A Polyhedral Perspective of the Lonely Runner Conjecture

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by

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Certificate

This is to certify that this dissertation entitled 'A Polyhedral Perspective of the Lonely Runner Conjecture' towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Hrishikesh V at Indian Institute of Science Education and Research, Pune under the supervision of Dr Avinash Bhardwaj, Assistant Professor, Department of Industrial Engineering and Operations Research, Indian Institute of Technology, Bombay and Dr Vishnu Narayanan, Associate Professor, Department of Industrial Engineering and Operations Research, Indian Institute of Technology, Bombay, during the academic year 2022-2023.

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This thesis is dedicated to my family.

Declaration

I hereby declare that the matter embodied in the report entitled 'A Polyhedral Perspective of the Lonely Runner Conjecture' are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr Avinash Bhardwaj and Dr Vishnu Narayanan and the same has not been submitted elsewhere for any other degree.

Hrishikes

Hrishikesh V

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Abstract

The main focus of this thesis is the ‘Lonely Runner Conjecture’, an open problem that has remained unsolved for over half a century. The problem comes in different flavours. As a result, solving the Conjecture provides us with new information in various fields of Mathematics.

First, we take a tour of Polyhedral theory and Discrete Geometry. On this tour, we will have a peek into concepts like ‘Polyhedra’, ‘Ehrhart theory’ and ‘Lattices’, and the field of ‘Geometry of Numbers’. Then, we go over the well-known results about the Conjecture. While doing so, we shall see a detailed description of the ‘Lonely Runner polyhedron’, and the results obtained using it. Finally, we make use of the various concepts that we learnt and obtain a few new results.

Contents

Introduction

Like with a multitude of problems in the world of Mathematics, the 'Lonely Runner Conjecture' is easy to state but very challenging to solve. The Conjecture states the following: "Consider n runners on a unit circular track, starting off from a common position. If they run with constant and pairwise-distinct speeds, then there exists a time t when the ith runner is lonely".
The Conjecture is important to us due to its relevance to various areas of Mathematics. If one is able to prove the Conjecture, then it would help us to have a better understanding of concepts like the chromatic number of special classes of graphs and the covering radius of polytopes among others.

The origin of the Conjecture, in 1967, was as a problem in Diophantine approximation. Later, equivalent reformulations as analytic and geometric problems respectively. It wasn't until 1996 that the Conjecture was formulated or named as we know it today.

The Conjecture has been approached from a multitude of directions. The one that is of interest to us is the Polyhedral theory approach. In [5], Beck, Hoşten and Schymura constructed the 'Lonely Runner polyhedron', the protagonist of this thesis. The polyhedron is defined as follows:

$$P(\mathbf{n}) := \left\{ \mathbf{x} \in \mathbb{R}^k : \frac{n_i - kn_j}{k+1} \leq n_j x_i - n_i x_j \leq \frac{kn_i - n_j}{k+1}, 1 \leq i < j \leq k \right\}$$

where $\mathbf{n} \in (\mathbb{Z}^+)^k$ and represents the speeds of the runners. The goal of this thesis is to study this polyhedron and show that it contains an integer point.

The initial stages of the thesis were all about reading and understanding the work done in [5]. In order to understand the content of the manuscript, the basics of Convex Geometry,

Polyhedral theory in particular, had to be clear. This was ensured by going through the first few chapters of [40]. This is the content of Chapters 1 and 2.
It was hard to improvise on the existing ideas, due to the difficulty in visualising polyhedra in higher dimensions. As a result, we had to look for other approaches.

As mentioned earlier, the aim is to show that the 'Lonely Runner polyhedron' contains an integer point. This problem can be thought of as an enumeration of integer points in polyhedra. It turns out that these are the kind of problems that are studied in the field of 'Ehrhart theory'. We studied this topic, from [3], hoping to get some ideas. This is the content of Chapter 3.
This new content wasn't of much help because the generating functions that we had to work with were hard to deal with in the case of the 'Lonely runner polyhedron'. We were back to square one, looking for new approaches.

Sophisticated techniques weren't of much help to us. We were back to the basics. We asked ourselves, 'What is the simplest non-trivial polyhedron that has a simple-enough characterization, and can be guaranteed to contain integer points'. It is easy to see that 'lines' is the answer to this question. We started looking for conditions on lines that contain an integer point. During our literature review, no result could be found, other than for lines in $\mathbb{R}^2$. So we sat down to characterize these lines ourselves. We were successful in doing this, for the class of lines whose direction ratios are rational vectors (check Theorem 3.1.3).

The result introduced us to a new tool, namely 'Point-lattices'. On studying lattices, from [12] and [37], we came to know about the field of 'Geometry of Numbers' and problems such as the 'Shortest Vector' and 'Closest Vector' problems. All of this is the content of Chapter 4, irrespective of whether they have been employed in our work or are being used in our ongoing work.

While looking through the work that had been done thus far, we observed that there was no single source that contained the history of the Conjecture, as well as the different approaches that had been taken. Chapter 5 is a brief review of the history and results of the Conjecture. Due to the relevance of the results in [5] with our work, we provide detailed descriptions of those results.

We end with Chapter 6 which includes the new results that we obtained during the course of the past year. Finally, we conclude with other approaches that can be considered, and with a Conjecture of our own, which if proven to be true, would imply the correctness of the ‘Lonely Runner Conjecture’.

Original Contributions

All the results in Chapter 6, Theorem 3.1.3 and Conjecture 7.0.1 are original contributions. Moreover, Chapter 5 is a complete and brief literature review of the ‘Lonely Runner Conjecture’.

Chapter 1

Preliminaries

1.1 Notation

We first familiarise ourselves with the notation that will be used throughout the thesis.

$\mathbb{R}$	:	The set of real numbers
$\mathbb{R}^n$	:	n-dimensional analogue of $\mathbb{R}$
$\mathbb{R}^+$	:	The set of positive real numbers
$(\mathbb{R}^+ \cup \{0\})^n$	:	The positive orthant in $\mathbb{R}^n$
$\mathbb{Z}$	:	The set of integers
$\mathbb{Z}^n$	:	n-dimensional analogue of $\mathbb{Z}$
$\mathbb{Z}^+$	:	The set of positive integers
$(\mathbb{Z}^+)^n$	:	n-dimensional analogue of $\mathbb{Z}^+$
$\mathbb{N}$	:	The set of natural numbers $(\mathbb{Z}^+ \cup \{0\})$
$(\mathbb{R}^n)^*$	:	The dual vector space corresponding to $\mathbb{R}^n$
$\mathbf{e}_i$	:	The *ith* standard unit vector
$\mathbf{e}$	:	Sum of standard unit vectors in $\mathbb{R}^n$
$\mathcal{B}(\mathbf{c}, r)$	:	The ball of radius r with centre at $\mathbf{c}$

$\#(S)$	:	Cardinality of S
$[n]$	:	The index set $\{1, \ldots, n\}$
$GCD(a, b)$	:	The greatest common divisor of a and b

Throughout this thesis, we shall work with the vector space $\mathbb{R}^n$.

1.2 Basic concepts from Convex Geometry

From our course on Linear Algebra, we have a good understanding of Linear combinations and Linear subspaces. Now, we learn a few relevant definitions.

Definition 1.2.1. *An* affine combination *of the points* $\mathbf{x_1}, \ldots, \mathbf{x_m} \in \mathbb{R}^n$ *is a linear combination* $\sum_{i=1}^{m} a_i \mathbf{x_i}$ *such that* $\sum_{i=1}^{m} a_i = 1$.

Definition 1.2.2. *A* convex combination *of the points* $\mathbf{x_1}, \ldots, \mathbf{x_m} \in \mathbb{R}^n$ *is a linear combination* $\sum_{i=1}^{m} a_i \mathbf{x_i}$ *such that* $a_i \geq 0$ *and* $\sum_{i=1}^{m} a_i = 1$.

Definition 1.2.3. *A* conical combination *of the points* $\mathbf{x_1}, \ldots, \mathbf{x_m} \in \mathbb{R}^n$ *is a linear combination* $\sum_{i=1}^{m} a_i \mathbf{x_i}$ *such that* $a_i \geq 0$.

Remark 1.2.1. *Convex combinations are those linear combinations that are both affine combinations and conical combinations.*

Definition 1.2.4. *A set* $S \subseteq \mathbb{R}^n$ *is an* affine subspace *(or flat) of* $\mathbb{R}^n$ *if it contains every affine combination of its points.*

Remark 1.2.2. *Every non-empty affine subspace* $S \subseteq \mathbb{R}^n$ *is the translate of a linear subspace* $L \subseteq \mathbb{R}^n$ *(i.e,* $S = \mathbf{v} + L$*, for some* $\mathbf{v} \in \mathbb{R}^n$*).*

Definition 1.2.5. *The* affine hull *of a set* $S \subseteq \mathbb{R}^n$*, written as* $aff(S)$*, is the intersection of all affine subspaces that contain* S*.*

Definition 1.2.6. *The* dimension of an affine subspace *is the dimension of the corresponding linear subspace.*

Remark 1.2.3. *A point is an affine subspace of dimension* 0*. Similarly, lines and planes in* $\mathbb{R}^n$ *are affine subspaces of dimension* 1 *and* 2 *respectively.*

Definition 1.2.7. *A set* $S \subseteq \mathbb{R}^n$ *is a* convex subset *of* $\mathbb{R}^n$ *if it contains every convex combination of its points.*

Definition 1.2.8. *The* convex hull *of a set* $S \subseteq \mathbb{R}^n$*, written as* $conv(S)$*, is the intersection of all convex sets that contain* S*.*

Definition 1.2.9. *The dimension of a convex set is the dimension of its affine hull.*

Definition 1.2.10. *A set* $S \subseteq \mathbb{R}^n$ *is a* polyhedral cone *(or cone) if it contains every conical combination of its points.*

Definition 1.2.11. *A set* $S \subseteq \mathbb{R}^n$ *is a* cone *if* $\forall\ \mathbf{x} \in S, \forall\ c \geq 0, c\mathbf{x} \in S$*.*

Definition 1.2.12. *The* conical hull *(or positive hull) of a set* $S \subseteq \mathbb{R}^n$*, written as* $cone(S)$*, is the intersection of all cones that contain* S*.*

Example 1. *Consider two points* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*. Then,* $aff(\{\mathbf{x}, \mathbf{y}\})$ *is the line through* $\mathbf{x}$ *and* $\mathbf{y}$*. Moreover,* $conv(\{\mathbf{x}, \mathbf{y}\}) = [\mathbf{x}, \mathbf{y}]$*, the line segment joining* $\mathbf{x}$ *and* $\mathbf{y}$*. If* $\mathbf{y} = c\mathbf{x}$ *for some* $c \geq 0$*, then* $cone(\{\mathbf{x}, \mathbf{y}\})$ *is the ray emanating from* $\mathbf{0}$ *and passing through* $\mathbf{x}$*. If not, then* $cone(\{\mathbf{x}, \mathbf{y}\})$ *is the region between the rays emanating from* $\mathbf{0}$ *and passing through* $\mathbf{x}$ *and* $\mathbf{y}$ *respectively.*

It becomes slightly harder to determine the hulls of three or more points, as there are many cases to consider.

In the above example, we had written $aff(\{\mathbf{x}, \mathbf{y}\})$ etc, and this is the right notation. However, for simplicity, we shall write $aff(\mathbf{x}, \mathbf{y})$.

Definition 1.2.13. n (≥ 1) *points are said to be* affinely independent *if their affine hull has dimension* $(n-1)$*.*

Definition 1.2.14. *A* hyperplane *in* $\mathbb{R}^n$ *is a set of the form* $H = \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b\right\}$*.*

Remark 1.2.4. *Hyperplanes in* $\mathbb{R}^n$ *are affine subspaces of dimension* $(n-1)$*.*

Definition 1.2.15. *The sets* $H^+ = \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \geq b\right\}$ *and* $H^- = \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \leq b\right\}$ *are called* positive *and* negative closed halfspaces *respectively.*

Definition 1.2.16. *The* minkowski sum *of sets $P, Q \subseteq \mathbb{R}^n$ is defined to be:*

$$P + Q := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in P, \mathbf{y} \in Q\}$$

An easy way to visualise the Minkowski sum of sets P and Q is to place the set P on every point of Q, and then take the union of the sets.

Example 2. *Let $P = conv\{(0,0),(1,0),(0,1)\}$ and $Q = conv\{(4,0),(4,1),(5,1),(5,0)\}$. Then, $P + Q$ is given by:*

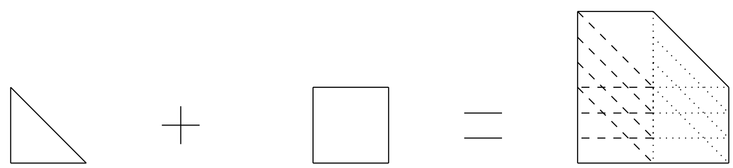


Figure 1.1: Minkowski sum of a triangle and a square

1.3 Basic concepts from Linear algebra and Number theory

Definition 1.3.1. Gram-Schmidt orthogonalization *is the process of determining an orthogonal basis $\left(\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_n}\right)$ from a set of linearly independent vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq \mathbb{R}^n$ such that $span(\mathbf{b}_1, \ldots, \mathbf{b}_i) = span\left(\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_i}\right)$ for all $i \in [n]$.*

The process/algorithm for Gram-Schmidt orthogonalization is as follows:

- $\overline{\mathbf{b}_1} := \mathbf{b}_1$, and,
- $\overline{\mathbf{b}_j} := \mathbf{b}_j - \sum\limits_{i<j} c_{ij}\overline{\mathbf{b}_i}$, where $c_{ij} := \dfrac{\mathbf{b}_j \cdot \overline{\mathbf{b}_i}}{\left|\left|\overline{\mathbf{b}_i}\right|\right|^2}$ for all $j = 2, \ldots, n$.

Definition 1.3.2. *A* diophantine equation *is a polynomial equation in two/more variables, with integer coefficients, such that the solutions are restricted to be integers.*

A linear Diophantine equation is a Diophantine equation in which the polynomial equation is a linear equation.

We now provide a theorem about linear Diophantine equations. Check [11] for the details of the proof.

Theorem 1.3.1. *The linear Diophantine equation* $ax + by = c$ *has a solution if and only if* $GCD(a, b)|c$*. Furthermore, if the system is feasible, then it has infinitely many solutions.*

1.4 Basic concepts from Combinatorics

Definition 1.4.1. *An* ordinary generating function $f(t)$ *is a formal power series*

$$f(t) := \sum_{n=0}^{\infty} a_n t^n$$

where the coefficients are given by the sequence $a_0, a_1, \ldots$.

Chapter 2

Polyhedral theory

Recently, the study of the 'Lonely Runner Conjecture' was given a new direction. It is now being studied from a Polyhedral theory perspective. My work in this thesis is about studying, what is called the 'Lonely Runner polyhedron'. In order to work with this polyhedron, we first must have a hold on basic Polyhedral theory. That is the aim of this chapter.

The content of the chapter is a literature review from [40].

2.1 Basic definitions

Definition 2.1.1. *An* H-polyhedron *is the intersection of finitely many closed halfspaces.*

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $P := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ is an H-polyhedron.

Example 3. $P = \left\{ \mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} \right\}$ *and*
$Q = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 5 \end{pmatrix} \right\}$ *represent H-polyhedra.*

Definition 2.1.2. *A* V-polyhedron *is a finitely generated convex-conical combination.*

Let $\mathbf{V} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{n\times l}$. Then $P := conv(\mathbf{V}) + cone(\mathbf{C})$ is a V-polyhedron.

Example 4. $P = conv(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2) + cone(\mathbf{e}_1)$ *represents a strip, that is unbounded in the positive x_1 direction.*

Definition 2.1.3. *A set $P \subseteq \mathbb{R}^n$ is* bounded *if it contains no rays.*

Definition 2.1.4. *An* H-polytope *is a bounded H-polyhedron.*

Example 5. $P = \left\{ \mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} 1 & -4 \\ -5 & 1 \\ 1 & 7 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} -7 \\ 6 \\ -10 \\ 1 \end{pmatrix} \right\}$ *represents an H-polytope.*

Definition 2.1.5. *A* V-polytope *is a bounded V-polyhedron.*

Let $\mathbf{V} \in \mathbb{R}^{n\times m}$. Then $P := conv(\mathbf{V})$ is a V-polytope.

Example 6. $P = conv\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \end{pmatrix} \right)$ *represents a V-polytope.*

2.2 Projection of H-polyhedra and Fourier-Motzkin elimination

Informally speaking, 'Fourier-Motzkin elimination' is the analogue of Gaussian elimination, for a system of inequalities. It can be thought of as an orderly elimination of the variables of the system of inequalities such that the solutions are unaltered.

Fourier-Motzkin elimination

Consider the system of inequalities $\mathbf{Ax} \leq \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. To eliminate x_k ($k \in [n]$) from the system, first, create the following sets. Define:

$$V^0 := \{i \in [m] : a_{ik} = 0\}; V^+ := \{i \in [m] : a_{ik} > 0\}; V^- := \{i \in [m] : a_{ik} < 0\}$$

Create a new system of inequalities as follows:

- For $i \in V^0$, include the ith inequality of the original system to the new system, and,
- For $(i,l) \in V^+ \times V^-$, include in the new system the inequality

$$a_{ik}\left(\sum_{j=1}^{n} a_{lj}x_j\right) - a_{lk}\left(\sum_{j=1}^{n} a_{ij}x_j\right) \leq a_{ik}b_l - a_{lk}b_i$$

The new system of inequalities has a solution if and only if the original system of inequalities had a solution.

Definition 2.2.1. *Let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leq \mathbf{b}\}$ be an H-polyhedron. For $k \in [n]$, the* projection of P along the kth direction *is $P_k := \{(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) : (x_1, \ldots, x_n) \in P\}$.*

Theorem 2.2.1. *If $P \subseteq \mathbb{R}^n$ is an H-polyhedron, then the projection P_k is an H-polyhedron.*

Proof. Let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leq \mathbf{b}\}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Apply Fourier-Motzkin elimination on the system of inequalities defining P. Let the new system of inequalities obtained be $\mathbf{A}'\mathbf{x}' \leq \mathbf{b}'$. Then, define $P' := \{\mathbf{x}' \in \mathbb{R}^{n-1} : \mathbf{A}'\mathbf{x}' \leq \mathbf{b}'\}$. It suffices to prove that $P' = P_k$.
Consider $\mathbf{y} \in P_k$. By the definition of P_k, there exists $\mathbf{x} \in P$ such that $\mathbf{Ax} \leq \mathbf{b}$ and the projection of $\mathbf{x}$ along the kth direction is $\mathbf{y}$. The inequalities defining P' are obtained from the inequalities that define P, and they are independent of x_k. As a result, these inequalities are satisfied by $\mathbf{y}$. Thus, $\mathbf{y} \in P'$, and hence, $P_k \subseteq P'$.
WLOG assume that $k = 1$ and let V^0, V^+ and V^- be as in Fourier-Motzkin elimination. Consider $\mathbf{x}' = (x_2, \ldots, x_n) \in P'$. We must show that there exists $x_1 \in \mathbb{R}$ such that $\mathbf{Ax} \leq \mathbf{b}$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Define $c_i := b_i - \sum_{j=2}^{n} a_{ij}x_j$ for $i \in [m]$. Note that $\mathbf{Ax} \leq \mathbf{b}$ can equivalently be written as:

$$a_{i1}x_1 \leq c_i, i \in [m]$$

The system of inequalities defining P_k contains the inequalities defining P that are obtained from V_0. Thus, it suffices to show that:

$$\exists\ x_1 \text{ s.t } a_{i1}x_1 \leq c_i, i \in V^+ \cup V^-$$

This is equivalent to:

$$\begin{aligned}
&\max_{l\in V^-}\frac{c_l}{a_{li}} \leq x_1 \leq \min_{i\in V^-}\frac{c_i}{a_{i1}}\\
\iff &\frac{c_l}{a_{l1}} \leq \frac{c_i}{a_{i1}}, (i,l) \in V^+ \times V^-\\
\iff &0 \leq a_{i1}c_l - a_{l1}c_i, (i,l) \in V^+ \times V^-\\
\iff &\mathbf{A}'\mathbf{x}' \leq \mathbf{b}'
\end{aligned}$$

And this holds since $\mathbf{x}' \in P'$. Thus, $P' \subseteq P_k$, thereby completing the proof. $\square$

2.3 Simple polytope families

There are a few simple and well-known families of polytopes. These can be generalized to any dimension.

