SOFT GRAVITONS AND STRUCTURE OF NULL INFINITY IN LOGARITHMICALLY ASYMPTOTIC FLAT SPACETIME

by

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CERTIFICATE

I hereby declare that the matter embodied in the report entitled **"Soft Gravitons and Structure of Null Infinity in Logarithmically Asymptotic Flat Spacetime"** are the results of the work carried out by me at the Department of Physics, **Chennai Mathematical Institute**, under the supervision **Prof. Alok Laddha** and the same has not been submitted elsewhere for any other degree.

Signature of ______________________

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DECLARATION

This is to certify that this dissertation entitled **“Soft Gravitons and Structure of Null Infinity in Logarithmically Asymptotic Flat Spacetime”** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Raikhik Das** at **Indian Institute of Science Education and Research, Pune** under the supervision of **Prof. Alok Laddha** at the physics department of **Chennai Mathematical Institute** during the academic year of 2022-23.

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LIST OF SYMBOLS

$\mathbb{R}$	Set of Real numbers
$\mathbb{Z}$	Set of Integers
$\mathcal{I}^+$	Future Null Infinity
$\mathcal{I}^-$	Future Null Infinity
c	speed of light
G	Newton's Universal Gravitational Constant
i^+	Future Timelike Infinity
i^-	Future Timelike Infinity
i^0	Spatial Infinity

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Abstract

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Recent developments in soft theorems and the rise of Celestial Holography have rejuvenated the interest in the asymptotic structure of spacetimes. Bondi, van der Burg, Metzner, and Sachs's work and Ashtekar's notion of asymptotic flatness assume that the future null infinity in the conformal metric and the physical metric is C^{∞}. Similarly, the absence of incoming radiation requires the past null infinity to be C^{∞}. But this condition cannot be put on spatial infinity due to the probable presence of isolated sources, which are represented by an important class of solutions. Despite the lack of differentiability of spatial infinity extending to the metric, the assumption of C^{∞} of the manifold along with the null infinity leads to the peeling property, given by: $C_{\mu\nu\rho\sigma} = O(\Omega)$. But Christodoulou and Klainerman argued that this peeling property is too restrictive and strong of a condition that eliminates plausible physical spacetime solutions. The shortcomings of the peeling property motivate the construction of the logarithmically asymptotic flat (LAF) spacetimes, which gives the relation: $C_{\mu\nu\rho\sigma} = O(\Omega \log \Omega)$ with Weyl tensor and its dual satisfying the relations: $C_{\mu\nu\rho\sigma} n^{\mu} n^{\rho} = O(\Omega)$ and ${}^{*}C_{\mu\nu\rho\sigma} n^{\mu} n^{\rho} = O(\Omega)$ where $n^{\mu} = g^{\mu\nu}\Omega_{,\nu}$. Upon taking this asymptotically logarithmic flat condition differentiability structure of the infinities change. In this thesis, we study the structure of future null infinity of asymptotically logarithmic flat (LAF) spacetime and if peeling is violated at $|u| \to \infty$ for massive particles, massless scalar, and massive scalar field on the Minkowski background.

Chapter 1

INTRODUCTION

Asymptotic symmetries, soft theorems, and the memory effect have been studied separately for years in the theoretical physics community. But recent insight into those seemingly different topics has led us to conclude that they are closely connected and effectively equivalent. Hence, studying one of them will lead to understanding the others. Further, recent interests in Celestial Holography and Soft theorems have re-energized the discussion on asymptotic characteristics of spacetime.

Studying isolated systems is of particular interest in general relativity. Although there cannot be any such system as genuinely isolated, it is reasonable to consider the spacetime at large distances from a compact source. It is beneficial to treat spacetime at large distances as asymptotically flat. Therefore, studying asymptotically flat (AF) spacetime is of particular interest to theoretical physicists.

One may be interested in studying isolated charge distributions in electromagnetism in special relativity. In this context, the fall-off rates of the charge-current density (j_μ) and electromagnetic field tensor ($F_{\mu\nu}$) can be used to define the "isolated system." For example, one can take a spatially compact charge-current source such that j_μ vanishes outside a timelike world tube, and $F_{\mu\nu} = O(1/r^2)$ as $r \to \infty$ at fixed t, and $F_{\mu\nu} = O(1/r)$ as $r \to \infty$ along any null geodesic. The asymptotic gauge fields (A_μ) and $F_{\mu\nu}$ can be expanded in a multipolar form using Maxwell's equations. In the absence of incoming radiation, one can have a simple relation between the multipole coefficients and the charge-current distribution.

In the attempt to derive similar results in general relativity, one encounters the problem of the precise definition of "isolated system" not being straightforward. In this case, there is no such background flat metric ($\eta_{\mu\nu}$) in terms in which one can write the fall-off rates of the curvature of spacetime metric ($g_{\mu\nu}$). To remedy this problem, one may define spacetime to be asymptotically flat. If there exists any coordinate system, $\{x^0, x^1, x^2, x^3\}$ such that $g_{\mu\nu} = \eta_{\mu\nu} + O(1/r)$ as $r \to \infty$, where $r = [(x^1)^2 + (x^2)^2 + (x^3)^2]$. This problem is solved by formulating asymptotic flatness.

Asymptotic flatness defines an asymptotically flat (AF) spacetime with an appropriate boundary represented by suitably added points at infinity.

Now with this framework, one can get multipolar moments of the gravitational wave outside the compact spatial support in stationary cases. But there does not exist a satisfactory multipolar expansion for gravitational waves in a non-stationary case.

There was a notion that the AF spacetimes are similar to the properties of flat spacetimes. So, the symmetry group for AF spacetimes will be identical to the flat spacetimes, i.e., the Poincare group. The first careful analysis of AF spacetime was carried out by Bondi, van der Burg, Metzner[1] and Sachs[2]. They specified the asymptotic fall-off requirements and showed that the symmetry group for AF spacetime is an infinite dimensional symmetry group called the BMS group, named after Bondi, van der Burg, Metzner, and Sachs. Later Penrose introduced the notion of "Asymptotic Simplcity" [3, 4] stating that the future null infinity ($\mathscr{I}^+$) and past null infinity ($\mathscr{I}^-$) are infinitely differentiable (C^∞) but the timelike and spatial infinity are not smooth. Geroch [5] provided an alternative notion of asymptotic flatness at spatial infinity (i^0) based on the behaviour of initial data on the Cauchy surface at large distances. Ashtekar and Hansen combined these two notions into one [6, 7].

1.1 Conformal infinity

We now have two problems to overcome for a useful formalism for analyzing gravitational radiation and other aspects of the distant gravitational field due to an isolated system.

1. The concept of asymptotic flatness requires a clear definition.

2. We need a meaningful notion of how to take limits at infinity and a precise framework for describing the mathematical entities these limits represent.

We propose a solution for problem 2. We take the Minkowski metric without any gravitational field.

The Minkowski metric has the following form in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.1}$$

Now, let's introduce retarded and advanced coordinates, respectively, as the following:

$$u = t - r \tag{1.2}$$

$$v = t + r \tag{1.3}$$

In new coordinates (u, v, θ, ϕ), the Minkowski metric components are

$$ds^2 = -dudv + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.4}$$

If we try to investigate at $v \to \infty$ keeping u fixed, we may run into complications in curved spacetime. A naive approach to solve this problem will be to express the metric in new coordinates where $V = \frac{1}{v}$. Then the new metric takes the form:

$$ds^2 = \frac{1}{V^2}dudV + \frac{1}{4}(\frac{1}{V} - u)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.5}$$

Here, for $v \to \infty$, we have $V = 0$, but for $v = 0$, the metric blows up. So, we cannot extend the metric to $v = 0$. This is nothing but a case of choosing bad coordinates.

However, lets consider an unphysical metric $\bar{g}_{ab}$ where $\bar{g}_{ab} = V^2\eta_{ab} = \frac{1}{v^2}\eta_{ab}$. η_{ab} is the Minkowski metric. Therefore $\bar{g}_{ab}$ is related to η_{ab} by a conformal transformation $\Omega = V$. So, the new metric $\bar{g}_{ab}$ can be written as

$$d\bar{s}^2 = dudV + \frac{1}{4}(1 - uV)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.6}$$

Now, we can do our tensor analysis in the region where $v \to \infty$ as $v \to \infty$ at a finite distance, but there are other issues with this new metric $\bar{g}_{ab}$. The conformal factor $V = \frac{1}{v}$ blows up at $v = 0$. Although we have extended our metric to future null infinity ($v \to \infty$ at fixed u), we cannot do that for past null infinity ($u \to -\infty$ at fixed v) or spatial infinity ($r \to \infty$ at fixed t). But all these can be resolved if we design the metric as

$$\tilde{g}_{ab} = \Omega^2\eta_{ab} \tag{1.7}$$

with $\Omega^2 = 4(1+v^2)^{-1}(1+u^2)^{-1}$.

Now, we define T, R for Minkowski spacetime by

$$T = \tan^{-1} v + \tan^{-1} u$$

$$R = \tan^{-1} v - \tan^{-1} u \tag{1.8}$$

The following inequality constrains the ranges of T, R.

$$-\pi < T + R < \pi \quad , \quad -\pi < T - R < \pi \quad , \quad 0 \leq R \tag{1.9}$$

The components of $\tilde{g}_{ab}$ in the coordinates (T, R, θ, ϕ) are given by

$$d\tilde{s}^2 = -dT^2 + dR^2 + \sin^2 R(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.10}$$

This is the natural Lorentz metric on $\mathbb{S}^3 \times \mathbb{R}$ which represents Einstein's static universe but with the restrictions, inequalities eq.[(1.9)].

Conformal infinity consists of $i^+, i^-, i^0, \mathcal{I}^+, \mathcal{I}^-$. With this, we can solve the problem[2] we previously had.

1.2 Asymptotically Flat Spacetime

As we have introduced the conformal structure of flat spacetime, let us introduce a metric $g_{\mu\nu}$ to define asymptotic flatness. One can do so in two ways:

1. We can use covariant objects involving the conformal factor used for Penrose compactification.
2. We can use an adapted coordinate system and specify fall-off conditions.

We will take the second route. We would define AF spacetime with respect to $\mathcal{I}^+$. This explains the radiation zone where gravitational waves and other null waves have an impact. Bondi, van der Burg, Metzner [1] and Sachs [2] addressed this problem in the 1960-s. They considered a family of null hypersurfaces labeled by constant u. By construction, the normal vectors to these hypersurfaces $n^\mu = g^{\mu\nu}\partial_\nu u$ are null. Therefore, $g^{uu} = 0$. The angular coordinates $x^A = (\theta, \phi) \ni$ the directional derivative along the normal n^μ is zero, $n^\mu \partial_\mu x^A = 0 \implies g^{uA} = 0$. The radial coordinate r is selected to be the luminosity distance: $\partial_r det(g_{AB}/r^2) = 0$. The coordinates $x^\mu = (u, r, x^A)$ are known as Bondi-Sachs coordinate system or Bondi gauge. Therefore, $g_{rr} = g_{rA} = 0$. The form that the line element takes is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{uu}du^2 + 2g_{ur}dudr + 2g_{uA}dudx^A + g_{AB}dx^A dx^B \tag{1.11}$$

Now, we can define the notion of asymptotic flatness. One can obtain Minkowski spacetime when $r \to \infty$ at constant u, x^A. The line element in retarded coordinates takes the form:

$$ds^2 = -du^2 - 2du dr + r^2 \gamma_{AB} dx^A dx^B, \tag{1.12}$$

where γ_{AB} is the metric on the unit 2-sphere. Therefore, one can demand

$$\lim_{r\to\infty} g_{uu} = \lim_{r\to\infty} g_{ur} = -1 \ , \ \lim_{r\to\infty} g_{uA} = 0 \ , \ \lim_{r\to\infty} g_{AB} = r^2\gamma_{AB} \tag{1.13}$$

The boundary conditions are very restrictive. A phase space with well-defined charges must be defined. But to keep all the physical spacetime, one cannot be too restrictive. The proposed conditions are

$$g_{uu} = -1 + \mathcal{O}(r^{-1}) \ , \ g_{ur} = -1 + \mathcal{O}(r^{-2}) \ , \ g_{uA} = \mathcal{O}(r^0) \ , \ g_{AB} = r^2\gamma_{AB} + \mathcal{O}(r). \tag{1.14}$$

From the fall-off conditions in eq.[(1.14)] the class of metrics we get are:

$$\begin{aligned} ds^2 = \quad & -du^2 - 2du dr + r^2\gamma_{AB} dx^A dx^B (\text{Minkowski}) \\ & + \frac{2m}{r} du^2 + r C_{AB} dx^A dx^B + D^B C_{AB} du dx^A \\ & \frac{1}{16r^2} C_{AB} C^{AB} du dr + \frac{1}{r} + \frac{1}{r}[\frac{4}{3}(N_A + u\partial_A m_B) - \frac{1}{8}\partial_A (C_{BC} C^{BC})] du dx^A \\ & + \frac{1}{4}\gamma_{AB} C_{CD} C^{CD} dx^A dx^B \\ & + (\text{Subleading terms}) \end{aligned} \tag{1.15}$$

All the indices in eq.[(1.15)] are raised with γ^{AB} and $\gamma^{AB} C_{AB} = 0$.

$m(u, x^A)$ is the Bondi mass aspect. $M(u) = \oint_{S^2_\infty} d^2\Omega \, m(u, x^A)$ is Bondi mass. C_{AB}, at the subleading order, is traceless and symmetric. $N_{AB} = \partial_u C_{AB}$ is called Bondi news tensor. $N_A u, x^A$, at the sub-subleading order, is called Bondi angular momentum aspect.

Now the metric in eq.[(1.15)] is not a solution for Einstein's equations. We have to put additional constraints to make the ansatz consistent with Einstein's equations:

$$\partial_u m = \frac{1}{4} D^A D^B N_{AB} - T_{uu} \tag{1.16}$$

$$\text{with } T_{uu} = \frac{1}{8} N_{AB} N^{AB} + 4\pi \lim_{r\to\infty} (r^2 T^M_{uu}) \tag{1.17}$$

and

$$\partial_u N_A = -\frac{1}{4} D^B (D_B D^C C_{CA} - D_A D^C C_{CB}) + u \partial_A (T_{uu} - \frac{1}{4} D^B D^C N_{BC}) - T_{uA} \tag{1.18}$$

$$\text{with } T_{uA} = 8\pi \lim_{r \to \infty} (r^2 T^M_{uA}) - \frac{1}{4} \partial_A (C_{BC} N^{BC}) + \frac{1}{4} D_B (C^{BC} N_{CA}) - \frac{1}{2} C_{AB} D_C N^{BC}. \tag{1.19}$$

1.3 Peeling Property

Though i^0 in AF spacetimes are not smooth, the "Asymptotic Symplicity" condition and Bondi, van der Burg, Metzner [1] and Sachs [2] approach assume no loss of differentiability at $\mathcal{I}^+$. For a C^∞ physical manifold, the Asymptotic Symplicity condition makes the conformal manifold C^∞. This assumption leads to the peeling property of $\mathcal{I}^+$: $C_{\mu\nu\sigma\gamma} = O(\Omega)$, where $C_{\mu\nu\sigma\gamma}$ and Ω are respectively the Weyl tensor and the conformal factor.

This model of AF spacetimes at $\mathcal{I}$ has provided a robust formalism for understanding gravitational radiation and its effect at $\mathcal{I}^+$. But several studies have provided evidence that peeling property at $\mathcal{I}^+$ does not hold in AF spacetimes. Scalar, electromagnetic, and gravitational field perturbation of Schwarzschild background violates peeling property at $\mathcal{I}$ [8]. The gravitational radiation from a collapsing gas cloud with Newtonian limit violates the peeling property at $\mathcal{I}$. Damour has shown that in the presence of quadrupole moment at far past with Newtonian limit in Schwarzschild background, peeling is violated at $\mathcal{I}^+$. Christodoulou [9] has demonstrated that in the presence of N massive particles at far fast with the Newtonian limit in Minkowski background, peeling is violated at $\mathcal{I}^+$ as $u \to -\infty$. Damour later showed that quadrupole moments at far past with Newtonian limit violate peeling at $\mathcal{I}^+$ as $u \to -\infty$ on Schwarzschildian background. In Christodoulou and Damour's analysis, $\log r/r^3$ term appears in the metric, where r is the radial distance. This $\log r/r^3$ term in the metric makes the future null infinity C^2 as $u \to -\infty$. Hence, the peeling is violated.

Peeling at future null infinity can also be studied from the point of view of soft theorems. In Ashoke Sen et al.'s work [10], one can see that due to particles at far past, $\log\omega$ terms appear in the analysis of soft gravitons. But in this analysis, Sen et al. use perturbative metric instead of pure flat metric as the background. In the case of Newtonian limit and Minkowski background, one can see the $\log\omega$ terms emerge only in the case of massive particles. But, Sen et al.'s analysis [10] demonstrates the existence of $\log\omega$ terms in the case of massive and massless particles. The $\log\omega$ terms, in the case of soft gravitons, leads us to the violation of peeling at $\mathcal{I}^+$. This analysis implies the violation of peeling at $\mathcal{I}^+$ as $u \to +\infty$ and $u \to -\infty$.

