Determinants of Representations of Hyperoctahedral Groups

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Certificate

This is to certify that this dissertation entitled Determinants of Representations of Hyperoctahedral Groups towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Debarun Ghosh at Indian Institute of Science Education and Research under the supervision of Steven Spallone, Associate Professor, Department of Mathematics , during the academic year 2016-2017.

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This thesis is dedicated to Mom and Dad.

Declaration

I hereby declare that the matter embodied in the report entitled Determinants of Representations of Hyperoctahedral Groups are the results of the work carried out by me at the Department of Mathematics, Name of the Institute, under the supervision of Steven Spallone and the same has not been submitted elsewhere for any other degree.

Debarun Ghosh

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Abstract

The Weyl group of type B_n, which we denote by $\mathbb{B}_n$, has four linear representations $\omega : \mathbb{B}_n \to \{\pm 1\}$. In this paper we count, for a given n and ω, the number $N_\omega(n)$ of irreducible representations π of $\mathbb{B}_n$ satisfying $\det \circ \pi = \omega$. Let $n \geq 6$. If n odd, then $N_1(n) > N_{\mathrm{sgn}^0}(n) > N_{\mathrm{sgn}^1}(n) = N_\epsilon(n)$. If n even, we have $N_1(n) > N_{\mathrm{sgn}^1}(n) > N_{\mathrm{sgn}^0}(n) = N_\epsilon(n)$.

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Introduction

The n-th hyperoctahedral group $\mathbb{B}_n$ is the Weyl group of type B_n. It has four linear representations $\omega : \mathbb{B}_n \to \{\pm 1\}$. Let (π, V) be an irreducible representation of $\mathbb{B}_n$. Let $\det : \mathrm{GL}_{\mathbb{C}}(V) \to \mathbb{C}^*$ be the determinant map, then the composition $\det \circ \pi$ is a linear representation of $\mathbb{B}_n$. In this paper we count, for a given n and ω, the number $N_\omega(n)$ of irreducible representations π of $\mathbb{B}_n$ satisfying $\det \circ \pi = \omega$.

In the first chapter, we recall the theory of 2-core towers which will be extensively used in the later sections. We also study the paper by Arvind Ayyer, Amritanshu Prasad and Steven Spallone, in which they give an analogous result for the Symmetric group S_n. All the irreducible representations of $\mathbb{B}_n$ are constructed with the help of Mackey Theory in chapter 2.

We find formulas for determinants of $\mathbb{B}_n$ in chapter 3. The question then reduces to determining the parities of two quantities $x_{\alpha,\beta}$ and $y_{\alpha,\beta}$ [Theorem 3.1.4], which is solved using the 2-core tower theory of partitions.

We give explicit formulas for $N_\omega(n)$ in Chapter 4. We also prove the following.

Theorem 0.0.1. Let $n \geq 6$. If n odd, then $N_1(n) > N_{\mathrm{sgn}^0}(n) > N_{\mathrm{sgn}^1}(n) = N_\epsilon(n)$. If n even, we have $N_1(n) > N_{\mathrm{sgn}^1}(n) > N_{\mathrm{sgn}^0}(n) = N_\epsilon(n)$.

Chapter 1

The determinant problem for S_n

1.1 The 2-core tower of a partition

For a given partition λ, the 2-core $\mathrm{core}_2(\lambda)$ has no hooks of length divisible by 2. The p-quotient $\mathrm{quo}_2(\lambda)$ contains all the information about hooks of length divisible by 2 in λ. The 2-quotient is a 2-tuple (λ_0, λ_1) of partitions, where the total number of cells is the number of 2-hooks whose rims were removed from λ to obtain $\mathrm{core}_2(\lambda)$. Consequently,

$$| \lambda | = | \mathrm{core}_2 \lambda | + 2(| \lambda_0 | + | \lambda_1 |).$$

The size of the partition λ_k in the 2-quotient is the number of nodes in the Young diagram of λ whose hook-lengths are multiples of 2, and whose hand nodes have content congruent to k modulo 2 (by definition, the content of the node $(i; j)$ is $j - i$). The partition λ can be recovered uniquely from $\mathrm{core}_2(\lambda)$ and $\mathrm{quo}_2(\lambda)$.

For a given partition λ, its 2-core tower is defined as follows: it has rows numbered by integers $0, 1, 2, \ldots$. The ith row of this tower has 2^i many 2-cores. The 0th row has the partition $\alpha_\phi := \mathrm{core}_2(\lambda)$. The first row consists of the partitions α_0, α_1, where, if $\mathrm{quo}_2\lambda = (\lambda_0; \lambda_1)$, then $\alpha_i = \mathrm{core}_2\lambda_i$. Let $\mathrm{quo}_2\lambda_i = (\lambda_{i0}, \lambda_{i1})$, and define $\alpha_{ij} = \mathrm{core}_2\lambda_{ij}$. Inductively, having defined partitions λ_x for a binary sequence x, define the partitions λ_{x0} and λ_{x1} by

$$\mathrm{quo}_2\lambda_x = (\lambda_{x0}, \lambda_{x1}),$$

and let $\alpha_{x\epsilon} = \text{core}_2\lambda_{x\epsilon}$ for $\epsilon = 0, 1$. Let $w_i(\lambda)$ denote the sum of the sizes of the partitions in the ith row of the 2-core tower of λ. Hence,

$$n = \sum_{i=0}^{\infty} w_i(\lambda) 2^i.$$

More details can be found in Olsson's monograph [7].

1.2 Review of Arvind Ayyer, Amritanshu Prasad and Steven Spallone's Paper

We briefy recall the theorems and results in the case of symmetric groups. More details can be found in the paper [2].

For the symmetric group S_n, the partitions of n parametrize the irreducible complex representations of S_n. Let (ρ, V_λ) be a complex finite dimensional representation and $\det : \text{GL}_{\mathbb{C}}(V_\lambda) \to \mathbb{C}^*$ denote the determinant function. The composition $\det \circ \rho : S_n \to \mathbb{C}^*$ is a linear character of S_n. Hence, it is either the trivial character or the sign character. We call (ρ, V) a chiral representation, if $\det \circ \rho$ is the sign character of S_n. If the representation (ρ, V_λ) is chiral, we shall say the corresponding partition λ is a chiral partition. Suppose n is a positive integer with the following binary expansion:

$$n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, \epsilon \in \{0, 1\}, 0 < k_1 < k_2 < \cdots < k_r.$$

Let f_λ denote the dimension of V_λ, for a given partition λ. For any integer m, let $v_2(m)$ denote the largest among integers v such that $2v$ divides m.

Theorem 1.2.1. [2, Theorem 6] A partition λ of n is chiral if and only if one of the following holds:

1. The partition λ satisfies

$$w_i(\lambda) = \begin{cases} 1 & \text{if } i \in \{k_1, k_2, \ldots, k_r\}, \text{ or if } \epsilon = 1 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

   and the unique non-trivial partition in the k_1th row of the 2-core tower of μ is α_x, where the binary sequence x of length k_1 begins with ϵ. In this case f_λ is odd.

2. For some $0 < v < k_1$,

$$w_i(\lambda) = \begin{cases} 2 & \text{if } i = k_1 - v, \\ 1 & \text{if } k_1 - v + 1 \leq i \leq k_1 - 1 \text{ or } i \in \{k_1, k_2, \ldots, k_r\}, \text{ or if } \epsilon = 1 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the two non-trivial partitions in the $(k-v)$th row of the 2-core tower of μ are α_x and α_y, for binary sequences x and y such that x begins with 0 and y begins with 1. In this case $v_2(f_\lambda) = v$.

3. We have $\epsilon = 1$ and the partition λ satisfies

$$w_i(\lambda) = \begin{cases} 3 & \text{if } i = 0, \\ 1 & \text{if } i \in \{1, \ldots, k_1 - 1, k_2, \ldots, k_r\}, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $v_2(f_\lambda) = k_1$.

As a consequence of the above theorem, we have the following result.

Theorem 1.2.2. [2, Theorem 1] The number of chiral partitions of n is given by

$$B(n) = 2^{k_2+\cdots+k_r}\left(2^{k_1-1} + \sum_{v=1}^{k_1-1} 2^{(v+1)(k_1-1)-\binom{v}{2}} + \epsilon 2^{\binom{k_1}{2}}\right).$$

We shall call a partition λ an odd partition if the dimension of the corresponding representation is odd.

Theorem 1.2.3. [5]If n has the binary expansion $n = a_0 + 2a_1 + 2^2a_2 + 2^3a_3 + \ldots$, with $a_i \in \{0, 1\}$, then

$$\alpha(n) = a_1 + 2a_2 + 3a_3 + \ldots.$$

The number of odd partitions of n is $A(n) := 2^{\alpha(n)}$.

We denote $\text{sum}(a, b)$ is neat, if there is no carry in adding a and b in binary. Similarly we denote $\text{sum}(a, b)$ is messy, if there is a carry.

Proposition 1.2.4. Let n, a and b be positive integers such that $a+b=n$. Then $A(n) \leq A(a)A(b)$, with equality holding if sum(a,b) is neat.

This can be easily seen by just writing down the binary expansions of a and b. Let $\tilde{B}(n) := \dfrac{B(n)}{A(n)}$, then

$$\tilde{B}(n) = \frac{\left(2^{k_1-1} + \sum_{v=1}^{k_1-1} 2^{(v+1)(k_1-1)-\binom{v}{2}} + \epsilon 2^{\binom{k_1}{2}}\right)}{2^{k_1}}.$$

Lemma 1.2.5. Let n be a positive integer.

1. If $n \equiv 0 \mod 4$, then $\tilde{B}(n) \geq \frac{3}{2}$.
2. If $n \equiv 1 \mod 4$, then $\tilde{B}(n) \geq \frac{7}{2}$.
3. If $n \equiv 2 \mod 4$, then $\tilde{B}(n) = \frac{1}{2}$.
4. If $n \equiv 3 \mod 4$, then $\tilde{B}(n) = 1$.

Proof. This follows from $\dfrac{\sum_{v=1}^{k_1-1} 2^{(v+1)(k_1-1)-\binom{v}{2}} + \epsilon 2^{\binom{k_1}{2}}}{2^{k_1}} \geq 0$, with equality holding iff $k_1 = 1$ and $\epsilon = 0$.