1. **Standard n-hypercube**
   We all know of squares and cubes. Their generalization in higher dimensions is the n-hypercube ($\mathcal{C}_n$). Within the set of all n-hypercubes, the most important is the standard n-hypercube ($\square_n$). It can be represented as:

   $$\square_n := \{\mathbf{x} \in \mathbb{R}^n : 0 \leq x_i \leq 1 \;\forall\; 1 \leq i \leq n\}$$

   as well as,

   $$\square_n := conv\{\{0,1\}^n\}$$

2. **Standard n-simplex**
   The n-simplex is a generalization of triangles and tetrahedra. If $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$ are $(n+1)$ affinely independent vectors, then the n-simplex generated by these vectors is $conv\{v_1, \ldots, v_{n+1}\}$. Among n-simplices, the most studied are the standard n-simplices ($\triangle_n$). They can be represented as:

   $$\triangle_n := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{e} \cdot \mathbf{x} = 1, x_i \geq 0 \;\forall\; 1 \leq i \leq n\}$$

as well as,

$$\triangle_n := conv\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$$

3. **n-dimensional pyramid**

   One might have guessed that these are a generalization of the pyramid. However, these do not exactly resemble the pyramids with a square bottom face that we know.
   In order to construct this new polytope, first consider $\square_{n-1}$ in $\mathbb{R}^n$. Next, add a new vertex at $\mathbf{e}_n$ and join it to all vertices of $\square_{n-1}$. Thus, these can be represented as:

$$Pyr_n := conv\{\square_{n-1}, \mathbf{e}_n\}$$

   The other, slightly more complicated representation is as:

$$Pyr_n := \{\mathbf{x} \in \mathbb{R}^n : 0 \leq x_1, \ldots, x_{n-1} \leq 1 - x_n \leq 1\}$$

4. **n-dimensional crosspolytope**

   This is a generalization of the octahedron. It can be represented as:

$$\begin{aligned} \diamond_n &:= \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| \leq 1\right\} \\ &:= conv\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\} \end{aligned}$$

2.4 Representation of polyhedra and cones

From the previous Section, one might wonder why each of those polytopes had two representations. A question that comes up automatically is whether the property is true for every polytope. A more general question would be to ask whether the property holds for every polyhedron. We shall get answers to these questions in this Section.

We now state results about the representation of polytopes, polyhedra and cones. Check [40] for further details.

Theorem 2.4.1. *$P \subseteq \mathbb{R}^n$ is a V-polytope if and only if it is an H-polytope.*

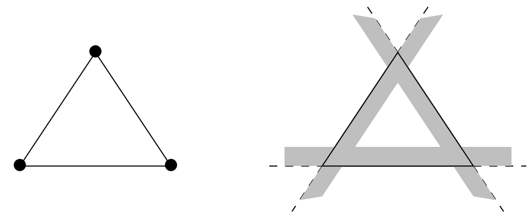


Figure 2.1: V and H representations of a triangle

Theorem 2.4.2. *$P \subseteq \mathbb{R}^n$ is a V-polyhedron if and only if it is an H-polyhedron.*

Theorem 2.4.3. *$P \subseteq \mathbb{R}^n$ is a cone if and only if it is a finite intersection of closed halfspaces.*

We have seen that alternate representations for polyhedra and cones exist. But do we really need to study both? To answer this, consider Integer programming. When given a problem, the constraints represent an H-polyhedron. To be able to solve the problem with ease, the aim, indirectly, is to obtain the V-polyhedron to make our work of determining the optimal solution simpler.

2.5 Faces of polyhedra

Definition 2.5.1. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A linear inequality $\sum_{i=1}^{n} a_i x_i \leq b$ is a* valid inequality *for P if it is satisfied by every point $\mathbf{x} \in P$.*

Definition 2.5.2. *A* face *of P is a set of the form $F = P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b \right\}$, where $\sum_{i=1}^{n} a_i x_i \leq b$ is a valid inequality for P.*

Note that P and $\emptyset$ are both faces of P. They are obtained from the valid inequalities $\sum_{i=1}^{n} 0x_i \leq 0$ and $\sum_{i=1}^{n} 0x_i \leq 1$ respectively.

Definition 2.5.3. *The* dimension *of a face F is the dimension of its affine hull.*

Remark 2.5.1. *Vertices, edges, ridges and facets are facets of dimension 0, 1, $(dim(P) - 2)$ and $(dim(P) - 1)$ respectively.*

Definition 2.5.4. *A face F is a* proper face *if it satisfies $F \subset P$.*

Theorem 2.5.1. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and F be a face of P. Then:*

(i) F is a polyhedron, and,

(ii) Every intersection of faces of P is a face of P.

Proof. Let F be defined by the valid inequality $\sum_{i=1}^{n} a_i x_i \leq b$. By the definition of a face:

$$F = P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b \right\}$$

Since P and the hyperplane are both polyhedra, they are intersections of finitely many closed halfspaces. Their intersection is a finite intersection of closed halfspaces, and thus, F is a polyhedron. Assume that $\sum_{i=1}^{n} c_i x_i \leq d$ is a valid inequality of P. Then:

$$G = P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} c_i x_i = d \right\}$$

Since both $\sum_{i=1}^{n} a_i x_i \leq b$ and $\sum_{i=1}^{n} c_i x_i \leq d$ are valid inequalities for P, we have that their sum, given by $\sum_{i=1}^{n} (a_i + c_i) x_i \leq (b + d)$, is a valid inequality of P. Furthermore,

$$\begin{aligned} F \cap G &= \left(P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b \right\} \right) \cap \left(P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} c_i x_i = d \right\} \right) \\ &= P \cap \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} (a_i + c_i) x_i = b + d \right\} \end{aligned}$$

Thus, $F \cap G$ is a face of P. □

Chapter 3

Enumeration of integer points in polyhedra

We had mentioned in the previous chapter that the 'Lonely Runner Conjecture' is now being studied using Polyhedral theory. In particular, the idea is to show that the 'Lonely Runner polyhedron' contains an integer point. Alternatively, this check can be done by explicitly counting the number of integer points in the polyhedron and showing that it is non-zero. In we chapter, we learn different techniques that can be used to count the number of integer points in different polyhedra.

Most of the content in this chapter is a literature review from [3]. Theorem 3.1.3 is my own contribution.

3.1 Integer points on lines

Lines, in $\mathbb{R}^n$, are the simplest non-trivial polyhedra that could contain integer point(s). So we ask, given a line, to determine the number of points that it passes through. We first consider the simplified version of the problem and consider lines in $\mathbb{R}^2$.

Theorem 3.1.1. *Let* $y = \frac{a}{b}x + \frac{c}{d}, a, c \in \mathbb{Z}, b, d \in \mathbb{Z}^+$, *where* $GCD(a,b) = 1$ *and* $GCD(c,d) = 1$. *If* $d|b$, *then the line passes through infinitely many integer points and none otherwise.*

Proof. First, assume that $d|b$. As a result, $b = dk$ where $k \in \mathbb{Z}^+$. Substitute this in the equation of the line. That gives:

$$y = \frac{a}{dk}x + \frac{c}{d} = \frac{a}{dk}x + \frac{ck}{dk}$$

Equivalently, the equation can also be written as:

$$(dk)y - ax = ck \tag{3.1}$$

Note that (3.1) is a linear Diophantine equation. Moreover, $GCD(dk, -a) = GCD(b, -a) = 1$. Thus, due to Theorem 1.3.1, the equation has infinitely many solutions. Therefore, the line passes through infinitely many integer points.

Now, assume that $b = dk + l$, where $0 < l < d$. Substituting this in the equation of the line gives:

$$\begin{aligned} y &= \frac{a}{dk+l} + \frac{c}{d} \\ d(dk+l)y &= (ad)x + c(dk+l) \\ d(dk+l)y - (ad)x &= c(dk+l) \end{aligned} \tag{3.2}$$

Note that $GCD(d(dk+l), -ad) = d \cdot GCD(dk+l, -a) = d \cdot GCD(b, -a) = d$. Since $GCD(c, d) = 1$, we have $d \nmid c(dk+l)$. Again, (3.2) is a linear Diophantine equation. However, in this case, the equation has no solutions. Therefore, the line does not pass through any integer point. $\square$

We provide the following theorem about the classification of lines in $\mathbb{R}^2$, without proof. Check [30] for more details.

Theorem 3.1.2. *Every line in $\mathbb{R}^2$ belongs to one of the following classes: (a) has a rational slope and does not pass through any integer point, (b) has a rational slope and passes through infinitely many integer points, (c) has an irrational slope and does not pass through any integer point, (d) has an irrational slope and passes through exactly one integer point, and, (e) parallel to the y-axis and passes through infinitely many integer points or no integer point.*

Now, we go back to studying lines in $\mathbb{R}^n$ $(n \geq 2)$. We looked at the literature for results similar to Theorem 3.1.1 for lines in $\mathbb{R}^n$. However, we weren't able to find anything. So we made it a priority to determine such a characterization.

Our first thought was to try and generalize the result and proof of Theorem 3.1.1. However, lines in $\mathbb{R}^3$ or higher do not have an equation as simple as the slope-intercept form of lines in $\mathbb{R}^2$. Instead, their equation is given by the parametric equation of a line or the symmetric equation of a line. This meant that we had to look for other options.

Definition 3.1.1. *The equation of a line in* symmetric form *is given by*

$$\frac{x_1 - a_1}{r_1} = \frac{x_2 - a_2}{r_2} = \dots \frac{x_n - a_n}{r_n} = t$$

where $\mathbf{a} = (a_1, \dots, a_n)$ *is a point on the line and* $\mathbf{r} = (r_1, \dots, r_n)$ *represents the direction ratios of the line.*

Remark 3.1.1. *Due to the parameter t in the equation of a line, $\mathbf{r}$ can be multiplied by a real number without affecting/changing the line. Due to this, $\mathbf{m} \in \mathbb{Q}^n$ can be converted to $\mathbf{m} \in \mathbb{Z}^n$.*

Looking at the symmetric form of a line, we asked, "What if we equate the terms corresponding to coordinates i and j, convert to equations similar to that in slope-intercept form and use the method from Theorem 3.1.1?" In order to do this, we would have to solve a system of linear Diophantine equations, and this is not easy. So we were back looking for other options.

Theorem 3.1.3. *Let $\mathbf{a} = \frac{p}{q}\mathbf{r} + \mathbf{s}, p \in \mathbb{Z}, q \in \mathbb{Z}^+, \mathbf{r}, \mathbf{s} \in \mathbb{Z}^n$. Then the line passing through $\mathbf{a}$ and having direction ratios $\mathbf{r}$ passes through infinitely many integer points.*

Proof. The equation of the line is given by:

$$\frac{x_1 - \left(\dfrac{pr_1}{q} + s_1\right)}{r_1} = \dots = \frac{x_n - \left(\dfrac{pr_n}{q} + s_n\right)}{r_n} \tag{3.3}$$

Equate the terms corresponding to the ith and nth coordinates. That gives:

$$\frac{x_i - \left(\dfrac{pr_i}{q} + s_i\right)}{r_i} = \frac{x_n - \left(\dfrac{pr_n}{q} + s_n\right)}{r_n}$$

The equation can be rewritten as follows:

$$x_i = \frac{r_i}{r_n} x_n + \frac{r_n s_i - r_i s_n}{r_n} \tag{3.4}$$

Note that $s_n + cr_n \in \mathbb{Z} \; \forall \; c \in \mathbb{Z}$. On substituting this in (3.4), we get that $x_i \in \mathbb{Z}$ for all $i \in [n]$. Thus, the line passes through infinitely many integer points. $\square$

One can verify that a characterization similar to that in Theorem 3.1.2 holds for lines in $\mathbb{R}^n$. In the general case, the classes will depend on both $\mathbf{a}$ and $\mathbf{r}$.

3.2 Pick's Theorem

In Section 3.1, we looked at lines (one-dimensional polyhedra). We were able to characterize lines based on the number of integer points that they pass through. What about two-dimensional polyhedra, simply called polygons? We simplify our question and consider polygons all of whose vertices are rational and are in $\mathbb{R}^2$. First, we shall solve this for a special class of polygons.

Definition 3.2.1. *An* integral polytope *is a polytope all of whose vertices have integral coordinates.*

Definition 3.2.2. *A* simple polygon *is a polygon that does not intersect itself and does not have holes.*

Theorem 3.2.1 (Pick's Theorem)**.** *Let P be a simple, integral polygon. If A, I and B denote the area of P, the number of integer points in $int(P)$ and the number of integer points on ∂P respectively, then $A = I + \frac{B}{2} - 1$.*

Proof. We give an outline of the proof. Check [29] for complete proof.
First, show that the theorem holds for primitive triangles (i.e, triangles whose integer points are their three vertices). An important point in this step is showing that primitive triangles have area $\frac{1}{2}$. This is based on showing that any partition of a coordinate rectangle (i.e, rectangle whose edges are all parallel to the coordinate axes) into primitive triangles requires the same number of primitive triangles. Finally, show that any simple, integral polygon can

be partitioned into primitive triangles. Again, these partitions are invariant of the number of primitive triangles that are required. Thus, the statement is true. $\square$

Remark 3.2.1. *Using Pick's Theorem, we can compute the total number of integer points contained in an integral polygon using the formula* $I + B = A + \frac{B}{2} + 1$.

Pick's Theorem doesn't extrapolate to polygons with rational vertices. Enumeration in the general case requires more sophisticated techniques and we study that next.

3.3 Integer-point enumerator and Ehrhart series

We are now back in $\mathbb{R}^n$. Let $P \subseteq \mathbb{R}^n$ be any polytope.

Definition 3.3.1. *For* $t \in \mathbb{R}^+$, *the* t^{th}-dilate of a polytope P *is given by*

$$tP := \{t\mathbf{x} : \mathbf{x} \in P\}$$

Definition 3.3.2. *The* integer-point enumerator *of the* t^{th} *dilate of a polytope* P *is :*

$$L_P(t) := \#(tP \cap \mathbb{Z}^n) = \#\left(P \cap \frac{1}{t}\mathbb{Z}^n\right)$$

Definition 3.3.3. *The* Ehrhart series *of* P *is the ordinary generating function*

$$Ehr_P(x) = 1 + \sum_{t \geq 1} L_P(t) x^t$$

where $L_p(t)$ *is the integer point enumerator.*

3.3.1 Standard n-hypercube

Let $P = \square_n$ and $t \in \mathbb{Z}$. Note that $tP \cap \mathbb{Z}^n = \{\mathbf{x} \in \mathbb{Z}^n : 0 \leq x_i \leq t \ \forall\ 1 \leq i \leq n\}$. Thus, we have $L_{\square_n}(t) = (t+1)^n$. Furthermore, we have:

$$Ehr_P(x) = 1 + \sum_{t \geq 1} (t+1)^n x^t = \sum_{t \geq 0} (t+1)^n x^t = \frac{1}{x} \sum_{t \geq 1} t^n x^t$$

The generating function isn't simplified. That can be done using what are called 'Eulerian numbers'. Check [3] for more details.

The n-hypercube was a simple example. Next, consider a slightly more complicated case.

3.3.2 Hyperplane restricted to positive orthant

Let $P = \left\{ \mathbf{y} \in (\mathbb{R}^+ \cup \{0\})^n : \sum_{i=1}^{n} m_i y_i = t^2, m_i \in \mathbb{Z}^+, i \in [n] \right\}$, where $t \in \mathbb{N}$. Then, we have $tP \cap \mathbb{Z}^n = \left\{ \mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} m_i x_i = t, m_i \in \mathbb{Z}^+, i \in [n] \right\}$. Assume that $GCD(m_1, \ldots, m_n) = 1$, because otherwise, a solution might not exist to the Diophantine equation for some values of t. This would make our work harder. Now, using properties of geometric series, we have:

$$\left(\frac{1}{1-z^{m_1}}\right) \ldots \left(\frac{1}{1-z^{m_n}}\right) = \left(\sum_{x_1 \geq 0} z^{m_1 x_1}\right) \ldots \left(\sum_{x_n \geq 0} z^{m_n x_n}\right) = \sum_{x_1 \geq 0} \ldots \sum_{x_n \geq 0} z^{m_1 x_1} \ldots z^{m_n x_n}$$

Note that the coefficient of z^t in the above power series gives the number of ways of writing t as a non-negative integer combination of $x_1, \ldots, x_{n-1}$ and x_n. For simpler calculations, the equation can be divided throughout by z^t, and then, the constant would give us the required quantity. Thus, we have:

$$L_P(t) = const\left(\left(\sum_{x_1 \geq 0} z^{m_1 x_1}\right) \ldots \left(\sum_{x_n \geq 0} z^{m_n x_n}\right) z^{-t}\right)$$

It is hard to determine $L_P(t)$ in the general case. Work has been done on solving the problem, and it is now known for a few cases. Check [3] for more information.

3.3.3 Standard n-simplex

Let $P = \triangle_n$ and $t \in \mathbb{Z}$. In order to use what we obtained in Subsection 3.3.2, we must transform the defining inequality of $t\triangle_n$ to an equation. So, we consider:

$$x_1 + \ldots + x_n + x_{n+1} = t$$

where $x_{n+1} \in \mathbb{N}$. Using the earlier result, we have:

$$\begin{aligned}
L_{\triangle_n}(t) &= const\left(\left(\sum_{x_1 \geq 0} z^{x_1}\right)\dots\left(\sum_{x_{n+1}\geq 0} z^{x_{n+1}}\right) z^{-t}\right) \\
&= const\left(\frac{1}{(1-z)^{n+1}z^t}\right) \\
&= const\left(\frac{1}{z^t}\sum_{i\geq 0}\binom{n+i}{n} z^i\right) \\
&= \binom{n+t}{n}
\end{aligned}$$

Furthermore,

$$Ehr_{\triangle_n}(x) = 1 + \sum_{t\geq 1}\binom{n+t}{n}x^t = \sum_{t\geq 0}\binom{n+t}{n}x^t = \frac{1}{(1-x)^{n+1}}$$

It is possible to evaluate $L_P(t)$ and $Ehr_P(z)$ for the remaining families of polytopes in a previous Section. Check [3] for more details.

3.3.4 Rational polytopes

Lemma 3.3.1. *If $S \subseteq \mathbb{R}^n$ is a set and $\mathbf{v} \in \mathbb{Z}^n$, then $\#(S \cap \mathbb{Z}^n) = \#((S + \mathbf{v}) \cap \mathbb{Z}^n)$.*

Proof. Consider $\mathbf{u} \in \mathbb{R}^n$. Observe that $\mathbf{u} \in S \iff (\mathbf{u}+\mathbf{v}) \in (S+\mathbf{v})$. Thus, there is a bijection between the points of S and $(S+\mathbf{v})$. As a consequence, there is a bijection between the integer points of S and $(S+\mathbf{v})$. The result follows. ☐

Due to Lemma 3.3.1, we consider a rational polytope in the non-negative orthant. Such a polytope can be defined by:

$$P = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

where $\mathbf{A} \in \mathbb{Z}^{m\times n}$ and $\mathbf{b} \in \mathbb{Z}^m$. Moreover, we have:

$$tP = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \mathbf{A}\mathbf{x} = t\mathbf{b}\}$$

Up until now, all polytopes in this Section were defined by one equation. However, in this case, we have $m \geq 1$ distinct equations. Thus, we require $\mathbf{z} = (z_1, \ldots, z_m)$ to be a vector. Let $\mathbf{A} = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$. Then, using similar arguments as earlier, we have:

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\ldots(1-\mathbf{z}^{\mathbf{c}_n})\mathbf{z}^{t\mathbf{b}}} = \left(\sum_{l_1 \geq 0} \mathbf{z}^{l_1 \mathbf{c}_1}\right) \ldots \left(\sum_{l_n \geq 0} \mathbf{z}^{l_n \mathbf{c}_n}\right) \frac{1}{\mathbf{z}^{t\mathbf{b}}}$$

Let $\mathbf{z}^{\mathbf{c}} := z_1^{c_1} \ldots z_m^{c_m}$. On multiplication, every term would have the form $l_1\mathbf{c}_1 + \ldots + l_n\mathbf{c}_n - t\mathbf{b} = \mathbf{A}\mathbf{l} - t\mathbf{b}$. Using this, we have:

$$L_P(t) = const\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\ldots(1-\mathbf{z}^{\mathbf{c}_n})\mathbf{z}^{t\mathbf{b}}}\right)$$

Finally, we have:

$$Ehr_P(x) = \sum_{t \geq 0}\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\ldots(1-\mathbf{z}^{\mathbf{c}_n})\mathbf{z}^{t\mathbf{b}}}\right) x^t$$

Note that the terms $(1-\mathbf{z}^{\mathbf{c}_i})$ are all independent of t, the summation variable. Using this fact, we have:

$$Ehr_P(x) = const\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\ldots(1-\mathbf{z}^{\mathbf{c}_n})}\sum_{t \geq 0}\frac{x^t}{\mathbf{z}^{t\mathbf{b}}}\right) = const\left(\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1})\ldots(1-\mathbf{z}^{\mathbf{c}_n})}\frac{1}{1-\frac{x}{\mathbf{z}^{\mathbf{b}}}}\right)$$

Chapter 4

Lattices and the Geometry of Numbers

We saw in the previous chapter that the 'Lonely Runner Conjecture' can be proven by showing that the 'Lonely Runner polyhedron' contains an integer point. The set of integer points is a special type of set, called a lattice. Moreover, every lattice is inherently related to the set of integer points. This gives us a reason to study lattices.

The study of lattices leads to the emergence of a new field in Mathematics, known today as the 'Geometry of Numbers'. We learn the two main theorems of this field. Finally, we study two well-known problems on lattices. We learn a few bounds related to these problems.

This chapter is a literature review. The content is obtained from [37] and [12].

