Spontaneous Collapse Models From A Coarse-grained Deterministic And Non-unitary Dynamics

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Certificate

This is to certify that this dissertation entitled Spontaneous Collapse Models From A Coarse-grained Deterministic And Non-unitary Dynamics towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Kakade Kartik Nanasaheb at Indian Institute of Science Education and Research under the supervision of Dr. Tejinder Pal Singh (TIFR, Mumbai and IUCAA, Pune), Professor, Department of Physics, during the academic year 2022-2023.

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This thesis is dedicated to my parents
for their endless support and encouragement

Declaration

I hereby declare that the matter embodied in the report entitled Spontaneous Collapse Models From A Coarse-grained Deterministic And Non-unitary Dynamics are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Tejinder Pal Singh (TIFR, Mumbai and IUCAA, Pune) and the same has not been submitted elsewhere for any other degree.

Kakade Kartik Nanasaheb

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Prajwal, I'm sure the first thing you would do when you open this thesis is search for your name in this page. Here you are. Thank you so much for cross-checking the calculations and helping to resolve my doubts.

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Abstract

We study the measurement process of a quantum system at Planck time resolution using trace dynamics. When a quantum system interacts with a classical apparatus the combined system is macroscopic and the non-unitarity in the trace dynamics equations become important. In quantum mechanics we have a unitary and deterministic evolution of the wavefunction and therefore superpositions are preserved with time. However, in trace dynamics since non-unitarity is important, we investigate the non-unitary and deterministic evolution of a quantum system. To gain insights into this different kind of evolution, we consider a simple two-level qubit system. The motivation for pursuing this study was trace dynamics formulation in which quantum theory comes out as an emergent phenomena. We present, in this thesis, an overview of standard quantum formalism, the measurement problem, spontaneous collapse models and trace dynamics.

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Chapter 1

Introduction

> *And here he had the idea. An idea that could only be had with the unfettered radicalism of the young. The idea that would transform physics in its entirety—together with whole of science and our very conception of the world. An idea, I believe, that humanity has not yet fully absorbed.*
>
> Carlo Rovelli
> while describing Heisenberg's discovery of matrix mechanics
> in *Helgoland, 2021*[1]

In the summer of 1925, when he was working late in the night on an isolated island called Helgoland, Heisenberg laid the mathematical foundations of quantum theory. Since then, quantum theory has flauntingly passed every test of experiments. It is known for the explanation of blackbody spectrum, the photoelectric effect, molecular orbital theory in chemistry, spectrum emitted by atoms, quantum electrodynamics, developments in standard model, nanotechnology, applications in condensed matter physics such as semi- and super- conductors, quantum information and quantum computation. It has wide applications and gives highly precise predictions. Yet, as I write this in the summer of 2023—almost one hundred years later—the meaning of quantum theory is not very clear[2]. It faces serious conceptual problems which are still being debated[3, 4, 5].

What causes physicists and philosophers to think that there is something wrong with quantum theory despite its exceptional performance in experiments? In this thesis, we discuss one of the major issues with quantum mechanics called the measurement problem. Before doing so, in Chapter 2 we will first review the standard formalism of quantum theory with an example of a two-level qubit system. We will particularly stress on the two different kinds of dynamical evolution present

in this standard formalism. Next, in Chapter 3, we discuss the problems with this standard formulation and how it gives rise to the problem of measurement. Again, we use the same two level system to illustrate the problem. Then, in the coming Chapter 4, we present the basic features of spontaneous collapse models which provide a potential solution to the measurement problem. The Ghirardi-Rimini-Weber model is presented with some examples to illustrate essential properties of collapse models in general. The QMUPL and the CSL models are briefly discussed along with the qubit example. Collapse models are solely based on experimental data and do not make assumptions about underlying mechanisms i.e. these models are phenomenological. Therefore, it interesting to study the underlying physical theories or mechanisms which could give a complete understanding of the physical processes that give rise to the behavior of collapse models. Trace dynamics is an example of such a theory and we briefly review the Chapter 5. Finally, we move on to the question addressed in this thesis which was motivated from the trace dynamics formalism.

Chapter 2

Time Evolution In Standard Quantum Mechanics

In this chapter we discuss the time evolution in the standard formalism of quantum mechanics. We illustrate the two contrasting dynamics with an example of a qubit and ask some fundamental questions that are raised due to the peculiar nature of the dynamics.

2.1 Dynamics

Quantum theory as we know it[6] is mainly based on two dynamical postulates. The first postulate describes the behaviour of of an isolated quantum system S. Whereas, the second postulate describes the behaviour of the quantum system S when a measurement is performed on it.

1. **Isolated evolution:** The state vector $|\psi\rangle$, in a Hilbert space $\mathscr{H}$, of an isolated quantum system evolves according to the Schrödinger equation

$$i\hbar\frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle \tag{2.1}$$

or in the density matrix formulation, according to the von-Neumann equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H},\rho] \tag{2.2}$$

where $\hat{H}$ is the Hamiltonian of the system S.

2. **Measurement:** When a macroscopic measurement device M, is brought in contact to mea-

sure an observable A, which is a Hermitian operator with spectral density $A = \sum_k \lambda_k |k\rangle \langle k|$ the state $|\psi\rangle$ of the system collapses randomly to one of the eigenstates of A, say $|k\rangle$

$$|\psi\rangle \to |\psi_k\rangle = \frac{\langle k|\psi\rangle}{\sqrt{p_k}} |k\rangle \qquad \rho \to \rho_k = \frac{|k\rangle \langle k|\rho|k\rangle \langle k|}{p_k} \tag{2.3}$$

where $\{|k\rangle\}$'s are the eigenstates of A, $\{\lambda_k\}$'s are the eigenvalues and are outcomes of the measurement. The probability that the outcome of the measurement is λ_k is given by the Born rule

$$p_k = \langle \psi|k\rangle \langle k|\psi\rangle = |\langle \psi|k\rangle|^2 = \mathrm{Tr}\left[\rho |k\rangle \langle k|\right] \tag{2.4}$$

Let us consider an illustrative example which we will frequently encounter in this thesis. Consider the Hamiltonian of a qubit (which is a simplified version of an atom and can be manipulated with precision controls in laboratories[7])

$$H = \hbar\omega\sigma_x = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

this Hamiltonian has two eigenstates $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalues $\lambda_+ = \hbar\omega$ and $\lambda_- = -\hbar\omega$ respectively.

Isolated evolution: Given the initial state $|\psi(0)\rangle$, to solve the Schrödinger equation (2.1), we write the time evolved state in the basis of eigenstates (eigenbasis) $\{|+\rangle, |-\rangle\}$ of H

$$|\psi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle \tag{2.6}$$

and then solve for the coefficients $c_+(t)$ and $c_-(t)$ to get

$$|\psi(t)\rangle = e^{-i\omega t} c_+(0)|+\rangle + e^{i\omega t} c_-(0)|-\rangle \tag{2.7}$$

where $c_+ = \langle\psi(0)|+\rangle$ and $c_- = \langle\psi(0)|-\rangle$. The Hamiltonian (2.5) has an effect of rotating the Bloch vector(a vector connecting origin and a point inside or on the Bloch sphere, see Fig. 2.1) about the axis which is aligned with the eigenvectors $|\pm\rangle$.

Measurement: If we choose to measure the energy of this qubit, the observable is $H = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-| = \hbar\omega|+\rangle\langle+| - \hbar\omega|-\rangle\langle-|$. The state (2.7) will collapse to one of the eigenstates $|+\rangle$ or $|-\rangle$ with probabilities $p_+ = |\langle\psi(0)|+\rangle|^2$ and $p_- = |\langle\psi(0)|-\rangle|^2$ respectively.

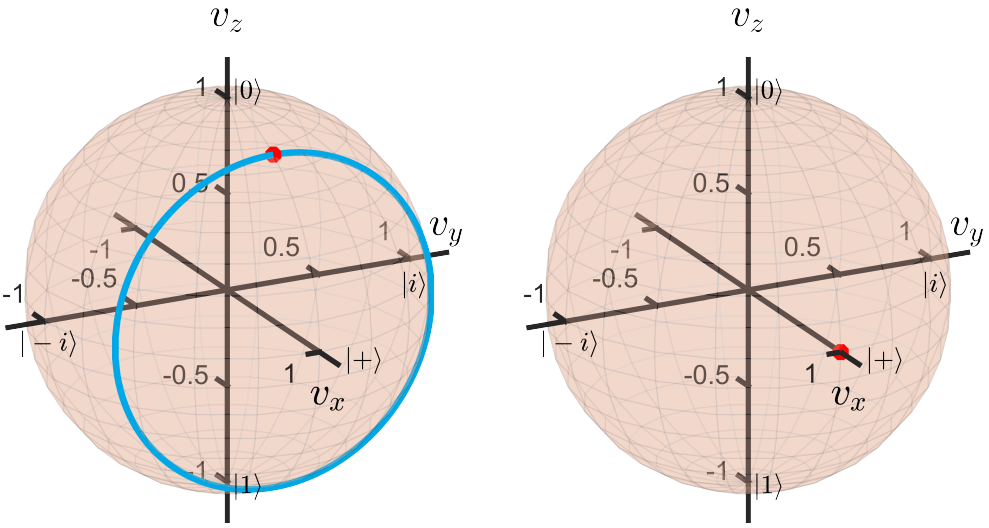

Figure 2.1: The figure on the left shows the trajectory of the Bloch vector given an initial state $|\psi(0)\rangle$ (shown in red) and the Hamiltonian $H = \hbar\omega\sigma_x$. Under the Schrödinger evolution (2.1), the vector rotates about the v_x-axis. On the right, is the Bloch vector just after measurement of the observable H (shown in red).

