well do these calculations for left wedge (in this case (5.77) should be modified to $\mathcal{L}_{-K}\phi = -i\omega\phi$). Then, any general element of the solution space for field equation can be written as

$$\phi(x) = \phi^{-}(x) + \phi^{+}(x) \tag{5.101}$$

where $\phi^{\pm}$ are positive frequency fields on the Cauchy segments I,II respectively. In terms of the "single-particle" Hilbert space, we may write this as

$$\mathcal{H}_R \cong_{iso} \mathcal{H}_{II} \oplus \mathcal{H}_I \tag{5.102}$$

$\mathcal{H}_R$ being the single particle Hilbert space corresponding to the total solution. The multi-particle description can be given in terms of Fock space [31] :

$$\mathfrak{F}_R \cong_{iso} \mathfrak{F}_{II} \otimes \mathfrak{F}_I \tag{5.103}$$

We are now in a position to look at the Quantum memory effect. Consider, two states $|\phi_1\rangle, |\phi_2\rangle \in \mathfrak{F}_R$. Let:

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + p|1\rangle_A|1\rangle_B) \tag{5.104}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle_C|0\rangle_D + q|1\rangle_C|1\rangle_D) \tag{5.105}$$

where we let $0 < p, q << 1$. Now define the product state

$$|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \tag{5.106}$$

We use Negativity as an entanglement measure for various sub-systems of Φ state (from definition, $N = \frac{1}{2}(||\rho^T|| - 1)$ where $||\cdot||$ is the trace-norm. ρ is the reduced density matrix corresponding to a particular subsystem we want to study) We note the following results:

Prior to scattering, A is maximally entangled with B, same for the modes C and D. The Negativity for these subsystems are

$$N_{A,B} = \frac{p}{1+p^2} \tag{5.107}$$

$$N_{C,D} = \frac{q}{1+q^2} \tag{5.108}$$

while $N_{A,D} = N_{B,C} = 0$. We could take $p = q = 1$, then each of these pairs A, B and C, D are maximally-entangled.

After scattering, the B and D modes will mix with the negative frequency modes in the left wedge giving rise to some non-trivial entanglement. The idea is to consider $|0\rangle_B|0\rangle_D$ as element of the Fock space after scattering, i.e. we make an ansatz

$$|0\rangle_B|0\rangle_D = \sum_{m=0,n=0}^{\infty} c_{mn}|m\rangle_{\bar{B}}|n\rangle_{\bar{D}} \tag{5.109}$$

The coefficients c_{mn} are simply given by the scalar products ${}_{\bar{B}}\langle m|0\rangle_B \times_{\bar{D}} \langle n|0\rangle_D$. To compute the scalar products, we recall some of the basic definitions pertaining to ladder-operators:

$$a_{B/D}|0\rangle_{B/D} = 0 \qquad a_{\bar{B}/\bar{D}}|0\rangle_{\bar{B}/\bar{D}} = 0 \tag{5.110}$$

$$|n\rangle_{B/D} = \frac{1}{\sqrt{n!}}(a_{B/D})^n|0\rangle_{B/D} \qquad |n\rangle_{\bar{B}/\bar{D}} = \frac{1}{\sqrt{n!}}(a_{\bar{B}/\bar{D}})^n|0\rangle_{\bar{B}/\bar{D}} \tag{5.111}$$

and the Bogoluibov relation

$${a^{k}}^{\dagger}_{B/D} = \sum_{k'}(\alpha_{k,k'}{a^{k'}}^{\dagger}_{\bar{B}/\bar{D}} - \beta_{k,k'}a^{k'}_{\bar{B}/\bar{D}}) \tag{5.112}$$