This thesis aims to study the structure of Logarithmically Asymptotic Flat (LAF) Spacetime at $\mathcal{I}^+$. Such spacetimes are not asymptotically simple, and instead of peeling, they follow partial peeling [11]:

$$C_{\mu\nu\sigma\gamma} = O(\Omega \log \Omega).$$

We will also attempt to study different systems and check if verify if they follow or violate the peeling property at $\mathcal{I}^+$ as $u \to \pm\infty$ with pure Minkowski background. We try to use Corvino's gluing construction to construct a metric with a C^2 differentiability. But we run into a few problems, so we try to study the system with N massive particles at far past with the Newtonian limit. Then we analyze soft gravitons with massless scalar and massive scalar fields on the Minkowski background. From this analysis, we try to enquire if peeling is violated in such cases.

In sec[2], we have shown the calculations and used methods in this thesis. In sec[2.1], we discussed Corvino's gluing construction, attempting to glue manifolds with different differentiabilities so that the future null infinity violates peeling. Then we turn to the analysis of linearized gravitational wave in the case of a spatially compact source in sec[2.2]. Next, we discuss the generation of non-artifact log terms in the metric at the future null infinity in sec[2.3]. Further, we study the concept of partial peeling and the metric components in LAF spacetimes. Then in sec[2.5.1], we go through some necessary mathematical tools and results needed in the further analysis in our thesis. Sec[2.7] discusses the effects of soft gravitons on the future null infinity as $u \to \pm\infty$. We show in sec[2.8] that a system with N massive particles at far past (far future) on the 4- dimensional Minkowski background violates peeling at $\mathcal{I}^+_-$ ($\mathcal{I}^+_+$). We demonstrate in sec[2.9.3] that a massless scalar field on the 4- dimensional Minkowski background does not violate peeling $\mathcal{I}^+$. We show in sec[2.10.3] that a massive scalar field on the 4- dimensional Minkowski background violates peeling at $\mathcal{I}^+_\pm$. Then we summarize the results and discuss the implications of the results in sec[3]. At last, we conclude our thesis in sec[4].

Chapter 2

METHODS AND CALCULATIONS

2.1 Corvino's Gluing construction:

A gluing construction by Corvino demonstrated the existence of non-trivial flat scalar flat metrics that behave like the Schwarzschild metric at large distances. This method glues an asymptotically flat (AF) metric g with the Schwarzschild metric on $B(0,2R_0) \setminus B(0,R_0)$ annulus and preserves the condition: $R(g) = 0$, where R(g) is Ricci scalar.

2.1.1 Creating Vacuum Spacetimes with Higher Degree of Differentiability at the Null Infinity:

One may think that a vacuum spacetime admitting conformal compactifications at null infinity with a higher degree of differentiability structure and a global $\mathcal{I}$ can be created by Corvino's construction. The Schwarszchildian or Kerrian metrics contain hyperboloidal hypersurfaces near i^0. In such cases, Friedrich's Stability theorem can yield asymptotically simple spacetimes if the initial data is close to Minkowski spacetime.

2.1.2 Problem in Creating Such Vacuum Spacetimes :

Suppose there is a sequence of data (g_i, K_i). In that case, the gluing radius in Corvino's construction can tend to infinity, resulting in the non-zero norm of hyperboloidal initial data for Friedrich's stability theorem.

2.1.3 Avoiding the Problem:

Imposing a parity condition on the initial data sets can avoid the problem. Here a slight variation of Corvino's construction is used. An extension can be produced across the boundary $S(0,R) = \partial B(0,R)$ for any fixed radius R. This can be done for small initial datasets regardless of whether they arise from the AF initial data sets. As a result, it is possible to create an asymptotically simple family of infinite-dimensional vacuum spacetimes in Penrose's notion. But this method creates C^k differentiable conformal compactification instead of C^∞ differentiable conformal compactification.

2.1.4 Extensions on Initial Data Sets:

Smith and Weinstein's Method:

Suppose, we have a vacuum initial data set (M,K,g)[1], where $\overline{M} = M \cup \partial M$. ∂M is compact boundary of M, with (K,g) extending smoothly, or in $C^k(\overline{M})$ to the boundary. In such a case, there exists an extension using the method of Smith and Weinstein [12]. But for Schwarzschildian extension and non-trivial extrinsic curvature, it is unclear how this method can be implemented.

Corvino and Schoen's Method:

The advantage of the results provided by Corvino and Schoen [13] is that they can obtain alternative extension without the assumptions of the mean outer convex boundary and vanishing K. The metric loses lesser differentiability in this method than in the Smith-Weinstein technique. This method gives C^k extensions of C^{2k+1} metrics, where $k \in \mathbb{Z}^+ \cup \{0\}$. Let's assume that $\exists (K,g) \in (C^{k+2} \times C^{k+3})(\overline{M})$, $k \geq 4$, $\ni (K,g)$ satisfies the vacuum constraints on manifold $\overline{M}$ with compact boundary. Let M_0 be a manifold $\ni \partial M_0$ is diffeomorphic to ∂M and M' be a manifold constructed by gluing M and M_0 across ∂M. Let x be any smooth function in the neighbourhood $\mathscr{W}$ of ∂M on M', with $\partial M = \{x = 0\}$ and $\nexists p \in \partial M \ni dx(p) = 0$ and $x > 0$ on M_0. Now, let $\mathscr{V} := \mathscr{W} \cap M_0$ be diffeomorphic to $\partial M \times [0, x_0]$, where x is coordinate along the $[0, x_0]$ factor.

Suppose, on M_0 $\exists (K_0, g_0) \in (C^{k+2} \times C^{k+3})(\overline{M}_0)$ satisfying the vacuum constraints. (K,g) and (K_0, g_0) may not match across ∂M. Standard techniques extend (K,g) to $(\hat{K}, \hat{g})$ on $M_0 \ni$

1. $(\hat{K}, \hat{g}) \in C^{k+2} \times C^{k+3}$

[1] K is $(0,2)$-symmetric tensor on manifold M and is discussed in the Appendix (B.7).

2. $(\hat{K}, \hat{g}) = (K_0, g_0)$ on $M_0 \setminus \mathscr{V}$

3.

$$||\hat{g} - g_0||_{C^{k+3}(\mathscr{V})} \leq C \sum_{i=0}^{k+3} ||\partial_x^i g|_{\partial M} - |\partial_x^i g_0|_{\partial M}|_{C^{k+3-i}(\partial M)} \tag{2.1}$$

$$||\hat{K} - K_0||_{C^{k+2}(\mathscr{V})} \leq C \sum_{i=0}^{k+2} ||\partial_x^i K|_{\partial M} - |\partial_x^i K_0|_{\partial M}|_{C^{k+2-i}(\partial M)} \tag{2.2}$$

4. $\forall\, 0 \leq i \leq k+1$

$$|(\hat{\nabla})^{(i)} \rho(\hat{K}, \hat{g})|_{\hat{g}} + |(\hat{\nabla})^{(i)} J(\hat{K}, \hat{g})|_{\hat{g}} \leq C(||\hat{g} - g_0||_{C^{k+3}(\mathscr{V})} + ||\hat{K} - K_0||_{C^{k+2}(\mathscr{V})}) x^{k+1-i} \tag{2.3}$$

$\rho \equiv$ scalar constraint operator

$J \equiv$ vector constraint operator

C is constant that may depend upon $||\hat{g} - g_0||_{L^\infty}$ and $||\hat{K} - K_0||_{L^\infty}$. The first extension is performed under the assumptions that $\nexists (Y, N) \ni P^*(Y, N) = 0$ on $\mathscr{V}$, where

$$P^*(Y, N) = \begin{pmatrix} 2(\nabla_{(i} Y_{j)}) - \nabla^l Y_l g_{ij} - K_{ij} N + \mathrm{tr}(K) N g_{ij} \\ \\ \nabla^l Y_l K_{ij} - 2K^l_{(i} \nabla_{j)} Y_l + K^q_l \nabla_q Y^l g_{ij} - \Delta N g_{ij} + \nabla_i \nabla_j N \\ +(\nabla^p K_{lp} g_{ij} - \nabla_l K_{ij}) Y^l - N \mathrm{Ric}(g)_{ij} + 2N K^l_i K_{jl} - 2N \mathrm{tr}(K) K_{ij} \end{pmatrix} \tag{2.4}$$

Zero Kernel on $\mathscr{V}$:

Indicating the existence of Killing vectors in the corresponding globally hyperbolic vacuum spacetime, non-trivial fields that fulfill $P*(Y, N) = 0$ are referred to as Killing initial data (KID) [14]. If (K, g) and its derivatives are sufficiently close to (K_0, g_0) on ∂M up to appropriate order as described in eq(2.1)-(2.2), $\exists$ vacuum initial dataset $(K_0 + \delta K, g_0 + \delta g) \in (C^k \times C^k)(\overline{\mathscr{V}}) \ni$ all of its derivatives up to order k coincides with those of (K, g) on $\{0\} \times \partial M$.

The construction mentioned above provides new non-trivial extensions in the following situation:

1. $(K, g) \in \{(K_\lambda, g_\lambda)\}$, where $\{(K_\lambda, g_\lambda)\}$ is a one parameter family of solutions of vacuum constraint equations on M.

2. $\lambda \to 0 \implies (K_\lambda, g_\lambda) \to (K_0|_M, g_0|_M)$ in $(C^{k+2} \times C^{k+3})(\overline{M})$

Non-zero Kernel:

The situation gets more complicated than 2.1.4 if the kernel is non-zero or the families of metrics near a metric have a non-zero kernel. We face such complications while trying to construct AF metrics with a small mass. Let's consider a case where $\overline{M} \underset{submanifold}{\subset} M' = \mathbb{R}^3 \ni \partial M$ is smooth ; $K_0 \equiv 0$ and $g_0 =$ Euclidean metric. The condition of $\overline{M}$, being a submanifold of M', can be made without the loss of generality in the following sense: Any 2-dimensional orientable manifold can be embedded into $\mathbb{R}^3$. Hence, a tubular neighborhood can also be embedded into $\mathbb{R}^3$. $\overline{M}$ can be embedded in $\mathbb{R}^3$ if $\overline{M}$ is replaced by a tubular neighbourhood $(-x_0, 0] \times \partial M$. Then, the closure of $\overline{M}$ will have two boundaries: $\{-x_0\} \times \partial M$ and $\{0\} \times \partial M$.But, we will ignore $\{-x_0\} \times \partial M$. $\{0\} \times \partial M$ will be considered the outer boundary of M seen from infinity. We will further assume that (k, g) is close to (K_0, g_0):

$$||g - g_0||_{C^{k+3}(M)} + ||K - K_0||_{C^{k+2}(M)} < \epsilon \tag{2.5}$$

and that

$$g(x) = g(-x) \quad , \qquad K(x) = -K(-x); \tag{2.6}$$

A family of initial data, constructed via the conformal method, is referred to as *parity-covariant*. One can create constructions preserving the parity-covariance, and here, we will only discuss such extensions.

Only $\hat{K} = K_0 = 0$ case is considered here for definiteness, even though the same argument can be applied for appropriately small K-s. All of the symmetry properties of the original data are preserved by Corvino's creations, which glue together "up to kernel" $\hat{g}$ with standard (non-translated) Schwarzschild metrics g_m, where $m \in (-\delta, \delta)$ with $\delta \leq \min(1, 1/R)$, maintaining parity-covariance of "solutions up to kernel" $(\hat{K} + \delta K_m = 0, \hat{g} + \delta g_m)$. Parity covariance implies that the center of mass of the resulting metric is zero. So, the disappearance of the integral over $R(\hat{g} + \delta g_m)$ over $\mathscr{V}$ is the only challenge to the metric's requirement to be scalar flat in the proof in Corvino[15]. Let m_0 ($\leq C\epsilon$) be the mass of (K, g), naively calculated using ADM integral over $S(0, R)$.

$$\frac{1}{16\pi}\int_{[0,x_0]\times\partial M} R(\hat{g} + \delta g_m) = m - m_0 + O(\epsilon^2) \tag{2.7}$$

If the reference Schwarzschild metric g_m has mass $m = m_0 - \epsilon$, the RHS in eq(2.7) will be a negative value. If $m = m_0 + \epsilon$, the RHS in eq(2.7) will be positive value. Since, LHS in eq(2.7) depends continuously on m, $\exists\; m \in (m_0 - \epsilon, m_0 + \epsilon) \ni$ LHS is zero.

If $K \neq 0$, one needs to choose a constant $0 \leq \lambda \leq 1$ and put the constraint on initial data sets such that

$$|\vec{p}_0|_\delta \leq \lambda m_0 \quad , \tag{2.8}$$

where $\vec{p}_0$ is the ADM momentum of (M, K, g).

Theorem: Let's assume that vacuum initial data sets $(K, g) \in C^{l+3} \times C^{l+4}$, on a compact submanifold $\overline{M}$ of $\mathbb{R}^3$, is parity-covariant, where $l \geq 3$. Suppose that $\exists \lambda \in [0,1] \ni$ eq(2.8) is satisfied. Let Ω be any bounded domain containing $\overline{M}$. Then $\exists\; \epsilon > 0 \ni$ if eq(2.5) holds good, $\exists$ a vacuum $C^l \times C^l$ extensions of (K, g) across the part of ∂M, which is homologous to large coordinate spheres in the asymptotically flat region, with the extensions being Kerrian outside Ω.

2.1.5 Initial Data with Non-connected Trapped Surfaces ("Many Black-holes Initial Data")

The following time-symmetric initial data for a vacuum spacetime can be created using the extension technique.

1. $\exists\mathscr{K}$, a compact set $\ni g$ is a Schwarzschild metric with parameter m on the connected component of $M \setminus \mathscr{K}$

2. On the Schwarzschild-Kruskal-Szekeres manifold, M comprises $2N+1$ such surfaces, with the metric being Schwarzschild in the neighborhood of each corresponding S. Let S denote the usual minimal sphere within the time-symmetric initial data.

By gluing $2N+1$ Schwarzschild metrics, (M,g) is constructed. An initial data set (M,g) contains $2N$ black holes.

Construction can be presented where one chooses two strictly positive radii $0 < 4R_1 < R_2 < \infty$ and $\vec{x}_i \in \Gamma_0(4R_1,R_2) := B(0,R_2)\setminus\overline{B(0,4R_1)}$ $\forall i \in \{1,2,\ldots,2N\}$. Then the radii r_i are chosen such that $B(\vec{x}_i,4r_i)$ are disjoint are pairwise disjoint and $B(\vec{x}_i,4r_i) \in \Gamma_0(4R_1,R_2)$ $\forall i$.

Lets set $\Omega := \Gamma_0(4R_1,R_2)\setminus(\bigcup_i \overline{B(\vec{x}_i,4r_i)})$ and assume that Ω is invariant under the parity map $\vec{x} \to -\vec{x}$. Let $\vec{M} = (m,m_0,m_1,\ldots,m_{2N})$ such that $2m < 2R_1$, $2m_0 < R_1$, $2m_i < r_i$ and lets construct the metric $g_{\vec{M}}$ in the following manner.

1. Schwarzschild metric $g_{\vec{M}}$, with mass m_0, is centred at 0 on $\Gamma_0(R_1,2R_1)$.

2. Schwarzschild metric $g_{\vec{M}}$, with mass m_0, is centred at 0 on $\Gamma_0(3R_1,R_2)\setminus(\bigcup_i \overline{B(\vec{x}_i,4r_i)})$.

3. The metric $g_{\vec{M}}$ interpolate between two Schwarzschild metric on $\Gamma_0(2R_1,3R_1)$.

4. Schwarzschild metric with mass m_i, is centred at $\vec{x}_i$ on $\Gamma_{\vec{x}_i}(r_i,2r_i) := B(\vec{x}_i,2r_i)\setminus\overline{B(\vec{x}_i,r_i)}$.

5. The metric interpolates between two metrics on $\Gamma_{\vec{x}_i}(2r_i,3r_i)$.

6. For $i = 1,\ldots,2N$, the masses mi are chosen, and the gluings are carried out in such a way that the resulting metric is symmetric under the parity map $\vec{x} \to -\vec{x}$.

Now, one may think of gluing manifolds with logarithmic terms to create a spacetime with a future null infinity that has differentiability less than 3, therefore violating peeling conditions. But it

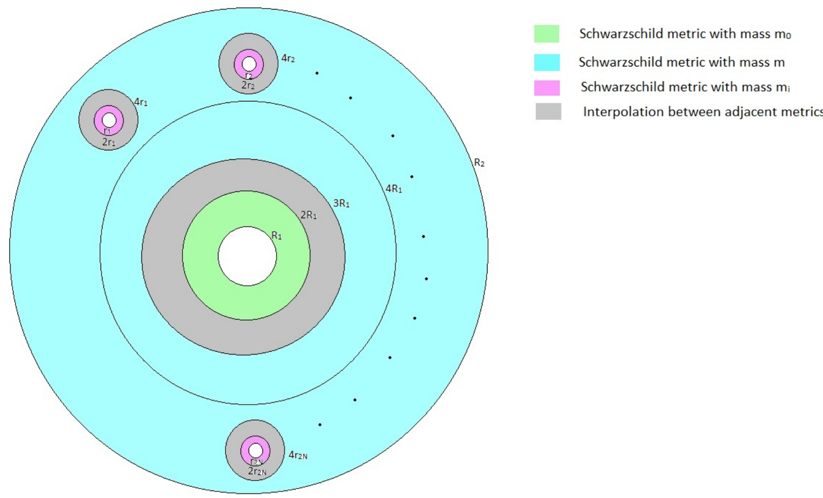

Figure 2.1: The diagram of the construction gluing $2N+1$ Schwarzschild metrics are glued together in sec[2.1.5]

is difficult to determine if some coordinate transformation exists that can eliminate the logarithmic terms in the metric. From this construction, it is also hard to get an intuitive picture of the physical conditions of such a spacetime.

