Topics in large N Chern-Simons matter theories

A thesis
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by

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Certificate

This is to certify that this dissertation entitled Topics in large-N Chern-Simons matter theories Here towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents work carried out by Kartik Sharma at Tata Institute of Fundamental Research, Mumbai under the supervision of Dr. Shiraz Minwalla, Senior Professor, Department of Theoretical Physics, TIFR Mumbai during the academic year 2022-2023.

Dr. Shiraz Minwalla

Committee:

Guide: Dr. Shiraz Minwalla
Expert: Dr. Sachin Jain

This thesis is dedicated to my parents and my sister

Declaration

I hereby declare that the matter embodied in the report entitled 'Topics in large N Chern-Simons matter theories' are the results of the work carried out by me at the Department of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai, under the supervision of Dr Shiraz Minwalla and the same has not been submitted elsewhere for any other degree.

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Abstract

The structure S-matrix in matter Chern-Simons theories is different from the S-matrix in trivially gapped theories. In this project we present a conjecture for the structure, compounding and crossing rules for matter Chern-Simons theories, which hold even at finite N and k. This is done by the use of the Worldline formalism and the inherent topological nature of the pure Chern-Simons theories. For $2 \to 2$ scattering of fundamental particles in matter Chern-Simons theories with gauge group $SU(N)_k$ and $U(N)_k$ theories, we find the crossing rules to be q-deformations, with $q = e^{2\pi/\kappa}$, of the classical crossing symmetry rules. These results are also consistent with the results of [4], which are obtained by direct Feynman diagram computations in large N.

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Chapter 1

Introduction

introd

The Scattering matrix is among the most important observables in quantum field theories. While the exact form of the matrix varies from theory to theory, the general structure of the S-matrix can be ascertained for large classes of QFTs from general physical principals. For example, for all trivially-gapped theories (theories with a mass gap which flow to the trivial quantum field theory in the IR), the S-matrix takes the general form

$$S = \mathcal{S}_{id} + \iota \tau \tag{1.1}$$

smat1

where $\mathcal{S}_{id}$ is the forward scattering part of the S-matrix(which includes a delta function signifying the localization of this part in forward direction) and the analytic part τ, which encodes the actually scattering amplitude information and is an analytic function of the various external momenta associated to the scattering process.

While S-matrices of trivially gapped are extremely well studied, by contrast the S-matrix of quantum field theories without a mass gap does not even have a universally accepted definition. The difficulty arises due to the IR infinities that plague gapless QFTs whose severity varies from theory to theory. [1] The systematic understanding of gapless quantum field theories has remained elusive, and is therefore not discussed in the present work.

There is, however, a third class of theories that interpolate between the gapless and the trivially gapped QFTs, the topologically gapped quantum field theories. These theories possess a mass gap but flow to a topological

[1]These IR issues have been tackled in some simple yet important cases such as by Faddeev and Kullish [1] in $D = 4$ QED. Kullish:1970ut

field theory in the IR limit.[2] The standard examples of TQFTs are the Chern-Simons field theories coupled to matter fields in $2+1$ dimensional spacetime.

The general aim of this work is to study the structure of S-matrix in Chern-Simons matter theories and specifically understand the crossing-symmetry and unitarity properties of the analytic part of the S-matrix in these theories.

1.1 Crossing and Unitarity in Quantum Field Theories

Crossing Symmetry is the property of S-matrix which relates the scattering amplitudes of closely related scattering processes in a relativistic quantum field theory. It states that the scattering amplitude of a process involving an incoming (or outgoing) particle is related to the scattering amplitude of an outgoing (or incoming) anti-particle via analytic continuation. More specifically we can obtain the anti-particle scattering amplitude by analytically continuing the particle scattering amplitudes to negative energies.

It is important to note that crossing symmetry is a non-trivial property which is not universally true.[3] An important consistency check of the validity of crossing symmetry is whether the analytically continued scattering matrix satisfies the unitarity condition.

The unitarity of the S-matrix can be written mathematically as

$$S^\dagger S = \mathcal{S}_{id} \tag{1.2}$$

Unlike the crossing symmetry, unitarity of the S-matrix is considered to be an integral property of any consistent quantum field theory. In trivially gapped theories, using the expansion (1.1), we can rewrite the unitarity equation (1.2) as

$$\iota(\tau^\dagger - \tau) = \tau^\dagger \tau \tag{1.3}$$

The above expression can be used to derive the Optical theorem.

It is natural to consider whether a similar analysis holds for Chern-Simons matter theories.

[2] Topological quantum field theories are QFTs where the correlation functions do not depend on the background metric of the spacetime. This causes the theories to have incredibly simple dynamical properties. This makes it tractable to define these quantum field theories rigorously as done by Witten in [2]

[3] for a brief review and appropriate references refer to the introduction of [3]

1.2 Crossing and Unitarity in Chern-Simons Matter Theories

crsunics

The unitarity and crossing properties of Chern-Simons matter theories with gauge groups $SU(N)_k$ and $U(N)_k$[4] which are coupled to fundamental fermion or boson matter are explored in [4] in the large N 't Hooft limit. [5]

In the paper the authors present explicit computations and conjectures for the analytic part of the scattering amplitudes for $2 \to 2$ fundamental-fundamental and fundamental-anti-fundamental scattering. It is well known that in a trivially gapped field theory with a $SU(N)$ global symmetry, the $2 \to 2$ scattering amplitudes for fundamental- fundamental and fundamental-anti-fundamental scattering are related via the crossing symmetry

$$\begin{aligned} \tau_I &= \frac{\tau_s(N+1) + \tau_a(N-1)}{2} \\ \tau_{adj} &= \frac{\tau_s - \tau_a}{2} \end{aligned} \tag{1.4}$$

crosym1

where τ_I and τ_{adj} are the analytic part of the fundamental-anti-fundamental scattering amplitude in the singlet and adjoint channel respectively. τ_s and τ_a are the symmetric and antisymmetric channel scattering amplitudes in the fundamental-fundamental scattering. A derivation of relations (1.4) is given in section (4.6) In the large N limit we see that the above relations become

$$\begin{aligned} \tau_I &= N\left(\frac{\tau_s + \tau_a}{2}\right) \\ \tau_{adj} &= \frac{\tau_s - \tau_a}{2} \end{aligned} \tag{1.5}$$

crosym2

In [4], it was realised that in order for unitarity conditions to be satisfied by $2 \to 2$ the crossing relations in the large N limit have to be changed by adding a correction factor and become

$$\begin{aligned} \tau_I &= N\left(\frac{\sin \pi\lambda}{\pi\lambda}\right)\left(\frac{\tau_s + \tau_a}{2}\right) \\ \tau_{adj} &= \frac{\tau_s - \tau_a}{2} \end{aligned} \tag{1.6}$$

crosym3

[4]where k denotes the level of the Chern-Simons gauge theory

[5]In the 't Hooft limit, $N \to \infty$ and $k \to \infty$ while $N/k \to \lambda$ is held fixed. λ is called the 't Hooft coupling.

This presents the following question: How does one derive these altered crossing relations and whether it is possible to derive these relations at arbitrary N and k.

The thesis is organised as follows: In chapter 2 we review some key aspects of pure Chern-Simons and Chern-Simons matter theories. Chapter 3 covers the proposed general structure of S-matrix in Chern-Simons matter theories. Then we review and reformulate the crossing symmetry problem in chapter 4, followed by the main formalism for crossing symmetry in chapter 5. We apply the formalism proposed in chapter 5 to the case of $2 \to 2$ scattering in chapter 6, followed by a discussion of results and future directions in chapter 7.

Chapter 2

Chern-Simons Matter Theories

csmt

The aim of this section is to give a brief introduction to Chern-Simon gauge theories and Chern-Simons matter theories. We shall also discuss briefly some results associated to these these theories that shall be relevant to us in the subsequent sections.

2.1 Chern-Simons Gauge Theories

Chern-Simons Gauge Theories are topological field theories in the $2+1$ dimension. The action is the integral of the Chern-Simons 3-form

$$S_{CS} = \frac{k}{4\pi} \int_M Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{2.1}$$

csaction

where A is a one-form gauge field in the adjoint representation of the gauge group G. k is called the level of the theory and must take integer values for the path integral to be single valued. These theories have no propagating field and are classically trivial. On the quantum mechanical level the theories have been solved exactly for arbitrary gauge group G and level k in the seminal work of Witten [5] (Witten:1988hf).

2.2 Wilson Lines and Knot Theory

Since the usual gauge invariant local observable spoil the general covariance of the theory, it is more appropriate to study Wilson line operators as observables in Chern-Simons theories. Let $\mathcal{C}$ be a closed curve inside the spacetime M. Then one can define the Wilson loop $W_R(\mathcal{C})$ as follows

$$W_R(\mathcal{C}) = Tr_R P \exp\left(\int_{\mathcal{C}} A_i dx^i\right) \tag{2.2}$$

where R is an irreducible representation of the gauge group G. Note that since the pure Chern-Simons theory is a topological field theory, the expectation values of Wilson loops give rise to a whole class of topological invariants. As the curve $\mathcal{C}$ can be considered a knot embedded in the space M, expectation value of Wilson loop (2.2) is a knot invariant.

In his seminal paper [5], WItten showed that the Jones polynomials, a class of Knot invariant polynomials can be obtained as Wilson loop expectation values of Chern-Simons gauge theories with gauge group $SU(2)$.

In order to compute correlation functions of WIlson loops, we must first quantize the Chern-Simons gauge theories, which was first done by Witten in [5].

2.3 Quantization of Chern-Simons gauge theories and relation to Wess-Zumino-Witten theory

One of the most important steps in exactly solving the pure Chern-Simons theory is its holographic duality to the Wess-Zumino-Witten theory. While the duality can be stated in a more general way, we shall instead recall here the most important result of this duality. If one canonically quantizes the Chern-Simons theory on a ball with Wilson-lines which have insertion in representation $R_1, \cdots R_m$ on the boundary spheres, the Hilbert space of the theory is isomorphic to the space of conformal blocks on the WZW theory with level k and the target group G being isomorphic to the gauge group of the Chern-Simons theory.

Thus there exists an inner product on the space of conformal blocks, which is known as the Witten inner product.

2.4 Level-Rank Duality

There is also a set of dualities between Chern-Simons theories called the level-rank dualities. These dualities posit that the following theories are equivalent to each other,

$$
\begin{aligned}
U(N)_{k,k} &\sim SU(|k|)_{-sgn(k)N} \\
U(N)_{k,\kappa} &\sim U(|k|)_{-sgn(k)N,-\kappa}
\end{aligned}
\tag{2.3}
$$

From here on, we denote the Chern-Simons theory with gauge group $U(N)_{k,k+N}$ at Type I theory and $U(N)_{k,k}$ as Type II theory. The Type II theory is level-rank dual to $SU(N)$ theory and Type I theory is level-rank dual to itself. These dualities provides a robust check for many relations in Chern-Simons matter theories and are also related to the conjectured Bose-Fermi dualities in matter Chern-Simons theories.

2.5 Chern-Simons matter theories

The main theory of interest to us is the Chern-Sinoms theory coupled to matter fields in a minimal fashion. Unlike the pure Chern-Simons theories matter Chern-Simons theories have not been solved exactly. However, attempts have been made to solve these theories atleast in the large N 't Hooft limit. [1] The aim of this thesis, based on [9] is to understand crossing symmetry in S-matrix in Chern-Simons matter theories. We now look at the structure of the S-matrix in Chern-Simons matter theories.

[1] we refer the reader to references [6], [7] [8] for a better picture of matter Chern-Simons theories

Chapter 3

Modification of the structure of S matrix

As previously mentioned, the S-matrix has the structure of trivially gapped theories takes the form (1.1). In the case of the Chern-Simons matter theories, this structure must be altered. The most direct evidence for this is given in [4], where by Feynman diagram computations, one finds that the S-matrix breaks up as follows

$$S = \cos(\pi\nu)\mathcal{S}_{id} + \iota\tau \tag{3.1}$$

Where ν is a phase factor which we shall explain shortly.