Using these relations, it may be shown that (see [32]) (5.109) can be re-written in the following form:

$$|0\rangle_B|0\rangle_D = N_1 e^{\frac{1}{2}\sum_{m,n,p}\beta^{\dagger}_{m,p}(\alpha^{-1})^{\dagger}_{p,n}a_{\bar{B}}{}^{\dagger}_{m}a_{\bar{D}}{}^{\dagger}_{n}}|0\rangle_{\bar{B}}|0\rangle_{\bar{D}} \tag{5.113}$$

where N_1 is the normalization factor. The state $|\Phi\rangle$ will contain number of higher order modes. For A, B subsystem, the negativity is given by

$$\begin{aligned} N_{A,B} = &\frac{p}{1+p^2} - \frac{p^2(1+2q^2)+q^2+p(1+5q^2)}{2(1+p^2)(1+q^2)}|\alpha^{(1)}_{B,D}|^2 \\ &- \frac{p[p^2q^2+1+2q^2+p(3+5q^2)](1+q^2)-2p^4q^4}{2p(1+p^2)(1+q^2)^2}|\beta^{(1)}_{B,D}|^2 \end{aligned}$$

Note that for small p, q, both mode mixing (α) and particle creation (β) will decrease the entanglement of the subsystem A, B.

$$N_{B,D} = \frac{2pq|\beta^{(1)}_{B,D}|}{(1+p^2)(1+q^2)}$$

Particle creation does increase the entanglement between B, D which was initially zero. Also note that the entanglement increase is faster in this case (which is linear) compared to subsystem A, B (which is quadratic).

Chapter 6

Conclusions and Outlook

As we noted in the previous chapter, the super-translation has non-trivial effect on entanglement of all sub-systems, while the contribution of super-rotation is selective : the negativity of B, D and A, C subsystems are unaffected by super-rotations in our perturbation calculation. The super-translation in Bogoluibov coefficients is time-independent, so there were no contribution from $V^{\rho} = -\partial_u f_H \rho$ (evaluated at $u = u_0$). It is interesting to note that the charges for zero modes (f_H =const) does determine the Bekenstein-Hawking entropy [11]:

$$\Delta Q = \frac{\kappa \Delta A}{8\pi G} \tag{6.1}$$

It essentially follows from using the Quasi-local charge definition near Event horizon. The zero-modes determine the surface-area of the cross-section. Although, strictly speaking the area of Rindler horizon is infinite. So it is only meaningful to talk about entropy per unit area $\Delta Q/\Delta A$. This particular property of particle production being only produced by super-translation near event horizon is also in agreement with Hawking's proposal [14] : information of infalling particles in a gravitational collapse is stored in a super-translation associated with shift of the horizon that the infalling particle caused. According to Hawking, the information is not lost but stored as deformities on event horizon, like a hologram. In future, we can repeat this calculation for fermionic and gauge fields to check if the proposal holds in those cases as well.

However, this semi-classical calculation relies on analytical solution of scalar-field, which severely restricts its domain of application in other space-times. This, is primarily because the Bogoluibov coefficients are calculated by taking inner-product on hyper-surfaces which extend through out the bulk. One possibility could be to check if we can invoke peeling property of the scalar field and compute the coefficients order by order in powers of r. As noted in the derivation, the fall off property was derived assuming large r approximation. To what extent this fall off property can be extended in the bulk is an open problem (a recent review regarding this issue has been discussed here [33]). However, this is not a problem if we just consider gravitational memory effect : we can comment about trajectories in the vicinity of event horizon without considering it's behavior in the bulk.

The shock wave construction that we have considered in our calculation is very generic: we can invoke the boundary conditions of our choice such that it induces the necessary symmetry transformation on prescribed hyper-surface (for our case, we had conformal boundary and event horizon as the two boundary surfaces). It essentially generalizes the original construction due to Hawking, Perry, Strominger [9] However, is this the most generic shock wave? The construction, as of now, is only perturbative, which can miss out interesting non-linear effects of the full Einstein's equation. One particular example includes Bondi mass-loss due to outgoing radiation, which does not appear at linearized level [3]. Thus, in near future, we hope to generalize this construction in a non-perturbative setting.

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