Classical Observables from Feynman Amplitudes

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Classical Observables from Feynman Amplitudes towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Aviral Aggarwalat Indian Institute of Science Education and Research under the supervision of Alok Laddha, Professor, Department of Physics , during the academic year 2022-2023.

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I dedicate this thesis to those who lost their loved ones in Covid-19 Pandemic

Declaration

I hereby declare that the matter embodied in the report entitled Classical Observables from Feynman Amplitudes are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Alok Laddha and the same has not been submitted elsewhere for any other degree.

Aviral

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Abstract

We study KMOC formalism developed to calculate classical observables from amplitudes in Quantum Field Theory. Amplitudes and S-matrix carry the information of interaction of any scattering process and thus the classically observed quantities should be attainable from them .We try to show some new calculations regarding the angular momentum impulses via this formalism which led to some interesting results. We studied the non-conservation of angular momentum classically and understand the phenomenon.We studied Faddev Kulish states that account for long range interactions of states and try to apply them to KMOC formalism.We then studied the seminal papers by Weinberg on soft photons and gravitons and look at the idea of these states arising from soft photons.Some results regarding the validity of classical soft photon was also reviwed in context of angular momentum radiated during collision.We then also study the S-matrix and its infrared divergence problem . We saw how the dressings of Faddev Kulish states cancel these divergences in the S-Matrix. The dressings are known to be coherent states of photons and the equivalence between S-matrix and these states has been explored.

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Introduction

Importance of scattering interactions between particles be it the interactions in the Large Hadron Coliiders or Gravitional wave detections of Black hole collision and nuetron star mergers, has increased in the light of recent discoveries by LIGO and LHC.
These new discoveries makes it important to study two body scattering problem and generate a theoretical results to understand and analyze the vast amounts of data.
Traditionally these were done using Post Newtonian solutions[5] , Effective field theory approach[7], numerical relativity [28].
We will primarily focus on studying these collisons using the scattering amplitude methods largely following the formalism developed in [22][10][26]. Scattering amplitudes have been studied to obtain the potential between two bodies in the past [8].The importance of Loop level diagrams in this context is also well known from [21].Starting with a paper [21] it proved that one must consider the loop level diagrams too in order to calculate classical quantities like stress-energy tensor , especially when we have two mass-less fields coupling.
This technique has clear advantages over the traditional approaches which have been used up untill now. Firstly, the gravitational scattering can be studied easily using the graviational amplitudes , these amplitudes were shown to be product of two Yang-Mills amplitude first introduced in [30][3]. This connection is called the double copy. Secondly, the classical approaches to calculating these observables has several issues like the conservation of momentum loss due to radiation can not be taken care of by using lorentz force laws. One must include the Lorrentz - Abraham force ([14]) to study system completely. This force has to be put by hand in the dynamical equations which comes with its own baggage of problems like causality violations and runaway solutions. This technique should be the solutions to these problems , and we will see how in this thesis.
Although the calculations for several observable were done in the seminal work by [22]. We shall look at a few other observables which were not studied using this technique like

the angular momentum impulse. We shall re-discover the problem of non conservation of angular momentum using these calculations which though known in many forms has been overlooked in the past. Analyzing the issue with the results will take us to delve into several other closely knitted concepts. One among these will be the problem of Infrared divergence of S-matrix , which we find is the major criminal for the discrepency in the results. These divergences has been shown while calculating the amplitudes. These divergences were fixed in the past by Faddev and Kulish by considering the possiblity of interacting asymptotic states. These dressed states make the S matrix finite and in principle solves the issue of angular momentum. We shall explore the ideas behind this approach .

Soft theorems are yet another important aspects which deals with the idea of infrared divergences first introduced by Wienberg in [35][32]. These are key to connecting the infrared divergences of amplitudes with the classical trajectory of the particles.

The Thesis is organised in the following way - In the first chapter we review the methodology of KMOC formalism. We shall back those results by classical calculations in second chapter . In the third Chapter we shall introduce Faddev Kulish states and look at their effect on our previous calculations and the ambiguity of IR divergences of S matrix. In the fourth chapter we will study the basics of soft theorems and look at their connection to IR divergence of S matrix , We shall look explicity at how these divergences are dealt with by using Faddev Kulish states [20].

Chapter 1

Preliminaries

We will overview the basic definitions and notations that will be helpful throughout the thesis. Let us first look at restoring the powers of h that are often emitted while doing the quantum field theory calculations. We define the quantity called wavenumber, $\bar{p}$ of any particle associated with its momentum p-

$$\bar{p} = \frac{p}{h}$$

to restore the power of h in amplitude .When $h \neq 1$ the mass dimension of masses and momenta of the particle are unchanged, nor is their change in dimension of polarisation vector .
Dimensionless coupling constant in electrodynamics and gravity scales as

$$k/\sqrt{h}$$

We will see how important this scaling becomes when we want to ensure the that the observables are classical in the subsequent sections.
Mode expansion formulas for the various fields and the commuatation relations are -
For complex scalar fields-

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{+ip\cdot x},\right)$$
$$\phi^\dagger(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(b(\mathbf{p})e^{-ip\cdot x} + a^\dagger(\mathbf{p})e^{+ip\cdot x},\right)$$

The creation and annihilation operators in the mode expansion , satisfy the commutation relations

$$\begin{aligned}
\left[b_{\vec{p}}, b_{\vec{q}}^{\dagger}\right] &= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \\
\left[c_{\vec{p}}, c_{\vec{q}}^{\dagger}\right] &= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \\
[b_{\vec{p}}, b_{\vec{q}}] = [c_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}] &= \left[b_{\vec{p}}, c_{\vec{q}}^{\dagger}\right] = 0
\end{aligned}$$

The Dirac field expansion is,

$$\begin{aligned}
\psi(x) &= \int d^3p \left(c_r(\mathbf{p})u_r(\mathbf{p})e^{tpx} + d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{-\mathrm{t}p}\right), \\
\bar{\psi}(x) &= \int d\,\tilde{p} \left(c^{\dagger}(\mathbf{p})d_r(\mathbf{p})e^{\mathrm{t}\boldsymbol{x}} + d_r(\mathbf{p})\eth_r(\mathbf{p})e^{-\mathrm{Ap}x}\right),
\end{aligned}$$

where $c^{\dagger}$(p) is the electron creation operator and $d^{\dagger}$(p) is the positron creation operator, while v_r(p) and v_r(p) are constant spinors satisfying the equations

$$\begin{aligned}
(p+m)u_r(p) &= v_r(p)(p+m) = 0 \\
(-p+m)v_r(\mathbf{p}) &= v_r(\mathbf{p})(-p+m) = 0 \\
v_r(\mathbf{p})u_n(\mathbf{p}) &= -v_r(\mathbf{p})v_r(\mathbf{p}) = \delta_{rx} \\
u_r(\mathbf{p})v_x(\mathbf{p}) &= 0.
\end{aligned}$$

The anticommutation relations for the creation and annihilation operators are

$$\begin{aligned}
\left\{c(\mathbf{p}), c^3\left(\mathbf{p}'\right)\right\} &= (2\pi)^3\left(2\omega_{\mathbf{p}}\right)\delta^5\left(\mathbf{p}-\mathbf{p}'\right), \\
\left\{d(\mathbf{p}), d^{\dagger}\left(\mathbf{p}'\right)\right\} &= (2\pi)^3\left(2\omega_{\mathbf{p}}\right)\delta^5\left(\mathbf{p}-\mathbf{p}'\right).
\end{aligned}$$

For Maxwell field-

$$A_\mu(x) = \int \widetilde{d^k k}\left(a_\mu(\mathbf{k})e^{ikx} + a_\mu^{\dagger}(\mathbf{k})e^{-ikx}\right).$$

1.0.1 Notations

In our notations $\hat{d}p$ implicitly has a factor of 2π; in general, $\hat{d}^n p$ is defined by

$$\hat{d}^n p \equiv \frac{d^n p}{(2\pi)^n}$$

We take only the positive-energy solutions of the delta functions of $p_i^2 - m_i^2$,this is indicated by the $(+)$ superscript in $\hat{\delta}^{(+)}$ whiile doing integration.

$$\hat{\delta}^{(+)}\left(p^2 - m^2\right) \equiv 2\pi\Theta\left(p^0\right)\delta\left(p^2 - m^2\right)$$

We shorten the notation for on-shell integrals over Lorentz-invariant phase space as -,

$$d\Phi\left(p_i\right) \equiv \hat{d}^4 p_i \hat{\delta}^{(+)}\left(p_i^2 - m_i^2\right)$$

Wedge Product of two vectors is denotes by -

$$A^\mu \wedge B^\nu$$

which is short for anti-symmteric cross product.

$$A^\mu \wedge B^\nu = A^\mu \times B^\nu - B^\mu \times A^\nu$$

Another important notation we will encounter several times is

$$D = (p_1.p_2)^2 - (m_1 m_2)^2$$

Chapter 2

Review of KMOC formalism

In this chapter we will review the methodolody for calculating classical observables from the methods of Feynman amplitude shown in papers [22] [10] , this is now called KMOC formalism .It is based in -in formalism of two wavefunctions. Classical observables like impulse , radiation can be studied via the scattering amplitudes .This approach has huge advantages over classical methods which we shall point out as we discuss the formalism.
Let us start by looking at various components in the formalism in the sections of this chapter.

