The Yamabe Problem

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by

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Certificate

This is to certify that this dissertation entitled The Yamabe Problem towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study carried out by Vantipalli Ritvik at Indian Institute of Science under the supervision of Dr. Swarnendu Sil, Assistant Professor, Department of Mathematics, Indian Institute of Science, Bengaluru, during the academic year 2022-2023.

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This thesis is dedicated to my parents.

Declaration

I hereby declare that the matter embodied in the report entitled The Yamabe Problem are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science, Bengaluru, under the supervision of Dr. Swarnendu Sil and the same has not been submitted elsewhere for any other degree.

V. Ritvik

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Abstract

In this thesis, we study the proof of the so-called Yamabe Problem. This problem was proposed by Yamabe in an attempt to solve the Poincaré conjecture eventually. The problem was to prove whether, given any compact Riemannian manifold $M_n (n \geq 3)$, a conformal change of metric exists such that the manifold has a constant scalar curvature. This geometric problem reduces to proving the existence of smooth, positive solutions to a semilinear elliptic PDE of the form

$$\Delta u + h(x)u = \lambda f(x)u^{2^*-1} \tag{0.0.1}$$

where $h, f \in C^\infty(M)$ and $f > 0$. In this thesis, we study the solution to Yamabe's problem. This includes studying many prerequisites such as Sobolev spaces, Regularity theory for uniformly elliptic equations, and a little Calculus of Variations. In the end, we study Lee-Parker's paper[7] for a solution to Yamabe's problem.

Contents

Introduction

The main aim of this thesis is to study the prerequisites necessary to understand the proof of solution to the Yamabe Problem. Yamabe problem is an important example of a non-linear PDE which has been solved. Initially, this question was posed by Yamabe[11]. In the same paper, he also attempted a proof, which was later found to be erroneous. Finally, the problem was solved due work of multiple authors such as Trudinger, Aubin, Hebey, Vaugon, Schoen, and Yau.

In Chapter 1, we define various geometric prerequisites necessary to understand the rest of the thesis. This includes definitions and properties of manifolds, Tangent spaces/bundles, connections, Riemannian metric, etc. This chapter is a combination of concepts from [1] and [6].

In Chapter 2, we study one of the most fundamental concepts in the study of PDEs. We define and study properties and theorems about the Sobolev spaces on Euclidean spaces ($\mathbb{R}^n$ and subdomains of $\mathbb{R}^n$) and compact Riemannian manifolds. In particular, we give a brief proof of the best Sobolev constant, which plays a critical role in solving the Yamabe problem. [3] and [1] are the books followed for all of the proofs in this chapter.

In Chapter 3, we study the regularity theory for uniformly elliptic PDEs. [4] is the primary reference for this chapter.

In the final chapter, we first understand the difficulty in the Yamabe problem by first solving the subcritical case and noticing why this approach fails in the critical case due to lack of compactness. We then prove the smoothness of the solution in both critical and sub-critical cases. The rest of the chapter is devoted to proving the existence of a solution in the critical case. A complete proof of this would require many more concepts, which cannot be completed in a year. So we assume a vital theorem known as the Positive mass theorem. All proofs in this chapter will be found in [1],[10] and [7].

Original Contribution

This thesis has no claims on any original results by the author. It is a presentation of a solution to the Yamabe Problem. While many survey articles are already available on this topic, these articles assume a familiarity with Sobolev spaced and Regularity theory. In this thesis, we provided an almost complete presentation of a solution to the Yamabe problem, which any Mathematics student can pick up and read after knowing a minimal amount of Functional analysis and geometry.

Chapter 1

Preliminaries

1.1 Manifold and Differentiable Manifold

Definition 1.1. A Manifold M_n is a second countable, Hausdorff topological space such that for any given point p in M_n, there exists an open neighborhood U of p which is homeomorphic to $\mathbb{R}^n$(or equivalently an open subset of $\mathbb{R}^n$).

Definition 1.2. A chart on a manifold M_n is a pair (U, ϕ), where U is an open set of M_n and ϕ a homeomorphism from U to an open subset of $\mathbb{R}^n$.

For any point $p \in U$, components of $\phi(p)$ are called the local coordinates w.r.t the chart (U, ϕ). Two charts (U_α, ϕ_α) and (U_β, ϕ_β) are called C^k compatible (smoothly compatible) if the transition map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a C^k (smooth[1]) diffeomorphism.

Definition 1.3. A collection $(U_i, \phi_i)_{i \in I}$ of charts which covers M_n i.e. $\bigcup_{i \in I} U_i = M_n$ is called an Atlas. An Atlas is said to be of class C^k (smooth) if any two given charts in the atlas are C^k (smoothly) compatible.

Two atlases are said to be C^k (smoothly) compatible if there union is still a C^k (smooth) atlas. Compatibility is an equivalence relation. An equivalence class of C^k (smooth) atlases is called a C^k (smooth) differentiable structure.

[1]Here by smooth we mean C^∞

Definition 1.4. A manifold with a C^k (smooth) differentiable structure is called a C^k differentiable (smooth) manifold.

When we say a chart on a differentiable manifold, we mean a chart belonging to an atlas of the differentiable structure.

1.2 Tangent Space and Tangent bundle

Definition 1.5. A derivation on M_n at point p is an $\mathbb{R}$-linear map $X : C^\infty(M_n) \to \mathbb{R}$ satisfying the product rule : $X(fg) = f(p)X(g) + X(f)g(p)$

It isn't hard to see that the set of derivations at a point form a vector space. On $\mathbb{R}^n$ we already know that any given directional derivative is a derivation, and we can also further prove any given derivation is a directional derivative.

On an abstract manifold, we don't have the notion of a geometric tangent plane in the ambient space to define tangent space. Noting the isomorphism between $T_p\Pi$ and space of derivations at p that we observed in $\mathbb{R}^n$ we define tangent space for a manifold.

Definition 1.6. The tangent space of M_n at point p, denoted by T_pM, is the space of derivations on M_n at point p. An element of T_pM is called tangent vector at p.

Definition 1.7. Let $F : W_d \to M_n$, we define push-forward associated with F as $F_* : T_pW \to T_{F(p)}M$ such that $F_*(X)(f) = X(f \circ F)$, where $X \in T_pW$ and $f \in C^\infty(M_n)$.

Let $i : U_n \hookrightarrow M_n$ be the inclusion map, where U_n is an neighborhood from the chart (U, ϕ) with the induced differentiable manifold structure from M_n. Note that a derivation's action on a function depends only on the definition of the function in the neighborhood of p, using this we can prove i_* is an isomorphism between T_pM and T_pU, which itself is isomorphic to $T_{\phi(p)}\phi(U)$ through ϕ_*. Using these two isomorphisms we can identify $\frac{\partial}{\partial x_i}|_{\phi(p)}$ which form a basis in $T_{\phi(p)}\phi(U)$ with tangent vectors in T_pM and denote this as $\frac{\partial}{\partial x_i}|_p$ or $\partial_i|_p$.

Definition 1.8. A curve is a map $\gamma : [a, b] \to M_n$. The velocity of the curve at t_0 is defined as $\dot{\gamma}(t_0) = \gamma_* \frac{d}{dt}|_{t_0}$.

Before defining the tangent bundle, we define a vector bundle.

Definition 1.9. Let E and M be smooth manifolds. A smooth surjection $\Pi : E \to M$ is a smooth vector bundle of rank r if

1. For every $p \in M$, the set E_p:=$\Pi^{-1}(p)$ is a real vector space of dimension r, called the 'fiber at p';
2. Every point $p \in M$ has an open neighborhood U such that there is a fiber–preserving diffeomorphism $\phi_U : \Pi^{-1}(U) \to U \times \mathbb{R}^r$ that restricts to a linear isomorphism $E_p \to \{p\} \times \mathbb{R}^r$ on each fiber.

Here E is called total space, M the base space, and the space E_p the fiber above p of the vector bundle. We often say E is a vector bundle over M. This definition effectively says that the total space is made of fibres which are vector spaces and locally the space E looks like $U \times \mathbb{R}^r$.

Let us define the manifold TM made up of tangent spaces, called the tangent bundle of M

$$TM := \{(p, v) | p \in M_n, v \in T_pM\} \tag{1.2.1}$$

We still need to define the manifold structure and differentiable structure. Given any point $\tilde{p} \in TM, \tilde{p} = (p, v)$. $p \in M_n$ so $\exists$ neighborhood U of p in M_n and a chart (U, ϕ). Define

$$\tilde{U} := \{(p, v) | p \in U, v \in T_pM\}. \tag{1.2.2}$$

Using the local coordinates from the chart, we can prove that $\tilde{U}$ is homeomorphic to $\phi(U) \times \mathbb{R}^n$. More precisely we give TM the topology such that these maps are homeomorphic. We define TU to be homeomorphic to $\phi(U) \times \mathbb{R}^n$ using the map

$$\Phi_U : (p, v) \to (\phi(p), v_1, v_2, ...), \tag{1.2.3}$$

where $v = \Sigma_i v_i \frac{\partial}{\partial x_i}$. We give TM the topology generated by TU as U runs over all coordinate open subsets[2] of M. The transitions maps corresponding to these charts are smooth[3] . Since the charts U cover M_n, charts $\tilde{U}$ cover TM and hence form a smooth atlas. This makes TM a smooth manifold. The projection map $\Pi : (p, v) \to p$ and Φ_U defined previously satisfy the conditions (1) and (2) in the definition of a vector bundle.

[2]that is the open sets U which have a corresponding chart (U, ϕ)

[3]assuming M_n is a smooth manifold

Definition 1.10. The vector bundle $\Pi : TM \to M$ is called the tangent bundle.

Since T_pM is a vector space, we naturally have its dual, the cotangent space denoted by T_p^*M, and tensor spaces $T_l^k(T_pM) = \otimes^k T_p^*M \otimes^l T_pM$. Using these spaces, we can define the cotangent bundle T^*M and tensor bundle T_l^kM in the same fashion as we defined the tangent bundle.

If $\Pi : E \to M$ is a vector bundle over M , a section of E is a map $F : M \to E$ such that $\Pi \circ F = Id_M$, in other words $F(p) \in E_p$ for all p. It is said to be a smooth section if it is smooth as a map between manifolds. We denote the space of all smooth sections of vector bundle by $\Gamma(E)$.

A smooth vector (tensor) field is defined as a smooth section of the tangent (tensor) bundle. We denote the space of all smooth vector fields by $\mathcal{T}(M)$ and the space of all (k,l) tensor fields which is a smooth section of T_l^kM by $\mathcal{T}_l^k(M)$. In addition, we denote the covariant k-tensor fields (i.e., smooth sections of $T_0^kM = T^kM$) by $\mathcal{T}^k(M)$.

Definition 1.11. Let $F : W_d \to M_n$. we define pull-back associated with F, denoted by F^*, as the dual map associated to the push-forward map. So $F^* : T^*_{F(p)}M \to T_p^*W$

1.3 Riemannian Metric

In the thesis, we will follow the Einstein summation convention. As per this convention, if an index occurs in a term, we will sum over that index. For example, if e_i forms a basis of tangent space at a point and v is a tangent there. Then

$$v = v^i e_i = \sum_{i=1}^{n} v^i e_i \tag{1.3.1}$$

Definition 1.12. A Riemannian metric on a smooth manifold M_n is a 2-tensor field $g \in \mathcal{T}^2(M)$, which is symmetric and positive definite.

Therefore given any $X, Y \in T_pM$ we have :

- $g(X,Y) = g(Y,X)$ and

- $g(X, X) > 0$, if $X \neq 0$

We would also use notation $g(X, Y) = \langle X, Y \rangle$ when the choice of the metric is clear from the context.

The metric is uniquely determined in local coordinates if we know $\langle \partial_i, \partial_j \rangle = g_{ij}$. So when defining a metric, it is sufficient to give g_{ij} in local coordinates.

The standard metric on $\mathbb{R}^n$ is given by

$$g = \delta_{ij} \tag{1.3.2}$$

We can prove that on a smooth manifold, there always exists a Riemannian metric. We first take the coordinate open sets U_i from chart (U_i, ϕ_i) and define a metric on U_i induced by the metric[4] on $\phi_i(U_i)$ and extend it to M_n using partitions of unity.

Definition 1.13. A smooth manifold M_n with a Riemannian metric g defined on it is called a riemannian manifold (M_n, g).

Given a metric g, let $g_{ij} = \langle \partial_i, \partial_j \rangle$ be the components of the metric matrix. Since g is symmetric we have $g_{ij} = g_{ji}$. Then we define g^{ij} to be the components of the inverse matrix of the metric matrix. Therefore $g_{ij}g^{jk} = \delta_i^k$. And similar to components of metric matrix we have $g^{ij} = g^{ji}$

Definition 1.14. Given a riemannian manifold (M_n, g) change of metric on M_n to $\tilde{g}$ is called conformal if $\tilde{g} = e^f g$ for some smooth function f.

Previously we defined the velocity of a curve on a differentiable manifold. Now on a Riemannian manifold, because of the metric, we can define the speed of the at a point.

Definition 1.15. Let $\gamma : [a, b] \to M_n$ be a curve. Speed of the curve at t_0 is defined as $\|\dot{\gamma}(t_0)\| = \sqrt{\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0) \rangle}$

Definition 1.16. Given a curve $\gamma : [a, b] \to M_n$, length of the curve is defined as $L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt$.

[4] $\phi_i(U_i)$ being subset of $\mathbb{R}^n$ has a natural metric

Definition 1.17. Given $p, q \in M_n$, we define $d(p,q) = \inf(L(\gamma))$ over all differentiable curves γ from p to q.

The metric defined by the above function over M_n makes M_n into a metric space, and the metric topology agrees with the original topology over the manifold. We shall always assume the M_n is complete as a metric space. Such Riemannian manifolds are called complete manifolds.

1.4 Connection and Covariant derivative

Definition 1.18. Let $\Pi : E \to M$ be a smooth vector bundle. A connection in E is a map

$$\nabla : \mathcal{T}(M) \times \Gamma(E) \to \Gamma(E) \tag{1.4.1}$$

written $(X, Y) \to \nabla_X Y$ satisfying the following properties :

1. $\nabla_X Y$ is $C^\infty(M_n)$-linear in X i.e.,
   $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for $f, g \in C^\infty(M_n)$
2. $\nabla_X Y$ is $\mathbb{R}$-linear in Y i.e.,
   $\nabla_X aY_1 + bY_2 = a\nabla_X Y_1 + b\nabla_X Y_2$ for $a, b \in \mathbb{R}$
3. ∇ satisfies the following product rule :
   $\nabla_X fY = X(f)Y + f\nabla_X(Y)$ for $f \in C^\infty(M_n)$

and $\nabla_X Y$ is called covariant derivative of Y in the direction of X.

As a special case of the notion of a connection, we have linear(affine) connection over a manifold $\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ satisfying conditions (1), (2), and (3).

Definition 1.19. The n^3 functions Γ^k_{ij} defined the following way :

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \tag{1.4.2}$$

are called the Christoffel symbols of ∇ w.r.t. the given local coordinates

If $X = X^i\partial_i$ and $Y = Y^i\partial_i$then $\nabla_X Y$ in terms of Christoffel symbols is

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij})\partial_k \tag{1.4.3}$$

A smooth vector field over curve $\gamma : [a, b] \to M_n$ is a smooth map from $[a, b]$ to TM. Let us denote the space of all smooth vector fields over curve γ by $\mathcal{T}(\gamma)$. A smooth vector field is called extensible if it is a restriction of a smooth vector field defined in a neighborhood the curve. We can now define covariant derivative of a vector field along a curve.

Definition 1.20. Given curve $\gamma : [a.b] \to M_n$ an let operator $D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ be satisfying the following properties:

1. It is $\mathbb{R}$-linear , i.e. given $V, W \in \mathcal{T}(\gamma)$ and $a, b \in \mathbb{R}$

$$D_t(aV + bW) = aD_t(V) + bD_t(W) \tag{1.4.4}$$

2. It satisfies the following product rule. Given any $f \in C^\infty([a, b]$ and $V \in \mathcal{T}(\gamma)$

$$D_t(fV) = \dot{f}V + fD_t(V) \tag{1.4.5}$$

3. If V is extensible to $\tilde{V}$, then $D_t(V) = \nabla_{\dot{\gamma}(t)}\tilde{V}$

For any $V \in \mathcal{T}(\gamma)$, $D_t(V)$ is then called the covariant derivative of V along curve γ. Given a linear connection on M_n, it is easy to prove that D_t exists and is unique. So covariant derivative along a curve is well-defined.

We have previously defined the velocity of a curve, now using covariant derivative, we can define acceleration.

Definition 1.21. Given curve $\gamma : [a, b] \to M_n$, acceleration of the curve , denoted by $\ddot{\gamma}(t)$, is defined as $\ddot{\gamma}(t) = D_t(\dot{\gamma}(t))$, i.e., the covariant derivative of the velocity vector field.

Previously we defined a connection on a manifold, noting that we could differentiate vector fields on $\mathbb{R}^n$. Along the same line, we can also differentiate tensor fields on $\mathbb{R}^n$. So now we would like to extend the linear connection to tensor fields on manifolds.

Definition 1.22. We can extend the notion of linear connection/covariant derivative w.r.t. to a vector field to tensor fields in the following way :

1. For $f \in \mathcal{T}^0(M)$, $\nabla_X(f) = X(f)$

2. ∇_X preserves the type of tensor field.

3. ∇_X commutes with contraction w.r.t. any pair of indices

4. $\nabla_X(F \otimes G) = \nabla_X F \otimes G + F \otimes \nabla_X G$

It can be proven that given a linear connection on a manifold, there exists a unique linear connection on tensor fields such that it agrees with linear connection on the manifold.

Definition 1.23. We can define total covariant derivative[5] of a tensor. Given (k, l)-tensor field F, we define a $(k+1, l)$ tensor field ∇F called total covariant derivative defined by

$$\nabla F(Y_1, \ldots, Y_k, X, \omega^1, \ldots, \omega^l) = \nabla_X F(Y_1, \ldots, Y_k, \omega^1, \ldots, \omega^l) \tag{1.4.6}$$

Definition 1.24. A linear connection is said to be compatible with a metric or a metric connection if for any given $X, Y, Z \in \mathcal{T}(M)$, we have

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{1.4.7}$$

This equivalent to the condition that $\nabla_X g \equiv 0$ for any $X \in \mathcal{T}(M)$.

Definition 1.25. The torsion of a connection is defined as the map $T : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ such that [6]

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{1.4.8}$$

A connection is called torsion-free if the torsion of the connection is zero.

[5]Sometimes it is simply called the covariant derivative of a tensor

[6]$[X, Y] = XY - YX \in \mathcal{T}(M)$ called the Lie bracket. So that for any $f \in C^\infty(M_n)$, $[X, Y](f) = X(Yf) - Y(Xf)$

Suppose a connection is torsion-free. Then $T(\partial_i, \partial_j) = 0$. Partial derivatives commute, so $[\partial_i, \partial_j] = 0$, and by definition of Christoffel symbols, we have,

$$T(\partial_i, \partial_j) = \nabla_i \partial_j - \nabla_j \partial_i = (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k = 0 \tag{1.4.9}$$

so

$$\Gamma_{ij}^k = \Gamma_{ji}^k \tag{1.4.10}$$

We can also prove this is a sufficient condition for a connection to be torsion-free by expressing the torsion in local coordinates and using the symmetry of Christoffel symbols in i, j. We would get that the torsion is zero given any vector fields X and Y.

Definition 1.26. A connection on a Riemannian manifold is called a Riemannian connection or Levi-Civita connection if it is torsion-free and a metric connection.