The physical reason for the appearance of the cos factor is by looking at the non-relativistic limit of the Chern-Simons theory. Since there are no propagating modes in the Chern-Simons field, the interaction of two particles interacting via the Chern-Simons field is a long-distance phase type interaction. In the non-relativistic limit this means that in the frame where one of the two particles is kept fixed the wave-function of the other particle acquires a phase as it moves around the first particle once. This is precisely the Aharonov-Bohm effect.

3.1 Aharonov-Bohm effect and forward scattering

The Aharonov-Bohm effect describes the quantum mechanics of a charged particle in the presence of a small solenoid. As is well known the particle wave-function acquires a phase factor $e^{i2\pi\nu}$ as it goes around the solenoid.[1] The scattering matrix of the Aharonov-Bohm effect is well know and takes the form of (3.1).

The factor in front of the forward scattering term $\mathcal{S}_{id}$ has a simple interpretation. Two different paths are possible for a forward scattering path along the two sides of the solenoid. These two paths have $e^{i\pi\nu}$ and the $e^{-i\pi\nu}$ phase factors as these must be added together to give the total term and each factor contributes an equal half to the final term (thus a factor of half in front of the terms) one gets the final factor to be

$$\frac{e^{i\pi\nu} + e^{-i\pi\nu}}{2} = \cos(\pi\nu) \tag{3.2}$$

A derivation of the Aharonov-Bohm S-matrix can be found in [10]. Based on the Aharonov-Bohm effect one can conclude that the Chern-Simons S-matrix will have form of (3.1). There is however, the question of what value ν in the Chern-Simons case.

3.2 Forward Scattering Phase in the Chern-Simons case

We consider the $2 \to 2$ scattering process $\alpha\beta \to \alpha\beta$ where particles α and β correspond to the representations R_α and R_β in the WZW theory dual to the Chern-Simons theory under consideration. Then if $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are the primary operators of these representation, the fusion rules are denoted as follows

$$\mathcal{O}_\alpha \mathcal{O}_\beta = \sum_M \mathcal{N}^M_{\alpha\beta} \mathcal{O}_M \tag{3.3}$$

fusion rules

Then we define $\nu^M_{\alpha\beta} = h_M - h_\alpha - h_\beta$ and the S matrix has the form

$$S_M = \cos(\pi\nu^M_{\alpha\beta})\mathcal{S}_{id} + \iota\tau_M \tag{3.4}$$

scat mat 2

The motivation for $\nu^M_{\alpha\beta}$ comes from the monodromy phase factors one encounters in the theory of rational conformal field theories. For a more through discussion the reader is advised to read the relevant sections in [9].

We now move to the discussion of crossing symmetry in Chern-Simons matter theories.

[1]The ν factor depend on the magnetic flux in the solenoid and it's exact form is not of interest

Chapter 4

Crossing symmetry in theories with a global symmetry

In this section we will will review crossing symmetry in trivially gapped theories with a continuous global symmetry. The material in this section is not original however the reader may find it's presentation unusual. This will however help us generalize the analysis to the case of our interest, the Chern-Simons matter theories.

4.1 Invariant Tensors and Invariant Maps

Consider the space

$$H = R_1 \otimes R_2 \cdots \otimes R_{i+j} \tag{4.1}$$

tensor

where the R_i are irreducible representations of the global symmetry lie group G. Consider a tensor residing in H

$$T^{\vec{m}_{a_1}\vec{m}_{a_2}\ldots\vec{m}_{a_{i+j}}} \tag{4.2}$$

tensor2

Here $\vec{m}_{a_k}$ are representation indices in R_k. Clearly we can group rotate the tensor, obtaining a new tensor,

$$T_G^{\vec{m}_{a_1}\vec{m}_{a_2}\ldots\vec{m}_{a_{i+j}}} = G^{\vec{m}_{a_1}}_{\vec{m}'_{a_1}} G^{\vec{m}_{a_2}}_{\vec{m}'_{a_2}} \ldots G^{\vec{m}_{a_{i+j}}}_{\vec{m}'_{a_{i+j}}} T^{\vec{m}'_{a_1}\vec{m}'_{a_2}\ldots\vec{m}'_{a_{i+j}}} \tag{4.3}$$

tensor3

where $G^{\vec{m}_{a_k}}_{\vec{m}'_{a_k}}$ is the matrix element associated with the rotation of the R_kth representation. The tensor T is defined to be a group invariant tensor if

$$T_G^{\vec{m}_{a_1}\vec{m}_{a_2}\ldots\vec{m}_{a_{i+j}}} = T^{\vec{m}_{a_1}\vec{m}_{a_2}\ldots\vec{m}_{a_{i+j}}} \tag{4.4}$$

tensor4

thus the unrotated and rotated tensors are equal. There is also the concept of an invariant map, closely related to the idea of invariant tensors. Consider the linear operator $\mathcal{O}$ that maps an initial Hilbert space H_{in}

$$H_{in} = R_1 \otimes R_2 \cdots \otimes R_i \tag{4.5}$$

hilb1

into the H_{out},

$$H_{out} = R^*_{i+1} \otimes R^*_{i+2} \cdots \otimes R^*_{i+j} \tag{4.6}$$

hilb2

Note that $H = H_{in} \otimes H^*_{out}$. Then $\mathcal{O}$ group invariant map if

$$\mathcal{O}\,|\chi\rangle = |\psi\rangle \rightarrow \mathcal{O}G\,|\chi\rangle = G\,|\psi\rangle \tag{4.7}$$

tensor5

This condition translates to $\mathcal{O} = G^\dagger \mathcal{O} G$. Now consider for each representation R_i having an orthogonal basis $|m_{a_i}\rangle$, clearly the action of group rotation becomes as follows

$$G\,|\vec{m}_{a_i}\rangle = (G)^{\vec{m}'_{a_i}}_{\vec{m}_{a_i}}\,|\vec{m}'_{a_i}\rangle \tag{4.8}$$

tensor6

Now in this basis an invariant map M_T can be written as

$$M_T = T^{\vec{m}_{a_1}\vec{m}_{a_2}\ldots\vec{m}_{a_{i+j}}}\,|\vec{m}_{a_1}\rangle \ldots |\vec{m}_{a_i}\rangle\,\langle\vec{m}_{a_{i+1}}| \ldots \langle\vec{m}_{a_{i+j}}| \tag{4.9}$$

If M_T is an invariant map, then T must be an invariant tensor.

Now if we consider a scattering process with the in-states Hilbert space H_{in} and the out-states being inside H_{out} the S-matrix will clearly break down into

$$S = \mathcal{S}_i M_{T_i} \tag{4.10}$$

map

where M_{T_i} span over a basis of invariant maps and $\mathcal{S}_i$ are the momentum dependence part of the S-matrix.

4.2 Crossing and Compounding and Unitarity

Now consider an invariant tensor T inside the Hilbert space H. Now separating the Hilbert space into the initial and final Hilbert spaces, in two different ways associates two different invariant maps to the same tensor T. Crossing symmetry claims that the momentum dependant functions that multiply these two tensors in the S-matrix are related to each other by analytic continuation.

In terms of equations suppose we separate the Hilbert space H into $H_{in} \otimes H^*_{out}$ and $H'_{in} \otimes H'^*_{out}$ spaces. Then the S-matrix can be written as

$$\begin{aligned} S &= \sum \mathcal{S}_i M_{T_i} \\ S &= \sum \tilde{\mathcal{S}}_i \tilde{M}_{T_i} \end{aligned} \tag{4.11}$$

map2

Now another operation we can perform using invariant maps is composition. Consider a map M_T from (4.5) to (4.6) and another map $M_{T'}$ from (4.6) into (4.12)

$$H'_{out} = {R'}_1 \otimes \ldots {R'}_p \tag{4.12}$$

hilb3

Note that T' is a tensor in the space $H_{out} \otimes H'^*_{out}$. The compounded tensor TT' can be obtained by multiplying the associated maps as

$$M_T M_{T'} = M_{TT'} \tag{4.13}$$

compouding

By explicit evaluation of the RHS of (4.13) TT" is given by multiplying the tensors T and T' and contracting the indices associated to space H_{out}. One can also verify that if T anf T' are invariant tensors then so is TT'. Now the unitarity condition $SS^\dagger = I$ becomes the following

$$\sum_{final\ states} \sum_{i,j} (\mathcal{S}^*_j \star \mathcal{S}_i) M^\dagger_{T_j} M_{T_i} = \mathcal{S}_{id} M_{id} \tag{4.14}$$

compounding2

Note that $\star$ is a convolution operation with the measure $\prod_i \frac{d^3 p_i}{(2\pi)^3}(2\pi)\delta(p_i^2 + m^2)$. It now important to choose a useful basis set for the invariant tensor to continue our analysis. The basis we shall look at is known as the projector basis.

4.3 Projector Basis

In order to construct the projector basis we first consider the Clebsch-Gordan decomposition of the tensor product

$$R_i \otimes \cdots \otimes R_i = \sum_a Q_a \tilde{R}_a \tag{4.15}$$

cb exp

where a runs over all unitary irreducible representations of the global symmetry group G and Q_a denotes the number of times the representation $\tilde{R}_a$ appear in the fusion product. When $Q_a \geq 1$, we work in an orthogonal basis in the space of tensors which transform representation R_a.

We work with states $|\vec{m}\rangle_{a,r}$ $(r = 1, .., Q_a)$. We also demand that the basis be an orthonormal one, with

$$ {}_{r,a}\langle \vec{m} | \vec{m}' \rangle_{\ a',r'} = \delta_{a,a'} \delta_{r,r'} \delta_{\vec{m},\vec{m}'} \tag{4.16}$$

orth

The projector basis is defined as

$$ P_a^{rr'} = \sum_{\vec{m}} |\vec{m}\rangle_{a,r'} \ {}_{a,r}\langle \vec{m}| \tag{4.17}$$

proj1

now $P_a^{rr'}$ and $(P_a^{rr'})^\dagger$ constitute the basis for invariant maps from $H_{in} \rightarrow H_{out}$ and from $H_{out} \rightarrow H_{in}$ respectively. Now this basis is useful because when when compounding of these operators is in some sense orthogonal as

$$ (P_a^{r_1 r_2})^\dagger P_{a'}^{r_3 r_4} = \delta_{aa'} \delta_{r_2,r_4} \sum_{\vec{m}} |\vec{m}\rangle_{a,r_1} \ {}_{a,r_3}\langle \vec{m}| = \delta_{aa'} \delta_{r_2,r_4} \hat{P}_a^{r_3 r_1} \tag{4.18}$$

orth2

Now $\hat{P}_a^{r_3 r_1}$ form a basis of invariant maps from $H_{in} \rightarrow H_{in}$. Because of this $\hat{P}_a^{r_1 r_3} = (\hat{P}_a^{r_3 r_1})^\dagger$. This makes the compounding equation as follows

$$ \hat{P}_a^{r_1 r_2} \times (\hat{P}_{a'}^{r_3 r_4}) = \delta_{aa'} \delta_{r_1 r_4} \hat{P}_a^{r_3 r_2} \tag{4.19}$$

orth3

The projectors also satisfy the following identity

$$ \sum_{r,a} \hat{P}_a^{rr} = M_{id} \tag{4.20}$$

proj2

where M_{id} is the identity operator on H_{in}. From here on we will use $T_a^{rr'}$ to denote the tensor associated to $\hat{P}_a^{r_1 r_2}$. We denote T_{id} as the tensor associated to M_{id}.