2.1 Incoming state

In any scattering experiment the incoming states are prepared in far past.They are described by $\phi_i(p_i)$ for the i^{th} particle in the momentum space. We require classical scattering of the particles therefore want $\phi_i(p_i)$ to have reasonably well defined momentum and position. This puts certain constraints on the kind of wavefunction we can assume which will be discussed in more detail in 2.5 .
For now, we take a general $\phi_i(p_i)$ and define the initial state, which is given by-

$$|\Psi>_{in}= \int d\phi(p_1)d\phi(p_2)\phi(p_1)\phi(p_2)e^{ib.p_1/h}|p_1p_2> \tag{2.1}$$

in momentum space.

Here $\phi_i(p_i)'s$ are as said the wavefunction of the classical particles , b^μ is the impact paramater, and it is integrated over on shell integration measure $d\phi(p_i)$ for both the particles look at 1.0.1 in the preliminaries for its defination and notation..
We should notice one thing that the wavefunction above has been written for both incoming particles relative to particle two . That is why we have a $e^{ib.p_1/h}$ in the expression. This will play an important conceptual role in our subsequent calculations.

2.2 Impulse on particle

To see how the formalism works, let us focus on calculating momentum impulse of the particles. Momentum impulse is the change in momentum of particle in any scattering. Assuming detectors at a sphere of a large radius as compared to the scattering region will measure the momentum of incoming and outgoing particle.
Let P_1^μ be the momentum operator for particle one.
The expectation value for outgoing momentum of particle one is given by -

$$< p_1^\mu >_{out} = _{out}< \psi|P_1^\mu|\psi >_{out} \tag{2.2}$$

Writing in terms of incoming states-

$$< p_1^\mu >_{out} = _{in}< \psi|S^\dagger P_1^\mu S|\psi >_{in} \tag{2.3}$$

'S' is the Scattering matrix for the two particle scattering which evolves the incoming state to outgoing state.
Expectation value of this momentum impulse is evaluated by the difference of the outgoing and incoming momentum.

$$< \Delta p_1^\mu > = \lim_{h\to 0} [_{in}< \Psi|S^\dagger P_1^\mu S|\Psi >_{in} - _{in} < \Psi|P_1^\mu|\Psi >_{in}] \tag{2.4}$$

Similarly one can find the impulse on particle two by replacing by momentum operator for particle two.
We can write the scattering matrix in terms of transfer matrix -

$$S = I + iT \tag{2.5}$$

Using the Unitarity property of S-matrix we get

$$< \Delta P^{\mu} >=< \psi|i[P,T]|\psi > + < \psi|T^{\dagger}[P,T]|\psi > \tag{2.6}$$

The first term in the above equation will be called I_1^{μ} and second I_2^{μ}.
We will substitute our wavefunction from 2.1 to 2.6 and try to get the impulse in terms of amplitude.
We get -

$$\begin{aligned} I_{(1)}^{\mu} = \int d\Phi\left(p_1\right) d\Phi\left(p_2\right) \hat{d}^4 q \hat{\delta}\left(2p_1 \cdot q + q^2\right) \hat{\delta}\left(2p_2 \cdot q - q^2\right) \Theta\left(p_1^0 + q^0\right) \Theta\left(p_2^0 - q^0\right) \\ \times e^{-ib\cdot q/\hbar} \phi_1\left(p_1\right) \phi_1^*\left(p_1 + q\right) \phi_2\left(p_2\right) \phi_2^*\left(p_2 - q\right) \\ \times iq^{\mu} A\left(p_1 p_2 \rightarrow p_1 + q, p_2 - q\right). \end{aligned} \tag{2.7}$$

$$\begin{aligned} I_{(2)}^{\mu} = \textstyle\sum_X \int \prod_{i=1,2} \quad & d\Phi\left(p_i\right) \hat{d}^4 w_i \hat{d}^4 q \hat{\delta}\left(2p_i \cdot w_i + w_i^2\right) \Theta\left(p_i^0 + w_i^0\right) \\ & \times \hat{\delta}\left(2p_1 \cdot q + q^2\right) \hat{\delta}\left(2p_2 \cdot q - q^2\right) \Theta\left(p_1^0 + q^0\right) \Theta\left(p_2^0 - q^0\right) \\ & \times \phi_1\left(p_1\right) \phi_2\left(p_2\right) \phi_1^*\left(p_1 + q\right) \phi_2^*\left(p_2 - q\right) \\ & \times e^{-ib\cdot q/\hbar} w_1^{\mu} \hat{\delta}^{(4)}\left(w_1 + w_2 + r_X\right) \\ & \times A\left(p_1, p_2 \rightarrow p_1 + w_1, p_2 + w_2, r_X\right) \\ & \times A^*\left(p_1 + q, p_2 - q \rightarrow p_1 + w_1, p_2 + w_2, r_X\right) \end{aligned} \tag{2.8}$$

Here $A(p_1 p_2 \rightarrow p_3, p_4)$ is the transition matrix element between incoming $|p_1, p_2>$ and outgoing $|p_3, p_4>$ states written as $< p_3, p_4|T|p_1, p_2 >$, q_i is the 'momentum mismatch' given by $q_i = p_i' - p_i$, ω is the loop momentum in I_2^{μ} and intermediate states are summed over in expression two as well.
We will skip the derivation of these results in the interest of time and space , (look at original paper for [22] for details) . One must look at the definition of integration measure from 1.0.1. Displaying the above amplitudes using Feynman diagram would look like.

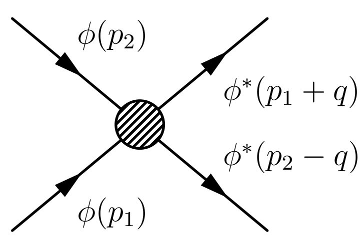

Figure 2.1: $A\left(p_1 p_2 \rightarrow p_1 + q, p_2 - q\right)$

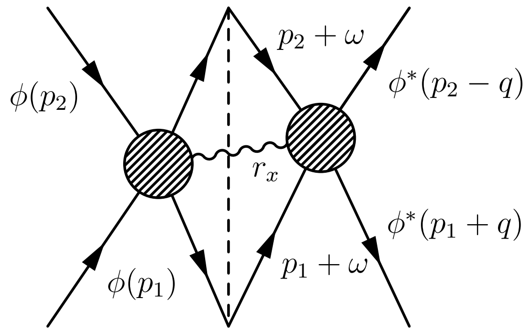

Figure 2.2: $A\left(p_1, p_2 \rightarrow p_1 + w_1, p_2 + w_2, r_X\right) \times A^*\left(p_1 + q, p_2 - q \rightarrow p_1 + w_1, p_2 + w_2, r_X\right)$ for I_2^μ

2.3 Radiation and Conservation of momentum

In the previous section, we saw how impulses is written in terms of amplitudes . We will now see how radiation is accounted for in this formalism. Radiation emitted is a well defined observable as it can be expressed(we will show how) in terms of Feynman amplitudes .
Let K^μ be the momentum of radiated particle. Following similar approach to calculating the impulse we find, that the expectation value of momentum for the radiated particles is

$$R^\mu =< K^\mu >=_{in}< \psi|T^\dagger K^\mu T|\psi >_{in} \tag{2.9}$$

as there is no incoming radiation there is only one term unlike expression 2.6.
Inserting complete set of states in between gives-

$$\begin{aligned} R^u &= \sum_X \int d\Phi(k) d\Phi\left(r_1\right) d\Phi\left(r_2\right) \left\langle\psi\left|T^\dagger\right| k r_1 r_2 X\right\rangle k_X^u \left\langle k r_1 r_2 X|T|\psi\right\rangle \\ &= \sum_X \int d\Phi(k) d\Phi\left(r_1\right) d\Phi\left(r_2\right) k_x^\mu \left|\left\langle k r_1 r_2 X|T|\psi\right\rangle\right|^2 \end{aligned}$$