Keeping this example in mind, we shall now discuss some key features of the two kinds of time evolutions we just demonstrated. The Schrödinger time evolution can be described using a time evolution operator U. Using this operator, we can relate the two states $|\psi(0)\rangle$ and $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle \tag{2.8}$$

We demand that this operator obey the following properties for certain reasons.

1. The operator $U(t,t_0)$ is linear.
   The states $|\psi(0)\rangle$ and $|\psi(t)\rangle$ belong to a vector space. We do not want to make things complicated unless necessary and linear operators on a vector space are the simplest to define. The superposition of two states is also a valid state of the quantum system.

2. The operator $U(t,t_0)$ is unitary.
   We need to preserve the normalization of the state vector during the evolution. This ensures that the sum of probabilities of measurement outcomes is one. The unitarity also ensures

that the expectation values of observables are well defined. In our example, the length of the Bloch vector remains conserved during the entire evolution.

3. Composition property $U(t_2,t_1)U(t_1,t_0) = U(t_2,t_0)$
   Time evolution from t_0 to some intermediate time t_1 and then from t_1 to t_2 should be physically equivalent to an evolution from t_0 directly to t_2. In fact, due to this property, the operators $U(t)$ form a continuous unitary group and the Hamiltonian is the generator of this group leading to a unitary time evolution. As seen in the example, the trajectories of the evolution are continuous.

To sum up, the Schrödinger evolution is a *deterministic*, *linear* and *unitary* evolution.

The time evolution of the wavefunction during measurement is however not as transparent. We can, nevertheless, infer some properties of this 'collapse evolution' using our example. Observe the following points (refer to Fig. 2.1).

1. Whether the qubit collapses to $|+\rangle$ or $|-\rangle$ is not determined beforehand. We can only assign probability of occurrence to each of the outcomes. This implies the collapse evolution should *involve probabilities* in some manner or the other.

2. Before measurement, the superposition of two states $|+\rangle$ and $|-\rangle$ is preserved due to linearity. After measurement, this superposition is destroyed and the qubit is present in one of the states. Since superposition is destroyed, the collapse evolution must be *non-linear*.

3. Unitarity also implies that the entire evolution before the measurement is time reversible. However, after measurement the final state cannot be reverse time-evolved to the original state (also, there is no trajectory which shows how the initial state evolves towards one of the eigenstates). Therefore the collapse evolution must also be *non-unitary*.

As we see, the Schrödinger evolution and the collapse evolution demonstrate contrasting properties. One is linear, deterministic and unitary while the other is non-linear, non-unitary and involves probabilities. This formulation of quantum mechanics—as a fundamental theory of nature—raises a number of questions. Some of them are

1. Why do we need to treat macroscopic measurement devices with drastically different laws?

2. Shouldn't the classical behaviour of measurement devices be derived using fundamental laws rather than defining it?

3. Where is the fine line, if at all, between the definitions of a microscopic and a macroscopic system?

4. Why do probabilities arise during measurements?

5. If we treat the measurement device M and the quantum system S using the same deterministic laws of Schrödinger why do we not observe the macroscopic superpositions experimentally?

6. What is the rate of the wavefunction collapse? Is it instantaneous?

7. How does a system transition from quantum to classical?

We shall elaborate more on these questions in the following chapters.

Chapter 3

The Quantum Measurement Problem

What goes wrong if we ignore the collapse postulate and treat the process of measurement as a mere quantum mechanical interaction between the measurement device M and the quantum system S? We get results which are experimentally false. We will largely follow arguments by Norsen[8], for more general treatment see [9]

3.1 Descriptions of the systems S and M

For the microscopic system S, we will consider the same qubit (or an atom) system example given in section 2.1. The Hilbert space of the system is $\mathbb{C}^2$ and the Hamiltonian is $H = \hbar\omega\sigma_x$ and has two possible energy states. Given the initial state of the system $|\psi(0)\rangle$ the time-evolved state under H is given by (2.7)

$$|\psi(t)\rangle = e^{-i\omega t}c_+(0)\,|+\rangle + e^{i\omega t}c_-(0)\,|-\rangle$$

with $c_+ = \langle\psi(0)|+\rangle$ and $c_- = \langle\psi(0)|-\rangle$

We have a macroscopic measurement device M, which measures the energy of the system S. This device has a pointer of mass M_0 ($\sim 10^{24}$; macroscopically large). We are going to treat this device quantum mechanically; by describing it using a wavefunction. Therefore, the Hilbert space for this system is $\mathscr{H}_M =$ a set of continuously differentiable and square integrable complex functions, which contains functions $\phi(x,t)$, where x denotes position of the pointer. We will treat the pointer as a free particle with a Gaussian wavefunction. At $t = 0$ this wavefunction is give by

$$\phi(x,0) = Ae^{-x^2/2\sigma^2} \tag{3.1}$$

where A is the normalization constant, σ is the standard deviation which is a small quantity here

so that the $\phi(x,0)$ is sharply peaked at $x=0$, the initial position of the pointer.

The Hamiltonian for the pointer is

$$H_{\text{device}} = -\frac{\hbar^2}{2M_0}\frac{\partial^2}{\partial x^2} \tag{3.2}$$

The combined Hilbert space for the $S+M$ system is $\mathbb{C}^2 \otimes \mathscr{H}_M$. Initial wavefunction of the total system is then given by[1]

$$|\Psi(0)\rangle = |\psi(0)\rangle\,|\phi(x,0)\rangle \tag{3.3}$$

This wavefunction will evolve under the total Hamiltonian given by

$$\hat{H}_{\text{total}} = \hat{H} + \hat{H}_{\text{device}} + \hat{H}_{\text{interaction}} \tag{3.4}$$

We are treating measurement as an interaction between two systems M and S and therefore we need to include the interaction Hamiltonian which couples the two systems. Otherwise, both systems will evolve independently. The choice of the interaction Hamiltonian depends on the characteristic property of the measurement device—the pointer must move in accordance with the eigenvalues of the observable, or else our device is of no use. To 'generate' displacements in the position of the pointer, we take $H_{\text{interaction}}$ proportional to the momentum operator $\hat{p}_x$ (generator of position). These displacements are not arbitrary but we want them proportional to the eigenvalues of the observable. So, $H_{\text{interaction}}$ can be taken to be proportional to the desired observable. Therefore, we consider the following

$$\hat{H}_{\text{interaction}} = \gamma \hat{A}\hat{p}_x \tag{3.5}$$

here γ is the strength of the interaction. In our case, we want to measure the energy of the particle, so $\hat{A} = \hat{H}$. The interaction Hamiltonian becomes

$$\hat{H}_{\text{interaction}} = -i\hbar\gamma\hat{H}\frac{\partial}{\partial x} \tag{3.6}$$

The dynamical equation for the evolution of the combined wavefunction $|\Psi(t)\rangle$ is the Schrödinger equation

$$i\hbar\frac{\partial\,|\Psi(t)\rangle}{\partial t} = \left[\hat{H} + \hat{H}_{\text{device}} + \hat{H}_{\text{interaction}}\right]|\Psi(t)\rangle \tag{3.7}$$

[1]Here we are abusing the notation. Instead of denoting $f(x) = \langle x|f\rangle$, we are directly writing the functional form in braket notation $|f(x)\rangle$. This is just to reflect the fact that the functions $f(x)$ are 'vectors' (and therefore, states) in the Hilbert space $\mathscr{H}_M$.

Let us consider the initial state $|+\rangle|\phi(0)\rangle$. The qubit is in the eigenstate of the observable H. Solving the above equation we get the

$$|\Psi(t)\rangle = e^{-i\omega t}|+\rangle\left|A(t)e^{-(x-\gamma\hbar\omega t)^2/2\sigma(t)^2}\right\rangle \tag{3.8}$$

the normalization factor A becomes $A/\sqrt{k}$ and the spread σ slightly increases to become $k\sigma$ where $k = 1 + \frac{i\hbar t}{M_0\sigma}$. Similarly, one can write the evolved state when the initial state is $|-\rangle|\phi(0)\rangle$.

This is what was needed. After some interaction-time t, the position of the pointer has moved to the right or left depending on the eigenvalue $\pm\hbar\omega$ and thus measuring it. So it when the initial state of the particle is the eigenstate of the observable, it is possible to describe the measurement process using the usual Schrödinger dynamics and not using any additional collapse postulate. But then, what is the problem?