4.3.1 Unitarity in Projector Basis

Now if we rewrite the S-matrix in terms of the projector basis

$$ S = \sum_{a,r_1,r_2} \mathcal{S}_a^{r_1 r_2} P_a^{r_1 r_2} \tag{4.21}$$

smat2

The unitarity equation becomes

$$\sum_{\text{final states}} \sum_{r} (\mathcal{S}_a^{r_1 r})^* \star \mathcal{S}_a^{r r_2} = \mathcal{S}_{id} \delta_{r_1, r_2} \tag{4.22}$$

smat3

In the special case when the initial particles are the same as the final particles, the S-matrix can be expanded as (1.1). Then we can also write

$$\mathcal{S}_a^{rr'} = \delta^{rr'} \mathcal{S}_{id} + i \tau_a^{rr'} \tag{4.23}$$

smat4

4.3.2 Crossing symmetry in the Projector Basis

Consider two different crossing frames with projector bases $\{P_a^{r,r'}\}$ and $\{\tilde{P}_a^{r,r'}\}$. We denote $\{T_a^{r,r'}\}$ and $\{\tilde{T}_a^{r,r'}\}$ as the tensor bases associated to the invariant maps $\{P_a^{r,r'}\}$ and $\{\tilde{P}_a^{r,r'}\}$ respectively. If the two tensor bases are related as follows

$$T_a^{rr'} = \sum_{ss'b} M_{arr'}^{bss'} \tilde{T}_b^{ss'} \tag{4.24}$$

tensor cross1

The matrix M is called a 6-j symbol and are a very well studied concept in group theory. Now crossing symmetry claims that

$$\sum_{rr'a} M_{arr'}^{bss'} \mathcal{S}_a^{rr'} = \tilde{\mathcal{S}}_b^{ss'} \tag{4.25}$$

tensor cross2

with the equality holding after the appropriate analytic continuation in cross ratios. Note that the definition of the projector basis is critically dependent on the orthonormal basis we choose. It is however important to note that the choice of this basis is not unique. For example multiplying each basis vector with a phase does not change our analysis. Regardless, it is useful to construct the basis more systematically.

4.4 Construction of projector Index Structures

Note that the definition of projector map $P_a^{rr'}$ in (4.17) depends on what basis states $|m\rangle_{a,r}$ we choose for each of the representations involved. In this section we define these states in a way that we shall find useful in the subsequent sections.

Consider fusion of representations R_a and R_b, whose decomposition is as follows

$$R_a \otimes R_b = N^c_{a,b} R_c \tag{4.26}$$

Now we pick an appropriate basis for the Clebsch-Gordan(CG) coefficients $(C^c_{a,b})$ so that the the states

$$|\vec{m}\rangle_{c,r} = \sum_{\vec{m}_a,\vec{m}_b} (C^r)^{\vec{m}_a,\vec{m}_b}_{\vec{m}_c} |\vec{m}\rangle_a |\vec{m}\rangle_b \tag{4.27}$$

obey orthonormality relations.

$${}_{c',r'}\langle\vec{m}'|\vec{m}\rangle_{c,r} = \delta_{\vec{m},\vec{m}'}\delta_{c,c'}\delta_{r,r'} \tag{4.28}$$

Plugging in (4.27) into (4.28) we get the condition

$$\sum_{\vec{m}_a,\vec{m}_b} (C^r)^{\vec{m}_a\vec{m}_b}_{\vec{m}_c} (C^{r'})^{*\ m'_{c'}}_{m_a,m_b} = \delta^{\vec{m}'_{c'}}_{\vec{m}_c}\delta_{cc'}\delta_{r,r'} \tag{4.29}$$

Because the dependence on m_c and $m'_{c'}$ in the previous equation is determined by group invariance, we don't lose any information by contracting away these indices. After doing this operation we get

$$\sum_{\vec{m}_a,\vec{m}_b\vec{m}_c} (C^r)^{\vec{m}_a\vec{m}_b}_{\vec{m}_c} (C^{r'})^{*\ m_c}_{m_a,m_b} = \delta_{cc'}\delta_{r,r'} D^{cl}_c \tag{4.30}$$

where D^{cl}_c is the dimension of the representation space R_c.

We can now find a orthonormal basis for a representation R_a in the tensor product of $R_1, \cdots, R_m$ decomposed into irreducible representations. The process is iterative, first obtain all the possible irreducible representations $R^{(1)}$ by doing a decomposition of tensor product $R_1 \otimes R_2$. We label the representations by r_1 and $R^{(1)}$ (r_1 labelling which copy of $R^{(1)}$ is being referred to here). We understand how to construct an orthonormal basis for $R^{(1)}$ from the preceding discussion.

Next we fuse $R^{(1)}$ with R_3, for which we get additional variables $R^{(2)}$ and r_2. We iterate this process until we fuse R_{m_1} with R_m and look at how many R_a representations we get. Each possible R_a representation is labelled by r which represents the variables $(r_1, r_2, \cdots, r_{m_1}, R^{(1)}, \cdots R^{(m-2)})$. An elegant way to depict this pictorially in fig (4.1).[1] For a more through discussion on the construction of these classical bases, the reader is encouraged to look at [9].

[1] This pictorial representation is inspired by the Moore-Seiberg notation found in [11]

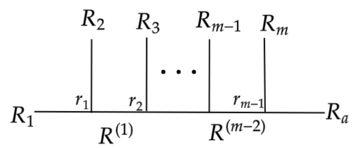

Figure 4.1: Diagrammatic representation of classical fusion

4.5 Explicit Example: $2 \to 2$ fundamental scattering in $SU(N)$

We consider the tensor product space

$$R_F \otimes R_F \otimes R_F^* \otimes R_F^* \tag{4.31}$$

where R_F is the fundamental representation of $SU(N)$ and R_F^* is the anti-fundamental representation. The space (4.31) can be separated in two distinct ways. First way is the fundamental-fundamental scattering in which we divide into

$$H_{in} = H_{out} = R_F \otimes R_F \tag{4.32}$$

The other way is the fundamental-antifundamental scattering where

$$H_{in} = H_{out} = R_F \otimes R_F^* \tag{4.33}$$

in both cases $H_{in} \otimes H_{out}^*$ is (4.31). Before we understand each crossing channel, it is useful to set some notational conventions.

We fix i and i' to be the indices associated to the first and second fundamental representation and j and j' as the indices associated to the two anti-fundamental representations in (4.31). When the fundamental state $|i\rangle$ transforms like a lower index, the anti-fundamental state $|j\rangle$ carries an upper index. Complex conjugation (Hermitian conjugate of tensors) flips the upper/lower index to lower/upper index. With this convention we can write a basis for the space of invariants of (4.31).

$$\begin{aligned} (T_d)_{ii'}{}^{jj'} &= \delta_i^j \delta_{i'}^{j'} \\ (T_e)_{ii'}{}^{jj'} &= \delta_i^{j'} \delta_{i'}^{j} \end{aligned} \tag{4.34}$$

The Hermitian conjugate of the basis is

$$\begin{aligned} (T_d^\dagger)^{ii'}{}_{jj'} &= \delta_j^i \delta_{j'}^{i'} \\ (T_e^\dagger)^{ii'}{}_{jj'} &= \delta_{j'}^i \delta_j^{i'} \end{aligned} \tag{4.35}$$

We will now construct the projector basis.

4.5.1 Fundamental-Fundamental Scattering

The Hilbert spaces in (4.32) can be decomposed as follows

$$R_F \otimes R_F = R_{Sym} \oplus R_{Asym} \tag{4.36}$$

ff decomposition1

The tensor product is a direct sum of the symmetric and antisymmetric representations, which are irreducible representations of the group $SU(N)$. The projector invariant basis is then

$$\begin{aligned} T_s = \frac{T_d + T_e}{2} &\implies (T_s)_{ii'}{}^{jj'} = \frac{\delta_i^j \delta_{i'}^{j'} + \delta_{i'}^j \delta_i^{j'}}{2} \quad \text{and} \quad (T_s^\dagger)^{ii'}{}_{jj'} = \frac{\delta_j^i \delta_{j'}^{i'} + \delta_{j'}^i \delta_j^{i'}}{2} \\ T_a = \frac{T_d - T_e}{2} &\implies (T_a)_{ii'}{}^{jj'} = \frac{\delta_i^j \delta_{i'}^{j'} - \delta_{i'}^j \delta_i^{j'}}{2} \quad \text{and} \quad (T_s^\dagger)^{ii'}{}_{jj'} = \frac{\delta_j^i \delta_{j'}^{i'} - \delta_{j'}^i \delta_j^{i'}}{2} \end{aligned} \tag{4.37}$$

ff basis

One can verify that under compounding these tensors behave exactly like projectors as shown in (4.38).

$$\begin{aligned} T_s^\dagger T_s &= \sum_{jj'} (T_s^\dagger)^{i_2 i_2'}{}_{jj'} (T_s)_{i_1 i_1'}{}^{jj'} = \frac{\delta_{i_1}^{i_2} \delta_{i_1'}^{i_2'} + \delta_{i_1}^{i_2'} \delta_{i_1'}^{i_2}}{2} = \hat{T}_s \\ T_a^\dagger T_a &= \sum_{jj'} (T_a^\dagger)^{i_2 i_2'}{}_{jj'} (T_a)_{i_1 i_1'}{}^{jj'} = \frac{\delta_{i_1}^{i_2} \delta_{i_1'}^{i_2'} - \delta_{i_1}^{i_2'} \delta_{i_1'}^{i_2}}{2} = \hat{T}_a \\ T_s^\dagger T_a &= T_a^\dagger T_s = 0 \end{aligned} \tag{4.38}$$

ff comp

Note that here since $H_{in} = H_{out}$, T_s and T_a live in the same space as $\hat{T}_s$ and $\hat{T}_a$ and we choose the phase of the projectors so that $T_{s/a} = \hat{T}_{s/a}$. Also clearly, $T_{s/a} = T_{s/a}^\dagger$.

4.5.2 Fundamental-Anti-fundamental Scattering

In (4.33), the tensor product decomposes into the singlet (denoted by R_I) and adjoint (denoted by R_{Adj}) representations as follows

$$R_F \otimes R_F^* = R_I \oplus R_{Adj} \tag{4.39}$$

fa decomposition

The projector invariant map basis is then

$$T_I = \frac{T_d}{N} \implies (T_I)_i{}^j{}_{i'}{}^{j'} = \frac{\delta_i^j \delta_{i'}^{j'}}{N} \quad \text{and} (T_I^\dagger)^i{}_j{}^{i'}{}_{j'} = \frac{\delta_j^i \delta_{j'}^{i'}}{N}$$
$$T_{Adj} = T_e - \frac{T_d}{N} \implies (T_{Adj})_i{}^j{}_{i'}{}^{j'} = \delta_i^{j'} \delta_{i'}^j - \frac{\delta_i^j \delta_{i'}^{j'}}{N} \quad \text{and} \quad (T_{Adj}^\dagger)^i{}_j{}^{i'}{}_{j'} = \delta_{j'}^i \delta_{i'}^j - \frac{\delta_j^i \delta_{j'}^{i'}}{N} \tag{4.40}$$

fa basis

The multiplication of the tensors gives us

$$T_I^\dagger T_I = \sum_{i,j} (T_I^\dagger)^i{}_j{}^{i_2}{}_{j_2} (T_I)_{i_1}{}^{j_1}{}_i{}^j = \frac{\delta_{j_2}^{i_2} \delta_{j_1}^{i_1}}{N} = \hat{T}_I$$
$$T_{Adj}^\dagger T_{Adj} = \sum_{i,j} (T_{Adj}^\dagger)^i{}_j{}^{i_2}{}_{j_2} (T_{Adj})_{i_1}{}^{j_1}{}_i{}^j = \delta_{j_2}^{i_1} \delta_{j_1}^{i_2} - \frac{\delta_{j_2}^{i_2} \delta_{j_1}^{i_1}}{N} = \hat{T}_{Adj} \tag{4.41}$$
$$T_I^\dagger T_{Adj} = T_{Adj}^\dagger T_I = 0$$

fa comp

As in the fundamental-fundamental scattering case, we choose the phases to ensure that $T_{I/Adj} = \hat{T}_{I/Adj}$. Moreover, $T_{I/Adj} = T_{I/Adj}^\dagger$.