In this above expression, X can be empty too (which we shall use later), and k_X^u is the sum of momentum of messengers andk^μ and the momenta of any intermediate state X.The phase space integral over k accounts for the sum over the helicities too.
Substituting the waveform of initial state from 2.1, we find that the expectation value of the

radiated momentum is given by,

$$
\begin{aligned}
& R^u=\sum_X \int d\Phi(k) d\Phi\left(r_1\right) d\Phi\left(r_2\right) k_x^\mu \mid \int d\Phi\left(p_1\right) d\Phi\left(p_2\right) e^{b \cdot p_1 / h'} \phi_1\left(p_1\right) \phi_2\left(p_2\right) \\
& \times \mid A\left(p_1, p_2 \rightarrow r_1, r_2, k, r_X\right) \delta^{4)}\left(p_1+p_2-r_1-r_2-k-r_X\right)\Big|^2 \\
& =\sum_X \int d\Phi(k) \prod_{1=1,2} d\Phi\left(r_1\right) d\Phi\left(p_i\right) d\Phi\left(p_1'\right) \phi_k\left(p_4\right) \phi_1^*\left(p_2'\right) k_X^\mu e^{d\left(p_1-p_1'\right) b} \\
& \times A\left(p_1, p_2 \rightarrow r_1, r_2, k, r_X\right) \hat{\delta}^{(4)}\left(p_1+p_2-r_1-r_2-k-r_X\right) \\
& \times A^*\left(p_1', p_2' \rightarrow r_1, r_2, k, r_X\right) \hat{\delta}^{(4)}\left(p_1'+p_2-r_1-r_2-k-r_X\right) .
\end{aligned}
\tag{2.10}
$$

We yet again introduced momentum transfer , $q_1=p_1'-p_1$, and change the variable of integrals from p_1' to integrals over the q_{r} as in previous sections .

$$
\begin{aligned}
R^\mu=\sum_X \int d\Phi(k) \prod_{t=1,2} & d\Phi\left(r_1\right) d\Phi\left(p_t\right) d^4 q \phi_1\left(p_1\right) \phi_2\left(p_2\right) \phi_1^*\left(p_1+q\right) \phi_2^*\left(p_2-q\right) \\
& \times \hat{\delta}\left(2 p_1 \cdot q+q^2\right) \hat{\delta}\left(2 q . p_2-q^2\right) \Theta\left(p_1^0+q^0\right) \Theta\left(p_2^0-q^0\right) \\
& \times k_X^u e^{-4t-q/1} \hat{\delta}^{(4)}\left(p_1+p_2-r_1-r_2-k-r_x\right) \\
& \times A\left(p_1, p_2 \rightarrow r_1, r_2, k, r_X\right) A^*\left(p_1+q, p_2-q \rightarrow r_1, r_2, k, r_X\right) .
\end{aligned}
$$

Although the main focus of this thesis will be calculating the impulse we will realise that radiation emitted can not be taken for granted and is an important aspects of these kinds of calculations as it helps in conservation of momentum. We will revisit the radiation in later sections.

2.4 Conservation of momentum

We will now see how the expectation value of radiated momentum is important for total momentum conservation in the scattering process. In this formalism, we do not require to take both Lorentz and ALD force by hand as it is already taken into account .This is one of the biggest strengths of KMOC formalism over classical calculations.
The sum of momentum impulse for both particles is (using 2.6)

$$< \Delta p_1^\mu > + < \Delta p_2^\mu > = < \psi | i[P_1 + P_2, T] | \psi > + < \psi | T^\dagger [P_1 + P_2, T] | \psi > \tag{2.11}$$

It is clear that the total momentum is time independent, written as -

$$[\Sigma P_i, T] = 0 \tag{2.12}$$

The first term in 2.11 would hence vanish . Which is justified by the fact that the first term only accounts for the exchange of momentum in between the particles .
The second term corresponds to radiation . Taking only other momentum in the system to be the radiated field.
We know from the previous equation that

$$[P_1 + P_2 + K, T] = 0 \tag{2.13}$$

hence

$$< \Delta P_1 > + < \Delta P_1 > =_{in} < \psi | T \dagger K^\mu T | \psi >_{in} = -R^\mu \tag{2.14}$$

This conservation of impulse is derived independent of the order at which the calculation is being done.
We should note here that the radiation for electromagnetic case takes place at $O(e^6)$ only (proved in 3.3), before that there is no momentum radiated in the fields, the two particles just exchange some momentum which we will calculate in 2.6.

2.5 Classical wavefunction

In this section we will discuss the various properties and constraints on our wavefunction for it to be eligible for KMOC formalism. We want the expectation value of the impulse to reach classical value as we take $h \to 0$.
We typically need two properties from our wavefunction . As they will be defined in the momentum space we want to spread of the wavefunction to not be very large , because we don't want the interaction with other particle to peer into the wavepacket's quantum properties.
The spread in position space of the particle is given by l_w. Assuming l_c to be the compton wavelength of the particle. For a non relativistic wavefunction we take a wavepacket with

minimum uncertainty given by

$$exp(\frac{-\boldsymbol{p}^2}{2hml_c/l_w^2}) = exp(\frac{-\boldsymbol{p}^2}{2m^2l_c^2/l_w^2}) \tag{2.15}$$

This suggests for relativistic wavepacket the dimensionless ratio

$$\zeta = (\frac{l_c}{l_w})^2$$

to be the parameter that takes us to the classical limit. The wavefunction will become sharply peaked as we reach the classical limit with peak corresponding to the classical value of momentum $p_i = m_i u_i$.

Looking the the expression for impulse we figure out that the particle state with momentum p and its conjugate should both represent the particle in the classical limit and their overlap should be of $O(1)$ with correction of $O(\zeta)$. Requiring the overlap to be $O(1)$ is equivalent to saying that $\phi^*(p+q)$ is not different from $\phi^*(p)$. It implies that derivative at p of the wavefunction is small or

$$\frac{q_0.u_i}{m\zeta} << 1$$

. q_0 is the characteristic value of q scaling the wavefunction to wavenumber would mean

$$q_o.u_i l_w << \sqrt{\zeta} \tag{2.16}$$

The delta function in our integrals would also need to be accounted for . The on shell delta function of p written in wavenumber form is

$$\delta(2p_1.q + q^2) = \frac{1}{hm_i}\delta(2\bar{q}.u_i + l_c\bar{q}^2) \tag{2.17}$$

The two additional dimensionless ratios $l_c\sqrt{-\hat{q}^2}$ and $\frac{q.u_i}{\sqrt{-\hat{q}^2}}$ have to be taken into account where $l_s = 1/\sqrt{-\bar{q}^2}$ is the scattering length .

These two ratios can be constrained as our delta function depends on them. Non relativistic limit suggests that these constraints go by -

$$\frac{l_c}{l_s} <= \sqrt{\zeta}$$

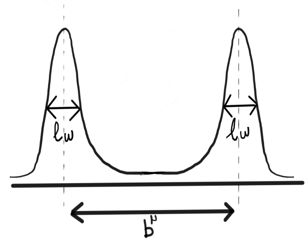

Figure 2.3: A qualitative representation of Two waveparticle and its width compared to impact parameter

$$q.u_i l_s <= \sqrt{\zeta}$$

Combining the lower eq and previous constraint , We obtain

$$l_w << l_S$$

which would mean that that spread of wavefunction should be much less than the scattering length of our experiment.
Also taking the above constraint with $\zeta << 1$ We get

$$l_w << l_s << \sqrt{-b^2}$$

. We should remember not to take $\zeta = 0$ limit blindly as we only want to take the leading order term in the limit , which might or might not be proportional to ζ
Following the constraints shown above it important to take into account to scaling of several other quantities in our impulse equation.

The general rule for classical limit goes like (see [22] for details) -

- We should scale the momentum transfers and loop momentum by $\frac{1}{h}$. and convert them into wavenumbers.
- All radiated momenta k and r_x must be scaled by $\frac{1}{h}$.

- Scale the coupling constant (as mentioned in 1)

- The singular terms must be expanded in powers of h using laurent expansion and we will generally see these terms cancelling out.

- Inside the on shell deta function one can neglect $\hat{q}^2$ term in isolation of any other singular term .

- Neglect the hq^0 term inside the positive-energy theta functions.