Now suppose the initial wavefunction of the particle is a *superposition* of the eigenstates $|\pm\rangle$ and the pointer has the same wavefunction as in the previous case then the combined initial wavefunction is

$$|\Psi(0)\rangle = (c_1|+\rangle + c_2|-\rangle)|\phi(x,0)\rangle \tag{3.9}$$

The Schrödinger equation (3.7) is *linear* and so the final solution will be the superposition of the solutions obtained by solving for individual terms. We have already solved the problem for individual terms, now we just have to sum them up. So we have,

$$|\Psi(t)\rangle = e^{-i\omega t}c_1(t)|+\rangle\left|A(t)e^{-(x-\gamma\hbar\omega t)^2/2\sigma(t)^2}\right\rangle + e^{i\omega t}c_2(t)|-\rangle\left|A(t)e^{-(x+\gamma\hbar\omega t)^2/2\sigma(t)^2}\right\rangle \tag{3.10}$$

This equation is of the form

$$|\Psi(t)\rangle = c_1(t)\left|\begin{matrix}\text{qubit in}\\ \text{state } +\end{matrix}\right\rangle\left|\begin{matrix}\text{pointer at}\\ \text{poisition } +\hbar\omega\end{matrix}\right\rangle + c_2(t)\left|\begin{matrix}\text{qubit in}\\ \text{state } -\end{matrix}\right\rangle\left|\begin{matrix}\text{pointer at}\\ \text{poisition } -\hbar\omega\end{matrix}\right\rangle \tag{3.11}$$

which is an entangled superposition. The density matrix $\rho = |\Psi\rangle\langle\Psi|$ of such a state has cross or interference terms.

This is where the problem lies. The above equation simply states that when quantum system S is in a superposition, the macroscopic pointer M is also in a superposition pointing at all possible measurement outcomes at same time. This is contrary to our observation that in any single run of such an experiment we only get one *definite* outcome; we either see the cat dead or alive but not both!

3.2 The meaning of it all?

What we did was the following: we discarded the collapse postulate and said that 'everything is quantum' and so there are no physical processes which don't obey the Schrödinger equation. So we treated the measurement as no other 'special' process (which needs additional structure of collapse in usual QM) but using the Schrödinger equation. We discussed a very specific modeling of the measurement device, however, other models lead to the same conclusions. For brief review see[10].

Treating the measurement process as a quantum mechanical interaction—doesn't matter which model you use—leads to a peculiar feature: the combined state of the system and device is an entangled superposition state of the form (3.11). The measurement outcomes are not definite and it predicts interference of macroscopic objects. This is a reflection of the fact that the Schrödinger equation is linear and unitary. In the standard formalism of quantum mechanics, a pure state (like equation (3.11)) corresponds to the an ensemble whose members are identically prepared and are indistinguishable. However, if we actually did perform the measurement in the laboratory, we do get single and unique outcome in every run of the experiment contradicting the indistinguishability of the ensemble. Yes, the frequencies are correctly predicted by the coefficients when the experiment is run many times, though it is not very clear what happens in an individual run. Logically then, it is then meaningless to talk about probabilities/frequencies of different outcomes given that theory does not predict any of the individual outcomes. In the words of Wigner[11], "The process of measurements cannot be described by the equations of quantum mechanics because their existence is in contradiction with its principles".

One can conclude the following from the above discussion

1. That we need to discard the hypothesis that 'everything is quantum' and there exists special processes which don't obey Schrödinger equation but need an entirely different treatment.
   OR

2. That the measurement process does indeed obey SE and that everything is indeed quantum but we need to *interpret* correctly the meaning of equation(3.11); what does smearing of pointer really mean?

Most of the interpretations of quantum mechanics[12] revolve around the above two points; either they try to interpret what does a cat both dead and alive mean or they are finding different methods to explain such special processes of measurement. In the next chapter, we shall see an interpretation which modifies the Schrödinger evolution by adding certain correction terms.

Chapter 4

Spontaneous Collapse Models

4.1 What qualifies as a collapse model?

Collapse model[13] is a class of dynamical models satisfying certain assumptions which not only solve the measurement problem but also have a clear ontology of what is actually going on. These models are phenomenological and unify the Schrödinger evolution and wavefunction collapse dynamics. As we saw in the previous chapter that linearity and unitarity of the Schrödinger equation are responsible for the measurement problem. We will then need to drop one or both of these assumptions to solve the problem.

To make a working collapse model we then need to have, in contrast with the standard formulation (section 2.1), the following three characteristics in the dynamics

1. Non-linearity
   We need to suppress the macroscopic linear superpositions. To achieve this we need the dynamics to be non-linear. This also makes the dynamics non-unitary.

2. Stochasticity
   To explain the Born rule (2.4) and the emergence of probabilities, we need stochasticity. Moreover, this is a very strong requirement in order to avoid superluminal signalling. We discuss the reasoning for this below.

3. Amplification mechanism
   We want to suppress macroscopic superpositions but we don't want to do that at microscopic level because that would contradict already established empirical evidences of quantum theory at the microscopic scale. Therefore, we need to have some mechanism which ensures

that in the microscopic limit collapse models agree with Schrödinger dynamics but also suppresses superpositions in the macroscopic limit to agree with classical Newtonian dynamics (See Fig. 4.1)

The collapse models are therefore *stochastic*, *non-linear*, and *non-unitary*.

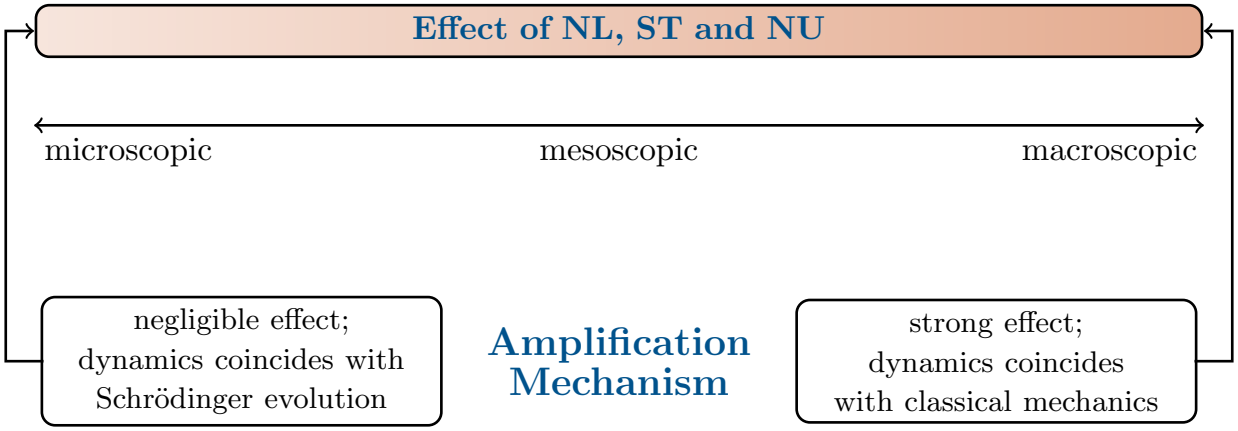

Figure 4.1: Amplification mechanism showing effect of Non-linear (NL), Stochastic (ST) and Non-unitary (NU) terms. The strength of these terms decreases in the microscopic limit and increases in the macroscopic limit.

Before we move on to discuss the specific models, we will comment on two technicalities. The first concerns the reduction of the off-diagonal terms in the density matrix and the second concerns stochasticity as a requirement for no faster-than-light communication[14]. We need to understand the first to appreciate the second.

Consider a homogenous ensemble of N macroscopic systems in a state which is a superposition of two orthogonal states $|0\rangle$ and $|1\rangle$ (similar to equation (3.11)). Which can be written in two formats—state vector and density matrix

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{4.1}$$

With regards to the three conditions mentioned above, we want to suppress the macroscopic superpostions. If we aim to evolve this density matrix ρ to another matrix ρ_{red}, by allowing the off-diagonal terms to decay we get

$$\rho_{\text{red}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.2}$$

but since the mapping of density matrix to state vectors is not one-to-one, the matrix ρ_{red} could correspond to the *superposed* state

$$N/2 \text{ systems in state} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad N/2 \text{ systems in state} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{4.3}$$

instead of the required non-superposed state with the same density matrix ρ_{red}

$$N/2 \text{ systems in state} |0\rangle \quad \text{and} \quad N/2 \text{ systems in state} |1\rangle \tag{4.4}$$

The modifications as described above should be done in the Schrödinger equation (2.1) itself and *not* in the density matrix von-Neumann equation (2.2) to ensure we end up in the required non-superposed state.