4.6 Crossing rules in global symmetry

pqr

Using (4.37) and (4.40), it can be verified that

$$T_s = \frac{(N+1)T_I + T_{Adj}}{2}$$
$$T_s = \frac{(N-1)T_I - T_{Adj}}{2} \tag{4.42}$$

crossing

Then if we write the fundamental-fundamental S-matrix as

$$(S)_{ii'}{}^{jj'} = \mathcal{S}_{id} T_{id} + (\tau_a)_{ii'}{}^{jj'} T_a + (\tau_s)_{ii'}{}^{jj'} T_s \tag{4.43}$$

crossing2

Similarly the fundamental-anti-fundamental S-matrix is

$$S_i{}^j{}_{i'}{}^{j'} = \mathcal{S}_{id} T_{id} + (\tau_I)_i{}^j{}_{i'}{}^{j'} T_I + (\tau_{Adj})_i{}^j{}_{i'}{}^{j'} T_{Adj} \tag{4.44}$$

crossing3

Now Crossing Symmetry claims that the S-matrix corresponding to the same invariant tensor are related by analytic continuation. Therefore,

$$\tau_I = \frac{((N+1)\tau_s + (N-1)\tau_a}{2}$$
$$\tau_{Adj} = \frac{\tau_s - \tau_a}{2} \tag{4.45}$$

crossing4

In the large N limit, (4.45) becomes (1.5).

Having derived the crossing relations in the global symmetry case, we now move to the analysis of the Chern-Simons matter theory.

Chapter 5

Crossing symmetry in Chern-Simons Matter Theories

5.1 World-Line Formalism

We start by understanding the path-integral of Chern-Simons matter theories. Instead of working in the flat spacetime, we regularize this space to a Lorentzian cylinder $\mathcal{C}$ (see figure (5.1)). We consider the spatial extant R and temporal extant T to be very large ($Rm >> 1$ and $Tm >> 1$) with the spatial extant being much larger than the temporal extant. The condition of $R >> T$ ensures that the wavefunction of the particles being scattered vanishes at the curved boundaries of the cylinder so we don't have to worry about scattered particles encountering the curved boundary. Since no particle reaches the curved boundary of the cylinder $\mathcal{C}$, the Hamiltonian vanished on the boundary. We can therefore shrink away the curved boundary, leaving us with a solid ball as depicted in fig (5.1).

We now use the worldline formalism[1] to evaluate the matter part of the path integral as sum over particle trajectories with bulk interactions. The schematic equation looks as follows

$$\begin{aligned} \int \mathcal{D}A_\mu \mathcal{D}\phi(\cdots) &= \int \mathcal{D}A_\mu \sum_{\text{Particle Trajectories}} (\cdots) \\ &= \sum_{\text{Particle Trajectories}} \int \mathcal{D}A_\mu (\text{Wilson Lines}) \end{aligned} \tag{5.1}$$

[1] For a review of the world line formalism we encourage the reader to refer to [12]

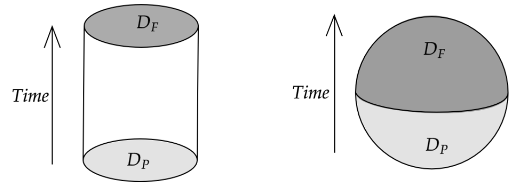

Figure 5.1: Lorentzian cylinder and Sphere pillbox

The Wilson lines that appear in the last line in general include Wilson lines that branch at interaction points. Now since $\int \mathcal{D}A_\mu$(Wilson Lines) is just the expectation value of Wilson lines in Pure Chern-Simons theory we get the path integral to assume the form

$$S \sim \sum_{\text{Ptcl.Trajectories}} \langle \text{Wilson Loops} \rangle_{\text{Pure CS Theory}} \tag{5.2}$$

worldline2

Because the pure Chern-Simons theory is a topological theory, the space of all particle trajectories can be decomposed into sectors/chambers of topologically equivalent trajectories. The path integral then takes the much simplified form

$$S \sim \sum_{\text{Chambers}} \langle \text{Wilson Loops} \rangle_{\text{Chamber}} \sum_{\text{Trajectories in Chamber}} e^{iS_L} \tag{5.3}$$

worldline3

where e^{iS_L} is the weight associated with each particle trajectory. Thus the path integral becomes a weighted sum of Wilson line expectation values with the Wilson lines starting and ending at specific boundary points (This is encoded in the boundary condition of the path integral). We can then write (5.3) more compactly as

$$S = \sum_{\text{Topologies t}} S_t W_t \tag{5.4}$$

worldline4

where W_t is the path integral of the pure Chern-Simons theory with Wilson line insertion in topology t. S_t is the S-matrix associated with the topology

t. By the seminal paper of Witten [5], we know that for each topology t,

$$W_t = \sum_i \alpha_t^i G_i \tag{5.5}$$

where G_i are running over the space of conformal blocks with insertions associated with boundary points of the Wilson lines. If we define

$$\mathcal{S}_i = \sum_i \alpha_t^i S_t \tag{5.6}$$

then by plugging in (5.5) into (5.4) we get

$$S = \sum_i \mathcal{S}_i G_i \tag{5.7}$$

Comparing (5.7) with (4.10) one finds that in the Chern-Simons case the role of invariant tensors is taken by conformal blocks. This correspondence is also highlighted in work of Moore and Seiberg [13]. Indeed in the large k limit [2] one finds that conformal blocks do indeed become the corresponding invariant tensors.

Until now, however, note that the conformal blocks remain ill defined and we do not understand how to 'compound' conformal blocks (like we compounded the invariant tensors). In the subsequent sections we elucidate on these matters.

5.2 Conformal block and Tangles of Wilson lines

As we have understood in the previous subsection, a 'tangle' of Wilson line insertion in a path integral is related to WZW-conformal blocks. Indeed one can represent each conformal block in the pictorial form (an example being fig.(5.2)). At first glance one might be puzzled by this assertion. This is because the conformal blocks are supposed to depend only the position of boundary insertions. There, however, seems to be an infinitely more information in the bulk of tangle diagrams. However, this line of thinking is flawed because the topological nature of the Chern-Simons theory ensures that the

[2] the classical limit

only worthwhile difference in the tangles are those that change the topology of the tangles. These moves are captured by the multi-sheeted nature of conformal blocks.

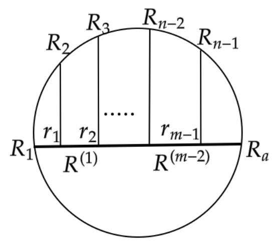

Figure 5.2: Wilson line representation of conformal blocks blocksq

One feature that the pictorial representation does not capture is the framing of Wilson lines. The framing for us however will be implicit in the following discussion. With a specific framing the Wilson line tangle diagrams are a faithful representation of conformal blocks. It is natural to ask what conformal blocks correspond to the classical projector maps. We shall discuss the systematic construction of the projector blocks later. Presently we propose a conjecture for how to compound conformal blocks using the tangle representation.

5.3 Compounding and Unitarity

For the unitarity condition $S^{\dagger}S = I$ to be satisfied when we plug in (cblock3 5.7), we must make sense of $G_j^{\dagger} \times G_i$. In order to do this we deal with a representation of conformal blocks in terms of a tangle of Wilson lines on a solid ball.

We then divide the insertions in each of the blocks into initial and final insertions. The initial insertions has the corresponding Hilbert space H_{in} and the final insertions correspond to Hilbert space H_{out}. We then flatten out the final half of the insertions on our conformal block as denoted in fig (prodflag1 5.3).

In order to do a hermitian conjugate of a conformal block, we reflect the block around its flat surface, and also reverse all the arrows along each Wilson line. Reversing the arrows is akin to complex conjugating all representation associated to boundary insertions. This is shown in the top left diagram of fig. (prodfig2 5.4).

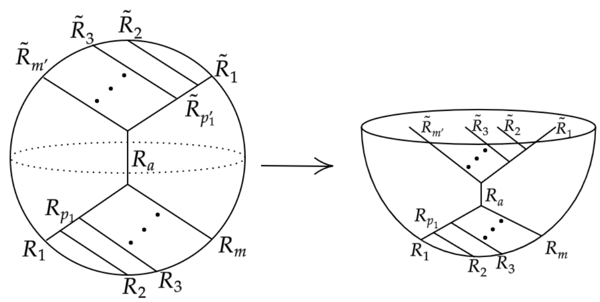

Figure 5.3: Flattening of the final half of insertions prodflag1

Finally $G_j^\dagger \times G_i$ is obtained by glueing the two final blocks as shown in fig. prodfig2 (5.4). Clearly the final diagram obtained is also representative of a block. As both G_I and G_j will have initial and final Hilbert spaces as H_{in} and H_{out}, the resulting block $G_j^\dagger \times G_i$ will have have H_{in} as both initial and final Hilbert space.

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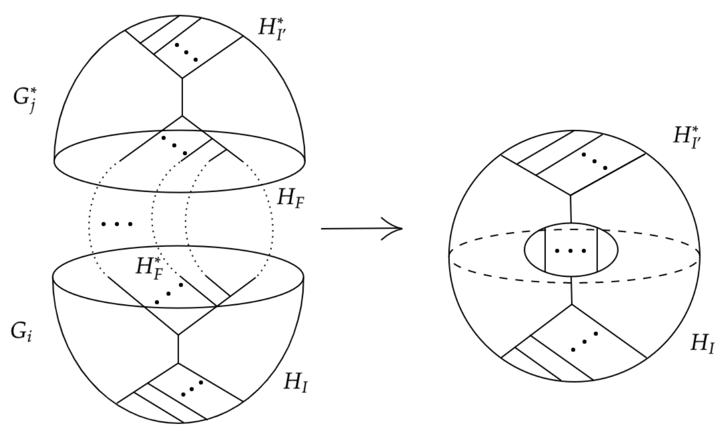

Figure 5.4: Block compounding prodfig2

The condition that the framing of Wilson lines in the compounded block should be the same as the framing of the blocks G_i and G_j ensure that our gluing method is unique[3].

It is also important to note another possible source of ambiguity in the compounding, related to the sheet structure of conformal blocks. As conformal blocks in general are multi-sheeted, a Wilson line tangle represents a conformal block along with a choice of sheet. One can move between sheets using monodromy operations which also affect the tangle representation as depicted in fig. (5.5). It should however be noted that there is no real ambiguity here due to sheet structure. This is because the monodromy of the resulting block encodes the monodromy degrees of freedom of the initial conformal blocks.

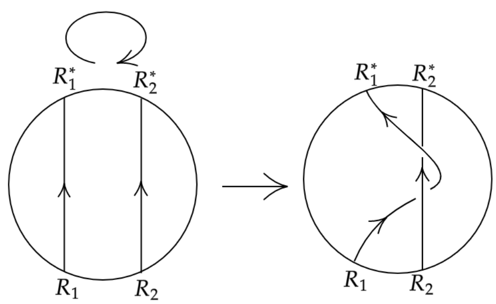

Figure 5.5: Monodromy operation on a simple identity block

5.4 Identity Block

For the unitarity equation $S^\dagger S = I$ to make sense we must define the symbol I. The identity block will have insertion in $H_{in} \otimes H_{in}^*$. Also the block must satisfy

$$I \times G = G \;\; \text{and} \;\; G' \times I = G' \tag{5.8}$$

The Wilson line representation of such a block is depicted in fig. (5.6). We shall denote the identity block from here on as G_{id}.

In Wilson line representation (6.6) is depicted in fig. (5.7).