4.1 Lattice

Definition 4.1.1. *Let* $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_m] \in \mathbb{R}^{n \times m}$ *be linearly independent vectors in* $\mathbb{R}^n$*. The* point lattice *generated by* $\mathbf{V}$ *is*

$$\Lambda(\mathbf{V}) = \{\mathbf{V}\mathbf{x} : \mathbf{x} \in \mathbb{Z}^m\} = \left\{ \sum_{i=1}^{m} x_i \mathbf{v}_i : x_i \in \mathbb{Z} \right\}$$

Definition 4.1.2. *A point-lattice is a discrete additive subgroup of* $\mathbb{R}^n$.

Theorem 4.1.1. *The two definitions are equivalent.*

Proof. Let the point-lattice Λ be generated by the linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Let $\mathbf{x} = \sum_{i=1}^{m} a_i \mathbf{v}_i$ and $\mathbf{y} = \sum_{i=1}^{m} b_i \mathbf{v}_i$ be elements of Λ. Then $(\mathbf{x} - \mathbf{y}) = \sum_{i=1}^{m} (a_i - b_i)\mathbf{v}_i \in \Lambda$. Thus, Λ is an additive subgroup. Next, complete the basis for $\mathbb{R}^n$. Let the new basis vectors be $\mathbf{w}_{m+1}, \ldots, \mathbf{w}_n$. Note that, $f(a_1, \ldots, a_n) : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(a_1, \ldots, a_n) = \sum_{i=1}^{m} a_i \mathbf{v}_i + \sum_{i=m+1}^{n} a_i \mathbf{w}_i$ is a continuous bijection. Moreover, f maps $\mathbb{Z}^m \times \{0\}^{n-m}$ to Λ. Since $\mathbb{Z}^m \times \{0\}^{n-m}$ is isomorphic to $\mathbb{Z}^n$, Λ is discrete iff $\mathbb{Z}^n$ is. Note that $\mathcal{B}(\mathbf{0}, r) \cap \mathbb{Z}^n = \{\mathbf{0}\}\ \forall\ 0 < r < 1$. Hence, $\mathbb{Z}^n$ is discrete.
We do not prove the other direction, only a heuristic. First, determine a lattice point $\mathbf{v}_1$ such that there is no lattice point in the line segment $(\mathbf{0}, \mathbf{v}_1)$. Alongside following the same procedure, to determine $\mathbf{v}_i$, we must ensure that $\mathbf{v}_i \notin \textit{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$, where $2 \leq i \leq n$. In some cases, it is could happen that there exists no $\mathbf{v}_{m+1}$ $(m + 1 \leq n)$ that satisfies the requirements. ☐

Example 7. $\Lambda([\mathbf{e_1}, \ldots, \mathbf{e_n}]) = \mathbb{Z}^n$ *is the simplest and most well-known point lattice. The set of even integers is another point-lattice.*

From here on, ‘lattices’ will always represent ‘point lattices’. A lattice can informally be thought of as a uniform arrangement of points in space. These sets have a variety of real-world applications, mainly in Cryptography, Crystallography and Error-correcting codes.

The definition of lattice seems very similar to that of a vector space. Since vector spaces are generated by a basis, can we expect something similar for lattices? Yes, lattices are generated by a basis, as the following definition shows.

Definition 4.1.3. *The matrix* $\mathbf{V}$ *in Definition 4.1.1 is called a* basis *of the lattice* $\Lambda(\mathbf{V})$.

Definition 4.1.4. *The* rank *of a lattice* $\Lambda(\mathbf{V})$ *is the cardinality of its basis* $\mathbf{V}$.

Definition 4.1.5. *The* dimension *of a lattice* $\Lambda(\mathbf{V})$ *is the length of every basis vector.*

The rank and dimension of the lattice in Definition 4.1.1 are m and n respectively.

Definition 4.1.6. *A lattice is* full-dimensional *if its rank equals its dimension.*

Definition 4.1.7. *An $n \times n$ matrix is unimodular if it has integer entries and its determinant is ± 1.*

Remark 4.1.1. *The determinant of the inverse of a unimodular matrix is ± 1.*

Along with having determinant ± 1, the inverse of a unimodular matrix has all entries in $\mathbb{Z}$. This is immediate using the relation between the inverse of a matrix and the minors of the matrix. Thus, the inverse of a unimodular matrix is unimodular.

Lemma 4.1.2. *If $\mathbf{U}$ is a unimodular matrix and $f : \mathbb{Z}^n \to \mathbb{Z}^n$ if given by $f(\mathbf{x}) = \mathbf{U}\mathbf{x}$, then f is a bijection.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$. Assume that $f(\mathbf{x}) = f(\mathbf{y})$. Due to Remark 4.1.1, $\mathbf{U}^{-1}$ exists. Multiplying by $\mathbf{U}^{-1}$ gives $\mathbf{U}^{-1}(\mathbf{U}\mathbf{x}) = \mathbf{U}^{-1}(\mathbf{U}\mathbf{y})$. Thus, $\mathbf{x} = \mathbf{y}$ and hence, f is injective.
Now, assume that $\mathbf{z} \in \mathbb{Z}^n$. Then, $\mathbf{z} = \mathbf{U}(\mathbf{U}^{-1}\mathbf{z}) = f(\mathbf{U}^{-1}\mathbf{z})$. Hence, f is surjective. □

We know that vector spaces do not have a unique basis. Moreover, the bases are related to each other by invertible matrix multiplication. Do similar properties hold for lattices? We see that they do, as the following theorem shows.

Theorem 4.1.3. *Let $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times n}$ be the basis of lattices $\Lambda(\mathbf{V})$ and $\Lambda(\mathbf{W})$ respectively. Then, $\Lambda(\mathbf{V}) = \Lambda(\mathbf{W})$ if and only if $\mathbf{W} = \mathbf{V}\mathbf{U}$, where $\mathbf{U} \in \mathbb{Z}^{n \times n}$ is a unimodular matrix.*

Proof. First, assume that $\mathbf{W} = \mathbf{V}\mathbf{U}$, with $\mathbf{U}$ being a unimodular matrix. Due to Lemma 4.1.2, we get:

$$\Lambda(\mathbf{W}) = \{\mathbf{W}\mathbf{x} : \mathbf{x} \in \mathbb{Z}^n\} = \{\mathbf{V}(\mathbf{U}\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^n\} = \{\mathbf{V}\mathbf{y} : \mathbf{y} \in \mathbb{Z}^n\} = \Lambda(\mathbf{V})$$

Now, assume that $\Lambda(\mathbf{V}) = \Lambda(\mathbf{W})$. Then, any column of $\mathbf{V}$ is an integral combination of the columns of $\mathbf{W}$, and vice versa. These integral coefficients generate matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{W} = \mathbf{V}\mathbf{A}$ and $\mathbf{V} = \mathbf{W}\mathbf{B}$. Then, we have:

$$\det(\mathbf{V}) = \det(\mathbf{W}\mathbf{B}) = \det(\mathbf{V}\mathbf{A}\mathbf{B}) = \det(\mathbf{V}) \cdot \det(\mathbf{A}) \cdot \det(\mathbf{B}) \tag{4.1}$$

Moreover, we have $\det(\mathbf{A}), \det(\mathbf{B}) \in \mathbb{Z}$. Combining this fact with (4.1) gives $\det(\mathbf{A}) = \det(\mathbf{B}) = \pm 1$. □

Theorem 4.1.3 shows that the rank of a lattice is basis-independent.

Definition 4.1.8. *Let* $\Lambda_1, \Lambda_2 \subseteq \mathbb{R}^n$ *be lattices.* Λ_1 *is a* sub-lattice *of* Λ_2 *if* $\Lambda_1 \subseteq \Lambda_2$.

Example 8. *The set of even integers and the set of multiples of four are lattices. Moreover, the latter is a sub-lattice of the former.*

We now have the first concept related to lattices that does not have an analogue for vector spaces.

Definition 4.1.9. *The* fundamental parallelepiped *of the lattice* $\Lambda(\mathbf{V})$ *is the polytope* $P(\mathbf{V}) = \{\mathbf{Vx} : \mathbf{x} \in [0,1)^n\}$.

Definition 4.1.10. *Let* $\Lambda = \Lambda(\mathbf{V}) \subseteq \mathbb{R}^n$ *for some basis* $\mathbf{V} \in \mathbb{R}^{n\times m}$. *The* determinant *of the lattice is defined by* $\det(\Lambda) = \sqrt{\det(\mathbf{V}^T\mathbf{V})}$.

Remark 4.1.2. *If* $\Lambda = \Lambda(\mathbf{V})$ *is a full-dimensional lattice, then* $\mathbf{V}$ *is a square matrix. Thus,* $\det(\Lambda) = \sqrt{\det(\mathbf{V}^T) \cdot \det(\mathbf{V})} = \sqrt{\det(\mathbf{V}) \cdot \det(\mathbf{V})} = |\det(\mathbf{V})|$.

We learnt earlier that lattices do not have a unique basis. Since the determinant of a lattice is defined in terms of a basis, is this quantity basis-independent? It is easy to see that this is the case using the relation between the bases of a lattice.

Lemma 4.1.4. *If* $P(\mathbf{V}) \subseteq \mathbb{R}^n$ *is a fundamental parallelepiped of the full-dimensional lattice* $\Lambda = \Lambda(\mathbf{V}) \subseteq \mathbb{R}^n$, *then* $vol(P(\mathbf{V})) = \det(\Lambda)$.

Proof. By the definition of a fundamental parallelepiped, we have:

$$P(\mathbf{V}) = \{\mathbf{Vx} : \mathbf{x} \in [0,1)^n\}$$

Note that $\mathbf{x} \to \mathbf{Vx}$ is a linear transformation. Using properties of linear transformations, we have $vol(P(\mathbf{V})) = vol([0,1)^n) \cdot |\det(\mathbf{V})| = |\det(\mathbf{V})| = \det(\Lambda)$. □

We shall now learn the concept of the dual of a lattice. Based on the trend in the chapter, one might feel that this concept is the analogue of the dual of a vector space. Unfortunately, that isn't the case.

Definition 4.1.11. *The* dual of a lattice Λ *is* $\Lambda^* := \{\mathbf{x} \in span(\Lambda) : \mathbf{x} \cdot \mathbf{y} \in \mathbb{Z} \ \forall \ \mathbf{y} \in \Lambda\} \subseteq (\mathbb{R}^n)^*$.

Remark 4.1.3. Λ^* *is a lattice. Moreover,* $(\Lambda^*)^* = \Lambda$.

Definition 4.1.12. *Let* $\mathbf{V} \in \mathbb{R}^{m \times n}$ *be a basis of* $\Lambda(\mathbf{V})$. *The* dual basis, $\mathbf{W}$, *is the basis that satisfies:*
(i) $span(\mathbf{W}) = span(\mathbf{V})$, *and, (ii)* $\mathbf{V}^T\mathbf{W} = \mathbf{I}$.

Theorem 4.1.5. *If* $\mathbf{V} \in \mathbb{R}^{n \times m}$ *is a basis of* $\Lambda \subseteq \mathbb{R}^n$, *then* $\mathbf{W} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}$ *is a basis of* Λ^*.

Proof. Consider $\mathbf{W}\mathbf{y} \in span(\mathbf{W})$. Then, $\mathbf{W}\mathbf{y} = \mathbf{V}((\mathbf{V}^T\mathbf{V})^{-1}\mathbf{y}) \in span(\mathbf{V})$. Thus, $span(\mathbf{W}) \subseteq span(\mathbf{V})$. Next, consider $\mathbf{V}\mathbf{x} \in span(\mathbf{V})$. Then, $\mathbf{V}\mathbf{x} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\mathbf{V}\mathbf{x} = \mathbf{W}(\mathbf{V}^T\mathbf{V}\mathbf{x}) \in span(\mathbf{W})$. Hence, $span(\mathbf{V}) \subseteq span(\mathbf{W})$.
Finally, note that $\mathbf{V}^T\mathbf{W} = \mathbf{V}^T\mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1} = \mathbf{I}$. □

Using the definition of the dual of a lattice and Theorem 4.1.5, it is easy to see that $\det(\Lambda) \cdot \det(\Lambda^*) = 1$ for any lattice Λ.

4.2 Geometry of Numbers

'Geometry of Numbers' is a unique and slightly new field of Mathematics. It was founded by a single high-impact result, namely Minkowski's Convex Body Theorem. Today, the field involves using geometric arguments to study problems from Number theory.

4.2.1 Minkowski's Convex Body Theorem

It is natural to expect the existence of an integer point (or lattice point) in a convex set of large volume. However, it wasn't until 1899 that a rigorous statement was made regarding this question. The result we are discussing is Minkowski's Convex Body Theorem.

It turns out that the 'Lonely Runner polyhedron' is large enough and has interesting properties. So, we are studying this theorem in the hope that it could be of some help.

Lemma 4.2.1. *If $n \in \mathbb{Z}^+$, $V > 1$ and $d > 0$, then $\exists\ t_0$ s.t $(t+1)^n V > (t+2d)^n\ \forall\ t \geq t_0$.*

Proof.

$$\lim_{t\to\infty} \frac{(t+1)^n V}{(t+2d)^n} = \lim_{t\to\infty} \frac{Vt^n + \sum\limits_{i=1}^{n} \binom{n}{i} t^i V}{t^n + \sum\limits_{i=1}^{n} \binom{n}{i} t^i (2d)^{n-i}} = V > 1$$

Since the limit of the ratio is more than one, $\exists\ t_0$ *s.t* $(t+1)^n V > (t+2d)^n\ \forall\ t \geq t_0$ □

Definition 4.2.1. *A set $S \subseteq \mathbb{R}^n$ is said to be* centrally symmetric *about $\mathbf{p} \in S$ if $\mathbf{p} + \mathbf{x} \in S \iff \mathbf{p} - \mathbf{x} \in S$.*

Theorem 4.2.2 (Minkowski's Theorem)**.** *Let $S \subseteq \mathbb{R}^n$ be a bounded, convex set and Λ be a full-dimensional lattice in $\mathbb{R}^n$. If S is centrally symmetric about $\mathbf{0}$ and $vol(S) > 2^n \cdot \det(\Lambda)$, then $S \cap (\Lambda \setminus \{\mathbf{0}\}) \neq \emptyset$.*

Proof. We prove it for $\Lambda = \mathbb{Z}^n$. The same proof can be generalized for any Λ.
Consider $S' = \frac{1}{2}S$. Note that $V = vol(S') = \left(\frac{1}{2}\right)^n \cdot vol(S) > \det(\mathbb{Z}^n) = 1$. Define $S'_{\mathbf{u}} := S' + \mathbf{u}$ where $\mathbf{u} \in \Lambda$.
Note that $\square_n$, the standard n-cube, is a fundamental parallelepiped of Λ. Now, consider $t\square_n$ with $t = \lceil t_0 \rceil$ and t_0 is as Lemma 4.2.1. Then, $\#(t\square_n \cap \Lambda) = (t+1)^n$. Moreover, the sum of volumes of the $(t+1)^n$ translates of S' centred at these lattice points is $(t+1)^n V$.
Let d be the maximum distance of any point of S' from $\mathbf{0}$. Then the $(t+1)^n$ translates of S' are contained in a cube of side length $(t+2d)$, and thus of volume $(t+2d)^n$. By the definition of t, we have $(t+1)^n V > (t+2d)^n$. Hence, the translates of S' overlap in the n-cube. Moreover, every translate of S' overlaps with some other translate.
Assume that $S'_{\mathbf{0}} \cap S'_{\mathbf{v}} = \mathbf{p}$, where $\mathbf{v} \in \Lambda$. Then, $(\mathbf{p} - \mathbf{v}) \in S'_{\mathbf{0}}$. Furthermore, $(\mathbf{v} - \mathbf{p}) \in S'_{\mathbf{0}}$ since $S'_{\mathbf{0}}$ is centrally symmetric about $\mathbf{0}$. Moreover, $S'_{\mathbf{0}}$ is convex. Thus, $S' = S'_{\mathbf{0}}$ contains $\frac{1}{2}\mathbf{v}$, the midpoint of $\mathbf{p}$ and $(\mathbf{v} - \mathbf{p})$. Therefore, $\mathbf{v} \in S$, thereby proving the result. □

Remark 4.2.1. *Consider $S = (-1, 1)^n$ and $\Lambda = \mathbb{Z}^n$. Note that $Vol(S) = 2^n \cdot \det(\Lambda)$, and S does not contain any non-zero integer points. Thus, the strict inequality in the statement.*

Remark 4.2.2. *If S is closed in addition to being bounded, due to its compactness, it suffices if the volume of S satisfies the weakened condition $vol(S) \geq 2^n \cdot \det(\Lambda)$.*

Remark 4.2.3. *Minkowski's Theorem has applications in Number theory. It can be used to prove Fermat's Theorem on the sum of two squares, Lagrange's four-square Theorem and Dirichlet's Theorem on simultaneous rational approximation among others.*

4.2.2 Blichfeldt's Theorem

The field of 'Geometry of Numbers' had been founded and the community was excited due to the wide variety of applications of the field. However, it took almost two decades before the next breakthrough. In 1914, Blichfeldt proved what can be thought of as the geometric analogue of the Pigeonhole principle. The result was so powerful that it led to proofs of results that couldn't be proved using Minkowski's Theorem.

Lemma 4.2.3. *Let* $A, B \subseteq \mathbb{R}^n$ *and* $\mathbf{x} \in \mathbb{R}^n$*. Then* $A \cap (B + \mathbf{x}) = ((A - \mathbf{x}) \cap B) + \mathbf{x}$*.*

Proof. Note that

$$\begin{aligned} & \mathbf{z} \in A \cap (B + \mathbf{x}) && (4.2) \\ \iff & (\mathbf{z} \in A) \wedge (\mathbf{z} \in (B + \mathbf{x})) \\ \iff & ((\mathbf{z} - \mathbf{x}) \in (A - \mathbf{x})) \wedge ((\mathbf{z} - \mathbf{x}) \in B) \\ \iff & (\mathbf{z} - \mathbf{x}) \in (A - \mathbf{x}) \cap B \\ \iff & \mathbf{z} \in ((A - \mathbf{x}) \cap B) + \mathbf{x} && (4.3) \end{aligned}$$

Combining (4.2) and (4.3) proves the result. □

Definition 4.2.2. *Let* S *be a set and* Σ *be a* σ*-algebra on* S*. A function* $m : \Sigma \to (\mathbb{R} \cup \{-\infty, \infty\})$ *is a* measure *if it satisfies:*

(a) $m(E) \geq 0 \ \forall \ E \in \sigma$*,*

(b) $m(\emptyset) = 0$*, and,*

(c) $\{E_i\}_{i=1}^{\infty}$ *is a countable collection of pairwise-disjoint sets in* $\Sigma \implies m\left(\coprod_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$*.*

Theorem 4.2.4 (Blichfeldt's Theorem)**.** *Let $S \subseteq \mathbb{R}^n$ be a bounded set and Λ be a full-dimensional lattice in $\mathbb{R}^n$. If $vol(S) > m \cdot det(\Lambda)$, then there exist at least $(m+1)$ distinct points in S, namely $\mathbf{x}_0, \ldots, \mathbf{x}_m$, such that $(\mathbf{x}_i - \mathbf{x}_j) \in \Lambda$ for $0 \leq i, j \leq m$.*

Proof. Let P be a fundamental parallelepiped of Λ and μ be a translation invariant measure on $\mathbb{R}^n$. Suppose that $vol(S) = \mu(S) > m \cdot det(\Lambda)$. Note that:

$$\begin{aligned} \mathbb{R}^n &= \coprod_{\mathbf{u} \in \Lambda} (P + \mathbf{u}) \\ S \cap \mathbb{R}^n &= S \cap \left(\coprod_{\mathbf{u} \in \Lambda} (P + \mathbf{u}) \right) \\ S &= \coprod_{\mathbf{u} \in \Lambda} (S \cap (P + \mathbf{u})) \\ &= \coprod_{\mathbf{u} \in \Lambda} (((S - \mathbf{u}) \cap P) + \mathbf{u}) \end{aligned} \tag{4.4}$$

where the third equality follows from the distributive property of union and intersection of sets while the last equality follows from the property in Lemma 4.2.3. Applying μ to (4.4) gives:

$$\begin{aligned} \mu(S) &= \mu \left(\coprod_{\mathbf{u} \in \Lambda} (((S - \mathbf{u}) \cap P) + \mathbf{u}) \right) \\ &= \sum_{\mathbf{u} \in \Lambda} \mu(((S - \mathbf{u}) \cap P) + \mathbf{u}) \\ &= \sum_{\mathbf{u} \in \Lambda} \mu((S - \mathbf{u}) \cap P) \end{aligned} \tag{4.5}$$

where the second equality follows from the additive property of measures while the last equality follows from the translation invariance property of μ.
We noted earlier that $\mu(S) > m \cdot det(\Lambda)$. Combining this with (4.5) gives:

$$\sum_{\mathbf{u} \in \Lambda} \mu((S - \mathbf{u}) \cap P) > m \cdot det(\Lambda) \tag{4.6}$$

Combining (4.6) with Lemma 4.1.4, we have:

$$\sum_{\mathbf{u} \in \Lambda} \mu((S - \mathbf{u}) \cap P) > m \cdot \mu(P) \tag{4.7}$$

Note that $S_{\mathbf{u}} := ((S - \mathbf{u}) \cap P) \subseteq P$ for $\mathbf{u} \in \Lambda$. From (4.7), we have that the sum of volumes of subsets of P is more than m times the volume of P. Thus, we can conclude that at least $(m+1)$ of the subsets must intersect. Let these subsets be $S_{\mathbf{z}_0}, \ldots, S_{\mathbf{z}_m}$, where $\mathbf{z}_0, \ldots, \mathbf{z}_m \in \Lambda$. Assume that $\mathbf{y} \in (S_{\mathbf{z}_0} \cap \ldots \cap S_{\mathbf{z}_m})$. By definition, there exists $\mathbf{x}_i \in S$ such that $\mathbf{y} = (\mathbf{x}_i - \mathbf{z}_i)$ for $0 \leq i \leq m$. Then, $(\mathbf{x}_i - \mathbf{x}_j) = (\mathbf{z}_i - \mathbf{z}_j) \in \Lambda$ for $0 \leq i, j \leq m$, thereby proving the result. □

Remark 4.2.4. *As with Minkowski's Theorem, the volume condition can be weakened if S is a closed set.*

4.2.3 Alternate proof of Minkowski's convex body Theorem

Proof. Consider the set $S' := \frac{1}{2}S$. Then, $vol(S') > \det(\Lambda)$. By Theorem 4.2.4, there exist distinct points $\frac{1}{2}\mathbf{x}, \frac{1}{2}\mathbf{y} \in S'$ such that $\left(\frac{1}{2}\mathbf{x} - \frac{1}{2}\mathbf{y}\right) \in (\Lambda \setminus \{\mathbf{0}\})$.
Note that $\mathbf{x}, \mathbf{y} \in S$. Furthermore, $-\mathbf{y} \in S$ since S is centrally symmetric about $\mathbf{0}$. Finally, using the convexity of S, we have $\left(\frac{1}{2}\mathbf{x} + \frac{1}{2}(-\mathbf{y})\right) \in S$, thereby proving the result. □

4.3 Lattice problems and relevant bounds

Lattice problems are a set of very hard-to-solve Optimization problems. Their hardness led to the development of new Cryptography schemes called 'Lattice-based Cryptosystems'. We shall study two such problems, namely the 'Shortest Vector problem' and the 'Closest Vector problem'.