Gisin and Polchinski independently showed to prevent superluminal signaling under some evolution $\rho(t) = \mathscr{E}[\rho(0)]$, any two equivalent density matrices must remain equivalent under the evolution $\mathscr{E}$[15, 16]. This can be understood intuitively (for details see[17]) by considering a Bell type Alice-Bob scenario. Consider two arbitrary but equivalent ensembles $\rho = \sum_i |\psi_i\rangle \langle\psi_i| = \sum_i |\phi_i\rangle \langle\phi_i|$. Alice and Bob are arbitrary far from each other and share the ensemble of states

$$|\text{shared}\rangle = \sum_i |\psi_i\rangle \otimes |v_i\rangle = \sum_i |\phi_i\rangle \otimes |w_i\rangle \tag{4.5}$$

where $|v_i\rangle$ and $|w_i\rangle$ are mutually orthogonal basis for Bob's Hilbert space (such a shared state can be prepared for any choice of arbitrary but equivalent ensembles $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$ (see Appendix B in [18] for the proof). Suppose we have an evolution of a kind that initially $\{|\psi_i(0)\rangle\} \equiv \{|\phi_i(0)\rangle\}$ (equivalent) but after some time $\{|\psi_i(t)\rangle\} \not\equiv \{|\phi_i(t)\rangle\}$ (not equivalent). Now if Bob chooses to measure the shared state in the basis $|v_i\rangle$, due to collapse Alice gets ensemble $|\psi_i\rangle$. If Bob chooses measure the shared state in the basis $|w_i\rangle$, Alice gets ensemble $|\phi_i\rangle$. Then according to the assumption, ensembles $|\psi_i\rangle$ and $|\phi_i\rangle$ which were previously equivalent would not remain so after some time. This would allow Alice to distinguish between the ensembles and thereby know the basis used by Bob (who is at arbitrary distance from Alice) for measurement, causing FTL communication. Therefore, in order to avoid superluminal signaling given some evolution, equivalent mixtures must remain equivalent under that evolution.

Now, for the case of collapse models we need the evolution to be non-linear. Consider a deterministic and non-linear evolution $\mathscr{E}$

$$|\psi(t)\rangle = \mathscr{E}[|\psi(0)\rangle] \tag{4.6}$$

which maps pure states to pure states. Consider similar to above two equivalent mixtures $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$ at $t = 0$. If the time evolved mixtures are equivalent for all times $t > 0$, then according to Davies' theorem[19] the evolution must be linear and unitary. Which contradicts our initial assumption that the evolution is non-linear. Therefore, the evolved mixtures must be inequivalent. Hence, a deterministic and non-linear evolution leads to superluminal signaling. Since we demand non-linear evolution in collapse models, owing to this result of Gisin, we must also have the evolution to be stochastic to avoid superluminal signaling.

4.2 Uniqueness of the collapse dynamics

In the previous section we saw that for the model to be a collapse model, we need to add non-linear and stochastic terms to the usual dynamics. However, the constraints discussed, along with norm preservation, allow only a unique form of the equation[20]. That is, given a process

$$d\psi_t = A(\psi_t)dt + \sum_{k=1}^{N} B_k(\psi_t)dW_{k,t} \tag{4.7}$$

what form of A and B, which are general non-linear operators, would generate a Lindblad type dynamics for $\rho_t = \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$?

Theorem: The above process for ψ_t leads to Lindblad type dynamics if and only if $N \geq n$ and the operators are of the form

$$A(\psi_t) = iH_0 - \frac{1}{2}\sum_{k=1}^{N}\left(L_k^\dagger L_k - 2l_{k,t}L_k + |l_{k,t}|^2\right) \tag{4.8}$$

$$B_k(\psi_k) = L_k - l_{k,t} \tag{4.9}$$

where L_k's are linearly independent operators called collapse operators and $l_{k,t} = \frac{1}{2}\left\langle \psi_t \middle| L_k^\dagger + L_k \middle| \psi_t \right\rangle$

The Lindblad equation for the corresponding density operator $\rho_t = \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$ is given by

$$\frac{d\rho_t}{dt} = -i[H_0, \rho_t] + \sum_{k=1}^{n}\left(L_k\rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_t\}\right) \tag{4.10}$$

Note that N is the number of terms in the summation of equation (4.9) and n is the number of terms in the summation of equation (4.10). See [20] for the details of the proof. As a special case,

when the collapse operators L_k are hermitian, $L_k^\dagger = L_k$, the process (4.7) becomes

$$d\psi_t = \left[-iHdt - \frac{1}{2}\sum_{k=1}^{N}(L_k - l_{k,t})^2 dt + \sum_{k=1}^{N}(L_k - l_{k,t})dW_{k,t} \right] \psi_t \tag{4.11}$$

with $l_{k,t} = \langle L_k \rangle = \langle \psi_t | L_k | \psi_t \rangle$

4.3 Some collapse models

In the spontaneous collapse models, the wave function collapse occurs spontaneously, like a natural phenomenon, and the presence of an observer is not necessary. We first describe the Ghirardi-Rimini-Weber model in some detail which is the simplest of all and captures the essential ideas of collapse models. Then we quickly discuss the QMUPL, the CSL spontaneous collapse models and give an example of a two level system.

4.3.1 GRW model

In the GRW model[21, 8], the wavefunction of a single quantum particle evolves according to Schrödinger equation and then spontaneously localizes at some random position. This can be modeled as a Poisson process. Suppose X is the random variable where $X =$ Number of localizations. Then $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ where $\lambda \equiv \lambda_{\text{GRW}} =$ average number of localizations per unit time. $\tau = 1/\lambda =$ average waiting time. This value of λ_{GRW} is postulated to be about $10^{-16}s^{-1}$. So the average waiting time is 3×10^8 years. Therefore for a single particle, spontaneous localizations are rare (about once in 300 million years). The wavefunction doesn't collapse during measurement as in the standard theory. No observer is involved in GRW theory. The collapse of the wavefunction is spontaneous; it is a natural phenomenon. And so λ, defined above, is a constant of nature (like $\hbar$) involved in this natural phenomenon. Position is fundamental in this theory; collapse occurs in position basis. This is because any measurement device displays the outcome using a pointer (or positional distribution of LEDs for that matter). So any other physical quantity is treated as contextual.

We define the following Gaussian function

$$L_r(x) = \frac{1}{(2\pi r_c^2)^{1/4}} e^{-(x-r)^2/4r_c^2} \tag{4.12}$$

with $\int |L_r(x)|^2 dx = 1$. The value of r_c is postulated to be around $10^{-7}m$ which is small compared

to macroscopic scale but large compared to atomic scales.

Let t_0 be the time of collapse, t_i be the time just before collapse and t_f be the time just after collapse. For the time between two collapses, the wavefunction evolves according to the Schrödinger equation. The wavefunction just after the collapse is given by

$$\psi(x,t_f) = \frac{L_r(x)\psi(x,t_i)}{\sqrt{\int |L_r(x)\psi(x,t_i)|^2 dx}} \tag{4.13}$$

where $N(r) = \sqrt{\int |L_r(x)\psi(x,t_i)|^2 dx}$ is the normalization. So $L_r(x)$ reshapes the wavefunction in such a way that it gets localized at the position r. This position r is randomly selected with probability distribution given by

$$P(r) = \int |L_r(x)\psi(x,t_i)|^2 dx \tag{4.14}$$

Given the postulated values of λ_{GRW} and r_c the amplification mechanism mentioned above can be derived along with Born rule and wavepacket reduction.

For treating multiple particles, we have two cases. First when the particles are not entangled and second when they are entangled. Consider two particles which are not entangled with a wavefunction given by

$$\psi(x_1,x_2,t_i) = \chi(x_1,t_i)\phi(x_2,t_i) \tag{4.15}$$

Which of the two particles suffers first localization is purely random in addition to when and where to localize. Let's suppose that particle 1 localizes first. Then the probability density that it localizes at point r is

$$P(r) = \int |L_r(x_1)\psi(x_1,x_2,t_i)|^2 dx_1 dx_2$$

Then the combined wavefunction just after collapse is given by

$$\psi(x_1,x_2,t_f) = \frac{L_r(x_2)\chi(x_1,t_i)}{\sqrt{P(r)}}\phi(x_2,t_i) \tag{4.16}$$

Thus, if the particles are initially not entangled, then after spontaneous collapse they remain so. The localization of one does not affect the other particle.

Now consider the case when two particles are entangled with the wavefunction given by

$$\psi(x_1,x_2,t_i) = \frac{1}{\sqrt{2}}\left(\delta(x_1-a)\delta(x_2+a) + \delta(x_1+a)\delta(x_2-a)\right) \tag{4.17}$$

Doing similar analysis we get the final wavefunction after the collapse as

$$\psi(x_1, x_2, t_f) = \delta(x_1 - a)\delta(x_2 + a) \tag{4.18}$$

which shows that if one of the entangled particle localizes spontaneously then the other particle has to forcibly also get localized. Localization of one affects other entangled particle.

Consequently, if there are $N(\sim 10^{24})$ entangled particles, all of them would spontaneously localize with the effective rate given by $N\lambda_{\text{GRW}}$ or with an average time $(1/N\lambda_{\text{GRW}})$ merely of the order of nano seconds. Therefore, macroscopic superpositions are suppressed very quickly.