We can prove that there is a unique Levi-Civita connection with the Christoffel symbols:

$$\Gamma_{ij}^l = \frac{1}{2}[\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}]g^{kl} \tag{1.4.11}$$

As an example, let us consider $\mathbb{R}^n$ with the standard metric and the connection D, which we will call Euclidean connection, defined on it. Here

$$D_{\partial_i}\partial_j = 0 = \Gamma_{ij}^k \tag{1.4.12}$$

So the connection is torsion-free. $g_{ij} = \delta_{ij}$ is constant over $\mathbb{R}^n$, hence $\nabla_X g \equiv 0$ for any $X \in \mathcal{T}(M)$. So the connection is compatible with metric. Therefore the Euclidean connection on $\mathbb{R}^n$ is a Levi-Civita connection.

1.5 Curvature

Definition 1.27. The curvature endomorphism w.r.t. a connection is a map $R : \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.5.1}$$

The curvature endomorphism is $C^\infty(M_n)$-multilinear and hence a $(3,1)$ tensor field called the curvature tensor. The components R^l_{ijk} in local coordinate system are given by

$$R(\partial_i, \partial_j)\partial_k = R^l_{kij}\partial_l \tag{1.5.2}$$

In local coordinates since ∂_i and ∂_j commute we have $[\partial_i, \partial_j] = 0$. Therefore,

$$R(\partial_i, \partial_j)\partial_k = \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k \tag{1.5.3}$$

Hence we have

$$R^l_{kij} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm} \tag{1.5.4}$$

Now if we take Euclidean connection, we have $\Gamma^k_{ij} = 0$, so $R^l_{kij} = 0$. We conclude that $\mathbb{R}^n$ has zero curvature tensor and consequently $R(X,Y)Z = 0$ for any $X, Y, Z \in \mathcal{T}(\mathbb{R}^n)$.

Definition 1.28. Let M_2 be a 2-dimensional Riemannian manifold with the Riemannian connection, the Gaussian curvature is defined to be $K = \langle R(X,Y)Y, X\rangle$.

Definition 1.29. From the curvature endomorphism, we can also define 4-tensor field defined by $R(X,Y,Z,T) = \langle R(Z,T)Y, X\rangle$. In local coordinates it is

$$R_{lkij} = g_{lm}R^m_{kij} \tag{1.5.5}$$

So we have $K = R(X,Y,X,Y)$

As you may have noticed, Gaussian curvature is defined only for 2-dimensional manifolds. As a generalization for higher dimensional manifolds we have sectional curvature.

Definition 1.30. If $X, Y \in T_pM$ such that they are orthonormal, the sectional curvature of the 2-dimensional subspace of T_pM spanned by X and Y is defined as $\sigma(X,Y) = R(X,Y,X,Y)$

Tensors with 4 indices are too difficult to work with. So we have defined several other curvatures which are easier to work with. We will define a few of them here.

Definition 1.31. Ricci(Ric) tensor/curvature is 2-tensor defined as the contraction of the $(3,1)$ Riemann curvature tensor. In terms of local coordinates we have

$$R_{ij} := R^k_{ikj} \tag{1.5.6}$$

Definition 1.32. Scalar curvature is defined as the trace of the Ricci tensor. In local coordinates, we have

$$S = g^{ij} R_{ij} \tag{1.5.7}$$

Definition 1.33. The Weyl tensor is defined on coordinate open sets as

$$W_{ijkl} = R_{ijkl} - \frac{1}{(n-2)}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) + \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il}) \tag{1.5.8}$$

It can be proven that the Weyl curvature is invariant under conformal maps, and it can be shown that this is the conformally invariant part of the curvature endomorphism. So it is clear that a locally conformally flat manifold will have Weyl curvature zero everywhere. The Converse is also true; if Weyl curvature is zero everywhere on the manifold, then the manifold is locally conformally flat.

1.6 Integration over Riemannian Manifolds

Denote $\Lambda^k(T_pM)$ to be the subspace of alternating tensors in $T^k(T_pM)$. Therefore we have $\omega(..., v_i, ..., v_j, ...) = -\omega(..., v_j, ..., v_i, ...)$ for any $\omega \in \Lambda^k(T_pM)$. We call it the space of exterior k-forms. Wedge product over $\Lambda^k(T_pM)$ is defined by the expression

$$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_k(v_1, .., v_k) = \det((\omega_i(v_j))) \tag{1.6.1}$$

Let $\Lambda^k(M)$ denote the vector bundle of exterior k-forms over M_n. We will call a smooth section of $\Lambda^k(M)$ a differential k-forms and the space of differential k-forms by $\Omega^k(M)$.

We now define the operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$. For $f \in \Omega^0(M) = \mathcal{T}^0(M)$ we have $df \in \Omega^1(M) = \mathcal{T}^1(M)$ such that $df(X) = X(f)$ for $X \in \mathcal{T}(M)$.We will define it using its action in local coordinates. For $\omega \in \Omega^k(M)$ its expression in local coordinates is

$$\omega = \sum_{i_1<...<i_k} \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{1.6.2}$$

$$d\omega = \sum_{i_1<...<i_k} d\omega_{i_1,...,i_k} \wedge dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{1.6.3}$$

$d\omega$ is called the differential of ω. The differential operator d has the following properties:

1. $d(\omega + \eta) = d\omega + d\eta$
2. If ω is a differential k-form and η is an differential l-form, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3. $d(d\omega) = 0$
4. $f^*(d\omega) = d(f^*\omega)$

Let E^n define the lower half space of $\mathbb{R}^n$ i.e. $E^n = \{(x^1, ..., x^n)|x^1 < 0\}$. And $\bar{E}^n$ its closure. We will now define a manifold with boundary in an analogous fashion to a manifold.

Definition 1.34. A Manifold with boundary M_n is a second countable, Hausdorff topological space such that for any given point p in M_n, there exists a neighborhood U of p which is homeomorphic to an open subset of $\bar{E}^n$.

The set points in M_n which have neighborhood homeomorphic to open set in $\mathbb{R}^n$ is called interior of the manifold. The rest of the points are boundary of the manifold denoted by ∂M

Just like on a manifold, we can define a differentiable structure, tangent space etc., on a manifold with boundary. So we will not elaborate on that.

Theorem 1.1. *If M_n is an oriented (smooth) n-manifold with with boundary and ∂M is non empty, then ∂M is a (smooth) $n-1$-manifold without boundary and with natural orientation induced from M_n*

We will first define integration of differential n-form $\mathbb{R}^n$ and then extend it to manifolds.

Suppose ω is a compactly supported differential n-form on $\mathbb{R}^n$. And U is an open set that contains the support of the differential form. Suppose $\omega = f dx^1 \wedge ... \wedge dx^2$.

$$\int_U \omega := \int_U f \tag{1.6.4}$$

That is, we are defining the integration of a differentiable n-form to be the integration of a function which we know from standard integration theory on $\mathbb{R}^n$.

Definition 1.35. Suppose ω is compactly supported differential n-form on an oriented differentiable(smooth) manifold M_n. And $(U_i, \phi_i)_{i\in I}$ is an atlas over $M_n$that is compatible with the orientation. $(\alpha_i)_{i\in I}$ be the partition of unity subordinate to the open sets U_i. And $\omega = f_i dx^1 \wedge ... \wedge dx^n$ w.r.t local coordinates over U_i. Then

$$\int_M \omega := \sum_{i\in I} \int_{\phi(U_i)} (f_i\alpha_i) \circ \phi_i^{-1} dx^1 \wedge ... \wedge dx^n \tag{1.6.5}$$

We can prove that the integral does not depend on the choice of atlas or particular partition of unity.

Let $i : \partial M \hookrightarrow M_n$ be the natural inclusion map. If ω be a differential $n-1$-form on M_n. Then $i^*\omega$ is a differential $n-1$-form over ∂M, we will identify $i^*\omega$ with ω for the following theorem.

Theorem 1.2 (Stoke's Theorem)**.** *Let M_n be an oriented differentiable manifold with bound- -ary and ω a differential $n-1$-form on it. Let ∂M be oriented manifold with natural orientation induced from M_n . Then*

$$\int_M d\omega = \int_{\partial M} \omega \tag{1.6.6}$$

Definition 1.36. Let M_n be an oriented Riemannian manifold. $\mathcal{A}$ an atlas compatible with the orientation. Let $(x^1, ..., x^n)$ be local coordinates w.r.t some chart in the atlas. We define a differential n-form $d\eta$ over M_n. In local coordinates, it has the following expression

$$d\eta = \sqrt{|g|} dx^1 \wedge ... \wedge dx^n \tag{1.6.7}$$

where $|g| = |det(g_{ij})|$. We will call this the Riemannian volume form.

We can check that $d\eta$ is well defined by checking that the volume form gives the same differential form even in different local coordinates in the same neighborhood.

In $\mathbb{R}^n$ gradient of a function f is vector field such that $\langle \text{grad} f, v_p \rangle = \frac{d}{dv} f|_p$. This we recognize as $v(f)$, where $v \in T_p\mathbb{R}^n$. We generalize this notion to manifolds.

Definition 1.37. For any $f \in \mathcal{T}^0(M)$, grad$f \in \mathcal{T}(M)$, such that $\langle \text{grad} f, X \rangle = df(X) = X(f)$

In local coordinates $\text{grad} f = g^{ij}(\partial_i f)\partial_j$

We know that on $\mathbb{R}^n$, $\text{grad}\, f = \nabla f$. On a manifold from the definition we used, ∇f denotes the covariant derivative[7] of f, which is a 1-form. But when the context is clear that ∇f is being used as vector fields instead of a form, we will use $\nabla f = \text{grad}\, f$.

We define interior multiplication of a differential k-form ω by X .This is denoted by $i_X\omega$, where $i_X\omega$ is a differential $k-1$-form defined by $i_X\omega(V_1, ..., V_{k-1}) = \omega(X, V_1, ..., V_{k_1})$

Definition 1.38. For $X \in \mathcal{T}(M)$, div$X \in \mathcal{T}^0(M)$, defined by $d(i_X d\eta) = \text{div} X d\eta$. We call it divergence of X.

$$i_X d\eta(V_1, ..., V_{n-1}) = \sqrt{|g|} dx^1 \wedge ... \wedge dx^n (X, V_1, .., V_{n-1}) \tag{1.6.8}$$

$$= \sqrt{|g|}(-1)^{i-1} X^i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n (V_1, ..., V_{n-1}) \tag{1.6.9}$$

Here $\widehat{}$ means we are ignoring that term in the wedge product. So we have

$$i_X d\eta = \sqrt{|g|}(-1)^{i-1} X^i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \tag{1.6.10}$$

Operating d to the above equation we get that $\text{div} X = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} X^i) = \partial_i X^i + X^l \Gamma^i_{li}$.

[7] which is also the exterior derivative

We will now define Laplacian on manifolds. On $\mathbb{R}^n$, $\Delta f = -\mathrm{div}(\mathrm{grad}(f))$(or just $\mathrm{div}(\mathrm{grad}(f))$ depending on convention). We will generalize this to manifolds.

Definition 1.39. Let f be a smooth function on M_n. $\Delta f := -\mathrm{div}(\mathrm{grad}(f))$. In local coordinates

$$\Delta f = -\frac{1}{\sqrt{|g|}}\partial_j(\sqrt{|g|}g^{ij}\partial_i f) \tag{1.6.11}$$

Let f be a compactly supported continuous function. We define integral of f over M_n as

$$\int_M f dV = \int_M f d\eta \tag{1.6.12}$$

Once we have defined the integral for compactly supported continuous functions, we can extend this to all functions and define a Lesbegue integral over the Riemannian manifold.

Chapter 2

Sobolev Spaces on $\mathbb{R}^n$ and closed Riemannian manifolds

2.1 Weak Derivatives and Sobolev Spaces on $\mathbb{R}^n$

Definition 2.1. Let $u, v \in L^1_{loc}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. v is called α^{th}-weak derivative of u if for every $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx \tag{2.1.1}$$

We denote α^{th}-weak derivative of u as $D^\alpha u = v$. Here α is a multi-index. So $\alpha = (\alpha_1, \ldots, \alpha_n)$.

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n} \tag{2.1.2}$$

We define Sobolev spaces over domains on $\mathbb{R}^n$.

Definition 2.2. $W^{k,p}(\Omega) = \{u : \Omega \to \mathbb{R} | u \in L^p(\Omega)$ and if the α^{th}-weak derivative $D^\alpha u \in L^p(\Omega)$ for all α such that $0 \leq |\alpha| \leq k\}$

Definition 2.3. We define a norm on $W^{k,p}(\Omega)$. Let $\phi \in W^{k,p}(\Omega)$

$$\|\phi\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \le k} \|D^\alpha \phi\|_{L^p(\Omega)} \quad 1 \le p \le \infty \tag{2.1.3}$$

Sobolev Space $W^{k.p}(\Omega)$ is a Banach space w.r.t. this norm.

Definition 2.4. $H^{k,p}(\Omega)$:= Closure of $\{u \in C^k(\Omega) | \|u\|_{W^{k,p}(\Omega)} < \infty\}$ w.r.t. the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ in $W^{k,p}(\Omega)$, when $1 \le p < \infty$.

According to a theorem of Meyers and Serrin we have $H^{k,p}(\Omega) = W^{k,p}(\Omega)$, for any open set $\Omega \in \mathbb{R}^n$.

Just like we defined $W^{k,p}(\Omega)$ and $H^{k,p}(\Omega)$ we can define $W_0^{k,p}(\Omega)$ or $H_0^{k,p}(\Omega)$.

Definition 2.5. $W_0^{k,p}(\Omega) = H_0^{k,p}(\Omega)$:= Closure of $C_c^k(\Omega)$ w.r.t the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ in $W^{k,p}(\Omega)$ when $1 \le p < \infty$. $W_0^{k,\infty} := W^{k,\infty}(\Omega) \cap W_0^{k,1}(\Omega)$ when $p = \infty$.

We will now look at some essential properties of Sobolev spaces.

1. $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are Banach spaces.

2. $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$.

Theorem 2.1 (Extension theorem)**.** *Assume Ω is bounded and $\partial\Omega$ is C^1. Select a bounded open set V such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n) \tag{2.1.4}$$

such that for each $u \in W^{1,p}(\Omega)$:

1. $Eu = u$ a.e on Ω

2. Eu has support in V

3. $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C(p, U, V)\|u\|_{W^{1,p}(\Omega)}$

It is clear that $W^{1,p}(\Omega)$ is a subspace of $L^p(\Omega)$. We would like to know more about this embedding. Let x and Y be two Banach spaces, such that $X \subset Y$. The inclusion map $i : X \hookrightarrow Y$ is continuous if and only if there exists a constant C such that for all $x \in X$, we have

$$\|x\|_Y \leq C\|x\|_X. \tag{2.1.5}$$

If this is true, we say that X is continuously embedded in Y. Furthermore, i is compact if each bounded sequence in X is precompact in Y. Then we say that X is compactly embedded in Y. We denote it by $X \subset\subset Y$.

Theorem 2.2 (Gagliardo-Nirenberg-Sobolev Inequality)**.** *Assume $1 \leq p < n$. There exists $C(p, n)$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p, n)\|Du\|_{L^p(\mathbb{R}^n)} \tag{2.1.6}$$

where $1/p^* = 1/p - 1/n$, for all $C_c^\infty(\mathbb{R}^n)$.

Using the density of $C_c^\infty(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$ this inequality essentially establishes that the embedding $W^{1,p}(\mathbb{R}^n)$ into $L^{p^*}(\mathbb{R}^n)$ is continuous. Also, now since $W^{1,p}$ is naturally continuously embedded in L^p, using the interpolation theorem gives the following.

Theorem 2.3 (Sobolev Embedding on $\mathbb{R}^n$)**.** *Assume $1 \leq p < n$ and $u \in W^{1,p}(\mathbb{R}^n)$. For $q \in [p, p^*]$ we have*

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{2.1.7}$$

The extension theorem can now be used to prove the Sobolev embedding for bounded domains.

Theorem 2.4 (Sobolev Embedding on bounded domains)**.** *Let Ω be a C^1 bounded domain of $\mathbb{R}^n$. Assume $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$. For $q \in [1, p^*]$ we have*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \tag{2.1.8}$$

In the case of $W_0^{1,p}$, there is a natural extension by zero. And this extension is continuous, so we have a special Sobolev type inequality for functions in $W_0^{1,p}(\Omega)$.

Theorem 2.5 (Poincaré- Sobolev Inequality)**.** *Let Ω be a bounded domain of $\mathbb{R}^n$. Assume $1 \leq p < n$ and $u \in W_0^{1,p}(\Omega)$. For $q \in [1, p^*]$ we have*

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \tag{2.1.9}$$

The following can be established by a slight variation in the proof of Gagliardo-Nirenberg inequality.

Theorem 2.6 (Sobolev embedding for the case $p = n$ on $\mathbb{R}^n$)**.** *Assume $p = n$ and $u \in W^{1,p}(\mathbb{R}^n)$. For $q \in [n, \infty)$ we have*

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{W^{1,n}(\mathbb{R}^n)} \tag{2.1.10}$$

Theorem 2.7 (Sobolev Embedding on bounded domains for the case $p = n$)**.** *Let Ω be a C^1 bounded domain of $\mathbb{R}^n$. Assume $p = n$ and $u \in W^{1,p}(\Omega)$. For $q \in [1, \infty)$ we have*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,n}(\Omega)} \tag{2.1.11}$$

We recall that the norm on Hölder space($C^{k,\gamma}(\bar{\Omega})$) is given by

$$\|u\|_{C^{k,\gamma}(\bar{\Omega})} = \sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{C(\bar{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_{C(\bar{\Omega})} \tag{2.1.12}$$

where

$$[u]_{C(\bar{\Omega})} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma} \tag{2.1.13}$$

Remark 2.1.1. *The definition of Hölder spaces changes slightly for Riemannian manifolds, where the Riemannian distance* $\mathrm{dist}(x, y)$ *replaces* $|x - y|$

Theorem 2.8 (Morrey's Inequality)**.** *Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{2.1.14}$$

for all $u \in C^1(\mathbb{R}^n)$, where $\gamma = 1 - n/p$.

Theorem 2.9 (Sobolev Embedding for bounded domains, $p > n$)**.** *Let Ω be a bounded domain of $\mathbb{R}^n$, and suppose $\partial\Omega$ is C^1. Assume $n < p \leq \infty$, and $u \in W^{1,p}(\Omega)$. Then for $\gamma = 1 - n/p$ we have*

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}. \tag{2.1.15}$$

In fact, for $0 \leq \alpha \leq \gamma$, $W^{1,p}(\Omega)$ continuously embeds into $C^{0,\alpha}(\bar{\Omega})$.

The General Sobolev inequalities follow from the above specific inequalities. The idea is to use Gagliardo-Nirenberg inequality or Morrey's inequality on high enough derivatives of u and drop the regularity by one.

Theorem 2.10 (General Sobolev Inequalities)**.** *Let Ω be a bounded domain of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(\Omega)$.*

1. *If $k < n/p$, then $u \in L^q(\Omega)$, where $1/q = 1/p - k/n$ and*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,q}(\Omega)} \tag{2.1.16}$$

2. *If $k > n/p$, then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{\Omega})$, where*

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, \text{ if } n/p \text{ is not an integer} \\ \text{any positive number} < 1, \text{ if } n/p \text{ is an integer} \end{cases} \tag{2.1.17}$$

   and

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{\Omega})} \leq C\|u\|_{W^{k,p}(\bar{\Omega})}. \tag{2.1.18}$$

Theorem 2.11 (Rellich-Kondrakov Compactness Theorem)**.** *Assume Ω is a bounded domain in $\mathbb{R}^n$ with a C^1 boundary. Suppose*

1. *$1 \leq p < n$, then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \tag{2.1.19}$$

   for each $q \in [1, p^*)$.