[3]Otherwise one could potentially perform Dehn twists before glueing in which case the compounding operation becomes ill defined

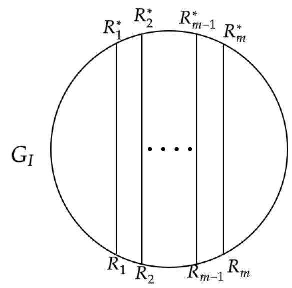

Figure 5.6: The Identity block

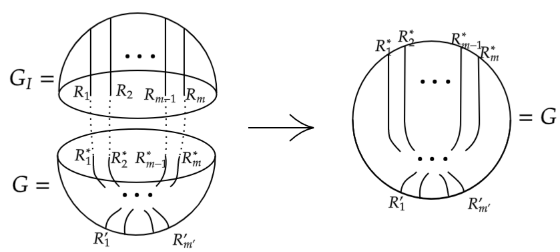

Figure 5.7: Compound the Identity block with a generic block

5.5 Projector Blocks

In the classical case it is a well known fact in group theory that all invariant tensors can be constructed using only the Clebsch-Gordan coefficients. Our strategy to construct conformal block analogues of projector invariant maps is similar.

The Wilson line tangles with trivalent vertices are studied by Witten in [14]. The analogue of Clebsch-Gordan coefficient $C_{\vec{m}_c}^{\vec{m}_a,\vec{m}_b}$ in the Chern-Simons case is a three-point block with initial insertion R_c and final insertions R_a^* and R_b^*. All three-point blocks will have a Wilson line tangle representation of the kind depicted in fig. (5.8).

We fix a choice of vertical framing for all blocks from here on.[4]

[4]For a better understanding of reasoning behind this framing the reader is encouraged to look at [14]

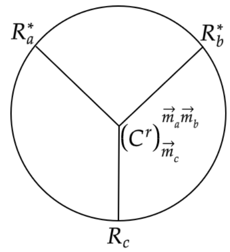

Figure 5.8: Three-point conformal block

3ptblc

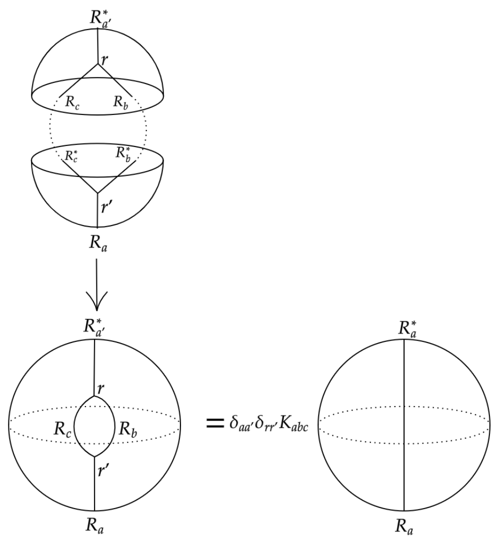

Figure 5.9: Compounding of three-point blocks

3ptorth

Note that at large enough k (where N^{WZW}_{abc} is equal to N^{cl}_{abc}) there is a one-one correspondence between CG-coefficients and three-point conformal blocks. Next we wish to make the basis of three point conformal blocks orthonormal analogous to (4.29). The analogue of contracting indices of the Clebsch-Gordan coefficients in Chern-Simons theory is the compounding procedure explained in the previous subsections.

Compounding of three point blocks is depicted in fig. (5.9). Next, note that since the LHS and RHS of fig. (5.10) have the same external insertions these conformal blocks lie in the same vector space of two-point conformal

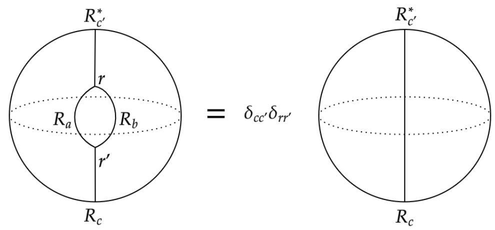

Figure 5.10: Compounding of three-point blocks 3ptorth2

blocks. It is well known that the vector space of two-point conformal blocks is one dimensional. So the innerproduct of these blocks with any two-point conformal block (like in fig. (5.12)) has an equivalent amount of information as the original equation. This statement is the Chern-Simons analogue of the classical statement that (4.29) and (4.30) carry an equal amount of information.

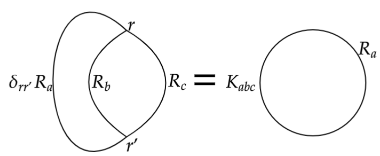

Figure 5.11: Compounding of three-point blocks 3ptorthn

Note that the expectation value of circular Wilson loop in in representation R_c is equal to the quantum dimension of the representation which is the analogue of the classical dimension that occurs in the RHS of (4.30).

The equation in fig. (5.12) also fixes the normalization of the three point blocks. Having made the choice of orthonormal basis of CG coefficients one can now construct an orthonormal basis for a more general class of conformal blocks, with representations $R_1, \cdots R_m$ fusing into R_a. This block in the Wilson line representation is depicted in fig.(5.2). Note that the abstract fusion of the classical case in fig. (4.1) is now a real Wilson line tangle in fig. (5.2).

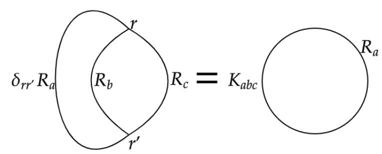

Figure 5.12: Compounding of three-point blocks 3ptorthn

The analogue of projector $P_a^{rr'}$ is depicted in fig. (5.13). We shall denote this block by $G_a^{rr'}$. The orthonormality relation of of the block will look like

$$(G_a^{r_1 r_2})^\dagger \times G_{a'}^{r_3 r_4} = \delta_{aa'} \delta_{r_2 r_4} \hat{G}_a^{r_3 r_1} \tag{5.9}$$ block1

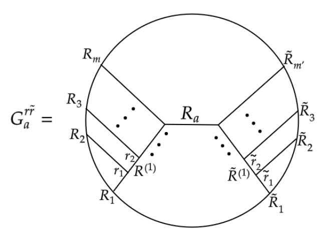

Figure 5.13: Analogue of projector block projblock

where $\hat{G}_a^{r_3 r_1}$ are blocks with insertions in $H_{in} \otimes H_{in}^*$. Analogous to the classical equation (4.19) we will have

$$\hat{G}_a^{r_1 r_2} \times \hat{G}_{a'}^{r_3 r_4} = \delta_{aa'} \delta_{r_2 r_4} \hat{G}_a^{r_3 r_1} \tag{5.10}$$ block2

and clearly $(\hat{G}_a^{r_1 r_2})^\dagger = \hat{G}_a^{r_2 r_1}$. To prove (5.9), we just need to show that the equation in fig.(5.14) holds. To show that fig.(5.14) is true, note that blocks on both LHS and RHS of fig. (5.14) are two-point blocks and as the space of two point blocks is unidimensional, fig.(5.14) is equivalent to fig. (5.15). To prove that the first equality in fig.(5.15) holds, we need only prove the equality in fig. (5.16). Then we can use fig. (5.16) recursively get the first equality in fig. (5.15).

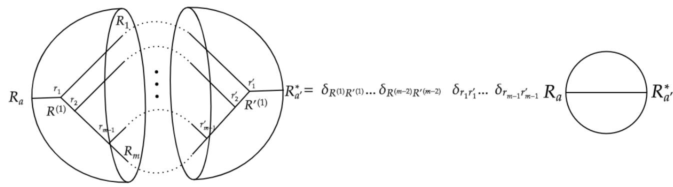

Figure 5.14: whyorthg2

Note that the second equality in fig. (5.15) is clear from fig. (5.12). Therefore we must only prove that fig. (5.16) holds. We use Witten's analysis in [14] to proceed with our proof. We first cut the LHS horizontally as shown in the diagram into two two=point conformal blocks. Algebraically, this move is akin to looking at a number as an innerproduct. We then insert the completeness relation depicted in fig. (5.17). The completeness relation also holds because of the unidimensional nature of the space of two-point blocks. Inserting the completion relation proves the equality in fig. (5.16). This shows that the projector blocks obey equations similar to the classical projector maps.

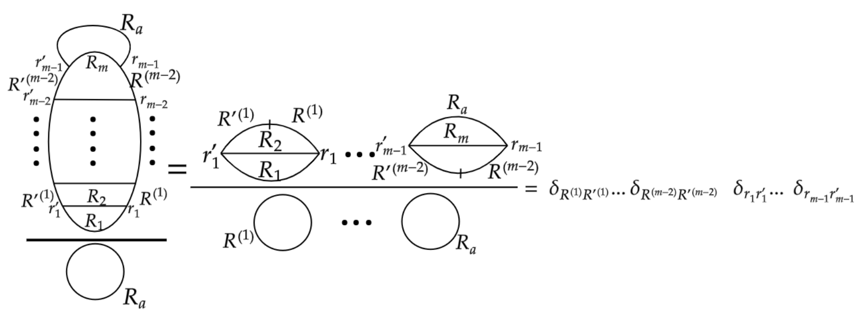

Figure 5.15: whyorthg3

5.6 Unitarity and Crossing Symmetry

As in the classical case, projector blocks on the space $H_{in} \otimes H^*_{in}$ obey the relation

$$\sum_{a,r} \hat{G}^{rr}_a = G_{id} \qquad (5.11)$$

uni1

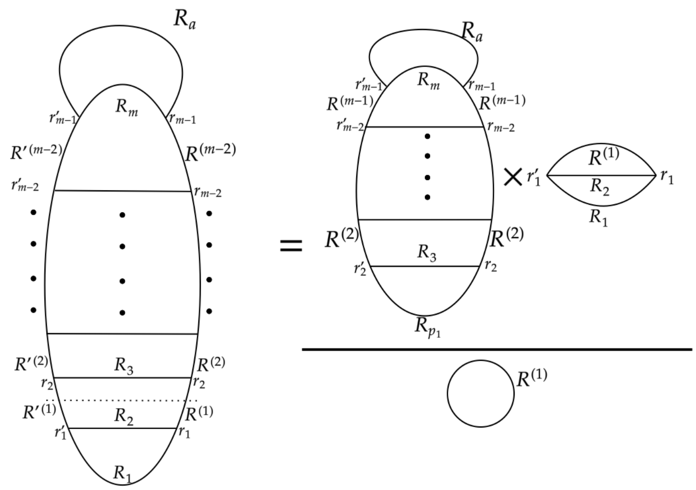

Figure 5.16: witten_proc1

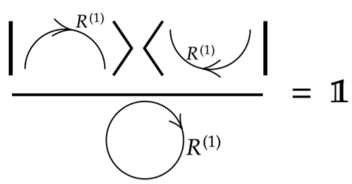

Figure 5.17: abc

For a proof of this statement we refer the reader to Mehta:2022lgq [9]. With this statement we can unitarity equations in the Chern-Simons case as follows:

First we expand the S-matrix as

$$S = \sum_{a,r,r'} \mathcal{S}_a^{rr'} G_a^{rr'} \tag{5.12}$$ uni2

Plugging (5.12) into the unitarity equation $S^\dagger S = \mathcal{S}_{id} G_{id}$, we get

$$
\begin{aligned}
& S^\dagger S = \mathcal{S}_{id} G_{id} \\
\implies & \sum_{\text{final states}} \left(\sum_{a,r,r'} (\mathcal{S}_a^{rr'})^* (G_a^{rr'})^\dagger \right) \times \left(\sum_{b,t,t'} (\mathcal{S}_b^{tt'})(G_b^{tt'}) \right) = \mathcal{S}_{id} \left(\sum_{c,r} \hat{G}_c^{rr} \right) \\
\implies & \sum_{\text{final states}} \sum_{a,r,t,r'} \left(\mathcal{S}_a^{rr'} \right)^* \star \left(\mathcal{S}_a^{rt'} \right) \hat{G}_a^{rt} = \sum_{a,r,t} \mathcal{S}_{id} \delta^{tr} \hat{G}_a^{rr}
\end{aligned}
\tag{5.13}
$$

uni3

This equation is the Chern-Simons analogue of (4.22). We now move on to understanding the crossing relations in these theories.

As in the classical case we can separate boundary insertions into initial and final. Each distinct separation gives us a basis for the space of conformal blocks with the aforementioned insertions, and the basis change between the blocks along with analytic continuation gives us the statement of crossing.