We will use large angle brackets to show classical limit -

$$<<f(p_1,p_2,..)>> = \int d\phi(p1)d\phi(p_2)|\phi_1(p_1)|^2|\phi_2(p_2)|^2 f(p_1,p_2,..) \tag{2.18}$$

Using the power counting 1 and 2.5 for various terms while taking the classical limit using the above equation we will look at the expression we get for both I_1 and I_2 calculations. For I_1 -

$$\begin{aligned} I^{\mu}_{(1),\mathrm{cl}} = i \Big\langle \iint \hat{d}^4 q \hat{\delta}\left(2p_1 \cdot q + q^2\right) \hat{\delta}\left(2p_2 \cdot q - q^2\right) \Theta\left(p_1^0 + q^0\right) \Theta\left(p_2^0 - q^0\right) \\ \times e^{-ib\cdot q/\hbar} q^{\mu} \mathcal{A}\left(p_1 p_2 \to p_1 + q, p_2 - q\right) \rangle\Big\rangle . \end{aligned} \tag{2.19}$$

Rescaling the quantities by the general rule mentioned above we get -

$$\begin{aligned} \Delta p_1^{\mu,(0)} \equiv I^{\mu,(0)}_{(1),\mathrm{cl}} = i \frac{g^2}{4} \Big\langle\Big\langle \hbar^2 \int \hat{d}^4 \bar{q} \hat{\delta}\left(\bar{q} \cdot p_1\right) \hat{\delta}\left(\bar{q} \cdot p_2\right) \\ \times e^{-ib\cdot\bar{q}} \bar{q}^{\mu} \overline{\mathcal{A}}^{(0)}\left(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}\right) \Big\rangle\Big\rangle . \end{aligned} \tag{2.20}$$

For I_2 we do the same thing and the net result taking both contributions is we get is

$$I^{\mu}_{\mathrm{cl}} = i \Big\langle\Big\langle \hbar^{-2} \int \hat{d}^4 q \hat{\delta}\left(2p_1 \cdot q + q^2\right) \hat{\delta}\left(2p_2 \cdot q - q^2\right) \Theta\left(p_1^0 + q^0\right) \Theta\left(p_2^0 - q^0\right) e^{-ib\cdot q/\hbar} \mathcal{I}^{\mu} \Big\rangle\Big\rangle$$

where the impulse kernel $\mathcal{I}^\mu$ is defined as,

$$\begin{aligned}\mathcal{I}^\mu \equiv\ & \hbar^2 q^\mu \mathcal{A}\left(p_1 p_2 \rightarrow p_1+q, p_2-q\right) \\ & -i\hbar^2 \sum_X \int \prod_{i=1,2} \hat{d}^4 w_i \hat{\delta}\left(2p_i \cdot w_i + w_i^2\right) \Theta\left(p_i^0 + w_i^0\right) \\ & \qquad \times w_1^\mu \hat{\delta}^{(4)}\left(w_1 + w_2 + r_X\right) \\ & \qquad \times \mathcal{A}\left(p_1 p_2 \rightarrow p_1 + w_1, p_2 + w_2, r_X\right) \\ & \qquad \times \mathcal{A}^*\left(p_1 + q, p_2 - q \rightarrow p_1 + w_1, p_2 + w_2, r_X\right).\end{aligned} \tag{2.21}$$

To summarize the result of this section , we have obtained the final expression for the classical value of impulse in terms of amplitude of diagrams. Note that we have taken out h^2 out of the expression and shall find that it cancels with the scaled propagator in the amplitudes . We can put in the value of the amplitude order by order and calculate the classical result. We again point out that it not just the tree level diagrams that contribute to the classical result , but higher orders as well.

2.6 Momentum impulse calculations

We shall discuss the formalism explicityl in the context of Scalar QED and Graviton scalar coupling . This formalism can be extended to spinning particles and massless particles described in [10] [26]. Let us study both the cases carefully.

2.6.1 Scalar QED

We start by working in scalar QED to explicitly apply the KMOC framework and calculate the impulse. The Lagrangian for Scalar QED is given by -

$$L = \frac{-1}{4} F^{\mu\nu} F_{\mu\nu} + \sum_{i=1,2} [(D^\mu \phi_i)^\dagger D_\mu \phi_i - m_i^2 \phi_i^\dagger \phi_i] \tag{2.22}$$

Since complex scalar fields contain charge we can use this property to describe the charge of classical particles. We will take the classical scattering of the charged particles under the influence of Coulomb force to be the classical analog of the quantum calculations.

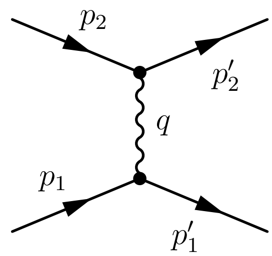

Figure 2.4: Scalar QED 2-2 scattering tree level Diagram

The feynman rules for Scalar QED are easy and are used the construct the feynman diagrams.Tree level diagram for Scalar QED is given in the **??**

Amplitude for the tree level diagram is -

$$\frac{4p_1.p_2 + q^2}{q^2}$$

Taking the relation between momentum mismatch and states momentum $q_1 = p_1' - p_1$ and $q_2 = p_2' + p_2$ Substituting this into 2.21. (Notice that only I_1 contributes at this level as I_2 is higher order term.)

$$\Delta p_1^\mu = i q_1 q_2 \int \frac{d^4q}{(2\pi)^4} \hat{\delta}\left(2p_1 \cdot q\right) \hat{\delta}\left(2p_2 \cdot q\right) e^{-ibq} \frac{4p_1.p_2 + q^2}{q^2} q^\mu \tag{2.23}$$

This integral can be solved (shown in .1) and the result we get is :-

$$\Delta p_1^\mu = -q_1 q_2 \frac{p_1.p_2 b^\mu}{\sqrt{(p_1.p_2)^2 - (m_1 m_2)^2} b^2} \tag{2.24}$$

Notice how the first order impulse is $O(e^2)$ as it should be.
Momentum impulse of second particle can also be found by interchanging p_1 and q with p_2 and $-q$ respectively . We will see that the sum of of impulse is zero which should be the case as it is a non-radiative process.

2.6.2 Gravity example

For the case of gravity we will take the lagrangian of gravitons coupled with scalar massive particles. We will take perturbative approach by expanding the metric in the backgroud of Minkowski.

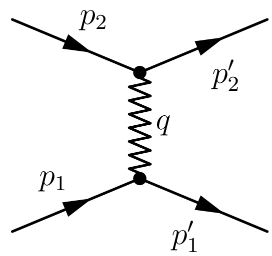

Figure 2.5: Scalar Graviton Tree level Diagram

The Lagrangian density for Scalar coupled to Graviton is(Look at [12] for perturbative Quantum Gravity basics) -

$$L = -R + \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi^2\right) \tag{2.25}$$

The feynman rules for calculating the tree level 2-2 scattering amplitudes are given in **??**. **??** is given by - The tree level amplitude in figure

$$-iM = i\frac{k}{2}[(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) - \eta_{\mu\nu}(p_1.p_2 - m^2)]i\frac{P_{\mu\nu\alpha\beta}}{2q^2}i\frac{k}{2}[(p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta) - \eta_\beta(p_1.p_2 - m^2)] \tag{2.26}$$

It equals

$$-i\frac{k^2[8(p_1.p_2)^2 - 4m_1^2m_2^2]}{8q^2} \tag{2.27}$$

Following similar approach as of previous section to calculate impulse . We need to replace the scalar QED amplitude with amplitude in equation 2.27 The final result after integration is given by -

$$\Delta p_1^\mu = -k^2\frac{(2(p_1.p_2)^2 - m_1^2m_2^2)b^\mu}{\sqrt{(p_1.p_2)^2 - (m_1m_2)^2} \times b^2} \tag{2.28}$$

This result matches the results with that of [19][27].

2.7 Angular momentum Impulse

Now we come to calculating angular momentum impulse of the particles at $0(e^2)$ using the same technique. The procedure for any general observable has been discussed and we

shall use the angular momentum operator instead of momentum operators in the previous calculations.
The calculations are much more subtle for this calculations and it is suggested to look at **??** for details as it is done first time in the context of KMOC formalism.
We have the same tree level diagram as **??** with the condition that $p_1^{'\mu} = p_1^\mu + q^\mu$.
The angular momentum operator for a particle with momentum p_i^μ we know and discussed in the 1.0.1 is :-

$$p_1^\mu \wedge \frac{\partial}{\partial p_1^\nu}$$

Using the similar expression as 2.6.

$$\Delta J^{\mu\nu} =< \psi|i[J^{\mu\nu}, T]|\psi > \tag{2.29}$$

It is clear that the angular momentum operator scales as $O(h^0)$ so unlike the momentum impulse we will scale our expression using the rules given in 2.6 but regard only the $O(h^{-1})$ terms as classical expression as the angular momentum operator does not scale with h .
Just like momentum impulse the expression simplifies to

$$\Delta J^{\mu\nu} = \int d^4q_1 d^4q_2 \hat{\delta}\left(q_1 \cdot p_1\right) \hat{\delta}\left(q_2 \cdot p_2\right) e^{-ib\cdot q} [p_1^\mu \wedge \frac{\partial}{\partial p_1^\nu} + p_1^{'\mu} \wedge \frac{\partial}{\partial p_1^{'\nu}}] (\delta^4\left(q_1+q_2\right) \frac{(p_1+p_1^{'}).(p_2+p_2^{'})}{q^2}) \tag{2.30}$$