2. *$p = n$, then*

$$W^{1,n}(\Omega) \subset\subset L^q(\Omega) \tag{2.1.20}$$

   for each $q \in [1, \infty)$

3. *$n < p$, then*

$$W^{1,p}(\Omega) \subset\subset C^{0,\alpha}(\bar{\Omega}) \tag{2.1.21}$$

   for each $\alpha \in [0, 1 - n/p)$

$W^{1,p}(\Omega)$ is not compactly embedded in the limiting space in all the above three cases. That is, the continuous embeddings

1. $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ when $p < n$,

2. $W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ when $p = n$,

3. $W^{1,p}(\Omega) \hookrightarrow C^{0,1-n/p}(\bar{\Omega})$ when $p > n$

are not compact.

We will give the standard counterexaple for this. Consider a non zero smooth function u compactly supported in unit ball B centered at origin. Let $u_\lambda(x) = \lambda^{\frac{n-p}{p}} u(\lambda x)$. This sequence of functions can be used to prove lack of compactness in all three cases. We will discuss how this sequence contradicts compactness in the first case.

We have $\|u\|_{L^p(B)} = \|u_\lambda\|_{L^{p^*}(B)}$. But as $\lambda \to \infty$ we can check that $\|u_\lambda\|_{L^p(B)} \to 0$ and $\|u_\lambda\|_{W^{1,p}(B)}$ is uniformly bounded. This should imply that there is subsequence converging to zero if the embedding is infact compact. But as we know $\|u\|_{L^p(B)} = \|u_\lambda\|_{L^{p^*}(B)} > 0$, the limit cannot be zero. This contradiction implies the lack of compactness.

2.2 Sobolev Spaces on Riemannian manifold

The above theorems on $\mathbb{R}^n$ and bounded domains of $\mathbb{R}^n$ have appropriate counterparts for Sobolev spaces on compact Riemannian manifolds. But first, we need to define Sobolev spaces on Riemannian manifolds.

Definition 2.6. Let (M_n, g) be a smooth Riemannian manifold and $\phi \in C^k(M_n)$, where integer $k \geq 0$. We define

$$|\nabla^k u|^2 = \nabla^{\alpha_1}\nabla^{\alpha_2}\dots\nabla^{\alpha_k} u \nabla_{\alpha_1}\nabla_{\alpha_2}\dots\nabla_{\alpha_k} u \tag{2.2.1}$$

Notice that α_i are multi indices of order 1, and we are following the Einstein summation convention.

In particular $|\nabla^0 u|^2 = |u|^2, |\nabla^1 u|^2 = |\nabla u|^2 = \nabla^\nu u \nabla_\nu u$.

$\nabla^k u$ will mean any kth (total) covariant derivative of u.

Notice that on $\mathbb{R}^n$

$$|\nabla^k u|^2 = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} |D^\alpha u|^2 \tag{2.2.2}$$

We define $\mathcal{C}^{k,p}(M_n) := \{u \in C^\infty(M_n) | |\nabla^l \phi| \in L^p(M_n)$ for integers l,k such that $0 \leq l \leq k$ and real $p \geq 1\}$

We now define Sobolev spaces on Riemannian manifolds as completion of the above space like we have Sobolev of spaces on open subsets of $\mathbb{R}^n$ as the closure of certain subspace of $C^k(\Omega)$.

Definition 2.7. The Sobolev Spaces $W^{k,p}(M_n) :=$ Completion of $\mathcal{C}^{k,p}(M_n)$ w.r.t. norm

$$\|u\|_{H^{k,p}} = \sum_{l=0}^{k} \|\nabla^l u\|_{L^p(M_n)} \tag{2.2.3}$$

$W_0^{k,p}(M_n)$ is defined similarly as the completion of $C_c^\infty(M_n)$ w.r.t. the above norm.

We can see $W^{k,p}(M_n)$ as subspaces of $L^p(M_n)$. Notice that if (u_n) is a Cauchy sequence in $\mathcal{C}^{k,p}(M_n)$, then it is also a Cauchy sequence in $L^p(M_n)$. So we can define $W^{k,p}(M_n)$ as completion of $\mathcal{C}^{k,p}(M_n)$ in $L^p(M_n)$. Suppose $u_n \to u \in W^{k,p}(M_n)$. We can define the norm of u to agree with the above norm by defining $|\nabla^l u| = \lim_{n\to\infty} |\nabla^l u_n|$. Now we can define $\|u\|_{W^{k,p}}$ the same way. We can check $W^{k,p}(M_n)$ is Banach space.

The general idea for proving these theorems on compact manifolds is to prove the theorems on coordinate charts using the theorems on $\Omega \in \mathbb{R}^n$ and combine them using partitions of unity. Or, more specifically, given any smooth function, we split it using partitions of unity and establish continuity or compactness for these partitioned functions. And then prove this extends for the entire function. Morrey's inequality has a slightly different proof, but it still can be established by using Morrey's inequality on $\Omega \in \mathbb{R}^n$.

We have proved the continuity of Sobolev embedding and established a Sobolev inequality. But the inequality doesn't have the optimal constant of inequality. This optimal inequality turns out to be central to the Yamabe problem. For $\mathbb{R}^n$, Talenti already obtained the value of the optimal constant and the extremizers. Aubin then proved that the same constant works as the optimal cosntant for not just $\mathbb{R}^n$ but all compact Riemannian manifolds. Here we will give Aubin's proof for the optimal constant on $\mathbb{R}^n$ and $\mathbb{S}^n$.

2.2.1 Aubin's Theorem

Theorem 2.12. *If $1 \leq p < n$, for all $u \in W^{1,p}(\mathbb{R}^n)$(same proof works for Sobolev spaces over $\mathbb{S}^n$ and $\mathbb{H}^n$)*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq K(n,p)\|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{2.2.4}$$

where $1/p^* = 1/p - 1/n$ and

$$K(n,p) = \frac{p-1}{n-p}\left[\frac{n-p}{n(p-1)}\right]^{1/p}\left[\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}\right]^{1/n} \tag{2.2.5}$$

for $1 < p < n$, and

$$K(n,1) = \frac{1}{n}\left[\frac{n}{\omega_{n-1}}\right]^{1/n}, \tag{2.2.6}$$

where $\Gamma(n)$ is the Gamma function and $\omega_n = \mathrm{vol}(\mathbb{S}^n)$.

$K(n,p)$ is the norm of embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$and it is by the functions of the form

$$u(x) = \left(\lambda + \|x\|^{p/(p-1)}\right)^{1-n/p} \tag{2.2.7}$$

where λ is a real number.

The proof involves broadly three steps

1. Approximating a bounded smooth functions by "nice" smooth functions with no dege- -nerate critical points.

2. Symmetrize these functions radially and prove that the problem reduces to proving the Sobolev inequality for these symmetrized functions

3. Sobolev inequality for these radially symmetric functions is essentially an inequality in functions of one variable. This is then proved using a lemma proved by Bliss.

The complete proof is available in Aubin[1]. We will now simply state the various propositions involved in the above three steps.

Proposition 2.2.1. *Let M_n be a Riemannian manifold. Given any bounded smooth functions $f : M_n \to \mathbb{R}$ and $\epsilon > 0$, there exists a smooth function g which has no degenerate critical* *points, such that $|f(x) - g(x)| < \epsilon$ for all $x \in M_n$. Furthermore, g can be chosen so that over any given a compact set K we have $|\nabla f - \nabla g| < \epsilon$.*

A point p is a critical point of a differentiable function f if $\nabla f(p) = 0$. A critical point is non-degenerate if the Hessian matrix $(\partial_i \partial_j f)$ at that point p is nonsingular. A nondegenerate critical point is isolated. This is obvious on $\mathbb{R}$. We have Hessian as the second derivative , which is non-zero at a critical point (where derivative = 0). Since the derivative of the derivative is non-zero, the critical point is isolated (derivative increases/decreases in the neighborhood of the critical point). We generalize this to higher dimensions. We have Morse lemma, which states that in the neighborhood of a nondegenerate critical point, there is a chart on which the function is of the form $f(x) = f(0) + x_1^2 + \ldots x_k^2 - x_{k+1}^2 - \ldots - x_n^2$. Using this, we can prove that the nondegenerate critical points are isolated.

Proposition 2.2.2. *Let $f \in C_c^\infty(M_n)$ such that $f \not\equiv 0$ and $\mathrm{supp} f = K$. There exists continuous functions (f_m) such that*

1. *$f_m \to f$ in $W^{1,p}(M_n)$*
2. *$\mathrm{supp} f_m = K_m \subset K$ and ∂K_m is $n-1$ dimensional submanifold of M_n*
3. *f_m is C^∞ on K_m*
4. *f_m has no degenerate critical points in K_m*

The proof of this proposition involves taking the approximating functions in the previous proposition and modifying them slightly so that the new functions approximate not just uniformly but in the Sobolev norm.

Proposition 2.2.3. *Let Σ be $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$. Consider a non-negative function $f \in C_c^\infty(\Sigma)$ with support K is such that ∂K is either empty or an $n-1$ dimensional submanifold. Also, suppose that f has only nondegenerate critical points on K. Choose $P \in \Sigma$. We will now define $g(r)$, a decreasing function on $[0, \infty)$. We define $g(r)$ such that*

$$\mu(\{Q | g[d(P,Q)] \geq a\}) = \mu(\{Q | f(Q) \geq a\}) = \psi(a). \tag{2.2.8}$$

So

$$g(r) = \sup\{a | \mu(B_r(P)) \geq \psi(a)\} \tag{2.2.9}$$

Then

$$\|\nabla g\|_{L^p(M_n)} \leq \|\nabla f\|_{L^p(M_n)} \quad \textit{for} \quad 1 \leq p < \infty \tag{2.2.10}$$

Proof. We perform Schwarz symmetrization on the function f. Because of how the function f is defined, f has finitely many critical points, and we can use the co-area formula.

So we can express the L^p norm of ∇f in terms of integrals on level sets. A clever application of Hölder's inequality and the isoperimetric inequality gives the required result. □

We can prove that g is Lipschitz continuous and hence absolutely continuous on $[0, \infty)$.

Proposition 2.2.4. *Let $g(r)$ be a decreasing function absolutely continuous on $[0, \infty)$, and equal to zero at infinity. Then :*

$$(\omega_{n-1})^{-1/n} \left(\int_0^\infty |g(r)|^{p^*} r^{n-1} dr \right)^{1/p^*} \leq K(n,p) \left(\int_0^\infty |g'(r)|^p r^{n-1} dr \right)^{1/p}, \tag{2.2.11}$$

where $K(n,p)$ is from Thm 2.12

The proof of this final proposition is a direct consequence of the following lemma by Bliss. An appropriate change of variables, i.e., $x = r^{(p-n)/(p-1)}$, immediately gives the above proposition.

Lemma 2.2.5 (Bliss)**.** *[2] Let p^*, p be constants such that $p^* > p > 1$, and let $f : [0, \infty) \to \mathbb{R}$, such that $f(x) \geq 0$ and*

$$J(f) = \int_0^\infty f^p dx \tag{2.2.12}$$

is given and finite. Then the integral

$$y(x) = \int_0^\infty f dx \tag{2.2.13}$$

is finite for all x and

$$I(f) = \int_0^\infty \frac{y^{p^*}}{x^{p^*-l}} dx \leq \tilde{K} \left(\int_0^\infty f^p dx \right)^{p^*/p} = \tilde{K} J^{p^*/p}, \tag{2.2.14}$$

where $p^*/n = l = p^*/p - 1$

$$\tilde{K} = \frac{1}{p^* - l - 1}\left[\frac{l\Gamma(p^*/l)}{\Gamma(1/l)\Gamma((p^*-1)/l)}\right]^l = \frac{n-p}{n(p-1)}\left[\frac{\Gamma(n)}{\Gamma(n/p)\Gamma(n-n/p+1)}\right]^{p^*/n} \tag{2.2.15}$$

and equality is attained for the function of the form

$$f = \frac{c}{(dx^l+1)^{(l+1)/l}} = \frac{c}{(dx^l+1)^{n/p}} \tag{2.2.16}$$

or

$$y = \frac{cx}{(dx^l+1)^{1/l}} \tag{2.2.17}$$

In solving the Yamabe problem, the constant $K(n,2)$ plays the central role.

Chapter 3

Regularity

The way we solve the Yamabe problem involves first proving the existence of a solution and then proving that this solution is smooth. In order to prove this smoothness, we need regularity theory. Regularity theory informs us about the regularity of a weak solution to a PDE. Being a weak solution already gives requires the solution to exist in some Sobolev space. For example a weak solution of $\Delta u = f$, where $f \in C_c^\infty$, belongs to $W^{1,2}(\Omega)$. But we can prove that this weak solution is, in fact, smooth.

3.1 L^2 Regularity

3.1.1 Caccioppoli inequality

Theorem 3.1 (Caccioppoli inequality)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution to $\Delta u = 0$ on Ω. That is*

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dx = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega) \tag{3.1.1}$$

Then for each $x_0 \in \Omega, 0 < \rho < R \leq dist(x_0, \partial\Omega)$ we have

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \leq \frac{c}{(R-\rho)^2} \int_{B_R(x_0) \setminus B_\rho(x_0)} |u - \lambda|^2 dx, \quad \forall \lambda \in \mathbb{R} \tag{3.1.2}$$

for some universal constant c.

The above inequality can be generalized for elliptic equations too.

Definition 3.1. Consider the partial differential equation over Ω

$$-\mathrm{div}(A\nabla(u)) = f - \mathrm{div}F, \tag{3.1.3}$$

where A is a linear operator over $\mathbb{R}^n$ for each $x \in \Omega$. This PDE is called uniformly elliptic if

$$\langle Av, v\rangle \geq \lambda \|v\|^2 \quad \forall v \in \mathbb{R}^n \tag{3.1.4}$$

for all $x \in \Omega$ and some constant $\lambda > 0$.

Theorem 3.2. *Let $u \in W^{1,2}(\Omega)$ be a weak solution to the uniformly elliptic equation*

$$-\mathrm{div}(A\nabla(u)) = f - \mathrm{div}F, \tag{3.1.5}$$

on Ω with $f, F \in L^2(\Omega)$ and $A \in L^\infty(\Omega)$. Then for any ball $B_R(x_0) \subset \Omega$ and $0 < \rho < R$ the following Caccioppoli inequality *holds:*

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \leq c\left\{\frac{1}{(R-\rho)^2}\int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\xi|^2 dx + R^2\int_{B_R(x_0)} f^2 dx + \int_{B_R(x_0)} |F|^2 dx\right\} \tag{3.1.6}$$

for any $\xi \in \mathbb{R}$ and some constant $c = c(\lambda, \Lambda)$. where

$$\langle Av, v\rangle \geq \lambda\|v\|^2 \quad \forall v \in \mathbb{R}^n \quad \textit{and} \quad \|A\|_{L^\infty(\Omega)} = \Lambda \tag{3.1.7}$$

Proof. The idea of the proof is the same as the proof for Caccioppoli inequality for harmonic functions. First we define a cut-off function $\eta \in C_c^\infty(\Omega)$ such that

1. $0 \leq \eta \leq 1$

2. $\eta \equiv 1$ on $B_\rho(x_0)$ and $\eta \equiv 0$ on $\Omega\setminus B_R(x_0)$

3. $|\nabla \eta| \leq \frac{2}{R-\rho}$

For now let us assume $f = 0$. We now consider the test function $\varphi = (u - \xi)\eta^2$. Using the fact that u is a weak solution and uniform ellipticity, we get

$$\begin{aligned} \lambda \int_{B_R(x_0)} \eta^2 |\nabla u|^2 dx \leq & - \int_{B_R(x_0)} 2\eta \langle A\nabla u, \nabla \eta \rangle (u - \xi) dx + \\ & \int_{B_R(x_0)} 2\langle F, \nabla \eta \rangle (u - \xi) + \int_{B_R(x_0)} \eta^2 \langle F, \nabla u \rangle dx \\ =: & (i) + (ii) + (iii) \end{aligned} \tag{3.1.8}$$

Now using the Young's inequality $2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$ and the properties of η we get

$$\begin{aligned} (i) &\leq \epsilon \Lambda \int_{B_R(x_0)} \eta^2 |\nabla u|^2 dx + \frac{4\Lambda}{\epsilon (R-\rho)^2} \int_{B_R(x_0) \setminus B_\rho(x_0)} |u - \xi|^2 dx \\ (ii) &\leq \frac{4}{(R-\rho)^2} \int_{B_R(x_0) \setminus B_\rho(x_0)} |u - \xi|^2 dx + \int_{B_R(x_0)} |F|^2 dx \\ (iii) &\leq \epsilon \Lambda \int_{B_R(x_0)} \eta^2 |\nabla u|^2 dx + \frac{1}{4\epsilon\Lambda} \int_{B_R(x_0)} |F|^2 dx \end{aligned} \tag{3.1.9}$$

Choosing $\epsilon = \frac{\lambda}{4\Lambda}$ and simplifying, we get the desired result. In case $f \neq 0$, translate the system to origin and consider the PDE:

$$-\Delta v = \bar{f} \quad \text{in } B_1(0) \tag{3.1.10}$$

$$v = 0 \quad \text{on } \partial B_1(0) \tag{3.1.11}$$

Let v be a weak solution of the PDE. Now using the weak formulation with v as test function and using Young's inequality and Poincaré inequality we get

$$\int_{B_1(0)} |\nabla v|^2 dx \leq \int_{B_1(0)} |\bar{f} v| dx \tag{3.1.12}$$

$$\leq C\epsilon \int_{B_1(0)} |\nabla v|^2 dx + \frac{1}{\epsilon} \int_{B_1(0)} |\bar{f}|^2 dx \tag{3.1.13}$$

So we have $\int_{B_1(0)} |\nabla v|^2 dx \leq C \int_{B_1(0)} |\bar{f}|^2 dx$ Now making a scaling argument we get

$$\int_{B_R(0)} |\nabla \bar{v}|^2 dx \leq CR^2 \int_{B_R(0)} |\bar{f}|^2 dx \tag{3.1.14}$$

Translating $\bar{v}$ bacl to original system and defining $\bar{F}$ as gradient of translated $\bar{v}$ and going to previous case, we get the Caccioppoli inequality. $\square$

The theorems of interior and boundary regularity depend on the following important properties of weak derivatives. Just like how the limit of the difference quotient gives the classical derivatives for smooth functions, these quotients also give weak derivatives. Except limit is not a pointwise limit, but in the sense of limit in a function space.

3.1.2 Difference Quotient

Definition 3.2 (Difference Quotient)**.** Given a function $u : \Omega \to \mathbb{R}^m$, an integer $s \in \{1, \dots, n\}$ and $h > 0$ we define the difference quotient

$$\tau_{h,s}u(x) := \frac{u(x + he_s) - u(x)}{h}, \quad \forall x \in \Omega_{s,h} := \{x \in \Omega | x + he_s \in \Omega\} \tag{3.1.15}$$

where $e_s = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$ with 1 in s-th position.

We can easily check the following properties hold if $u \in W^{1,p}(\Omega)$

1. $\tau_{h,s}u(x) \in W^{1,p}(\Omega_{s,h})$ for each h fixed.
2. $\tau_{h,s}\nabla u = \nabla \tau_{h,s}u$
3. If u or v is compactly supported in Ω we have
$$\int_\Omega u\tau_{h,s}v dx = -\int_\Omega v\tau_{-h,s}u dx \tag{3.1.16}$$
4. Leibniz's Rule holds

Proposition 3.1.1. *Let $1 < p < \infty$ and $\Omega_0 \subset\subset \Omega$. Then*

1. *There is constant $c(n)$ such that, for every $u \in W^{1,p}(\Omega)$ and $s = 1, \dots, n$ we have*
$$\|\tau_{h,s}u\|_{L^p(\Omega_0)} \leq c\|\nabla u\|_{L^p(\Omega)}, \quad |h| < \frac{dist(\Omega_0, \partial\Omega)}{2} \tag{3.1.17}$$

2. *If $u \in L^p(\Omega)$ and there exists $L \geq 0$ such that, for every $h < dist(\Omega_0, \partial\Omega)$, $s = 1, \dots, n$ we have*
$$\|\tau_{h,s} u\|_{L^p(\Omega_0)} \leq L \tag{3.1.18}$$
then $u \in W^{1,p}(\Omega_0)$, $\|\nabla u\|_{L^p(\Omega_0)} \leq L$ and $\tau_{h,s} \to \nabla_s u$ in $L^p(\Omega_0)$ as $h \to 0$

3.1.3 Interior Regularity

Theorem 3.3. *Let $u \in W^{1,2}\Omega$ be a weak solution of the uniformly elliptic equation*

$$-\text{div}(A\nabla u) = f - \text{div}F \tag{3.1.19}$$

where $f \in L^2(\Omega)$, $F \in W^{1,2}(\Omega)$ and A is lipschitz on Ω. Then $u \in W^{2,2}_{loc}(\Omega)$ and for any relatively compact subset Ω_0 of Ω we have

$$\|D^2 u\|_{L^2(\Omega_0)} \leq c(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega)}) \tag{3.1.20}$$

where c depends on Ω_0,Ω, ellpiticity and lipschitz constants of A.