2.2 Gravitational Radiation from Post-Newtonian Sources and Spatial Compact Support

2.2.1 Vacuum Field Equations with Nonlinear Iteration

Einstein's Field Equations

The famous Einstein-Hilbert action is following.

$$I_{\mathrm{EH}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R + I_{\mathrm{mat}}\left[\Psi, g_{\alpha\beta}\right] \quad , \tag{2.9}$$

where $-g \equiv \det(g)$, $\Psi \equiv$ is the matter field, $R \equiv$ Ricci scalar, $G \equiv$ Universal gravitaional constant and $c \equiv$ speed of light.

If we vary the Einstein-Hilbert action with respect to the space-time covariant metric $g_{\alpha\beta}$, we arrive at Einstein's field equations.

$$E^{\alpha\beta}\left[g, \partial g, \partial^2 g\right] = \frac{8\pi G}{c^4}T^{\alpha\beta}[\Psi, g] \tag{2.10}$$

$E^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}$ and $T^{\alpha\beta} \equiv \frac{2}{\sqrt{-g}}\delta I_{\mathrm{mat}} / \delta g_{\alpha\beta}$ in eq(2.10) are respectively the Einstein's curvature tensor and stress-energy tensor. Among these ten equations in eq(2.10), the contracted Bianchi Identity governs four equations. Contracted Bianchi identity gives the evolution matter system,

$$\nabla_\mu E^{\alpha\mu} = 0 \quad \Longrightarrow \quad \nabla_\mu T^{\alpha\mu} = 0. \tag{2.11}$$

We can also vary the matter action with respect to Ψ and obtain the matter equations. The remaining six equations in eq(2.10) place six independent constraints on the ten components of the metric $g_{\alpha\beta}$. The rest of the four can be fixed by coordinate choice.

If we choose the harmonic coordinates or de Donder coordinates, we get

$$h^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta} - \eta^{\alpha\beta}, \tag{2.12}$$

where $g_{\alpha\beta}$ is the covariant metric satisfying $g^{\alpha\mu}g_{\mu\beta} = \delta^\alpha_\beta$ and $\eta_{\alpha\beta}$ is the auxiliary Minkowski metric $\eta^{\alpha\beta} \equiv \mathrm{diag}(-1,1,1,1)$. The harmonic coordinate condition is responsible for the rest of the four constraints:

$$\partial_\mu h^{\alpha\mu} = 0 \tag{2.13}$$

The formulation of our coordinate system is given a preferable Minkowskian structure by eq(2.13), where the covariant Minkowski metric is $\eta_{\alpha\beta}$. It is quite convenient to examine gravitational waves as perturbations of space-time propagating on the constant background metric $\eta_{\alpha\beta}$ using the coordinate condition in eq(2.13).

Einstein's field equations can be expressed as inhomogeneous flat d'Alembertian equations in harmonic coordinates.

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta} \tag{2.14}$$

where $\Box \equiv \Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The source term $\tau^{\alpha\beta}$ can be interpreted as the stress-energy pseudo-tensor ($\tau^{\alpha\beta}$ is a Lorentz-covariant tensor) of the matter fields (i.e., described by $T^{\alpha\beta}$) and the gravitational field (i.e., given by the gravitational source term $\Lambda^{\alpha\beta}$).

$$\tau^{\alpha\beta} = |g|T^{\alpha\beta} + \frac{c^4}{16\pi G}\Lambda^{\alpha\beta} \tag{2.15}$$

Including all the non-linear terms, $\Lambda^{\alpha\beta}$ in harmonic coordinates takes the following from:

$$\begin{aligned}\Lambda^{\alpha\beta} = & -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ & - g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ & + \frac{1}{8}\left(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu}\right)(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\,\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}\end{aligned} \tag{2.16}$$

From the expression in eq(2.16), one can see that the terms are at least quadratic in gravitational field strength. $\Lambda^{\alpha\beta}$ can be expressed as follows

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}[h,h] + M^{\alpha\beta}[h,h,h] + L^{\alpha\beta}[h,h,h,h] + \ldots. \tag{2.17}$$

One can directly compute the various terms in eq(2.17). As an example

$$\begin{aligned}N^{\alpha\beta} = & - h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \frac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu} - \frac{1}{4}\partial^\alpha h\partial^\beta h + \partial_\nu h^{\alpha\mu}\left(\partial^\nu h^\beta_\mu + \partial_\mu h^{\beta\nu}\right) \\ & - 2\partial^{(\alpha}h_{\mu\nu}\partial^\mu h^{\beta)\nu} + \eta^{\alpha\beta}\left[-\frac{1}{4}\partial_\tau h_{\mu\nu}\partial^\tau h^{\mu\nu} + \frac{1}{8}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h_{\nu\tau}\partial^\nu h^{\mu\tau}\right]\end{aligned} \tag{2.18}$$

The condition in eq(2.13) is equivalent to the matter equations of motion. The conservation of the total pseudo-tensor $\tau^{\alpha\beta}$

$$\partial_\mu \tau^{\alpha\mu} = 0 \quad \Longleftrightarrow \quad \nabla_\mu T^{\alpha\mu} = 0. \tag{2.19}$$

Using the following four hypotheses, one can look for approximations to the solutions of the field eq(2.13)-(2.14):

1. The harmonic-coordinate radial distance is $r = |\mathbf{x}|$, and the matter stress-energy tensor $T^{\alpha\beta}$ is of spatially compact support and enclosed within some time-like world tube, say $r \leqslant a$. According to eq(2.19), the gravitational source term is divergence-free outside the source domain when $r > a$.

$$\partial_\mu \Lambda^{\alpha\mu} = 0 \quad (\text{ when } r > a) \tag{2.20}$$

2. $T^{\alpha\beta}(x) \in C^\infty\left(\mathbb{R}^3\right)$ for $r \leq a$.

3. In terms of the small parameter, the source is post-Newtonian. In order to determine the inner post-Newtonian field and the source's outer near zone, respectively, we assume that the approach of matching asymptotic expansions and the outer multipolar decomposition are valid.

4. We assume that the gravitational field was stationary before some finite instant $-\mathcal{T}$ in the past, namely

$$\frac{\partial}{\partial t}\left[h^{\alpha\beta}(\mathbf{x},t)\right]=0 \quad \text{when } t\leqslant -\mathcal{T}. \tag{2.21}$$

Because the no incoming radiation criterion guarantees that the matter source is an isolated system that doesn't receive radiation from infinity, the fourth hypothesis is put into place. At past null infinity, we should impose the no-incoming radiation condition.

Due to eq(2.21), the differential equation in eq(2.14) can be written as follows:

$$h^{\alpha\beta}=\frac{16\pi G}{c^4}\Box^{-1}_{\text{ret}}\,\tau^{\alpha\beta} \tag{2.22}$$

where the retarded inverse d'Alembertian integral operator is

$$\left(\Box^{-1}_{\text{ret}}\,\tau\right)(\mathbf{x},t)\equiv -\frac{1}{4\pi}\iiint_{\mathbb{R}^3}\frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\tau\left(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c\right) \tag{2.23}$$

Linearized vacuum equations

The linearized perturbation of the metric in Post-Minkowskian expansion can be written as

$$h^{\alpha\beta}_{\text{ext}}=Gh^{\alpha\beta}_{(1)}+\mathcal{O}\left(G^2\right), \tag{2.24}$$

where "ext" in the subscript represents the metric exterior to the compact source.

Hence, the conditions in eq(2.13),(2.14) combined with eq(2.24) gives

$$\Box h^{\alpha\beta}_{(1)}=0, \tag{2.25a}$$

$$\partial_\mu h^{\alpha\mu}_{(1)}=0. \tag{2.25b}$$

We can use symmetric-trace-free (STF) harmonics to implement the multipolar-post-Minkowskian (MPM) approach in order to explain the multipole expansion [16] and to search for a specific algorithm for the approximation scheme [**Blanchet Damour**]. The retarded multipolar waves represent the series of equations that make up the solution of (2.25).

$$h^{\alpha\beta}_{(1)}=\sum_{\ell=0}^{+\infty}\partial_L\left(\frac{\mathrm{K}^{\alpha\beta}_L(t-r/c)}{r}\right), \tag{2.26}$$

where $r = |\mathbf{x}|$, and $\mathrm{K}_L^{\alpha\beta} \equiv \mathrm{K}_{i_1 \ldots i_\ell}^{\alpha\beta} \ni \mathrm{K}_L(u) \in C^\infty(\mathbb{R})$, where $u \equiv t - r/c$. $K_L(u) = \text{constant}$, when $t \leqslant -\mathcal{T}$. Since for a monopolar wave, $\Box(\mathrm{K}_L(u)/r) = 0$.

The most general solution of eq(2.25) outside the time-like world tube enclosing the source and stationary in the past [eq(2.21)] is

$$h_{(1)}^{\alpha\beta} = k_{(1)}^{\alpha\beta} + \partial^\alpha \varphi_{(1)}^\beta + \partial^\beta \varphi_{(1)}^\alpha - \eta^{\alpha\beta} \partial_\mu \varphi_{(1)}^\mu . \tag{2.27}$$

The first term depends on two symmetric trace-free (STF) tensorial multipole moments, $I_L(u)$ and $J_L(u)$, which are arbitrary functions of time except for conservation laws of the monopole: $I = \text{constant}$, and dipoles: $I_i = \text{constant}$; $J_i = \text{constant}$.

$$k_{(1)}^{00} = -\frac{4}{c^2} \sum_{\ell \geqslant 0} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r} \mathrm{I}_L(u) \right) \tag{2.28a}$$

$$k_{(1)}^{0i} = \frac{4}{c^3} \sum_{\ell \geqslant 1} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left(\frac{1}{r} \mathrm{I}_{iL-1}^{(1)}(u) \right) + \frac{\ell}{\ell+1} \epsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} \, \mathrm{J}_{bL-1}(u) \right) \right\}, \tag{2.28b}$$

$$k_{(1)}^{ij} = -\frac{4}{c^4} \sum_{\ell \geqslant 2} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-2} \left(\frac{1}{r} \mathrm{I}_{ijL-2}^{(2)}(u) \right) + \frac{2\ell}{\ell+1} \partial_{aL-2} \left(\frac{1}{r} \epsilon_{ab(i} \mathrm{J}_{j)bL-2}^{(1)}(u) \right) \right\} \tag{2.28c}$$

A linearized gauge transformation is represented by the other terms, with gauge vector $\varphi_{(1)}^\alpha$ parametrized by four other multipole moments, say $\mathrm{W}_L(u), \mathrm{X}_L(u), \mathrm{Y}_L(u)$ and $\mathrm{Z}_L(u)$.

The MPM Solution

By Theorem2.2.1, One can present the most general solution of the linearized equations in the exterior of the source as

$$h_{\text{ext}}^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_{(n)}^{\alpha\beta} . \tag{2.29}$$

Upon substitue the Post-Minkowskian ansatz eq(2.30) in eq(2.14), we can get

$$\Box h_{(n)}^{\alpha\beta} = \Lambda_{(n)}^{\alpha\beta} \left[h_{(1)}, h_{(2)}, \ldots, h_{(n-1)} \right], \tag{22a}$$

$$\partial_\mu h_{(n)}^{\alpha\mu} = 0. \tag{22b}$$

In more details, eq(2.30) reads

$$\Box h^{\alpha\beta}_{(2)} = N^{\alpha\beta}\left[h_{(1)}, h_{(1)}\right], \tag{2.31a}$$

$$\Box h^{\alpha\beta}_{(3)} = M^{\alpha\beta}\left[h_{(1)}, h_{(1)}, h_{(1)}\right] + N^{\alpha\beta}\left[h_{(1)}, h_{(2)}\right] + N^{\alpha\beta}\left[h_{(2)}, h_{(1)}\right], \tag{2.31b}$$

$$\begin{aligned}\Box h^{\alpha\beta}_{(4)} = {} & L^{\alpha\beta}\left[h_{(1)}, h_{(1)}, h_{(1)}, h_{(1)}\right] \\ & + M^{\alpha\beta}\left[h_{(1)}, h_{(1)}, h_{(2)}\right] + M^{\alpha\beta}\left[h_{(1)}, h_{(2)}, h_{(1)}\right] + M^{\alpha\beta}\left[h_{(2)}, h_{(1)}, h_{(1)}\right] \\ & + N^{\alpha\beta}\left[h_{(2)}, h_{(2)}\right] + N^{\alpha\beta}\left[h_{(1)}, h_{(3)}\right] + N^{\alpha\beta}\left[h_{(3)}, h_{(1)}\right], \end{aligned} \tag{2.31c}$$

$$\vdots$$

Now, we want to expand $\Lambda^{\alpha\beta}_{(n)}$ into multiple contributions, with a singularity at $r = 0$, and satisfies the d'Alembertian equation when $r > 0$. We can obtain such a solution by the following method of [17]. One can first regularize the source term $\Lambda^{\alpha\beta}_{(n)}$ by multiplying it by the factor r^B, where $r = |\mathbf{x}|$ is the spatial radial distance and B is a complex number, $B \in \mathbb{C}$. Let's assume that $\Lambda^{\alpha\beta}_{(n)}$ is composed of multipolar pieces with maximal multipolarity $\ell_{\max}$.

$$I^{\alpha\beta}(B) \equiv \Box^{-1}_{\text{ret}}\left[\widetilde{r}^{B}\Lambda^{\alpha\beta}_{(n)}\right], \tag{2.32}$$

where $\Box^{-1}_{\text{ret}}$ stands for the retarded integral defined by eq(2.23). For convenience, the regularizing factor is made dimensionless by introducing some arbitrary constant length scale r_0.

$$\widetilde{r} \equiv \frac{r}{r_0}. \tag{2.33}$$

We can write the Laurent expansion of $I^{\alpha\beta}(B)$ when $B \to 0$ in the following way.

$$I^{\alpha\beta}(B) = \sum_{p=p_0}^{+\infty} \iota^{\alpha\beta}_{p} B^{p}, \tag{2.34}$$

Here, $p \in \mathbb{Z}$, and $\iota_p^{\alpha\beta} \equiv \iota_p^{\alpha\beta}(\mathbf{x},t)$. When $p_0 \leqslant -1$, there are poles; and $-p_0$, which depends on n, referring to the maximal order of these poles. One can equate the different powers of B by applying the d'Alembertian onto both sides of eq(2.34) and get

$$p_0 \leqslant p \leqslant -1 \quad \Longrightarrow \quad \Box\iota_p^{\alpha\beta} = 0, \tag{2.35a}$$

$$p \geqslant 0 \quad \Longrightarrow \quad \Box\iota_p^{\alpha\beta} = \frac{(\ln r)^p}{p!}\Lambda_{(n)}^{\alpha\beta}. \tag{2.35b}$$

Hence, we get that $\Box\iota_0^{\alpha\beta} = \Lambda_{(n)}^{\alpha\beta}$. Now, let's say that $u_{(n)}^{\alpha\beta} \equiv \iota_0^{\alpha\beta}$. Therefore

$$u_{(n)}^{\alpha\beta} = \mathcal{FP}_{B=0}\Box_{\text{ret}}^{-1}\left[\widetilde{r}^B\Lambda_{(n)}^{\alpha\beta}\right], \tag{2.36a}$$

$$w_{(n)}^{\alpha} = \partial_\mu u_{(n)}^{\alpha\mu} = \mathcal{FP}_{B=0}\Box_{\text{ret}}^{-1}\left[B\widetilde{r}^B\frac{n_i}{r}\Lambda_{(n)}^{\alpha i}\right]. \tag{2.36b}$$

Now, we define $v_{(n)}^{\alpha\mu}$ such that $\partial_\mu v_{(n)}^{\alpha\mu} = -w_{(n)}^{\alpha}$. And then, if we pose

$$h_{(n)}^{\alpha\beta} = u_{(n)}^{\alpha\beta} + v_{(n)}^{\alpha\beta}, \tag{2.37}$$

eqs(2.30) are satisfied.

Generality of the MPM solution

The most general solution of Einstein's field equations in the vacuum region outside an isolated source, admitting some MPM expansions, is given by (in the harmonic coordinates)

$$h_{ext}^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_{(n)}^{\alpha\beta}\left[\mathrm{I}_L,\ \mathrm{J}_L,\ldots,\mathrm{Z}_L\right]. \tag{2.38}$$

It is dependent on two sets of arbitrary STF-tensorial functions of time $\mathrm{I}_L(u)$ and $\mathrm{J}_L(u)$ (satisfying the conservation laws).

2.2.2 Structure of Near-zone and far-zone

In the near-zone (when $r \to 0$), the expansion of the post-Minkowskian exterior metric has the following general structure:$\forall N \in \mathbb{N}$,

$$h_{(n)}(\mathbf{x},t) = \sum \hat{n}_L r^m (\ln r)^p F_{L,m,p,n}(t) + o\left(r^N\right), \tag{2.39}$$

where $m \in \mathbb{Z}$, with $m_0 \leqslant m \leqslant N$ (and m_0 becoming increasingly negative as n grows), $p \in \mathbb{N}$ with $p \leqslant n-1$. The functions $F_{L,m,p,n}$ are multilinear functionals of the source multipole moments $\mathrm{I}_L, \ldots, \mathrm{Z}_L$.