Chapter 2

Mackey Theory for $\mathbb{B}_n$

Let $\mathbb{B}_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ denote the n-th hyperoctahedral group. Let ρ_λ be an irreducible representation of S_n. The normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ has two S_n invariant characters; the trivial character, and the character ϵ whose restriction to each factor $(\mathbb{Z}/2\mathbb{Z})$ is nontrivial. Consider two irreducible representations of $\mathbb{B}_n$ namely,

$$\rho^0_\lambda(x;\omega) = \rho_\lambda(\omega) \quad \text{and} \quad \rho^1_\lambda(x;\omega) = \epsilon(x)\rho_\lambda(\omega),$$

where $x \in (\mathbb{Z}/2\mathbb{Z})^n$ and $\omega \in S_n$. Let $\rho_{\alpha\beta}$ be defined as,

$$\rho_{\alpha\beta} = \mathrm{Ind}^{\mathbb{B}_n}_{\mathbb{B}_a \times \mathbb{B}_b} \rho^0_\alpha \boxtimes \rho^1_\beta.$$

In order to find all the irreducible representations, we are going to need the following theorems by Clifford (see [4], Theorem 6.2, Theorem 6.5).

Theorem 2.0.1. Let $H \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Let ξ be an irreducible constituent of χ_H and suppose $\xi = \xi_1, \ldots, \xi_t$, are the distinct conjugates of ξ in G. Then

$$\chi_H = e \sum_{i=1}^{t} \xi_i,$$

where $e = [\chi_H, \xi]$.

Definition 2.0.2. Let G be a group. Let $H \triangleleft G$ and $\xi \in \text{Irr}(H)$. Then

$$N_G(\xi) = \{g \in G \mid \xi^g = \xi\}$$

is the normalizer of ξ in G.

Theorem 2.0.3. Let $H \triangleleft G$, $\xi \in \text{Irr}(H)$, and $T = N_G(\xi)$. Let

$$\mathcal{A} = \{\psi \in \text{Irr}(T) \mid [\psi_H, \xi] \neq 0\},$$

$$\mathcal{B} = \{\chi \in \text{Irr}(G) \mid [\chi_H, \xi] \neq 0\}.$$

Then

1. If $\psi \in \mathcal{A}$, then ψ^G is irreducible;
2. The map $\psi \mapsto \psi^G$ is a bijection of $\mathcal{A}$ onto $\mathcal{B}$;
3. If $\psi^G = \chi$, with $\psi \in \mathcal{A}$, then ψ is the unique irreducible constituent of χ_T which lies in $\mathcal{A}$;
4. If $\psi^G = \chi$, with $\psi \in \mathcal{A}$, then $[\psi_H, \xi] = [\chi_H, \xi]$.

Let

$$\rho_{\alpha\beta} = \text{Ind}_{\mathbb{B}_a \times \mathbb{B}_b}^{\mathbb{B}_n} \rho_\alpha^0 \boxtimes \rho_\beta^1.$$

Lemma 2.0.4. If $\rho_{\alpha\beta}$ is defined as above, it is an irreducible representation of $\mathbb{B}_n$.

Proof. $V_n := C_2^n$ is an obvious normal subgroup of $\mathbb{B}_n$. Let $\xi_{a,b} \in \text{Irr}(V_n)$ be defined as:

$$\xi_{a,b}(v_1, \ldots, v_a, v_{a+1}, \ldots, v_{a+b}) = (-1)^{v_{a+1}+\cdots+v_{a+b}}.$$

Then

$$N_G(\xi_{a,b}) = C_2^n \rtimes (S_a \times S_b) \equiv \mathbb{B}_a \times \mathbb{B}_b.$$

Certainly $\rho_\alpha^0 \boxtimes \rho_\beta^1 \in \text{Irr}(N_g(\xi_{a,b}))$. Now let $w_a \in V_\alpha$ and $w_b \in V_\beta$,

$$(\rho_\alpha^0 \boxtimes \rho_\beta^1)_{V_n}(w_a, w_b) = \text{Id}_{V_\alpha} \boxtimes \epsilon(w_b)\text{Id}_{V_\beta}$$

$$\chi_{(\rho^0_\alpha \boxtimes \rho^1_\beta)_{v_n}}(w_a, w_b) = f_\alpha f_\beta \epsilon(w_b) = f_\alpha f_\beta \xi_{a,b}(w_a, w_b)$$

Hence $[(\rho^0_\alpha \boxtimes \rho^1_\beta)_{V_n}, \xi_{a,b}] \neq 0$, therefore $\rho^0_\alpha \boxtimes \rho^1_\beta \in \mathcal{A}$ and $\mathrm{Ind}^{\mathbb{B}_n}_{\mathbb{B}_a \times \mathbb{B}_b} \rho^0_\alpha \boxtimes \rho^1_\beta$ is an irreducible representation of $\mathbb{B}_n$.

Lemma 2.0.5. All the irreducible representations of $\mathbb{B}_n$ arise this way.

Proof. Let π be an irreducible representation of $\mathbb{B}_n$. Then π_{V_n} contains a linear functional σ. We can identify σ with a row vector having a number of 0's and b number of 1's. Thus σ is conjugate to some $\xi_{a,b}$. Using Theorem 2.0.1, we can deduce π_{V_n} contains $\xi_{a,b}$. π must be an induction of some τ from $\mathbb{B}_a \times \mathbb{B}_b$. We know that, τ must be a tensor product of τ_α from $\mathbb{B}_a$ and τ_β from $\mathbb{B}_b$. By Theorem 2.0.3, τ contains a trivial representation. By Theorem 2.0.1, and as τ_{v_n} is trivial, it must be of the form ρ^0_α. Similarly for τ_b, we can prove it is of the form ρ^1_β. Hence τ is of the form $\rho_{\alpha,\beta}$.

Then

$$\{\rho_{\alpha\beta} | \alpha \vdash a, \beta \vdash b, a + b = n\}$$

is a complete set of representatives for the set of isomorphism classes of irreducible representations of $\mathbb{B}_n$. The dimension $f_{\alpha\beta}$ of the representation space $V_{\alpha\beta}$ is

$$f_{\alpha\beta} = \dim V_{\alpha\beta} = \frac{n!}{a!b!} f_\alpha f_\beta.$$

More details can be found about the irreducible representations in [6].

Chapter 3

Solomon Formulas for $\mathbb{B}_n$

3.1 Formula for determinants

There are four one-dimensional representations of $\mathbb{B}_n$, which we denote as 1, ϵ, sgn^0 and sgn^1. Let $(x;\omega) \in \mathbb{B}_n$ where $x \in (\mathbb{Z}/2\mathbb{Z})^n$ and $\omega \in S_n$.
The projection to S_n followed by the sign character is sgn^0, i.e. $\mathrm{sgn}^0(x;\omega) = \mathrm{sgn}(\omega)$. Similarly $\epsilon(x;\omega) = \epsilon(x)$, where ϵ is the character whose restriction to each factor $(\mathbb{Z}/2\mathbb{Z})$ is nontrivial. Finally, sgn^1 is the product of ϵ and sgn^0, i.e. $\mathrm{sgn}^1(x;\omega) = \epsilon(x)\mathrm{sgn}(\omega)$.
The determinant of each irreducible representation is equal to one of them. Let $e_1 := (1, 0, \ldots, 0; 1_{S_n}))$ and $s_1 := (\vec{0}; (12))$. Let $\chi_{\alpha\beta} := \chi_{\rho_{\alpha\beta}}$ and $\chi_\lambda := \chi_{\rho_\lambda}$. The following table lists the characters of the one dimensional representations at two particular conjugacy classes.

$\rho_{\alpha\beta}$	$\chi_{\rho_{\alpha\beta}}(e_1)$	$\chi_{\rho_{\alpha\beta}}(s_1)$
1	1	1
sgn^0	1	-1
ϵ	-1	1
sgn^1	-1	-1

Let,

$$g_\lambda = \frac{f_\lambda - \chi_\lambda((12))}{2}. \tag{3.1}$$

Let H be a subgroup of the finite group G and let $g_1, \dots, g_m$ be representatives for the distinct left cosets of H in G.

Proposition 3.1.1. (Frobenius character formula) If ψ is the character afforded by V then the induced character is given by

$$\mathrm{Ind}_H^G(\psi)(g) = \sum_{i=1}^{m} \psi(g_i^{-1} g g_i)$$

where $g \in G/H$ and ψ is extended by 0 to G.

Using Proposition 3.1.1, we can determine the following formulas.

Proposition 3.1.2. For given α and β, we have

$$\chi_{\alpha\beta}(e_1) = f_\alpha f_\beta \left[2\binom{n-1}{a-1, b} - \binom{n}{a, b} \right].$$

Proof. To apply the Frobenius character formula, we need to know the distinct left coset representatives of $S_{a+b}/(S_a \times S_b)$. Let $\underline{n} = \{1, 2, \dots, n\}$; take its power set and represent it as $\wp(\underline{n})$. Take $\underline{a} = (1, 2, \dots, a) \in \wp(\underline{n})$ and consider the action of S_n on $\wp(\underline{n})$. Then by the orbit-stabiliser theorem,

$$\mathrm{Orb}(\underline{a}) \cong S_{a+b}/(S_a \times S_b) \text{ as } G\text{-sets}$$

Now by Proposition 3.1.1,

$$\mathrm{Ind}_{\mathbb{B}_a \times \mathbb{B}_b}^{\mathbb{B}_n}(\rho_\alpha^0 \boxtimes \rho_\beta^1)(e_1) = \sum_{i=1}^{m} (\rho_\alpha^0 \boxtimes \rho_\beta^1)(g_i^{-1} e_1 g_i)$$

Consider the map g_i which is given by:

$$e_1 \mapsto g_i^{-1} e_1 g_i,$$

$$g_i : \begin{pmatrix} 1 \\ \vdots \\ a \\ a+1 \\ \vdots \\ b+a \end{pmatrix} \longmapsto \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_a} \\ y_{i_1} \\ \vdots \\ y_{i_b} \end{pmatrix} .$$

Let $X_a := (x_{i_1}, \ldots, x_{i_a})^T$ and $X_b := (y_{i_1}, \ldots, y_{i_b})^T$. Hence

$$\#\{g_i \mid 1 \in X_a\} = \#\{g_i \mid 1 \notin X_b\} = \binom{n-1}{a-1, b},$$

is the number of g_i's such that $(\rho_\alpha^0 \boxtimes \rho_\beta^1)(g_i^{-1} e_1 g_i) = 1$. The number of g_i's such that $(\rho_\alpha^0 \boxtimes \rho_\beta^1)(g_i^{-1} e_1 g_i) = -1$ is equal to $\left(\binom{n}{a,b} - \binom{n-1}{a-1,b}\right)$. Hence

$$\begin{aligned} \chi_{\alpha\beta}(e_1) &= f_\alpha f_\beta \left[\binom{n-1}{a-1, b} - \left(\binom{n}{a, b} - \binom{n-1}{a-1, b}\right)\right] \\ &= f_\alpha f_\beta \left[2\binom{n-1}{a-1, b} - \binom{n}{a, b}\right]. \end{aligned}$$