Consider two divisions of insertions into initial and final. We expand the S-matrix as

$$S = \sum_{a,r,r'} \mathcal{S}_a^{rr'} G_a^{rr'} \tag{5.14}$$

cs cross1

and

$$S = \sum_{a,r,r'} \tilde{\mathcal{S}}_a^{rr'} \tilde{G}_a^{rr'} \tag{5.15}$$

cs cross2

where $G_a^{rr'}$ and $\tilde{G}_a^{rr'}$ are both bases of blocks for different divisions of insertions. Therefore there exists a map between these bases

$$G_a^{rr'} = \sum_{s,s',b} M_{bss'}^{arr'} \tilde{G}_b^{ss'} \tag{5.16}$$

cs cross3

Crossing symmetry is the assertion that

$$\tilde{\tau}_b^{ss'} = \sum_{s,s',b} M_{bss'}^{arr'} \tau_a^{rr'} \tag{5.17}$$

cs cross4

where the appropriate analytic continuations have been performed. The coefficient $M_{bss'}^{arr'}$ are matrix elements which may be computed using the innerproduct on conformal blocks. As

$$|G_a^{rr'}\rangle = \sum_{s,s',b} M_{bss'}^{arr'} |\tilde{G}_b^{ss'}\rangle \tag{5.18}$$

cs cross5

then we have

$$M^{arr'}_{bss'} = \frac{\langle \tilde{G}^{ss'}_b | G^{rr'}_a \rangle}{\langle \tilde{G}^{ss'}_b | \tilde{G}^{ss'}_b \rangle} \tag{5.19}$$

Now the inner product $\langle \tilde{G}^{ss'}_b | \tilde{G}^{ss'}_b \rangle$ is equal to the quantum dimension of the representation R_b , labelled as D^k_b. For a reasoning of this fact we refer the reader to [9].

$$M^{arr'}_{bss'} = \frac{\langle \tilde{G}^{ss'}_b | G^{rr'}_a \rangle}{D^k_b} \tag{5.20}$$

Note that the innerproduct being used here is the one introduced by Witten in [14]. We use this innerproduct more explicitly in the subsequent sections. We now look at an example of $2 \to 2$ scattering in Chern-Simons matter theories.

Chapter 6

Scattering in fundamental-fundamental and fundamental-anti-fundamental insertions

We now study the fundamental-fundamental and fundamental-anti-fundamental scattering in Chern-Simons matter theories of Type I, Type II and $SU(N)$. We will also work out some checks of the our result.

6.1 Fundamental-Fundamental scattering

We first look at the two fundamentals scattering into two fundamentals. The basis for this this process is show in fig. (6.1) We denote the symmetric and anti-symmetric blocks as G_s and G_a respectively. It is also clear (by taking the appropriate reflection of blocks) that $G_s^\dagger = G_s$ and $G_a^\dagger = G_a$. It should also be noted that the two blocks are orthogonal as depicted in fig. (6.2).

The argument for the orthogonality goes as follows, we compound the two blocks to get the block in the upper right hand corner. We can then scoop out the two-point block, which amounts to factoring the block path integral into an inner product. Since the two-point block has different external insertions it is zero, and thus the whole inner-product, that is the entire block, must be zero.

Note that the conformal blocks G_s and G_a become the classical invariant

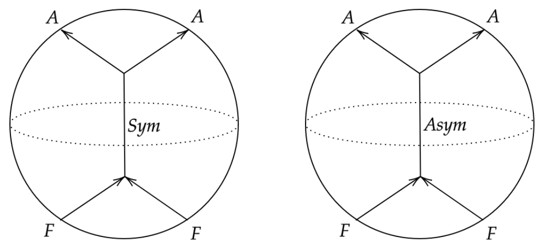

Figure 6.1: Block basis for fundamental-fundamental scattering sasym

tensors in the limit $k \to \infty$. Next we look at the compounding of the blocks G_s and G_a with themselves respectively. This compounding is depicted in

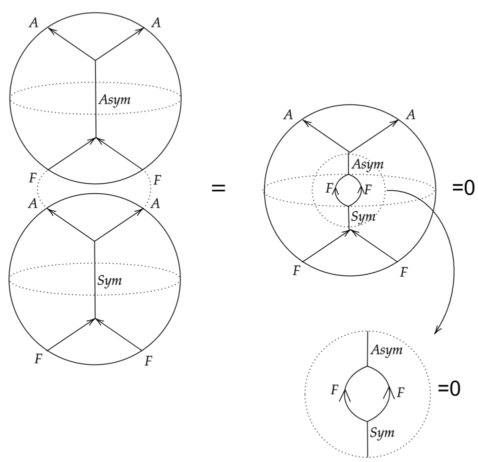

Figure 6.2: Orthogonality of symmetric and snti-symmetric blocks sasymorth

sasymsasym

the fig.(6.3). We see that compounding of $G^{\dagger}_{s/a}$ with $G_{s/a}$ equals $\hat{G}_{s/a}$ times a numerical factor $\alpha_{Sym/Asym}$.

Note that as $H_{in} = H_{out}$ in fundamental-fundamental scattering, the

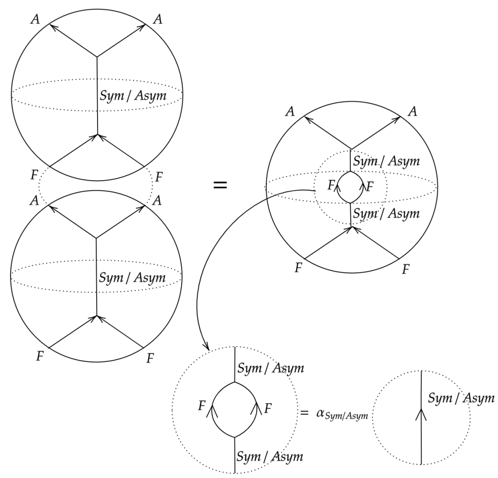

Figure 6.3: Compounding of symmetric/anti-symmetric block with itself sasymsasym

suitably normalized conformal blocks must satisfy $\hat{G}_{s/a} = G_{s/a}$. Note that this normalization condition is non-trivial to the give a concrete definition of the projector blocks.

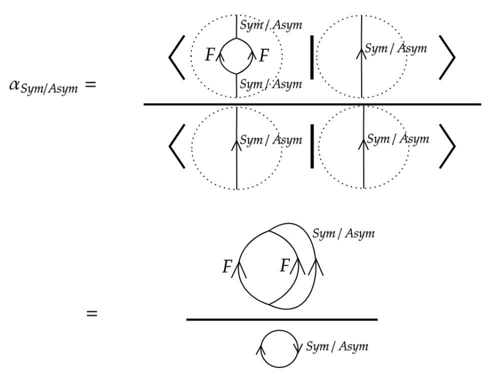

Figure 6.4: Evaluation of $\alpha_{Sym/Asym}$ aingadjnorm

In order to evaluate $\alpha_{Sym/Asym}$ like in the previous section we need only take an inner product with the two point identity conformal block as done in fig(6.4). aingadjnorm In order to make the blocks into projectors, one of two thing could be done. The first is to note that the diagram in the numerator of fig.(6.4) depends on the value we set for the three-point conformal blocks. If we demand the appropriate normalization, we get find that $\alpha_{Sym/Asym} = 1$.

Alternatively, we can explicitly multiply the three-point blocks with the square root of $1/\alpha_{Sym/Asym}$. We choose the latter, as it makes our calculations explicit. The normalized four-point blocks are depicted in fig (6.5).

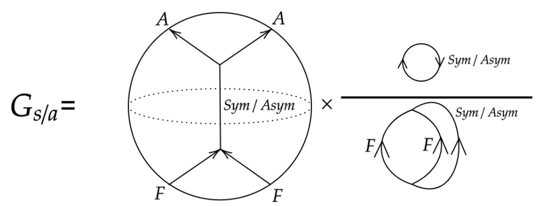

Figure 6.5: Normalized symmetric and anti-symmetric blocks

With this normalization, by (5.11) we can write

$$G_{id} = G_s + G_a \tag{6.1}$$

To check the validity of (6.1), we first write (6.1) in vector form as

$$|G_{id}\rangle = |\theta\rangle = |G_s\rangle + |G_a\rangle \tag{6.2}$$

By taking the inner product of (6.2) with $\langle G_{s/a}|$ gives us the condition

$$\langle G_{s/a}|\theta\rangle = \langle G_{s/a}|G_{s/a}\rangle \tag{6.3}$$

As previously stated,

$$\langle G_{s/a}|G_{s/a}\rangle = D^k_{s/a} \tag{6.4}$$

Both (6.3) and (6.4) by fig. (6.6). Note that details about the inner product used can be found in [2].

We now move to the other channel of fundamental-anti-fundamental scattering.

6.2 Fundamental-Anti-fundamental scattering

An orthogonal basis in the fundamental-anti-fundamental scattering channel is depicted in fig.(6.7). We denote the Singlet and Adjoint blocks by G_I and

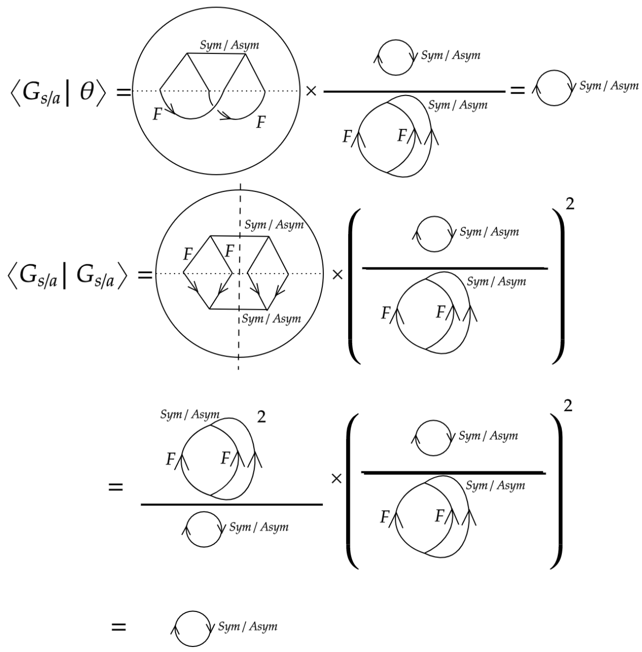

Figure 6.6: Computation of $\langle G_{s/a}|G_{s/a}\rangle$ and $\langle G_{s/a}|\theta\rangle$ identity

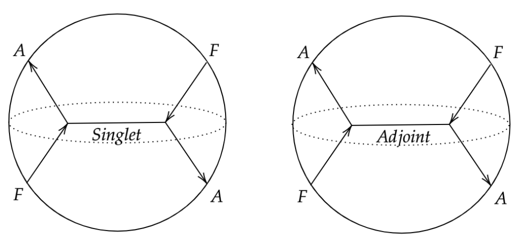

Figure 6.7: Singlet and Adjoint blocks sadj

G_{Adj} respectively. The two blocks are orthogonal by an argument identical to the one in the previous subsection, as is depicted in fig. (6.8). We next normalize the blocks to make them into projectors as we did earlier. The compounding of blocks G_I and G_{Adj} with themselves, respectively, we get the equation in fig. (6.9). As in the previous subsection, we can find the value

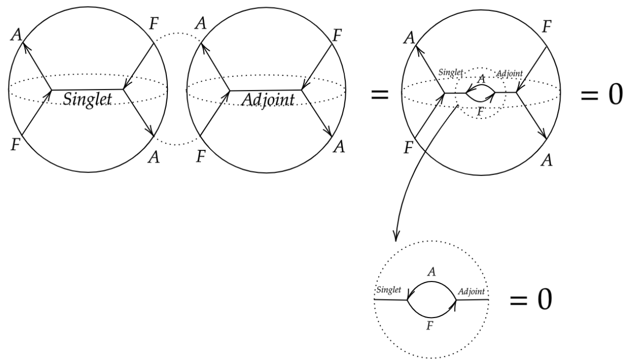

Figure 6.8: Orthogonality of Singlet and Adjoint blocks sadjorth

of $\alpha_{I/Adj}$ by taking an innerproduct with the identity two-point function as shown in fig.(6.10). We can then normalize the blocks G_I and G_{Adj} to make them into projector blocks. The normalized blocks are depicted in fig.(6.11).