Near-Zone Expansion

If we restore the powers of c in eq(2.39) and use the fact that $r \longrightarrow \frac{r}{c}$, the structure of post-Minkowskian expansion ($c \longrightarrow \infty$) is

$$h_{(n)}(c) \simeq \sum_{p,q \in \mathbb{N}} \frac{(\ln c)^p}{c^q}, \tag{2.40}$$

where $p \leqslant n-1$ (and $q \geqslant 2$).

Far-Zone Expansion

The structure of the far-zone expansion at future null infinity can be obtained by paralleling the near-zone expansion's structure. When $r \longrightarrow +\infty$ with $u = t - r/c = \text{constant}$: $\forall N \in \mathbb{N}$

$$h_{(n)}(\mathbf{x}, t) = \sum \frac{\hat{n}_L (\ln r)^p}{r^k} G_{L,k,p,n}(u) + O\left(\frac{1}{r^N}\right) \tag{2.41}$$

where $k, p \in \mathbb{N}$, with $1 \leqslant k \leqslant N$, and were, similar to the near-zone expansion, some powers of logarithms, such that $p \leqslant n-1$, appear. The studies of Bondi et al. [1], Sachs [2] and Penrose [3, 4] have shown the absence of the logarithmic terms in the metric if other coordinate systems are chosen. Hence, logarithmic terms can be concluded to be the "artifact" terms.

2.3 Non-artifact Log Terms

For the differentiability of $\mathcal{I}^+$ to be finite, the perturbation to the Minkowski metric needs to have non-artifact $\log r$ terms. The non-artifact $\log r$ terms in the metric will come due to the physical system rather than the coordinate choice.

2.3.1 Christodoulou's Analysis

Christodoulou [9] demonstrated that massive particles at far past with Newtonian limit on Minkowski background violate peeling at $\mathcal{I}^+$.

He showed that there exists a limit such that $\lim_{u\to-\infty} u^2 N_{\mu\nu}(x) \neq 0$. We will define $\Xi^- = \lim_{u\to-\infty} u^2 N_{\mu\nu}(x)$. We can show that

$$\lim_{r\to\infty} \partial_u (r^4\beta) = \frac{\mathcal{D}^{(3)}\Xi^-}{|u|}, \tag{2.42}$$

where third order differential operator on unit sphere $\mathbf{S}^2$ is represented by $\mathcal{D}^{(3)}$. If we integrate eq.[(2.42)], we get

$$\lim_{u\to-\infty} (r^4\beta)(u_1(t),t) - (r^4\beta)(u_2(t),0) \sim \int_{u_1(t)}^{u_2(t)} \frac{\mathcal{D}^{(3)}\Xi^-}{|u|} du \sim (\log r - \log|u|)\mathcal{D}^{(3)}\Xi^-. \tag{2.43}$$

Therefore,

$$\lim_{\mathcal{I}^+, u\to-\infty} \beta = B^* \frac{\log r - \log|u|}{r^4}, \tag{2.44}$$

where B^* depends on the quadrupole distribution.

The $\log r$ terms appearing in eq.[(2.44)] is not an artifact term. Instead, it appears due to the system considered.

2.3.2 Damour's Analysis

Sachs [2] pointed out in his work that Bondi, Metzner, and Sachs's work is conducted taking the linear perturbation to the Minkowski metric into account. Their analysis results in the absence of non-artifact $\log r$ terms. But Damour pointed out that if non-linear effects are taken into account, non-artifact $\log r$ terms may appear in the metric. He claimed that monopole $\times$ quadrupole terms would generate non-artifact $\log r$ terms. He demonstrated that massive particles at far past with

Newtonian limit on a Schwarzschild background generate non-artifact $\log r$ terms in the metric at $\mathcal{I}^+$ and makes the future null infinity C^2, resulting in the violation of peeling at $\mathcal{I}^+$.

2.4 Partial Peeling

As previously discussed, instead of very strong and restrictive peeling property represented by $C_{\mu\nu\rho\sigma} = O(\Omega)$, logarithmically asymptotic flat spacetime solutions are considered with weak or partial peeling property [11]:

$$C_{\mu\nu\rho\sigma} = O(\Omega \ln(\Omega))$$

with partial peeling property: $C_{\mu\nu\rho\sigma} n^\mu n^\rho = O(\Omega)$ and ${}^*C_{\mu\nu\rho\sigma} n^\mu n^\rho = O(\Omega)$, where $n^\mu = g^{\mu\nu}\Omega_{,\nu}$.

2.4.1 Asymptotic Behavior due to Peeling Property

We can arrive at the peeling property upon solving the initial value problem on $\mathscr{I}$ with the assumption of $\mathscr{I}$ being C^∞. Now, we use conformal Bondi frame formalism, entailing null coordinate system $x^\alpha = (u, r, x^A)$, where r is the inverse luminosity distance. The conformal factor is $\Omega = r = \frac{1}{\tilde{r}}$. g^{01}, g^{11} and g^{1A} are non-vanishing contravariant components of conformal metric tensor and g^{AB} has unit determinant. So, $g_{AB}g^{BC} = \delta^C_A$ implies $g^{AB}g_{AB,1} = 0$.

The boundary condition on the metric at $\mathscr{I}$ where $r = 0$ becomes $g^{11} = g^{11}_{,1} = g^{1A} = g^{1A}_{,1} = g^{01}_{,1} = 0$; $g^{01} = 1$ and $g_{AB} = q^{AB}$, where q_{AB} is unit 2-sphere. Assuming C^∞ on $\mathscr{I}$ to imply strong peeling property, we can arrive at the asymptotic form:

$$g_{AB} = q_{AB} + c_{AB}r + c^{DE}c_{DE}q_{AB}\frac{r^2}{4} + C^0 r^3 \tag{2.45}$$

where c_{AB} is independent of r. C^0 represents the generic fields in neighborhood of $\mathscr{I}$ but at the hypersurface where r = constant the fields are represented by C^∞.

2.4.2 Weaker Asymptotic Form

Instead of the standard asymptotic form in (2.45), let's consider a weaker asymptotic behavior:

$$g_{AB} = q_{AB} + c_{AB}r + d_{AB}r^2 + j_{AB}r^3 \ln r + C^0 r^3 \tag{2.46}$$

where c_{AB} and d_{AB} are independent of r. Due to unit determinant condition on g_{AB}, we get $q^{AB}d_{AB} = c^{AB}c_{AB}/2$ and $q^{AB}j_{AB} = 0$.

The evolution by Einstein's equations preserves the asymptotic behavior in (2.46). Einstein's equations produce the following relations (2.47)-(2.51) [11, 18] in the conformal Bondi frame formalism.

$$g_{01}g^{01}_{,1} = -(r/8)g^{AB}_{,1}g_{AB,1} \tag{2.47}$$

$$r[g_{DC}g^{01}(g^{1C}g_{01})_{,1}]_{,1} - 2g_{DC}g^{01}(g^{1C}g_{01})_{,1} = rK_D \tag{2.48}$$

where

$$K_D = r^{-2}(r^2 g_{01}g^{01}_{,D})_{,1} - (g^{AB}_{,1}g_{BD})_{,A} + (1/2)g_{AB,1}g_{AB,D} \tag{2.49}$$

$$(g_{01}g^{11}/r^3)_{,1} = K/r^2 \tag{2.50}$$

where

$$K = (1/2)g_{01}g^{AB}R_{AB} - (2/r)(g_{01}g^{1A})_{,A} \tag{2.51}$$

<u>*Note*</u>: $g^{AB}R_{AB}$ has neither g^{11} nor u-derivatives.

Upon subjecting the boundary conditions at $\mathscr{I}$ and eq(2.46), from eq(2.47)-(2.51), we can calculate the contravariant conformal metric components.

$$g^{01} = e^{-2\beta} \tag{2.52}$$

where $\beta = -r^2 c^{AB}c_{AB}/32 + C^0 r^3 \ln r$

$$g^{1A} = -(r^2/2)c^{AB}_{:B} - (2/3)r^3 N^A + (2/3)(d^{AB} - dq^{AB}/2)_{:B} r^3 \ln r + C^0 r^4 \ln r \tag{2.53}$$

where $N^A(u, x^A)$ is the angular moment aspect.

Asymptotic calculation gives the relation: $K = -1 + C^0 r^2 \ln r$.

$$g^{11} = r^2 - 2Mr^3 + C^0 r^4 \ln r \tag{2.54}$$

where $M(u, x^A)$ is the mass aspect.

$$g_{AB,0} = N_{AB} r + (1/2) q_{AB} c^{DE} N_{DE} r^2 + (1/6)[2(d_{AB} - dq_{AB}/2) - (d_{AB} - dq_{AB}/2)^{:C}_{C}] r^3 \ln r + C^0 r^3 \tag{2.55}$$

where $N_{AB}(u, x^D)$ is the news tensor.

2.5 Fourier Transforms

Now, we will study soft gravitons for different systems to study the structure of null infinity with the corresponding set-ups. For that purpose, we will be developing some necessary mathematical tools. In this section, we will go through some useful results that we will be using in the following sections.

2.5.1 Different Fourier Transforms

We will be working in 4-dimensional spacetime. The coordinate system we choose is Cartesian coordinate system, $x \equiv (t, \vec{x}) \equiv (x^0, x^1, x^2, x^3)$. We have defined different kinds of Fourier transformation in eq.[(2.56), (2.57), (2.58)]:

$$\hat{F}(k) \equiv \int d^4x e^{-ik\cdot x} F(t,\vec{x}) \tag{2.56}$$

$$\bar{F}(t,\vec{k}) \equiv \int d^3x e^{-i\vec{k}\cdot\vec{x}} F(t,\vec{x}) \tag{2.57}$$

$$\tilde{F}(\omega,\vec{x}) \equiv \int dt e^{-i\omega t} F(t,\vec{x}). \tag{2.58}$$

Inversely, we can also define the inverse Fourier transform as follows:

$$F(t,\vec{x}) \equiv \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \hat{F}(k) \tag{2.59}$$

$$F(t,\vec{x}) \equiv \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \bar{F}(t,\vec{k}) \tag{2.60}$$

$$F(t,\vec{x}) \equiv \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{F}(\omega,\vec{x}). \tag{2.61}$$

2.5.2 Radiative Field at Large distances

Let us assume,

$$\Box F(x) = -j(x) \quad , \quad \text{where } \Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta. \tag{2.62}$$

The retarded solution of eq.[(2.62)] is given in eq.[(2.63)].

$$F(x) = -\int d^4x' G_r(x,x') j(x'), \tag{2.63}$$

where $G_r(x,x')$ is the retarded Green's function:

$$G_r(x,x') = \int \frac{dl^4}{(2\pi)^4} \frac{e^{il\cdot(x-x')}}{(l_0+i\epsilon)^2-\vec{l}^2}. \tag{2.64}$$

Using eq.[(2.58)], we get:

$$\begin{aligned}
\tilde{F}(\omega,\vec{x}) \quad &= -\int dt e^{-i\omega t}\int d^4x'\int \frac{dl^4}{(2\pi)^4}\frac{e^{il\cdot(x-x')}}{(l_0+i\epsilon)^2-\vec{l}^2}j(x') \\
&= -\int d^4x'\int \frac{dl^4}{(2\pi)^3}\frac{e^{il_0x'^0+i\vec{l}\cdot(\vec{x}-\vec{x}')}}{(l_0+i\epsilon)^2-\vec{l}^2}j(x')\int \frac{dt}{2\pi}e^{-i(\omega-l_0)t} \\
&= -\int d^4x'\int \frac{dl^3}{(2\pi)^3}\frac{e^{i\omega x'^0+i\vec{l}\cdot(\vec{x}-\vec{x}')}}{(\omega+i\epsilon)^2-\vec{l}^2}j(x') \\
&= -\int d^4x'\int \frac{dl_\perp^2}{(2\pi)^2}\frac{dl_\parallel}{(2\pi)}\frac{e^{i\omega x'^0+il_\parallel|\vec{x}-\vec{x}'|}}{(\omega+i\epsilon)^2-\vec{l}_\perp^2-l_\parallel^2}j(x') \\
&= i\int d^4x'\int \frac{dl_\perp^2}{(2\pi)^2}\frac{e^{i\omega x'^0+i\sqrt{(\omega+i\epsilon)^2-\vec{l}_\perp^2}|(\vec{x}-\vec{x}')|}}{2\sqrt{(\omega+i\epsilon)^2-\vec{l}_\perp^2}}j(x') \\
&= i\int d^4x'\frac{e^{i\omega x'^0+i(\omega+i\epsilon)|(\vec{x}-\vec{x}')|}}{2(\omega+i\epsilon)}\frac{(\omega+i\epsilon)}{2\pi i|(\vec{x}-\vec{x}')|}j(x') \\
&\simeq \frac{e^{i\omega|\vec{x}|}}{4\pi|\vec{x}|}\int d^4x'e^{-ik\cdot x'}j(x'). \qquad (2.65)
\end{aligned}$$

In eq.[(2.65)], we have assumed that $|\vec{x}| >> |\vec{x}'|$.

We get to the third step from the second step of eq.[(2.65)], using $\int \frac{dt}{2\pi}e^{-i(\omega-l_0)t} = \delta(l_0-\omega)$. In the fourth step, we have broken $\vec{l}$ into components parallel and perpendicular to $(\vec{x}-\vec{x}')$, respectively $\vec{l}_\perp$ and $l_\parallel$. Now, we can see that $(\omega+i\epsilon)^2-\vec{l}_\perp^2-l_\parallel^2 = 0$ has two roots: $\pm\sqrt{(\omega+i\epsilon)^2-\vec{l}_\perp^2}$. We arrive at the fifth step doing the contour integration. For large $|(\vec{x}-\vec{x}')|$ in the fifth step, the exponent is a rapidly varying function of $\vec{l}_\perp$. So, we can integrate over $\vec{l}_\perp$ using saddle point approximation.

Hence, under the assumption that $|\vec{x}| >> |\vec{x}'|$, we can write combining eq.[(2.56)] and eq.[(2.65)],

$$\tilde{F}(\omega,\vec{x}) = \frac{e^{i\omega r}}{4\pi r}\int d^4x'e^{-ik\cdot x'}j(x'), \qquad (2.66)$$

where $r = |\vec{x}|$, $n_i = \frac{x_i}{r}$ and $k \equiv \omega(1,\vec{n})$.

2.5.3 Late time and Early time behavior from Fourier Transform

In the analysis of soft gravitons, we will often have to handle functions $\tilde{F}(\omega,\vec{x})$ that are non-analytic as $\omega \to 0$. Different terms in $\tilde{F}(\omega,\vec{x})$ will be proportional to $\omega^{(\zeta-1)}(\log\omega)^\kappa$, where

$\zeta, \kappa = \{0, 1, 2, 3, ...\}$. We expect $\lim_{\omega\to 0}\tilde{F}(\omega,\vec{x})$ to be corresponding to $\lim_{|t|\to\infty}F(t,\vec{x})$, by the principle of Fourier transformation. Now, we wish to derive the precise correspondance between $\lim_{\omega\to 0}\tilde{F}(\omega,\vec{x})$ and $\lim_{|t|\to\infty}F(t,\vec{x})$. We do all this analysis for constant $\vec{x}$. So, we will not display the $\vec{x}$ dependence.

Case 1: $\zeta = \kappa = 0$

Firstly, we will analyze the singularities of the form $\frac{1}{\omega}$ for small ω. Let us assume the function of the form: $\tilde{F}(\omega) = Ce^{i\omega\phi}\frac{1}{\omega}f(\omega)$, where C and ϕ is constant, and $f(\omega)$ is an function of ω, such that $f(\omega)$ is smooth at $\omega = 0$ $f(0) = 1$.

$$F(t) = \int \frac{d\omega}{2\pi}e^{-i\omega t}\tilde{F}(\omega) = C\int \frac{d\omega}{2\pi}e^{-i\omega u}\frac{1}{\omega}f(\omega) \quad , \quad \text{where } u \equiv t - \phi. \tag{2.67}$$

Now, to the do the integration in eq.[(2.67)] around $\omega = 0$, we use the following:

$$\delta(\omega) = -\lim_{\epsilon\to 0}\left(\frac{1}{\omega - i\epsilon} - \frac{1}{\omega + i\epsilon}\right). \tag{2.68}$$

Let us assume,

$$F_{\pm}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\frac{1}{\omega \pm i\epsilon}f(\omega). \tag{2.69}$$

From eq.[(2.68)] and eq.[(2.69)], we get:

$$\begin{aligned} F_{-}(t) - F_{+}(t) \quad &= \frac{C}{2\pi}\int d\omega e^{-i\omega u}\frac{1}{\omega - i\epsilon}f(\omega) - \frac{C}{2\pi}\int d\omega e^{-i\omega u}\frac{1}{\omega + i\epsilon}f(\omega) \\ &= \frac{C}{2\pi}\int d\omega e^{-i\omega u}\delta(\omega)f(\omega) \\ &= \frac{C}{2\pi} \end{aligned} \tag{2.70}$$

$F_{-}(t) - F_{+}(t)$ is just a constant and is of no practical use in our further calculation.

To do the integration in eq.[(2.69)], we close the contour in the lower (upper) half plane for positive (negative) u and pick up the residues at the poles.