Notice the plus sign in the primed and unprimed momentum contrary to the momentum impulse . The result can be found by using the product rule and considering only the $O(h^{-1})$ terms, neglecting the higher order . the first term corresponds to

$$\delta^4\left(q_1+q_2\right) [p_1^\mu \wedge \frac{\partial}{\partial p_1^\nu} A^0 + p_1^{'\mu} \wedge \frac{\partial}{\partial p_1^{'\nu}} A^0] \tag{2.31}$$

from the product rule which we will give the expression-

$$= \delta^4\left(q_1+q_2\right) \frac{4p_1 \wedge p_2}{q^2} \tag{2.32}$$

and from the second term in the product rule -

$$\int d^4q_1 \delta(p_2.q)[A(p_1 \wedge p_2)p_1.p_2/D \tag{2.33}$$

Look at the .3 for detailed calculations . Combining both the term in 17 and 25.

$$\Delta J^{\mu\nu} = \frac{(p_1 \wedge p_2)^{\mu\nu} m_1^2 m_2^2}{((p_1.p_2)^2 - m^4)^{3/2}} log(-b^2\epsilon^2) \tag{2.34}$$

Here ϵ is the IR cutoff . We get the result exactly matching with the classical calculations [17][4] which we will show in the next chapter explicitly.
The most important point to note here is that if we exchange the values of p_1 andp_2 we get the same expression with a negative sign. Therefore the sum of angular momentum impulse of two particles don't cancel each other .
Let us look at the expression in the COM frame of reference. The momentum for particle one is:-

$$p_1^\mu = (E_1, 0, 0, p_Z) \tag{2.35}$$

While for particle two is

$$p_2^\mu = (E_2, 0, 0, -p_Z) \tag{2.36}$$

Taking the wedge product we see that $\Delta J^{ij} = 0$ but $\Delta J^{0i} \neq 0$.Which means that some components of Angular momentum impulse are not conserved , which is surprising . We go on to explore this idea of non -conservation of angular momentum in scattering from different point of views further in the thesis.
Let us first look at some higher order calculation via KMOC first.

2.8 Order e^4 momentum impulse

In this section we will give a flavour of how the next to leading order calculations are done for momentum impulse and how and the loop level diagrams give the classical contributions. For these the contribution comes from two kinds of terms I_1 and I_2 defined in the above section.
We have to be careful in considering the powers of 'h' in these calculations. The contributions comes from triangle box and cut box diagrams the other contributions are zero.(For detailed steps of these calculations look at [22])[2]) Let us discuss each of the diagrams and their contributions.

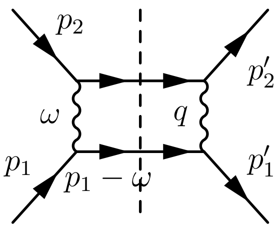

Figure 2.6: Cut Box Diagram of $O(e)^4$

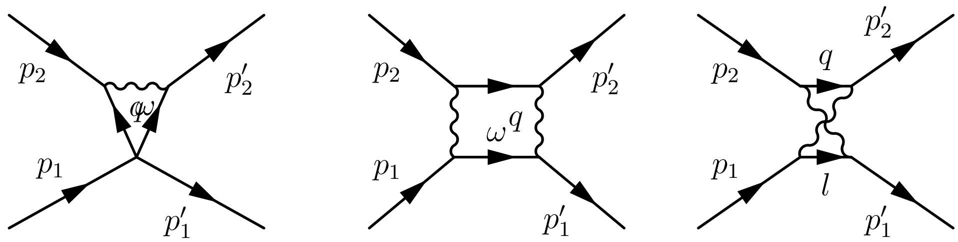

Figure 2.7: Triangle , Box and Cross Box diagrams respectively

Cut box arise due to I_2 terms from the kernel 2.21 The amplitude is given by -

$$\begin{aligned}\dot{B}^\mu =& -i\frac{Q_1^2Q_2^2}{\hbar^2}\int \hat{d}^4\bar{w}\hat{\delta}\left(2p_1\cdot\bar{w}+\hbar\bar{w}^2\right)\hat{\delta}\left(2p_2\cdot\bar{w}-\hbar\bar{w}^2\right)\frac{\bar{w}^\mu}{\bar{w}^2(\bar{w}-\bar{q})^2} \\ &\times\left(2p_1+\hbar\bar{w}\right)\cdot\left(2p_2-\bar{w}\hbar\right)\left(2p_1+\hbar\bar{q}+\hbar\bar{w}\right)\cdot\left(2p_2-\hbar\bar{q}-\hbar\bar{w}\right).\end{aligned} \tag{2.37}$$

We expand the above expression via laurent expansion to look at the leading order power in 'h'.We then perform the integration over ω and q.

The final result is -

$$\tilde{\mathcal{I}}_3^\mu = g^4\frac{\left(Q_1Q_2p_1\cdot p_2\right)^2}{8\pi^2\mathcal{D}^2|b|^2}\left[\left(m_1^2+p_1\cdot p_2\right)p_2^\mu-\left(m_2^2+p_1\cdot p_2\right)p_1^\mu\right]. \tag{2.38}$$

For triangle diagram the contributions are - The amplitude of the diagram is -

$$-2Q_1^2Q_2^2\int \hat{d}^{D^2}\frac{(2p_1+\ell)\cdot(2p_1+q+\ell)}{\ell^2(\ell-q)^2\left(2p_1-\ell+\ell^2+i\ell\right)}. \tag{2.39}$$

Performing the integration we get,

$$\tilde{\mathcal{I}}_1^\mu = -\frac{g^4}{32\pi}\left(m_1+m_2\right)\frac{\left(Q_1Q_2\right)^2}{\sqrt{D}}\frac{b^\mu}{|b|^3} \tag{2.40}$$

The box and cross boxed diagrams give vanishing contributions to the momentum impulse(Appendix of [2]). The total contributions is just the addition of 2.40 and 2.38 giving

$$\Delta p_1^{(1)\mu} = -\frac{g^4}{32\pi^2|b|^3}\frac{(Q_1Q_2)^2}{\mathcal{D}}\left[\pi\sqrt{\mathcal{D}}\,(m_1+m_2)\,b^\mu + 4\frac{(p_1\cdot p_2)^2\,(p_1+p_2)^2\,|b|}{\mathcal{D}}p^\mu\right]$$

where we have used

$$p^\mu = \frac{m_1m_2}{(p_1+p_2)^2}\left[\left(\frac{m_2}{m_1}+\frac{p_1\cdot p_2}{m_1m_2}\right)p_1^\mu - \left(\frac{m_1}{m_2}+\frac{p_1\cdot p_2}{m_1m_2}\right)p_2^\mu\right]$$

The final result obtained can also be derived from the classical calculations hence proving the robustness of KMOC technique.

We will also look at the the radiation emitted using KMOC , It can be use to show the classical soft photon theorem via KMOC . It was done in [25].
Classical radiation of particles can be calculated using the above equations. One has to compute the five point amplitude of the scattering . The radiation kernel is defined as -

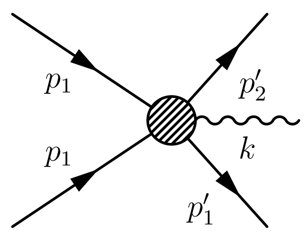

Figure 2.8: 5-Point Radiation Diagram

$$\begin{aligned}\mathcal{R}^{(0)}(\bar{k}) = 4\int \hat{d}^4\bar{w}_1\hat{d}^4\bar{w}_2\hat{\delta}\,(2p_1\cdot\bar{w}_1)\,\hat{\delta}\,(2p_2\cdot\bar{w}_2)\,\hat{\delta}^{(4)}\left(\bar{k}-\bar{w}_1-\bar{w}_2\right)e^{i\bar{w}_1\cdot b}\\ \times\left\{\frac{Q_1^2Q_2}{\overline{w}_2^2}\left[-p_2\cdot\varepsilon + \frac{(p_1\cdot p_2)\,(\bar{w}_2\cdot\varepsilon)}{p_1\cdot\bar{k}} + \frac{\left(p_2\cdot\bar{k}\right)(p_1\cdot\varepsilon)}{p_1\cdot\bar{k}}\right.\right.\\ \left.\left.-\frac{\left(\bar{k}\cdot\bar{w}_2\right)(p_1\cdot p_2)\,(p_1\cdot\varepsilon)}{\left(p_1\cdot\bar{k}\right)^2}\right] + (1\leftrightarrow 2)\right\},\end{aligned} \tag{2.41}$$

The 5 Point amplitude can be found by either taking all of the possible 5 Point diagrams and calculating them or one can use an easier way of combining the 2 point and 3 point vertex (Look at [1] for details).
This corresponds to classical current of moving charges particles depending on what order

we calculate the amplitude.This expression is also given in [22] . Radiation was shown to be equal to the one obtained from subleading classical soft photon theorem in [25].
We shall discuss more about the classical radiation of angular momentum in the next section.