Definition 4.3.1. *Given a lattice Λ, the* length of the shortest vector *is defined to be* $\lambda_1(\Lambda) := \inf\{||\mathbf{x}|| : \mathbf{x} \in \{\Lambda \setminus \{\mathbf{0}\}\}\}$.

It is not necessary that we restrict ourselves to the length of the shortest vector. Instead, we could consider the ith shortest vector.

Definition 4.3.2. *Let $\Lambda \subseteq \mathbb{R}^n$ be a full-dimensional lattice. For $1 \leq i \leq n$, the i*th successive minimum, *$\lambda_i(\Lambda)$, is defined to be the smallest $r > 0$ such that Λ contains at least i linearly independent vectors of length at most r.*

Mathematically, the above can be written as:

$$\lambda_i(\Lambda) := \inf\{r \geq 0 : \dim(span(\Lambda \cap \overline{\mathcal{B}}(\mathbf{0}, r))) \geq i\}$$

Remark 4.3.1. *By definition of $\lambda_i(\Lambda)$, we have $0 < \lambda_1(\Lambda) \leq \lambda_2(\Lambda) \leq \ldots \leq \lambda_n(\Lambda)$.*

Note that, by the definition of successive minima, we have that $\forall\ i \in [n]$ there exists $\mathbf{v}_i \in \Lambda$ such that $||\mathbf{v}_i|| = \lambda_i(\Lambda)$. Thus, the successive minima of a lattice are achieved.

4.3.1 Shortest vector problem (SVP)

Consider a lattice Λ and fix a norm. The shortest vector problem asks for the shortest non-zero vector of Λ. A modification of SVP asks for the length of the shortest vector, and this is more relevant to Mathematicians.
It is known that both these versions of the problem are hard to solve. Moreover, what matters to us, with regard to the 'Lonely Runner Conjecture', is whether the length of the shortest vector is within a certain quantity. For this reason, we study bounds on the shortest vector.

Theorem 4.3.1. *Let $\Lambda \subseteq \mathbb{R}^n$ be a full-dimensional lattice. If $\left(\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_n}\right)$ are the Gram-Schmidt vectors corresponding to the basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of Λ, then $\lambda_1(\Lambda) \geq \min\limits_{i=1,\ldots,n} \left|\left|\overline{\mathbf{b}_i}\right|\right|$.*

Proof. Let $\mathbf{x} = \sum\limits_{i=1}^{n} a_i \mathbf{b}_i$ be a lattice point. Let $k \in [n]$ be the largest index such that $a_k \neq 0$. Define the subspace $S := span\{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\} = span\left\{\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_{k-1}}\right\}$. Then, since $a_{k+1}, \ldots, a_n = 0$, we have $\mathbf{x} \in \left(S + a_k \overline{\mathbf{b}_k}\right)$. Hence, $||\mathbf{x}|| \geq d(\mathbf{x}, S) = |a_k| \cdot \left|\left|\overline{\mathbf{b}_k}\right|\right| \geq \left|\left|\overline{\mathbf{b}_k}\right|\right| \geq \min\limits_{i \in [n]} \left|\left|\overline{\mathbf{b}_i}\right|\right|$. Combining this with $\lambda_1(\Lambda) \geq ||\mathbf{x}||\ \forall\ \mathbf{x} \in (\Lambda \setminus \{\mathbf{0}\})$ proves the result. □

As mentioned earlier, we are interested in whether the length of the shortest vector is less than some known distance. So it makes sense that we study upper bounds on the shortest vector.

Lemma 4.3.2. *If $\mathcal{B}_n(\mathbf{0}, r) \subseteq \mathbb{R}^n$, then $vol(\mathcal{B}_n(\mathbf{0}, r)) \geq \left(\dfrac{2r}{\sqrt{n}}\right)^n$.*

Proof. Note that $\left[-\frac{r}{\sqrt{n}}, \frac{r}{\sqrt{n}}\right]^n \subseteq \mathcal{B}_n(\mathbf{0}, r)$. Moreover, $vol\left(\left[-\frac{r}{\sqrt{n}}, \frac{r}{\sqrt{n}}\right]^n\right) = \left(\frac{2r}{\sqrt{n}}\right)^n$. Thus, $vol(\mathcal{B}_n(\mathbf{0}, r)) \geq \left(\frac{2r}{\sqrt{n}}\right)^n$. □

Theorem 4.3.3. *If $\Lambda \subseteq \mathbb{R}^n$ is a full-dimensional lattice, then $\lambda_1(\Lambda) \leq \sqrt{n} \cdot (\det(\Lambda))^{\frac{1}{n}}$.*

Proof. Consider $B := \mathcal{B}_n(\mathbf{0}, \sqrt{n} \cdot (\det(\Lambda))^{\frac{1}{n}})$. Due to Lemma 4.3.2, $vol(B) \geq 2^n \cdot \det(\Lambda)$. Since B is a closed set, by Remark 4.2.2, B contains a non-zero lattice point. The length of this vector is bounded above by the radius of B. Thus, $\lambda_1(\Lambda) \leq \sqrt{n} \cdot (\det(\Lambda))^{\frac{1}{n}}$. □

Applying Theorem 4.3.3 to Λ and Λ^* and then using the relation between the determinants of Λ and Λ^* gives $\lambda_1(\Lambda) \cdot \lambda_1(\Lambda^*) \leq n$.

Theorem 4.3.4. *If $\Lambda \subseteq \mathbb{R}^n$ is a full-dimensional lattice, then $\left(\prod_{i=1}^{n} \lambda_i(\Lambda)\right)^{\frac{1}{n}} \leq \sqrt{n} \cdot (\det(\Lambda))^{\frac{1}{n}}$*

Proof. Let $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \Lambda$ be the vectors that attain the successive minima. Let $\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_n}$ be the corresponding Gram-Schmidt vectors. Consider the ellipsoid:

$$E := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} \left(\frac{\mathbf{x} \cdot \overline{\mathbf{b}_i}}{\lambda_i(\Lambda) \, ||\overline{\mathbf{b}_i}||} \right)^2 \leq 1 \right\}$$

Let $int(E)$ be the interior of E. Consider $\mathbf{x} \in (\Lambda \setminus \{\mathbf{0}\})$. Let $k \in [n]$ be the largest index such that $\lambda_k(\Lambda) \leq ||\mathbf{x}||$. Then, $\mathbf{x} \in span\{\mathbf{b}_1, \ldots, \mathbf{b}_k\} = span\left\{\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_k}\right\}$ (by Definition 1.3.1). This gives:

$$\begin{aligned}
\sum_{i=1}^{n} \left(\frac{\mathbf{x} \cdot \overline{\mathbf{b}_i}}{\lambda_i(\Lambda) \, ||\overline{\mathbf{b}_i}||} \right)^2 &= \sum_{i=1}^{k} \left(\frac{\mathbf{x} \cdot \overline{\mathbf{b}_i}}{\lambda_i(\Lambda) \, ||\overline{\mathbf{b}_i}||} \right)^2 \\
&\geq \frac{1}{(\lambda_k(\Lambda))^2} \sum_{i=1}^{k} \left(\mathbf{x} \cdot \frac{\overline{\mathbf{b}_i}}{||\overline{\mathbf{b}_i}||} \right)^2 \\
&= \frac{||\mathbf{x}||^2}{(\lambda_k(\Lambda))^2} \\
&\geq 1
\end{aligned}$$

where the first inequality follows from $\lambda_1(\Lambda) \leq \ldots \leq \lambda_k(\Lambda)$, the following equality follows from the fact that $\left\{ \frac{\overline{\mathbf{b}_1}}{||\overline{\mathbf{b}_1}||}, \ldots, \frac{\overline{\mathbf{b}_k}}{||\overline{\mathbf{b}_k}||} \right\}$ is a set of orthonormal spanning vectors for $\mathbf{x}$ and the final inequality follows from the definition of k. Thus, we have $\mathbf{x} \notin int(E)$. Since $int(E) \cap (\Lambda \setminus \{0\})$, by Theorem 4.2.2, we have:

$$
\begin{aligned}
2^n \cdot \det(\Lambda) &\geq vol(int(E)) \\
&= vol(E) \\
&= vol(\mathcal{B}_n(\mathbf{0}, 1)) \cdot \prod_{i=1}^{n} \lambda_i(\Lambda) \\
&\geq \left(\frac{2}{\sqrt{n}}\right)^n \cdot \prod_{i=1}^{n} \lambda_i(\Lambda)
\end{aligned}
$$

where the second equality follows from the formula for the volume of an n-dimensional ellipsoid and the last inequality follows from Lemma 4.3.2. Rearranging the terms proves the result. □

4.3.2 Closest vector problem (CVP)

Given a lattice $\Lambda \subseteq \mathbb{R}^n$ and a target vector $\mathbf{t} \in \mathbb{R}^n$, the closest vector problem asks for the lattice point that is closest to $\mathbf{t}$. As with the shortest vector problem, a modification of this problem asks for the length of the closest vector.

We now provide a result about a bound on the distance of a target vector from a given lattice. Check [27] for more details and the proof.

Theorem 4.3.5. *Let $\Lambda \subseteq \mathbb{R}^n$ be a full-dimensional lattice, and $\mathbf{t}$ be the target vector. If $\mathbf{b}_1, \ldots, \mathbf{b}_n$ is a basis of Λ and $\overline{\mathbf{b}_1}, \ldots, \overline{\mathbf{b}_n}$ are the corresponding Gram-Schmidt vectors, then $d(\mathbf{t}, \Lambda) \leq \frac{1}{2} \left(\sum_{i=1}^{n} ||\overline{\mathbf{b}_i}||^2 \right)^{\frac{1}{2}}$.*

Chapter 5

The Lonely Runner Conjecture: History

This chapter shall be a literature review of the 'Lonely Runner Conjecture'. We provide proofs for the results mentioned in the latter part of the Chapter, while most of the Chapter is an ensemble of past results with relevant references. We shall try to keep it as complete as possible.

5.1 Introduction

In 1967, Jörg M. Wills[39] proposed the following:

Conjecture 5.1.1. *Let $n \geq 2$ be an integer. Then there exist pairwise-distinct irrational numbers $a_1, \ldots, a_n$ and real number $b_1, \ldots, b_n$ such that the system of inequalities*

$$|qa_i - p_i - b_i| \leq \frac{n-1}{2(n+1)}, i \in [n]$$

cannot be solved such that $p_i \in \mathbb{Z}$ for all $i \in [n]$ and $q \in \mathbb{Z}$.

This was a Number theoretic statement involving Diophantine approximation. Wills showed that the statement holds for $n = 2$. Then, in 1971, Betke and Wills[6] proposed the following:

Conjecture 5.1.2. *Let $d(x, \mathbb{Z})$ represent the distance of a real number x to the nearest integer. For each $n \in \mathbb{Z}^+$, define:*

$$\kappa(n) = \inf \sup_{q \in \mathbb{Z}} \min_{1 \leq i \leq n} d(q\alpha_i, \mathbb{Z})$$

where the infimum is over all n-tuples $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{R} \setminus \mathbb{Q})^n$. Then, $\kappa(n) \geq \dfrac{1}{n+1}$.

This new statement has more of an analytic flavour and doesn't seem related to Conjecture 5.1.1. However, in [6], it was shown that Conjectures 5.1.1 and 5.1.2 are equivalent. Then, T.W Cusick[17], in 1974 provided a completely new statement, which seems unrelated to both the previous statements. It goes as follows:

Conjecture 5.1.3. *Let $C := \left(\square_n - \dfrac{1}{2}\mathbf{e}\right) \subseteq \mathbb{R}^n$. Consider the set:*

$$\Delta C := \left\{ \frac{1}{n+1} C + \left(m_1 + \frac{1}{2}, \ldots, m_n + \frac{1}{2} \right) : m_i \in \mathbb{N}, i \in [n] \right\}$$

Then any line through the origin and the positive orthant intersects ΔC.

In [17], the equivalence of Conjectures 5.1.2 and 5.1.1 was shown. Moreover, Conjecture 5.1.3 was shown to be true for $n = 3$. Alongside this proof, Cusick gave a new proof for $n = 2$. Furthermore, a characterization of a simple family of critical lines (i.e, lines that do not intersect the interior of any cube) is provided, besides the examples of critical lines.

Conjecture 5.1.3 and other similar statements by Cusick, are what were termed 'View-Obstruction problems' and they were more of Geometry than Number theory.
It wasn't until 1996[8] that the Conjecture attained its name. Before getting to the nomenclature, consider the following:

Conjecture 5.1.4. *Consider $(l+1)$ pairwise-distinct positive real numbers $m_0, \ldots, m_l$. For any $0 \leq i \leq l$, there is a real number t such that the distance $t(m_j - m_i)$ to the nearest integer is at least $\dfrac{1}{l+1}$, for all $0 \leq j \leq l, j \neq i$.*

Imagine yourself, and l other friends, running on a circular track of unit circumference. Moreover, each one of you starts from a common point on the track and runs with a constant speed of m_i rounds per unit time. The Conjecture then states that you shall be lonely $\Big($i.e, at least $\dfrac{1}{l+1}$ distance from every other runner$\Big)$ sometime in the future, and so will each of your friends. This is how the Conjecture acquired its title, 'The Lonely Runner Conjecture'.

Example 9. *Consider $m_0 = e$, $m_1 = \pi$ and $m_2 = \varphi$. First, with $t = 3.3$, we see that the distance of $t(\pi - e)$ and $t(\varphi - e)$ to their nearest integers are at least $\frac{1}{3}$. Similarly, the statement holds for $t(e - \pi)$ and $t(\varphi - pi)$ for $t = 1$, and for $t(e - \varphi)$ and $t(\pi - \varphi)$ with $t = \sqrt{5}$.*

It is the $(m_j - m_i)$s that matter and not the m_i's themselves. Consequently, the speed of a runner can be subtracted from all the speeds. This leaves a runner stationary and possibly, some runners with negative speeds. However, running with a negative speed is equivalent to running in the opposite direction, with the same magnitude of speed. Combining the stationarity of a runner with the symmetry of the track, all resultant speeds can be assumed to be positive. A similar argument can be made for the time t as well. Thus, an equivalent version of the Conjecture is:

Conjecture 5.1.5. *Given k positive real numbers $n_1, n_2, \ldots, n_k$, there is a non-negative real number t such that the distance of each tn_i, $1 \leq i \leq k$ to its nearest integer is at least $\dfrac{1}{k+1}$.*

From here on, $\mathbf{n}$ will always represent a vector of speeds of the k runners. Furthermore, we shall assume that $n_1 \geq n_2 \geq \ldots \geq n_k > 0$.
Let $d(x, \mathbb{Z})$ be the distance of a real number x to its nearest integer. It is easy to see that:

$$d(x, \mathbb{Z}) = \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$$

Lemma 5.1.6. *Let $x \in \mathbb{R}$ and $y \in \left[0, \dfrac{1}{2}\right]$. Then, $d(x, \mathbb{Z}) \geq y \iff \{x\} \in [y, 1-y]$.*

Proof. It follows easily from the definition of $d(x, \mathbb{Z})$ and the definition of $\{x\}$. $\square$

Using Lemma 5.1.6 the Conjecture can be mathematically written as:

$$\exists\ t \in \mathbb{R}^+ \cup \{0\} \text{ s.t } \{tn_i\} \in \left[\frac{1}{k+1}, \frac{k}{k+1}\right] \text{ for all } i \in [k]$$

The fractional part in this formulation is the main reason behind the Conjecture remaining an open problem.

Now, we define a concept that will be used throughout.

Definition 5.1.1. *t is a* suitable time *for* $\mathbf{n}$ *if at time t all runners are in the region* $\left[\frac{1}{k+1}, \frac{k}{k+1}\right]$.

Next, we have a look at an example for this version of the Conjecture.

Example 10. *Consider* $\mathbf{n} = (9, 7, 6, 1)$*. Each of the runners starts from a common position, and they run in the clockwise direction. Up to* $t = \frac{11}{10}$*, the fastest runner runs a total distance* $\left(9 \times \frac{11}{10}\right) = 9.9$ *rounds. So the position of the fastest runner at* $t = \frac{11}{10}$ *is* $\frac{9}{10}$*th of a round. This isn't a part of the interval* $\left[\frac{1}{5}, \frac{4}{5}\right]$*. Hence,* $\frac{11}{10}$ *is not a suitable time for* $\mathbf{n}$*.*

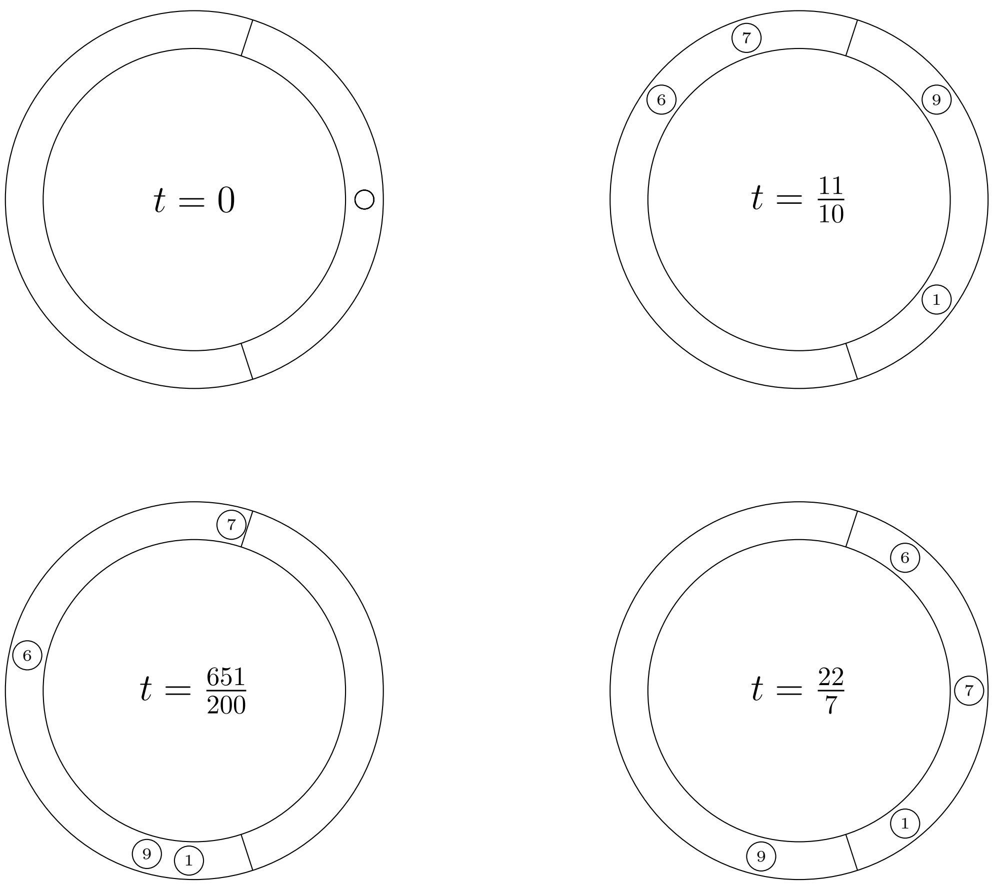


Figure 5.1: Positions of runners with speeds $(9, 7, 6, 1)$ at different times

Similar calculations show that $t = \dfrac{22}{7}$ *is not a suitable time, while* $t = \dfrac{651}{200}$ *is. These conclusions can be verified from Figure 5.1.*

It is easy to see that the latest formulation of the Conjecture holds for $k = 1, 2$. In case of $k = 1$, let $t = \dfrac{1}{2n_1}$ and the statement holds. A similar argument, on a case-by-case basis, proves the Conjecture for $k = 2$. For higher values of k, the number of cases is too high and thus, this method isn't very helpful.