4.3.2 The QMUPL model

The Quantum Mechanics with Universal Position Localization (QMUPL) model has same axioms has a GRW but here the dynamics is described by a stochastic differential equation.

$$d\psi_t = \left[-\frac{i}{\hbar} H dt - \frac{\lambda}{2}(q - \langle q \rangle_t)^2 dt + \sqrt{\lambda}(q - \langle q \rangle_t) dW_t \right] \psi_t \tag{4.19}$$

where q is the position operator and $\langle q \rangle_t = \langle \psi_t | q | \psi_t \rangle$. This model explicitly displays the modification terms to the Schrödinger equation. The first term on the RHS is the usual deterministic Schrödinger evolution, the second term is the deterministic nonlinear term and the third term is a stochastic noise term with dW_t as the standard Weiner process. Here λ is called collapse constant and determines the strength of stochastic and nonlinear terms. This collapse constant is proportional to the mass of the particle $\lambda = \frac{m}{m_0}\lambda_0$ with $\lambda_0 = 10^{-2} m^{-2} s^{-1}$ and m_0 is the reference mass usually taken to be mass of a nucleon. Drawback of this model is that it is valid only for distinguishable particles.

4.3.3 The CSL model

The GRW and the QMUPL models have discrete time evolution, the noise field depends only on time and not on space and are valid for only distinguishable particles. To overcome these difficulties we have a more advanced model called continuous spontaneous localization(CSL). In this model the time evolution is continuous, the noise term depends on space as well as time, and it includes identical particles as well. The dynamics is described by the following equation which

is quite similar to the QMUPL equation

$$d\psi_t = \left[-\frac{i}{\hbar} H dt - \frac{\Gamma}{2m_0^2} \int d\mathbf{x} [M(\mathbf{x}) - \langle M(\mathbf{x}) \rangle_t]^2 dt + \frac{\sqrt{\Gamma}}{m_0} \int d\mathbf{x} [M(\mathbf{x}) - \langle M(\mathbf{x}) \rangle_t] dW_t(\mathbf{x}) \right] \psi_t \tag{4.20}$$

Γ is a positive coupling constant which determines the strength of collapse process. $M(\mathbf{x})$ is called the smeared mass density operator given by

$$M(\mathbf{x}) = \sum_i m_i N_i(\mathbf{x}), \qquad N_i(\mathbf{x}) = \int d\mathbf{y} g(\mathbf{y} - \mathbf{x}) \psi_i^\dagger(\mathbf{y}) \psi_i(\mathbf{y}) \tag{4.21}$$

$\psi_i^\dagger(\mathbf{y})$, $\psi_i(\mathbf{y})$ are creation and annihilation operators of particle type i in space point $\mathbf{y}$. These operators allow for inclusion of identical particles. The smearing function given by

$$g(\mathbf{x}) = \frac{1}{(\sqrt{2\pi} r_c)^3} e^{-\mathbf{x}^2/2r_c^2} \tag{4.22}$$

As an example of the CSL model, we will consider our good-old two level system qubit system. We consider the same Hamiltonian (2.5) $H_0 = \hbar\omega\sigma_x$. We wish to measure σ_z observable of the system and so choose one collapse operator $L = \sigma_z$. Then the CSL equation for this system is

$$d|\psi_t\rangle = \left[-i\omega\sigma_x dt - \frac{\Gamma}{2}(\sigma_z - \langle\sigma_z\rangle_t)^2 dt + \sqrt{\Gamma}(\sigma_z - \langle\sigma_z\rangle_t) dW_t \right] |\psi_t\rangle \tag{4.23}$$

for the details of the analysis, see[22]. We have reproduced their results in Fig 4.2 below.

4.4 Experimental tests

Since spontaneous collapse occur according to the time and length scales determined by λ and r_c, these models predict data slightly differently than the standard quantum theory. This makes spontaneous collapse models experimentally testable. Numerous experiments are being carried out to observe deviations from quantum theory[23].

These experiments can be classified into two categories: interferometric and non-interferometric. Interferometric experiments are a direct test of the validity of the superposition principle. In these experiments, a spatial superposition is prepared and is allowed to interfere. Then interference pattern is measured. If the fringes are observed, the superposition principle is valid for that system, else it is not. These experiments are difficult to control as superpositions may break down due to environmental noise.

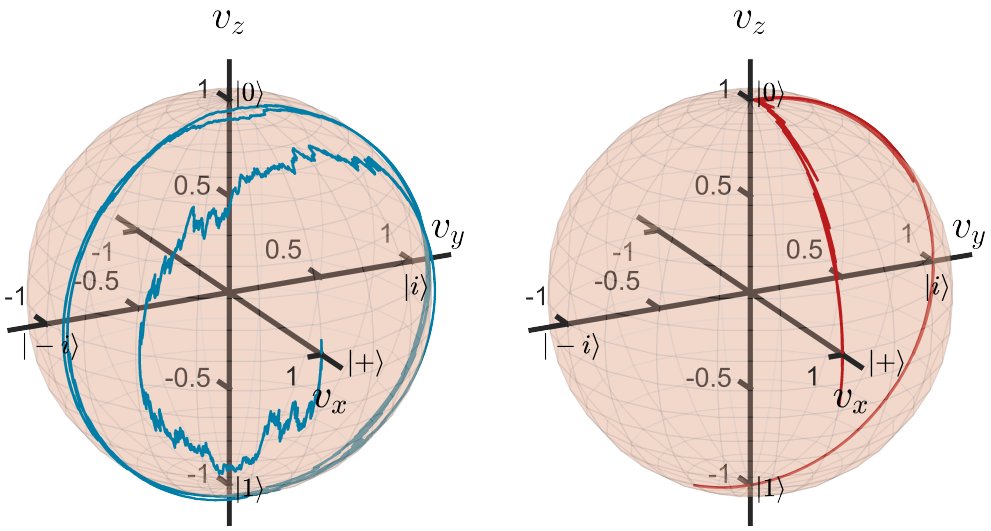

Figure 4.2: A two level system showing one realization of the CSL stochastic differential equation. On left side we have $\Gamma = 0.05s^{-1}$ which is small and takes some time for the system to reach the eigenstate. On the right we have $\Gamma = 10s^{-1}$, which drives the system to one of the eigenstates(here$|0\rangle$) fairly quickly. The figure on the right also illustrates the fact that in CSL models, the system does not stay in the eigenstate after collapse; it fluctuates around the state and jumps to the other state after some time. Results computationally reproduced from [22].

Non-interferometric experiments look for deviations from usual quantum theory due to the white noise term $W_t(\mathbf{x})$ in the collapse equation. These experiments do not depend on the preparation of a superposition state and therefore are more helpful. Suppose a particle evolves in some potential, then due to the noise term, the dynamics is stochastic. Due to the random acceleration of this particle, some effects such as violation of energy conservation is seen. Such signatures are being detected in non-interferometric experiments. See Fig. 4.3

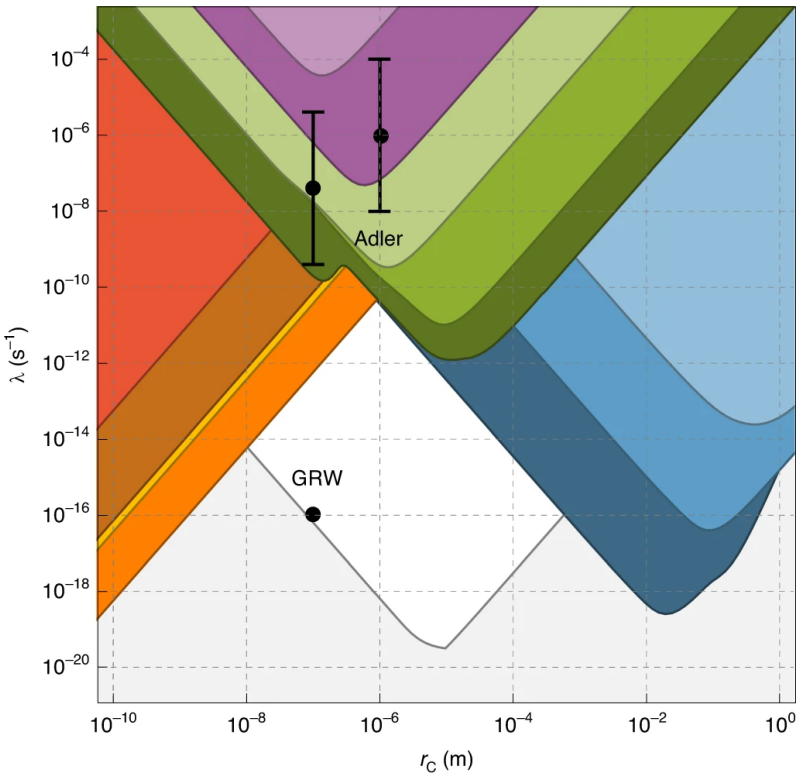

Figure 4.3: A phase diagram of the parameters λ and r_c (discussed in section 4.3.1). This figure shows values of these two parameters which are ruled out by various non-interferometric experiments. These include cantilever-based experiments, gravitational wave experiments from detectors AURIGA, LISA Pathfinder and LIGO, optomechanical systems, spontaneous X-ray emission tests, phonon excitation experiments, heating rate of Neptune and cold-atom experiments. Different colours indicate different experiments. The light grey region shows the values excluded from the theoretical requirement of the amplification mechanism (that macroscopic superpositions cannot exist for long time). White color shows the region which remains to be tested.[23] *Reproduced with permission from Springer Nature*

The spontaneous collapse models which were discussed are phenomenological models and the modification of the dynamics seems ad-hoc. What is the theoretical understanding of these models that could explain, for example, the origin of random noise? Are there any strong theoretical reasons for this stochastic modifications of quantum mechanics? In other words, could the collapse dynamics be derived from some 'first principles'?