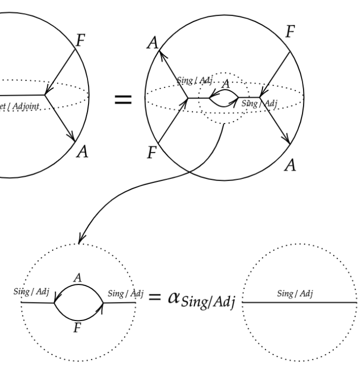

Figure 6.9: $G^{\dagger}_{I/Adj} \times G_{I/Adj} = \alpha_{I/Adj} \hat{G}_{I/Adj}$ ssadjadj

The normalized blocks as defined in fig.(6.11) must satisfy the identity

$$\tilde{G}_{id} = G_I + G_{Adj} \tag{6.5}$$ ide

Where the $\tilde{G}_{id}$ is the identity blocks. Like the analogous equation (6.1) in the previous subsection, this equation can also be proved directly by similar methods.

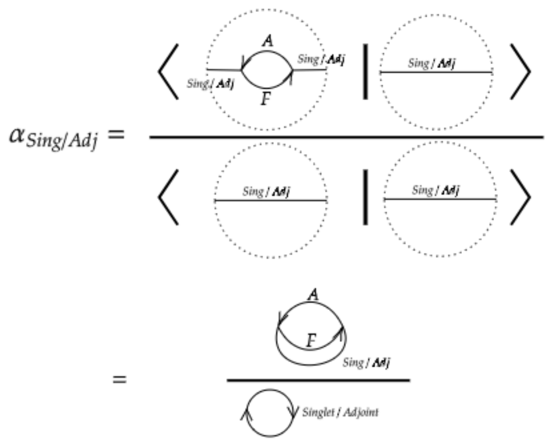

Figure 6.10: Evaluating $\alpha_{I/Adj}$ singadjnorm

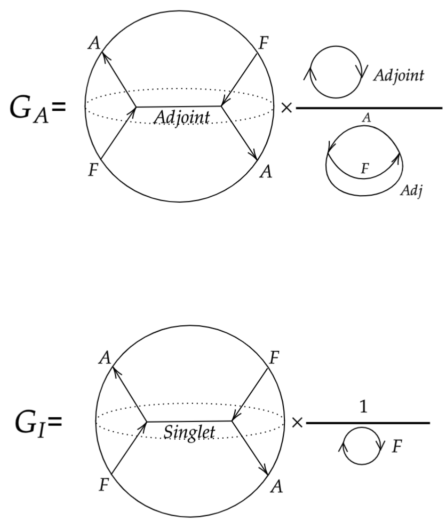

Figure 6.11: Normalized blocks G_{Adj} and G_I adjn

Note that although $\tilde{G}_{id}$ is indeed the identity block, but it is the identity block associated to the space $H_{in} \otimes H^*_{out}$, with $H_{in} = H_{out} = R_F \otimes R^*_F$, which is distinct from the identity block G_{id} in (6.1). Both blocks belong to the same space of conformal blocks and are indeed related to each other by a monodromy operation.

Also note that the norm of the projector blocks is equal to the quantum dimension of the associated representation. So we have,

$$\langle G_I | G_I \rangle = 1 \qquad \text{and} \qquad \langle G_{Adj} | G_{Adj} \rangle = D^k_{Adj} \tag{6.6}$$

Finally, we should also note that the unnormalized block G_I is equal to the identity conformal block $\tilde{G}_{id}$. This is because the Wilson line insertion associated to the singlet channel in fig. (6.7) is only there for clarity. Since the singlet representation is the trivial representation, in the path integral form of the conformal block there is no Wilson line insertion for the same. Thus if we remove the singlet Wilson line, we get the identity conformal block.

6.3 Crossing Symmetry

In order to obtain the crossing symmetry we first need to relate the fundamental-fundamental and fundamental-anti-fundamental conformal blocks to each other. We can write

$$\begin{aligned} G_s &= \alpha^I_s G_I + \alpha^{Adj}_s G_{Adj} \\ G_a &= \alpha^I_a G_I + \alpha^{Adj}_a G_{Adj} \end{aligned} \tag{6.7}$$

Using the norm of G_I and G_{Adj} we see that

$$\alpha^I_{a/s} = \langle G_{Adj} | G_{a/s} \rangle \quad \text{and} \quad \alpha^{Adj}_{a/s} = \frac{\langle G_{Adj} | G_{a/s} \rangle}{D^k_{Adj}} \tag{6.8}$$

In order to find the inner product between $G_{s/a}$ and $G_{I/Adj}$, we must glue the ket state of $G_{s/a}$ to the bra state $G_{I/Adj}$. To do this we deform the conformal blocks as we have done in fig.(6.12). For $G_{s/a}$, we move both the fundamental insertions and the left most anti-fundamental insertion to the top half, which we then flatten.

For the conformal blocks in the other channel, we first exchange the fundamental and anti-fundamental insertions, then moving the insertions and flattening the final half as before.

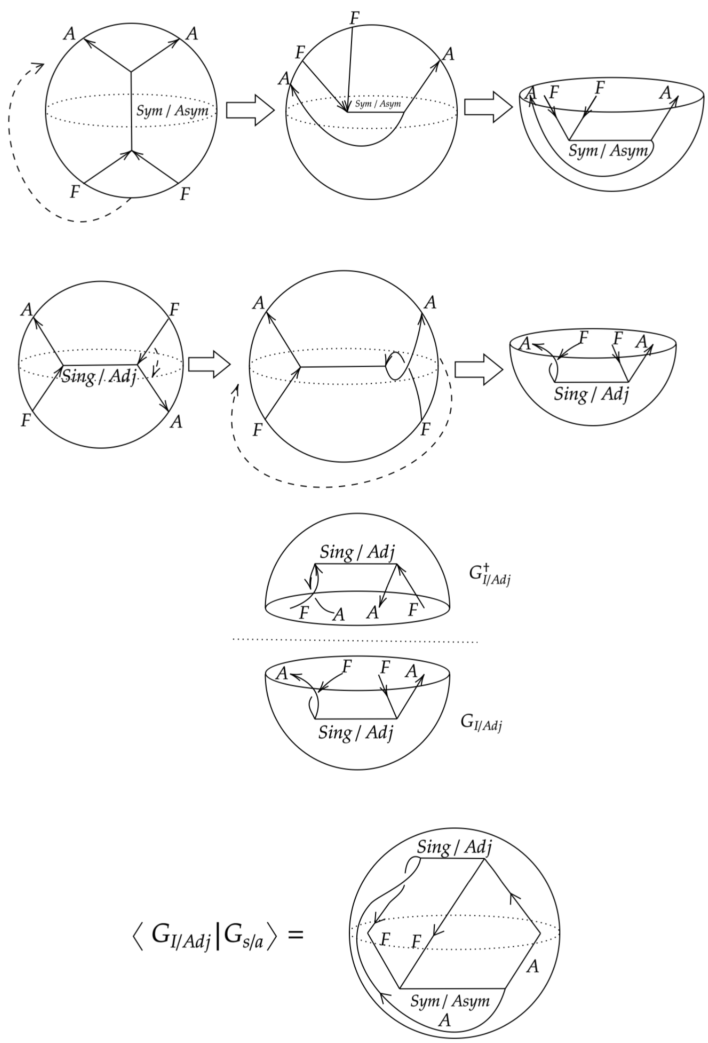

Figure 6.12: Manipulation of blocks to take the inner product innprod

We then take a reflection of this block and complex conjugate all the Wilson line representations, which amounts to making the ket vector into a bra vector (or equivalently taking the hermitian conjugate of the block).

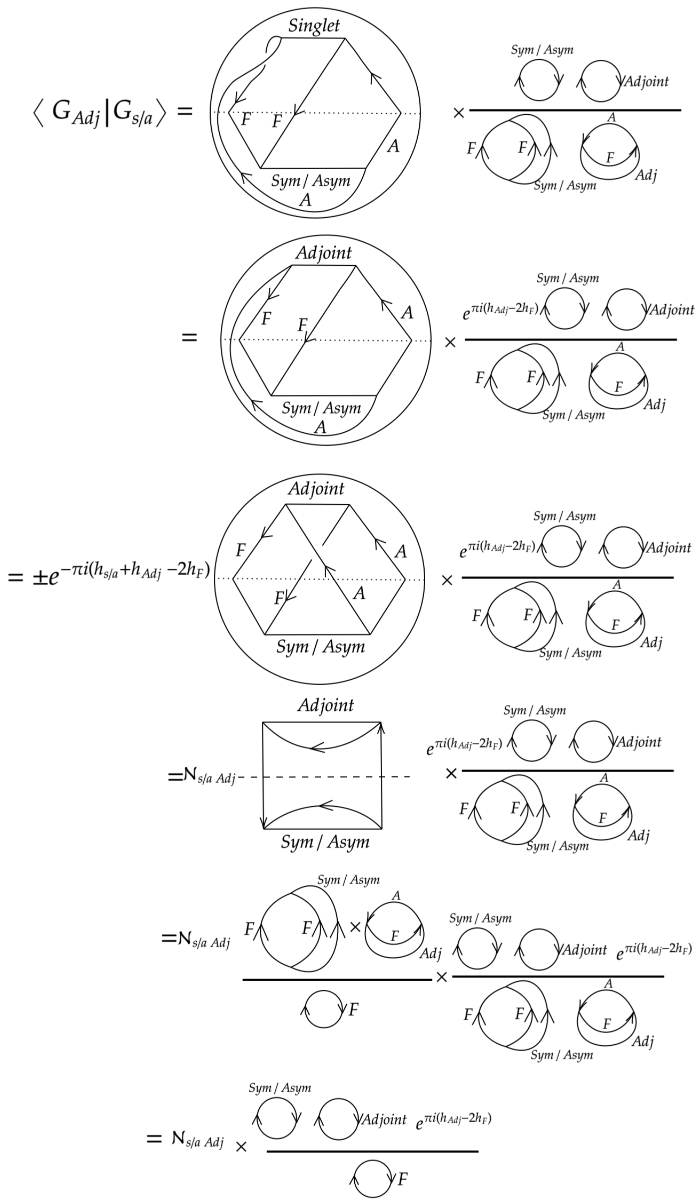

Figure 6.13: sasymadj

Finally we join the resulting two sets of blocks together as shown in (6.12). innprod

Note that the exchange of insertions is necessary because it makes the relative position of insertion the same as in the fundamental-fundamental scattering channel.