Next, we go through a tour of the history of the Conjecture, seeing a variety of approaches that have been used to tackle the problem. More importantly, it will be a brief literature review of most of the known results.

5.2 Known results and approaches

The proofs for $k = 1, 2$ were such that they couldn't be modified to prove the Conjecture for $k \geq 3$. Cusick[19], in 1982, gave a new proof for $k = 2$. This was the first such proof that was extendable to higher dimensions. In the same paper, the new procedure was used to prove the Conjecture for $k = 3$.
Later, in 1983, Cusick and Pomerance[18], with some computer assistance, proved the Conjecture for $k = 4$. Then, Biennia et al[8], in their famous paper provided a simpler proof using nowhere-zero flows.
Next, it was Bohman, Holzman and Kleitman[10], who in 2001 thought of the Conjecture as a covering problem and proved the Conjecture for $k = 5$. In the same paper, they provided the first reduction of the Conjecture. For a long time, the problem was always attempted under the assumption that the speeds are real numbers. They showed that it suffices to prove the Conjecture for $\mathbf{n} \in (\mathbb{Q}^+)^k$. However, their result had a drawback. The reduction to the case of rational vectors in k dimensions was conditional on the correctness of the Conjecture for rational vectors in $(k-1)$ dimensions. Soon, in 2004, Renault[35] gave a simpler proof of the Conjecture for $k = 5$.
Finally, Barajas and Serra[1] studied the regular chromatic number of distance graphs and proved the $k = 6$ case. Moreover, they provide a new proof for $k = 4$.

Recently, in 2017, Schymura and Malikiosis[28] provided equivalent versions of the Con-

jecture in terms of the motion of Billiards balls in k-hypercubes, lines in higher dimensional tori and the covering radius of lattice zonotopes. Using these, they proved that the Conjecture holds for all $\mathbf{n}$ if it holds for $\mathbf{n} \in (\mathbb{Z}^+)^k$. Moreover, it is easy to see that the Conjecture is true for $a\mathbf{n}$, where $a \in \mathbb{Z}^+$ if it is true for $\mathbf{n} \in (\mathbb{Z}^+)^k$ satisfying $GCD(n_1, \ldots, n_k) = 1$.
A large portion of the results that were proven earlier wrongly assumed that Wills had reduced the work to prove the Conjecture for $\mathbf{n} \in (\mathbb{Z}^+)^k$. In fact, the reduction by Wills was conditional on the Conjecture being true for all vectors in lower dimensions. However, with this result in [28], the wrong assumption in those papers was taken care of.

Every monotonic sequence of positive real numbers has what is called its *lacunarity*. It is defined as follows:

$$Lacunarity := \inf_{i \geq 1} \begin{cases} \dfrac{n_i}{n_{i+1}} & \text{if } n_i \geq n_{i+1}, i \geq 1 \\ \dfrac{n_{i+1}}{n_i} & \text{if } n_i \leq n_{i+1}, i \geq 1 \end{cases}$$

We say a sequence is L-lacunary if the infimum is at least L. Every $\mathbf{n}$ can be thought of as a finite sequence. Pandey[32], in 2009, proved that $\dfrac{2(k+1)}{k-1}$-lacunary integer sequences are instances. The same year, Barajas and Serra[2] improved the result when they showed that 2-lacunary integer sequences are instances. Then, in 2011, Dubickas[23] showed that $\left(1 + \dfrac{33\log(k)}{k}\right)$-lacunary integer sequences are instances if k is very large. Later, in 2016, Czerwiński[21] showed that $\mathbf{n}$ is an instance if $(\mathbf{n} \setminus \{n_1\})$ is $\dfrac{k+1}{8e}$-lacunary. Recently, Beck, Hoşten and Schymura[5] showed that $\mathbf{n}$ is an instance if $(\mathbf{n} \setminus \{n_1, n_k\})$ is $\dfrac{2k}{k-1}$-lacunary and $GCD(n_{k-1}, n_k) \leq \dfrac{k-1}{k+1}(n_{k-1} - n_k)$. In order to prove the Conjecture using this line of thought, one would either have to show that 1-lacunary integer sequences are instances. Otherwise, it can be done by showing that the complement set of sequences from any of the above results are instances.

Another relevant quantity is the *gap of loneliness*. It is the quantity $\kappa(k)$ from Conjecture 5.1.2. It is easy to see that $\kappa(k) \geq \dfrac{1}{2k}$. In 1994, Chen[13] improved the lower bound to $\dfrac{1}{2k - 1 + \frac{1}{2k-3}}$. Then, in 1999, Chen and Cusick[14] showed that the bound can be increased to $\dfrac{1}{2k-3}$ when $k \geq 4$ and $(2k-3)$ is a prime number. In 2016, Perarnau and

Serra[33] provided a new proof for the existing bound, alongside improving the bound to $\dfrac{1}{2k-2+o(1)}$, for sufficiently large k. Recently, Terence Tao[38] gave the best-known bound of $\left(\dfrac{1}{2k}+o\left(\dfrac{\log k}{k^2(\log\log k)^2}\right)\right)$, for sufficiently large k. If one wishes to prove the Conjecture using this approach, they would have to get the bound up to $\dfrac{1}{k+1}$.
Kravitz[31] considered a stronger version of the Conjecture. It was conjectured that $\kappa(k)$ belongs to $T(k) := \left\{\dfrac{s}{ks+1} : s \in \mathbb{Z}^+\right\}$ or $\kappa(k) \geq \dfrac{1}{k+1}$ (with the latter being the condition in the statement of the 'Lonely Runner Conjecture'). The statement was proven unconditionally for $k \leq 3$. Moreover, it was shown to be true for $k = 4, 6$ under the assumption that the fastest runner is very fast compared to the other runners. However, in a recent International gathering, Fan and Sun[24] provided a counterexample of this stronger version in the case of $k = 4$. Furthermore, the stronger version was amended and it now states that $\kappa(k)$ belongs to $S(k) := \left\{\dfrac{s}{ks+1} : s \in \mathbb{Z}^+, l \in [k]\right\}$ or $\kappa(k) \geq \dfrac{1}{k+1}$. Very recently, Giri and Kravitz[25] studied $S(k)$ and showed that the set of accumulation points of $S(k)$ is $S(k-1)$.

Alongside the result mentioned earlier, Tao[38] also showed that it suffices to prove the Conjecture under the assumption that the speeds are all of the order $n^{O(n^2)}$. Moreover, it was shown that the Conjecture is true if all speeds are at most $1.2k$. Furthermore, it was noted that a desired goal is to increase the multiplier from 1.2 to 2, establishing it as a sufficient condition to prove the Conjecture. Recently, Bohman and Peng[9] showed that it is possible to get the multiplier arbitrarily close to 2 when the number of runners is sufficiently large. Very recently, Pomerance[34] proved a slightly stronger result.

For a long time, there have been thoughts on whether the 'Lonely Runner Conjecture' has been put forth in the most general way possible. As per [5], Wills recently Conjectured that the runners need not start from a common point, and still there would be a time when all of them are at least $\dfrac{1}{k+1}$ distance from a fixed origin. This statement, as with the 'Lonely Runner Conjecture' is easy to prove for $k = 1$. Along with the new statement, a proof was provided for $k = 2$ in [5]. In 2020, Cslovjecsek, Malikiosis, Naszódi and Schymura[16] studied the covering radius of polytopes and proved that the new statement holds for $k = 3$. More recently, in 2022, Rifford[36] defined the view-obstruction version for the new statement and provided new proofs for $k = 2, 3$.

It has been known for some time that $(k, k-1, \ldots, 1)$ is a tight instance (the bound

from the Conjecture is attained) for all $k \in \mathbb{Z}^+$. However, it wasn't known whether there are any other instances. Goddyn and Wong[26] studied these special instances. Using computer codes, they obtained an infinite family of new tight instances (with integer speeds) and characterized most of them.
Czerwiński and Grytczuk[22] proved that every $\mathbf{n}$ is almost lonely (i.e, there exists a time t such that there is at most one runner in the forbidden region of the track).
Czerwiński[20] considered random sets of speeds with a stronger version of the Conjecture. Using Fourier Analysis, it was proven that random speed sets are very lonely (i.e, each of the runners is very close to the point that is diametrically opposite to the starting position) with probability tending to one.
Chow and Rimanić[15] formulated the analogous version of the Conjecture in terms of function fields, and proved it for certain cases.

Henceforth, in addition to $n_1 \geq n_2 \geq \ldots \geq n_k > 0$, we shall assume that $\mathbf{n} \in (\mathbb{Z}^+)^k$. The rest of the chapter is a detailed explanation of results that have been obtained using Polyhedral theory, especially the 'Lonely Runner polyhedron'.

5.3 Entry of Polyhedral theory

Consider an arbitrary k-dimensional cube whose form is as in Conjecture 5.1.3. It would be defined as:

$$\begin{aligned}\mathcal{C}(\mathbf{m}) &:= \mathbf{m} + \left[\frac{1}{k+1}, \frac{k}{k+1}\right]^k \\ &= \left\{\mathbf{x} \in \mathbb{R}^k : m_i + \frac{1}{k+1} \leq x_i \leq m_i + \frac{k}{k+1} \ \forall\, i \in [k]\right\}\end{aligned}$$

where $\mathbf{m} \in \mathbb{N}^k$.
Next, consider the line that has direction ratios $\mathbf{n}$ and passes through the origin, where $\mathbf{n} \in (\mathbb{Z}^+)^k$. This line would be defined by:

$$l := \frac{x_1}{n_1} = \ldots = \frac{x_k}{n_k}$$

The view-obstruction version of the Conjecture states that $l \cap \mathcal{C}(\mathbf{m}) \neq \emptyset$ for some $\mathbf{m} \in \mathbb{N}^k$. Due to a previous result, we have $\mathbf{n} \in \mathcal{S}(\mathbf{m})$, where $\mathcal{S}(\mathbf{m})$ is the conic (or non-negative) span of $\mathcal{C}(\mathbf{m})$.

What are the spanning vectors of $\mathcal{S}(\mathbf{m})$? It is easy to see that each spanning vector passes through a vertex of $\mathcal{C}(\mathbf{m})$. So we assume the spanning vectors to be all vectors of the form:

$$(k+1)\mathbf{m} + \{1, k\}^k$$

Using these, the generating hyperplanes of $\mathcal{S}(\mathbf{m})$ can be determined. Then:

$$\begin{aligned}
\mathcal{S}(\mathbf{m}) &= \left\{\mathbf{x} \in \mathbb{R}^k : ((k+1)m_i + 1)x_j \leq ((k+1)m_j + k)x_i, 1 \leq i, j \leq k\right\} \\
&= \left\{\mathbf{x} \in \mathbb{R}^k : \frac{(k+1)m_j + 1}{(k+1)m_i + k} \leq \frac{x_j}{x_i} \leq \frac{(k+1)m_j + k}{(k+1)m_i + 1}, 1 \leq i < j \leq k\right\} \quad (5.1) \\
&= \left\{\mathbf{x} \in \mathbb{R}^k : \frac{1}{(k+1)x_j} - \frac{k}{(k+1)x_i} \leq \frac{m_i}{x_i} - \frac{m_j}{x_j} \leq \frac{k}{(k+1)x_j} - \frac{1}{(k+1)x_i}, 1 \leq i < j \leq k\right\} \quad (5.2)
\end{aligned}$$

where the inequalities in (5.1) are obtained by combining inequalities corresponding to the ordered pairs (i, j) and (j, i), and removing the trivial inequalities obtained from $i = j$. Now, consider the following polyhedron.

$$P(\mathbf{n}) := \left\{\mathbf{x} \in \mathbb{R}^k : \frac{1}{(k+1)n_j} - \frac{k}{(k+1)n_i} \leq \frac{x_i}{n_i} - \frac{x_j}{n_j} \leq \frac{k}{(k+1)n_j} - \frac{1}{(k+1)n_i}, 1 \leq i < j \leq k\right\} \quad (5.3)$$

$$= \left\{\mathbf{x} \in \mathbb{R}^k : \frac{n_i - kn_j}{k+1} \leq n_j x_i - n_i x_j \leq \frac{kn_i - n_j}{k+1}, 1 \leq i < j \leq k\right\}$$

The polyhedron $P(\mathbf{n})$ is called the 'Lonely Runner polyhedron'.

On comparing (5.2) and (5.3), we observe that:

$$\mathbf{m} \in P(\mathbf{n}) \iff \mathbf{n} \in \mathcal{S}(\mathbf{m}) \quad (5.4)$$

Combining (5.4) with Conjecture 5.1.3, we get the following:

Theorem 5.3.1. *Let* $\mathbf{n} \in (\mathbb{Z}^+)^k$. *Then, the following are equivalent:*

(i) $\mathbf{n}$ *is a Lonely Runner instance,*

(ii) $\exists\ \mathbf{m} \in \mathbb{N}^k$ *such that* $\mathbf{n} \in \mathcal{S}(\mathbf{m})$*, and,*

(iii) $P(\mathbf{n}) \cap \mathbb{Z}^k \neq \emptyset$.

Most of the results in the rest of this thesis shall be based on showing (*iii*). First, we give an alternate proof of the Conjecture for $k = 2$.

Theorem 5.3.2. *If* $\mathbf{n} \in (\mathbb{Z}^+)^2$*, then it is an instance.*

Proof. Let $\mathbf{n} = (n_1, n_2)$. Due to a result mentioned in Section 5.2, we can assume that $GCD(n_1, n_2) = 1$. In this case, the 'Lonely Runner polyhedron' is given by:

$$P(\mathbf{n}) = \left\{ \mathbf{x} \in \mathbb{R}^2 : \frac{n_1 - 2n_2}{3} \leq n_2 x_1 - n_1 x_2 \leq \frac{2n_1 - n_2}{3} \right\} \tag{5.5}$$

The length of the interval in (5.5) is:

$$l = \frac{2n_1 - n_2}{3} - \frac{n_1 - 2n_2}{3} = \frac{n_1 + n_2}{3}$$

Note that $l \geq 1$ when $n_1 \geq 2$ and $n_2 \geq 1$ and thus, the interval is at least a unit long. This ensures the existence of an integer in the interval.
By Bezout's Lemma, $\exists\ y_1, y_2 \in \mathbb{Z}$ such that $1 = n_2 y_1 - n_1 y_2$. Furthermore, $m = n_2(my_1) - n_1(my_2) = n_2 x_1 - n_1 x_2$ for any $m \in \mathbb{Z}$. Thus, the integer in the interval can be written as $n_2 x_1 - n_1 x_2$ for $x_1, x_2 \in \mathbb{Z}$ and hence, $P(\mathbf{n}) \cap \mathbb{Z}^2 \neq \emptyset$ for $\mathbf{n} \neq (1, 1)$.
The only case that remains is $n_1 = n_2 = 1$. Here, the polyhedron is given by:

$$P((1,1)) = \left\{ \mathbf{x} \in \mathbb{R}^2 : -\frac{1}{3} \leq x_1 - x_2 \leq \frac{1}{3} \right\}$$

Observe that 0 is in the interval. Moreover, $\mathbf{0}$ satisfies the compound inequality. Thus, $\mathbf{0} \in P((1,1))$. Therefore, $P(\mathbf{n}) \cap \mathbb{Z}^2 \neq \emptyset\ \forall\ \mathbf{n} \in (\mathbb{Z}^+)^2$. □

We now use the 'Lonely Runner polyhedron' to obtain new families of instances.

Theorem 5.3.3. *If* $\mathbf{n}$ *satisfies* $n_1 \leq k n_k$*, then it is an instance.*

Proof. The polyhedron is given by:

$$P(\mathbf{n}) = \left\{ \mathbf{x} \in \mathbb{R}^k : \frac{n_i - kn_j}{k+1} \leq n_j x_i - n_i x_j \leq \frac{kn_i - n_j}{k+1}, 1 \leq i < j \leq k \right\}$$

Since $n_1 \geq n_i$ for $i \geq 1$ and $n_j \geq n_k$ for $j \leq k$, we have $\frac{n_1 - kn_k}{k+1} \geq \frac{n_i - kn_j}{k+1}$ for $1 \leq i < j \leq k$. Moreover, due to the hypothesis, we have $0 \geq \frac{n_1 - kn_k}{k+1}$. Combining the inequalities results in:

$$\frac{n_i - kn_j}{k+1} \leq 0 \; \forall \; 1 \leq i < j \leq k \tag{5.6}$$

Now, consider the interval obtained from the ordered pair (i, j). The lower endpoint is a negative quantity due to the above equation, whereas the right endpoint is always positive since $n_i \geq n_j$ for $i < j$. Thus, 0 lies in the interval. Moreover, $\mathbf{0}$ satisfies each of the inequalities of $P(\mathbf{n})$ and thus, we have $\mathbf{0} \in P(\mathbf{n})$. Therefore, $\mathbf{n}$ is an instance. □

Theorem 5.3.4. *If there exists a positive integer $\leq k+1$ that does not divide any of $n_1, \ldots, n_k$, then $\mathbf{n}$ is an instance.*

Proof. Assume that p is an integer satisfying the conditions in the theorem statement. Note that $\left\{ \frac{n_i}{p} \right\} \in \left[\frac{1}{p}, \frac{p-1}{p} \right] \subseteq \left[\frac{1}{k+1}, \frac{k}{k+1} \right] \; \forall \; i \in [k]$. Thus, $\mathbf{n}$ is an instance due to the statement of Conjecture 5.1.5. □

Corollary 5.3.5. *If all components of $\mathbf{n}$ are odd integers, then it is an instance.*

Proof. Since all components of $\mathbf{n}$ are odd, 2 does not divide any $n_i, i \in [k]$. The result holds due to Theorem 5.3.4. □

Theorem 5.3.6. *Let $E := \{j \in [k] : n_j$ is even$\}$ and $O := [k] \backslash E$. If $\mathbf{n}$ satisfies $\max\{n_j : j \in O\} \leq \frac{k-1}{2} \min\{n_j : j \in E\}$ and $\max\{n_j : j \in E\} \leq k \min\{n_j : j \in E\}$, then it is an instance.*

Proof. Define

$$m_j := \begin{cases} \frac{n_j}{2} & \text{if } n_j \in E, \\ \frac{n_j - 1}{2} & \text{if } n_j \in O \end{cases}$$

We show that $\mathbf{n}$ satisfies (5.1). First, consider $i, j \in O$. In this case,

$$\frac{(k+1)m_j+1}{(k+1)m_i+k} = \frac{(k+1)\dfrac{n_j-1}{2}+1}{(k+1)\dfrac{n_i-1}{2}+k} = \frac{(k+1)n_j-(k-1)}{(k+1)n_i+(k-1)} \leq \frac{n_j}{n_i}$$

is true unconditionally. Similarly, it can be shown that the other inequality is satisfied unconditionally. Next, consider the case where $i \in O$ and $j \in E$. It is easy to see that the left inequality in (5.1) is true when $\max\{n_j : j \in O\} \leq \frac{k-1}{2} \min\{n_i : i \in E\}$ while the other inequality is true unconditionally. Next, we assume that $i \in E$ and $j \in O$. The proof is similar to the previous case. Finally, assume that $i, j \in E$. The left inequality in this case is true when $\max\{n_i : i \in E\} \leq k \min\{n_j : j \in E\}$ while the right inequality is satisfied when $\max\{n_j : j \in E\} \leq k \min\{n_i : i \in E\}$. Therefore, $\mathbf{n}$ is an instance. $\square$

Corollary 5.3.7. *If $\mathbf{n}$ is such that each of $n_2, \ldots, n_k$ is odd, then it is an instance.*

Proof. First, assume that n_1 is odd. Then, $\mathbf{n}$ is an instance due to Corollary 5.3.5. Now, assume that n_1 is even. Then, due to Theorem 5.3.6, $\mathbf{n}$ is an instance. Thus, the result holds. $\square$

5.3.1 Projection of $P(\mathbf{n})$

Theorem 5.3.3 required the strong condition that $n_1 \leq kn_k$. We see whether this condition can be relaxed and whether new families of instances can be obtained.

First, consider the orthogonal projection of $P(\mathbf{n})$ along the first l coordinates, where $l \leq k$. This new polyhedron, say $Q_l(\mathbf{n})$ can be thought of as the set of all points in $span(\mathbf{e}_{l+1}, \ldots, \mathbf{e}_k)$ that satisfy the defining inequalities of $P(\mathbf{n})$. With this, the polyhedron is defined by:

$$Q_l(\mathbf{n}) := \left\{(x_{l+1}, \ldots, x_k) : \frac{n_i - kn_j}{k+1} \leq n_j x_i - n_i x_j \leq \frac{kn_i - n_j}{k+1}, l+1 \leq i < j \leq k\right\}$$

This polyhedron contains the origin if and only if $n_{l+1} \leq kn_k$.