Chapter 5

Trace Dynamics

Trace dynamics forms the basis of the 'first principles' that we commented on at the end of the previous chapter. Considering the history of physics, most theories eventually replaced by their higher order correct theories. Quantum theory might be no exception. Moreover, the ad-hoc quantization procedure, the measurement problem, problems in unification of quantum with gravity, the cosmological constant problem only add to the motivation and to the belief that quantum theory might be a very good approximation to a deeper theory. The recent important result[24] makes this belief even stronger. In this chapter we attempt to summarize the spirit of trace dynamics with the help of an example of harmonic oscillator. Complete details can be seen in Adler's book[25] which unfortunately is the only reference and demands a lot of blood, sweat and tears. For a brief review see[26].

In classical Lagrangian and Hamiltonian dynamics, the dynamics is on a phase space with the phase space coordinates being generalized coordinates q_i and their conjugate momenta p_i. These phase space coordinates commute. Trace dynamics formalism is analogous to classical Lagrangian or Hamiltonian dynamics with a fundamental difference: They phase space coordinates, the degrees of freedom, are matrices and they are noncommutative in general. It is assumed that these commutation relations of these degrees of freedom are arbitrary; that is they are not necessarily the usual commutation relations ($[q,p] = i\hbar$). The usual commutation relations in fact emerge from the statistical mechanics of this matrix dynamics. The trace dynamics formalism can be understood in a sequence of three steps. Step 1 Classical Dynamics, Step 2 Statistical Thermodynamics, Step 3 Statistical Fluctuations.

5.1 Degrees of freedom and concept of trace derivative

The fundamental degrees of freedom are matrices with Grassmann numbers as matrix elements. These matrices do not commute in general. If θ_i and θ_j are two Grassmann numbers then they satisfy $\theta_i\theta_j + \theta_j\theta_i = 0$ and $\theta_i^2 = \theta_j^2 = 0$. This property divides the degrees of freedom into two disjoint sectors of Bosonic sector containing even grade elements and Fermionic sector containing odd grade elements.

Consider a harmonic oscillator with the generalised coordinate q and momentum p. In trace dynamics, these phase space variables are matrices. The commutation relation between these variables is not specified and is taken to be arbitrary. However, the eigenvalues of these matrices are the values of the classical dynamical variables obtained through extremization of the action. To develop a action principle with matrix degrees of freedom, we need the concept of variation of a polynomial with respect to a matrix called trace derivative.

Trace derivative Consider a polynomial P which is made out of non commuting matrix variables. The trace of the matrix P is a complex number. Then derivative of this complex number $\mathbf{P} \equiv \mathrm{Tr}P$ with respect to matrix variable O can be defined by varying and then cyclically permuting such that in each term the factor δO is on the right.

$$\delta\mathbf{P} = \mathrm{Tr}\frac{\delta\mathbf{P}}{\delta O}\delta O \tag{5.1}$$

Note that we will take O as bosonic or fermionic but not mixed and we construct P such that it is always an even grade.

5.2 Hamiltonian dynamics of matrix variables

Using trace derivative we construct a Lagrangian and Hamiltonian dynamics of Grassmann valued matrices. The equations for the action, the Lagrangian and the Hamiltonian look similar to the ones in classical mechanics.

$$\mathrm{Tr}S = \int d\tau\ \mathrm{Tr}L \quad ; \quad \frac{\delta \mathrm{Tr}L}{\delta q_r} - \frac{d}{d\tau}\frac{\delta \mathrm{Tr}L}{\dot{q}_r} = 0 \tag{5.2}$$

$$\mathrm{Tr}H = \mathrm{Tr}\left[\sum_r p_r\dot{q}_r - L\right] \quad ; \quad \frac{\delta \mathrm{Tr}H}{\delta q_r} = -\dot{p}_r \quad \frac{\delta \mathrm{Tr}H}{\delta p_r} = \varepsilon_r\dot{q}_r \tag{5.3}$$

where $\varepsilon_r = \pm 1$ depending on whether q_r and p_r are bosonic or fermionic.

The Hamiltonian in the theory have a global unitary invariance ($q_i \to U^\dagger q_i U$, $p_i \to U^\dagger p_i U$) and so there exist a unique non trivial conserved charge called the Adler-Millard charge.

$$\tilde{C} = \sum_{i\in B}[q_i, p_i] - \sum_{i\in F}\{q_i, p_i\} \tag{5.4}$$

the commutator sum is over bosonic(B) and anticommutator sum is over fermionic(F) degrees of freedom. $\tilde{C}$ plays a vital role in the emergence of the usual commutation relations.

In our example of harmonic oscillator, the Hamiltonian, Hamilton's equations and the Adler-Millard charge are

$$\mathrm{Tr}H = \mathrm{Tr}\left[\frac{p^2}{2m} + \frac{1}{2}kq^2\right] \quad ; \quad \frac{\delta \mathrm{Tr}H}{\delta q} = kq = -\dot{p} \quad \frac{\delta \mathrm{Tr}H}{\delta p} = \frac{p}{m} = \dot{q} \quad ; \quad \tilde{C} = [q, p] \tag{5.5}$$

here we assume that q and p are bosonic, therefore $\varepsilon = 1$ and the anticommutator is zero.

We will now do a change of variables by transforming q and p into their self-adjoint and anti-self-adjoint parts $q = q^s + q^{as}$ and $p = p^s + p^{as}$. This is needed so that we can derive the Schrödinger equation. The self-adjoint and anti-self-adjoint components of the trace Hamiltonian are

$$\mathrm{Tr}H^s = \mathrm{Tr}\left[\frac{(p^s)^2}{2m} + \frac{1}{2}k(q^s)^2 + \frac{(p^{as})^2}{2m} + \frac{1}{2}k(q^{as})^2\right] \tag{5.6}$$

$$\mathrm{Tr}H^{as} = \mathrm{Tr}\left[\frac{2p^s p^{as}}{2m} + \frac{2kq^s q^{as}}{2}\right] \tag{5.7}$$

The self-adjoint and anti-self-adjoint components of the trace Adler-Millard charge are

$$\tilde{C}^{as} = [q^s, p^s] + [q^{as}, p^{as}] \quad ; \quad \tilde{C}^s = [q^s, p^{as}] + [q^{as}, p^s] \tag{5.8}$$

An important point to note that the Hamilton's equations in trace dynamics is in general non-unitary. The Hamilton's equations for the harmonic oscillator example cannot be written in the form $i[G, q]$ or $i[G, p]$ for any generator G. Therefore the dynamics is non-unitary.

5.3 Statistical Thermodynamics

We then formulate a statistical mechanics of these degrees of freedom by constructing a canonical ensemble with constraints determined by the conserved quantities such as the trace Hamiltonian, Adler-Millard charge and a quantity called fermion number. We coarse-grain the previous dynamics over many Plank times and quantum theory emerges in the thermodynamic equilibrium. This

can be done only if the TrH^{as} and $\tilde{C}^s$ are negligible. Then Adler-Millard charge is equipartitioned and since $\tilde{C}$ is anti-self-adjoint, its canonical average can be written in the form

$$\langle \tilde{C} \rangle_{AV} = i_{eff}\hbar \tag{5.9}$$

where $i_{eff} = i\,\mathrm{diag}(1,-1,1,-1,\ldots,1,-1)$. These commutation relations are obtained for statistically averaged dynamical variables. Heisenberg equations of motion and the Schrodinger equation can also be derived

$$[q_{r\,eff}, p_{r\,eff}] = i_{eff}\hbar \qquad r \in B \tag{5.10}$$

$$\{q_{r\,eff}, p_{r\,eff}\} = i_{eff}\hbar \qquad r \in F \tag{5.11}$$

$$\hbar \dot{x}_{r\,eff} = i_{eff}[H_{eff}, x_{r\,eff}] \tag{5.12}$$

where eff denotes the coarse-grained effective variables.

In our example, we neglect TrH^{as} and $\tilde{C}^s$. The Adler-Millard charge is equipartitioned over the two degrees of freedom with the equipartitioned value equal to $i_{eff}\hbar$. This gives the commutation relations $[q^s, p^s] = i_{eff}\hbar$ and $[q^{as}, p^{as}] = i_{eff}\hbar$. These effective variables obey Heisenberg equations of motions. Using the commutation relations, we can now write in the position representation and thus obtain the Schrödinger equation.