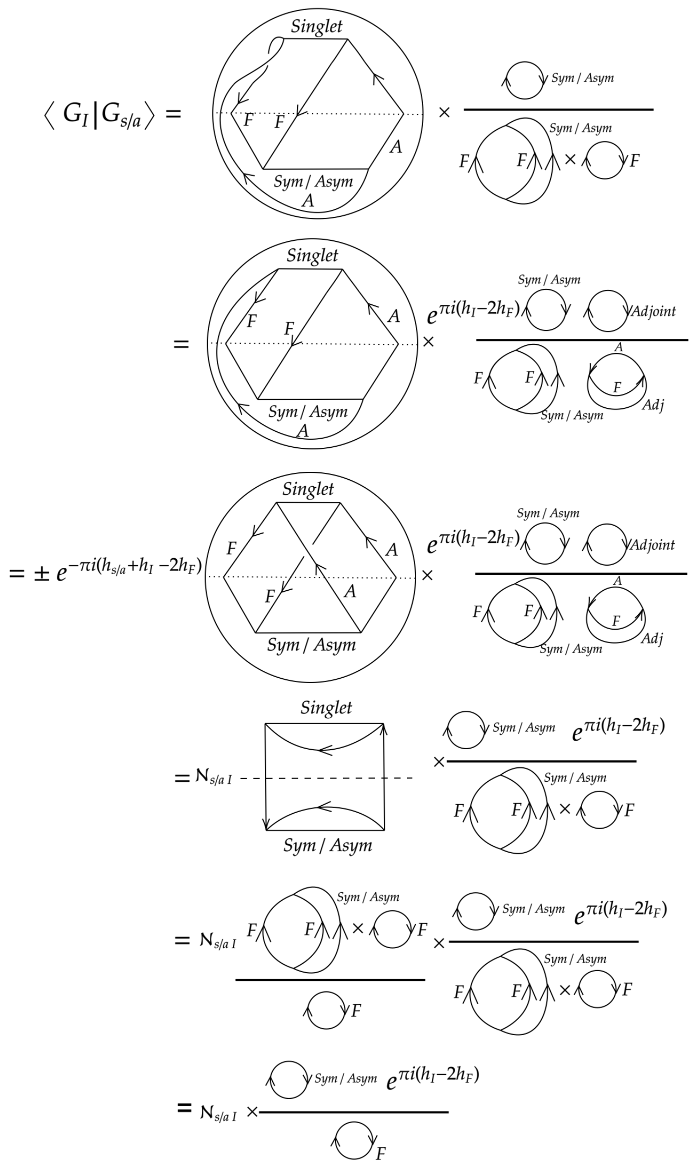

Figure 6.14:

sasymsing

Fig.(6.13) and fig.(6.14) depict the evaluation of the innerproduct of normalized conformal blocks. Here we use the formalism developed by Witten in [14] . To go from step one to step two in figures (6.13) and (6.14), we untwist the three-point vertex which results in a phase factor. Note however that half twist is not a unique choice. We can make a clockwise half twist, an anti-clockwise half twist, a one and a half twist etc. There are an infinite number of such choices. We will return to this issue later. Presently, however, we focus on evaluating the inner product with the choice we have made.

Going from step two to step three, we pull the Wilson line over the rest of the tangle and then use the phase factor twists to get a tetrahedron. The tetrahedron can then be evaluated using skein relations and this is done explicitly in [14].

Using skein relations, we go from step three to step four in figures (6.13) and (6.14). Beyond this we can use methods already introduces to evaluate the innerproduct. We cut the diagram along the horizontal line and insert a completion relation to get step five. Now the factor $\mathcal{N}_{s/a\ I/Adj}$ is evaluated for $SU(N)$ in [14]. We generalize this result to type I and type II theories as well by replacing the $SU(N)$ skein relations with the skein relations of Type I and Type II. As reader who are familiar with [14] will note the rest of the derivation is identical. For $SU(2)$, Type I and Type II theories we get

$$\mathcal{N}_{s/a\ I/\mathrm{Adj}} = \frac{e^{-\frac{\pi i}{2}(4h_F-h_s-h_a)} e^{\pi i(h_{s/a}+h_{I/\mathrm{Adj}}-2h_F)} \left(e^{\pi i \frac{h_s-h_a}{2}} - e^{-\pi i \frac{h_s-h_a}{2}}\right)}{1-e^{-\pi i(4h_F-h_s-h_a)} e^{2\pi i(h_{s/a}+h_{I/\mathrm{Adj}}-2h_F)}} \tag{6.9}$$

With this one finds the final answer for the inner product to be

$$\alpha_{s/a}^{I/Adj} = \frac{\mathcal{N}_{s/a\ I/\mathrm{Adj}} D^k_{s/a}}{D^k_F} \tag{6.10}$$

Plugging in the values, we get the relation between conformal blocks of different channels as

$$\begin{aligned} G_s &= \frac{G_I e^{-2\pi i h_F} \lfloor N+1 \rfloor_q + G_{\mathrm{Adj}} e^{\pi i (h_{\mathrm{Adj}}-2h_F)}}{\lfloor 2 \rfloor_q} \\ G_a &= \frac{G_I e^{-2\pi i h_F} \lfloor N-1 \rfloor_q - G_{\mathrm{Adj}} e^{\pi i (h_{\mathrm{Adj}}-2h_F)}}{\lfloor 2 \rfloor_q} \end{aligned} \tag{6.11}$$

Where the notation $\lfloor a \rfloor_q$ denotes the q-deformed numbers which are denoted by

$$\lfloor a \rfloor_q = \frac{q^{\frac{a}{2}} - q^{\frac{-a}{2}}}{q^{\frac{1}{2}} - q^{\frac{-1}{2}}} \quad \text{where} \quad q = e^{\frac{2\pi i}{\kappa}} \tag{6.12}$$

crs3

where $\kappa = sgn(k)(|k| + N)$. Note that this result is the q-deformation of the classical result (4.42). With (6.11) we can write the crossing relations,

$$\begin{aligned} \tau_I^{(0)} &= e^{\pi i(h_I - 2h_F)} \left(\tau_s \left(\frac{\lfloor N+1 \rfloor_q}{\lfloor 2 \rfloor_q} \right) + \tau_a \left(\frac{\lfloor N-1 \rfloor_q}{\lfloor 2 \rfloor_q} \right) \right) \\ \tau_{\text{Adj}}^{(0)} &= e^{\pi i(h_{\text{Adj}} - 2h_F)} \left(\frac{\tau_s - \tau_a}{\lfloor 2 \rfloor_q} \right) \end{aligned} \tag{6.13}$$

Smattrq

with the appropriate analytic continuation having been done. Now recall that in the previous section while taking the inner product of the conformal blocks we had mentioned that an infinite number of choices for the twists exist. So if we define

$$\nu_I = 2h_F - h_I \quad \text{and} \quad \nu_{Adj} = 2h_F - h_{Adj} \tag{6.14}$$

phases1

Then in general we can write the crossing relation as

$$\begin{aligned} \tau_I^{(n)} &= e^{(2n-1)\pi i \nu_I} \left(\tau_s \left(\frac{\lfloor N+1 \rfloor_q}{\lfloor 2 \rfloor_q} \right) + \tau_a \left(\frac{\lfloor N-1 \rfloor_q}{\lfloor 2 \rfloor_q} \right) \right) \\ \tau_{\text{Adj}}^{(n)} &= e^{(2n-1)\pi i \nu_{\text{Adj}}} \left(\frac{\tau_s - \tau_a}{\lfloor 2 \rfloor_q} \right) \end{aligned} \tag{6.15}$$

Smattrqt

by putting $n = 0$ in (6.15) one can verify that (6.13) holds. The reasoning behind why this ambiguity occurs is unclear. However, it is important to note that this ambiguity has no effect on any observables because we wish to look at the probability density not scattering amplitudes. We can therefore simply absorb the phases into the definition of scattering amplitudes as

$$\begin{aligned} \tau_I^{(n)} &= e^{i(2n-1)\pi\nu_I} \tau_I, \\ \tau_{\text{Adj}}^{(n)} &= e^{i(2n-1)\pi\nu_{\text{Adj}}} \tau_{\text{Adj}}, \end{aligned} \tag{6.16}$$

staun

With this redefinition we can rewrite the crossing relations in their final form

$$\begin{aligned} \tau_I &= \left(\tau_s \left(\frac{\lfloor N+1 \rfloor_q}{\lfloor 2 \rfloor_q}\right) + \tau_a \left(\frac{\lfloor N-1 \rfloor_q}{\lfloor 2 \rfloor_q}\right)\right) \\ \tau_{\text{Adj}} &= \left(\frac{\tau_s - \tau_a}{\lfloor 2 \rfloor_q}\right) \end{aligned} \tag{6.17}$$

6.3.1 Crossing result in the classical and the 't Hooft limit

In the classical limit, where we take $k \to \infty$, keeping N finite, we find that the q-deformed numbers go to classical numbers and therefore we get back the classical crossing relation (1.4). This result is expected since in the large k limit we expect the crossing to work like in the classical global symmetry case.

Next, if we take the 't Hooft limit, where we take $N \to \infty$ and $k \to \infty$, while keeping $\frac{N}{k} = \lambda$, one finds the relations (1.6). This explains the extra correction factor first conjectured in [4]. Therefore our result matches the direct Feynman diagram computations in the large N limit.

These limits serve as very strong checks for our conjecture. We look at another check of our conjecture using the level-rank duality.

6.3.2 Duality

Recall that Chern-Simon theories are dual to each other under the level-rank dualities. As the result (6.15) applies equally well to $SU(N)$, Type I and Type II $U(N)_k$ theories, the crossing relation must be consistent under duality. Under the level rank duality one finds

$$N' = |k| \quad k' = -sgn(k)N, \quad \kappa' = -\kappa, \quad q' = q^{-1} \tag{6.18}$$

Then under duality one can easily show that

$$\lfloor N-1 \rfloor_q \to \lfloor |k|+1 \rfloor_{q^{-1}} = \lfloor N-1 \rfloor_q \tag{6.19}$$

and the phases change under duality to

$$e^{-2\pi i h'_F} = -e^{-2\pi i h_F}, \quad e^{\pi i \left(h'_{\text{Adj}} - 2h'_F\right)} = e^{\pi i \left(h_{\text{Adj}} - 2h_F\right)} \tag{6.20}$$

along with an exchange of the conformal blocks $G_s \leftrightarrow -G_a$ we find that the crossing relations agree under duality. Note that under level rank duality, the symmetric and anti-symmetric representations change places. The negative sign of the anti-symmetric block is due to the matter field exchanging between boson and fermion.

Chapter 7

Results and discussion

In this thesis, a conjecture of the crossing rules for Chern-Simons matter theory is presented. We hypothesise that the structure of S-matrix in Chern-Simons matter theories to be of the form

$$S = \cos(\pi\nu)G_{id} + \iota\sum_i \tau_i G_i \tag{7.1}$$

where the G_is are conformal blocks, the Chern-Simons analogue of the classical invariant tensors. We propose the structure of projector conformal blocks and their compounding rules in chapter 5. With this information one can write down the unitarity equation and crossing symmetry relations. Then in chapter 6, we apply our formalism to the case of $2 \to 2$ scattering in Chern-Simons matter theories of Type I, Type II and $SU(N)$ and find the Chern-Simons crossing relations (6.17) to be exactly the q-deformations of the classical crossing relations (1.5). This result also explains the crossing relations for Chern-Simons matter theory in the 't Hooft limit found in [4], which had an extra correction factor.

While our proposed formalism, covered in chapter 5, is motivated by physical considerations and also passes various non-trivial checks in the special case of $2 \to 2$ scattering, we would like derive the formalism possibly from the path integral formalism.

Another possibility that we have not touched upon is the role of Quantum Groups. Quantum Groups are associative algebras which are deformation of classical Groups, which possess many properties of the classical Groups. Among these properties is the ability to take tensor products of representations to produce new representations and associate invariant tensors to these representations. The relation between the Quantum Group and pure Chern-Simons gauge theory is well known and it was shown by Faddeev et al. that Wess-Zumino-Witten theories possess a Quantum group symmetry. It is therefore an appealing proposition to try an mirror the classical crossing symmetry formalism in the Chern-Simons case using quantum groups instead of classical groups. We have not explored this possibility here.

In the future we would like to generalise our formalism to general topologically gapped theories, which we hope will provide us insight into scattering theory of even gapless theories.

The most direct implications of this work is to understand the crossing symmetry in the mass-deformed ABJM theory. The ABJM theory is a 3-dimensional supersymmetric analogue of the Chern-Simons theories. The ABJM theory has gained prominence in recent years due to its conjectured duality to M-theory on $AdS_4 \times S^7$ via the AdS/CFT correspondence.

A final point that we would like to clear up better is the role of phases in our crossing relations. The physical origin of the phase ambiguity mentioned earlier remains elusive at present and a better understanding of the phases is another goal we would like to pursue in the future.

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