Proposition 3.1.3. For a given α and β, we have

$$\chi_{\alpha\beta}(s_1) = \binom{n-2}{a-2, b} f_\beta \chi_\alpha((12)) + \binom{n-2}{a, b-2} f_\alpha \chi_\beta((12)).$$

Proof. By Proposition 3.1.1,

$$\chi_{\alpha\beta}(s_1) = \sum_{i=1}^{m} (\rho_\alpha^0 \boxtimes \rho_\beta^1)(g_i^{-1}(12)g_i).$$

We know that $g_i^{-1}(12)g_i = (g_i^{-1}(1) g_i^{-1}(2))$. Consider the map g_i which is given by:

$$(12) \mapsto (g_i^{-1}(12)g_i),$$

$$g_i : \begin{pmatrix} 1 \\ \vdots \\ a \\ a+1 \\ \vdots \\ b+a \end{pmatrix} \longmapsto \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_a} \\ y_{i_1} \\ \vdots \\ y_{i_b} \end{pmatrix} .$$

Thus,

$$\#\{g_i \mid 1 \in X_a \text{ and } 2 \in X_a\} = \#\{g_i \mid 1 \notin X_b \text{ and } 2 \notin X_a\} = \binom{n-2}{a-2, b}$$

and

$$\#\{g_i \mid 1 \in Y_a \text{ and } 2 \in Y_a\} = \#\{g_i \mid 1 \notin Y_b \text{ and } 2 \notin Y_b\} = \binom{n-2}{a, b-2}.$$

Hence

$$\chi_{\alpha\beta}(s_1) = \binom{n-2}{a-2, b} f_\beta \chi_\alpha((12)) + \binom{n-2}{a, b-2} f_\alpha \chi_\beta((12)).$$

Let $x_{\alpha\beta}$ and $y_{\alpha,\beta}$ denote the multiplicity of -1 as eigenvalues of $\rho_{\alpha\beta}(e_1)$ and $\rho_{\alpha\beta}(s_1)$ respectively. Then, similar to the formulas of Solomon for determinants of S_n (see [10], Exercise 7.55), we have:

$$\begin{aligned} x_{\alpha\beta} &= \frac{\chi_{\alpha\beta}(1,1,\ldots,1;1_{S_n}) - \chi_{\alpha\beta}(e_1)}{2} \\ &= f_\alpha f_\beta \binom{n-1}{a, b-1}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} y_{\alpha\beta} &= \frac{\chi_{\alpha\beta}(1,1,\ldots,1;1_{S_n}) - \chi_{\alpha\beta}(s_1)}{2} \\ &= \frac{f_\alpha f_\beta \binom{n}{a,b} - \binom{n-2}{a-2,b} f_\beta \chi_\alpha((12)) - \binom{n-2}{a,b-2} f_\alpha \chi_\beta((12))}{2}. \end{aligned}$$

By Equation 3.1, we have

$$\begin{aligned} y_{\alpha\beta} &= \frac{\binom{n-2}{a-2,b} f_\beta \chi_\alpha((12)) + \binom{n-2}{a,b-2} f_\alpha \chi_\beta((12)) - f_\alpha f_\beta \binom{n}{a,b}}{2} \\ &= \frac{f_\alpha f_\beta \left[\binom{n-2}{a-2,b} + \binom{n-2}{a,b-2} - \binom{n}{a,b}\right]}{2} - \binom{n-2}{a-2,b} f_\alpha g_\alpha - \binom{n-2}{a,b-2} f_\beta g_\beta \qquad (3.3) \\ &= f_\alpha f_\beta \binom{n-2}{a-1,b-1} + f_\beta g_\alpha \binom{n-2}{a-2,b} + f_\alpha g_\beta \binom{n-2}{a,b-2}. \end{aligned}$$

Theorem 3.1.4. For a given $x_{\alpha\beta}$ and $y_{\alpha\beta}$,

$$\det \circ \rho_{\alpha\beta} = \epsilon^{x_{\alpha\beta}} \cdot \left(\mathrm{sgn}^0\right)^{y_{\alpha\beta}}.$$

The study of the determinants of the representations of the hyperoctahedral group then reduces to finding the parities of $x_{\alpha\beta}$ and $y_{\alpha\beta}$.

3.2 Parity of $x_{\alpha\beta}$

Using the theory of 2-core towers of partitions developed for example in ([2], [5], [7]), we were able to determine the parity of x. That is, for which bipartitions (α, β) of n, x is odd or even; and count the number of such partitions of a given n. Let $X(n) = \{(\alpha, \beta) \models n \mid x_{\alpha\beta} \text{ is odd}\}$. Recall, $\binom{n}{a,b}$ is odd iff there are no carries while adding a and b in binary. Suppose n is a positive integer with the following binary expansion:

$$n = 2^{k_0} + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, 0 \leq k_1 < k_2 < \cdots < k_r.$$

Define $\mathrm{bin}(n) = k_0, \ldots, k_r$, a set of cardinality $\nu(n)$. Thus, $\binom{n}{a,b}$ is odd iff $\mathrm{bin}(a)$ and $\mathrm{bin}(b)$ are disjoint.

For a partition λ of n, let $\gamma = \mathrm{core}_2(\lambda)$ and $(\alpha; \beta) = \mathrm{quo}_2(\lambda)$. Let $g = |\gamma|$, $a = |\alpha|$, and $b = |\beta|$. Let $2n$ have the following binary expansion:

$$2n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, 0 < k_1 < k_2 < \cdots < k_r.$$

Lemma 3.2.1. Suppose λ is a partition of $2n$ with trivial 2-core and f_λ is odd. Let the

2-core tower of λ satisfy the following conditions:

$$w_i(\lambda) = \begin{cases} 1 & \text{if } i \in \{k_1, k_2, \ldots, k_r\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\binom{n-1}{a,b-1}$ is odd iff the unique non-trivial partition in the k_1th row is in the right subtree of the tower. On the other hand, $\binom{n-1}{a,b-1}$ is even iff the unique non-trivial partition in the k_1th row is in the left subtree of the tower.

Proof. **Case** $k_1 = 1$: If the unique non-trivial partition in the 1st row is in the right subtree, then a is even and b is odd. The conditions on w_i guarantee that $\text{bin}(a)$ and $\text{bin}(b-1)$ are disjoint. If the unique non-trivial partition in the 1st row is in the left subtree, then a is odd and b is even. Thus $\text{bin}(a)$ and $\text{bin}(b-1)$ are not disjoint.
Case $k_1 > 1$: In this case both a and b are even. We know,

$$\text{bin}(b-1) = \{0, 1, \ldots, v_2(b) - 1\} \cup (\text{bin}(b) - \{v_2(b)\}).$$

If the unique non-trivial partition in the k_1th row is in the right subtree, then $k_1 = v_2(b) - 1$ and $k_2 \leq v_2(a) + 1$. Thus $\text{bin}(a)$ and $\text{bin}(b-1)$ are disjoint. If the unique non-trivial partition in the k_1th row is in the left subtree, then $k_1 = v_(a) - 1$ and $k_2 \leq v_2(b) + 1$. Hence $\text{bin}(a)$ and $\text{bin}(b-1)$ are not disjoint.

Lemma 3.2.2. Given λ a partition of $2n$ with trivial 2-core. For some $0 < v < k_1$, let the 2-core tower of λ satisfy the following conditions:

$$w_i(\lambda) = \begin{cases} 2 & \text{if } i = k_1 - v, \\ 1 & \text{if } k_1 - v + 1 \leq i \leq k_1 - 1 \text{ or } i \in \{k_1, k_2, \ldots, k_r\}, \\ 0 & \text{otherwise,} \end{cases}$$

and the two non-trivial partitions in the $(k_1 - v)$th row are one on the left and one on the right subtree. Then $\binom{n-1}{a,b-1}$ is odd.

Proof. Case 1:
Suppose $k_1 - v = 1$. In the 2-core tower, this means $\alpha_0 = 1$ and $\alpha_1 = 1$. Both a and b are odd; hence $b-1$ is even and as the number of 1's that occur in other rows of the 2-core tower is just 1, there are no carries while adding a and $b-1$.
Case 2:

Suppose $k_1 > 2$ and $v \neq k_1 - 1$. Then a and b are even, but both have 1 at the $(k_1 - v)$th position in the binary expansion. Thus, $b - 1$ is odd and looking at the binary expansion of $b - 1$, from the right, it will be all 1's $k_1 - v + 1$ times followed by a zero at the $(k_1 - v)$th place, followed by some nonzero part. Hence there won't be any carries while adding a and $b - 1$.

Recall the following from [1, Lemma 6].

Lemma 3.2.3. The partition λ is odd if and only if $|\gamma| \leq 1$ (so γ is $\emptyset$ or (1)), the sets of place values where 1 appears in the binary expansions of g, $2a$ and $2b$ are disjoint, and α and β are odd.

These three conditions completely determine $x_{\alpha\beta} \mod 2$. Let $\mathrm{Bip} := \{(\alpha, \beta) \models n\}$ and $\mathrm{Bip}_w = \{(\alpha, \beta) \models n \mid \det \circ \rho = w\}$. Let $\mathrm{C}_n = \{\alpha \mid \alpha \vdash n, \alpha \text{ is chiral}\}$.

Lemma 3.2.4. For a given n,

$$N_\epsilon(n) + N_{\mathrm{sgn}^1}(n) = B(2n). \tag{3.4}$$

Proof. We know $\mathrm{Bip}_\epsilon \cup \mathrm{Bip}_{\mathrm{sgn}^1} = \{(\alpha, \beta) \mid (\alpha, \beta) \models n, x_{\alpha,\beta} \equiv 1 (\mathrm{mod}\ 2)\}$. Consider $\phi : \mathrm{Bip}_\epsilon \cup \mathrm{Bip}_{\mathrm{sgn}^1} \to \mathrm{C}_{2n}$, where $\phi(\alpha, \beta)$ is the unique partition whose 2-core is trivial and whose 2-quotient is (α, β). Then from the previous lemmas and Theorem 1.2.1, it follows that ϕ is a bijective map.