Now, we get:

$$F_+(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\frac{1}{\omega + i\epsilon}f(\omega) = \begin{cases} iC + O(e^{-u}) & \text{for } u > 0 \\ O(e^{-u}) & \text{for } u < 0 \end{cases}$$
$$F_-(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\frac{1}{\omega - i\epsilon}f(\omega) = \begin{cases} O(e^{-u}) & \text{for } u > 0 \\ iC + O(e^{-u}) & \text{for } u < 0 \end{cases} \tag{2.71}$$

The result in eq.[(2.71)] can be summarized as following:

$$F_\pm(t) = iCH(\pm u) + O(e^{-u}) \tag{2.72}$$

$H(u)$ in eq.[(2.70)] is the Heavyside Theta function defined as:

$$H(u) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The $O(e^{-u})$ contribution in eq.[(2.71)] comes from the poles of $f(\omega)$. The step function $H(u)$ implies a jump in $e_{\mu\nu}$ at $\mathcal{I}^+$ from $u \to -\infty$ to $u \to \infty$. This is called the memory effect.

Case 2: $\zeta = \kappa = 1$

Now, we will analyze the singularities of the form $\log\omega$ for small ω. Let us assume the function of the form: $\tilde{F}(\omega) = Ce^{i\omega\phi}(\log\omega)f(\omega)$, where C and ϕ is constant, and $f(\omega)$ is an function of ω, such that $f(\omega)$ is smooth at $\omega = 0$ $f(0) = 1$.

Now, we define:

$$F_\pm(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\log(\omega \pm i\epsilon)f(\omega). \tag{2.73}$$

Therefore,

$$F_{-}(t) - F_{+}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\log(\omega - i\epsilon) f(\omega) - \frac{C}{2\pi}\int d\omega e^{-i\omega u}\log(\omega + i\epsilon) f(\omega) \tag{2.74}$$

To do the calculation in eq.[(2.74)], we integrate both sides of the eq.[(2.68)].

$$\begin{aligned} \int d\omega \delta(\omega) &\sim -\lim_{\epsilon\to 0}\int d\omega \left(\frac{1}{\omega - i\epsilon} - \frac{1}{\omega + i\epsilon}\right) \\ \Longrightarrow \lim_{\epsilon\to 0}(\log(\omega + i\epsilon)) &- (\log(\omega - i\epsilon)) = 2\pi i H(-\omega). \end{aligned} \tag{2.75}$$

From eq.[(2.74)] and eq.[(2.75)], we get:

$$\begin{aligned} F_{+}(t) - F_{-}(t) = &\quad iC\int d\omega e^{-i\omega u} H(-\omega) f(\omega) \\ = &\quad iC\int_{-\infty}^{0} d\omega e^{-i\omega u} f(\omega) \\ \simeq &\quad -\frac{C}{u} \quad , \quad \text{for } |u| \to \infty. \end{aligned} \tag{2.76}$$

We have suppresed the $O(e^{-u})$ terms that arise due to the poles of $f(\omega)$.

Therefore,

$$\begin{aligned} F_{+}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\log(\omega + i\epsilon) f(\omega) &= \begin{cases} -\frac{C}{u} & \text{for } u \to \infty \\ 0 & \text{for } u \to -\infty \end{cases} \\ F_{-}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\log(\omega - i\epsilon) f(\omega) &= \begin{cases} 0 & \text{for } u \to \infty \\ \frac{C}{u} & \text{for } u \to -\infty \end{cases} \end{aligned} \tag{2.77}$$

Case 3: $\zeta = 2$ and $\kappa = 1$

Now, we will analyze the singularities of the form $\log\omega$ for small ω. Let us assume the function of the form: $\tilde{F}(\omega) = Ce^{i\omega\phi}\omega(\log\omega)f(\omega)$, where C and ϕ is constant, and $f(\omega)$ is an function of ω, such that $f(\omega)$ is smooth at $\omega = 0$ $f(0) = 1$.

Now, we define:

$$\begin{aligned} F_{\pm}(t) = &\quad \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\log(\omega \pm i\epsilon)f(\omega) \\ = &\quad \frac{iC}{2\pi}\frac{d}{du}\int d\omega e^{-i\omega u}\log(\omega \pm i\epsilon)f(\omega). \end{aligned} \tag{2.78}$$

Using eq.[(2.77)] and eq.[(2.78)], we get:

$$F_{+}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\log(\omega + i\epsilon)f(\omega) = \begin{cases} i\frac{C}{u^2} & \text{for } u \to \infty \\ 0 & \text{for } u \to -\infty \end{cases}$$

$$F_{-}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\log(\omega - i\epsilon)f(\omega) = \begin{cases} 0 & \text{for } u \to \infty \\ -i\frac{C}{u^2} & \text{for } u \to -\infty \end{cases} \tag{2.79}$$

Case 4: $\zeta = \kappa = 2$

Now, we will analyze the singularities of the form $\log\omega$ for small ω. Let us assume the function of the form: $\tilde{F}(\omega) = Ce^{i\omega\phi}\omega\{(\log\omega)\}^2 f(\omega)$, where C and ϕ is constant, and $f(\omega)$ is an function of ω, such that $f(\omega)$ is smooth at $\omega = 0$ $f(0) = 1$.

Now, we define:

$$\begin{aligned} F_{\pm}(t) = &\quad \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\{\log(\omega \pm i\epsilon)\}^2 f(\omega) \\ = &\quad \frac{iC}{2\pi}\frac{d}{du}\int d\omega e^{-i\omega u}\{\log(\omega \pm i\epsilon)\}^2 f(\omega) \end{aligned} \tag{2.80}$$

and

$$G(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\{\log(\omega+i\epsilon)\}\{\log(\omega-i\epsilon)\}f(\omega). \tag{2.81}$$

$$F_{+}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\{\log(\omega+i\epsilon)\}^2 f(\omega) = \begin{cases} -2iC\frac{\log|u|}{u^2} & \text{for } u\rightarrow\infty \\ 0 & \text{for } u\rightarrow-\infty \end{cases}$$

$$F_{-}(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\log(\omega-i\epsilon)f(\omega) = \begin{cases} 0 & \text{for } u\rightarrow\infty \\ 2iC\frac{\log|u|}{u^2} & \text{for } u\rightarrow-\infty \end{cases} \tag{2.82}$$

$$G(t) = \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega\{\log(\omega+i\epsilon)\}\{\log(\omega-i\epsilon)\}f(\omega) = \begin{cases} -iC\frac{\log|u|}{u^2} & \text{for } u\rightarrow\infty \\ iC\frac{\log|u|}{u^2} & \text{for } u\rightarrow-\infty \end{cases}. \tag{2.83}$$

Case 5: $\zeta = 1,2,3,...$ **and** $\kappa = 0$

Now, we will analyze ω^n for small ω, where $n = 0,1,2,.....$ Let us assume the function of the form: $\tilde{F}(\omega) = Ce^{i\omega\phi}\omega^n f(\omega)$, where C and ϕ is constant, and $f(\omega)$ is an function of ω, such that $f(\omega)$ is smooth at $\omega = 0$ $f(0) = 1$.

Now, we define:

$$\begin{aligned} F(t) = & \quad \frac{C}{2\pi}\int d\omega e^{-i\omega u}\omega^n f(\omega) \\ = & \quad \frac{C}{2\pi}(-i)^n\frac{d^n}{du^n}\int d\omega e^{-i\omega u}f(\omega) \\ = & \quad (-i)^n\frac{d^n}{du^n}\{O(e^{-u})\} \end{aligned} \tag{2.84}$$

2.6 Gravitational Waves

Let us assume that we have a metric $g_{\mu\nu}$ such that

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}, \tag{2.85}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ part represents the perturbation to $\eta_{\mu\nu}$.

Now, we define:

$$e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta} \tag{2.86}$$

Therefore,

$$h_{\mu\nu} = e_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}e_{\alpha\beta} \tag{2.87}$$

In de-Donder coordinates or harmonic coordinates, Einstein's equations of general relativity for $g_{\mu\nu}$ reduces to:

$$\Box e_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{2.88}$$

where $\Box \equiv \eta_{\alpha\beta\partial_\alpha\partial_\beta}$, G is Newton's universal constant and $T_{\mu\nu}$ is the stress-energy tensor.

For convenience, we will work in units for which $8\pi G = 1$. Therefore, eq.[(2.88)] takes the form:

$$\Box e_{\mu\nu}(x) = -T_{\mu\nu}(x). \tag{2.89}$$

From eq.[(2.89)] and eq.[(2.66)], we get:

$$\tilde{e}_{\mu\nu}(\omega, \vec{x}) = \frac{e^{i\omega r}}{4\pi r}\int d^4x' e^{-ik\cdot x'}T_{\mu\nu}(x'), \tag{2.90}$$

where r is large distance.

From eq.[(2.61)] and eq.[(2.90)], we get:

$$\begin{aligned} e_{\mu\nu}(t,\vec{x}) = & \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{e}_{\mu\nu}(\omega,\vec{x}) \\ = & \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^{i\omega r}}{4\pi r} \int d^4x' e^{-ik\cdot x'} T_{\mu\nu}(x') \\ = & \int \frac{d\omega}{2\pi} \frac{e^{-i\omega u}}{4\pi r} \int d^4x' e^{-ik\cdot x'} T_{\mu\nu}(x') \quad , \quad \text{where } u = t - r. \end{aligned} \tag{2.91}$$

2.7 Effect of Soft Gravitons on Future Null Infinity

From 2.5.3, we can conclude that $e_{\mu\nu}(\omega,\vec{x})$ with $\omega \to 0$, corresponds to $lim_{|t|\to\infty} T_{\mu\nu}(t,\vec{x})$. Hence, soft gravitons, i.e. $h_{\mu\nu}$ with $\omega \to 0$, corresponds to $lim_{|t|\to\infty} T_{\mu\nu}(t,\vec{x})$. The soft gravitons, generated due to the contribution of $\lim_{t\to-\infty} T_{\mu\nu}(t,\vec{x})$, influence the structure of $\mathcal{I}^+$ as $u \to -\infty$. Similarly, the soft gravitons, generated due to the contribution of $\lim_{t\to\infty} T_{\mu\nu}(t,\vec{x})$, influence the structure of $\mathcal{I}^+$ as $u \to \infty$. Now, we will discuss how different terms in $e_{\mu\nu}(\omega,\vec{x})$ at soft limit influences the structure of $\mathcal{I}^+$.

Our main goal here is to verify if peeling is violated at null infinity. To do that, we turn to Christodoulou's analysis. Let us assume that $N_{\mu\nu}(x)$ is the news tensor. According to Christodoulou's analysis, if $u^2 N_{\mu\nu}(x)$ has a finite non-zero limit as $u \to \pm\infty$ at $\mathcal{I}^+$, then the peeling will be violated at $\mathcal{I}^+$ as $u \to \pm\infty$. Now, combined with our analysis, at due to soft limits $\lim_{u\to\pm\infty} u^2 \partial_u (r e_{\mu\nu}(t,\vec{x})) \neq 0$ for the violation of peeling at $\mathcal{I}^+$ as $u \to \pm\infty$. So for violation of peeling, we have to show the following:

$$\lim_{u\to\pm\infty} u^2 \partial_u \int \frac{d\omega}{2\pi} \frac{e^{-i\omega u}}{4\pi} \int d^4x' e^{-ik\cdot x'} T_{\mu\nu}(x') \neq 0 \tag{2.92}$$

Therefore, our work is reduced to checking if

$$\lim_{u\to\pm\infty} u^2 \partial_u \int \frac{d\omega}{2\pi} e^{-i\omega u} \omega^{(\zeta-1)} (\log\omega)^{\kappa} \neq 0 \quad , \quad \text{where}\zeta,\kappa = \{0,1,2,3,....\}. \tag{2.93}$$

If such a limit as in eq.[(2.93)] exists, peeling will be violated at $\mathcal{I}^+$ as $u \to \pm\infty$.

2.7.1 Effect of $\frac{1}{\omega}$ Terms at Soft Limit

If we compare eq.[(2.72)] in 2.5.3 with eq.[(2.91)], we can conclude that if there is a term that is proportional to $\frac{1}{\omega}$ in $\tilde{e}_{\mu\nu}(\omega,\vec{x})$ at the soft limit, there will be a term proportional to $H(u)$ in $e_{\mu\nu}(t,\vec{x})$. Here, $H(u)$ is the Heavyside theta function. Because of the presence of this step function in $e_{\mu\nu}(t,\vec{x})$, there is a jump in $e_{\mu\nu}(t,\vec{x})$ at $\mathcal{I}^+$ as we go from $u \to -\infty$ to $u \to \infty$.

From this, we can see how $\frac{1}{\omega}$ term in $\tilde{e}_{\mu\nu}(\omega,\vec{x})$ at the soft limit is responsible for the gravitational memory effect at $\mathcal{I}^+$. But for this case, the limit as in eq.[(2.93)] is 0. So, the $\frac{1}{\omega}$ term does not violate the peeling.

2.7.2 Effect of $\log\omega$ Terms at Soft Limit

If we compare eq.[(2.72)] in 2.5.3 with eq.[(2.91)], we can conclude that if there is a term that is proportional to $\frac{1}{\omega}$ in $\tilde{e}_{\mu\nu}(\omega,\vec{x})$ at the soft limit, there will be a term proportional to $\frac{1}{u}$ in $e_{\mu\nu}(t,\vec{x})$.

The limit in eq.[(2.93)] in this case is as following:

$$\lim_{u\to\pm\infty} u^2\partial_u \int \frac{d\omega}{2\pi} e^{-i\omega u} \log\omega = \lim_{u\to\pm\infty} u^2\partial_u(\mp\frac{1}{u}) = \pm 1 \tag{2.94}$$

Therefore, $\log\omega$ term in $\tilde{e}_{\mu\nu}(\omega,\vec{x})$ at soft limit is responsible for violation of peeling at $\mathcal{I}^+$ as $u \to \pm\infty$.

2.7.3 Effect of Other Terms at Soft Limit

Doing a similar analysis as the terms above at the soft limit for the other terms in $\tilde{e}_{\mu\nu}(\omega,\vec{x})$, we find out that the limit in eq.[(2.93)] is 0. Hence, they are not responsible for the violation of peeling.

2.8 Soft Gravitons for Massive Particles

Now, our goal is to verify that N massive particles at far past and far future violate the peeling property at the future null infinity. If we can show that $\lim_{r\to\infty}\tilde{e}_{\mu\nu}(\omega,\vec{x})$ has $\log\omega$ terms at the soft limit, from 2.7.2, we will be able to conclude that N massive particles at far past and far future violate the peeling property at $\mathcal{I}^+$ as $u \to -\infty$ and $u \to \infty$, respectively.

2.8.1 Set up at Far Past

Let us consider a system of N massive particles at past infinity, and the distance between each pair of particles then is large. Hence, in such a system, we can assume the Newtonian limit. The i-th particle has mass $m_{(i)}$ and the wordline: $r_i(\sigma) = \vec{b}_{(i)} + \frac{\vec{p}_{(i)}}{m_{(i)}}\sigma$, where $\vec{b}_{(i)}$, $\vec{p}_{(i)}$ are constants, for $i \in \{1,2,3,......,N\}$.

As the system is in Newtonian limit,

$$\begin{aligned} \vec{a}_{(i)} = & -\sum_{\substack{i\neq j \\ j=1}}^{N} \frac{Gm_{(j)}}{r_{ij}}\hat{r}_{ij} \quad , \quad \text{where } \vec{r}_{ij} = \vec{r}_{(i)} - \vec{r}_{(j)} \\ = & -\sum_{\substack{i\neq j \\ j=1}}^{N} \frac{Gm_{(j)}}{(\vec{p}_{(i)} - \vec{p}_{(j)})^2\sigma^2}\hat{r}_{ij}, \end{aligned} \tag{2.95}$$

where$\vec{a}_{(i)}$ is the acceleration of the i-th particle at far past. From eq.[(2.95)], we can see that the trajectory of the i-th particle takes the following form:

$$\vec{r}_{(i)}(\sigma) = \vec{b}_{(i)} + \frac{\vec{p}_{(i)}}{m_{(i)}}\sigma + \vec{c}_{(i)}\log\sigma. \tag{2.96}$$

$\vec{c}_{(i)}$ can also be calculated from the correction to the trajectories and the velocity vectors with the help of the leading term in $\tilde{e}_{\mu\nu}(\omega,\vec{x})$.

2.8.2 Stress Energy Tensor for Massive Particles at Far Past

We will be working in de-Donder coordinates or Harmonic coordinates. In this coordinate, the stress-energy tensor due to massive particles at far past will be as follows:

$$T_{\mu\nu}(x) = \sum_{i=1}^{N} m_{(i)} \int d\sigma \delta^{(4)}(x - r_{(i)}(\sigma)) \frac{dr_{(i)\mu}(\sigma)}{d\sigma}\frac{dr_{(i)\nu}(\sigma)}{d\sigma}. \tag{2.97}$$

2.8.3 Soft Gravitons for Massive Particles at Far Past

Now, we will calculate $\tilde{e}_{\mu\nu}(\omega,\vec{x})$ for stress energy tensor in eq.[(2.97)].

$$\tilde{e}_{\mu\nu}(\omega,\vec{x}) = \frac{e^{i\omega r}}{4\pi r}\int dx'^4 e^{-ik\cdot x'} T_{\mu\nu}(x') \tag{2.98}$$

where $k^\mu \equiv \omega(1,\vec{n})$ with $n_i = \frac{x_i}{r}$.