Statistical Fluctuations In trace dynamics, the anti-self-adjoint component of the trace Hamiltonian becomes significant due to entanglement of many degrees of freedom[27]. Due to this, superposition states decay rapidly leading to spontaneous collapse. Since the anti-self-adjoint component of the trace Hamiltonian has become significant, the dynamics is now again non-unitary and deterministic. Taking motivation from this fact, we have modelled a deterministic and non-unitary dynamics to see its effect on superposition of states. This should ideally give the collapse models after coarse-graining,

Chapter 6

Model of A Two-level System

6.1 Description of the model

We aim to understand the deterministic, non-unitary and norm preserving evolution of wavefuntion. The norm preservation is neither an ad hoc imposition in our model nor the result of the "probabilistic interpretation" of quantum theory. Rather, the norm preservation is asserted from the octonion thoery developed by Prof. TP Singh. Accordingly, we model our system by considering a general Hamiltonian of the form

$$H = H_0 + i\gamma A \tag{6.1}$$

where $H_0^\dagger = H_0$, $A^\dagger = A$, $i = \sqrt{-1}$ and γ is the *coupling constant*, a real free parameter which couples H_0 and A. Then, the total Hamiltonian H is not self-adjoint. Such Hamiltonians would give rise to a non-unitary evolution. The physical meanings of H_0, A and γ are discussed in the later sections. The anti-selfadjoint part (iA) of the total Hamiltonian is of particular interest in the context of trace dynamics/octonionic theory. The non-unitary evolution does not preserve norm of the state vector $|\phi\rangle$, in general. Therefore, we impose norm preservation by hand. However, note that even though the norm preservation appears an ad-hoc treatment, it can be reasoned using octonionic theory as mentioned earlier.

Let $|\phi\rangle$ be a state which follows the evolution given by (1) without any imposition of the norm preservation. Then defining other state $|\psi\rangle = \frac{|\phi\rangle}{\sqrt{\langle\phi||\phi\rangle}}$ and imposing norm preservation, we get a

modified Schrödinger equation

$$\frac{d|\psi\rangle}{dt} = [-iH_0 + \gamma(A - \langle A\rangle)]|\psi\rangle \tag{6.2}$$

where $\langle A\rangle = \langle\psi|A|\psi\rangle$. The first term is the usual Schrödinger evolution. The second term gives the modified dynamics which is non-unitary. The peculiar form of the second term $A - \langle A\rangle$ is essential for norm preservation; proof of which is straightforward. In the density matrix formulation the evolution equation becomes

$$\frac{d\rho}{dt} = -i[H_0, \rho] + \gamma\{A, \rho\} - 2\gamma \mathrm{Tr}(\rho A)\rho \tag{6.3}$$

where, $\rho = |\psi\rangle\langle\psi|$ and, $[.,.]$ and $\{.,.\}$ have the usual meaning of commutator and anticommutator.

For a two-level qubit system, H_0, ρ, A are 2×2 self-adjoint matrices. We choose to write these matrices in the eigenbasis of A. Without loss of generality, λ_0, $\lambda_1 \in \mathbb{R}$ are eigenvalues of A such that $\lambda_0 > \lambda_1$. Note that we exclude the case of degeneracy ($\lambda_0 = \lambda_1$), the reasoning for which will be clear when we discuss the physical meaning of A. The matrices in the mentioned basis are then given as

$$H_0 = \begin{pmatrix} a_0 & b_{0r} + ib_{0i} \\ b_{0r} - ib_{0i} & d_0 \end{pmatrix} \qquad \rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \qquad A = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \tag{6.4}$$

where x, y, z are three components of the Bloch vector $\vec{v} = (x, y, z)$ and a_0, b_{0r}, b_{0i}, d_0 are real numbers. Using equation (6.3) for the matrices in equation (6.4) we get the evolution equation for the three components of the Bloch vector.

$$\dot{x} = -(a_0 - d_0)y - 2b_{0i}z - \gamma(\lambda_0 - \lambda_1)xz \tag{6.5}$$

$$\dot{y} = (a_0 - d_0)x - 2b_{0r}z - \gamma(\lambda_0 - \lambda_1)yz \tag{6.6}$$

$$\dot{z} = 2b_{0i}x + 2b_{0r}y - \gamma(\lambda_0 - \lambda_1)(z^2 - 1) \tag{6.7}$$

6.2 Results

These results provide an answer to the question we asked at the beginning: *How does the state of the system evolve under a deterministic, non unitary, and norm-preserving evolution?* For strong

coupling between H_0 and A, the system tends to the eigenstates of A. When $\gamma \gg 1$ or $\gamma \ll -1$ the differential equations (6.5), (6.6), (6.7) become

$$\begin{aligned}
\dot{x} &= -\gamma(\lambda_0 - \lambda_1)xz \\
\dot{y} &= -\gamma(\lambda_0 - \lambda_1)yz \\
\dot{z} &= -\gamma(\lambda_0 - \lambda_1)(z^2 - 1)
\end{aligned}$$

The solution for the z-component is

$$z = \frac{1 - \exp(-2\gamma(\lambda_0 - \lambda_1)t)}{1 + \exp(-2\gamma(\lambda_0 - \lambda_1)t)} \tag{6.8}$$

For $\gamma \gg 1$, $z \to 1$ and $\dot{x} = -\gamma(\lambda_0 - \lambda_1)x$, $\dot{y} = -\gamma(\lambda_0 - \lambda_1)y$ which imply $x, y \to 0$. For $\gamma \ll -1$, $z \to -1$ and similarly $x, y \to 0$. Therefore for the case of strong coupling ($\gamma \gg 1$ or $\gamma \ll -1$) the system tends to the eigenstates $|0\rangle$ or $|1\rangle$ of A in finite time. The system is driven to to the states $|0\rangle$ or $|1\rangle$ depending on the sign of γ. For positive γ, the system is driven to to $|0\rangle$ whereas for negative γ it is driven to $|1\rangle$. This result is valid irrespective of the initial state if the system. Numerical evaluation performed (See Fig. 6.1) for a special case using Runge-Kutta method are shown in the Bloch sphere representation figure. The trajectories show the evolution of the state with time for different values of γ.

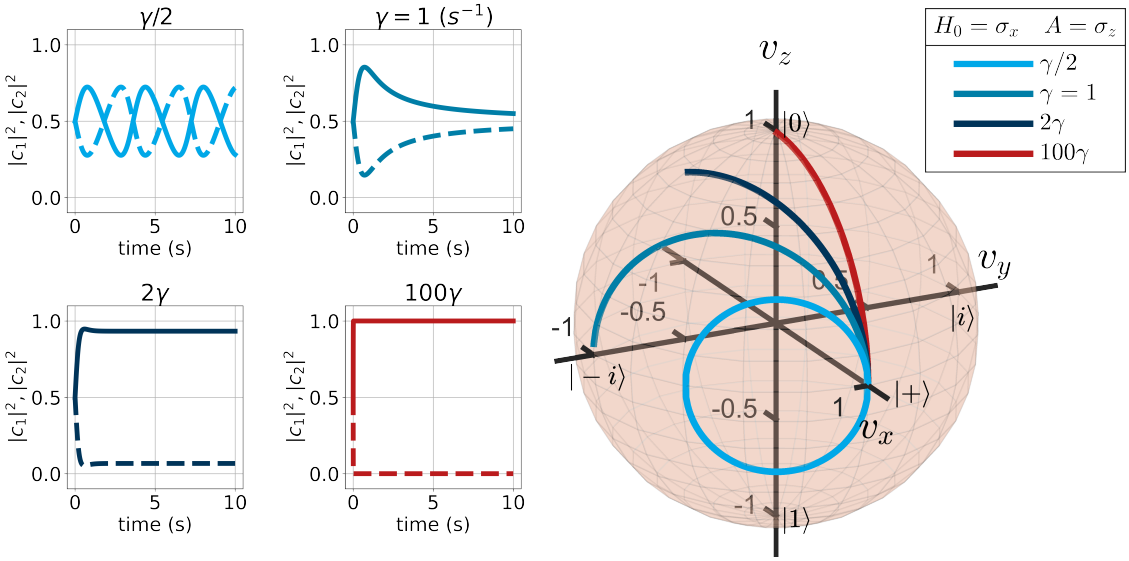

Figure 6.1: The figure shows numerical evolution of the state vector under the Hamiltonian $H = \sigma_x + i\gamma\sigma_z$. Different values of γ were considered. For large values of γ it is seen that the state vector converges to the eigenstate of σ_z (here $|0\rangle$). When gamma is negative, the vector will converge to the other eigenstate $|1\rangle$. The plots on the left show the values of $|c_1|^2$(solid) and $|c_2|^2$(dashed) which in the colloquial sense are the probabilities of finding the state in $|0\rangle$ and $|1\rangle$ respectively.