3.3 Parity of $y_{\alpha\beta}$

By Equation 3.3 we have

$$y_{\alpha\beta} = f_\alpha f_\beta \binom{n-2}{a-1, b-1} + f_\beta g_\alpha \binom{n-2}{a-2, b} + f_\alpha g_\beta \binom{n-2}{a, b-2}.$$

To study the above equation further, we must understand the correlation between the parities of f_λ and g_λ for various λ. Take μ to be a partition of m. Let m have the following binary expansion:

$$m = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, \epsilon \in \{0, 1\}, 0 < k_1 < k_2 < \cdots < k_r.$$

The two lemmas stated below follow directly from Theorem 1.2.1.

Lemma 3.3.1. If the following conditions are satisfied, then $g_\mu \equiv f_\mu + 1 \mod 2$.

1. Let m be even. Suppose, the 2-core tower of μ satisfies the following conditions:

$$w_i(\mu) = \begin{cases} 2 & \text{if } i = k_1 - v, \\ 1 & \text{if } k_1 - v + 1 \leq i \leq k_1 - 1 \text{ or } i \in \{k_1, k_2, \ldots, k_r\}, \\ 0 & \text{otherwise,} \end{cases}$$

   and the two non-trivial partitions in the $(k_1 - v)$th row are one on the left and one on the right subtree.

2. Let m be odd. Suppose, the 2-core tower of μ satisfies the following conditions:

$$w_i(\mu) = \begin{cases} 3 & \text{if } i = 0, \\ 1 & \text{if } i \in \{1, \ldots, k_1 - 1, k_2, \ldots, k_r\}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.3.2. Suppose, the 2-core tower of μ satisfies the following conditions:

$$w_i(\mu) = \begin{cases} 1 & \text{if } i \in \{k_1, k_2, \ldots, k_r\}, \\ 0 & \text{otherwise,} \end{cases}$$

and the unique non-trivial partition in the k_1th row is in the left subtree of the tower. Then $g_\mu \equiv f_\mu \mod 2$.

□

Corollary 3.3.3. 1. The number of partitions λ of n, such that f_λ and g_λ are both odd is $\frac{1}{2}A(n)$.

2. The number of partitions λ of n, such that f_λ is odd and g_λ is even is $\frac{1}{2}A(n)$.

3. The number of partitions λ of n, such that f_λ is even and g_λ is odd is $B(n) - \frac{1}{2}A(n)$.

Proof. These follow directly by Lemmas 3.3.1, 3.3.2 and Theorem 1.2.2, .

3.4 Involutions on the set of Bipartitions

Recall

$$x_{\alpha,\beta} = f_\alpha f_\beta \binom{n-1}{a, b-1}$$

and

$$y_{\alpha,\beta} = f_\alpha f_\beta \binom{n-2}{a-1, b-1} + f_\alpha g_\beta \binom{n-2}{a, b-2} + f_\beta g_\alpha \binom{n-2}{a-2, b}.$$

Proposition 3.4.1. Consider the map $\iota_1 : \mathrm{Bip}_\epsilon \to \mathrm{Bip}$ defined by $\iota_1(\alpha, \beta) := (\alpha', \beta)$. Then the parity of $x_{\alpha,\beta}$ is preserved and the parity of $y_{\alpha,\beta}$ changes iff $\binom{n-2}{a-1,b-1}$ is odd.

Proof. We know that

$$x_{\alpha',\beta} = f_\alpha f_\beta \binom{n-1}{a, b-1}$$

and

$$y_{\alpha',\beta} = f_{\alpha'} f_\beta \binom{n-2}{a-1, b-1} + f'_\alpha g_\beta \binom{n-2}{a, b-2} + f_\beta g_{\alpha'} \binom{n-2}{a-2, b}.$$

We know that $x_{\alpha,\beta}$ be odd. Then $x_{\alpha',\beta}$ is also odd and

$$y_{\alpha',\beta} \equiv \binom{n-2}{a-1, b-1} + g_{\alpha'} \binom{n-2}{a-2, b} + g_\beta \binom{n-2}{a, b-2} (\text{mod } 2).$$

Case 1: Let $\binom{n-2}{a-1,b-1}$ be odd and $\binom{n-2}{a,b-2}$ be even. Hence

$$y_{\alpha,\beta} \equiv 1 + g_\alpha \binom{n-2}{a-2, b} (\text{mod } 2)$$

and

$$y_{\alpha',\beta} \equiv 1 + g_{\alpha'} \binom{n-2}{a-2, b} (\text{mod } 2).$$

Since f_α is odd, we know $g_{\alpha'} \equiv 1 + g_\alpha (\text{mod } 2)$. Thus $y_{\alpha,\beta}$ changes parity under the map ι_1 iff $\binom{n-2}{a-2,b}$ is odd.

Case 2: Let $\binom{n-2}{a-1,b-1}$ be odd and $\binom{n-2}{a,b-2}$ be even. Hence

$$y_{\alpha,\beta} \equiv g_\alpha \binom{n-2}{a-2, b} + g_\beta$$

and

$$y_{\alpha',\beta} \equiv g_{\alpha'}\binom{n-2}{a-2,b} + g_\beta.$$

Similarly, $y_{\alpha,\beta}$ changes parity under the map ι_1 iff $\binom{n-2}{a-2,b}$ is odd.

Proposition 3.4.2. Consider the map $\iota_1 : \mathrm{Bip}_{\mathrm{sgn}^0} \to \mathrm{Bip}$ defined by $\iota_1(\alpha, \beta) := (\alpha', \beta)$. Then the parity of $x_{\alpha,\beta}$ is preserved. The parity of $y_{\alpha,\beta}$ is preserved if f_β is even or g_α doesn't change under conjugation, otherwise the parity changes iff $\binom{n-2}{a-2,b}$ is odd.

Proof. We know that $x_{\alpha,\beta}$ be even and $y_{\alpha,\beta}$ is odd. Hence, either one of f_α or f_β is even or $\binom{n-1}{a,b-1}$ is even. Consider f_α is even. Thus,

$$y_{\alpha,\beta} \equiv g_\alpha\binom{n-2}{a-2,b} \pmod 2,$$

and

$$y_{\alpha',\beta} \equiv g_{\alpha'}\binom{n-2}{a-2,b} \pmod 2.$$

Hence the parity changes iff $\binom{n-2}{a-2,b}$ is odd. Consider f_β is even, then

$$y_{\alpha,\beta} \equiv g_\beta\binom{n-2}{a,b-2}.$$

Hence, there is no change in parity. Let f_α and f_β be odd, and $\binom{n-1}{a,b-1}$ is even. Then,

$$y_{\alpha,\beta} \equiv \binom{n-2}{a-1,b-1} + g_\beta\binom{n-2}{a,b-2} + g_\alpha\binom{n-2}{a-2,b} \pmod 2,$$

and

$$y_{\alpha',\beta} = \binom{n-2}{a-1,b-1} + g_\beta\binom{n-2}{a,b-2} + g_{\alpha'}\binom{n-2}{a-2,b} \pmod 2.$$

So there is a change in parity iff $\binom{n-2}{a-2,b}$ is odd.

Proposition 3.4.3. Consider the map $\iota_2 : \mathrm{Bip}_\epsilon \to \mathrm{Bip}$ defined by $\iota_2(\alpha, \beta) := (\alpha, \beta')$. In this case the parity of $x_{\alpha,\beta}$ is preserved and the parity of $y_{\alpha,\beta}$ changes iff $\binom{n-2}{a-1,b-1}$ is even and $\binom{n-2}{a,b-2}$ is odd.

Proof. We have $x_{\alpha,\beta'} \equiv f_\alpha f_{\beta'}\binom{n-2}{a,b-1}$ and thus there is no change in parity.

Case 1: Let $\binom{n-2}{a-1,b-1}$ be odd and $\binom{n-2}{a-2,b}$ be even. Then

$$y_{\alpha,\beta} \equiv 1 + g_\alpha \binom{n-2}{a-2,b} \pmod 2$$

and

$$y_{\alpha,\beta'} \equiv 1 + g_\alpha \binom{n-2}{a-2,b} \pmod 2.$$

Thus there is no change in parity.
Case 2: Let $\binom{n-2}{a-1,b-1}$ be even and $\binom{n-2}{a-2,b}$ is odd. Then

$$y_{\alpha,\beta} \equiv g_\beta + g_\alpha \binom{n-2}{a-2,b} \pmod 2$$

and

$$y_{\alpha,\beta'} \equiv g_{\beta'} + g_\alpha \binom{n-2}{a-2,b} \pmod 2.$$

Thus the parity changes.

Proposition 3.4.4. Consider the map $\iota_2 : \mathrm{Bip}_{\mathrm{sgn}^0} \to \mathrm{Bip}$ defined by $\iota_2(\alpha, \beta) := (\alpha, \beta')$. Then the parity of $x_{\alpha,\beta}$ is preserved. The parity of $y_{\alpha,\beta}$ is preserved if f_α is even or g_β doesn't change under conjugation, otherwise the parity changes iff $\binom{n-2}{a,b-2}$ is odd.

This follows directly from Proposition 3.4.2

Proposition 3.4.5. Consider the map $\iota_3 : \mathrm{Bip} \to \mathrm{Bip}$ defined by $\iota_3(\alpha, \beta) := (\beta, \alpha)$. The parity of $x_{\alpha,\beta}$ changes iff the 2-core tower of $\phi(\alpha, \beta)$ satisfies Lemma 5.1 while the parity of $y_{\alpha,\beta}$ is preserved.

This follows directly from Lemma 5.1 and the fact that $y_{\alpha,\beta}$ is symmetric with respect to α and β. Combining the above lemmas we get the following.

Corollary 3.4.6. Consider the map $\iota_4 : \mathrm{Bip}_\epsilon \to \mathrm{Bip}$ defined by $\iota_4(\alpha, \beta) := (\beta', \alpha')$. Then $\iota_4(\alpha, \beta) \in \mathrm{Bip}_\epsilon$ iff the 2-core tower of $\phi(\alpha, \beta)$ satisfies Lemma 5.2 and one of the following is true:

1. $\binom{n-2}{a-2,b}$, $\binom{n-2}{a-1,b-1}$ and $\binom{n-2}{a,b-2}$ are odd, even and odd respectively,

2. $\binom{n-2}{a-2,b}$, $\binom{n-2}{a-1,b-1}$ and $\binom{n-2}{a,b-2}$ are even, odd and even respectively.

Corollary 3.4.7. let $\iota_4 : \mathrm{Bip}_{\mathrm{sgn}^0} \to \mathrm{Bip}$ defined as above. Then $\iota_4(\alpha, \beta) \in \mathrm{Bip}_{\mathrm{sgn}^0}$ iff exactly one of f_α or f_β is even and the first non-trivial row in α or β contains two [1]'s symmetrically.