Now, project $P(\mathbf{n})$ onto the first l coordinates, where $l \leq k$. Let this new polyhedron be $P_l(\mathbf{n})$. Note that $\mathbf{p}' = (p_1, \ldots, p_l, 0, \ldots, 0) \in P_l(\mathbf{n})$ if $\mathbf{p} = (p_1, \ldots, p_k) \in P(\mathbf{n})$.

Consider a generating inequality of $P(\mathbf{n})$. Assume that $i, j \leq l$. The corresponding generating inequality of $P_l(\mathbf{n})$ remains unchanged in this case. Next, consider $i \leq l$ and $j > l$. In order to determine the corresponding generating inequality, we must substitute $x_j = 0$. This gives:

$$\frac{n_i - kn_j}{k+1} \leq n_j x_i \leq \frac{kn_i - n_j}{k+1}$$

Finally, assume that $i, j > l$. The generating inequality then becomes:

$$\frac{n_i - kn_j}{k+1} \leq 0 \leq \frac{kn_i - n_j}{k+1}$$

The right inequality above holds unconditionally, while the left inequality holds if and only if $n_{l+1} \leq kn_k$. So, assuming that $n_{l+1} \leq kn_k$, the projected polyhedron is defined by:

$$\begin{aligned} P_l(\mathbf{n}) := \Big\{ \mathbf{x} \in \mathbb{R}^l : \frac{n_i - kn_j}{k+1} \leq n_j x_i - n_i x_j \leq \frac{kn_i - n_j}{k+1}, 1 \leq i < j \leq l, \\ \frac{n_i - kn_j}{k+1} \leq n_j x_i \leq \frac{kn_i - n_j}{k+1}, 1 \leq i \leq l < j \leq k \Big\} \end{aligned}$$

With this, if it is shown that $P_l(\mathbf{n})$ contains an integer point, then it implies that $P(\mathbf{n})$ contains an integer point in the case where $\mathbf{n}$ satisfies $n_{l+1} \leq kn_k$. Note that this is a weaker assumption compared to $n_1 \leq kn_k$.

Theorem 5.3.8. *If $k \geq 3$ and $\mathbf{n}$ satisfies $n_2 \leq (k-2)n_k$, then it is an instance.*

Proof. Consider the projection onto the first coordinate. This polyhedron is $P_1(\mathbf{n})$. Next, note that $n_2 \leq (k-2)n_k < kn_k$. Thus, it suffices to show that $P_1(\mathbf{n}) \cap \mathbb{Z} \neq \emptyset$. Observe that $P_1(\mathbf{n}) = \left[\frac{n_1}{(k+1)n_k} - \frac{k}{k+1}, \frac{kn_1}{(k+1)n_2} - \frac{1}{k+1} \right]$. The length, l, of the interval is:

$$\begin{aligned} l &= \left(\frac{kn_1}{(k+1)n_2} - \frac{1}{k+1} \right) - \left(\frac{n_1}{(k+1)n_k} - \frac{k}{k+1} \right) \\ &= \frac{n_1}{k+1} \left(\frac{k}{n_2} - \frac{1}{n_k} \right) + \frac{k-1}{k+1} \\ &\geq \frac{n_1}{k+1} \frac{2}{n_2} + \frac{k-1}{k+1} \\ &\geq \frac{2}{k+1} + \frac{k-1}{k+1} \\ &= 1 \end{aligned} \tag{5.7}$$

where the first inequality follows from the hypothesis while the latter follows from $n_1 \geq n_2$. Since the interval is more than a unit long, it will contain an integer point. Hence, $\mathbf{n}$ is an instance. $\square$

Theorem 5.3.9. *If $k \geq 3$ and $\mathbf{n}$ satisfies $n_3 \leq (k-2)n_k$ and $n_2 \geq kn_3$, then it is an instance.*

Proof. Consider the projection onto the first two coordinates, given by $P_2(\mathbf{n})$. Since $n_3 \leq (k-2)n_k < kn_k$, it suffices to prove that $P_2(\mathbf{n})$ contains an integer point. The defining inequalities of this polyhedron are given by:

$$\frac{n_1}{(k+1)n_k} - \frac{k}{k+1} \leq x_1 \leq \frac{kn_1}{(k+1)n_3} - \frac{1}{k+1}$$
$$\frac{n_2}{(k+1)n_k} - \frac{k}{k+1} \leq x_2 \leq \frac{kn_2}{(k+1)n_3} - \frac{1}{k+1}$$
$$\frac{n_1 - kn_2}{k+1} \leq n_2x_1 - n_1x_2 \leq \frac{kn_1 - n_2}{k+1}$$

Let $\left(\frac{n_2}{(k+1)n_k} - \frac{k}{k+1}\right) = a$ and $\left(\frac{kn_2}{(k+1)n_3} - \frac{1}{k+1}\right) = b$.

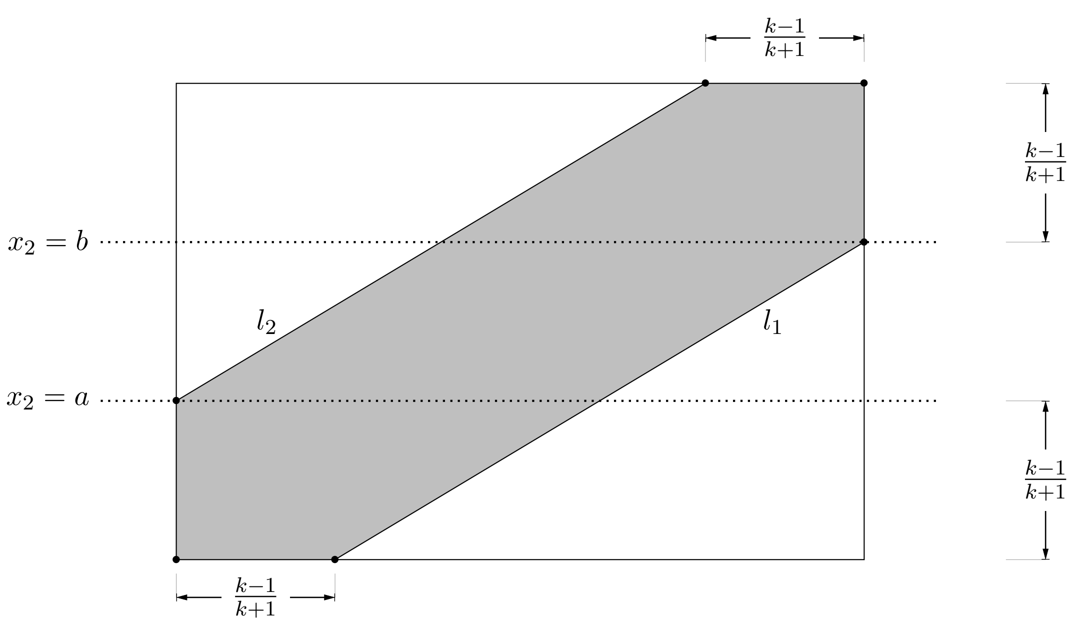


Figure 5.2: $P_2(\mathbf{n})$ under the assumption that $n_3 \leq kn_k$

The width of the horizontal strip bounded by lines $x_2 = a$ and $x_2 = b$ is:

$$\begin{aligned}
w &= \frac{k}{k+1}\left(\frac{n_2}{n_3} - 1\right) - \frac{1}{k+1}\left(\frac{n_2}{n_k} - 1\right) \\
&= \frac{n_2}{k+1}\left(\frac{k}{n_3} - \frac{1}{n_k}\right) - \frac{k-1}{k+1} \\
&\geq \frac{n_2}{k+1}\frac{2}{n_3} - \frac{k-1}{k+1} \\
&\geq \frac{2k}{k+1} - \frac{k-1}{k+1} \\
&= 1
\end{aligned}$$

where the first inequality follows from $n_3 \leq (k-2)n_k$ while the latter follows from $n_2 \geq kn_3$. Since the width is at least a unit, there exists a horizontal lattice line that passes through the strip. Now, consider the width of the strip along this line. It is given by the horizontal distance between the lines l_1 and l_2 and can be computed as:

$$\begin{aligned}
l &= \frac{\dfrac{kn_1 - n_2}{k+1} - \dfrac{n_1 - kn_2}{k+1}}{n_2} \\
&= \frac{(k-1)(n_1 + n_2)}{(k+1)n_2} \\
&\geq \frac{2(k-1)n_2}{(k+1)n_2} \\
&= \frac{2(k-1)}{k+1} \\
&= 1 + \frac{k-3}{k+1} \\
&\geq 1
\end{aligned}$$

where the first inequality follows from $n_1 \geq n_2$ and the latter follows from $k \geq 3$. Since the width along the lattice line is more than a unit, it ensures the existence of an integer point on the lattice line. Therefore, $\mathbf{n}$ is an instance. $\square$

Recently, there has been a new manuscript by Beck and Schymura[4] that uses Polyhedral theory, from a slightly different perspective. They reformulated and studied the Conjecture using the concept of Zonotopes, the Minkowski sum of line segments.

Chapter 6

The Lonely Runner Conjecture: New Results

This chapter is about my work during the course of this year. A large portion of the content is available in [7].

6.1 Basic results about suitable times

Theorem 6.1.1. $t = \dfrac{k}{(k+1)n_1}$ *is a suitable time for* $\mathbf{n}$ *if and only if* $\mathbf{n}$ *satisfies* $n_1 \leq kn_k$.

Proof. Assume that $n_1 \leq kn_k$. Since $n_k \leq n_i \leq n_1$ for all $i \in [k]$, we have:

$$n_1 \leq kn_k \leq kn_i \leq kn_1$$

On dividing throughout by kn_1, we get:

$$\frac{1}{k} \leq \frac{n_i}{n_1} \leq 1 \text{ for all } \in [k]$$
$$\frac{1}{k+1} \leq \frac{k}{(k+1)n_1} n_i \leq \frac{k}{k+1} \text{ for all } i \in [k] \tag{6.1}$$

where the second line is obtained by multiplying the first line throughout by $\dfrac{k}{k+1}$. Note that each of the quantities is a proper fraction, since $n_i \leq n_1$. Thus, $\left\{\frac{k}{(k+1)n_1}n_i\right\} = \frac{k}{(k+1)n_1}n_i$. Combining this with (6.1), we have:

$$\frac{1}{k+1} \leq \left\{\frac{k}{(k+1)n_1}n_i\right\} \leq \frac{k}{k+1} \text{ for all } i \in [k]$$

Thus, by definition, $\frac{k}{(k+1)n_1}$ is a suitable time. Now, assume that $n_1 > kn_k$. On dividing by $(k+1)n_1$, we have,

$$0 < \frac{k}{(k+1)n_1}n_k < \frac{1}{k+1}$$
$$0 < \left\{\frac{k}{(k+1)n_1}n_k\right\} < \left\{\frac{1}{k+1}\right\}$$

It follows that $\left\{\frac{k}{(k+1)n_1}n_k\right\} \notin \left[\frac{1}{k+1}, \frac{k}{k+1}\right]$ and, consequently, $\frac{k}{(k+1)n_1}$ is not a suitable time. $\square$

This theorem provides a suitable time for all $\mathbf{n}$ that satisfies $n_1 \leq kn_k$. Thus, we have an alternate proof to Theorem 5.3.3.

It is easy to see that the positions of the runners would be the same at $t \in [0, 1)$ and $(n + t)$, where $n \in \mathbb{Z}^+$. This can be verified using the properties of the fractional part function. Intuitively, due to the periodic nature of circular motion, the runners would be back at the starting position at time $t = 1$. As a result, the condition required in the Conjecture would already have been attained before time $t = 1$, or it will never be attained. Thus, it suffices to consider $t \in [0, 1)$. However, can this interval be shortened?

Lemma 6.1.2. *Given any* $\mathbf{n}$, $t \in [0, 1]$ *is a suitable time if and only if* $1 - t$ *is.*

Proof. Let t be a suitable time. It follows that $\{n_i t\} = b_i$ for some $b_i \in \left[\frac{1}{k+1}, \frac{k}{k+1}\right]$, $i \in [k]$. Observe that, since $t \leq 1$, the i^{th} runner will have completely covered at most $(n_i - 1)$ rounds, where $i \in [k]$. Thus, $n_i t = a_i + b_i$ where $a_i \in \mathbb{N}$ and $a_i \leq n_i - 1$. Now, consider the position of the i^{th} runner at time $1-t$.

$$\begin{aligned} n_i(1-t) &= n_i - n_i t = n_i - (a_i + b_i) = (n_i - a_i - 1) + (1 - b_i) \\ \{n_i(1-t)\} &= \{(n_i - a_i - 1) + (1 - b_i)\} = \{1 - b_i\}. \end{aligned}$$

Furthermore, for $i \in [k]$, $b_i \in \left[\frac{1}{k+1}, \frac{k}{k+1}\right]$ implies that $(1-b_i) \in \left[\frac{1}{k+1}, \frac{k}{k+1}\right]$. We have, $\{n_i(1-t)\} \in \left[\frac{1}{k+1}, \frac{k}{k+1}\right]$ for all $i \in [k]$. Thus $(1-t)$ is a suitable time as well.

A similar argument can be made by assuming $(1-t)$ to be a suitable time, which shows $1-(1-t) = t$ is a suitable time as well. The result follows. □

Theorem 6.1.3. $\mathbf{n}$ *is a Lonely Runner instance if and only if there is a suitable time* $t \leq \frac{1}{2}$.

Proof. Sufficiency follows from the definition of suitable time. To see the necessity, assume that $\mathbf{n}$ is a Lonely Runner instance. Thus there is a suitable time $s \in [0,1]$. Consider the function

$$t = \begin{cases} s & \text{if } s \leq \frac{1}{2} \\ 1-s & \text{otherwise} \end{cases}$$

Lemma 6.1.2 yields that both s and $1-s$ are suitable times, implying that at least one of s and $(1-s)$ is at most $\frac{1}{2}$. The result follows. □

6.2 New families of instances

We improvise on the work done using $P_1(\mathbf{n})$ and $P_2(\mathbf{n})$, and obtain two new families of instances.

Theorem 6.2.1. *If* $\mathbf{n}$ *satisfies* $n_2 \leq kn_k$ *and* $n_1 \mod ((k+1)n_k) \in [n_k, kn_k]$, *then* $\mathbf{n}$ *is a Lonely Runner instance.*

Proof. Since it has been assumed that $n_2 \leq kn_k$, it suffices to show the existence of an integral point in $P_1(\mathbf{n})$. From (5.7), we know that the length, l, of the interval $P_1(\mathbf{n})$ is:

$$\begin{aligned}
l &= \frac{n_1}{k+1}\left(\frac{k}{n_2} - \frac{1}{n_k}\right) + \frac{k-1}{k+1} \\
&= \frac{n_1}{k+1}\left(\frac{kn_k - n_2}{n_2 n_k}\right) + \frac{k-1}{k+1} \\
&\geq \frac{k-1}{k+1} \qquad (6.2)
\end{aligned}$$

where the inequality follows $n_2 \leq kn_k$ and the positivity of n_1, n_2 and n_k.

Assume that $n_1 = a(k+1)n_k + b$, where $a \in \mathbb{N}$ and $0 \leq b < (k+1)n_k$. More specifically, we have $n_k \leq b \leq kn_k$ since $n_1 \mod ((k+1)n_k) \in [n_k, kn_k]$. With this assumption, we have:

$$\begin{aligned}
\frac{n_1}{(k+1)n_k} - \frac{k}{k+1} &= \frac{a(k+1)n_k + b}{(k+1)n_k} - \frac{k}{k+1} \\
&= a + \frac{b - kn_k}{(k+1)n_k} \\
&= a - 1 + \frac{b + n_k}{(k+1)n_k} \\
&\leq a - 1 + \frac{kn_k + n_k}{(k+1)n_k} \\
&= a \qquad (6.3)
\end{aligned}$$

Additionally, using (6.2) and the definition of the length of an interval, we have:

$$\begin{aligned}
\frac{kn_1}{(k+1)n_2} - \frac{1}{k+1} &\geq \frac{n_1}{(k+1)n_k} - \frac{k}{k+1} + \frac{k-1}{k+1} \\
&= \frac{a(k+1)n_k + b}{(k+1)n_k} - \frac{k}{k+1} + \frac{k-1}{k+1} \\
&= a + \frac{b - kn_k}{(k+1)n_k} + \frac{k-1}{k+1} \\
&\geq a + \frac{n_k - kn_k}{(k+1)n_k} + \frac{k-1}{k+1} \\
&= a \qquad (6.4)
\end{aligned}$$

From (6.3) and (6.4), we have $\frac{n_1}{(k+1)n_k} - \frac{k}{k+1} \leq a \leq \frac{kn_1}{(k+1)n_2} - \frac{1}{k+1}$. The result follows. $\square$

Remark 6.2.1. *A few examples of speed vectors that are Lonely Runner instances due to*

Theorem 6.2.1 and not due to previous known results are:

$$(17, 16, 7, 6, 5, 4, 2), (18, 16, 7, 6, 5, 4, 3, 2) \text{ and } (20, 18, 8, 7, 6, 5, 4, 3, 2).$$

We shall now see a series of Lemmas that we require in order to determine the other family of instances.

Lemma 6.2.2. *Consider convex sets $S_1, S_2 \subseteq \mathbb{R}^2$ and the strip $\{\mathbf{x} \in \mathbb{R}^2 : l \leq x_2 \leq u\}$. If*

$$S_1 \cap S_2 \cap \{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\} \neq \emptyset \ \forall \ a \in [l, u]$$

then $(S_1 \cup S_2) \cap \{\mathbf{x} \in \mathbb{R}^2 : l \leq x_2 \leq u\}$ is a convex set.

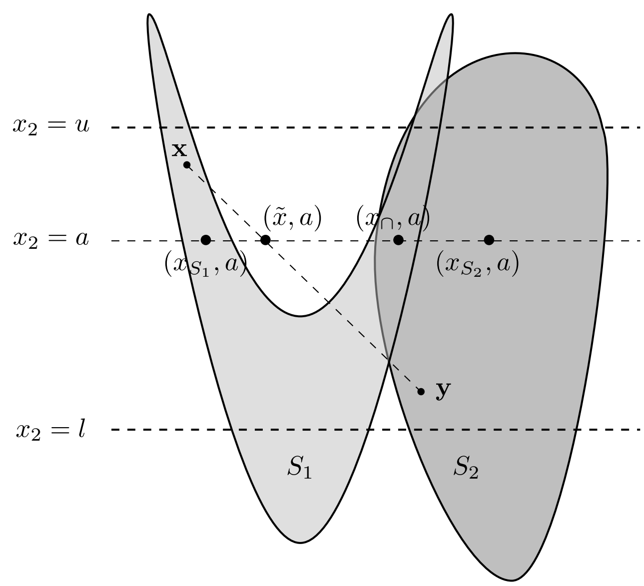


Figure 6.1: Depicting contradiction for Lemma 6.2.2 when either S_1 or S_2 is non-convex.

Proof. Let $U = (S_1 \cup S_2) \cap \{\mathbf{x} \in \mathbb{R}^2 : l \leq x_2 \leq u\}$. Assume that U is non-convex. Thus, there exists points $\mathbf{x}, \mathbf{y} \in U$ such that a point in $conv(\mathbf{x}, \mathbf{y})$ (i.e, the line segment $[\mathbf{x}, \mathbf{y}]$), is not contained in U. In particular, $\exists \ \tilde{\mathbf{x}} = (\tilde{x}, a) \in conv(\mathbf{x}, \mathbf{y})$ and $\tilde{\mathbf{x}} \notin U$.

Since $\mathbf{x}, \mathbf{y} \in U$ and $\tilde{\mathbf{x}} = (\tilde{x}, a) \in conv(\mathbf{x}, \mathbf{y})$, it follows that $a \in [l, u]$. Furthermore, consider three additional points $(x_{S_1}, a) \in S_1 \setminus S_2$, $(x_{S_2}, a) \in S_2 \setminus S_1$ and $(x_\cap, a) \in S_1 \cap S_2$ such that either $x_{S_1} < \tilde{x} < x_\cap$ or $x_\cap < \tilde{x} < x_{S_2}$ (these points will exist because of our assumptions). If $x_{S_1} < \tilde{x} < x_\cap$, then since both $(x_{S_1}, a), (x_\cap, a) \in S_1$, their convex combination $(\tilde{x}, a) \in S_1$

(since S_1 is convex), and consequently $(\tilde{x}, a) \in U$, which poses a contradiction. Alternatively, if $x_\cap < \tilde{x} < x_{S_2}$, then since both $(x_\cap, a), (x_{S_2}, a) \in S_2$, their convex combination $(\tilde{x}, a) \in S_2$ (since S_2 is convex), and consequently $(\tilde{x}, a) \in U$, which also poses a contradiction. The result follows. □

Definition 6.2.1. *Consider $S \subseteq \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n$. The width of S in the direction of $\mathbf{a}$ is defined as $w_S(\mathbf{a}) = \max_{\mathbf{x}\in S}(\mathbf{a}\cdot\mathbf{x}) - \min_{\mathbf{x}\in S}(\mathbf{a}\cdot\mathbf{x})$.*

The width notation shall be used constantly throughout the rest of this Section.