Now, from eq.[(2.97)] and eq.[(2.98)], we get:

$$
\begin{aligned}
&\tilde{e}_{\mu\nu}(\omega,\vec{x}) \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\int dx'^4 e^{-ik\cdot x'} T_{\mu\nu}(x') \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\int dx'^4 e^{-ik\cdot x'} \sum_{i=1}^{N} m_{(i)} \int d\sigma \delta^{(4)}(x - r_{(i)}(\sigma)) \frac{dr_{(i)\mu}(\sigma)}{d\sigma}\frac{dr_{(i)\nu}(\sigma)}{d\sigma} \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\sum_{i=1}^{N} m_{(i)} \int d\sigma e^{-ik\cdot r_{(i)}(\sigma)} \frac{dr_{(i)\mu}(\sigma)}{d\sigma}\frac{dr_{(i)\nu}(\sigma)}{d\sigma} \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\sum_{i=1}^{N} m_{(i)} \int d\sigma e^{ik_0\sigma - i\vec{k}\cdot(\vec{b}_{(i)} + \frac{\vec{p}_{(i)}}{m_{(i)}}\sigma + \vec{c}_{(i)}\log\sigma)} \frac{dr_{(i)\mu}(\sigma)}{d\sigma}\frac{dr_{(i)\nu}(\sigma)}{d\sigma} \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\sum_{i=1}^{N} m_{(i)} e^{-i\vec{k}\cdot\vec{b}_{(i)}} \int d\sigma e^{ik_0\sigma - i\vec{k}\cdot(\vec{b}_{(i)} + \frac{\vec{p}_{(i)}}{m_{(i)}}\sigma + \vec{c}_{(i)}\log\sigma)} (\frac{p_{(i)\mu}}{m_{(i)}} + \frac{c_{(i)\mu}}{\sigma})(\frac{p_{(i)\mu}}{m_{(i)}} + \frac{c_{(i)\mu}}{\sigma}) \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\sum_{i=1}^{N} m_{(i)} e^{-i\vec{k}\cdot\vec{b}_{(i)}} \int d\sigma e^{ik_0\sigma - i\vec{k}\cdot(\vec{b}_{(i)} + \frac{\vec{p}_{(i)}}{m_{(i)}}\sigma)} (1 - i\vec{k}\cdot\vec{c}_{(i)}\log\sigma)(\frac{p_{(i)\mu}}{m_{(i)}} + \frac{c_{(i)\mu}}{\sigma})(\frac{p_{(i)\mu}}{m_{(i)}} + \frac{c_{(i)\mu}}{\sigma}) \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\Bigg(-\sum_{a=1}^{N} p_{(a)\mu}p_{(a)\nu}\frac{1}{ik\cdot p_{(a)}} \\
& + \frac{\log\omega}{4\pi}\Big\{ \sum_{a=1}^{N}\sum_{\substack{a\neq b \\ b=1}}^{N} \frac{p_{(a)}\cdot p_{(b)}\{\frac{3}{2}p_{(a)}^2 p_{(b)}^2 - (p_{(a)}\cdot p_{(b)})^2\}}{\{(p_{(a)}\cdot p_{(b)})^2 - p_{(a)}^2 p_{(b)}^2\}^{\frac{3}{2}}} \frac{k_\rho p_{(a)\mu}}{k\cdot p_{(a)}}(p_{(a)}^\rho p_{(b)_\nu} - p_{(b)\nu}p_{(a)}^\rho) \\
& + \sum_{a=1}^{N}\sum_{b=1}^{N} \frac{k\cdot p_{(b)}}{k\cdot p_{(a)}} p_{(a)\mu}p_{(a)\nu} \Big\}\Bigg) + \mathcal{O}(1).
\end{aligned}
\tag{2.99}
$$

In eq.[(2.99)], the leading term is proportional to $\frac{1}{\omega}$, which is responsible for the memory effect, and the subleading term is proportional to $\log\omega$, which leads to the violation of peeling at $\mathcal{I}^+$ as $u \to -\infty$.

2.8.4 Soft Gravitons for Massive Particles at Far Future

Let us consider a system of N massive particles at future timelike infinity, and the distance between each pair of particles then is large. Hence, in such a system, we can assume the Newtonian limit. The i-th particle has mass $m_{(i)}$ and the word line: $r_i(\sigma) = \vec{b'}_{(i)} + \frac{\vec{p'}_{(i)}}{m_{(i)}}\sigma$, where $\vec{b'}_{(i)}$, $\vec{p'}_{(i)}$ are constants, for $i \in \{1,2,3,......,N\}$.

If we carry out the similar analysis as 2.7.1, 2.7.2 and 2.7.3 for this case we get the following result:

$$\begin{aligned}
&\tilde{e}_{\mu\nu}(\omega,\vec{x}) \\
= \quad & \frac{e^{i\omega r}}{4\pi r}\Bigg(-\sum_{a=1}^{N} p'_{(a)\mu}p'_{(a)\nu}\frac{1}{ik\cdot p'_{(a)}} \\
&+\frac{\log\omega}{4\pi}\Big\{\sum_{a=1}^{N}\sum_{\substack{a\neq b\\ b=1}}^{N}\frac{p'_{(a)}\cdot p'_{(b)}\{\frac{3}{2}p'^2_{(a)}p'^2_{(b)}-(p'_{(a)}\cdot p'_{(b)})^2\}}{\{(p'_{(a)}\cdot p'_{(b)})^2-p'^2_{(a)}p'^2_{(b)}\}^{\frac{3}{2}}}\frac{k_\rho p'_{(a)\mu}}{k\cdot p'_{(a)}}(p'^{\rho}_{(a)}p'_{(b)_\nu}-p'_{(b)\nu}p'^{\rho)}_{(a)} \\
&+\sum_{a=1}^{N}\sum_{b=1}^{N}\frac{k\cdot p'_{(b)}}{k\cdot p'_{(a)}}p'_{(a)\mu}p'_{(a)\nu}\Big\}\Bigg)+\mathcal{O}(1). \qquad (2.100)
\end{aligned}$$

Similar to the far past case, in eq.[(2.100)] the leading term is proportional to $\frac{1}{\omega}$, which is responsible for the memory effect, and the subleading term is proportional to $\log\omega$, which leads to the violation of peeling at $\mathcal{I}^+$ as $u \to \infty$.

2.9 Soft Gravitons for Massless Scalar Field

This section will verify if massless scalar fields on the Minkowski background violate the peeling property at the future null infinity. If we can show that $\lim_{r\to\infty}\tilde{e}_{\mu\nu}(\omega,\vec{x})$ has $\log\omega$ terms at the soft limit, from 2.7.2, we will be able to conclude that massless scalar fields violate the peeling property at $\mathcal{I}^+$ as $u \to -\infty$ and $u \to \infty$, respectively.

2.9.1 Massless Scalar Field Phase Space

The data on outgoing null geodesics can describe the massive scalar field space. The chosen coordinates for this description are

$$u = t - r \quad , \quad r. \qquad (2.101)$$

In the new coordinates, the line element takes the form as follows:

$$ds^2 = -du^2 - 2dudr + r^2 d\Omega^2, \qquad (2.102)$$

where $d\Omega^2$ is the unit sphere metric.

As $r \rightarrow \infty$, the massless scalar field behaves as follows:

$$\varphi(u,r,\hat{n}) = \frac{\varphi^{(1)}(u,\hat{n})}{r} + \frac{\varphi^{(2)}(u,\hat{n})}{r^2} + O(\frac{1}{r^3}). \tag{2.103}$$

2.9.2 Strees Energy Tensor for a Massless Scalar Field at Large Distances

We will treat all the tensors in de-Donder coordinates. So, let us first define:

$$\partial_\mu r = \tilde{n}_\mu \quad , \quad \partial_\mu u = -n_\mu, \quad \partial_\mu \hat{n}_i = \frac{1}{r}\eta^{\perp}_{\mu i}, \tag{2.104}$$

where $\eta^{\perp}_{\mu\nu} = \eta_{\mu\nu} + n_\mu n_\nu - n_\mu \tilde{n}_\nu - n_\nu \tilde{n}_\mu$ with $\hat{n}_i = \frac{x_i}{r}$, $n = (1, \vec{\hat{n}})$ and $\tilde{n} = (0, \vec{\hat{n}})$.

For a general function $f(u,r,\hat{n})$,

$$\begin{aligned} \partial_\mu f(\tau,\rho,\hat{n}) = & \quad \partial_\mu u \frac{\partial f}{\partial u} + \partial_\mu r \frac{\partial f}{\partial r} + \frac{1}{r}\eta^{\perp}_{\mu i}\partial^i_\perp f \\ = & \quad -n_\mu \frac{\partial f}{\partial u} + \tilde{n}_\mu \frac{\partial f}{\partial r} + \frac{1}{r}\eta^{\perp}_{\mu i}\partial^i_\perp f \end{aligned} \tag{2.105}$$

The stress-energy tensor for a massless scalar field is

$$T_{\mu\nu}(x) = (\partial_\mu \varphi \partial_\nu \varphi) - \eta_{\mu\nu}\mathcal{L}, \tag{2.106}$$

where the Lagrangian of the massless scalar field is $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi$.

The components of the stress-energy tensor for the massless scalar field as $r \rightarrow \infty$ take the following form:

$$T^R_{\mu\nu}(x) = \frac{A(u,\hat{n})}{r^2} n_\mu n_\nu + \frac{B(u,\hat{n})}{r^3} n_\mu n_\nu + \frac{G(u,\hat{n})_{(\mu} n_{\nu)}}{r^3} + \frac{H(u,\hat{n})_{\mu\nu}}{r^3} + O(\frac{1}{r^4}). \tag{2.107}$$

2.9.3 Soft Gravitons at future null infinity due to Massless Scalar Field

$$\tilde{e}_{\mu\nu}(\omega,\vec{x}) = \frac{e^{i\omega r}}{4\pi r}\int dx'^4 e^{-ik\cdot x'} T_{\mu\nu} = \tilde{e}^1_{\mu\nu}(\omega,\vec{x}) + \tilde{e}^2_{\mu\nu}(\omega,\vec{x}), \tag{2.108}$$

where

$$\tilde{e}^1_{\mu\nu}(\omega,\vec{x}) = \frac{e^{i\omega r}}{4\pi r}\int_{r>r_0} dx'^4 e^{-ik\cdot x'} T^R_{\mu\nu}(x')$$

and

$$\tilde{e}^2_{\mu\nu}(\omega,\vec{x}) = \frac{e^{i\omega r}}{4\pi r}\int_{r\leq r_0} dx'^4 e^{-ik\cdot x'} T_{\mu\nu}(x')$$

.

If we substitute eq.[(2.107)] in$\tilde{e}^1_{\mu\nu}(\omega,\vec{x})$, we get

$$\begin{aligned}
\tilde{e}^1_{ij}(\omega,\vec{x}) = & \frac{e^{i\omega r}}{4\pi r}\int dx'^4 e^{-ik\cdot x'}(\frac{A(u',\hat{n}')}{r'^2}n'_i n'_j + \frac{B(u',\hat{n}')}{r'^3}n'_i n'_j + \frac{G(u',\hat{n}')_{(i}n'_{j)}}{r'^3} + \frac{H(u',\hat{n}')_{ij}}{r'^3} + O(\frac{1}{r'^4})) \\
= & \frac{e^{i\omega r}}{4\pi r}\int r^2 du' dr' d\hat{n}' e^{i\omega u' + i\omega r' - i\omega \vec{n}\cdot\vec{n}' r'} \\
& (\frac{A(u',\hat{n}')}{r'^2}n'_i n'_j + \frac{B(u',\hat{n}')}{r'^3}n'_i n'_j + \frac{G(u',\hat{n}')_{(i}n'_{j)}}{r'^3} + \frac{H(u',\hat{n}')_{ij}}{r'^3} + O(\frac{1}{r'^4})) \\
= & \frac{e^{i\omega r}}{4\pi r}\int du' dr' d\hat{n}' e^{i\omega u' + i\omega r' - i\omega \vec{n}\cdot\vec{n}' r'} \\
& (A(u',\hat{n}')n'_i n'_j + \frac{B(u',\hat{n}')}{r'}n'_i n'_j + \frac{G(u',\hat{n}')_{(i}n'_{j)}}{r'} + \frac{H(u',\hat{n}')_{ij}}{r'} + O(\frac{1}{r'^2})) \\
= & \frac{e^{i\omega r}}{4\pi r}\int du' dr' d\hat{n}' e^{i\omega u' + i\omega r' - i\omega \vec{n}\cdot\vec{n}' r'} A(u',\hat{n}')n'_i n'_j \\
& \frac{e^{i\omega r}}{4\pi r}\int du' dr' d\hat{n}' e^{i\omega u' + i\omega r' - i\omega \vec{n}\cdot\vec{n}' r'}(\frac{B(u',\hat{n}')}{r'}n'_i n'_j + \frac{G(u',\hat{n}')_{(i}n'_{j)}}{r'} + \frac{H(u',\hat{n}')_{ij}}{r'}) \\
& \frac{e^{i\omega r}}{4\pi r}\int du' dr' d\hat{n}' e^{i\omega u' + i\omega r' - i\omega \vec{n}\cdot\vec{n}' r'}(O(\frac{1}{r'^2}))
\end{aligned} \tag{2.109}$$

We have used the following relations in the second step of eq.[(2.109)]:

$$dx'^4 \to r^2 du' dr' d\hat{n}' \quad , \quad k\cdot x' = -(\omega u' + \omega r' - \omega\vec{n}\cdot\vec{n}' r'), \tag{2.110}$$

where $k^\mu \equiv \omega(1,\vec{n})$.

Now, let us calculate the first term in the last step of eq.[(2.109)]:

$$\begin{aligned}
& \frac{e^{i\omega r}}{4\pi r}\int du' d\hat{n}' \int_{r_0}^{\infty} dr' e^{i\omega u' + i\omega r' - i\omega \vec{n}\cdot\vec{n}' r'} A(u',\hat{n}') n'_i n'_j \\
= \quad & -\frac{e^{i\omega r}}{4\pi r}\int du' d\hat{n}' \frac{e^{i\omega u' + i\omega(1-\vec{n}\cdot\vec{n}')r_0}}{i\omega(1-\vec{n}\cdot\vec{n}')} A(u',\hat{n}') n'_i n'_j \\
= \quad & i\frac{e^{i\omega r}}{4\pi r}\int du' d\hat{n}' \frac{1+\omega(u'+(1-\vec{n}\cdot\vec{n}')r_0)}{\omega(1-\vec{n}\cdot\vec{n}')} A(u',\hat{n}') n'_i n'_j \\
= \quad & i\frac{e^{i\omega r}}{4\pi r}\frac{1}{\omega}\int du' d\hat{n}' \frac{1}{(1-\vec{n}\cdot\vec{n}')} A(u',\hat{n}') n'_i n'_j + O(1).
\end{aligned} \tag{2.111}$$

In the second step of the eq.[(2.111)], we have ignored the terms at infinity.

Now let's calculate the second term in the last step of eq.[(2.109)].

$$\begin{aligned}
& \frac{e^{i\omega r}}{4\pi r}\int du' d\hat{n}' \int_{r_0}^{\infty} dr' e^{i\omega u' + i\omega r' - i\omega \vec{n}\cdot\vec{n}' r'} \left(\frac{B(u',\hat{n}')}{r'} n'_i n'_j + \frac{G(u',\hat{n}')_{(i}n'_{j)}}{r'} + \frac{H(u',\hat{n}')_{ij}}{r'}\right) \\
= \quad & O(1)
\end{aligned}$$

In eq.[(2.112)], we have used the result from eq.[(2.112)].

$$\begin{aligned}
& \int_{r_0}^{\infty} dr' \frac{e^{i\omega(1-\vec{n}\cdot\vec{n}')r'}}{r'} f(u',\hat{n}') \\
\simeq \quad & \int_{r_0}^{L} dr' \frac{e^{i\omega(1-\vec{n}\cdot\vec{n}')r'}}{r'} f(u',\hat{n}') \quad , \quad \text{where we have regularized the integral and set } L >> r_0 \\
= \quad & \int_{\omega(1-\vec{n}\cdot\vec{n}')r_0}^{\omega(1-\vec{n}\cdot\vec{n}')L} dz' \frac{e^{iz'}}{z'} f(u',\hat{n}') \quad , \quad \text{assume } z' = \omega(1-\vec{n}\cdot\vec{n}')r' \\
= \quad & \int_{\omega(1-\vec{n}\cdot\vec{n}')r_0}^{\omega(1-\vec{n}\cdot\vec{n}')L} dz' \frac{\sum_{n=0}^{\infty}\frac{(iz')^n}{n!}}{z'} f(u',\hat{n}') \\
= \quad & f(u',\hat{n}')\{\log z + \sum_{n=1}^{\infty}\frac{(iz')^n}{n\cdot n!}\}\Big|_{\omega(1-\vec{n}\cdot\vec{n}')r_0}^{\omega(1-\vec{n}\cdot\vec{n}')L} \\
\simeq \quad & -f(u',\hat{n}')\{\log(\frac{L}{r_0}) + \sum_{n=1}^{\infty}\frac{(i\omega(1-\vec{n}\cdot\vec{n}')^n(L^n - r_0^n)}{n\cdot n!}\},
\end{aligned} \tag{2.112}$$

where in the last step, we have ignored the contribution at infinity.

The third term in the last step of eq.[(2.109)] is of no interest to us as it will result in at least $O(1)$ terms.