Chapter 4

Counting Representations of Given Determinant

4.1 Counting in terms of (a, b)

Let

$$N_\omega(n) = \#\{(\alpha, \beta) \models n \mid \det \circ \rho_{\alpha\beta} = \omega\}$$

and

$$N_\omega(a, b) = \#\{\alpha, \beta) \mid \alpha \vdash a, \beta \vdash b, \det \circ \rho_{\alpha\beta} = \omega\} \text{ for } \omega \in \{1, \epsilon, \mathrm{sgn}^0, \mathrm{sgn}^1\}.$$

It follows that

$$N_\omega(n) = \sum_{a+b=n} N_\omega(a, b).$$

Recall that,

$$x_{\alpha\beta} = f_\alpha f_\beta \binom{n-1}{a, b-1},$$

and

$$y_{\alpha\beta} = f_\alpha f_\beta \binom{n-2}{a-1, b-1} + f_\beta g_\alpha \binom{n-2}{a-2, b} + f_\alpha g_\beta \binom{n-2}{a, b-2}.$$

Lemma 4.1.1. The following conditions are equivalent:

1. $\binom{n-2}{a-1,b-1}$ is odd, $\binom{n-2}{a,b-2}$ is even, and $\binom{n-2}{a-2,b}$ is even,

2. n is even and $\binom{n-2}{a-1,b-1}$ is odd.

This is immediate from considering the binary expansions of a, b and n.
Recall $A(n) = 2^{\alpha(n)}$, is the number of odd partitions of n.

Proposition 4.1.2. For given a and b, we have

$$N_\epsilon(a,b) = \begin{cases} 0 \ \text{ if } \binom{n-1}{a,b-1} \text{ is even,} \\ 0 \ \text{ if } n \text{ is even and } \binom{n-2}{a-1,b-1} \text{ is odd,} \\ \frac{1}{2}A(a)A(b) \ \text{ otherwise.} \end{cases} \tag{4.1}$$

Proof. For $x_{\alpha\beta}$ to be odd, we need f_α, f_β and $\binom{n-1}{a,b-1}$ to be odd. We know,

$$\binom{n-1}{a,b-1} = \binom{n-2}{a-1,b-1} + \binom{n-2}{a,b-2}.$$

So there are two cases we need to consider.
Case 1: If $\binom{n-2}{a-1,b-1}$ is odd and $\binom{n-2}{a,b-2}$ is even, then

$$y_{\alpha\beta} \equiv 1 + g_\alpha \binom{n-2}{a-2,b} \pmod 2.$$

For $y_{\alpha\beta}$ to be even, we need $g_\alpha \binom{n-2}{a-2,b}$ to be odd, thus

$$\begin{aligned} N_\epsilon(a,b) &= \#\{f_\alpha \text{ is odd, } g_\alpha \text{ is odd}\}\#\{f_\beta \text{ is odd}\} \\ &= 2^{\alpha(a)-1}2^{\alpha(b)} \\ &= \frac{1}{2}A(a)A(b). \end{aligned}$$

Case 2: If $\binom{n-2}{a-1,b-1}$ is even and $\binom{n-2}{a,b-2}$ is odd, then

$$y_{\alpha\beta} \equiv g_\alpha \binom{n-2}{a-2,b} + g_\beta \pmod 2.$$

For $y_{\alpha\beta}$ to be even, we need $g_\alpha \binom{n-2}{a-2,b}$ and g_β to be both odd or both even. We again have two possibilities.

Case 2a: If $\binom{n-2}{a-2,b}$ is even, then

$$\begin{aligned} N_\epsilon(a,b) &= \#\{f_\beta \text{ is odd, } g_\beta \text{ is odd}\}\#\{f_\alpha \text{ is odd}\} \\ &= 2^{\alpha(a)}2^{\alpha(b)-1} \\ &= \frac{1}{2}A(a)A(b). \end{aligned}$$

Case 2b: If $\binom{n-2}{a-2,b}$ is odd, then

$$\begin{aligned} N_\epsilon(a,b) &= \#\{f_\alpha \text{ is odd, } g_\alpha \text{ is even}\}\#\{f_\beta \text{ is odd, } g_\beta \text{ is even}\} \\ &\quad + \#\{f_\alpha \text{ is odd, } g_\alpha \text{ is odd}\}\#\{f_\beta \text{ is odd, } g_\beta \text{ is odd}\} \\ &= A(a)A(b) - A(a)\frac{1}{2}A(b) - \frac{1}{2}A(a)A(b) + 2\cdot\frac{1}{2}A(a)\frac{1}{2}A(b) \\ &= \frac{1}{2}A(a)A(b). \end{aligned}$$

Proposition 4.1.3. For given a and b, we have

$$N_{\mathrm{sgn}^1}(a,b) = \begin{cases} 0 \ \text{ if } \binom{n-1}{a,b-1} \text{ is even,} \\ A(a)A(b) \text{ if } n \text{ is even and } \binom{n-2}{a-1,b-1} \text{ is odd,} \\ \frac{1}{2}A(a)A(b) \ \text{ otherwise.} \end{cases} \tag{4.2}$$

Proof. For $x_{\alpha\beta}$ to be odd, we need f_α, f_β and $\binom{n-1}{a,b-1}$ to be odd. Again there are two cases to consider.

Case 1: If $\binom{n-2}{a-1,b-1}$ is odd and $\binom{n-2}{a,b-2}$ is even, then

$$y_{\alpha\beta} \equiv 1 + g_\alpha \binom{n-2}{a-2,b} \pmod 2.$$

For $y_{\alpha\beta}$ to be odd, we need $g_\alpha\binom{n-2}{a-2,b}$ to be even.

Case 1a: If $\binom{n-2}{a-2,b}$ is even, then $N_{\mathrm{sgn}^1}(a,b) = A(a)A(b)$.

Case 1b: If $\binom{n-2}{a-2,b}$ is odd, then

$$\begin{aligned} N_{\mathrm{sgn}^1}(a,b) &= \#\{f_\alpha \text{ is odd, } g_\alpha \text{ is even}\}\#\{f_\beta \text{ is odd}\} \\ &= \left[\#\{f_\alpha \text{ is odd}\} - \#\{f_\alpha \text{ is odd, } g_\alpha \text{ is odd}\}\right]\#\{f_\beta \text{ is odd}\} \\ &= \frac{1}{2}A(a)A(b). \end{aligned}$$

Case 2: If $\binom{n-2}{a-1,b-1}$ is even and $\binom{n-2}{a,b-2}$ is odd, then

$$y_{\alpha\beta} \equiv g_\alpha \binom{n-2}{a-2,b} + g_\beta \pmod 2.$$

For $y_{\alpha\beta}$ to be odd, we have two cases:
Case 2a: If $g_\alpha\binom{n-2}{a-2,b}$ is even and g_β is odd, then

$$y_{\alpha\beta} \equiv g_\beta \pmod 2.$$

Thus, we have

$$\begin{aligned} N_{\mathrm{sgn}^1}(a,b) &= \#\{f_\beta \text{ is odd}, g_\beta \text{ is odd}\}\#\{f_\alpha \text{ is odd}\} \\ &= \frac{1}{2}A(b)A(a). \end{aligned}$$

Case 2b: If $g_\alpha\binom{n-2}{a-2,b}$ is odd and g_β is even, then

$$y_{\alpha\beta} \equiv g_\beta + g_\alpha \pmod 2.$$

Thus, we have

$$\begin{aligned} N_{\mathrm{sgn}^1}(a,b) = &\#\{f_\beta \text{ is odd}, g_\beta \text{ is odd}\}\#\{f_\alpha \text{ is odd}, g_\alpha \text{ is even}\} \\ &+ \#\{f_\beta \text{ is odd}, g_\beta \text{ is even}\}\#\{f_\alpha \text{ is odd}, g_\alpha \text{ is odd}\} \\ = &\frac{1}{2}A(b)A(a). \end{aligned}$$

Corollary 4.1.4. Given a positive odd integer n, $N_{\mathrm{sgn}^1}(n) = N_\epsilon(n)$.

Proof. This follows immediately from Theorems 4.1.2 and 4.1.3. We can even construct a bijective map. Recall $\mathrm{Bip}_{\mathrm{sgn}^1} = \{(\alpha,\beta) \mid (\alpha,\beta) \models n, \det\circ\rho_{\alpha,\beta} = \mathrm{sgn}^1\}$ and $\mathrm{Bip}_\epsilon = \{(\alpha,\beta) \mid (\alpha,\beta) \models n, \det\circ\rho_{\alpha,\beta} = \epsilon\}$. Then

$$\begin{aligned} \phi : \mathrm{Bip}_{\mathrm{sgn}^1} &\to \mathrm{Bip}_\epsilon \\ (\alpha,\beta) &\mapsto (\alpha',\beta'), \end{aligned}$$

where α' and β' are respective conjugate partitions. It can be easily verified that $x_{\phi(\alpha,\beta)} \equiv x_{\alpha,\beta} \pmod 2$ and $y_{\phi(\alpha,\beta)} \equiv 1 + y_{\alpha,\beta} \pmod 2$.

Proposition 4.1.5. Let $s_1 = B(a)A(b)$ and $s_2 = B(b)A(a)$. For given a and b, we have

$\binom{n-2}{a-2,b}$	$\binom{n-2}{a-1,b-1}$	$\binom{n-2}{a,b-2}$	$N_{\text{sgn}^0}(a,b)$
even	even	even	0
even	odd	even	0
odd	even	even	s_1
even	odd	odd	s_2
odd	odd	odd	$s_1 + s_2 - \frac{1}{2}A(a)A(b)$
odd	odd	even	$s_1 - \frac{1}{2}A(a)A(b)$
even	even	odd	$s_2 - \frac{1}{2}A(a)A(b)$
odd	even	odd	$s_1 + s_2 - A(a)A(b)$

Proof. For $x_{\alpha\beta}$ to be even, we need at least one of f_α, f_β and $\binom{n-1}{a,b-1}$ to be even. Let $\binom{n-1}{a,b-1}$ be even.