6.3 Interpretation and Discussion

This project is tiny part of a large research program carried out by Prof. T P Singh. This program seeks to unify quantum theory and gravity using a theoretical formulation of octonionic theory. For a recent review of the program see[28]. This formulation is based on trace dynamics in which quantum theory is emergent from more fundamental dynamics. To understand our results, we need this context of trace dynamics. We first discuss the concept of time in this formulation.

6.3.1 Connes Time

We treat quantum mechanics on classical spacetime. Doing this we assume that quantum system can coexist with classical spacetime. This is a problem for the following reason. In the Schrödinger equation, time is treated as a parameter. But this time is a part of the 4D spacetime manifold whose geometry depends on the classical bodies. These objects are fundamentally quantum and obey Schrödinger equation. But that is what we started with. We need to remove the classical spacetime manifold and treat time differently. In trace dynamics, there is natural time parameter τ, which is conjugate to the Adler-Millard charge. This time parameter is not the coordinate time (t)

in special relativity. This parameter τ called Connes time is a feature of Connes' non-commutative geometry[29]. This Connes time is measured in the units of Planck time τ_p. All the evolution considered in trace dynamics is with respect to this Connes time parameter.

6.3.2 Time of arrival

Naively, time of arrival is the time —in terms of Connes time—before which the system and environment start interacting (Refer to Fig. 6.2). In trace dynamics(TD), quantum mechanics is derived by averaging over smaller Planck time intervals and is a thermodynamic equilibrium of trace dynamics. That is, according to trace dynamics observations of quantum mechanics are over a coarse-grained time scale. When a quantum system arrives at a measuring apparatus the time of arrival is also observed over a coarse-grained scale. In quantum mechanics, therefore, this time of arrival is not important and we get a random distribution of outcome over many measurements. However in trace dynamics, which provides a more accurate description of the measurement process, time of arrival is important. Depending on the time of arrival, different measurement outcomes are possible. When quantum system interacts with a classical apparatus, the Hamiltonian of the combined system becomes of the form (6.1); the imaginary part becomes increases and becomes significant. Depending on the time of arrival, this macroscopic system at Planck scale resolution will be in different microstates (according to coarse-grained QM, since it is a thermodynamics equilibrium of TD, all these microstates are equivalent). Since within a Planck time, the measuring system is changing very rapidly due to large number of degrees of freedom, it is important to know the time of arrival up to Planck time to determine the further outcomes of the measurement. We therefore propose that γ is function of time of arrival which decides whether the imaginary part of the Hamiltonian is switched on with positive or negative γ.

6.3.3 Coarse-graining

We make an ansatz for coarse-graining the time of arrival, $\gamma \to \sqrt{\Gamma} W(t)$, where $W(t)$ is the Weiner process. Taking norm-preservation into consideration, we obtain the following map which give the correct CSL differential equation.

$$m(X) = \sqrt{\Gamma} X W(t) - \Gamma \left(X^2 - \langle X^2 \rangle \right) \tag{6.9}$$

where, $X = A - \langle A \rangle$ and Γ is the CSL parameter.

Because of the coarse-graining, the non-linear deterministic equation becomes stochastic and thus avoids superluminal signalling[15] in the emergent theory.

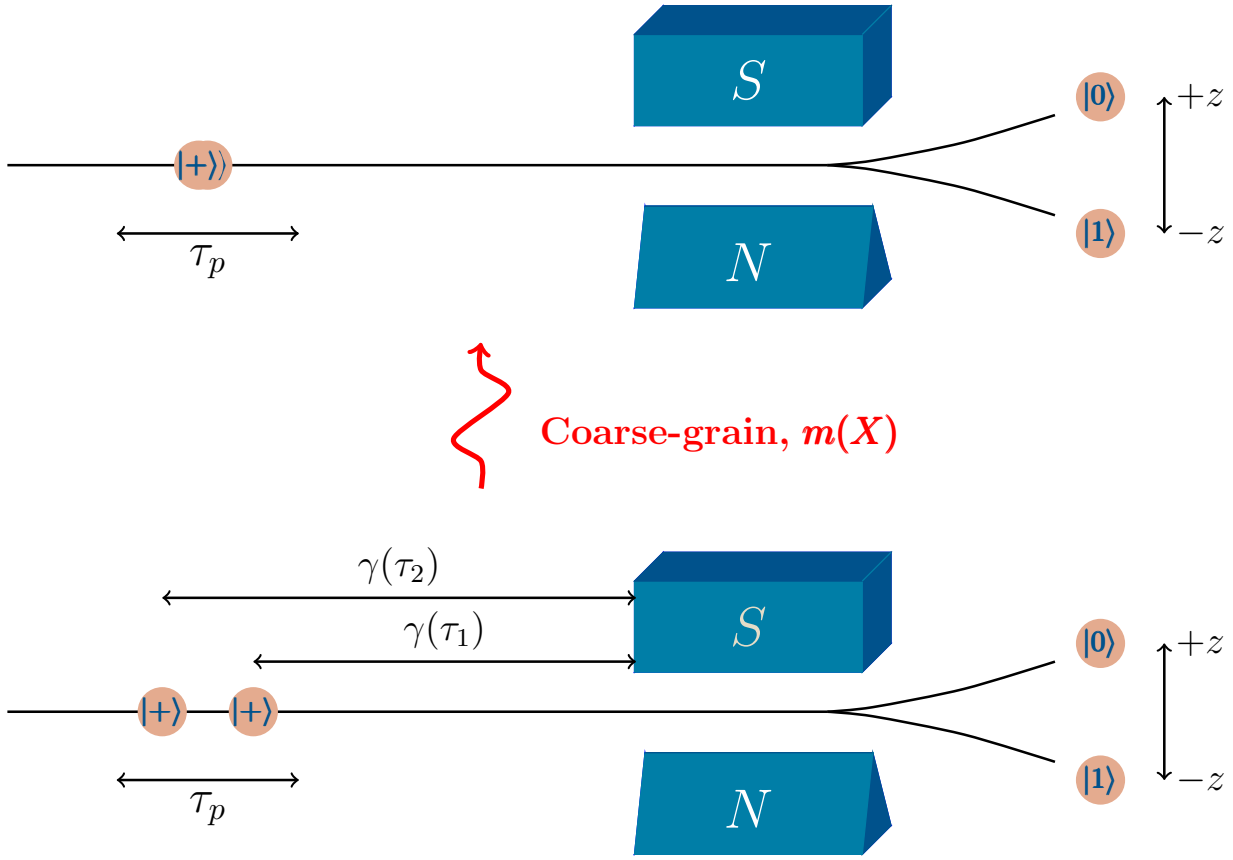

Figure 6.2: The figure shows a schematic Stern-Gerlach measurement at two different time scales. The figure above is the coarse-grained view which is obtained by averaging over many Plank time (τ_p) intervals. The figure on the bottom is a fine-grained view which is resolved more than Planck scale. Two qubits are prepared in an identical state $|+\rangle$. This is a macroscopic state according to trace dynamics. Therefore, it cannot be much resolved in the coarse-grained view and the outcomes appear random. However at higher resolution, the qubits are in a different trace dynamical microstates and hence have different times of arrival, τ_1 and τ_2 (in terms of Connes time). We propose that the coupling in (6.1) depends on these times of arrival and decide whether γ is positive or negative thereby giving different outcomes.

6.3.4 Conclusion and future prospects

We have developed a toy model for gaining insights into how trace dynamics could explain collapse models. In this model, the evolution is deterministic and non-unitary but norm-preserving. The Hamiltonian of this system is not Hermitian but has an anti-self-adjoint part A which is responsible for non-unitary evolution. We have coupled the original Hamiltonian H_0 to with a coupling γ. It

is observed that for large value of γ the state vector is always driven to the eigenvectors of A in a finite time. This forms the major part of our results. This result is straightforward to generalize to any Hermitian operator A with all unique eigenvalues. The model does not work when there is degeneracy in A's spectrum. This is because as seen in equation (6.5) if $\lambda_0 = \lambda_1$, the differential equations reduce to as if $A = 0$.

This toy model, as discussed previously, makes a tiny contribution to a larger research program on unification. It gives insights into how a measurement process might actually work using time of arrival and coarse-graining concept. Of course, the model is not perfect and the following improvements are required and are a part of future work.

1. The time of arrival concept is not precisely defined. We will need a precise definition of this time to make certain calculations possible.

2. On a philosophical note, since coarse-graining plays an important role here (it is not present in any interpretation of QM), it would be interesting to check its implications on the nature of observation and the observer.

3. Considering the implications of these results, this toy model would help in actual derivation of collapse models from the fundamental Lagrangian developed in[27].

4. The map (6.9) is just an ansatz, a rigorous derivation of coarse-graining needs to be done from first principles of trace dynamics. Upon attempting to do this, we realised that this is a difficult task and would need a significant time.

5. Doing the above derivation will also help in understanding the origin, and more importantly the spectrum, of the noise that drives collapse models.

6. This derivation of spectrum of noise would help in directing or steering the experiments carried out to test collapse models; the experimentalists would know what to look for.

Thus, this toy model is of importance to the future of collapse models and of quantum theory.

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