Now, we will calculate $\tilde{e}^2_{\mu\nu}(\omega,\vec{x})$. To do so, we write

$$\begin{aligned} k_\alpha \tilde{e}^{2\alpha\beta}(\omega,\vec{x}) = & -\frac{e^{i\omega r}}{4\pi r}\int_{r\le r_0} dx'^4 \{\frac{\partial}{\partial x'^\alpha} e^{-ik\cdot x'}\} T^{\alpha\beta}(x') \\ = & \int d\hat{n}' \int du' r'^2 \hat{n}'_\alpha e^{-ik\cdot x'} T^{R\alpha\beta}(x')|_{r'=r_0}, \end{aligned} \tag{2.113}$$

where integration by parts was done in the second step. We have picked up the boundary term at $r' = r_0$ and used the conservation law: $\partial_\alpha T_R^{\alpha\beta}(x') = 0$.

The sum of the incoming flux and outgoing momentum flux is equal. Therefore,

$$r'^2 \int d\hat{n}' \int du' \hat{n}'_\alpha T^{R\alpha\beta} x'|_{r'=r_0} = 0. \tag{2.114}$$

Using eq.[(2.114)] in eq.[(2.113)], we get

$$k_\alpha \tilde{e}^{2\alpha\beta}(x) = \frac{e^{i\omega r}}{4\pi r} r'^2 \int d\hat{n}' \int du' \hat{n}'_\alpha k\cdot x' T^{R\alpha\beta}(x')\Big|_{r'=r_0} + O\left(\omega^2\right). \tag{2.115}$$

We can take the solution of eq.[(2.115)] to be

$$\tilde{e}^{2\alpha\beta}(x) = \frac{e^{i\omega r}}{4\pi r} r'^2 \int d\hat{n}' \int du' \hat{n}'_\gamma x'^\alpha T^{R\gamma\beta}(x')\Big|_{r'=r_0} + O(\omega). \tag{2.116}$$

But $\tilde{e}^{2\alpha\beta}(x)$ in eq.[(2.116)] is not symmetric. To symmetrize $\tilde{e}^{2\alpha\beta}(x)$, we use the angular momentum conservation:

$$r'^2 \int d\hat{n}' \int du' \hat{n}'_\gamma \left[-x'^\alpha T^{R\gamma\beta}(x') + x'^\beta T^{R\gamma\alpha}(x')\right]\Big|_{r'=r_0} = 0. \tag{2.117}$$

The symmetrized $\tilde{e}^{2\alpha\beta}(x)$ is as following:

$$\tilde{e}^{2\alpha\beta}(x) = \frac{1}{2}\frac{e^{i\omega r}}{4\pi r} \int d\hat{n}' \int du' r'^2 \hat{n}'_\gamma \left[x'^\alpha T^{R\gamma\beta}(x') + x'^\beta T^{R\gamma\alpha}(x')\right]\Big|_{r'=r_0} + O(\omega). \tag{2.118}$$

In eq.[(2.118)], we can see that $\tilde{e}^{2\alpha\beta}(x) = O(1)$. Hence, leading and subleading contribution in $\lim_{r\to\infty} \tilde{e}^{\alpha\beta}(x)$ comes from only $\tilde{e}^{1\alpha\beta}(x)$.

From the analysis above, we can conclude that due to the absence of $\log\omega$ term in $\tilde{e}_{ij}(\omega,\vec{x})$, peeling is not violated at $\mathcal{I}^+$ as $u \to \pm\infty$ for a system with a massless scalar field on the Minkowski background.

2.10 Soft Gravitons for Massive Scalar Field

In this section, we will verify if massive scalar fields on the Minkowski background violate the peeling property at the future null infinity. If we can show that $\lim_{r\to\infty} \tilde{e}_{\mu\nu}(\omega,\vec{x})$ has $\log\omega$ terms at the soft limit, from 2.7.2, we will be able to conclude that massive scalar fields violate the peeling property at $\mathcal{I}^+$ as $u \to -\infty$ and $u \to \infty$, respectively.

2.10.1 Massive Scalar Field Phase Space

The data on a unit hyperboloid describing timelike infinity can describe the massive scalar field space. The chosen coordinates for this description are

$$\tau = \sqrt{t^2 - r^2} \quad , \quad \rho = \frac{r}{\sqrt{t^2 - r^2}}. \tag{2.119}$$

In the new coordinates, the line element takes the form as follows:

$$ds^2 = -d\tau^2 + \frac{\tau^2}{1+\rho^2} d\rho^2 + \rho^2 \tau^2 d\Omega^2, \tag{2.120}$$

where $d\Omega^2$ is the unit sphere metric.

As $|\tau| \to \infty$, the massive scalar field behaves as following:

$$\varphi(\tau,\rho,\hat{n}) = \frac{\varphi^{(1)}(\rho,\hat{n})}{\tau^{3/2}} e^{-im\tau} + \frac{\varphi^{(2)}(\rho,\hat{n})}{\tau^{5/2}} e^{-im\tau} + O(\frac{1}{\tau^{7/2}}) \tag{2.121}$$

$$\varphi^*(\tau,\rho,\hat{n}) = \frac{\varphi^{(1)*}(\rho,\hat{n})}{\tau^{3/2}} e^{im\tau} + \frac{\varphi^{(2)*}(\rho,\hat{n})}{\tau^{5/2}} e^{im\tau} + O(\frac{1}{\tau^{7/2}}) \tag{2.122}$$

2.10.2 Strees Energy Tensor for a Massive Scalar Field

We will treat all the tensors in de-Donder coordinates. So, let us first define:

$$\partial_0\tau = \sqrt{1+\rho^2} \quad , \quad \partial_i\tau = -\rho n_i \quad , \quad \partial_0\rho = -\frac{\rho\sqrt{1+\rho^2}}{\tau} \quad , \quad \partial_i\rho = \frac{1+\rho^2}{\tau}n_i \quad , \quad \partial_\mu n_i = \frac{1}{\rho\tau}\eta^{\perp}_{\mu i}, \tag{2.123}$$

where $\eta^{\perp}_{\mu\nu} = \eta_{\mu\nu} + n_\mu n_\nu - n_\mu \tilde{n}_\nu - n_\nu \tilde{n}_\mu$ with $\hat{n}_i = \frac{x_i}{r}$, $n = (1, \vec{\hat{n}})$ and $\tilde{n} = (0, \vec{\hat{n}})$.

For a general function $f(\tau,\rho,\hat{n})$,

$$\partial_\mu f(\tau,\rho,\hat{n}) = \partial_\mu\tau\frac{\partial f}{\partial\tau} + \partial_\mu\rho\frac{\partial f}{\partial\rho} + \frac{1}{\tau\rho}\eta^{\perp}_{\mu i}\partial^i_\perp f \tag{2.124}$$

The stress-energy tensor for a massive scalar field is

$$T_{\mu\nu}(x) = (\partial_\mu\varphi\partial_\nu\varphi^* + \partial_\mu\varphi^*\partial_\nu\varphi) - \eta_{\mu\nu}\mathcal{L} \tag{2.125}$$

The spatial components of stress-energy tensor for the massive scalar field as $\tau \to -\infty$ takes the following form:

$$T^R_{ij}(x) = \frac{A(\rho,\hat{n})}{\tau^3}n_i n_j + \frac{B(\rho,\hat{n})}{\tau^4}n_i n_j + \frac{G(\rho,\hat{n})_{(i}n_{j)}}{\tau^4} + \frac{H(\rho,\hat{n})_{ij}}{\tau^4} + \mathcal{O}(\frac{1}{\tau^5}). \tag{2.126}$$

2.10.3 Soft Gravitons at future null infinity

We want to focus on the soft limit. So, we will be concentrating on eq.[(2.127)].

$$\lim_{\omega\to 0}\tilde{e}_{ij}(\omega,\vec{x}) = \frac{e^{i\omega r}}{4\pi r}\int dx'^4 e^{-ik\cdot x'}T^R_{ij}(x') \tag{2.127}$$

If we substitute eq.[(2.126)] in eq.[(2.127)], we get

$$\begin{aligned}
\lim_{\omega\to 0}\tilde{e}_{ij}(\omega,\vec{x}) = & \frac{e^{i\omega r}}{4\pi r}\int dx'^4 e^{ik\cdot x'}(\frac{A(\rho',\hat{n}')}{\tau'^3}n'_i n'_j + \frac{B(\rho',\hat{n}')}{\tau'^4}n'_i n'_j + \frac{G(\rho',\hat{n}')_{(i}n'_{j)}}{\tau'^4} + \frac{H(\rho',\hat{n}')_{ij}}{\tau'^4} + O(\frac{1}{\tau'^5})) \\
= & \frac{e^{i\omega r}}{4\pi r}\int \frac{\tau'^3\rho'^2}{\sqrt{1+\rho'^2}}d\tau' d\rho' d\hat{n}' e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')} \\
& (\frac{A(\rho',\hat{n}')}{\tau'^3}n'_i n'_j + \frac{B(\rho',\hat{n}')}{\tau'^4}n'_i n'_j + \frac{G(\rho',\hat{n}')_{(i}n'_{j)}}{\tau'^4} + \frac{H(\rho',\hat{n}')_{ij}}{\tau'^4} + O(\frac{1}{\tau'^5})) \\
= & \frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\tau' d\rho' d\hat{n}' e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')} \\
& (A(\rho',\hat{n}')n'_i n'_j + \frac{B(\rho',\hat{n}')}{\tau'}n'_i n'_j + \frac{G(\rho',\hat{n}')_{(i}n'_{j)}}{\tau'} + \frac{H(\rho',\hat{n}')_{ij}}{\tau'} + O(\frac{1}{\tau'^2})) \\
= & \frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\tau' d\rho' d\hat{n}' e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')} A(\rho',\hat{n}')n'_i n'_j \\
& \frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\tau' d\rho' d\hat{n}' e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')}(\frac{B(\rho',\hat{n}')}{\tau'}n'_i n'_j + \frac{G(\rho',\hat{n}')_{(i}n'_{j)}}{\tau'} + \frac{H(\rho',\hat{n}')_{ij}}{\tau'}) \\
& \frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\tau' d\rho' d\hat{n}' e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')}(O(\frac{1}{\tau'^2})) \qquad (2.128)
\end{aligned}$$

We have used the following relations in the second step of eq.[(2.128)]:

$$dx'^4 \to \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\tau' d\rho' d\hat{n}' \quad , \quad k\cdot x' = -\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}'), \qquad (2.129)$$

where $k^\mu \equiv \omega(1,\vec{n})$.

Let us verify the behavior of $\mathcal{I}^+$ as $u\to-\infty$. So, we will calculate the first term in the last step of eq.[(2.128)][2]:

$$\begin{aligned}
& \frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}' \int_{-\infty}^{-\tau_0} d\tau' e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')} A(\rho',\hat{n}')n'_i n'_j \\
= & -\frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}' \frac{e^{-i\omega\tau'_0(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')}}{\omega} A(\rho',\hat{n}')n'_i n'_j \\
= & -\frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}' \frac{1-i\omega\tau'_0(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')}{\omega} A(\rho',\hat{n}')n'_i n'_j \\
= & -\frac{e^{i\omega r}}{4\pi r}\frac{1}{\omega}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}' A(\rho',\hat{n}')n'_i n'_j + O(1). \qquad (2.130)
\end{aligned}$$

[2]As we are interested in the structure of $\mathcal{I}^+$ as $u\to-\infty$, we will take $-\tau_0\to-\infty$.

In the second step of the eq.[(2.130)], we have ignored the terms at infinity.

Now let's calculate the second term in the last step of eq.[(2.128)].

$$\begin{aligned} & \frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}'\int d\tau' e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')}\left(\frac{B(\rho',\hat{n}')}{\tau'}n'_i n'_j+\frac{G(\rho',\hat{n}')_{(i}n'_{j)}}{\tau'}+\frac{H(\rho',\hat{n}')_{ij}}{\tau'}\right) \\ = \quad & \frac{e^{i\omega r}}{4\pi r}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}' \log\{\omega(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')\}(B(\rho',\hat{n}')n'_i n'_j+G(\rho',\hat{n}')_{(i}n'_{j)}+H(\rho',\hat{n}')_{ij}) \\ = \quad & \frac{e^{i\omega r}}{4\pi r}(\log\omega)\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}'(B(\rho',\hat{n}')n'_i n'_j+G(\rho',\hat{n}')_{(i}n'_{j)}+H(\rho',\hat{n}')_{ij})+\mathcal{O}(1) \qquad (2.131) \end{aligned}$$

In eq.[(2.131)], we are only interested in the limit $\tau \to -\infty$. So,

$$\int d\tau' \frac{e^{i\omega\tau'(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')}}{\tau'} = \log\{\omega(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}')\}, \qquad (2.132)$$

where $\omega(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}') \to 0$. We have discussed this in Appendix[C]. Now, we can see that $(\sqrt{1+\rho'^2}-\rho'\vec{n}\cdot\vec{n}') > 0$, always. Hence, $\omega \to 0$.

The third term in the last step of eq.[(2.128)] is of no interest to us as it will result in at least $\mathcal{O}(1)$ terms.

If we want to verify the behavior of $\mathcal{I}^+$ as $u \to -\infty$, we can take the limit of integration from τ_0 to ∞ while integrating over $d\tau$. Here, $\tau_0 \to \infty$. In this case, the first term in eq.[(2.128)] will be

$$-\frac{e^{i\omega r}}{4\pi r}\frac{1}{\omega}\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}' A(\rho',\hat{n}')n'_i n'_j+\mathcal{O}(1),$$

and the second term would be

$$-\frac{e^{i\omega r}}{4\pi r}(\log\omega)\int \frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho' d\hat{n}'(B(\rho',\hat{n}')n'_i n'_j+G(\rho',\hat{n}')_{(i}n'_{j)}+H(\rho',\hat{n}')_{ij})+\mathcal{O}(1).$$

From the analysis above, we can conclude that due to the presence of $\log\omega$ term in $\tilde{e}_{ij}(\omega,\vec{x})$ for $\tau_0 \to \pm\infty$, peeling is violated at $\mathcal{I}^+$ as $u \to \pm\infty$ for a system with a massive scalar field on the Minkowski background.

Chapter 3

RESULTS AND DISCUSSION

3.1 Results

In 2.2, we see that with a compact spatial source in the Minkowski background, the gravitational wave does not violate peeling at the future null infinity. Hence, the spacetime is asymptotically simple, and the future null infinity is C^∞.

Then we turn to the analysis of soft gravitons due to different systems on the Minkowski background to investigate the structure of the future null infinity. We show in 2.7.1 that the $\frac{1}{\omega}$ terms in $\lim_{\omega\to 0}\tilde{e}_{\mu\nu}(\omega,\vec{x})$ are responsible for the memory effect at $\mathcal{I}^+$. We also demostrate in 2.7.2 that the $\log\omega$ terms in $\lim_{\omega\to 0}\tilde{e}_{\mu\nu}(\omega,\vec{x})$ generate the terms proportional to $\frac{1}{u}$ in $\lim_{r,|u|\to\infty} h_{\mu\nu}(x)$ at leading order. We show in 2.3.1 how the terms proportional to $\frac{1}{u}$ in $\lim_{r,|u|\to\infty} h_{\mu\nu}(x)$ at leading order generate non-artifact $\log r$ terms and make the future null infinity C^2. Therefore, the $\log\omega$ terms in $\lim_{\omega\to 0}\tilde{e}_{\mu\nu}(\omega,\vec{x})$ responsible for the violation of peeling at $\mathcal{I}^+$ as $u\longrightarrow\pm\infty$ Hence, as long as we can demonstrate the presence of the $\log\omega$ terms in $\lim_{\omega\to 0}\tilde{e}_{\mu\nu}(\omega,\vec{x})$ for a system, we can conclude that peeling is violated at $\mathcal{I}^+$ as $u\longrightarrow\pm\infty$ for that corresponding system.

In 2.8.3, we have shown that for N massive particles at far past with the Newtonian limit,

$$\begin{aligned} &\tilde{e}_{\mu\nu}(\omega,\vec{x}) \\ = \quad & -\frac{e^{i\omega r}}{4\pi r}\sum_{a=1}^{N} p_{(a)\mu}p_{(a)\nu}\frac{1}{ik\cdot p_{(a)}} \\ &+\frac{e^{i\omega r}}{4\pi r}\frac{\log\omega}{4\pi}\Big\{\sum_{a=1}^{N}\sum_{\substack{b=1\\ a\neq b}}^{N}\frac{p_{(a)}\cdot p_{(b)}\{\frac{3}{2}p_{(a)}^2p_{(b)}^2-(p_{(a)}\cdot p_{(b)})^2\}}{\{(p_{(a)}\cdot p_{(b)})^2-p_{(a)}^2p_{(b)}^2\}^{\frac{3}{2}}}\frac{k_\rho p_{(a)\mu}}{k\cdot p_{(a)}}(p^{\rho}_{(a)}p_{(b)_\nu}-p_{(b)\nu}p^{\rho}_{(a)}) \\ &+\sum_{a=1}^{N}\sum_{b=1}^{N}\frac{k\cdot p_{(b)}}{k\cdot p_{(a)}}p_{(a)\mu}p_{(a)\nu}\Big\}+O(1), \end{aligned} \tag{3.1}$$

where the leading term is proportional to $\frac{1}{\omega}$, which is responsible for the memory effect, and the subleading term is proportional to $\log\omega$, which leads to the violation of peeling at $\mathcal{I}^+$ as $u \to -\infty$.