Chapter 3

Classical calculation

We will now verify the calculations done the above section by calculating the results we get from classical calculations.The approach will follow the paper [15]. The approach is discussed in the following subsection.

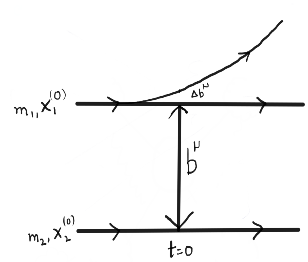

Figure 3.1: Classical particle trajectory correction to leading order

3.1 Methodology for classical calculations

We will consider two body classical scattering under a large impact parameter. We will try to iteratively find the solution to coupled differential equations.

We assume that the trajectories of the particles can be expanded in the power of the coupling constant .At each order , starting from the lowest, one can calculate the the electromagnetic fields produces by the particles following that trajectory. Knowing these fields allows us to compute the first order forces on the particles using the lorentz force laws. Hence the next higher order deviations in the tranjectory can be found. Iterating this procedure allows us to compute to any desired perturbative order.
Needless to say the calculations become harder at each step and unsolvable by hand.
More formally, we expand the trajectories as,

$$x_1(t) = b_1 + u_1(t) + \Delta^1 x_1(t) + \Delta^2 x_1(t)..., \tag{3.1}$$

$$x_2(t) = u_2(t) + \Delta^1 x_2(t) + \Delta^2 x_2(t)... \tag{3.2}$$

where Δ^n is e^{2n} in order and b_1 is the impact parameter.

Let the trajectory be only of leading order .Considering these trajectory we will be able to calculate electromagnetic force on one particle due to the other one by the procedure shown below.
The Maxwell's field equation reads

$$\partial_\mu F^{\mu\nu}(x) = J^\nu(x) \tag{3.3}$$

For Force on first particle due to second one, we take the current of second particle.

$$J_2^\mu(x) = eQ_2 \int d\tau \delta^4(x - x_2(\tau)) u_2^\mu(\tau) \tag{3.4}$$

$u_2^\mu(\tau)$ is the derivative of the particle trajectory with respect to τ Putting this value in 3.3 and using the Lorentz gauge condition would evaluate to -

$$\partial^2 A_2^\mu = eQ_2 \int d\tau \delta^4(x - x_2(\tau)) v_2^\mu(\tau) \tag{3.5}$$

Going into fourier space this would give

$$\partial^2 A_2^\mu = -\frac{eQ_2}{q^2} u_2^\mu \delta(q.u_2) \tag{3.6}$$

The field strength corresponding to this field is -

$$F_2^{\mu\nu}(x) = ieQ_2 \int d^4q\delta(q.u_2)e^{-q.x(\tau)}\frac{q^\mu u_2^\nu - u_2^\mu q^\nu}{q^2} \tag{3.7}$$

Plugging in the value of $x(\tau)$ as the equation of first particle because we want to calculate the force on first particle -

$$m_1\frac{\partial p_1^\mu}{\partial t} = iq_1q_2 \int \frac{d^4q}{(2\pi)^4}\hat{\delta}\left(2p_2 \cdot q\right)e^{-i(b+u_1.\tau)q}\frac{q^\mu u_2.u_1 - u_2^\mu q.u_1}{q^2} \tag{3.8}$$

Integration over time from $-\infty$ to ∞ would lead to classical momentum impulse which equals-

$$\Delta p_1^\mu = iq_1q_2 \int \frac{d^4q}{(2\pi)^4}\hat{\delta}\left(2p_1 \cdot q\right)\hat{\delta}\left(2p_2 \cdot q\right)e^{-ibq}\frac{4p_1.p_2 + q^2}{q^2}q^\mu \tag{3.9}$$

We see that this equation matches exactly with our result from KMOC 2.23.

Using this, one can also find the correction to the particle trajectory . The integration has to be carried out till only a finite time t in order to get velocity as a function of time. The integral must converge at $t \to -\infty$ so we replace $q.u$ in the exponential to $q.u + i\epsilon$.

$$= m\Delta p_1^\mu(t) = iq_1q_2 \int \frac{d^4q}{(2\pi)^4}\hat{\delta}\left(2p_2 \cdot q\right)e^{-i(b.q}\frac{q^\mu u_2.u_1 - u_2^\mu q.u_1}{q^2}\int_{-\infty}^{t} dte^{-i(q.u+i\epsilon)t} \tag{3.10}$$

$$= m\Delta p_1^\mu(t) = iq_1q_2 \int \frac{d^4q}{(2\pi)^4}\hat{\delta}\left(2p_2 \cdot q\right)e^{-i(b.q}\frac{q^\mu u_2.u_1 - u_2^\mu q.u_1}{q^2(q.u + i\epsilon)} \tag{3.11}$$

The leading correction to the position of the particle is given by integrating once more, with the result:

$$m\Delta x_1^\mu(t) = iq_1q_2 \int \frac{d^4q}{(2\pi)^4}\hat{\delta}\left(2p_2 \cdot q\right)e^{-i(b+u_2.x)q}\frac{q^\mu u_2.u_1 - u_2^\mu q.u_1}{q^2(q.u + i\epsilon)^2} \tag{3.12}$$

Correction to trajectory is important to calculate the angular momentum impulse. Calculation of Angular momentum impulse lassically upto leading order can be done now .
Initial angular momentum in this system for the first particle is calculated below . One must take caution in defining the origin because angular momentum is a origin quantity.

The origin is taken to be particle 2's trajectory at the t=0. This is where our earlier discussion of wavefunction written in with respect to particle 2 come into account . This choice of orgin becomes key because of our wavefunction structure for matching of the results. Initial angular momentum is-

$$J^{\mu\nu}_{in} = (b \wedge p_{in})^{\mu\nu} \tag{3.13}$$

Final angular momentum is -

$$J^{\mu\nu}_{out} = ((b + \Delta b)^{\mu}) \wedge p_{fin})^{\mu\nu} \tag{3.14}$$

Notice the shift in the impact parameter.
Substituting p_{fi} from 3.11 and subtracting 3.13 gives

$$\Delta J^{\mu\nu}_1 = b^{\mu}\Delta p^{\nu}_1 + z_1(0)^{\mu} p^{\nu}_1 - \mu \leftrightarrow \nu \tag{3.15}$$

Substituting the values from the calculated values gives -

$$\Delta J^{\mu\nu} = \frac{(p_1 \wedge p_2)^{\mu\nu} m_1^2 m_2^2}{((p_1.p_2)^2 - m^4)^{3/2}} log(-b^2\epsilon^2) \tag{3.16}$$

For details of the calculation, see [25].
It matches with our result from the KMOC formalism The procedure and calculation for momentum and angular momentum impulse in the electromagnetic case is elaborated much more in [29] (Look at this for detailed explanation and higher order calculations).We will take some results for the higher order calculations from this paper in the subsequent sections.

3.2 Back reaction in electrodynamics

We will take time before proceeding, to discuss one of the most important advantages of KMOC over classical calculations. In every classical calculation at higher orders we have to put in by hand the ALD force along with the usual lorrentz force to get the correct answer. We will breiefy discuss the existence of this ALD force also called back reaction . Although this is discussed in many standard book but we will largely follow the approach and notations of [31]. The setup is similar to our classical calculations section. The electromagnetic potential can be derived from the current like was done in section 3.1. We got

$$A_\mu(x) = \int d^4x' G_{\mu\nu}(x - x') J^\nu(x') \tag{3.17}$$

We take any potential at a point to be generated by the current in an otherwise vacuum background, we must take the retarded greens function in the above equation. The retarded greens function is given by -

$$G^+(x - x') = \frac{\theta(t' - t)\delta((x - x')^2)}{2\pi} \tag{3.18}$$

Field equation is then given by -

$$A^+_\mu(x) = \int d^4x' G^+_{\mu\nu}(x - x') J^\nu(x') \tag{3.19}$$

The current is parametrized by a variable 's' given by -

$$J^\mu = e \int ds \frac{\partial z^\mu}{\partial s} \delta^4(x' - z(s)) \tag{3.20}$$

The field is -

$$A^+_\mu(x) = \int d^4x' G^+_{\mu\nu}(x - x') ds \frac{\partial z^\mu}{\partial s} \delta^4(x' - z(s)) \tag{3.21}$$

Doing the integration over x' gives

$$A^+_\mu(x) = \int G^+_{\mu\nu}(x - z(s)) \frac{\partial z^\mu}{\partial s} \tag{3.22}$$