Proof. Since u is a weak solution, we have

$$\int_\Omega \langle A\nabla u, \nabla\varphi\rangle dx = \int_\Omega f\varphi dx + \int_\Omega \langle F, \nabla\varphi\rangle dx, \quad \forall\varphi \in W^{1,2}_0(\Omega). \tag{3.1.21}$$

We once again assume $f = 0$. Now let us use the test function $\varphi(x + he_s)$ we get

$$\int_\Omega \langle A(x + he_s)\nabla u(x + he_s), \nabla\varphi\rangle dx = \int_\Omega \langle F(x + he_s), \nabla\varphi\rangle dx \tag{3.1.22}$$

Subtracting the above two equations

$$\int_\Omega \langle A(x + he_s)\nabla\tau_{h,s}u, \nabla\varphi\rangle dx + \langle\tau_{h,s}A\nabla u, \nabla\varphi\rangle dx = \int_\Omega \langle\tau_{h,s}F, \nabla\varphi\rangle dx \tag{3.1.23}$$

Notice $\tau_{h,s}u$ is a solution to weak formulation of the uniformly elliptic equation on Ω_0. So we apply Caccioppoli inequality on some $B_{4R}(x_0) \subset \Omega_0$

$$\begin{aligned}\int_{B_R(x_0)} |\tau_{h,s}\nabla u|^2 dx \leq \frac{c}{R^2}\int_{B_{2R}(x_0)} |\tau_{h,s}u|^2 dx + c\int_{B_{2R}(x_0)} |\tau_{h,s}A|^2|\nabla u|^2 dx \\ + c\int_{B_{2R}(x_0)} |\tau_{h,s}F|^2 dx\end{aligned} \tag{3.1.24}$$

As $h \to 0$ the three terms are bounded (since A is lipschitz and $u, F \in W^{1,2}(\Omega)$)hence the difference quotient is L^2 implying $\nabla u \in W^{1,2}(B_R(x_0))$. Taking $h \to 0$ and Caccioppoli, we get

$$\int_{B_R(x_0)} |D^2 u|^2 dx \leq c_1(R, L)\int_{B_{4R}(x_0)} |u|^2 dx + \int_{B_{2R}(x_0)} |DF|^2 dx \tag{3.1.25}$$

where L is lipschitz constant of A. Now we can cover the domain Ω_0 with finitely many such balls to get the desired result.

For the case $f \neq 0$ we take weak solution $\tilde{u}$ of the equation $-\Delta\tilde{u} = f$ and notice that $f = -\text{div}\tilde{F}$, where $\tilde{F} = \nabla\tilde{u}$. Now we can reduce this problem to the previous case. $\square$

Theorem 3.4. *Assume that $u \in W^{1,2}(\Omega)$ is a weak solution to the unfiormly elliptic equation*

$$-\text{div}(A\nabla u) = f - \text{div}F \tag{3.1.26}$$

and for some integer $k > 0$ we have $A \in C^{k,1}(\Omega)$, $f \in W^{k,2}(\Omega)$ and $F \in W^{k+1,2}(\Omega)$. Then $u \in W^{k+2,2}_{loc}(\Omega)$ and for any relatively compact set Ω_0 of Ω, we have

$$\|D^{k+2}u\|_{L^2(\Omega_0)} \leq c(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)} + \|DF\|_{W^{k,2}(\Omega)}) \tag{3.1.27}$$

where c depends on Ω_0, Ω and the lipschitz constant of $D^k A$

Proof. The proof goes by induction on k. The case $k = 0$ is already proved in the previous theorem. As the induction hypothesis, we assume the theorem to prove is valid for $k-1$ and confirm it is true for k. Choose the test function $\varphi = \frac{\partial\psi}{\partial x_s}$ for $1 \leq s \leq n$ and some $\psi \in C_c^\infty(\Omega)$. By integration of parts, we get

$$\int_\Omega \left\langle \frac{\partial(A\nabla u)}{\partial x_s}, \nabla\psi \right\rangle dx = \int_\Omega \frac{\partial f}{\partial x_s}\psi dx + \int_\Omega \left\langle \frac{\partial F}{\partial x_s}, \nabla\psi \right\rangle dx \tag{3.1.28}$$

which becomes

$$\int_\Omega \left\langle A\nabla\left(\frac{\partial u}{\partial x_s}\right), \nabla\psi\right\rangle dx = \int_\Omega \frac{\partial f}{\partial x_s}\psi dx + \int_\Omega \left\langle \frac{\partial F}{\partial x_s} - \frac{\partial A}{\partial x_s}\nabla u, \nabla\psi\right\rangle dx. \tag{3.1.29}$$

Consider a set $\tilde{\Omega}$ such that $\Omega_0 \subset\subset \tilde{\Omega} \subset\subset \Omega$. With this formulation it is clear that $\partial u/\partial x_s$ is weak solution to uniformly elliptic PDE with $\tilde{f} = \partial f/\partial x_s \in W^{k-1,2}(\tilde{\Omega})$ and $\frac{\partial F}{\partial x_s} - \frac{\partial A}{\partial x_s}\nabla u \in W^{k,2}(\tilde{\Omega})$. From induction hypothesis we get $\partial u/\partial x_s \in W^{k+1,2}(\Omega_0)$ and hence $u \in W^{k+2,2}(\Omega_0)$. The inequality also follows easily from considering the same PDE and using the induction hypothesis. □

Remark 3.1.1. *From the above theorem, we can prove using the Sobolev embedding that if A, F, f are $C^\infty(\Omega)$, then the weak solution is also $C^\infty(\Omega)$.*

3.1.4 Boundary Regularity

With the assumptions of the previous theorem, we can prove that the solution is also $W^{k+2,2}(\Omega)$ and not just locally. That is, the regularity of the solution holds till the boundary of the domain and not just in the interior.

Theorem 3.5. *Let the hypothesis of the previous theorem be in force. In addition assume that $\partial\Omega$ is C^{k+2} and $u - g \in W_0^{1,2}(\Omega)$ for a given $g \in W^{k+2,2}(\Omega)$. Then $u \in W^{k+2,2}(\Omega)$, we have*

$$\|D^{k+2}u\|_{L^2(\Omega)} \leq c(\|f\|_{W^{k,2}(\Omega)} + \|DF\|_{W^{k,2}(\Omega)} + \|g\|_{W^{k+2,2}(\Omega)}) \tag{3.1.30}$$

Proof. Replacing u by $u - g$ we see no loss of generality, so we may assume $u \in W_0^{1,2}(\Omega)$. The basic idea of most of the proof is already in the previous theorem. Because we want a result applicable to all of Ω, we have some balls which intersect the boundary. We flatten this boundary using C^k diffeomorphism. We can then redefine the coefficients of transformed PDE so that the weak solution u transforms into a weak solution $\tilde{u}$ of transformed PDE. Clearly $\tilde{u} \in W^{k+2,2}(\tilde{D})$ iff $u \in W^{k+2,2}(D)$.

In the previous proof, we translated the equation and used a differece quotient to establish the regularity. In this, we also have a boundary, so translating PDE is not possible in the direction normal to the boundary. So we get estimates for all partial derivatives except $\partial^2 u/\partial x_n \partial x_n$. Here we use the weak formulation to isolate $\partial^2 u/\partial x_n \partial x_n$ on one side and use duality to get an estimate over its norm. This estimate would include on the RHS $\|u\|_{L^2}$ which can again be bounded using Poincaré's inequality. Substituting $u-g$ in place of u we get

$$\|D^2(u-g)\|_{L^2(\Omega)} \leq c(\|\nabla(u-g)\|_{L^2(\Omega)} + \|\tilde{f}\|_{L^2(\Omega)} + \|D\tilde{F}\|_{L^2(\Omega)}) \tag{3.1.31}$$

i.e,

$$\|D^2u\|_{L^2(\Omega)} \leq c(\|\nabla u\|_{L^2(\Omega)} + \|\nabla g\|_{W^{1,2}(\Omega)} + \|\tilde{f}\|_{L^2(\Omega)} + \|D\tilde{F}\|_{L^2(\Omega)}) \tag{3.1.32}$$

Taking $u-g$ as the test function in the weak formulation and Hölder and Poincaré inequality, we get

$$\lambda\|\nabla u\|^2_{L^2(\Omega)} \leq \frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{c}{\lambda}\left(\|F\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|\nabla g\|^2_{L^2(\Omega)}\right) \tag{3.1.33}$$

□

3.2 Schauder Estimates

We will now define two spaces that are helpful in characterizing Hölder continuous functions, namely Morrey and Companato spaces.

In the following section, we will only consider spaces with the following property: Let $\Omega \subset \mathbb{R}^n$. There exists a constant $A > 0$ such that for all $x_0 \in \Omega, \rho < \operatorname{diam}\Omega$ we have

$$|B_\rho(x_0) \cap \Omega| \geq A\rho^n \tag{3.2.1}$$

This property is satisfied by the smooth domains we deal with in this thesis.

Definition 3.3. Set $\Omega(x_0,\rho) := \Omega \cap B_\rho(x_0)$ and for every $1 \leq p < \infty, \lambda \geq 0$ we define the Morrey space $L^{p,\lambda}(\Omega)$

$$L^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) \middle| \sup_{\substack{x_0\in\Omega \\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p dx < \infty \right\} \tag{3.2.2}$$

with the norm

$$\|u\|^p_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p dx \tag{3.2.3}$$

and the Companato space $\mathcal{L}^{p,\lambda}(\Omega)$

$$\mathcal{L}^{p,\lambda}(\Omega) := \left\{ u \in L^p(\Omega) \middle| \sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p dx < \infty \right\} \tag{3.2.4}$$

where $u_{x_0,\rho} = \fint_{\Omega_{x_0,\rho}} u dx$ with the seminorm

$$[u]^p_{p,\lambda,\Omega} := \sup_{\substack{x_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p dx \tag{3.2.5}$$

and the norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} = [u]_{p,\lambda} + \|u\|_{L^p(\Omega)} \tag{3.2.6}$$

Following are some essential properties of Campanato and Morrey spaces.

Proposition 3.2.1. *For $0 \leq \lambda < n$ we have $L^{p,\lambda}(\Omega) \cong \mathcal{L}^{p,\lambda}(\Omega)$.*

Theorem 3.6 (Campanato)**.** *For $n < \lambda \leq n + p$ and $\alpha = \frac{\lambda - n}{p}$ we have $\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega})$ and the Hölder seminorm*

$$[u]_{C^{0,\alpha}(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \tag{3.2.7}$$

is equivalent to $[u]_{p,\lambda,\Omega}$. If $\lambda > n + p$ and $u \in \mathcal{L}^{p,\lambda}(\Omega)$ then u is constant.

We will now prove specific decay estimates which are an essential tool in establishing the Campanato estimates.

Proposition 3.2.2. *Let A be a constant matrix and satisfy uniform ellipticity condition. Then there exists a constant $c(n, \lambda, \Lambda)$ such that any solution $u \in W^{1,2}_{loc}(\Omega)$ of*

$$\operatorname{div}(A\nabla u) = 0 \quad \text{in } \Omega \tag{3.2.8}$$

satisfies

$$\int_{B_\rho(x_0)} |u|^2 dx \leq c \left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |u|^2 dx \tag{3.2.9}$$

and

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx \leq c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx \tag{3.2.10}$$

for arbitrary balls $B_\rho(x_0) \subset\subset B_R(x_0) \subset\subset \Omega$. This proposition extends to all the higher partial derivatives of u, as they also satisfy the PDE.

Proof. Both inequalities are trivial for $\rho \geq R/2$. (Choose $c \geq 2^n$ or 2^{n+2}). So we will assume $\rho < R/2$.

Let us prove the first inequality. By L^2 regularity we have for $k \geq 1, u \in W^{k,2}_{\text{loc}}(\Omega)$ and

$$\|u\|_{W^{k,2}(B_{R/2})} \leq c(k, R, n, m, \lambda, \Lambda)\|u\|_{L^2(B_R)}, \tag{3.2.11}$$

Thus for k large enough, we have (using the Sobolev Embedding theorem)

$$\begin{aligned}
\int_{B_\rho(x_0)} |u|^2 dx &\leq \omega_n \rho^n \sup_{B_\rho(x_0)} |u|^2 \\
&\leq \omega_n \rho^n \sup_{B_{R/2}(x_0)} |u|^2 \\
&\leq c_1(n, R)\rho^n \|u\|^2_{W^{k,2}(B_{R/2}(x_0))} \\
&\leq c_2(R, n, m, \lambda, \Lambda)\rho^n \|u\|^2_{L^2(B_R(x_0))}.
\end{aligned} \tag{3.2.12}$$

A simple scaling argument (in an appropriately translated domain, notice $u(Rx)$ is a solution in $B_1(0)$ if $u(x)$ is a solution in $B_R(0)$) proves

$$c_2(R, n, m, \lambda, \Lambda) = \frac{1}{R^n} c(n, m, \lambda, \Lambda). \tag{3.2.13}$$

The second inequality follows from the first by applying the first inequality to partial derivatives $D_s u$ together with Cacciopoli and Poincaré:

$$\begin{aligned}
\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx &\leq c_1 \rho^2 \int_{B_\rho(x_0)} |\nabla u|^2 dx \\
&\leq c_2 \rho^2 \left(\frac{\rho}{R}\right)^n \int_{B_{R/2}(x_0)} |\nabla u|^2 dx \\
&\leq c_3 \rho^2 \left(\frac{\rho}{R}\right)^n \frac{1}{R^2} \int_{B_R(x_0)} |u - u_{x_0,\rho}|^2 dx.
\end{aligned} \tag{3.2.14}$$

□

Now we state a lemma that is very useful for obtaining Campanato estimates.

Lemma 3.2.3. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-negative and non-decreasing function satisfying*

$$\phi(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \epsilon\right]\phi(R) + BR^{\beta} \tag{3.2.15}$$

for some $A, \alpha, \beta > 0$, with $\alpha > \beta$ and for all $0 < \rho \leq R \leq R_0$, where $R_0 > 0$ is given. Then there exists constants $\epsilon_0 + \epsilon_0(A, \alpha, \beta)$ and $c = c(A, \alpha, \beta)$ such that if $\epsilon_0 \leq \epsilon_0$, we have

$$\phi(\rho) \leq c\left[\frac{\phi(R)}{R^{\beta}} + B\right]\rho^{\beta} \tag{3.2.16}$$

for all $0 \leq \rho \leq R \leq R_0$.

Proof. Refer to Lemma **5.13** in [4]. □

Theorem 3.7 (Interior Campanato Estimates for constant coefficients)**.** *Let $u \in W^{1,2}_{loc}(\Omega)$ be a solution to*

$$\operatorname{div}(A\nabla u) = -\operatorname{div} F \tag{3.2.17}$$

with A constant and satisfying uniform ellipticity. If $F \in \mathcal{L}^{2,\mu}_{loc}(\Omega), 0 \leq \mu < n+2$, then $\nabla u \in \mathcal{L}^{2,\mu}_{loc}(\Omega)$, and

$$\|\nabla u\|_{\mathcal{L}^{2,\mu}(K)} \leq c\left(\|\nabla u\|_{L^2(\Omega)} + [F]_{\mathcal{L}^{2,\mu}(\tilde{\Omega})}\right), \tag{3.2.18}$$

for every compact $K \subset\subset \tilde{\Omega} \subset\subset \Omega$, with $c(n, m, K, \tilde{\Omega}, \lambda, \Lambda, \mu)$

Proof. To use the decay estimates, we split $u = v + w$ where

$$\operatorname{div}(\nabla v) = 0 \quad \text{in } B_R(x_0) \tag{3.2.19}$$

$$v = u \quad \text{on } \partial B_R(x_0), \tag{3.2.20}$$

so that we can use decay estimates over ∇v

Because we want estimates using Campanato seminorm, we will use decay estimates with the term $|u - u_{x_0,\rho}|$. So we get

$$\int_{B_\rho(x_0)} |\nabla v - (\nabla v)_{x_0,\rho}|^2 dx \le c \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2 dx. \tag{3.2.21}$$

Using the decay estimates for the first inequality and decomposition of u repeatedly, we get

$$\begin{aligned}
&\int_{B_\rho(x_0)} |\nabla u - (\nabla u)_{x_0,\rho}|^2 dx \\
=&\int_{B_\rho(x_0)} |\nabla v - (\nabla v)_{x_0,\rho} + \nabla w - (\nabla w)_{x_0,\rho}|^2 dx \\
\le& c_1 \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2 dx + 2\int_{B_\rho(x_0)} |\nabla w - (\nabla w)_{x_0,\rho}|^2 dx \\
\le& c_2 \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 dx + c_3\int_{B_R(x_0)} |\nabla w - (\nabla w)_{x_0,\rho}|^2 dx \\
\le& c_2 \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 dx + c_3\int_{B_R(x_0)} |\nabla w|^2 dx.
\end{aligned} \tag{3.2.22}$$

We will now estimate $\int_{B_R(x_0)} |\nabla w|^2 dx$. Observe

$$\begin{aligned}
\int_{B_R(x_0)} \langle A\nabla w, \nabla\varphi\rangle dx =& \int_{B_R(x_0)} \langle F, \nabla\varphi\rangle dx & (3.2.23)\\
=& \int_{B_R(x_0)} \langle (F - (F)_{x_0,R}), \nabla\varphi\rangle dx & (3.2.24)
\end{aligned}$$

for every $\varphi \in W_0^{1,2}(B_R(x_0))$. Choose $\varphi = w \in W_0^{1,2}$ as a test function and use ellipticity. We get

$$\begin{aligned}
\lambda \int_{B_R(x_0)} |\nabla w|^2 dx \le& \int_{B_R(x_0)} \langle A\nabla w, \nabla w\rangle dx \\
=& \int_{B_R(x_0)} \langle F - (F)_{x_0,R}, \nabla w\rangle dx \\
\le& \left(\int_{B_R(x_0)} |F - (F)_{x_0,R}|^2 dx\right)^{\frac{1}{2}} \left(\int_{B_R(x_0)} |\nabla w|^2 dx\right)^{\frac{1}{2}},
\end{aligned} \tag{3.2.25}$$

thus,

$$\int_{B_R(x_0)} |\nabla w|^2 dx \le c_3 \int_{B_R(x_0)} |F - (F)_{x_0,R}|^2 dx \le [F]^2_{2,\lambda,\tilde{\Omega}} R^\mu. \tag{3.2.26}$$

Using Lemma 3.2.3 with $\alpha = n+2$ and $\beta = \mu$

$$\phi(\rho) \leq c\left[\left(\frac{\rho}{R}\right)^{\mu} \phi(R) + B\rho^{\mu}\right] \leq c_1 \left[\left(\frac{\rho}{\mu}\right)^{\mu} \|\nabla u\|^2_{L^2(\Omega)} + [F]^2_{2,\lambda,\tilde{\Omega}}\rho^{\mu}\right]. \tag{3.2.27}$$

Now covering K with balls of radius ρ, we get the desired result 3.2.18. □

We have know that $\mathcal{L}^{p,\lambda} \cong C^{0,\alpha}$ when $n < \lambda < n+p$. So as a corollary, we get Schauder estimates for constant coefficient equations using the Campanato estimates.