Case 1: If $\binom{n-2}{a-1,b-1}$ is even and $\binom{n-2}{a,b-2}$ is even, then

$$y_{\alpha\beta} \equiv f_\beta g_\alpha \binom{n-2}{a-2,b} \pmod{2}.$$

For $y_{\alpha\beta}$ to be odd, we need $f_\beta g_\alpha \binom{n-2}{a-2,b}$ to be odd, thus

$$\begin{aligned} N_{\text{sgn}^0}(a,b) &= \#\{f_\beta \text{ is odd}\}\left[\#\{f_\alpha \text{ is odd}, g_\alpha \text{ is odd}\} + \#\{f_\alpha \text{ is even}, g_\alpha \text{ is odd}\}\right] \\ &= B(a)A(b). \end{aligned}$$

Case 2: If $\binom{n-2}{a-1,b-1}$ and $\binom{n-2}{a,b-2}$ is odd, then

$$y_{\alpha\beta} \equiv f_\alpha f_\beta + f_\beta g_\alpha \binom{n-2}{a-2,b} + f_\alpha g_\beta \pmod{2}.$$

Case 2a: If $\binom{n-2}{a-2,b}$ is even, then

$$\begin{aligned} y_{\alpha\beta} &\equiv f_\alpha f_\beta + f_\alpha g_\beta \pmod{2} \\ &\equiv f_\alpha \left[f_\beta + g_\beta\right]. \end{aligned}$$

Hence,

$$\begin{aligned} N_{\text{sgn}^0}(a,b) &= \#\{f_\alpha \text{ is odd}\}\left[\#\{f_\beta \text{ is odd}, g_\beta \text{ is odd}\} + \#\{f_\beta \text{ is even}, g_\beta \text{ is odd}\}\right] \\ &= B(b)A(a). \end{aligned}$$

Case 2b: If $\binom{n-2}{a-2,b}$ is odd, then

$$y_{\alpha\beta} \equiv f_\alpha f_\beta + f_\beta g_\alpha + f_\alpha g_\beta \pmod{2}.$$

If f_α and f_β are both odd, then $y_{\alpha\beta} \equiv 1 + g_\alpha + g_\beta \pmod{2}$ and

$$\begin{aligned} N_1 =& \#\{f_\alpha \text{ is odd}, g_\alpha \text{ is even}\}\#\{f_\beta \text{ is odd}, g_\beta \text{ is even}\} \\ &+ \#\{f_\alpha \text{ is odd}, g_\alpha \text{ is odd}\}\#\{f_\beta \text{ is odd}, g_\beta \text{ is odd}\}. \end{aligned}$$

If f_α is odd and f_β is even, then $y_{\alpha\beta} \equiv g_\beta \pmod{2}$ and

$$N_2 = \#\{f_\alpha \text{ is odd}\}\#\{f_\beta \text{ is even}, g_\beta \text{ is odd}\}.$$

Similarly, if f_β is odd and f_α is even, then $y_{\alpha\beta} \equiv g_\alpha \pmod{2}$ and

$$N_3 = \#\{f_\beta \text{ is odd}\}\#\{f_\alpha \text{ is even}, g_\alpha \text{ is odd}\}.$$

Thus, we have

$$\begin{aligned} N_{\text{sgn}^0}(a,b) &= N_1 + N_2 + N_3 \\ &= B(b)A(a) + B(a)A(b) - \frac{1}{2}A(a)A(b). \end{aligned}$$

Let $\binom{n-1}{a,b-1}$ be odd; thus one of f_α and f_β has to be even.
Case 1: If $\binom{n-2}{a-1,b-1}$ is even and $\binom{n-2}{a,b-2}$ is odd, then

$$y_{\alpha\beta} \equiv f_\beta g_\alpha \binom{n-2}{a-2,b} + f_\alpha g_\beta \pmod{2}.$$

Case 1a: If $\binom{n-2}{a-2,b}$ is odd, then

$$y_{\alpha\beta} \equiv f_\beta g_\alpha + f_\alpha g_\beta \pmod{2}.$$

If f_α is even and f_β is odd, then $y_{\alpha\beta} \equiv g_\alpha \pmod{2}$. Similarly if f_β is even and f_α is odd, then $y_{\alpha\beta} \equiv g_\beta \pmod 2$. Thus,

$$\begin{aligned} N_{\mathrm{sgn}^0}(a,b) &= \#\{f_\beta \text{ is odd}\}\#\{f_\alpha \text{ is even, } g_\alpha \text{ is odd}\} + \#\{f_\alpha \text{ is odd}\}\#\{f_\beta \text{ is even, } g_\beta \text{ is odd}\} \\ &= A(b)\left[B(a) - \frac{1}{2}A(a)\right] + A(a)\left[B(b) - \frac{1}{2}A(b)\right] \\ &= A(b)B(a) + A(a)B(b) - A(a)A(b). \end{aligned}$$

Case 1b: If $\binom{n-2}{a-2,b}$ is even, then

$$y_{\alpha\beta} \equiv f_\alpha g_\beta \pmod 2.$$

Thus,

$$\begin{aligned} N_{\mathrm{sgn}^0}(a,b) &= \#\{f_\alpha \text{ is odd}\}\#\{f_\beta \text{ is even, } g_\beta \text{ is odd}\} \\ &= A(a)\left[B(b) - \frac{1}{2}A(b)\right]. \end{aligned}$$

Case 2: If $\binom{n-2}{a-1,b-1}$ is odd and $\binom{n-2}{a,b-2}$ is even, then

$$y_{\alpha\beta} \equiv f_\alpha f_\beta + f_\beta g_\alpha \binom{n-2}{a-2,b} \pmod 2.$$

Case 2a: If $\binom{n-2}{a-2,b}$ is even, then

$$y_{\alpha\beta} \equiv f_\alpha f_\beta \pmod 2.$$

Since one of f_α or f_β has to be even,

$$N_{\mathrm{sgn}^0}(a,b) = 0.$$

Case 2b: If $\binom{n-2}{a-2,b}$ is odd, then

$$\begin{aligned} y_{\alpha\beta} &\equiv f_\alpha f_\beta + f_\beta g_\alpha \pmod 2 \\ &\equiv f_\beta\left[f_\alpha + g_\alpha\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned} N_{\mathrm{sgn}^0}(a,b) &= \#\{f_\beta \text{ is odd}\}\left[\#\{f_\alpha \text{ is odd}, g_\alpha \text{ is odd}\} + \#\{f_\alpha \text{ is even}, g_\alpha \text{ is odd}\}\right] \\ &= B(a)A(b). \end{aligned}$$

4.2 Counting in terms of n

Let $\nu(n)$ denote the number of 1's in the binary expansion of n.

Lemma 4.2.1. For a given even n, let $k = \mathrm{ord}_2 n$. Then

$$\alpha(n-1) = \alpha(n) + \binom{k}{2} - k.$$

Proof. Suppose n is a positive integer with the following binary expansion:

$$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, 0 < k_1 < k_2 < \cdots < k_r.$$

By definition, $k = k_1$ and $\alpha(n) = k + k_2 + \cdots + k_r$. The binary expansion of $n-1$ will have 1's till $(k_1 - 1)$th place from the right, a zero at the k_1th place and rest remains unchanged. Thus, $\alpha(n-1) = 1 + 2 + \cdots + (k-1) + k_2 + \cdots + k_r$. Hence $\alpha(n-1) = \alpha(n) + \binom{k}{2} - k$.

Theorem 4.2.2. For a given n, we have

$$N_\epsilon(n) = \begin{cases} \frac{1}{2}A(n-1) \displaystyle\sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is messy}}} 2^{k-\binom{k}{2}} & \text{if } n \text{ is even, where } k = \mathrm{ord}_2 b, \\ 2^{\nu(n-1)-1}A(n-1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose n is odd and a and b are positive integers such that $a + b = n$. So there are two cases we need to consider.
Case 1: If a is odd and b is even, then $N_\epsilon(a,b) = 0$. This is because $\mathrm{sum}(a, b-1)$ is always messy.
Case 2: If a is even and b is odd, then $\mathrm{sum}(a, b-1)$ is neat if $\mathrm{bin}(a)$ and $\mathrm{bin}(b-1)$ are

disjoint. The number of choices for a and b is $2^{\nu(n-1)}$. Hence,

$$\begin{aligned}
N_\epsilon(n) &= \sum_{\substack{a+b=n \\ \text{sum}(a,b-1) \text{ is neat}}} \frac{1}{2}A(a)A(b) \\
&= \sum_{\substack{a+b=n \\ \text{sum}(a,b-1) \text{ is neat}}} \frac{1}{2}A(a)A(b-1) \\
&= \sum_{\substack{a+b=n \\ \text{sum}(a,b-1) \text{ is neat}}} \frac{1}{2}A(n-1) \\
&= 2^{\nu(n-1)-1}A(n-1).
\end{aligned}$$

Suppose n is even and a and b are positive integers as above.

Case 1: If a and b are both odd, then $N_\epsilon(a,b) = 0$. This is because $\text{sum}(a-1,b-1)$ is neat when $\text{sum}(a,b-1)$ is neat.

Case 2: Let a and b be both even. Let $k = \text{ord}_2(b)$. By the previous proposition, we know if b is even, then $\alpha(b-1) = \alpha(b) + \binom{k}{2} - k$. Hence,

$$\begin{aligned}
N_\epsilon(n) &= \sum_{\substack{a+b=n \\ \text{sum}(a,b-1) \text{ is neat} \\ \text{sum}(a-1,b-1) \text{ is messy}}} \frac{1}{2}A(a)A(b) \\
&= \sum_{\substack{a+b=n \\ \text{sum}(a,b-1) \text{ is neat} \\ \text{sum}(a-1,b-1) \text{ is messy}}} \frac{1}{2}A(a)A(b-1)2^{k-\binom{k}{2}} \\
&= \frac{1}{2}A(n-1) \sum_{\substack{a+b=n \\ \text{sum}(a,b-1) \text{ is neat} \\ \text{sum}(a-1,b-1) \text{ is messy}}} 2^{k-\binom{k}{2}}.
\end{aligned}$$

From this and Lemma 3.2.4, we can easily calculate $N_{\text{sgn}^1}(n) = B(2n) - N_\epsilon(n)$. Recall $\tilde{B}(n) = \dfrac{B(n)}{A(n)}$.

Theorem 4.2.3. For a given n, we have the following cases.