This field exerts force on the particle . the partial derivative of the above equation is given by

$$\partial_\alpha \equiv \int ds \frac{x - z_\alpha \cdot z}{x^\mu - z^\mu \cdot z} \frac{\partial G^+_{\mu\nu}(x - z(s))}{\partial s} \tag{3.23}$$

We will use integration by parts. We see that at the point of the particle this field blows up to infinity to counter this we take another greens function given by

$$G^R = 1/2(G^+ - G^-) \tag{3.24}$$

Using this we get from

$$F^{\mathrm{R}}_{\alpha\beta} = e\int_{-\infty}^{+\infty} dsG^{\mathrm{R}}(x-z)\frac{d}{ds}\left[\frac{(x_\alpha - z_\alpha)\,\dot{z}_\beta - (x_\beta - z_\beta)\,\dot{z}_\alpha}{(x^\mu - z^\mu)\,\dot{z}_\mu}\right],$$

since We want field on the trajectory of the particle we take the point. $x = z\,(s_0)$, defining $u = s - s_0$.

$$\begin{aligned} F^{\mathrm{R}}_{\alpha\beta}\,(z\,(s_0)) =& \frac{e}{4\pi}\int_{-\infty}^{+\infty} du\,\mathrm{sgn}(u)\delta\left(u^2\right) \\ & \times\frac{d}{du}\left[\frac{(z_\alpha\,(s_0) - z_\alpha(s))\,\dot{z}_\beta(s) - (z_\beta\,(s_0) - z_\beta(s))\,\dot{z}_\alpha(s)}{(z^\mu\,(s_0) - z^\mu(s))\,\dot{z}_\mu(s)}\right], \end{aligned}$$

All we need to do is to expand the particle position in terms of new parameters like ,

$$z(s) - z\,(s_0) = u\dot{z}\,(s_0) + \frac{u^2}{2}\ddot{z}\,(s_0) + \frac{u^3}{6}\dddot{z}\,(s_0) + \ldots \tag{3.25}$$

We find that the only terms surviving are order u^2. Performing the 'u' integrations we find that the ALD reaction force is given by

$$f^\mu_{\mathrm{LD}} = e\eta^{\mu\alpha}F^{\mathrm{R}}_{\alpha\beta}\dot{z}^\beta = -\frac{2}{3}\frac{e^2}{4\pi}\left(\dddot{z}^\mu + \ddot{z}^2\dot{z}^\mu\right) \tag{3.26}$$

This is important approach to ALD reaction .As it gives important insights into how to deal with effects of self force etc., which are important in field integrations . We emphasize here that the KMOC formalism does not require to put this force by hand unlike classical calculations which is needed for conservation of momentum.

3.3 Radiated angular momentum

In this section we shall derive the formulas for the the radiation of momentum and angular momentum in the fields classically.It follows a general approach of radiation calculation which can be looked at from [24]. The result obtained will provide us insights into the radiation of momentum and angular momentum.

The Energy momentum tensor of EM field is given as

$$T^{\mu\nu} = -F^{\mu}{}_{\rho}F^{\nu\rho} + \frac{\eta^{\mu\nu}}{4}F^{\rho\sigma}F_{\rho\sigma}$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is the field strength.This can be derived using the lagrangian of the EM field and also the arguments given in [35]. The Gauge field can be obtained in terms of the current using the greens function as done in the previous sections. In momentum space, the gauge field is given by

$$A_\mu(x) = \int \widetilde{dk}\left(G_{\mu\nu}\mathcal{J}^\nu(k)e^{-ik\cdot x} + \text{ c.c. }\right) \tag{3.27}$$

where $\mathcal{J}^\nu(k)$ is the conserved current satisfying $k_\mu\mathcal{J}^\mu(k) = 0$, and $G_{\mu\nu}$is the Green's function. We are interested in the momentum and angular momentum the fields carry to infinity. We assume our detectors to cover the scattering region in shape of a sphere of finite radius . We then take the sphere radius to be very large . The flux of momentum on that sphere will be the momentum seen as radiation by the detectors .

The radiated momentum is given by -

$$P^\mu = \int d^3x T^{\mu 0} \tag{3.28}$$

The radiated angular momentum is -

$$J^{\mu\nu} = \int d^3x x^{[\mu}T^{\nu]0} \tag{3.29}$$

Substituting 3.27 gives the formula for radiated momentum and angular momentum .

$$\begin{aligned} P^\mu &= \int \widetilde{dk}k^\mu\left(-\mathcal{J}^{*\rho}(k)\mathcal{J}_\rho(k)\right), \\ J^{\mu\nu} &= \int \widetilde{dk}\left(-\mathcal{J}^{*\rho}(k)\mathcal{L}^{\mu\nu}\mathcal{J}_\rho(k) - i\mathcal{J}^{*[\mu}(k)\mathcal{J}^{\nu]}(k)\right) \end{aligned} \tag{3.30}$$

This formula can further be modified to retarded coordinates in position space (u=t-r) to -

$$J_k^{\text{rad}} = \frac{\epsilon_{kij}}{16\pi G}\int du d\Omega\left[A_i\partial_u A_j - \frac{1}{2}x^i\partial_j f_{ab}\partial_u f_{ab}\right].$$

Note that the integrand radiation is bilinear in $f_{ij}$and in $\dot{f}_{ij} \equiv \partial_u f_{ij}$. By contrast, the radiated energy-momentum, P_i^{rad} , is quadratic in $\dot{f}_{ij}$, namely

$$P^\mu_{\text{rad}} = \frac{1}{32\pi G} \int dud\Omega \left[\partial_u f_{ab} \partial_u f_{ab}\right] n^\mu,$$

It was shown in [11] using this equation,the order of radiation, where $n^\mu = (1, x^i/r)$.The gauge fields and current can be expanded in the power of the coupling constant .

$$A_i(u,\theta,\phi) = eA_i^{(1)}(\theta,\phi) + e^2 A_i^{(2)}(u,\theta,\phi) + O\left(e^3\right),$$

where the $(O(e))$ contribution is independent of the retarded time u that $\dot{A}_i$ is of order $O\left(e^2\right)$. Because of the second term we can see that angular momentum is radiated at a lower order than the momentum radiation.

$$\Delta A_i(\theta,\phi) \equiv \int_{-\infty}^{+\infty} du \partial_u A_i = \left[A_i\right]_{u - -\infty}^{u - +\infty}$$

This term is also known as the memory effect and we can clearly see its relation to angular momentum radiated . We shall discuss this effect again in detail and see how it arises from the asymptotic symmetry and the associated charge.

3.4 Electromagnetic "Scoot"

The paper by [17] first concretely described the phenomenon of electromagnetic "Scoot". We will look at the setup here and state the results obtained from the paper.

Consider a scattering of two classical charged particles 1 and 2. To leading order, the particles move in straight lines. We work in a frame where one particle is at rest. We will take particle 2 to be at rest at the origin and denote this frame with a prime. Choosing the initial momentum of particle one to be in the z direction and the transverse separation to be in the x direction.

The leading-order trajectories are -

$$r_1' = (b, 0, vt'); r_2' = (0, 0, 0) \tag{3.31}$$

b^μ is the impact parameter which is solely in the x direction. Since the calculations in this paper is done in 3 dimensions only and time is used as parameter they defined the $0i^{th}$ component of the 4 angular momentum to be another quantity called the mass moment given by -

$$N_{\text{mech}} = \sum_I E_I r_I - t \sum_I p_I \tag{3.32}$$

Given this trajectory one can calculate the zeroth order electric and magnetic fields at any point in space and use the Lorentz equations to determine the leading order trajectory of the particles.
We then transform the trajectories in the frame of Center of momentum and shifting the origin to the Center of mass. We will here directly show the result from the paper , explicit calculation will be shown using analogous technique of Green's function which is more relevant for our approach .
One must also consider the various quantities like momentum , energy and angular momentum of the field along with the particle to take the full picture into account.
The calculation for field has more subtle issues like self energy which we will bypass here but should be kept in mind.

The various quantities for the electromagnetic fields can be calculated using the following formulas

$$E_{F\times} = \frac{1}{8\pi} \int \mathcal{E}_\times d^3x \tag{3.33}$$

$$p_{F\times} = \frac{1}{4\pi} \int S_\times d^3x \tag{3.34}$$

$$L_{F\times} = \frac{1}{4\pi} \int x \times S_\times d^3x \tag{3.35}$$

$$N_{F\times} = \frac{1}{8\pi} \int \mathcal{E}_\times x d^3x - \boldsymbol{p}_{F\times} t \tag{3.36}$$

Here S_x is the Poynting vector of the fields.