Corollary 3.1. *In addition to the hypothesis in previous theorems, let us assume that $F \in C^{k,\sigma}(\bar{\Omega}), k \geq 1, 0 < \sigma < 1$, then $u \in C^{k+1,\sigma}_{loc}(\Omega)$ and*

$$\|u\|_{C^{k+1,\sigma}(K)} \leq c\left(\|\nabla u\|_{L^2(\Omega)} + \|F\|_{C^{k,\sigma}(\bar{\Omega})}\right), \tag{3.2.28}$$

with $c(n, K, \Omega, \lambda, \Lambda, \sigma)$.

Proof. Using the L^2-regularity we have $u \in W^{k+1,2}_{\text{loc}}(\Omega)$ so we can differentiate the equation k times. If γ is a multi-index with $|\gamma| \leq k$, then we get

$$\operatorname{div}(\nabla(D_\gamma u)) = -\operatorname{div}(D_\gamma F). \tag{3.2.29}$$

Now we can use the preceding theorem and equivalence of Campanato and Hölder spaces to get the desired result. □

Theorem 3.8 (Interior Morrey Estimates for continuous coefficinets)**.** *Let $u \in W^{1,2}_{loc}(\Omega)$ be a solution to*

$$div(A(x)\nabla u) = -div(F), \tag{3.2.30}$$

with $A \in C(\bar{\Omega})$ and satisfying uniform ellipticity. Then if $F \in L^{2,\lambda}_{loc}(\Omega)$ for some $0 \leq \lambda < n$, we have $\nabla u \in L^{2,\lambda}_{loc}(\Omega)$ and following estimate

$$\|\nabla u\|_{L^{2,\lambda}(K)} \leq c\left(\|\nabla u\|_{L^2(\tilde{\Omega})} + \|F\|^2_{L^{2,\lambda}(\tilde{\Omega})}\right) \tag{3.2.31}$$

holds for every compact $K \subset\subset \tilde{\Omega} \subset\subset \Omega$, where $c = c(n, m, \lambda, \Lambda, K, \tilde{\Omega}, \omega)$ and ω is the modulus of continuity of A in $\tilde{\Omega}$:

$$\omega(R) := \sup_{\substack{x,y \in \tilde{\Omega} \\ |x-y| \leq R}} |A(x) - A(y)|. \tag{3.2.32}$$

Proof. In the previous proof, u was split into two functions, $u = v + w$, where v solves the homogenous part, and w solves the RHS. We will do a similar split here, except v solves only a constant homogenous equation. Fix $x_0 \in K$ and $B_R(x_0) \subset \tilde{\Omega}$ and write,

$$\begin{aligned} \operatorname{div}(A(x_0)\nabla u) &= -\operatorname{div}((A(x) - A(x_0))\nabla u + F) \\ &=: -\operatorname{div}(G) \end{aligned} \tag{3.2.33}$$

This is referred to as Korn's freezing trick. As in the constant coefficient case, we split $u = v + w$, where v solves the homogenous part. The rationale for doing the freezing is to apply the decay estimates on v. With the same computation as in the previous theorem, we obtain

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \le c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |\nabla u|^2 dx + c\int_{B_R(x_0)} |\nabla u - \nabla v|^2 dx \tag{3.2.34}$$

Now we have

$$\begin{aligned} \int_{B_R(x_0)} |\nabla w|^2 dx &\le c\int_{B_R(x_0)} |G|^2 dx \\ &\le c\int_{B_R(x_0)} |F|^2 dx + c\omega(R)^2 \int_{B_R(x_0)} |\nabla u|^2 dx. \end{aligned} \tag{3.2.35}$$

Combining the above two inequalities gives

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \le A\left\{\left(\frac{\rho}{R}\right)^n + \omega(R)^2\right\} \int_{B_R(x_0)} |\nabla u|^2 dx + c_1\|F\|_{L^{2,\lambda}(\tilde{\Omega})} R^\lambda, \tag{3.2.36}$$

Lemma 3.2.3 applied with $\phi(\rho) = \int_{B_\rho(x_0)} |\nabla u|^2 dx, \alpha = n, \beta = \lambda$ and choose $r \le R_0$ so that $\omega(R_0)$ is small enough yields the result. □

Theorem 3.9 (Interior Schauder estimates for Hölder continuous coefficients)**.** *Let $u \in W^{1,2}_{loc}(\Omega)$ be a solution to*

$$\operatorname{div}(A\nabla u) = -\operatorname{div}(F) \tag{3.2.37}$$

with $A \in C^{0,\sigma}_{loc}(\Omega)$ satisfying the uniform ellipticity condition for some $\sigma \in (0,1)$. If $F \in C^{0,\sigma}_{loc}(\Omega)$, then we have $\nabla u \in C^{0,\sigma}_{loc}(\Omega)$. Moreoever for every compact $K \subset\subset \tilde{\Omega} \subset\subset \Omega$

$$\|\nabla u\|_{C^{0,\sigma}(K)} \le c\left(\|\nabla u\|_{L^2(\tilde{\Omega})} + \|F\|_{C^{0,\sigma}(\tilde{\Omega})}\right). \tag{3.2.38}$$

$c = c(K, \tilde{\Omega}, \lambda, \|A\|_{C^{0,\sigma}(\Omega)})$

Proof. The proof will be similar to the continuous case. We now have extra information about the modulus of continuity, $\omega(R) \leq cR^\sigma$. The Campanato theorem shows that Hölder functions are in Campanato spaces, so we can use the decay estimates using the difference form. We define u, v, w, and G as in the previous proof.

$$\begin{aligned} \int_{B_\rho(x_0)} |\nabla u - (\nabla u)_{x_0,\rho}|^2 dx \leq & c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 dx \\ & + c\int_{B_R(x_0)} |\nabla w|^2 dx. \end{aligned} \tag{3.2.39}$$

Using a calculation similar to the one in the constant coefficients case we get

$$\int_{B_R(x_0)} |\nabla w|^2 dx \leq c_1 \int_{B_R(x_0)} |F - F_{x_0,R}|^2 dx + c_1 \omega(R)^2 \int_{B_R(x_0)} |\nabla u|^2 dx. \tag{3.2.40}$$

Using the previous theorem we get that $\nabla u \in L^{2,n-\epsilon}_{\text{loc}}(\Omega)$ for every $\epsilon > 0$. Therefore

$$\begin{aligned} \phi(\rho) := & \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,\rho}|^2 dx \\ \leq & \, c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 dx \\ & + c_1 \underbrace{\int_{B_R(x_0)} |F - F_{x_0,R}|^2 dx}_{[F]^2_{2,n+2\sigma} R^{n+2\sigma}} + \underbrace{\omega(R)^2}_{c_2 R^{2\sigma}} \underbrace{\int_{B_R(x_0)} |\nabla u|^2 dx}_{c(\epsilon) R^{n+2\sigma-\epsilon}} \\ \leq & \, c\left(\frac{\rho}{R}\right)^{n+2} \phi(R) + BR^{n+2\sigma-\epsilon}. \end{aligned} \tag{3.2.41}$$

Which by Lemma 3.2.3 implies $\nabla u \in \mathcal{L}^{2,n+2\sigma-\epsilon}_{\text{loc}}(\Omega) \cong C^{0,\sigma-\epsilon/2}_{\text{loc}}(\Omega)$. This implies ∇u is locally bounded so

$$\int_{B_R(x_0)} |\nabla u|^2 dx \leq \omega_n \sup_{B_R(x_0)} |\nabla u|^2 R^n. \tag{3.2.42}$$

This gives us a better estimate

$$\phi(\rho) \leq c\left(\frac{\rho}{R}\right)^{n+2} \phi(R) + BR^{n+2\sigma}. \tag{3.2.43}$$

Once again, using Lemma 3.2.3 we get

$$\phi(\rho) \leq c\left(\frac{\phi(R)}{R^{n+2\sigma}} + B\right)\rho^{n+2\sigma}. \tag{3.2.44}$$

Therefore we conclude $\nabla u \in \mathcal{L}^{2,n+2\sigma}_{\text{loc}}(\Omega) \cong C^{0,\sigma}_{\text{loc}}(\Omega)$ and the final estimate follows by covering.

□

Theorem 3.10 (Generalisation for higher derivatives). *Assume that $u \in W^{1,2}_{loc}(\Omega)$ is a solution to*

$$\operatorname{div}(A\nabla u) = f - \operatorname{div}(F), \tag{3.2.45}$$

where $k \geq 1$ and

1. *$A \in C^k_{loc}(\Omega)$ (resp. $C^{k,\sigma}_{loc}(\Omega)$ for some $0 < \sigma < 1$),*
2. *$D^k F \in L^{2,\lambda}_{loc}(\Omega)$, for some $\lambda < n$ (resp. $\mathcal{L}^{2,\lambda}_{loc}(\Omega)$, $n \leq \lambda \leq n + 2\sigma$),*
3. *$D^{k-1} f \in L^{2,\lambda}_{loc}(\Omega)$, for some $\lambda < n$ (resp. $\mathcal{L}^{2,\lambda}_{loc}(\Omega), n \leq \lambda \leq n + 2\sigma$).*

Then $D^{k+1}u \in L^{2,\lambda}_{loc}(\Omega)$ (resp. $\mathcal{L}^{2,\lambda}_{loc}(\Omega)$).

In particular if $A \in C^{k,\sigma}_{loc}(\Omega)$, $F \in C^{k,\sigma}_{loc}(\Omega)$ and $f \in C^{k-1,\sigma}_{loc}(\Omega)$, then $u \in C^{k+1.\sigma}_{loc}(\Omega)$.

All the above theorems also have a corresponding boundary regularity theorem. But for those, we need decay estimates for half-balls. The idea is again to prove the estimate locally and then use a covering argument. We first flatten the boundary as in the case of L^2 Regularity. Use half-balls to prove the estimate for domains near the boundary there and come back to the actual domain. The computation within the proof remains similar to the one in interior regularity.

Theorem 3.11 (Boundary regularity- Schauder estimates). *Let $u \in W^{1,2}(\Omega)$ be a solution to*

$$\begin{cases} \operatorname{div}(A(x)\nabla u) = -\operatorname{div}(F) & \text{in } \Omega \\ u - g \in W^{1,2}_0(\Omega) \end{cases} \tag{3.2.46}$$

with $A \in C^{k,\sigma}(\bar{\Omega})$ satisfying uniform ellipticity, $F \in C^{k,\sigma}(\bar{\Omega}), g \in C^{k+1,\sigma}(\bar{\Omega}), \sigma \in (0,1)$. Then we have $u \in C^{k+1,\sigma}(\bar{\Omega})$ and

$$\|u\|_{C^{k+1,\sigma}(\bar{\Omega})} \leq c(\Omega, \sigma, \lambda, \|A\|_{C^{k,\sigma}(\bar{\Omega})})\{\|F\|_{C^{k,\sigma}(\bar{\Omega})} + \|g\|_{C^{k+1,\sigma}(\bar{\Omega})}\}, \tag{3.2.47}$$

where λ is ellipticity constant.

3.3 L^p Regularity

Definition 3.4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f be a measurable function. The distribution function of f $\lambda_f(t) : [0, \infty) \to \mathbb{R}_0^+$ is defined as

$$\lambda_f(t) = \mu(\{x||f(x)| > t\}) \tag{3.3.1}$$

Theorem 3.12 (Layer Cake Representaion)**.** *Let ν be a borel measure on $[0, \infty)$ and define*

$$\phi(t) = \nu([0, t)). \tag{3.3.2}$$

Then for any positive measurable function f on Ω we have

$$\int_\Omega \phi(f(x))d\mu(x) = \int_0^\infty \lambda_f(t)d\nu(t) \tag{3.3.3}$$

Proof.

$$\int_\Omega \phi(f(x))d\mu(x) = \int_\Omega \int_0^\infty \chi_{\{t<f(x)\}}d\nu(t)d\mu(x) = \int_0^\infty \int_\Omega \chi_{\{t<f(x)\}}d\mu(x)d\nu(t) \tag{3.3.4}$$

$$\int_0^\infty \int_\Omega \chi_{\{t<f(x)\}}d\mu(x)d\nu(t) = \int_0^\infty \lambda_f(t)d\nu(t) \tag{3.3.5}$$

□

As a special case of the formula, we can express the L^p norm in terms of the distribution function. Let $d\nu(t) = pt^{p-1}dt$ so that $\phi(t) = t^p$. Then the L^p norm is

$$\|f\|^p_{L^p(\Omega)} = \int_\Omega |f(x)|^p d\mu(x) = \int_0^\infty pt^{p-1}\lambda_f(t)dt \tag{3.3.6}$$

Proposition 3.3.1 (Chebyshev's Inequality)**.** *Let $g \in L^1(\Omega)$. Then for any $s > 0$ we have*

$$s\lambda_g(t) \leq \|g\|_{L^1(\Omega)} \tag{3.3.7}$$

Proof.

$$s\lambda_g(t) = \int_\Omega s\chi_{\{|g|>s\}} d\mu \le \int_\Omega |g|\chi_{\{|g|>s\}} d\mu \le \|g\|_{L^1(\Omega)} \tag{3.3.8}$$

□

Taking $s = t^p$ and $g = |f|^p$ we the following

$$t^p \lambda_f(t) \le \|f\|^p_{L^p(\Omega)} \tag{3.3.9}$$

So we get that for any $f \in L^p(\Omega)$

$$\sup_{t>0} t^p \lambda_f(t) < \infty \tag{3.3.10}$$

The converse is not necessarily true, i.e., there exists a function with finite supremum which is not L^p integrable. For example consider the function $f = 1/|x|$ defined on unit ball B in $\mathbb{R}^n$. $\lambda_f(t) = c_n t^{-n}$. Clearly $f \in L^n_w(B)$. But we can check that $f \notin L^n(B)$.

Definition 3.5. We define the weak L^p space , $L^p_w(\Omega)$, as the set of functions

$$L^p_w(\Omega) := \left\{ f \middle| \|f\|_{L^p_w(\Omega)} = \sup_{t>0} t^p \lambda_f(t) < \infty \right\} \tag{3.3.11}$$

In the case $p = \infty$ we define $L^\infty_w(\Omega) := L^\infty(\Omega)$

Note that $\|f\|_{L^p_w(\Omega)}$ is not a norm. Given any $q < p$ and a finite measure space Ω we have

$$L^p(\Omega) \subset L^p_w(\Omega) \subset L^q(\Omega) \tag{3.3.12}$$

To prove $L^p_w(\Omega) \subseteq L^q(\Omega)$ just observe $\lambda_f(t) \le \min\{|\Omega|, t^{-p}\|f\|_{L^p_w(\Omega)}\}$ and use the layer cake representation.

Let T be an operator that sends measurable functions to measurable functions. We say that an operator T is Q-subadditive/quasi-linear if

$$|T(f+g)| \le Q(|T(f)| + |T(g)|), \tag{3.3.13}$$

where Q is inddependent of f and g. An operatot T is of the *weak-(p,q) type* if for any

$f \in L^p(\Omega)$, $Tf \in L^q_w(\Omega)$ and there a constant $A_{p,q}$ such that

$$\|Tf\|_{L^q_w(\Omega)} \leq A_{p,q}\|f\|_{L^p}(\Omega) \tag{3.3.14}$$

Strong-(p,q) type on the other hand means

$$\|Tf\|_{L^q(\Omega)} \leq A_{p,q}\|f\|_{L^p}(\Omega) \tag{3.3.15}$$

Theorem 3.13 (Marcinkiewicz's Interpolation Theorem)**.** *Let T be a Q-subadditve operator that is both weak-(p_0, p_0) type and weak-(p_1, p_1) type for $1 \leq p_0 < p_1 \leq \infty$. Then T is strong-(p, p) type for any p such that $p_0 < p < p_1$.*

To prove the Stampacchia Interpolation theorem, we need Calderon-Zygmund decomposi- -tion theorem and John-Nirenberg lemmas I-II. We will now give proof of the decomposition theorem and state the John-Nirenberg lemmas. The John-Nirenberg lemmas are also used in proving that the Campanato space $\mathcal{L}^{p,n} \cong BMO$.

Theorem 3.14 (Calderon-Zygmund decomposition)**.** *Let Q be an n-dimensional cube in $\mathbb{R}^n$ and let f be a non-negative function in $L^1(Q)$. Fix a parameter $t > 0$ in such a way that*

$$⨍_Q f(x)dx \leq t. \tag{3.3.16}$$

Then there exists a counatable family $\{Q_i\}_{i\in I}$ of cubes in the dyadic decomposition of Q such that

1. *$t < ⨍_{Q_i} f dx \leq 2^n t$ for every $i \in I$;*
2. *$f(x) \leq t$ for a.e. $x \in Q \setminus \cup_{i\in I} Q_i$.*

Proof. The plan is to divide a given cube into cubes of half the size and remove the cubes which satisfy the first condition.

This bisection divides the cube into 2^n subcubes. We will now choose the cubes which satisfy

$$⨍_P f(x)dx > t \tag{3.3.17}$$

to belong to family $\{Q_i\}$, and if the cube doesn't satisfy this condition, we will continue subdivision. Continuing this process infinitely, let $\mathcal{Q} := \{Q_i\}$ be the family of all cubes with an average greater than t. Suppose Q_i is a cube. Then Q_i came from the subdivision of some cube $\bar{Q}_i$ whose average is less than t. Using this, we can conclude the mass in Q_i can at most be $2^n t|Q_i|$. So the average over Q_i is between t and $2^n t$. So $\mathcal{Q}$ is the set of all subcubes in the decomposition which satisfy the first condition.

If $x \in Q \setminus \cup_{i\in I} Q_i$, then the average in cubes containing x as the size of the cube goes to zero is at most t. Using the Lebesgue differentiation theorem, we get $f(x) \leq t$ for almost all x. ☐

Now we will define BMO space or the space of functions of bounded mean oscillation.

Definition 3.6 ($BMO(Q_0)$)**.** Let Q_0 be an n-dimensional cube in $\mathbb{R}^n$. We say that a function $u \in L^1 Q_0$ belongs to the space of functions with bounded mean oscillation $BMO(Q_0)$ if

$$|u|_* := \sup \fint_Q |u - u_Q| dx < \infty, \tag{3.3.18}$$

where supremum is over all n-subcubes $Q \subset Q_0$, whose sides are parallel to Q_0, and u_Q is average of u over Q.

Theorem 3.15 (John-Nirenberg lemma I)**.** *There are constants $c_1, c_2 > 0$ depending only on n, such that*

$$|\{x \in Q \big| |u(x) - u_Q| > t\}| \leq c_1 \exp\left(-c_2 \frac{t}{|u|_*}\right) \cdot |Q| \tag{3.3.19}$$

for all cubes $Q \subset Q_0$ with sides parallel to those of Q_0, all $u \in BMO(Q_0)$ and all $t > 0$.