For notational brevity, we shall henceforth refer to $P_2(\mathbf{n})$ as Q. The generating inequalities of Q are:

$$\frac{n_1}{(k+1)n_k} - \frac{k}{k+1} \leq x_1 \leq \frac{kn_1}{(k+1)n_3} - \frac{1}{k+1}$$
$$\frac{n_2}{(k+1)n_k} - \frac{k}{k+1} \leq x_2 \leq \frac{kn_2}{(k+1)n_3} - \frac{1}{k+1}$$
$$\frac{n_1 - kn_2}{k+1} \leq n_2x_1 - n_1x_2 \leq \frac{kn_1 - n_2}{k+1}$$

In addition to the generating inequalities of Q, consider the following lines (Figures 6.2a and 6.2b).

$$l_1 : n_2x_1 - n_1x_2 = \frac{kn_1 - n_2}{k+1} \qquad l_2 : n_2x_1 - n_1x_2 = \frac{n_1 - kn_2}{k+1}$$

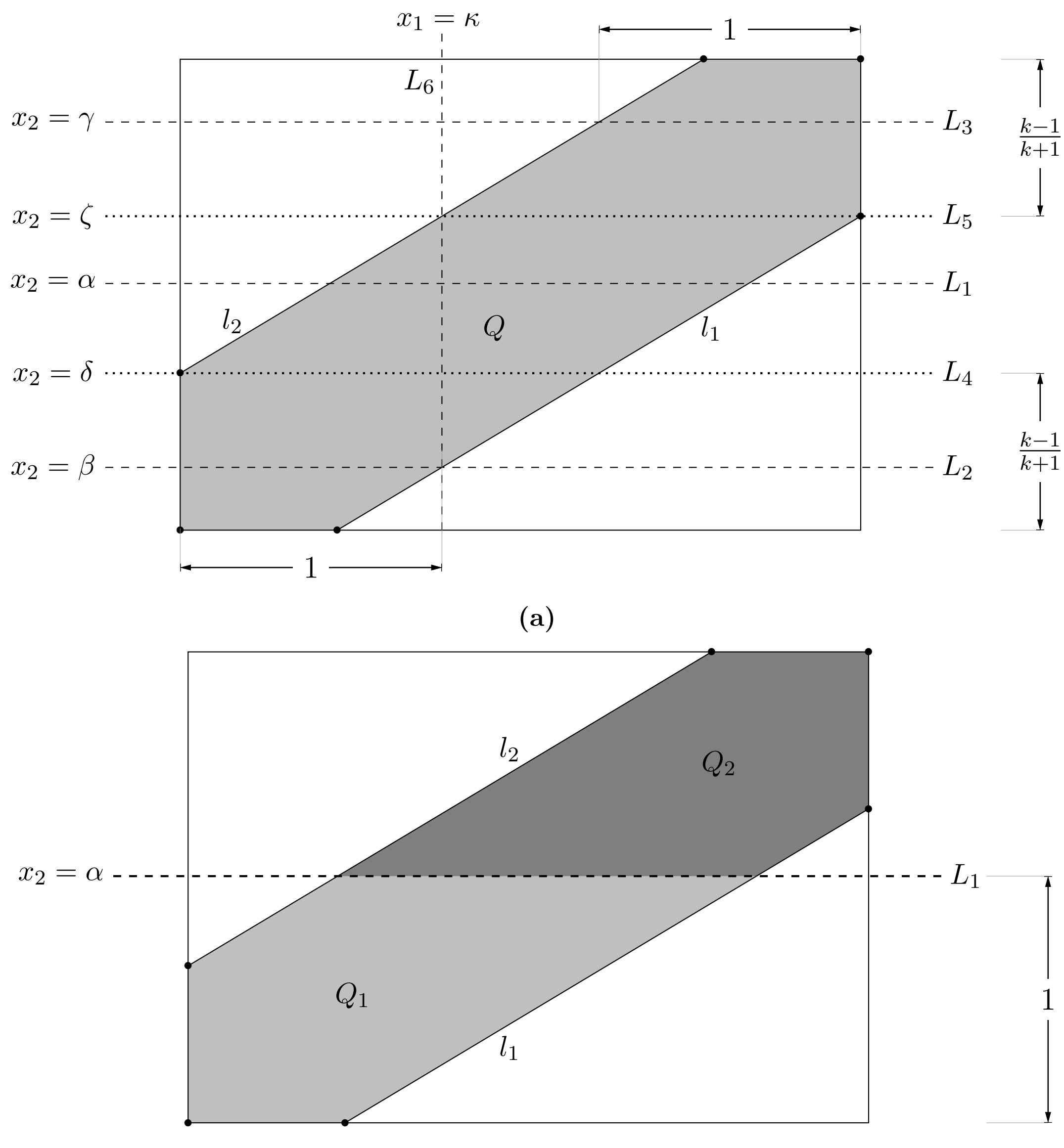
$$L_1 : x_2 = \alpha = \frac{n_2}{(k+1)n_k} + \frac{1}{k+1} \qquad L_2 : x_2 = \beta = \frac{n_2}{(k+1)n_k} + \frac{2n_2}{(k+1)n_1} - \frac{k}{k+1}$$

$$L_3 : x_2 = \gamma = \frac{kn_2}{(k+1)n_3} - \frac{2n_2}{(k+1)n_1} - \frac{1}{k+1} \qquad L_4 : x_2 = \delta = \frac{1}{k+1}\left(\frac{n_2}{n_k} - 1\right)$$

$$L_5 : x_2 = \zeta = \frac{k}{k+1}\left(\frac{n_2}{n_3} - 1\right) \qquad L_6 : x_1 = \kappa = \frac{n_1}{(k+1)n_k} + \frac{1}{k+1}$$

Lemma 6.2.3. *If $\mathbf{n}$ satisfies $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \geq k+1$, then $w_Q(\mathbf{e}_1) \geq 1$ and $w_Q(\mathbf{e}_2) \geq 1$.*

Proof. The lines $x_2 = \zeta + \frac{k-1}{k+1}$ and $x_2 = \delta - \frac{k-1}{k+1}$ represent diametrically opposite facets (top and bottom edges respectively) of Q. Then, $w_Q(\mathbf{e}_2)$ is the distance between these facets



(a)

(b)

Figure 6.2: Polyhedron Q under the assumption $n_2\left(\dfrac{k}{n_3} - \dfrac{1}{n_k}\right) \geq k + 1$

of Q.

$$\begin{aligned} w_Q(\mathbf{e}_2) &= \left(\frac{kn_2}{(k+1)n_3} - \frac{1}{k+1}\right) - \left(\frac{n_2}{(k+1)n_k} - \frac{k}{k+1}\right) \\ &= \frac{n_2}{k+1}\left(\frac{k}{n_3} - \frac{1}{n_k}\right) + \frac{k-1}{k+1} \qquad (6.5) \\ &\geq \frac{k+1}{k+1} + \frac{k-1}{k+1} \\ &\geq 1 \end{aligned}$$

where the first inequality follows from our assumption and the latter follows from $k \geq 1$. The lines $x_1 = \frac{kn_1}{(k+1)n_3} - \frac{1}{k+1}$ and $x_1 = \frac{n_1}{(k+1)n_k} - \frac{k}{k+1}$ represent diametrically opposite facets (right and left edges respectively) of Q. $w_Q(\mathbf{e}_1)$ is the distance between these facets and it can be expressed as

$$\begin{aligned} w_Q(\mathbf{e}_1) &= \left(\frac{kn_1}{(k+1)n_3} - \frac{1}{k+1}\right) - \left(\frac{n_1}{(k+1)n_k} - \frac{k}{k+1}\right) \\ &= \frac{n_1}{k+1}\left(\frac{k}{n_3} - \frac{1}{n_k}\right) + \frac{k-1}{k+1} \\ &\geq \frac{n_2}{k+1}\left(\frac{k}{n_3} - \frac{1}{n_k}\right) + \frac{k-1}{k+1} \\ &= w_Q(\mathbf{e}_2) \\ &\geq 1 \end{aligned}$$

where the first inequality follows from $n_1 \geq n_2$, the following equality from (6.5) and the last inequality from $w_Q(\mathbf{e}_2) \geq 1$. The result follows. □

From Lemma 6.2.3, we obtain that $Q_1 := Q \cap \{\mathbf{x} \in \mathbb{R}^2 : x_2 \leq \alpha\} \neq \emptyset$ and $Q_2 := Q \cap \{\mathbf{x} \in \mathbb{R}^2 : x_2 \geq \alpha\} \neq \emptyset$. Consider $Q_3 := Q_2 - \mathbf{e}_2$ and $Q_4 := Q_1 \cup Q_3$. Observe that, by definition, $w_{Q_1}(\mathbf{e}_2) = 1$ (Figure 6.2b).

Lemma 6.2.4. *If* $\mathbf{n}$ *satisfies* $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \geq k+1$, *then* $w_{Q_3}(\mathbf{e}_2) \geq \frac{k-1}{k+1}$.

Proof. Q_3 is a translate of Q_2. Thus, it suffices to show that $w_{Q_2}(\mathbf{e}_2) \geq \frac{k-1}{k+1}$. We have,

$$\begin{aligned}
w_{Q_2}(\mathbf{e}_2) &= \left(\zeta + \frac{k-1}{k+1}\right) - \alpha \\
&= \left(\frac{kn_2}{(k+1)n_3} - \frac{1}{k+1}\right) - \left(\frac{n_2}{(k+1)n_k} + \frac{1}{k+1}\right) \\
&= \frac{n_2}{k+1}\left(\frac{k}{n_3} - \frac{1}{n_k}\right) - \frac{2}{k+1} \\
&\geq \frac{k+1}{k+1} - \frac{2}{k+1} \\
&= \frac{k-1}{k+1}
\end{aligned}$$

where the inequality follows from our assumption. The result follows. $\square$

Lemma 6.2.5. *If* $\mathbf{n}$ *satisfies* $2n_1 \leq (k-1)n_2$ *and* $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \geq k+1$, *then*

$$\{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\} \cap Q_1 \cap Q_3 \neq \emptyset \ \forall \ a \in [\alpha - 1, \delta].$$

Proof. Consider $a \in [\alpha - 1, \delta]$, and the line $x_2 = a$. From Lemma 6.2.4 and the fact that $w_{Q_1}(\mathbf{e}_2) = 1$, we have $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\} \cap Q_1 \neq \emptyset$ and $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\} \cap Q_3 \neq \emptyset$.

Since $\alpha - 1 \leq a \leq \delta < \zeta$, the lines $x_2 = a$ and l_1 will intersect. Let A be the point of intersection of these lines. Specifically,

$$A = \left(\frac{kn_1 - n_2}{(k+1)n_2} + \frac{n_1 a}{n_2}, a\right)$$

By definition, $A \in Q_1$. Additionally, any point on the line $x_2 = a$, such that $\kappa - 1 \leq x_1 \leq \frac{kn_1 - n_2}{(k+1)n_2} + \frac{n_1 a}{n_2}$ will be in Q_1 as well.

Let B be the point of intersection of $x_2 = a + 1$ with l_2. Observe that $B \in Q_2$. In particular,

$$B = \left(\frac{n_1 - kn_2}{(k+1)n_2} + \frac{n_1(a+1)}{n_2}, a+1\right)$$

Since $Q_3 = Q_2 - \mathbf{e}_2$, $\exists$ $B' \in Q_3$ such that $B' = B - \mathbf{e}_2$. In particular,

$$B' = \left(\frac{n_1 - kn_2}{(k+1)n_2} + \frac{n_1(a+1)}{n_2}, a\right)$$

As noted earlier, to prove $B' \in Q_1$ it suffices to show that

$$\kappa - 1 \le \frac{n_1 - kn_2}{(k+1)n_2} + \frac{n_1(a+1)}{n_2} \le \frac{kn_1 - n_2}{(k+1)n_2} + \frac{n_1 a}{n_2}.$$

We have,

$$\begin{aligned}
\left(\frac{n_1 - kn_2}{(k+1)n_2} + \frac{n_1(a+1)}{n_2}\right) - \left(\frac{kn_1 - n_2}{(k+1)n_2} + \frac{n_1 a}{n_2}\right) &= \frac{n_1}{n_2}\left(\frac{2}{k+1}\right) - \frac{k-1}{k+1} \\
&= \frac{1}{k+1}\left(\frac{2n_1}{n_2} - (k-1)\right) \\
&\le 0 \qquad (6.6)
\end{aligned}$$

where the last inequality follows from our assumption. Additionally, note that

$$\begin{aligned}
\left(\frac{n_1 - kn_2}{(k+1)n_2} + \frac{n_1(a+1)}{n_2}\right) - (\kappa - 1) &= \left(\frac{n_1 - kn_2}{(k+1)n_2} + \frac{n_1(a+1)}{n_2}\right) - \left(\frac{n_1}{(k+1)n_k} - \frac{k}{k+1}\right) \\
&\ge \left(\frac{n_1 - kn_2}{(k+1)n_2} + \frac{n_1 \alpha}{n_2}\right) - \left(\frac{n_1}{(k+1)n_k} - \frac{k}{k+1}\right) \\
&= \left(\frac{n_1}{(k+1)n_2} + \frac{n_1}{(k+1)n_k} + \frac{n_1}{(k+1)n_2}\right) - \frac{n_1}{(k+1)n_k} \\
&= \frac{2n_1}{(k+1)n_2} \\
&> 0 \qquad (6.7)
\end{aligned}$$

where the second inequality is immediate from $\alpha - 1 \le a$ and the last inequality follows from the positivity of n_1, n_2 and k. From (6.6) and (6.7) it follows that $B' \in Q_1$. Combining with $B' \in \{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\}$ and $B' \in Q_3$ yields the result. $\square$

Lemma 6.2.6. *If* $\mathbf{n}$ *satisfies* $k \ge 3$, $2n_1 \le (k-1)n_2$ *and* $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \ge k+1$, *then*

$$w_{\{\mathbf{x}\in\mathbb{R}^2 : x_2 = a\}\cap Q_4}(\mathbf{e}_1) \ge 1 \;\forall\; a \in [\alpha - 1, \alpha].$$

Proof. First, consider $a \in [\delta, \alpha]$. Then, $w_{\{\mathbf{x}\in\mathbb{R}^2 : x_2 = a\}\cap Q_4}(\mathbf{e}_1)$ is at least as much as the distance between l_1 and l_2 in the x_1-direction.

$$\begin{aligned}
w_{\{\mathbf{x}\in\mathbb{R}^2:x_2=a\}\cap Q_4}(\mathbf{e}_1) &= \frac{1}{n_2}\left(\frac{kn_1-n_2}{k+1}-\frac{n_1-kn_2}{k+1}\right) \\
&= \frac{k-1}{k+1}\frac{n_1+n_2}{n_2} \\
&\geq 2\frac{k-1}{k+1} \\
&\geq 1
\end{aligned} \tag{6.8}$$

where the first inequality follows from $n_1 \geq n_2$ while the latter follows from $k \geq 3$.

Now, consider $a \in [\alpha-1, \delta]$. Since Q_1 and Q_3 are convex, as a result of Lemma 6.2.5 and Lemma 6.2.2, $Q_4 \cap \{\mathbf{x} \in \mathbb{R}^2 : \alpha - 1 \leq x_2 \leq \delta\}$ is convex.

Since l_1 is a facet of Q_2 and $Q_3 = Q_2 - \mathbf{e}_2$, it follows that $l_1' := l_1 - \mathbf{e}_2$ is a facet of Q_3. In particular, we have $l_1' : n_2x_1 - n_1(x_2+1) = \dfrac{kn_1-n_2}{k+1}$.

Let C be the point of intersection of l_1' and $x_2 = \alpha - 1$. Then:

$$C = \left(\frac{n_1}{n_2} + \frac{n_1}{(k+1)n_k} - \frac{1}{k+1}, \frac{n_2}{(k+1)n_k} - \frac{k}{k+1}\right)$$

It follows that $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = \alpha - 1\} \cap Q_4 = [(\kappa - 1, \alpha - 1), C]$. We, thus, have

$$\begin{aligned}
w_{\{\mathbf{x}\in\mathbb{R}^2:x_2=\alpha-1\}\cap Q_4}(\mathbf{e}_1) &= \left(\frac{n_1}{n_2} + \frac{n_1}{(k+1)n_k} - \frac{1}{k+1}\right) - \left(\frac{n_1}{(k+1)n_k} - \frac{k}{k+1}\right) \\
&= \frac{n_1}{n_2} + \frac{k-1}{k+1} \\
&> 1
\end{aligned} \tag{6.9}$$

where the inequality follows from $n_1 \geq n_2$ and $\dfrac{k-1}{k+1} > 0$.

In case $\zeta - \alpha \geq \dfrac{k-1}{k+1}$, l_1' would intersect $x_2 = \delta$. This point, say D, would be given by:

$$D = \left(\frac{2kn_1}{(k+1)n_2} + \frac{n_1}{(k+1)n_k} - \frac{1}{k+1}, \frac{1}{k+1}\left(\frac{n_2}{n_k} - 1\right)\right)$$

Consequently, $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = \delta\} \cap Q_4 = [(\kappa - 1, \delta), D]$. We, then, have:

$$\begin{aligned} w_{\{\mathbf{x}\in\mathbb{R}^2:x_2=\delta\}\cap Q_4}(\mathbf{e}_1) &= \left(\frac{2kn_1}{(k+1)n_2} + \frac{n_1}{(k+1)n_k} - \frac{1}{k+1}\right) - \left(\frac{n_1}{(k+1)n_k} - \frac{k}{k+1}\right) \\ &= \frac{2kn_1}{(k+1)n_2} + \frac{k-1}{k+1} \\ &> 1 \end{aligned} \tag{6.10}$$

where the inequality follows from $n_1 \geq n_2$ and $k \geq 3$. If $\zeta < \alpha + \frac{k-1}{k+1}$, then l_1' would not intersect $x_2 = \delta$. Consequently, we have

$$\{\mathbf{x} \in \mathbb{R}^2 : x_2 = \delta\} \cap Q_4 = \left[(\kappa - 1, \delta), \left(\frac{kn_1}{(k+1)n_3} - \frac{1}{k+1}, \delta\right)\right].$$

Combining this with Lemma 6.2.3, we have:

$$w_{\{\mathbf{x}\in\mathbb{R}^2:x_2=\delta\}\cap Q_4}(\mathbf{e}_1) = w_{Q_4}(\mathbf{e}_1) = w_Q(\mathbf{e}_1) > 1 \tag{6.11}$$

Since $Q_4 \cap \{\mathbf{x} \in \mathbb{R}^2 : \alpha - 1 \leq x_2 \leq \delta\}$ is convex, it follows from (6.9), (6.10) and (6.11) that

$$w_{\{\mathbf{x}\in\mathbb{R}^2:x_2=a\}\cap Q_4}(\mathbf{e}_1) > 1 \;\forall\; a \in [\alpha - 1, \delta] \tag{6.12}$$

The result follows from (6.8) and (6.12). $\square$

Lemma 6.2.7. *Define* $Q_5 := Q \cap \{\mathbf{x} \in \mathbb{R}^2 : \beta \leq x_2 \leq \gamma\}$. *If* $\mathbf{n}$ *satisfies* $k \geq 4$, $2n_1 > (k-1)n_2$ *and* $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \geq k + 1$, *then* $w_{Q_5}(\mathbf{e}_2) > 1$.

Proof. It is immediate from the definition of Q_5 that $w_{Q_5}(\mathbf{e}_2) = \gamma - \beta$.

$$\begin{aligned} w_{Q_5}(\mathbf{e}_2) &= \left(\frac{kn_2}{(k+1)n_3} - \frac{2n_2}{(k+1)n_1} - \frac{1}{k+1}\right) - \left(\frac{n_2}{(k+1)n_k} + \frac{2n_2}{(k+1)n_1} - \frac{k}{k+1}\right) \\ &= \frac{kn_2}{(k+1)n_3} - \frac{n_2}{(k+1)n_k} - \frac{4n_2}{(k+1)n_1} + \frac{k-1}{k+1} \\ &> \frac{kn_2}{(k+1)n_3} - \frac{n_2}{(k+1)n_k} - \frac{8}{(k-1)(k+1)} + \frac{k-1}{k+1} \\ &= \frac{kn_2}{(k+1)n_3} - \frac{n_2}{(k+1)n_k} - \frac{8}{(k-1)(k+1)} + 1 - \frac{2}{k+1} \end{aligned}$$

where the inequality follows from $2n_1 > (k-1)n_2$. After rearrangement, we get:

$$\begin{aligned} w_{Q_5}(\mathbf{e}_2) &= \frac{1}{k+1}\left(\frac{kn_2}{n_3} - \frac{n_2}{n_k} - 2\right) + 1 - \frac{8}{(k-1)(k+1)} \\ &\geq \frac{k-1}{k+1} + 1 - \frac{8}{(k-1)(k+1)} \\ &= 1 + \frac{(k-1)^2 - 8}{(k-1)(k+1)} \\ &> 1 \end{aligned}$$

where the second follows from $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \geq k+1$ and the final follows from $k \geq 4$. The result follows. $\square$

Now, we are ready for the main theorem.