1. If $n = 4k + 1$, then

$$N_{\mathrm{sgn}^0}(n) = A(n) \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a-2,b+2) \text{ is neat}}} \{2\tilde{B}(b+2) - \frac{1}{2}\}$$

$$+ \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a-1,b+1) \text{ is neat} \\ \mathrm{sum} a-2,b+2 \text{ is neat}}} A(a)A(b+2)\left\{\tilde{B}(b+2) + \tilde{B}(a) - \frac{1}{2}\right\}.$$

2. If $n = 4k + 2$, then

$$N_{\mathrm{sgn}^0}(n) = A(n) \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat}}} \left\{\tilde{B}(b+2) + \tilde{B}(a+2) - \frac{1}{2}\right\}.$$

3. If $n = 4k + 3$, then

$$N_{\mathrm{sgn}^0}(n) = A(n) \sum_{\substack{a+b=n-3 \\ \mathrm{sum}(a,b) \text{ is neat}}} \left\{\tilde{B}(b+3) + \tilde{B}(a+3) + \tilde{B}(b+2) + \tilde{B}(a+2) - 1\right\}.$$

4. If $n = 4k$, then

$$N_{\mathrm{sgn}^0}(n) = \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a+2,b-2) \text{ is neat}}} A(b)A(a+2)\{\tilde{B}(b) + \tilde{B}(a+2) - 1\}$$

$$+ \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a+2,b-2) \text{ is messy}}} A(n)\{2\tilde{B}(a+2) - \frac{1}{2}\}.$$

Proof. Let $n = 4k + 1$. The only non-zero contribution comes from conditions 3, 4, 5, 6 and 7 of Proposition 4.1.5. Each of these conditions corresponds to a unique 3-tuple (code) in the $(n-2)$nd row of the Pascal's triangle mod 2. For example, the third condition of Proposition 4.1.5 corresponds to the code 100. In this case, we are looking for the codes 100, 011, 111, 110 and 001. Hence,

$$
\begin{aligned}
N_{\text{sgn}^0}(n) = & \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat} \\ \text{sum}(a-1,b+1) \text{ is messy} \\ \text{sum}(a-2,b+2) \text{ is messy}}} \{B(b+2)A(a) - \frac{1}{2}A(a)A(b+2) + B(b+2)A(a)\} \\
& + \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat} \\ \text{sum}(a-1,b+1) \text{ is neat} \\ \text{sum}(a-2,b+2) \text{ is messy}}} \{B(b+2)A(a) - \frac{1}{2}A(a)A(b+2) + B(b+2)A(a)\} \\
& + \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat} \\ \text{sum}(a-1,b+1) \text{ is neat} \\ \text{sum}(a-2,b+2) \text{ is neat}}} \{B(a)A(b+2) - \frac{1}{2}A(a)A(b+2) + B(b+2)A(a)\} \\
& = A(n) \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat} \\ \text{sum}(a-2,b+2) \text{ is neat}}} \{2\tilde{B}(b+2) - \frac{1}{2}\} \\
& + \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat} \\ \text{sum}(a-1,b+1) \text{ is neat} \\ \text{sum} a-2,b+2 \text{ is neat}}} A(a)A(b+2) \left\{ \tilde{B}(b+2) + \tilde{B}(a) - \frac{1}{2} \right\}.
\end{aligned}
$$

Let $n = 4k+2$. The only non-zero contribution comes from conditions 3 and 7 of Proposition 4.1.5. Hence,

$$
\begin{aligned}
N_{\text{sgn}^0}(n) &= \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat}}} \{B(b+2)A(a) - \frac{1}{2}A(a)A(b+2) + B(a+2)A(b)\} \\
&= \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat}}} \{\tilde{B}(b+2)A(a)A(b+2) + \tilde{B}(a+2)A(b)A(a+2) - \frac{1}{2}A(a)A(b+2)\} \\
&= \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat}}} \{A(n)\tilde{B}(b+2) + A(n)\tilde{B}(a+2) - \frac{1}{2}A(n)\} \\
&= A(n) \sum_{\substack{a+b=n-2 \\ \text{sum}(a,b) \text{ is neat}}} \left\{ \tilde{B}(b+2) + \tilde{B}(a+2) - \frac{1}{2} \right\}.
\end{aligned}
$$

Let $n = 4k+3$. For every 1 in the $(n-3)$nd row, the required codes occur exactly once.

Noting this and proceding as above, we have

$$\begin{aligned}
N_{\mathrm{sgn}^0}(n) = & \sum_{\substack{a+b=n-3 \\ \mathrm{sum}(a,b) \text{ is neat}}} \{B(b+3)A(a) - \frac{1}{2}A(a)A(b+3) + B(b+2)A(a+1) + B(a+2)A(b+1) \\
& - \frac{1}{2}A(a+2)A(b+1) + B(a+3)A(b)\} \\
& = \sum_{\substack{a+b=n-3 \\ \mathrm{sum}(a,b) \text{ is neat}}} \{\tilde{B}(b+3)A(a)A(b+3) + \tilde{B}(b+2)A(a+1)A(b+2) \\
& + \tilde{B}(a+2)A(b+1)A(a+2) + \tilde{B}(a+3)A(b)A(a+3) - A(n)\} \\
& = A(n) \sum_{\substack{a+b=n-3 \\ \mathrm{sum}(a,b) \text{ is neat}}} \left\{\tilde{B}(b+3) + \tilde{B}(a+3) + \tilde{B}(a+2) + \tilde{B}(b+2) - 1\right\}.
\end{aligned}$$

Similarly for $n = 4k$, we have

$$\begin{aligned}
N_{\mathrm{sgn}^0}(n) = & \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a+2,b-2) \text{ is neat}}} B(a+2)A(b) + B(b)A(a+2) - A(a+2)A(b) \\
& + \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a+2,b-2) \text{ is messy}}} B(a+2)A(b) + B(a+2)A(b) - \frac{1}{2}A(b)A(a+2) \\
& = \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a+2,b-2) \text{ is neat}}} \left\{\tilde{B}(a+2) + \tilde{B}(b) - 1\right\} A(b)A(a+2) \\
& + \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a+2,b-2) \text{ is messy}}} \left\{2\tilde{B}(a+2) - \frac{1}{2}\right\} A(n).
\end{aligned}$$

Using the Theorems 4.2.2 and 4.2.3, we compute the following values:

n	$N_1(n)$	$N_{\text{sgn}^0}(n)$	$N_\epsilon(n)$	$N_{\text{sgn}^1}(n)$
1	1	0	1	0
2	1	1	1	2
3	2	4	2	2
4	4	4	4	8
5	8	20	4	4
6	33	8	8	16
7	46	32	16	16
8	69	28	28	60
9	116	168	8	8
10	417	16	16	32
11	624	64	32	32
12	909	64	64	128
13	1322	320	64	64
14	2153	128	128	256
15	2932	512	256	256

4.3 Proof of Theorem 0.0.1

Proposition 4.3.1. Let n be a positive integer. If $n \equiv 3 \pmod 4$, then $\tilde{B}(n) = 1$ and

$$\sum_{\substack{a+b=n-3 \\ \text{sum}(a,b) \text{ is neat}}} \tilde{B}(b+3) = 2^{\nu(n-3)}\tilde{B}(n).$$

Proof. Let $n = 4k+3$, then $n-3 = 4k$. Let a and b be positive integers such that $a+b = n-3$ and $\text{sum}(a,b)$ is neat, then a and b are both forced to be multiples of 4. Recall $\tilde{B}(n)$ is a function of k_1 and ϵ. Thus $\tilde{B}(b+3) = \tilde{B}(n) = 1$.

By [8, page 56, 9.6.2], we have

$$p(n) \geq \frac{n^{k-1}}{k!(k-1)!} \tag{4.3}$$

for all n and $k \le n$. Stirling approximation gives

$$k! \le \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} e^{\frac{1}{12k}},$$

and so

$$(k!)^2 \le (2\pi) k^{2k+1} e^{-2k} e^{\frac{1}{6k}}. \tag{4.4}$$

Lemma 4.3.2. We have

$$p(n) \ge (2\pi)^{-1} e^{2\sqrt{n}-2-(\frac{1}{6\sqrt{n}})} n^{-\frac{3}{2}}.$$

Proof. By (4.3), we have

$$p(n) \ge \frac{n^{[\sqrt{n}]-1}}{[\sqrt{n}]!^2}.$$

By (4.4), we have

$$\begin{aligned}([\sqrt{n}]!)^2 &\le (2\pi)[\sqrt{n}]^{2[\sqrt{n}]+1} e^{-2[\sqrt{n}]} e^{\frac{1}{6[\sqrt{n}]}} \\ &\le (2\pi) n^{[\sqrt{n}]+\frac{1}{2}} e^{-2[\sqrt{n}]+\frac{1}{6[\sqrt{n}]}}\end{aligned}$$

The lemma follows.

Lemma 4.3.3. We have

$$\frac{1}{2}n < A(n) \le n^{\frac{1}{2}(\log_2 n+1)}.$$

Proof. We have

$$\begin{aligned}\log_2 n - 1 < [\log_2 n] \le \alpha(n) &\le 1 + 2 + \cdots + [\log_2 n] \\ &= \frac{[\log_2 n]([\log_2 n]+1)}{2} \\ &\le \frac{1}{2}\log_2 n(\log_2 n + 1)\end{aligned}$$

Since $A(n) = 2^{\alpha(n)}$, this gives the lemma.

Proposition 4.3.4. For $n \ge 546$ we have

$$p(n) \ge 5nA(n).$$

Proof. By the previous, it is enough to show that

$$(2\pi)^{-1}e^{2\sqrt{n}-2-(\frac{1}{6\sqrt{n}})}n^{-\frac{3}{2}} \geq 5n^{\frac{1}{2}(\log_2 n+1)}.$$

Taking logs, this equivalent to showing

$$2\sqrt{n} \geq \log(2\pi) + 2 + \frac{1}{6\sqrt{n}} + (\log n)\left(\frac{1}{2}\log_2 n + 2\right) + \log(5).$$

This is evidently true for $n \geq 546$.

Note: Computation shows that this inequality is true $n \geq 64$.

Corollary 4.3.5. For a positive integer $n \geq 546$, let $P_2(n)$ denote the number of bipartitions of n. Then

$$P_2(n) > 2P(N) > 10nA(n).$$

Lemma 4.3.6. Let $n \geq 1$.

1. $\nu(n) \leq \log_2 n + 1$. So $2^{\nu(n)} \leq 2n$.
2. If $n+1$ is not a power of 2, then $\nu(n) \leq \log_2 n$. Thus $2^{\nu(n)} \leq n$.

Proposition 4.3.7. For $n \geq 546$, the following holds true:

1. If $n \equiv 1 \pmod 4$, then $\dfrac{P_2(n)}{2^{\nu(n)}A(n)\{\tilde{B}(n)+\frac{1}{4}\}} > 5$.
2. Otherwise $\dfrac{P_2(n)}{2^{\nu(n-1)}A(n-1)} > 5$.