Corollary 3.2. *For every $1 \leq p < \infty$ the Campanato space $\mathcal{L}^{p,n}(Q_0)$ is isomorphic to $BMO(Q_0)$.*

Proof. Using the John-Nirenberg lemma and layer-cake formula for L^p norm, we get

$$\int_Q |u - u_Q|^p dx \leq p \cdot c_1 \int_0^\infty t^{p-1} \exp(-\frac{c_2}{|u|_*} t) |Q| dt \tag{3.3.20}$$

$$= C(n,p) |u|_*^p |Q|. \tag{3.3.21}$$

This proves $BMO(Q_0) \hookrightarrow \mathcal{L}^{p,n}(Q_0)$. Using Jensen's inequality we get that $\mathcal{L}^{p,n}(Q_0) \hookrightarrow BMO(Q_0)$ ☐

Theorem 3.16 (John-Nirenberg lemma II). *Let $u \in L^1(Q_0)$ and suppose that for some $p \in [1, \infty]$ we have*

$$K_p(u) := \left(\sup_{\Delta \in \{\Delta\}} \sum_{Q_i \in \Delta} |Q_i| \left(\fint_{Q_i} |u - u_{Q_i}| \right)^p \right)^{\frac{1}{p}} < \infty \tag{3.3.22}$$

where $\{\Delta\}$ denotes the collection of all finite decompositions Δ of the cube Q_0 into subcubes Q_i with sides parallel to the axes. Then the function $u - u_{Q_0}$ (hence also u) belongs to $L^p_w(Q_0)$ and for all $t > 0$

$$\left| \left\{ x \in Q_0 \middle| |u(x) - u_{Q_0}| > t \right\} \right| \leq c(n,p) \left(\frac{K_p(u)}{t} \right)^p. \tag{3.3.23}$$

Now we can prove the Stampacchia Interpolation theorem.

Theorem 3.17 (Stampacchia Interpolation Theorem). *Let $1 \leq p < \infty$ and let T be a linear operator of strong type (p,p) and bounded from L^∞ into BMO, i.e.,*

$$\|Tu\|_{L^p} \leq c_1 \|u\|_{L^p}, \quad \textit{for every } u \in L^p(Q_0) \tag{3.3.24}$$

and

$$\|Tu\|_* \leq c_2 \|u\|_{L^\infty}, \quad \textit{for every } u \in BMO(Q_0). \tag{3.3.25}$$

Then T maps continuously $L^q(Q_0)$ into $L^q(Q_0)$ for all $q \in (p, \infty)$.

Proof. We will define a different operator T_Δ and prove it is strong-(p,p) and strong-(∞,∞). And using Marcinkiewicz's theorem, prove T_Δ is strong-(q,q). And then, use the John-Nirenberg lemma to prove T is strong (q,q).

Let $\Delta = \{Q_i\}$ be some subdivision of Q_0. Define

$$(T_{\Delta u})(x) := \fint_{Q_i} |Tu - (Tu)_{Q_i}| dx, \quad \text{for } x \in Q_i. \tag{3.3.26}$$

Then T_Δ is strong-(p,p) type

$$\|T_\Delta u\|_{L^p(Q_0)}\|^p_{L^p(Q_0)} = \sum_{Q_i\in\Delta} |Q_i| \left(\fint_{Q_i} |Tu - (Tu)_{Q_i}| dx \right)^p \tag{3.3.27}$$

$$\leq \sum_{Q_i\in\Delta} \int_{Q_i} |Tu - (Tu)_{Q_i}|^p dx \tag{3.3.28}$$

$$\leq 2^{p-1} \sum_{Q_i\in\Delta} \int_{Q_i} [|Tu|^p + |(Tu)_{Q_i}|^p] dx \tag{3.3.29}$$

$$\leq 2^p \sum_{Q_i\in\Delta} \int_{Q_i} |Tu|^p dx \tag{3.3.30}$$

$$= 2^p \|Tu\|^p_{L^p(Q_0)} \leq c_1 \|u\|^p_{L^p(Q_0)}. \tag{3.3.31}$$

We can also prove T_Δ is strong (∞,∞) type. Suppose $u \in L^\infty(Q_0)$ we have

$$\|T_\Delta u\|_{L^\infty(Q_0)} \leq |Tu|_* \leq c_2 \|u\|_{L^\infty(Q_0)}. \tag{3.3.32}$$

Clearly T_Δ is quasi-linear, so using Marcinkiewicz's theorem

$$\|T_\Delta u\|_{L^r(Q_0)} \leq c \|u\|_{L^r(Q_0)} \tag{3.3.33}$$

for all $r \in (p,\infty)$. We can prove that this constant c depends only on p, r, c_1 and c_2.. The (p,p) operator norm and (∞,∞) operator norms have uniform bounds depending on p, c_1, and c_2, c depends on these bounds and r.

Now we can use John-Nirenberg lemma II. We have

$$K_r(Tu) = \sup_{\Delta\in\{\Delta\}} \|T_\Delta u\|_{L^r(Q_0)} \leq C\|u\|_{L^r(Q_0)} < \infty, \tag{3.3.34}$$

therefore $Tu \in L^r_w(Q_0)$ and T is of weak (r,r) type for each $r \in (p,\infty)$. Now using Marcinkiewicz's theorem, T is of strong (q,q) for all $q \in (p,r)$ and so for every $q \in (p,\infty)$. □

Theorem 3.18 (L^p Regularity for constant and Hölder coefficients)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of the Dirichlet problem*

$$\begin{cases} div(A\nabla u) = div(F) \\ u \in W^{1,2}_0(\Omega) \end{cases} \tag{3.3.35}$$

where the PDE satisfies uniform ellipticity condition and $F \in L^p(\Omega)$ and $g \in W^{1,p}(\Omega)$ for some $p \geq 2$. Then $\nabla u \in L^p(\Omega)$ and

$$\|\nabla u\|_{L^p(\Omega)} \leq c\|F\|_{L^p(\Omega)}, \tag{3.3.36}$$

for some constant $c(\Omega, p, \lambda, |A|)$

Proof. Consider the map $\nabla u \to F$

$$T : L^2(\Omega) \to L^2(\Omega) \tag{3.3.37}$$

This map is continuous since

$$\lambda \int_\Omega |\nabla u|^2 dx \leq \int_\Omega \langle A\nabla u, \nabla u\rangle dx = \int_\Omega \langle F, \nabla u\rangle dx \tag{3.3.38}$$

$$\left(\int_\Omega |F|^2 dx\right)^{1/2} \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}. \tag{3.3.39}$$

From the Campanato estimates for constant coefficients and Hölder continuous coefficients, we have

$$[\nabla u]_{\mathcal{L}^{2,n}} \leq c\left(\|\nabla u\|_{L^2} + \|F\|_{\mathcal{L}^{2,n}}\right) \tag{3.3.40}$$

$$|\nabla u|_* \leq c_1[\nabla u]_{\mathcal{L}^{2,n}} \leq c_2\left(\|\nabla u\|_{L^2} + \|F\|_{\mathcal{L}^{2,n}}\right) \tag{3.3.41}$$

Since $\|\nabla u\|_{L^2} \leq \bar{c}\|F\|_{L^2} \leq \bar{c}\|F\|_{\mathcal{L}^{2,n}}$, we have

$$|\nabla u|_* \leq c_3\|F\|_{\mathcal{L}^{2,n}} \leq c_4\|F\|_{L^\infty}. \tag{3.3.42}$$

This proves T is continuous from L^∞ into $BMO(\omega)$.

Stamppachia's interpolation theorem now yields the L^p regularity. □

The L^p regularity theory also applies for $1 < p < 2$, but the proof is different from the above case.

Theorem 3.19 (L^p regularity for the case $1 < p < 2$)**.** *Let Let $u \in W^{1,2}(\Omega)$ be a weak* *solution of the Dirichlet problem*

$$\begin{cases} div(A(x)\nabla u) = div(F) \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{3.3.43}$$

where the PDE satisfies uniform ellipticity condition and $F \in L^p(\Omega)$ *and* $g \in W^{1,p}(\Omega)$ *for some* $1 < p < 2$*. Then* $\nabla u \in L^p(\Omega)$ *and*

$$\|\nabla u\|_{L^p(\Omega)} \leq c\|F\|_{L^p(\Omega)}, \tag{3.3.44}$$

for some constant $c(\Omega, p, \lambda, |A|)$

Proof. The proof involves expressing L^p norm of ∇u as the norm of the linear map over it's dual space.

$$\|\nabla u\|_{L^p} = \sup_{\|G\|_{L^{p'}} \leq 1} \int_\Omega \langle \nabla u, G \rangle \tag{3.3.45}$$

Now we perform "Helmholtz Decompositon" such that $G = A^*(x)\nabla\varphi + \tilde{G}$, where $\operatorname{div}\tilde{G} = 0$, and $\varphi \in W_0^{1,p'}(\Omega)$. We get such a decomposition by solving

$$-\operatorname{div}(A^*(x)\nabla\varphi) = \operatorname{div}(G) \quad \text{in } \Omega \tag{3.3.46}$$

$$\varphi = 0 \quad \text{on } \partial\Omega \tag{3.3.47}$$

and defining $\tilde{G} = G - A^*(x)\nabla\varphi$. So

$$\begin{aligned} \|\nabla u\|_{L^p} &= \sup_{\|G\|_{L^{p'}} \leq 1} \int_\Omega \langle \nabla u, A^*\nabla\varphi \rangle && (3.3.48) \\ &= \sup_{\|G\|_{L^{p'}} \leq 1} \int_\Omega \langle A(x)\nabla u, \nabla\varphi \rangle && (3.3.49) \\ &\leq \sup_{\|G\|_{L^{p'}} \leq 1} \|F\|_{L^p}\|\nabla\varphi\|_{L^{p'}} && (3.3.50) \\ &\leq \sup_{\|G\|_{L^{p'}} \leq 1} c\|G\|_{L^{p'}}\|F\|_{L^p} && (3.3.51) \\ &\leq c\|F\|_{L^p} && (3.3.52) \end{aligned}$$

Note the penultimate inequality comes from using $L^{p'}$ estimates as $p' > 2$. □

Chapter 4

The Yamabe Problem

4.1 The Yamabe Problem

Let us now state the Yamabe problem.

Let (M_n, g) be a compact smooth Riemannian manifold of dimension $n \geq 3$ and scalar curvature S. Does there exist a metric g', conformal to g, such that (M_n, g') has a constant scalar curvature S'?

If we consider the conformal change $g' = u^{4/(n-2)}g$, with $u \in C^\infty$ and $u > 0$, the scalar curvature satisfies the equation:

$$\frac{4(n-1)}{n-2}\Delta u + Su = S' u^{(n+2)/(n-2)} \tag{4.1.1}$$

Let $4\frac{n-1}{n-2}\Delta + S = \Delta_{\text{conf}}$, so the equation is now

$$\Delta_{\text{conf}} u = S' u^{(n+2)/(n-2)} = S' u^{2^*-1} \tag{4.1.2}$$

So now the Yamabe problem is finding smooth, positive function u solving the above PDE. Therefore, we need first to prove the existence of a solution and then the smoothness and positivity of the solution if it exists.

We will see that if exponent in the RHS of the PDE is $q-1$, $q<2^*$(subcritical) instead of 2^*-1, we can prove the existence and smoothness of solution using standard analysis. But the critical case(the Yamabe problem) is much more challenging.

But we can prove existence of solution for some compact manifolds using direct methods. Let us define the Yamabe functional:

$$I(u)=\frac{\int_M 4\frac{n-1}{n-2}|\nabla u|^2 dv+\int_M Su^2 dv}{(\int_M u^{2^*}dv)^{2/2^*}}, \tag{4.1.3}$$

and its infimum, called the Yamabe invariant :

$$\mu(M)=\inf_{u\in W^{1,2},u\neq 0} I(u) \tag{4.1.4}$$

We can also easily check that the Yamabe invariant is a conformally invariant.

Theorem 4.1. *[9] If $u\in W^{1,2}(M)$ is minimizer of $I(v)$ with $\|u\|_{L^{2^*}}=1$, then u satisfies $\Delta_{\text{conf}}u=\lambda u^{2^*-1}$ for $\lambda=\mu(M)$.*

Proof. Let $\varphi\in C_c^\infty(M)$. Then

$$\begin{aligned}
0 &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} I(u+\epsilon\varphi) && (4.1.5)\\
&= \frac{\int_M 8\frac{n-1}{n-2}\langle\nabla u,\nabla\varphi\rangle+2Su\varphi dv}{\|u\|^2_{L^{2^*}}}-\frac{2\int_M 4\frac{n-1}{n-2}|\nabla|^2+Su^2 dv}{\|u\|^2_{L^{2^*}}}\frac{\int_M u^{2^*-1}\varphi dv}{\|u\|^2_{L^{2^*}}} && (4.1.6)\\
&= \frac{2}{\|u\|^2_{L^{2^*}}}\int_M \varphi\left(-4\frac{n-1}{n-2}+Su-I(u)\frac{u^{2^*-1}}{\|u\|^{2^*-2}_{L^{2^*}}}\right) && (4.1.7)
\end{aligned}$$

Therefore u is a weak solution for the PDE with $\lambda=I(u)/\|u\|^{2^*-2}_{L^{2^*}}$. Since u is a minimizer we have $\lambda=\mu(M)$. □

Notice that if $S\leq 0$, substituting $u=1$ we get $\mu(M)\leq 0$. We will now prove the existence of solution when $\mu(M)\leq 0$. So that we will prove existence of solution for manifolds with negative or zero scalar curvature.

Theorem 4.2 (Existence when $\mu(M)\leq 0$)**.** $\Delta_{\text{conf}}=S'u^{2^*-1}$ *has a solution when* $\mu(M)\leq 0$.

Proof. In order to prove existence when $\mu(M)\leq 0$, we will use direct methods to prove the existence of a minimizer of $I(\varphi)$. Let $E(\varphi) = I(\varphi)\|\varphi\|^2_{L^{2^*}}$. We can prove that u is a minimizer of $I(\varphi)$ iff it is a minimizer of $\bar{E}(\varphi) = E(\varphi) - \mu(M)\|\varphi\|^2_{L^{2^*}}$. Now suppose $\mu(M) \leq 0$. Clearly $\bar{E}(\varphi) \geq 0$ and let u_k be a minimizing sequence of $\bar{E}(\varphi)$. $\bar{E}(u_k) \leq C$ and hence $\|u_k\|_{W^{1,2}(M)} \leq C$, by Sobolev inequality. So we have $u_k \rightharpoonup u$ in $W^{1,2}$ and L^{2^*} by Banach-Alaglu theorem, and $u_k \to u$ in L^2 by Rellich-Kondrakov theorem. Since $\mu(M) \leq 0$ by w.l.s.c of norms we have $E(u) \leq \liminf E(u_k)$ and so u is a minimizer. □

From here on we would assume that $\mu(M) > 0$.

Consider the manifold $\mathbb{S}^n$. The stereographic projection gives us a conformal mapping from the sphere onto $\mathbb{R}^n$. In the transformed metric, we have the scalar curvature to be zero. So the Yamabe functional is

$$\tilde{I} = \frac{4\frac{n-1}{n-2}\|\nabla u\|_{L^2}}{\|u\|^2_{L^{2^*}}} \tag{4.1.8}$$

The infimum of this will just be in terms of the optimal Sobolev constant $K(n, 2)$. We have $\mu(\mathbb{S}^n) = 4\frac{n-1}{n-2}/K^2(n, 2)$. We know that this infimum is attained on $\mathbb{R}^n$, which implies that we have non-trivial solutions to Yamabe problem on the sphere, We can even construct the extremizers without referring back to Aubin's or Talenti's proof. The construction is as follows. First do a stereographic projection on to $\mathbb{R}^n$. Perform a dialation by $\alpha > 0$. Now reverse the Stereographic projection. Using $\mathbb{R}^n$ as cover for $\mathbb{S}^n \setminus P$, with bijection given by stereographic projection, if we calculate the conformal factor on the sphere we get

$$u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha}\right)^{(2-n)/2}. \tag{4.1.9}$$

This metric on sphere is a conformal diffeomorphism and it minimizes the Yamabe functional on sphere. In fact, we have that the only metrics on sphere which minimize the Yamabe functional are the ones which are obtained by conformal diffeomorphism.

Theorem 4.3. *[7] The Yamabe functional on $(\mathbb{S}^n, \bar{g})$ is minimized by constant multiples of the standard multiples and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard metric on $\mathbb{S}^n$ that have constant scalar curvature.*

4.2 Existence and Regularity of Yamabe Problem

Theorem 4.4 (Existence in subcritical case)**.** *On a smooth Riemannian manifold M_n of dimension $n \geq 3$, let us consider the PDE*

$$\Delta u + h(x)u = \lambda f(x)u^{q-1}, \tag{4.2.1}$$

where $2 < q < 2^*$, $h(x)$ and $f(x)$ are C^∞ functions on manifold, with $f(x)$ everwhere strictly positive. This PDE has a weak solution $u \in W^{1,2}(M_n)$ for some λ.

Proof. Consider the functional

$$I_q(u) = \frac{\int_M |\nabla u|^2 dv + \int_M h(x)u^2 dv}{(\int_M f(x)u^q dv)^{2/q}}, \tag{4.2.2}$$

where $u \not\equiv 0$ and $0 \leq u \in W^{1,2}(M)$. Define $\mu_q = \inf I_q(u)$.

We will prove that this infimum is attained for $2 < q < 2^*$. Simple application of Hölder inequality shows us $I_q(u)$ is bounded below. Let u_i be a minimizing sequence such that $\int_M f u_i^q dv = 1$. We will prove that this minimizing sequence is bounded in $W^{1,2}$.

$$\|u_i\|^2_{W^{1,2}} = \|u_i\|^2_{L^2} + \|\nabla u_i\|^2_{L^2} = I_q(u_i) - \int_M h(x)u_i^2 + \|u_i\|^2_{L^2}. \tag{4.2.3}$$

$$\|u\|^2_{W^{1,2}} \leq \mu_q + 1 + (1 + \|h(x)\|_{L^\infty})\|u\|^2_{L^2} \tag{4.2.4}$$

$$\|u_i\|^2_{L^2} \leq V^{1-2/q}\|u_i\|^2_{L^2} \leq V^{1-2/q}[\inf f(x)]^{-2/q} \tag{4.2.5}$$

Now we can prove there exists a function attaining the infimum such that

$$I_q(u) = \mu_q \quad \text{and} \quad \int_M f(x)u^q dv = 1 \tag{4.2.6}$$

Since $2 < q < 2^*$, $W^{1,2}$ is compactly embedded in L^q. So we have a subsequence such that $u_i \to u$ in L^q and $u_i \rightharpoonup u$ in $W^{1,2}$.

The strong convergence in L^q ensures the constraint is satisfied by the limit u. w.l.s.c. of the norm gives us

$$\|u\|_{W^{1,2}} \leq \liminf \|u_i\|_{W^{1,2}}. \tag{4.2.7}$$

In addition, we also have $u_i \to u$ in L^2. These two combined gives us $I_q(u) \leq \mu_q$ and therefore, $I_q(u) = \mu_q$.

We can verify that this minimizer is a weak solution to PDE

$$\Delta u + h(x)u = \lambda f(x) u^{q-1}. \tag{4.2.8}$$

□

Theorem 4.5 (Regularity in Subcritical case)**.** *Let $u \in W^{1,2}(M)$ be a solution to*

$$\frac{4(n-1)}{n-2}\Delta u + Su = S' u^{q-1} \tag{4.2.9}$$

where Δ is the Laplace-Beltrami operator over the Riemannian manifold and $q \in (2, 2^*)$. Then $u \in C^\infty(M)$.

Proof. We have proved that there exists a function u and some λ which solves the PDE 4.2.1. Choosing h and f as an appropriately we can have $\lambda = S'$ and u to be solution of $\Delta_{\text{conf}} u = S' u^{q-1}$. Choose $p_1 = 2^*$. Since $u \in W^{1,2}(M)$, the Sobolev embedding theorem gives $u \in L^{p_1}(M)$. Hence $f \in L^{p_1/(q-1)}(M)$. From L^p regularity we get that $u \in W^{2,p_1/(q-1)}(M)$. Using Sobolev embedding again, we that

$$u \in L^{p_2}(M), \text{ where } p_2 = \frac{np_1}{n(q-1) - 2p_2} \tag{4.2.10}$$

if $n(q-1) > 2p_1$, or $u \in L^s(M)$ for all s if $n(q-1) \leq 2p_1$. Continuing this process, we get $u \in L^s(M)$ for all s. So we have $u \in W^{2,s}(M)$ for all s. By Sobolev embedding we also get that $u \in C^1(M)$, and since $q > 2$ we have $u^{q-1} \in W^{1,s}(M)$ for all s. Using L^p regularity we get $u \in W^{3,s}(M)$ for all s. So $u \in C^2(M)$. We can now apply the maximum principle to prove $u > 0$ and the smoothness also follows. □

Theorem 4.6 (Regularity in critical case)**.** *[5] Let $u \in W^{1,2}(M)$ be a solution to*

$$\Delta_{\mathrm{conf}} u = S' u^{2^*-1} \tag{4.2.11}$$

Then $u \in C^\infty(M)$.