Theorem 6.2.8. $\mathbf{n}$ *is a Lonely Runner instance if it satisfies* $k \geq 4$ *and* $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \geq k+1$.

Proof. Combining $n_2\left(\frac{k}{n_3} - \frac{1}{n_k}\right) \geq k+1$ with the positivity of n_2, n_k and k immediately yields that $n_3 < kn_k$. Thus, it suffices to show that $Q = P_2(\mathbf{n})$ is not integer lattice-free. In particular, if there exists an integer point in Q then $\mathbf{n}$ is a Lonely Runner instance.

Assume that $2n_1 \leq (k-1)n_2$. By definition of L_1, we have $w_{Q_4}(\mathbf{e}_2) = 1$. Thus, $\exists\, a \in \mathbb{Z}$ such that $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\} \cap Q_4 \neq \emptyset$. Lemma 6.2.6 suggests $w_{\{\mathbf{x}\in\mathbb{R}^2 : x_2 = a\}\cap Q_4}(\mathbf{e}_1) \geq 1$. It follows that there exists an integer point in Q_4 and consequently in Q.

Conversely, assume that $2n_1 > (k-1)n_2$. Lemma 6.2.7 yields $w_{Q_5}(\mathbf{e}_2) > 1$. Thus, $\exists\, a \in \mathbb{Z}$ such that $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\} \cap Q_5 \neq \emptyset$. Furthermore, by definition of L_2, L_3 and Q_5, we have $w_{\{\mathbf{x}\in\mathbb{R}^2 : x_2 = a\}\cap Q_5}(\mathbf{e}_1) \geq 1$. It follows that Q_5 and thus Q contains an integer point. $\square$

Remark 6.2.2. *A few examples of vectors of speeds that are Lonely Runner instances due to Theorem 6.2.8 and not due to previous known results (including Theorem 6.2.1) are:*

$$(20, 14, 8, 6, 5, 4, 2), (24, 14, 10, 9, 8, 6, 5, 2) \text{ and } (23, 18, 15, 10, 8, 7, 6, 4, 2).$$

It turns out that it is hard to visualise and work with polyhedra $P_l(\mathbf{n})$ for $l \geq 3$. Thus, only $P_1(\mathbf{n})$ and $P_2(\mathbf{n})$ were used. The projection methodology can be made use of further by considering integer points other than the origin in $Q_l(\mathbf{n})$ (defined in Subsection 5.3.1). However, this modification makes the inequalities a little messy to work with. So, we pursue other ideas.

6.3 Properties of the Lonely Runner polyhedron

Lemma 6.3.1. *Any line in $\mathbb{R}^k$ with direction ratios $\mathbf{n}$ is parallel to each of the facets of $P(\mathbf{n})$.*

Proof. Consider any facet of $P(\mathbf{n})$. The normal vector of this hyperplane is given by $\mathbf{N} = n_j\mathbf{e_i} - n_i\mathbf{e_j}$. Then, $\mathbf{n} \cdot \mathbf{N} = n_i(n_j) + n_j(-n_i) = 0$. Since the dot product is zero, the normal vector of the hyperplane and the direction vector of the line are parallel to each other. The result follows since the above holds for each of the facets of $P(\mathbf{n})$. $\square$

Remark 6.3.1. *Let l be a line in $\mathbb{R}^k$, with direction ratios $\mathbf{n}$. Then, due to Theorem 6.3.1, either $l \subseteq P(\mathbf{n})$ or $l \cap P(\mathbf{n}) = \emptyset$.*

Lemma 6.3.2. *$P(\mathbf{n})$ is centrally symmetric about $-\frac{1}{2}\mathbf{e}$.*

Proof. Let $\mathbf{v} = (v_1, ..., v_k)$ be an arbitrary point. With this, we get:

$$\begin{aligned}
&-\frac{1}{2}\mathbf{e} + \mathbf{v} \in P(\mathbf{n}) \\
&\iff \frac{n_i - kn_j}{k+1} \leq n_j\left(-\frac{1}{2} + v_i\right) - n_i\left(-\frac{1}{2} + v_j\right) \leq \frac{kn_i - n_j}{k+1} \quad \forall\ 1 \leq i < j \leq k \\
&\iff -\frac{(k-1)(n_i+n_j)}{2(k+1)} \leq n_jv_i - n_iv_j \leq \frac{(k-1)(n_i+n_j)}{2(k+1)} \quad \forall\ 1 \leq i < j \leq k \\
&\iff -\frac{(k-1)(n_i+n_j)}{2(k+1)} \leq -n_jv_i + n_iv_j \leq \frac{(k-1)(n_i+n_j)}{2(k+1)} \quad \forall\ 1 \leq i < j \leq k \\
&\iff \frac{n_i - kn_j}{k+1} \leq n_j\left(-\frac{1}{2} - v_i\right) - n_i\left(-\frac{1}{2} - v_j\right) \leq \frac{kn_i - n_j}{k+1} \quad \forall\ 1 \leq i < j \leq k \\
&\iff -\frac{1}{2}\mathbf{e} - \mathbf{v} \in P(\mathbf{n})
\end{aligned}$$

The result follows from the definition of central symmetry. $\square$

Remark 6.3.2. *The proof of Lemma 6.3.2 can be generalized to show that $P(\mathbf{n})$ is centrally symmetric about every point on the line* $L := \dfrac{x_1 + \frac{1}{2}}{n_1} = \ldots = \dfrac{x_k + \frac{1}{2}}{n_k}$.

From Remark 6.3.2, we can conclude that there is a line that is contained in $P(\mathbf{n})$ and hence, $P(\mathbf{n})$ is unbounded.

Lemma 6.3.3. *$P(\mathbf{n})$ contains a k-hypercube, whose facets are parallel to the coordinate hyperplanes, and has size at least* $\dfrac{k-1}{k+1}$.

Proof. First, consider the k-hypercube

$$\mathcal{C} = \left[-\frac{k}{k+1}, -\frac{1}{k+1}\right]^k$$

By definition, we have:

$$-\frac{k}{k+1} \leq x_i, x_j \leq -\frac{1}{k+1}, 1 \leq i, j \leq k$$

Then:

$$\frac{n_i - kn_j}{k+1} \leq n_j x_i - n_i x_j \leq \frac{kn_i - n_j}{k+1}, 1 \leq i, j \leq k$$

It follows that $\mathcal{C} \subseteq P(\mathbf{n})$. Thus, the result follows. $\square$

It can be verified that $P(\mathbf{n})$ does not contain any translate of the standard k-hypercube. Thus, it isn't easy to determine an integer point in $P(\mathbf{n})$. Hence, alternate approaches are required.

One approach that comes to mind is the use of Ehrhart theory. We immediately stumble into a hole. Due to the unboundedness of $P(\mathbf{n})$, we cannot directly use Ehrhart theory to estimate the number of integer points. We can get over this problem by bounding $P(\mathbf{n})$ between two parallel hyperplanes, say $x_k = 0$ and $x_k = a$ for some large $a \in \mathbb{R}$. However, the number of hyperplanes generating this polyhedron is very large. As a result, the number of variables in the Ehrhart series is very large, thereby making it cumbersome and hard to work with.

6.4 Other new results

So far, we haven't used any of the concepts that we had learnt in Chapter 4. We make use of some of them now and see how the rest can be used in the hope of making some progress.

Lemma 6.4.1. *Let $p \in \mathbb{Z}^+$ and $\mathbf{n} \in \mathbb{Z}^k$. Then, the set of points, $\Lambda_{\mathbf{n},p}$, defined by*

$$\Lambda_{\mathbf{n},p} := \left\{ \mathbf{x} \in \mathbb{R}^k : \mathbf{x} = \frac{a}{p}\mathbf{n} + \mathbf{b}, 0 \le a \le p-1, a \in \mathbb{Z}, \mathbf{b} \in \mathbb{Z}^k \right\}$$

is a lattice.

Proof. First, let $a = 0$ and $\mathbf{b} = \mathbf{0}$. Then, $\mathbf{x} = \mathbf{0} + \mathbf{0} = \mathbf{0} \in \Lambda_{\mathbf{n},p}$. Next, consider two points $\mathbf{y} = \frac{a_1}{p}\mathbf{n} + \mathbf{b_1}$ and $\mathbf{z} = \frac{a_2}{p}\mathbf{n} + \mathbf{b_2}$, where $0 \le a_1, a_2 \le p-1$, a_1, $a_2 \in \mathbb{Z}$ and $\mathbf{b_1}$, $\mathbf{b_2} \in \mathbb{Z}^k$. Their sum is given by $\mathbf{y} + \mathbf{z} = \left(\frac{a_1}{p}\mathbf{n} + \mathbf{b_1}\right) + \left(\frac{a_2}{p}\mathbf{n} + \mathbf{b_2}\right) = \frac{a_1 + a_2}{p}\mathbf{n} + (\mathbf{b_1} + \mathbf{b_2})$. Note that $(a_1 + a_2) \in \mathbb{Z}$ and $(\mathbf{b_1} + \mathbf{b_2}) \in \mathbb{Z}^k$. If $0 \le a_1 + a_2 \le p-1$, then $\mathbf{y} + \mathbf{z} \in \Lambda_{\mathbf{n},p}$. If not, then $p \le a_1 + a_2 \le 2p-2$. Then, $\mathbf{y} + \mathbf{z} = \frac{a_1 + a_2 - p}{p}\mathbf{n} + (\mathbf{b_1} + \mathbf{b_2} + \mathbf{n})$, and thus, $\mathbf{y} + \mathbf{z} \in \Lambda_{\mathbf{n},p}$. Finally, let $\mathbf{u}, \mathbf{v} \in \Lambda_{\mathbf{n},p}$. If $\mathbf{u} \ne \mathbf{v}$, then $\exists\, i \in [k]$ s.t $|u_i - v_i| \ge \frac{1}{p}$. As a result, $\mathcal{B}\left(\mathbf{v}, \frac{1}{2p}\right) = \mathbf{v}$. Therefore, $\Lambda_{\mathbf{n},p}$ is a discrete additive subgroup or lattice. □

Lemma 6.4.2. *If $p_1, p_2, m \in \mathbb{Z}^+$ such that $p_2 = mp_1$, then $\Lambda_{\mathbf{n},p_1} \subseteq \Lambda_{\mathbf{n},p_2}$.*

Proof. Consider $\mathbf{x} = \frac{a}{p_1}\mathbf{n} + \mathbf{b} \in \Lambda_{\mathbf{n},p_1}$. Note that $\mathbf{x} = \frac{am}{mp_1}\mathbf{n} + \mathbf{b} = \frac{am}{p_2}\mathbf{n} + \mathbf{b} \in \Lambda_{\mathbf{n},p_2}$, where $0 \le am \le m(p_1 - 1) \le mp_1 - 1 = p_2 - 1$. Thus, we have $\mathbf{x} \in \Lambda_{\mathbf{n},p_2}$ and the result holds. □

As mentioned earlier, $P(\mathbf{n})$ does not contain any translate of the standard k-hypercube which happens to be a fundamental parallelepiped of $\Lambda_{\mathbf{n},1}$. Due to Lemma 6.4.2, we expect that the translates of a fundamental parallelepiped of $\Lambda_{\mathbf{n},p}$ are contained in the translates of a fundamental parallelepiped of $\Lambda_{\mathbf{n},1}$. So we try to show that $P(\mathbf{n})$ contains a translate of a fundamental parallelepiped of $\Lambda_{\mathbf{n},p}$. This requires us to identify a basis of $\Lambda_{\mathbf{n},p}$.

We provide a reformulation of Theorem 3.1.3 in terms of lattices.

Theorem 6.4.3. *Let l be a line in $\mathbb{R}^k$, with direction ratios $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$. If there exists $p \in \mathbb{Z}^+$ such that $l \cap \Lambda_{\mathbf{n},p} \ne \emptyset$, then $l \cap \Lambda_{\mathbf{n},1} \ne \emptyset$.*

Now, we have a new sufficiency condition to prove the 'Lonely Runner Conjecture'.

Corollary 6.4.4. *If* $P(\mathbf{n}) \cap \left(\bigcup_{p \geq 1} \Lambda_{\mathbf{n},p} \right) \neq \emptyset$*, then* $P(\mathbf{n}) \cap \Lambda_{\mathbf{n},1} \neq \emptyset$*.*

Proof. Consider any line with direction ratios $\mathbf{n}$. Assume that $P(\mathbf{n}) \cap \left(\bigcup_{p \geq 1} \Lambda_{\mathbf{n},p} \right) \neq \emptyset$. Due to Theorem 6.4.3 and Remark 6.3.1, we have $P(\mathbf{n}) \cap \Lambda_{\mathbf{n},1} \neq \emptyset$. $\square$

Does Corollary 6.4.4 make our work any easier? Is it actually easier to show the existence of a lattice point of $\Lambda_{\mathbf{n},p}$ in $P(\mathbf{n})$? The following Lemma provides a heuristic argument for why this can be thought of as being true.

Lemma 6.4.5. *Let* $p \in \mathbb{Z}^+$*. Then* $\#([0,1)^k \cap \Lambda_{\mathbf{n},p}) = p$*. Moreover, if* $GCD(n_i, p) = 1$ *for all* $i \in [k]$*, then* $\#((0,1)^k \cap \Lambda_{\mathbf{n},p}) = p - 1$*.*

Proof. Consider $a \in \{0, \dots, p-1\}$ and $\mathbf{b} = \left(-\left\lfloor \frac{an_1}{p} \right\rfloor, \dots, -\left\lfloor \frac{an_k}{p} \right\rfloor \right)$. Then, we have $\frac{a}{p}\mathbf{n} + \mathbf{b} = \left(\frac{an_1}{p}, \dots, \frac{an_k}{p} \right) + \left(-\left\lfloor \frac{an_1}{p} \right\rfloor, \dots, -\left\lfloor \frac{an_k}{p} \right\rfloor \right) = \left(\left\{ \frac{an_1}{p} \right\}, \dots, \left\{ \frac{an_k}{p} \right\} \right) \in [0,1)^k$.
Thus, $\#([0,1)^k \cap \Lambda_{\mathbf{n},p}) \geq p$.
Now, consider $\mathbf{b}' \in \mathbb{Z}^k$ such that $\mathbf{b}' \neq \mathbf{b}$. Then, $\mathbf{b}' = \mathbf{b} + \mathbf{c}$, where $\mathbf{c} \in (\mathbb{Z}^k \setminus \{\mathbf{0}\})$. As a result, we have $\frac{a}{p}\mathbf{n} + \mathbf{b}' = \left(\frac{a}{p}\mathbf{n} + \mathbf{b} \right) + \mathbf{c}$. Note that c_i is a non-zero integer for some $i \in [k]$. Combining this with $\frac{a}{p}\mathbf{n} + \mathbf{b} \in [0,1)^k$, we get that $\frac{a}{p}\mathbf{n} + \mathbf{b}' \notin [0,1)^k$. Therefore, $\#([0,1)^k \cap \Lambda_{\mathbf{n},p}) = p$.
Let $a \in [p-1]$. Since $1 \leq a < p$, we have $GCD(a,p) < p$. Furthermore, we have $GCD(n_i, p) = 1 \ \forall \ i \in [k]$. As a result, we have $GCD(an_i, p) < p$ and thus, $p \nmid an_i \ \forall \ i \in [k]$. Hence, $0 < \left\{ \frac{an_i}{p} \right\} < 1 \ \forall \ i \in [k]$. The result follows. $\square$

From Lemma 6.4.5, we have that the fundamental parallelepiped of $\mathbb{Z}^k$ contains p lattice points. Moreover, its interior contains $(p-1)$ lattice points when p is co-prime to each of the speeds. Since p is not a fixed quantity, we can choose a value as per our needs. So by choosing p to be a prime number larger than n_1, we can always ensure the existence of $(p-1)$ lattice points in the interior of $\square_k$. We must show that at least one of these lies in

the k-hypercube that is centred at $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and has size $\frac{k-1}{k+1}$.
It might seem that we have a lot of freedom now and so, it should be easy to prove the Conjecture. However, a large fraction of $\square_k$ is unavailable for use as k increases. Thus, we need more in order to get anything meaningful.

Lemma 6.4.6. *Let $p, s \in \mathbb{Z}^+$. Define*

$$\mathcal{D}_{\mathbf{r},s} := \left[\frac{r_1}{s} + \frac{1}{s(k+1)}, \frac{r_1}{s} + \frac{k}{s(k+1)}\right] \times \ldots \times \left[\frac{r_k}{s} + \frac{1}{s(k+1)}, \frac{r_k}{s} + \frac{k}{s(k+1)}\right]$$

where $r_i \in \mathbb{N}$. If $(\Lambda_{\mathbf{n},p} \cap \mathcal{D}_{\mathbf{r},s}) \neq \emptyset$, then $(\Lambda_{\mathbf{n},p} \cap \mathcal{D}_{\mathbf{0},1}) \neq \emptyset$.

Proof. Let $\mathbf{v} \in (\Lambda_{\mathbf{n},p} \cap \mathcal{D}_{\mathbf{r},s})$. Then $s\mathbf{v} \in (\Lambda_{\mathbf{n},p} \cap \mathcal{D}_{\mathbf{r},1})$ and $(s\mathbf{v} - \mathbf{r}) \in (\Lambda_{\mathbf{n},p} \cap \mathcal{D}_{\mathbf{0},1})$. The result follows. $\square$

As a result of Lemma 6.3.3, our aim is to show that:

$$\left(\bigcup_{\mathbf{r}\in\mathbb{Z}^k} \bigcup_{s\in\mathbb{Z}^+} \mathcal{D}_{\mathbf{r},s}\right) \cap \left(\bigcup_{p\geq 1} \Lambda_{\mathbf{n},p}\right) \neq \emptyset$$

From visualizations that were made for $k = 2, 3$, we observed that a very large fraction of $\square_k$ is now used up. However, it is hard to quantify this fraction.

Showing the existence of a lattice point in the union of the k-hypercubes seems to be dependent on being able to determine a basis for $\Lambda_{\mathbf{n},p}$. We first looked at the literature for how to determine a basis of a lattice. It seems that lattice-based problems assume the existence of a basis, and so there is no question of determining a basis from scratch. As a result, we were on the hunt for a method/algorithm to determine a basis. Nothing materialized despite a lot of effort.

Chapter 7

Conclusion

This thesis was a literature review of known results of the 'Lonely Runner Conjecture', a description of our work and the theory needed to understand that work.

We started off by looking at some basic Convex Geometry concepts, in Chapter 1. We used them in Chapter 2 to study Polyhedra. Then, in Chapter 3, we tried to enumerate the number of integer points in different polyhedra. Chapter 4 was about studying lattices. Moreover, we came across a new field of Mathematics, namely 'Geometry of Numbers', and we learnt two of the most important results of the field. We completed the Chapter by looking at some bounds related to the 'Shortest Vector' and 'Closest Vector' problems.

In Chapter 5, we had a review of most of the known results about the Conjecture, with the focus being on work from [5]. We finished off by providing our new results.

There haven't been many approaches that have used Probability to obtain new results. An attempt could be made at showing that the lattice $\Lambda_{\mathbf{n},p}$, intersects the k-hypercube $\left[\frac{1}{k+1}, \frac{k}{k+1}\right]^k$ with probability 1. An important fact that could be of use is that each of the coordinates has an identical distribution when $GCD(p, n_i) = 1$ for all $i \in [k]$.

Another approach could be to write an Integer Linear Program with the objective function being the distance (using ∞-norm) of a line, with direction ratios $\mathbf{n}$, from the centre of an arbitrary integer translate of the standard k-hypercube. Show that the minimum possible value of the objective function is at most $\frac{k-1}{k+1}$.

One final approach would be to prove our Conjecture about suitable times. Proving this Conjecture ensures that the 'Lonely Runner Conjecture' is true.

Conjecture 7.0.1. *For any* $\mathbf{n}$ *with* $gcd(\mathbf{n}) = 1$*, there is always a suitable time of the form*

$$\frac{m}{2^{\lceil \ln_2(n_1)+1 \rceil}(k+1)n_1}$$

for some natural number m*, where* $\lceil \cdot \rceil$ *denotes the ceiling function.*

There are two reasons for why we believe Conjecture 7.0.1 to be true: First, considering $m = k2^{\lceil \ln_2(n_1)+1 \rceil}$ yields a suitable time for the family of vectors satisfying $n_1 \leq kn_k$. Additionally, we have computationally verified the existence of a suitable time given by Conjecture 7.0.1 for all possible $(\mathbf{n}, k)$ such that $n_1 \leq 32$, and $\gcd(\mathbf{n}) = 1$. In particular, we have $2^{32} - 1 = 4294967295$ speed vectors, $\mathbf{n}$, with $n_1 \leq 32$. Of these, 4294900694 are co-prime vectors (satisfying $\gcd(\mathbf{n}) = 1$). Further among these, 2646877074 ($\approx 61.62\%$) different $\mathbf{n}$ are characterized lonely runner instances due to the known results, including Theorems 6.2.1 and 6.2.8. If true, conjecture 7.0.1 thus yields a characterization for the remaining 38.38% of speed vectors.

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