In 2.8.4, we have shown that for N massive particles at far future with the Newtonian limit,

$$\begin{aligned}
&\tilde{e}_{\mu\nu}(\omega,\vec{x}) \\
= \quad & -\frac{e^{i\omega r}}{4\pi r}\sum_{a=1}^{N} p'_{(a)\mu}p'_{(a)\nu}\frac{1}{ik\cdot p'_{(a)}} \\
&+\frac{e^{i\omega r}}{4\pi r}\frac{\log\omega}{4\pi}\Big\{\sum_{a=1}^{N}\sum_{\substack{a\neq b\\ b=1}}^{N}\frac{p'_{(a)}\cdot p'_{(b)}\{\frac{3}{2}p'^2_{(a)}p'^2_{(b)}-(p'_{(a)}\cdot p'_{(b)})^2\}}{\{(p'_{(a)}\cdot p'_{(b)})^2-p'^2_{(a)}p'^2_{(b)}\}^{\frac{3}{2}}}\frac{k_\rho p'_{(a)\mu}}{k\cdot p'_{(a)}}(p'^{\rho}_{(a)}p'_{(b)_\nu}-p'_{(b)\nu}p'^{\rho)}_{(a)} \\
&+\sum_{a=1}^{N}\sum_{b=1}^{N}\frac{k\cdot p'_{(b)}}{k\cdot p'_{(a)}}p'_{(a)\mu}p'_{(a)\nu}\Big\}+O(1),
\end{aligned} \tag{3.2}$$

where the leading term is proportional to $\frac{1}{\omega}$, which is responsible for the memory effect, and the subleading term is proportional to $\log\omega$, which leads to the violation of peeling at $\mathcal{I}^+$ as $u \to \infty$.

In 2.9.3, we have shown that for a massless scalar field on 4-dimensional Minkowski background

$$\begin{aligned}
&\lim_{\omega\to 0}\tilde{e}_{\mu\nu}(\omega,\vec{x}) \\
= \quad & i\frac{e^{i\omega r}}{4\pi r}\frac{1}{\omega}\int du'd\hat{n}'\frac{1}{(1-\vec{n}\cdot\vec{n}')}A(u',\hat{n}')n'_in'_j+O(1),
\end{aligned}$$

where the leading term is proportional to $\frac{1}{\omega}$, which is responsible for the memory effect. But there is no term proportional to $\log\omega$. Hence, the peeling is not violated at $\mathcal{I}^+$.

In 2.10.3, we have shown that for a massive scalar field on 4-dimensional Minkowski background

$$\begin{aligned}
&\lim_{\omega\to 0}\tilde{e}_{\mu\nu}(\omega,\vec{x}) \\
= \quad & -\frac{e^{i\omega r}}{4\pi r}\frac{1}{\omega}\int\frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho'd\hat{n}'A(\rho',\hat{n}')n'_in'_j \\
&\pm\frac{e^{i\omega r}}{4\pi r}(\log\omega)\int\frac{\rho'^2}{\sqrt{1+\rho'^2}}d\rho'd\hat{n}'(B(\rho',\hat{n}')n'_in'_j+G(\rho',\hat{n}')_{(i}n'_{j)} \\
&+H(\rho',\hat{n}')_{ij})+O(1) \quad , \quad \text{at } \mathcal{I}^+_\mp,
\end{aligned} \tag{3.3}$$

where the leading term is proportional to $\frac{1}{\omega}$, which is responsible for the memory effect, and the subleading term is proportional to $\log\omega$, which leads to the violation of peeling at $\mathcal{I}^+$ as $u \to \pm\infty$.

3.2 Discussion

We aim to construct a physical spacetime that is not asymptotically simple and violates the peeling property at the future null infinity. For that, we need the metric at the future null infinity to be C^k, where $k < 3$. We attempt to create a spacetime with Corvino's gluing construction. With Corvino's gluing construction, we can glue different manifolds with different differentiability as long as the metrics and the extrinsic curvatures match at the gluing boundary of manifolds. To understand Corvino's gluing construction, we go through an application of the construction done by Chrusciel and Delay [19]. Chrusciel and Delay constructed non-trivial vacuum solutions of Einstein's equations. They glued $(2N+1)$ Schwarzschild metrics to create a spacetime with $2N$ blackholes. Similarly, if we want to create a spacetime that is C^2 at the future null infinity, we can glue a C^2 metric to some other metric. And to construct a C^2 metric, we can introduce $\log r$ terms in the metric. But the problems here in such a construction are:

1. It is very hard to interpret such manifolds physically.
2. It is very hard to detect if the $\log r$ terms can be eliminated from the metric with a coordinate transformation.

If the $\log r$ terms can be eliminated from the metric with a coordinate transformation, we can conclude that the log terms are not the consequence of physical conditions, rather they are just characteristics of the coordinate choice or the "artifacts of the coordinates."

When we look into the analysis of the gravitational waves for compact spatial support, we run into log terms in the metric while working in de-Donder coordinates or harmonic coordinates. But Bondi, Metzner [1] and Sachs's [2] analysis shows that such log terms do not appear in the Bondi gauge. Hence, we can see that the log terms in the metric of linear perturbation to the Minkowski metric in the harmonic gauge are just the "artifacts" of de-Donder coordinates.

We turn to soft graviton theorems to verify the existence of non-artifact $\log r$ terms in the metric at the future null infinity. We demonstrate that if non-linear effects are taken into account, then massive particles at far past or far future can also violate peeling at future null infinity. But we later go on to show that even if we do not consider non-linear effects, a system with a system with massive scalar field can violate peeling at the future null infinity. In these cases, we see that the violation of peeling happens due to the non-artifact $\log r$ terms present in $e_{\mu\nu}(x)$. But in case of massless scalar fields, peeling is not violated.

In our analysis, we have used the Minkowski metric as the background. But Sen et al. [10] used the perturbation of the Minkowski metric as the background to derive the soft graviton theorem for N-particles. The perturbation to the Minkowski metric generates log terms in the trajectories of the particles. In our analysis, the Newtonian limit generates log terms in the trajectories of the particles. If we look at it another way, we also consider the non-linear effect if we consider the Newtonian limit. Sen et al. [10] show the existence of log terms even for massless particles, but here we will only analyze the case for massive particles and move on to scalar fields in 4-dimension.

At soft limit for N-massive particles at past infinity or far future, massless scalar field and massive scalar field on Minkowski background the terms in $\tilde{e}_{ij}(\omega, \vec{x})$, the leading terms in all cases are proportional to $\frac{1}{\omega}$. From the discussion in 2.7.1, we can conclude that these leading terms generate the memory effect in all cases.

The subleading term in the case of N-massive particles at past infinity and far future and massive scalar field is proportional to $\log \omega$. Because of this $\log \omega$ term, there exists a non-zero limit for $\lim_{u \to \pm\infty} u^2 N_{\mu\nu}(x)$ 2.7.2, where $N_{\mu\nu}(x)$ is the news tensor. Hence, these generate non-artifact $\log r$ terms in $e_{ij}(x)$ and the metric at $\mathcal{I}^+$ becomes C^2. Therefore, the peeling is violated at the future null infinity, and the spacetime is not asymptotically simple for the massive particles and massive scalar field case in our analysis.

But the subleading term in the case of a massless scalar field is of order 1. Hence, $\lim_{u \to \pm\infty} u^2 N_{\mu\nu}(x) = 0$. Therefore, the peeling is not violated at the future null infinity, and the spacetime is asymptotically simple for a massless scalar field on the Minkowski background.

Chapter 4

CONCLUSION AND OUTLOOK

In this thesis, we try to investigate the structure of the future null infinity of asymptotically flat spacetime. We have mainly focused on spacetimes that are not asymptotically simple and violate Penrose's peeling property at future null infinity. We first try to use Corvino's gluing construction to construct a spacetime that violates the peeling property. But upon realizing the difficulty in recognizing if the logarithmic terms are artifacts or non-artifacts of the chosen coordinates, we turn to the analysis of soft gravitons. From the characteristics of soft gravitons, we infer the structure of the future null infinity as $|u| \to \infty$ in 4-dimensional spacetimes.

We have demonstrated that the peeling is violated at $\mathcal{I}^+$ as $u \to \pm\infty$ for N massive particles at far past or far future with the Newtonian limit on Minkowski background with dimensions$= 4$ and the violation of peeling can be demostrated due to a similar system on Minkowski background with dimensions> 4. Then we go on to show that peeling holds good for a massless scalar field on 4-dimensional Minkowski background. In contrast to the massless case, peeling is violated at the future null infinity as $u \to \pm\infty$ for a massive scalar field on 4-dimensional Minkowski background.

In the case of massless and massive scalar fields, the analysis can be replicated for Minkowski backgrounds with dimensions > 4. A similar calculation can also be done for other fields (e.g., Abelian gauge field, non-Abelian gauge field, vector fields, spinor fields, tensorial fields), and one can analyze the behavior of $\mathcal{I}^+$. One can investigate the asymptotic symmetries of LAF spacetimes and the charges corresponding to those symmetries instead of asymptotically simple spacetimes. One can also further investigate Celestial Holography corresponding to LAF spacetimes.

Appendix A

WEYL TENSOR

Lets assume, we have a metric $g_{\mu\nu}$ on n-dimensional spacetime.For $n \geq 3$,

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{(g_{\alpha\delta}R_{\gamma\beta} + g_{\gamma\beta}R_{\alpha\delta} - g_{\alpha\gamma}R_{\delta\beta} - g_{\delta\beta}R_{\alpha\gamma})}{(n-2)} + \frac{(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})R}{(n-2)(n-1)}, \tag{A.1}$$

where $C_{\alpha\beta\gamma\delta} \equiv$ Weyl tensor, $R_{\alpha\beta\gamma\delta} \equiv$ Riemann tensor, $R_{\alpha\beta} \equiv$ Ricci tensor and $R \equiv$ Ricci scalar on metric $g_{\mu\nu}$.

From eq.[(A.1)] we can get the following relations in eq.[(A.2)]:

$$\begin{aligned} &C_{\alpha\beta\gamma\delta} = -C_{\alpha\beta\delta\gamma} = -C_{\beta\alpha\gamma\delta} = C_{\gamma\delta\alpha\beta} \\ &C_{\alpha\beta\gamma\delta} + C_{\alpha\delta\beta\gamma} + C_{\alpha\gamma\delta\beta} = 0 \\ &C^{\alpha}_{\beta\alpha\delta} = 0. \end{aligned} \tag{A.2}$$

Now if $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ are conformally related such that

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}, \tag{A.3}$$

then

$$\tilde{C}^{\alpha}_{\beta\gamma\delta} = C^{\alpha}_{\beta\gamma\delta}, \tag{A.4}$$

where $C_{\alpha\beta\gamma\delta}$ and $\tilde{C}_{\alpha\beta\gamma\delta}$ are the weyl tensors on $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, respectively.

As seen in eq.[(A.4)], the conformal invariance of Weyl tensors, it is very helpful to study conformal methods involving Weyl tensors.

Appendix B

THE 3 + 1 DECOMPOSITION OF GENERAL RELATIVITY:

B.1 Submanifold:

If $\mathscr{N} \subseteq \mathscr{M}$ and $\mathscr{N}$ inherits a manifold structure from $\mathscr{M}$, $\mathscr{N}$ is called a submanifold of $\mathscr{M}$.

B.2 Foliations of Space-time:

In our discussion, we assume the space-time $(M, g_{\mu\nu})$ to be globally hyperbolic. We assume the topology of the spacetime is $\mathbb{R}\times S$, where S is a three-dimensional orientable manifold.

Definition of Foliation: A space-time is called foliated by the hypersurfaces S_t if the the hypersurfaces S_t are non-intersecting and $M = \bigcup_{t\in\mathbb{R}} S_t$, where we have taken $S_t = \{t\} \times S$.

B.3 Time Function:

It is often conveniently assumed that the hypersurfaces S_t are the level surfaces of a scalar function t, which is interpreted as a global time function.

The covector $\omega_\mu = \nabla_\mu t$ is normal to the hypersurface S_t.

$$\nabla_{[\mu}\omega_{\nu]} = \nabla_{[\mu}\nabla_{\nu]}t = 0$$

Hence, ω_a is closed.

B.4 Lapse Function:

The lapse function α is defined via:

$$g^{\mu\nu}\omega_\mu\omega_\nu = -\frac{1}{\alpha^2}$$

.The lapse is the measure of proper time elapsed between neighboring time slices along the direction given by the normal vector $\omega^\mu = g^{\mu\nu}\omega_\nu$. From the definition of lapse function, we can see that ω^μ is timelike (as $\alpha \neq 0 \implies \omega^\mu\omega_\mu < 0$), hypersurfaces S_t are spacelike.

Now, if we define *unit normal* as $n_\mu = -\alpha\omega_\mu$, $n^\mu n_\mu = -1$. One may think of n^a as the four-velocity, always orthogonal to the hypersurfaces S_t.

B.5 Intrinsic Metric of an Hypersurface:

The spacetime metric $g_{\mu\nu}$ induces a three-dimensional metric h_{ij} on the hypersurfaces S_t. The relation between g_{ab} and h_{ab} is given by

$$h_{ab} \equiv g_{ab} + n_a n_b$$

. The tensor h_{ab} is purely spatial. Hence, it has no component along n^a. This fact can be seen by contracting h_{ab} with the normal n^a:

$$n^a h_{ab} = n^a g_{ab} + n_a n^a n_b = n_b - n_b = 0$$

The inverse 3- metric can be obtained by raising the indices.

$$h^{ab} = g^{ab} + n^a n^b$$

The 3-metric h_{ab} can project all geometric along the direction of n^a. h_{ab} can decompose tensor in purely spacelike part which lies on S_t and purely timelike part which is normal to S_t. The spatial part of the tensor $T^{ab\ldots m}_{np\ldots w}$ is

$$(T^\perp)^{ab\ldots m}_{np\ldots w} = (h^a_{a'} h^b_{b'} \ldots h^m_{m'})(h^{n'}_n h^{p'}_p \ldots h^{w'}_w) T^{a'b'\ldots m'}_{n'p'\ldots w'}$$

where $h^a_b = \delta^a_b + n^a n_b$.

B.6 Covariant Derivatives on Hypersurfaces:

The 3-metric h_{ij} defines a unique kind of covariant derivative D_i. Here, D_a is torsion-free and compatible with the metric h_{ab}. The action of D_a on (m,n)-rank of tensor is demonstrated below:

$$D_a T^{b_1 b_2 \ldots b_m}_{c_1 c_2 \ldots c_n} = h^d_a (h^{b_1}_{b'_1} h^{b_2}_{b'_2} \ldots h^{b_m}_{b'_m})(h^{c'_1}_{c_1} h^{c'_2}_{c_2} \ldots h^{c'_n}_{c_n}) \nabla_d T^{b^p_1 rime b'_2 \ldots b'_m}_{c'_1 c'_2 \ldots c'_n}$$

The covariant derivative D_a for the 3-metric h_{ij} is associated with the spatial Christoffel symbols:

$$\gamma^{\mu}_{\nu\lambda} = \frac{1}{2}h^{\mu\rho}(\partial_\nu h_{\rho\lambda} + \partial_\lambda h_{\nu\rho} - \partial_\rho h_{\nu\lambda})$$

Now, we can associate a curvature tensor r^a_{bcd} to the covariant derivative D_a.

$$[D_a, D_b]v^c = r^c_{dab}v^d$$

Hence, $r^c_{dab}n^d = 0$.

Similarly, we can define the Ricci tensor and scalar for the metric h_{ij} as

$$r_d b \equiv r^c_{dcb} \quad , \qquad r \equiv h^{ab}r_{ab}$$

B.7 The Extrinsic Curvature:

The implications of Einstein's equations on the hypersurfaces can be understood by decomposing R^a_{bcd} into spatial parts, which involves r^a_{bcd}. r^a_{bcd} measures the intrinsic curvature of the hypersurfaces. The rest of the information is in the *extrinsic curvature*.

The extrinsic curvature is defined in the following way:

$$K_{ab} \equiv -h^c_a h^d_b \nabla_{(c}n_{d)} = -h^c_a h^d_b \nabla_c n_d \quad ,$$

since n^a is rotation free. We can see that the extrinsic curvature is $(0,2)$-rank symmetric tensor and purely spatial. The extrinsic curvature can also be presented as

$$K_{ab} = -\nabla_a n_b - n_a a_b \quad \text{and} K$$

where $a_b = n^c\nabla_c n_b$.

One can define the mean extrinsic curvature as

$$K \equiv h^{ab}K_{ab} \quad .$$

Appendix C

INVERSE FOURIER TRANSFORMATION OF log TERMS

When $\omega \to 0$,

$$\begin{aligned}
&\int \frac{d\omega}{2\pi} e^{\pm i\omega u} C \log(\omega) f(\omega) \\
&= C \int \frac{d\omega}{2\pi} e^{\pm i\omega u} \log(\omega \pm i\epsilon) f(\omega) \\
&\to \mp \frac{C}{u}, \text{ for } u \to \pm\infty,
\end{aligned} \tag{C.1}$$

where C is a constant and $f(\omega)$ is an arbitrary function of ω.

By reverse analysis, we can say that when $u \to -\infty$,

$$\int du e^{\pm i\omega u} \frac{C}{u} \to C \log(\omega), \text{ for } \omega \to 0. \tag{C.2}$$

Here, u is just a dummy variable. This we can replace u by some other variable, such as $\tau'(\sqrt{1+\rho'^2} - \rho' \vec{n} \cdot \vec{n}')$. We have done this in eq.[(2.132)].

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