Proof. The proof given in the sub-critical case doesn't work in the critical case. However, the bootstrapping argument still works if we can prove that $u \in L^s(M)$ for $s > 2^*$. The function u satisfies

$$\int_M \left(\frac{4(n-1)}{n-2} \langle \nabla u, \nabla \varphi \rangle + S u \varphi \right) dv = S' \int_M |u|^{2^*-1} \varphi dv \tag{4.2.12}$$

for all $\varphi \in W^{1,2}(M)$. We plan to choose an appropriate test function φ.

$$G_L(t) = \begin{cases} |t|^{2^*-1} & \text{if } |t| \leq L \\ \frac{2^*}{2} L^{2^*-2}|t| - \frac{2^*-2}{2} L^{2^*-1} & \text{if } |t| > L \end{cases} \tag{4.2.13}$$

$$F_L(t) = \begin{cases} |t|^{2^*/2} & \text{if } |t| \leq L \\ \frac{2^*}{2} L^{(2^*-2)/2}|t| - \frac{2^*-2}{2} L^{2^*/2} & \text{if } |t| > L. \end{cases} \tag{4.2.14}$$

Clearly $G_L(u)$ is uniformly Lipshitz continuous function of u and hence $G_L(u) \in W^{1,2}(M)$. Similarly we have $F_L(u) \in W^{1,2}(M)$. Observe that G and F are zero when $u \leq 0$ and that

$$(F_L'(t))^2 \leq \frac{2^*}{2} G_L'(t), \quad (F_L(t))^2 \geq t G_L(t), \tag{4.2.15}$$

$$F_L(t) \leq t^{2^*/2}, \quad G_L(t) \leq t^{2^*-1}. \tag{4.2.16}$$

Let us now use

$$\varphi = G_L(u) = G_L \tag{4.2.17}$$

as test function. Hence we get,

$$\frac{4(n-1)}{n-2} \int_M \langle \nabla u, \nabla G_L \rangle dv + \int_M S u G_L dv = S' \int_M u^{2^*-1} G_L dv. \tag{4.2.18}$$

Since $G_L \le u^{2^*-1}$ and $u \in L^{2^*}(M)$, we have

$$\left|\int_M SuG_l dv\right| \le C_1', \text{ and } |S'| \le C_2' \tag{4.2.19}$$

so,

$$\int_M G_L' |\nabla u|^2 dv \le C_1 + C_2 \int_M u^{2^*-1} G_L dv. \tag{4.2.20}$$

Using $(F_L'(t))^2 \le \frac{2^*}{2} G_L'(t)$ and $tG_L(t) \le F_L(t)$, we get that

$$\int_M |\nabla F_L|^2 dv \le C_1 + C_2 \int_M u^{2^*-2} F_L^2 dv. \tag{4.2.21}$$

Given $K > 0$, let

$$K^- = \{x \text{ s.t. } u(x) \le K\}, \tag{4.2.22}$$

$$K^+ = \{x \text{ s.t. } u(x) \ge K\} \tag{4.2.23}$$

Using Hölder's inequality and Sobolev inequality for the embedding $W^{1,2}(M) \hookrightarrow L^{2^*}(M)$,

$$\begin{aligned}
\int_M u^{2^*-2} F_L^2 dv &= \int_{K^-} u^{2^*-2} F_L^2 dv + \int_{K^+} u^{2^*-2} F_L^2 dv && (4.2.24)\\
&\le \int_{K^-} u^{2^*-2} F_L^2 dv + \left(\int_{K^+} u^{2^*} dv\right)^{2/n} \left(\int_{K^+} F_L^{2^*} dv\right)^{2/2^*} && (4.2.25)\\
&\le \int_{K^-} u^{2^*-2} F_L^2 dv + \epsilon(K) \left(\int_M F_L^{2^*}\right)^{2/2^*} && (4.2.26)\\
&\le \int_{K^-} u^{2^*-2} F_L^2 dv + C_3 \epsilon(K) \int_M (|\nabla F_L|^2 + F_L^2) dv && (4.2.27)
\end{aligned}$$

where $\epsilon(K) = (\int_{K^+} u^{2^*} dv)^{2/n}$, $C_3 > 0$ is a constant independent on K and L. Since $u \in L^{2^*}(M)$,

$$\lim_{K \to \infty} \epsilon(K) \to 0. \tag{4.2.28}$$

We fix K such that $C_2C_3\epsilon(K) < 2/2^*$. When $L > K$,

$$\int_{K^-} u^{2^*-2}F_L^2 dv \leq K^{2(2^*-1)}V(M). \tag{4.2.29}$$

Since $u \in L^{2^*}(M)$, and since $F_L)(t) \leq t^{2^*/2}$

$$\int_M F_L^2 dv \leq C_4. \tag{4.2.30}$$

Therefore it is clear that there exists $C_5, C_6 > 0$ independent of L, and $C_6 < 1$, such that

$$\int_M |\nabla F_L|^2 dv \leq C_5 + C_6 \int_M |\nabla F_L|^2 dv. \tag{4.2.31}$$

Hence,

$$\int_M |\nabla F_L|^2 dv \leq \frac{C_5}{1 - C_6}. \tag{4.2.32}$$

This gives us that $F_L \in W^{1,2}(M)$. Using Sobolev emebedding we get $F_L \in L^{2^*}(M)$ and

$$\int_M F_L^{2^*} dv \leq C_7 \tag{4.2.33}$$

where $C_7 > 0$ and doesn't depend on L. Taking $L \to \infty$, it follows that $u \in L^{(2^*)^2/2}(M)$. Since $(2^*)^2/2 > 2^*$ we increased the regularity. $\square$

Now we will prove a theorem that is crucial in establishing existence in the critical case. It is also through this we will see how the optimal Sobolev constant plays a role in the Yamabe problem.

Theorem 4.7 (Concentration-Compactness Lemma 2). *Suppose $u_m \rightharpoonup u$ weakly in $W^{1,2}(\mathbb{R}^n)$ and $\mu_m = |\nabla u_m|^2 dx \rightharpoonup \mu$, $\nu_m = |u_m|^{2^*} dx \rightharpoonup \nu$ weakly in the sense of measures where ν and ν are bounded non-negative measures on $\mathbb{R}^n$. Then we have :*

1. *There exists some at most countable set J, a family $\{x^j | j \in J\}$ of distinct points in $\mathbb{R}^n$, and a family $\{\nu^j | j \in J\}$ of positive numbers such that*

$$\nu = |u|^{2^*} dx + \sum_{j \in J} \nu^j \delta_{x^j}, \tag{4.2.34}$$

where δ_x is the Dirac-delta mass of mass 1 concentrated at $x \in \mathbb{R}^n$.

2. *In addition, we have*

$$\mu \geq |\nabla u|^2 dx + \sum_{j \in J} \mu^j \delta_{x^j} \tag{4.2.35}$$

for some family $\{\mu^j | j \in J\}, \mu^j > 0$ satisfying

$$\tilde{K}^2 \left(\nu^j\right)^{2/2^*} \leq \mu^j, \quad \textit{for all } j \in J. \tag{4.2.36}$$

where $\tilde{K}$ is the best Sobolev constant for inequality corresponding to the embedding $W^{1,2} \hookrightarrow L^{2^*}$

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq \tilde{K}^{-1} \|u\|_{W^{1,2}(\mathbb{R}^n)} \tag{4.2.37}$$

so that $K(n,2) = \tilde{K}^{-1}$.

Proof. Let $v_m = u_m - u \in W^{1,2}(\mathbb{R}^n)$. Then $v_m \rightharpoonup 0$ weakly in $W^{1,2}$ and by Brezis-Lieb lemma[8] we have that if $\int_{\mathbb{R}^n} |\varphi_m|^{2^*} \leq C$ and $u_m \to u$ pointwise a.e. then

$$\int_{\mathbb{R}^n} \left| |\varphi_m|^{2^*} - |\varphi|^{2^*} - |\varphi_m - \varphi|^{2^*} \right| dx = o(1) \tag{4.2.38}$$

where $o(1) \to 0$ as $m \to \infty$ .Let f be any bounded continuous function. We have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f(|u_m|^{2^*} - |u|^{2^*} - |u_m - u|^{2^*}) dx \right| &\leq \sup |f| \int_{\mathbb{R}^n} \left| |u_m|^{2^*} - |u|^{2^*} - |u_m - u|^{2^*} \right| dx && (4.2.39) \\ &= o(1) && (4.2.40) \end{aligned}$$

So we have

$$\begin{aligned} \omega_m := \nu_m - |u|^{2^*} dx &= (|u_m|^{2^*} - |u|^{2^*}) dx && (4.2.41) \\ &= |u_m - u|^{2^*} dx + o(1) = |v_m|^{2^*} dx + o(1). && (4.2.42) \end{aligned}$$

Define $\lambda_m := |\nabla u_m|^2 dx$ and assume that $\lambda_m \rightharpoonup \lambda$, while $\omega_m \rightharpoonup \omega = \nu - |u|^{2^*} dx$ weakly in sense of measure.

Choose $\xi \in C_0^\infty(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} |\xi|^{2^*} d\omega = \lim_{m\to\infty} \int_{\mathbb{R}^n} |\xi|^{2^*} d\omega_m = \lim_{m\to\infty} \int_{\mathbb{R}^n} |v_m \xi|^{2^*} dx \tag{4.2.43}$$

$$\leq \tilde{K}^{-2^*} \liminf_{m\to\infty} \left(\int_{\mathbb{R}^n} |\nabla(v_m\xi)|^2 dx \right)^{2^*/2} \tag{4.2.44}$$

$$\leq \tilde{K}^{-2^*} \liminf_{m\to\infty} \left(\int_{\mathbb{R}^n} |\xi|^2 |\nabla v_m|^2 dx \right)^{2^*/2} \tag{4.2.45}$$

$$= \tilde{K}^{-2^*} \left(\int_{\mathbb{R}^n} |\xi|^2 d\lambda \right)^{2^*/2}. \tag{4.2.46}$$

We arrived at the final inequality in the following way

$$\left(\int_{\mathbb{R}^n} |\nabla(v_m\xi)|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^n} |\xi|^2 |\nabla v_m|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^n} |v_m|^2 |\nabla \xi|^2 dx \right)^{\frac{1}{2}}. \tag{4.2.47}$$

The second term goes to zero as $m \to \infty$ as $|\nabla\xi||v_m| \to 0$ in L^2.

$$\int_{\mathbb{R}^n} |\nabla\xi|^2 |v_m|^2 dx \leq \|\nabla\xi\|_{L^\infty} \int_{\mathrm{supp}(\xi)} |v_m|^2 dx. \tag{4.2.48}$$

$W^{1,2}(\mathrm{supp}(\xi))$ compactly embeds into $L^2(\mathrm{supp}(\xi))$ and $v_m \rightharpoonup 0$ in $W^{1,2}(\mathrm{supp}(\xi))$, so it goes to zero in L^2. So finally, we have a reverse Hölder inequality:

$$\tilde{K}^2 \left(\int_{\mathbb{R}^n} |\xi|^{2^*} d\omega \right)^{2/2^*} \leq \int_{\mathbb{R}^n} |\xi|^2 d\lambda \tag{4.2.49}$$

holds for all $\xi \in C_0^\infty(\mathbb{R}^n)$. Now let us decompose ω into diffused and atomic parts. Let $\{x^j | j \in J\}$ be the atoms of ω and we have $\omega = \omega_0 + \sum_{j\in J} \nu^j \delta_{x^j}$ where ω_0 has no atoms.

Given any open set Ω, we can approximate the characteristic function using compactly supported smooth functions. Using this, we get

$$\tilde{K}^2 \omega(\Omega)^{2/2^*} \leq \lambda(\Omega). \tag{4.2.50}$$

Let Ω be open set such that $\lambda(\Omega) \leq \tilde{K}^2$ then

$$1 \geq \tilde{K}^{-2}\lambda(\Omega) \geq \omega(\Omega)^{2/2^*} \geq \omega(\Omega) \geq \omega_0(\Omega). \tag{4.2.51}$$

So ω_0 is absolutely continuous w.r.t. λ. So there is an $\lambda - L^1$ function f such that $d\omega_0 = f d\lambda$. For λ a.e. away from atoms of λ and ω we have

$$f(x) = \lim_{r\to 0} \frac{\omega_0(B_r(x))}{\lambda(B_r)(x)} \leq \lim_{r\to 0} \tilde{K}^{-2^*} \lambda(B_r(x))^{2^*/2-1} = 0. \tag{4.2.52}$$

Given any radon measure we can prove it can have only countably many atoms. So there are only countably many atoms of λ, none of which are atoms of ω_0, and since f is zero λ a.e. outside of the atoms, we get that ω_0 is the identically zero measure.

So

$$\omega = \nu - |u|^{2^*} dx = \sum_{j\in J} \nu^j \delta_{x^j}. \tag{4.2.53}$$

For any x^j choose a ξ such that $\xi(x^j) = 1$ and $\xi = 0$ outside a small ball around x^j. Now using the reverse Hölder inequality 4.2.49, we get

$$\tilde{K}^2 (\nu^j)^{2/2^*} \leq \lambda(\{x^j\}). \tag{4.2.54}$$

That is $\lambda \geq \sum_{j\in J} \tilde{K}^2 (\nu^j)^{2/2^*} \delta_{x^j}$.

We have

$$\int_{\mathbb{R}^n} \xi d\lambda = \lim_{m\to\infty} \int_{\mathbb{R}^n} \xi |\nabla v_m|^2 dx \tag{4.2.55}$$

$$= \lim_{m\to\infty} \int_{\mathbb{R}^n} \xi(|\nabla u|^2 + |\nabla u_m|^2 - 2\langle \nabla u, \nabla u_m\rangle) dx \tag{4.2.56}$$

$$= -\int_{\mathbb{R}^n} \xi |\nabla u|^2 dx + \int_{\mathbb{R}^n} \xi d\mu. \tag{4.2.57}$$

Hence we get

$$\mu \geq |\nabla u|^2 dx + \sum_{j\in J} \tilde{K}^2 (\nu^j)^{2/2^*} \delta_{x^j}. \tag{4.2.58}$$

□

Although we gave the above theorem on $\mathbb{R}^n$, the same proof works on a compact closed manifold.

Theorem 4.8 (Existence in critical case)**.** *On a smooth closed Riemannian manifold M_n of* *dimension $n \geq 3$, let us consider the PDE*

$$\Delta_{\mathrm{conf}} u = S' u^{2^*-1}. \tag{4.2.59}$$

This PDE has a weak solution $u \in W^{1,2}$ if $\mu(M) < \mu(\mathbb{S}^n)$.

Proof. Let $\{u_k\}$ be a minimizing sequence for $\mu_{2^*}(M) = \mu(M)$. WLOG, we can assume that $\|u_k\|_{L^{2^*}} = 1$. Upto a subsequence, we have

1. $u_k \to u$ in $L^2(M)$,
2. $u_k \rightharpoonup u$ in $W^{1,2}(M)$,
3. $u_k \rightharpoonup u$ in $L^{2^*}(M)$.

Let $\|u\|_{L^{2^*}}^{2^*} = t \in [0,1]$. We notice that if $t = 1$, we have in norms and weak convergence. This implies we have strong convergence in L^{2^*}. This will imply that the minimizer is attained, and we are done. Now by Concentration- Compactness Lemma 2, we have

$$\mu(M) = \lim_{k\to\infty} I(u_k) \geq \int_M \frac{4(n-1)}{n-2}|\nabla u|^2 + Su^2 + \frac{4(n-1)}{n-2}\tilde{K}^2(M)\sum_{j\in J}(\nu^j)^{2/2^*}. \tag{4.2.60}$$

Note $I(u) = \left(\int_M \frac{4(n-1)}{n-2}|\nabla u|^2 + Su^2 dv\right)/t^{2/2^*} \geq \mu(M)$. Now we use the fact that for all compact manifolds M_n $\frac{4(n-1)}{n-2}\tilde{K}^2(M) = \mu(\mathbb{S}^n)$. So,

$$\begin{aligned}
\mu(M) &\geq t^{2/2^*}\mu(M) + \mu(\mathbb{S}^n)\sum_{j\in J}(\nu^j)^{2/2^*} && (4.2.61)\\
&\geq t^{2/2^*}\mu(M) + \mu(\mathbb{S}^n)(1-t)^{2/2^*}\left(\sum_{j\in J}\frac{\nu^j}{1-t}\right)^{2/2^*} && (4.2.62)\\
&= t^{2/2^*}\mu(M) + \mu(\mathbb{S}^n)(1-t)^{2/2^*} && (4.2.63)
\end{aligned}$$

Now, since $\mu(\mathbb{S}^n) > \mu(M)$ and applying Jensen's inequality, we have

$$\begin{aligned}
\mu(M) &\geq t^{2/2^*}\mu(M) + \mu(\mathbb{S}^n)(1-t)^{2/2^*} && (4.2.64)\\
&\geq \mu(M)(t^{2/2^*} + (1-t)^{2/2^*}) && (4.2.65)\\
&\geq \mu(M) && (4.2.66)
\end{aligned}$$

This proves t is 0 or 1. It cannot be zero, as the second inequality will then be a strict one. So $t = 1$. $\square$

4.3 Existence when M_n is not locally conformally flat and $n \geq 6$

Lemma 4.3.1. *Suppose $k > -n$. Then as $\alpha \to 0$,*

$$I(\alpha) = \int_0^{\epsilon} r^k u_\alpha^2 r^{n-1} dr \tag{4.3.1}$$

is bounded above and below by positive multiples of α^{k+2} if $n > k+4$, $\alpha^{k+2}\log(1/\alpha)$ if $n = k+4$, and α^{n-2} if $n < k+4$.

Proof. The subsitution $\sigma = r/\alpha$ gives

$$I(\alpha) = \alpha^{k+2} \int_0^{\epsilon/\alpha} \sigma^{k+n-1}(\sigma^2+1)^{2-n} d\sigma \tag{4.3.2}$$

Observe that $\sigma^2 \leq 2\sigma^2$ for $\sigma \geq 1$, so $I(\alpha)$ is bounded sbove and below by positive multiples of

$$\alpha^{k+2}\left(C + \int_1^{\epsilon/\alpha} \sigma^{k+3-n} d\sigma\right). \tag{4.3.3}$$

The expression in parentheses is bounded if $n > k+4$; it is comparable to α^{n-k-4} if $n < k+4$, and to $\log(1/\alpha)$ if $n = k+4$. $\square$

Theorem 4.9 (Conformal Normal Coordinates)**.** *Let M_n be a Riemannian manifold and $P \in M_n$. For each $N \geq 2$ there is a conformal metric g on M such that*

$$\det g_{ij} = 1 + O(r^N), \tag{4.3.4}$$

where $r = |x|$ in g-normal coordinates at P. In these coordinates, if $N \geq 5$, the scalar curvature of g satisfies $S = O(r^2)$ and $\Delta S = \frac{1}{6}|W|^2$ at P.

Theorem 4.10. *If M has dimension $n \geq 6$ and is not locally conformally flat then $\mu(M) < \mu(\mathbb{S}^n)$.*

Proof. Let $\{x^i\}$ be conformal normal coordinates in a neighborhood of $P \in M_n$.