Proof. Case 1: Let $n \equiv 1 \pmod 4$. We know $B(n) \leq \frac{5}{2}A(n-2)$, then

$$\begin{aligned} 2^{\nu(n)}A(n)\left\{\tilde{B}(n)+\frac{1}{4}\right\} &= 2^{\nu(n)}\left\{B(n)+\frac{1}{4}A(n)\right\} \\ &< 2^{\nu(n)-2}\{10A(n-2)+A(n)\}. \end{aligned}$$

Thus,

$$\begin{aligned}\frac{2^{\nu(n)}A(n)\left\{\tilde{B}(n)+\frac{1}{4}\right\}}{P_2(n)} &< \frac{2^{\nu(n)-2}\{10A(n-2)+A(n)\}}{P_2(n)}\\ &= \frac{2^{\nu(n)-2}\times 10A(n-2)}{P_2(n)}+\frac{2^{\nu(n)-2}A(n)}{P_2(n)}\\ &< \frac{2^{\nu(n)-2}\times 10A(n-2)}{10(n-2)A(n-2)}+\frac{2^{\nu(n)-2}A(n)}{10nA(n)}\\ &< \frac{2^{\nu(n)-2}}{(n-2)}+\frac{2^{\nu(n)-2}}{10n}.\end{aligned}$$

Now using Lemma 4.3.6

$$\begin{aligned}\frac{2^{\nu(n)}A(n)\left\{\tilde{B}(n)+\frac{1}{4}\right\}}{P_2(n)} &< \frac{n}{4(n-2)}+\frac{1}{40}<\frac{11}{40}\\ &< \frac{1}{3}.\end{aligned}$$

Hence the result follows.

Case 2: We know $P_2(n) > P_2(n-1) \geq 10(n-1)A(n-1)$. It follows from Lemma 4.3.6

$$\frac{P_2(n)}{2^{\nu(n-1)}A(n-1)} > \frac{10(n-1)}{2^{\nu(n-1)}} > 5.$$

Proof. (Theorem 0.0.1) We have already proven $N_{\mathrm{sgn}^1}(n) \geq N_\epsilon(n)$, with equality holding iff n is odd. Let $n = 4k+3$. We know,

$$N_{\mathrm{sgn}^1}(n) = 2^{\nu(n-1)-1}A(n-1)$$

and

$$\begin{aligned}N_{\mathrm{sgn}^0}(n) &= A(n)\sum_{\substack{a+b=n-3\\ \mathrm{sum}(a,b)\text{ is neat}}}\left\{\tilde{B}(b+3)+\tilde{B}(a+3)+\tilde{B}(b+2)+\tilde{B}(a+2)-1\right\}\\ &= A(n)\sum_{\substack{a+b=n-3\\ \mathrm{sum}(a,b)\text{ is neat}}}\left\{1+1+\frac{1}{2}+\frac{1}{2}-1\right\}\\ &= 2^{\nu(n-3)+1}A(n).\end{aligned}$$

As $n = 4k+3$, we have $A(n) = A(n-1)$ and $2^{\nu(n-1)} = 2^{\nu(n-3)+1}$. Hence,

$$\frac{N_{\mathrm{sgn}^0}(n)}{N_{\mathrm{sgn}^1}(n)} = 2.$$

From Proposition 4.3.7, it follows that

$$\begin{aligned}\frac{N_1(n)}{N_{\mathrm{sgn}^0}(n)} &= \frac{P_2(n)}{N_{\mathrm{sgn}^0}(n)} - \frac{2N_\epsilon(n)}{N_{\mathrm{sgn}^0}(n)} - 1 \\ &= \frac{P_2(n)}{N_{\mathrm{sgn}^0}(n)} - 2 > 1.\end{aligned}$$

Let $n = 4k+2$. We know

$$N_\epsilon(n) = \frac{1}{2}A(n-1) \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is messy}}} 2^{k-\binom{k}{2}},$$

where $k = \mathrm{ord}_2(b)$. In our case, $k = 1$. Hence,

$$N_\epsilon(n) = \frac{A(n-1)2^{\nu(n-1)}}{2}$$

Recall,

$$N_{\mathrm{sgn}^0}(n) = A(n) \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat}}} \left\{ \tilde{B}(b+2) + \tilde{B}(a+2) - \frac{1}{2} \right\}.$$

The conditions on a and b force $a \equiv 2(\mod 4)$ and $b \equiv 2(\mod 4)$. Hence,

$$N_{\mathrm{sgn}^0}(n) = 2^{\nu(n-2)-1}A(n).$$

The equality follows as $A(n) = 2A(n-1)$ and $\nu(n-1) = \nu(n-2)+1$.

$$N_{\mathrm{sgn}^1}(n) = \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is messy}}} \frac{1}{2}A(a)A(b) + \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is neat}}} A(a)A(b)$$
$$= \frac{1}{2}A(n-1) \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is messy}}} 2^{k-\binom{k}{2}} + A(n-1) \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is neat}}} 2^{k'-\binom{k'}{2}}.$$

In our case, $k = 1$ and $k' = 0$. Hence,

$$N_{\mathrm{sgn}^1}(n) = \frac{A(n-1)2^{\nu(n)}}{2} + A(n-1)2^{\nu(n)-1}$$
$$= 2N_\epsilon(n).$$

From Proposition 4.3.7, it follows that

$$\frac{N_1(n)}{N_{\mathrm{sgn}^1}(n)} = \frac{P_2(n)}{N_{\mathrm{sgn}^1}(n)} - \frac{2N_\epsilon(n)}{N_{\mathrm{sgn}^1}(n)} - 1$$
$$= \frac{P_2(n)}{N_{\mathrm{sgn}^1}(n)} - 2 > 1.$$

Let $n = 4k$, $X = \{(\alpha, \beta) \models n \mid \det \circ \rho_{\alpha,\beta} = \epsilon\}$ and $Y = \{(\alpha, \beta) \models n \mid \det \circ \rho_{\alpha,\beta} = \mathrm{sgn}^0\}$. Define ι, a map from X to Y, as $\iota(\alpha, \beta) = (\beta', \alpha')$. Let $X_0 = \{x \in X \mid \iota(x) \in X\}$ and similarly $Y_0 = \{y \in Y \mid \iota(y) \in Y\}$. The elements of X_0 are (α, β), such that there are two [1]'s appearing symmetrically in the first nontrivial row of the $2-$core tower, and $\mid \alpha \mid$ and $\mid \beta \mid$ must also satisfy the conditions listed in Corollary 1.4. The elements of Y_0 are (α, β) such that there are two [1]'s appearing in the same subtree in the first nontrivial row of the $2-$core tower. If the [1]'s occur in the ith row, then

$$|X_0| = |Y_0| = \binom{2^{i-1}}{2} \times \#\{\text{possible } 2^i\text{-quotients for partitions of } 2n\}.$$

Thus $|X| = |Y|$, i.e. when $n = 4k$, $N_\epsilon(n) = N_{\mathrm{sgn}^0}(n)$. As for $N_{\mathrm{sgn}^1}(n)$, we have

$$\begin{aligned}
N_{\mathrm{sgn}^1}(n) &= \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is messy}}} \frac{1}{2}A(a)A(b) + \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is neat}}} A(a)A(b) \\
&= \frac{1}{2}A(n-1) \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is messy}}} 2^{k-\binom{k}{2}} + A(n-1) \sum_{\substack{a+b=n \\ \mathrm{sum}(a,b-1) \text{ is neat} \\ \mathrm{sum}(a-1,b-1) \text{ is neat}}} 2^{k'-\binom{k'}{2}} \\
&< \frac{1}{2}2^{\nu(n-1)-1}A(n-1) \times 2 + 2^{\nu(n-1)-1}A(n-1) \\
&= 2^{\nu(n-1)}A(n-1).
\end{aligned}$$

From Proposition 4.3.7, it follows that

$$\begin{aligned}
\frac{N_1(n)}{N_{\mathrm{sgn}^1}(n)} &= \frac{P_2(n)}{N_{\mathrm{sgn}^1}(n)} - \frac{2N_\epsilon(n)}{N_{\mathrm{sgn}^1}(n)} - 1 \\
&> \frac{(n-1)P(n)}{N_{\mathrm{sgn}^1}(n)} - 2 \\
&> 1.
\end{aligned}$$

Similarly let $n = 4k+1$, we have

$$N_{\mathrm{sgn}^1}(n) = 2^{\nu(n-1)-1}A(n-1),$$

and

$$\begin{aligned}
N_{\mathrm{sgn}^0}(n) = A(n) &\sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a-2,b+2) \text{ is neat}}} \{2\tilde{B}(b+2) - \frac{1}{2}\} \\
&+ \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a-1,b+1) \text{ is neat} \\ \mathrm{sum} a-2,b+2 \text{ is neat}}} A(a)A(b+2)\left\{\tilde{B}(b+2) + \tilde{B}(a) - \frac{1}{2}\right\} \\
&> A(n) \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a-2,b+2) \text{ is neat}}} \{2\tilde{B}(b+2) - \frac{1}{2}\}.
\end{aligned}$$

We know $\tilde{B}(n) \geq \frac{1}{2}$, hence $N_{\mathrm{sgn}^0}(n) > A(n)2^{\nu(n)-2}$.

As $n = 4k+1$, we have $A(n) = A(n-1)$ and $2^{\nu(n-1)-1} = 2^{\nu(n)-2}$. Thus

$$\frac{N_{\mathrm{sgn}^0}(n)}{N_{\mathrm{sgn}^1}(n)} > 1.$$

For the second part of the inequality, we need a greater bound on $N_{\mathrm{sgn}^0}(n)$. Hence,

$$\begin{aligned}
N_{\mathrm{sgn}^0}(n) = A(n) & \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a-2,b+2) \text{ is neat}}} \{2\tilde{B}(b+2) - \frac{1}{2}\} \\
&+ \sum_{\substack{a+b=n-2 \\ \mathrm{sum}(a,b) \text{ is neat} \\ \mathrm{sum}(a-1,b+1) \text{ is neat} \\ \mathrm{sum}(a-2,b+2) \text{ is neat}}} A(a)A(b+2)\left\{\tilde{B}(b+2) + \tilde{B}(a) - \frac{1}{2}\right\} \\
&< A(n)2^{\nu(n)-1}\left\{\tilde{B}(n) - \frac{1}{2}\right\} + A(n)2^{\nu(n)-1}\{1 + \frac{1}{2} - \frac{1}{2}\} \\
&= A(n)2^{\nu(n)}\{\tilde{B}(n) + \frac{1}{4}\}.
\end{aligned}$$

From Proposition 4.3.7, it follows that

$$\begin{aligned}
\frac{N_1(n)}{N_{\mathrm{sgn}^0}(n)} &= \frac{P_2(n)}{N_{\mathrm{sgn}^0}(n)} - \frac{2N_\epsilon(n)}{N_{\mathrm{sgn}^0}(n)} - 1 \\
&= \frac{P_2(n)}{N_{\mathrm{sgn}^0}(n)} - 2 > 1.
\end{aligned}$$

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