Let $a = 4\frac{n-1}{n-2}$. The functions u_α satisy $a\|\nabla u_\alpha\|^2_{L^2} = \mu(\mathbb{S}^n)\|u_\alpha\|^2_{L^p}$ on $\mathbb{R}^n$. Choose a smooth radial function η, such that it is supported in $B_{2\epsilon}$ and identically 1 in B_ϵ and $0 \le \eta \le 1$ everywhere else. Consider the function $\varphi = \eta u_\alpha$. Since φ is a function of $r = |x|$ alone we have

$$\int_{\mathbb{R}^n} a|\nabla\varphi|^2 dx = \int_{B_{2\epsilon}} \left(a\eta^2|\nabla u_\alpha|^2 + 2a\eta u_\alpha\langle\nabla\eta, \nabla u_\alpha\rangle + au_\alpha^2|\nabla\eta|^2\right) dx \tag{4.3.5}$$

$$\le \int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx + C\int_{A_\epsilon} \left(u_\alpha|\partial_r u_\alpha| + u_\alpha^2\right) dx, \tag{4.3.6}$$

where A_ϵ denotes the annulus $B_{2\epsilon} \setminus B_\epsilon$. Using the expression of u_α we can estimate $u_\alpha \le \alpha^{(n-2)/2}r^{2-n}$ and $|\partial_r u_\alpha| \le (n-2)\alpha^{(n-2)/2}r^{1-n}$. Therefore for a fixed ϵ, the second term in the integral inequality is $O(\alpha^{n-2})$ as $\alpha \to 0$.. As for the first term,

$$\int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx = \mu(\mathbb{S}^n)\left(\int_{B_\epsilon} u_\alpha^{2^*} dx + \int_{\mathbb{R}^n - \setminus B_\epsilon} u_\alpha^{2^*} dx\right)^{2/2^*} \tag{4.3.7}$$

$$\le \mu(\mathbb{S}^n)\left(\int_{B_{2\epsilon}} \varphi^{2^*} dx + \int_{\mathbb{R}^n\setminus B_\epsilon} \alpha^n r^{-2n} dx\right)^{2/2^*} \tag{4.3.8}$$

$$= \mu(\mathbb{S}^n)\left(\int_{B_{2\epsilon}} \varphi^{2^*} dx\right)^{2/2^*} + O(\alpha^n). \tag{4.3.9}$$

On a compact manifold, let φ be defined as ηu_α in normal coordinates $\{x^i\}$ in a neighbor- -hood of $P \in M_n$, extended smoothly by zero over the manifold. Since φ is a radial function and $g^{rr} \equiv 1$ in normal coordinates, we have $|\nabla\varphi|^2 = |\partial_r\varphi|^2$. Since $dV_g = (1 + O(r))dx$ in normal coordinates, the previous calculation gives

$$E(\varphi) = \int_{B_{2\epsilon}} \left(a|\nabla\varphi|^2 + S\varphi^2\right) dV_g \tag{4.3.10}$$

$$\leq (1 + C\epsilon)\left(\mu(\mathbb{S}^n)\|\varphi\|_{L^{2^*}}^2 + C\alpha^{n-2} + C\int_0^{2\epsilon}\int_{\mathbb{S}_r} u_\alpha^2 r^{n-1} d\omega dr\right). \tag{4.3.11}$$

Since $dV_g \equiv dx$ in conformal normal coordinates, the term $(1 + C\epsilon)$ is absent giving

$$E(\varphi) \leq \mu(\mathbb{S}^n)\|\varphi\|_{L^{2^*}}^2 + C\alpha^{n-2} + \int_{B_{2\epsilon}} S\varphi^2 dx. \tag{4.3.12}$$

But now in conformal normal coordinates $S = O(r^2)$ and $\Delta S(P) = \frac{1}{6}|W(P)|^2$, so

$$\int_{B_{2\epsilon}} S\varphi^2 dx \leq \int_{B_\epsilon} S u_\alpha^2 dx + C\int_{A_\epsilon} u_\alpha^2 dx \tag{4.3.13}$$

$$= \int_0^\epsilon \int_{\mathbb{S}_r} \left(\frac{1}{2}S_{,ij}x^i x^j + O(r^3)\right) u_\alpha^2 d\omega_r dr + O(\alpha^{n-2}) \tag{4.3.14}$$

$$= \int_0^\epsilon \left(-Cr^2|W(P)|^2 + O(r^3)\right) u_\alpha^2 r^{n-1} dr = O(\alpha^{n-2}). \tag{4.3.15}$$

Using Lemma 4.3.1 we get

$$E(\varphi) \leq \begin{cases} \mu(\mathbb{S}^n)\|\varphi\|_{2^*}^2 - C|W(P)|^2\alpha^4 + o(\alpha^4) & \text{if } n > 6, \\ \mu(\mathbb{S}^n)\|\varphi\|_{2^*}^2 - C|W(P)|^2\alpha^4 \log(1/\alpha) + O(\alpha^4) & \text{if } n = 6. \end{cases} \tag{4.3.16}$$

If M_n is not locally conformally flat, we can choose P so that $|W(P)|^2 > 0$, and then $I(\varphi) < \mu(\mathbb{S}^n)$ for α sufficiently small and $n \geq 6$. Thus $\mu(M_n) < \mu(\mathbb{S}^n)$. □

4.4 Existence in rest of the cases

Lemma 4.4.1. *Suppose $\mu(M) > 0$. Then at each $P \in M$ the Green function Γ_P for $4\frac{n-1}{n-2}\Delta + S = \Delta_{\text{conf}}$ exists and is strictly positive.*

Using Stereographic projection from $\mathbb{S}^n$ to $\mathbb{R}^n$, we can transfer Yamabe functional to $\mathbb{R}^n$ where analysis is more straightforward. We do a similar thing by defining a generalized stereographic projection for compact manifolds.

Definition 4.1. Suppose (M_n, g) is a compact Riemannian manifold with $\mu(M) > 0$. For $P \in M$ define the metric $\hat{g} = G^{2^*-2}g$ on $\hat{M} = M \setminus \{P\}$, where

$$G = (n-2)\omega_n a \Gamma_P. \tag{4.4.1}$$

The manifolds $(\hat{M}, g)$ together with the natural map $\sigma : M \setminus \{P\} \to \hat{M}$ is called the stereographic projection of M from P.

The image manifold of a stereographic projection has a special geometric structure called asymptotically flat.

Definition 4.2. A Riemannian manifold N with C^∞ metric g is called asymptotically flat of order $\tau > 0$ if there exists a decompostion $N = N_0 \cup N_\infty$ (with N_0 compact) and a diffeomorphism $N_\infty \to \mathbb{R}^n \setminus B_R$ for some $R > 0$, satisfying:

$$g_{ij} = \delta_{ij} + O(\rho^{-\tau}), \quad \partial_k g_{ij} = O(\rho^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = O(\rho^{-\tau-2}), \tag{4.4.2}$$

as $\rho = |z| \to \infty$ in the coordinates $\{z^i\}$ induced on N_∞. The coordinates $\{z^i\}$ are called asymptotic coordinates.

Although it looks like the definition depends on the asymptotic coordinates, it can be proven that the asymptotic flat structure is determined by the metric alone.

Fix a point $P \in M_n$. Choose the local coordinates to be the conformal normal coordinate system. We will explicitly describe the asymptotically flat structure of the stereographic projection $(\hat{M}, \hat{g})$.

Remark 4.4.1. *We write $f = O'(r^k)$ to mean $f = O(r^k)$ and $\nabla f = O(r^{k-1})$. Similarly we define O''. The set of smooth functions that vanish to order k at P is denoted by $\mathcal{C}_k$. $\mathcal{P}_k$ is the space of homogeneous polynomials of degree k.*

Lemma 4.4.2. *Let G be given by 4.4.1. In conformal normal coordinates $\{x^i\}$ at P, G has* *an asymptotic expansion*

$$G(x) = r^{2-n}\left(1 + \sum_{k=4}^{n} \psi_k(x)\right) = c \log r + O''(1), \tag{4.4.3}$$

where $r = |x|, \psi_k \in \mathcal{P}_k$, *and the* log *term appears only if* n *is even. The leading terms are:*

1. *if* $n = 3, 4, 5$, *or* M *is conformally flat in a neighborhood of* P,

$$G = r^{2-n} + A + O''(r) \quad (A = \textit{constant}); \tag{4.4.4}$$

2. *if* $n = 6$,

$$G = r^{2-n} - \frac{1}{288a}|W(P)|^2 \log r + O''(1); \tag{4.4.5}$$

3. *if* $n \geq 7$,

$$G = r^{2-n}\left[1 + \frac{1}{12a(n-4)}\left(\frac{r^4}{12(n-6)}|W(P)|^2 - S_{,ij}(P)x^i x^j r^2\right)\right] + O''(r^{7-n}). \tag{4.4.6}$$

Now we can prove that the asymptotically flat structure of $\hat{g}$ can be derived immediately from this lemma.

Theorem 4.11. *The metric* $\hat{g}$ *is asymptotically flat of order 1 if* $n = 3$, *order 2 if* $n \geq 4$, *and order* $n-2$ *if* M_n *is conformally flat near* P. *In inverted conformal normal coordinates, it has the expansion*

$$\hat{g}_{ij}(z) = \gamma^{2^*-2}(z)\left(\delta_{ij} + O''(\rho^{-2})\right) \tag{4.4.7}$$

where, in the three cases of Lemma 4.4.2

1. $\gamma(z) = 1 + A\rho^{2-n} + O''(\rho^{1-n}) \quad (A = \textit{constant})$

2. $\gamma(z) = 1 + \frac{1}{288a}|W(P)|^2\rho^{-4}\log\rho + O''(\rho^{-4})$;

3. $\gamma(z) = 1 + \frac{1}{12a(n-4)}\rho^{-6}\left(\frac{\rho^2}{12(n-6)}|W(P)|^2 - S_{,ij}(P)z^i z^j\right) + O''(\rho^-5)$.

Proof. Let $\{x^i\}$ be conformal normal coordinates on a neighborhood U of P, and define inverted conformal normal coordinates $z^i = r^{-2}x^i$ on $U \setminus \{P\}$. With $\rho = |z| = r^-1$ we have

$$\partial/\partial z^i = \rho^{-2}\left(\delta_{ij} - 2\rho^{-2}z^i z^j\right)\partial/\partial x^j. \tag{4.4.8}$$

If we write $\gamma = r^{n-2}G$, the components of $\hat{g}$ in z-coordinates are

$$\begin{aligned}\hat{g}_{ij}(z) &= \gamma^{2^*-2}\rho^4 g(\partial/\partial z^i, \partial/\partial z^j) && (4.4.9)\\ &= \gamma^{2^*-2}(\delta_{ik} - 2\rho^{-2}z^i z^k)(\delta_{jl} - 2\rho^{-2}z^j z^l)g_{kl}(\rho^{-2}z) && (4.4.10)\\ &= \gamma^{2^*-2}\left(\delta_{ij} + O''(\rho^{-2})\right). && (4.4.11)\end{aligned}$$

If M_n is conformally flat near P, $g_{kl} \equiv \delta_{kl}$ in conformal normal coordinates, so $\hat{g}_{ij} = \gamma^{2^*-2}\delta_{ij}$ in that case. Noting that the expansion for G gives the corresponding expansion for γ, we will get the desired theorem.

□

In the locally conformally flat case, we could define a function locally and provide a bound on the Yamabe invariant of the manifold. But in the case of the locally conformally flat case, the local geometry resembles that of the sphere. And on the sphere no function will have the Yamabe functional lower than $\mu(\mathbb{S}^n)$. That is why attempt to construct a function locally with Yamabe quotient less than $\mu(\mathbb{S}^n)$ will fail. In order to overcome this we need to define a function which carries some information about the global geometry. So in this case we define "distortion coefficient" which does exactly that for asymptotically flat manifolds.

We will now construct a test function on $\hat{M}$ and express its Yamabe quotient in terms of a number determined by the global geometry of $\hat{M}$.

Fix a large radius $R > 0$, let $\rho(z) = |z|$ in inverted conformal coordinates (extended to a smooth, positive function on $\hat{M}$), and let $\hat{M}_\infty = \{\rho > R\}$. Define φ on $\hat{M}$ by

$$\varphi(z) = \begin{cases} u_\alpha(z) & \rho(z) \geq R, \\ u_\alpha(R) & \rho(z) \leq R, \end{cases} \tag{4.4.12}$$

with $\alpha \gg R$ to be detremined later.

Since φ is a function of only the radial variable ρ, the behavior of Yamabe quotient as $\alpha \to \infty$ depends on the "average" behavior of the metric $\hat{g}$ over large spheres. As a measure of this average behaviour we introduce the "distortion coefficient".

Define

$$h(\rho) = \omega_\rho^{-1} \int_{S_\rho} \gamma^{(2^*+2)/2} d\omega_\rho. \tag{4.4.13}$$

The expansion of γ then gives an asymptotic expansion as $\rho \to \infty$:

$$h(\rho) = \begin{cases} 1 + (\nu/k)\rho^{-k} + O''(\rho^{-k-1}) & \text{if } n \neq 6, \\ 1 + (\nu/4)\rho^{-4} \log \rho + O''(\rho^{-4}) & \text{if } n = 6, \end{cases} \tag{4.4.14}$$

and therefore, since the $(n-1)$-form $d\omega_\rho/\omega_\rho$ is homegeneous of degree zero,

$$\frac{a}{2} \int_{S_\rho} \partial_\rho \gamma \frac{d\omega_\rho}{\omega_\rho} = h'(\rho) + O(\rho^{-2k-1}) \tag{4.4.15}$$

$$= \begin{cases} -\beta\rho^{-k-1} + O(\rho^{-k-2}) & \text{if } n \neq 6, \\ -\beta\rho^{-5} \log \rho + O(\rho^{-5}) & \text{if } n = 6. \end{cases} \tag{4.4.16}$$

We call the constant β, defined using conformal normal coordinates, the distortion coefficient of $\hat{g}$. Its geometric meaning at infinity is analogous to the scalar curvature at a finite point. It is this constant which determines the values of the Yamabe quotient for large α.

Proposition 4.4.3. *Let φ be defined as above. There are positive constants C and k such that*

$$E(\varphi) \leq \mu(\mathbb{S}^n)\|\varphi\|_{L^{2^*}}^2 - C\beta\alpha^{-k} + O(\alpha^{-k-1}) \tag{4.4.17}$$

if $n \neq 6$ or M is conformally flat near P,

$$E(\varphi) \leq \mu(\mathbb{S}^n)\|\varphi\|_{L^{2^*}}^2 - C\beta\alpha^{-4} \log \alpha + O(\alpha^{-4}) \tag{4.4.18}$$

if $n = 6$ and M is not conformally flat near P. Thus if $\beta > 0$, φ can be chosen so that the Yamabe quotient is less than $\mu(\mathbb{S}^n)$.

Proof. Since the scalar curvature of $\hat{g}$ is zero, the energy $E(\varphi)$ is

$$E(\varphi) = \int_{\hat{M}} a|\nabla\varphi|^2 dV_{\hat{g}} = \int_{\hat{M}_\infty} a\hat{g}^{\rho\rho}(\partial_\rho u_\alpha)^2 dV_{\hat{g}} = \int_{\hat{M}_\infty} a(\partial_\rho u_\alpha)^2 \gamma^2 dz. \tag{4.4.19}$$

Letting A_L denote the annulus $\{R \leq \rho \leq L\}$ and integrating by parts using the Euclidean Laplacian gives

$$\int_{A_L} a(\partial_\rho u_\alpha)^2\gamma^2 dz$$
$$= \int_{A_L} au_\alpha \Delta_0 u_\alpha \gamma^2 dz - \int_{A_L} au_\alpha \partial_\rho u_\alpha \partial_\rho(\gamma^2) dz - \int_{S_R \cup S_L} au_\alpha \partial_\rho u_\alpha \gamma^2 \partial_\rho \rfloor dz. \quad (4.4.20)$$

Since γ is bounded , we have that the integral over S_L is $O(L^{2-n})$ for fixed α, and thus vanishes as $L \to \infty$. Similarly, the integral over S_R is $O(\alpha^{-n})$. Using Hölder's inequality on the first integral we get

$$\int_{A_L} au_\alpha \Delta_0 u_\alpha \gamma^2 dz = 4n(n-1)\int_{A_L} u_\alpha^{2^*-2}(u_\alpha\gamma)^2 dz \quad (4.4.21)$$
$$\leq 4n(n-1)\left(\int_{A_L} u_\alpha^{2^*} dz\right)^{1-2/2^*}\left(\int_{A_L} u_\alpha^{2^*}\gamma^{2^*} dz\right)^{2/2^*} \quad (4.4.22)$$
$$\leq 4n(n-1)\|u_\alpha\|_{L^{2^*}}^{2^*-2}\left(\int_{\hat{M}} \varphi^{2^*} dV_{\hat{g}}\right)^{2/2^*} \quad (4.4.23)$$
$$= \mu(\mathbb{S}^n)\|u_\alpha\|_{L^{2^*}}^2. \quad (4.4.24)$$

The important term is the second term in 4.4.20. After letting $L \to \infty$ it becomes

$$-\int_R^\infty au_\alpha \partial_\rho u_\alpha \int_{S_\rho} \partial_\rho(\gamma)^2 d\omega_\rho d\rho \quad (4.4.25)$$

If $n \neq 6$ or M_n is conformally flat near P,

$$a\int_{S_\rho} \partial_\rho(\gamma^2) d\omega_\rho = 4(h'(\rho) + O(\rho^{-2k-1}))\omega_\rho \quad (4.4.26)$$
$$= -4(\beta\rho^{-k-1} + O(\rho^{-k-2}))\omega_\rho. \quad (4.4.27)$$

The change of variables $\sigma = \rho/\alpha$ shows that if $2-n < k < n$

$$C^{-1}\alpha^{-k+1} \leq \int_R^\infty \rho^{-k}\left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{1-n}\rho^{n-1}d\rho \leq C\alpha^{-k+1}. \quad (4.4.28)$$

Thus the second term in 4.4.20 is

$$-4\int_R^\infty \rho\alpha^{-1}\left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{1-n}\left(\beta\rho^{-k-1}+O(\rho^{-k-2})\right)\omega_\rho dr$$
$$\leq -C\beta\alpha^{-k}+O(\alpha^{-k-1}). \quad (4.4.29)$$

Combining the results of the above calculations, we obtain 4.4.17. If $n=6$, we use the inequality instead

$$C^{-1\alpha^{-k+1}}\log\alpha \leq \int_R^\infty \rho^{-k}\log\rho\left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{1-n}\rho^{n-1}d\rho \leq C\alpha^{-k+1}\log\alpha, \quad (4.4.30)$$

and a similar analysis yields 4.4.18 □

The above calculation reduces the solution of the Yamabe problem in the case $\mu(M)>0$ to determining the sign of β. We have

$$\mu(M)=\inf_{\psi\in C_c^\infty(\hat{M})}\frac{E(\psi)}{\|\psi\|_{L^{2^*}}^2}, \quad (4.4.31)$$

and so approximating our test function φ by a function $\psi\in C_c^\infty(\hat{(}M))$, we find that $\mu(M)<\mu(\mathbb{S}^n)$ if $\beta>0$. So we proved the following theorem

Theorem 4.12. *If (M,g) is a compact Riemannian manifold of dimension $n\geq 3$ with $\mu(M)<\mu(\mathbb{S}^n)$ if there is a generalized stereographic projection $\hat{M}$ of M with strictly positive distortion coefficient β.*

It can be proven that this distortion coefficient $\beta=\frac{1}{2}m(\hat{g})$, where $m(\hat{g})$ is the so-called ADM mass of an asymptotically flat manifold, when $n<6$ or M conformally flat in the neighborhood of the point w.r.t which we did the stereographic projection. According to the "Positive mass theorem," this mass is positive. Proving this theorem is beyond the scope of this thesis, so we would assume it.

With all of these assumptions, we have that the distortion coefficient is positive; hence we have proved the existence of solutions in the last case.

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