The results for the quantities using the particle trajectory is given by -

$$E_1 = \frac{m_1 + \gamma m_2}{E_0}\left(m_1 - \frac{m_2}{E_0}\frac{q_1 q_2}{\gamma v |t|}\right) + O\left(t^{-2}\right)$$

$$\boldsymbol{p}_1 = \left(\mu\gamma v - \frac{q_1 q_2}{\gamma^2 v^2}\frac{\left(m_1 + \gamma m_2\right)^2}{E_0^2 |t|}\right)\hat{z}$$

$$+\Theta(t)\frac{2q_1 q_2}{bv}\hat{x} + O\left(t^{-2}\right)$$

$$\boldsymbol{L}_1 = -\mu b\gamma v \frac{m_2\left(m_2 + \gamma m_1\right)}{E_0^2}\hat{\boldsymbol{y}} + O\left(t^{-2}\right)$$

$$\boldsymbol{N}_1 = \mp\frac{q_1 q_2}{\gamma^2 v^2}\left(\log\frac{2\gamma v E_0 |t|}{\left(m_1 + \gamma m_2\right) b} - 1\right)\hat{z} + O\left(t^{-2}\right)$$

Looking at the results carefully we can analyze that the mass momentum is ill defined as it blows up at early and late times and it is only the sum of mass moment for the three particles that can be evaluated as we also got in the KMOC result.

Unlike Momentum and energy change at early and late times that conserved we observe that The Mass moment of the particles are not. The result we obtained matches with the result of [17] only discrepency is that our result is in the invariant form but here it is derived in a particular frame.

A more general approach was shown in the recent paper by [4] and our results matches exactly theirs.

Chapter 4

Faddev kulish States

It is important to realise that the problem of ""Scoot"" lies in the exchange of angular momentum between the particles and fields at late or asymptotic times. It tells us that it is then important to have the information of the field at late times and not just velocities of the particles. Therefore this suggests a fundamental problem in our concept of taking the asymptotic states as free states . This brings us to the concept of Faddev-Kulish States described in [23].

4.1 Faddev Kulish Derivation

Here we take reference from [13] , and see the exact need of faddev kulish states and their form. Let us look at how the hamiltonian behaves at aysmptotic times. The interction potential for QED is given by

$$V = -\int d^3x L_I = -\int d^3x e\bar{\psi} A\psi \equiv -\int d^3x J^\mu A_\mu, \tag{4.1}$$

where J^μ is the conserved current .

We can expand the fields in the above equation using mode expansion. We see that using the expansion we will get eight terms of two types.

- terms containing only two creation operators or only two annihilation operators
- terms containing a mix of creation and annihilation operators.

The terms of first kind will have the exponential of form

$$= \int d^3x\widetilde{d^3k}\widetilde{d^3p}\widetilde{d^3p'} \exp\left(-i\left(p^0 + p'^0 \pm k^0\right)t\right) \exp\left(-i\left(\mathbf{p} + \mathbf{p'} - \mathbf{k}\right)\mathbf{x}\right) \tag{4.2}$$

Since we integrate over k we can send k $\to -$k in the terms with photon creation operators.. Integrating over d^3x will then result in a delta function,

$$\begin{aligned}
&= \int \widetilde{d^3k}\widetilde{d^3p}\widetilde{d^3p'} \exp\left(-i\left(p^0 + p'^0 \pm k^0\right)t\right) \delta^3\left(\mathbf{p} + \mathbf{p'} - \mathbf{k}\right) \\
&= \int \widetilde{d^3k}\widetilde{d^3p}\widetilde{d^3p'} \exp\left(-i\left[\sqrt{\mathbf{p}^2 + m^2} + \sqrt{\mathbf{p'}^2 + m^2} \pm k_0\right]t\right) \delta^3\left(\mathbf{p} + \mathbf{p'} - \mathbf{k}\right) \qquad (4.3) \\
&= \int \widetilde{d^3k}\widetilde{d^3p} \exp\left(-i\left[\sqrt{\mathbf{p}^2 + m^2} + \sqrt{(\mathbf{p} - \mathbf{k})^2 + m^2} \pm k_0\right]t\right)
\end{aligned}$$

As $|t| \to \infty$ the exponential will oscillate rapidly and and the integral will be cancelled for the first term. However for the second term this rapid oscillating can be suppressed by $k \to 0$. Thus it is only the second term which will contribute to the potentail at large times.

Using the exponential of the second term and commutation relations of the fields we get the asymptotic potential to be -

$$V_I^{as}(t) = -e \int \widetilde{d^3k}\widetilde{d^3p} p^\mu \rho(\mathbf{p}) \left[a_\mu(\mathbf{k}) e^{i\frac{ip}{p0}t} + a_\mu^\dagger(\mathbf{k}) e^{-i\frac{pk_0}{p0}t}\right] \tag{4.4}$$

where $\rho(\mathbf{p}) \equiv d_r^\dagger(\mathbf{p}) d_r(\mathbf{p}) - c_r^\dagger(\mathbf{p}) c_r(\mathbf{p})$ is the charged matter density operator.

In order to discuss the impact of this statement we will look at the IR divergence problem of S matrix. We can either take the Faddev kulish states to be evolved using traditional S matrix or use the new matrix acting on the free fock states.
The traditional approach to S matrix is following - We take the asymptotic states to be evolving with the free hamiltonian , i.e. the free asymptotitc states We therefore define a hard Moller operator defined as

$$\Omega_{+-} = lim_{t \to 0} e^{iHt} e^{-iH_0 t} \tag{4.5}$$

The incoming and outgoing states are represented by

$$|\psi >= \Omega_+|\psi_{out} >= \Omega_-|\psi_{in} > \tag{4.6}$$

Hence

$$|\psi_{out} >= S|\psi_{in} > \tag{4.7}$$

and

$$S = \Omega_+\Omega_- \tag{4.8}$$

The dyson S matrix from is given by

$$S(v_I) = Te^{-i\int_{-\infty}^{\infty} dtV_I} \tag{4.9}$$

But here the assumption that V_I vanishes as $t \to \infty$ is faulty As shown in the previous section this S matrix is Infrared divergent . Let us look at the newer approach by Faddev and Kulish . We define a new evolution operator such that

$$i\frac{\partial U(t)}{\partial t} = U(t)V_i^{as} \tag{4.10}$$

The time evolution equation for the free states is given by

$$S(v_A) = Te^{-i\int_{-\infty}^{\infty} dt(V_A - V_A^{as}} (4.11)$$

This S matrix is not the usual Dyson S matrix , but is related by -

$$S(v_A) = U^{-1}S_DU \tag{4.12}$$

or equivalently the moller operators are modified to

$$\Omega_{+-} = lim_{t\to 0}e^{iHt}e^{-iH_{as}t} \tag{4.13}$$

S matrix element takes the form

$$< f|S_A(V_A - V_I)|i >=_F K < f|S_D|i >_F K \tag{4.14}$$

where

$$|I >_F K = U(t)|I > \tag{4.15}$$

U is given in the eq [] we can taylor expand this term and due to time ordering operator we have-

$$U(t) = exp(-i\int^{t} dt' V^{I}_{as}(t') - 1/2\int^{t} dt' \int^{t'} ds[V^{I}_{as}(t'), V^{I}_{as}(s)]) \tag{4.16}$$

We can plug in the asymptotic potential from 4.4 and use the commutation relations. Hence re writing this in the known format we have -

$$U(t) = e^{R(t)}e^{i\phi(t)} \tag{4.17}$$

where

$$\begin{aligned} R(t) &= -i\int^{t} dt' V^{as}_{I} = e\int \widetilde{d^3k}\widetilde{d^3p}\frac{p^{\mu}}{pk}\left(a^{\dagger}_{\mu}(\mathbf{k})e^{-i\frac{pk}{p^0}t} - a_{\mu}(\mathbf{k})e^{i\frac{pk}{p^0}t}\right)\rho(\mathbf{p}) \\ \Phi(t) &= \frac{i}{2}\int^{t} dt' \int^{t'} dsQ\left(t', s\right) \\ &= -\frac{e^2}{4\pi}\int \widetilde{d^3p}\widetilde{d^3q} : \rho(\mathbf{p})\rho(\mathbf{q}) : \frac{pq}{\sqrt{(pq)^2 + m^4}} \operatorname{sgn} t \ln\frac{|t|}{t_0} \end{aligned} \tag{4.18}$$

There action of these exponential factors will be discussed in subsequent section. These states render the S-matrix to be finite (Proved in 5.4) .

4.1.1 Faddev-Kulish Derivation from Wilson line perspective

We shall now use the the concept of Wilson line to get a clearer and more physical picture of what these dressings are [18] [13]. The gauge field under lorentz gauge is given as

$$\Box A_{\mu} = J_{\mu}(x) \tag{4.19}$$

J^{μ} being the electric current. Solution is given using Greens's function as:-

$$A_{\mu}(x) = A^{in}_{\mu}(x) + \int d^4y G_{ret}(x-y)J_{\mu}(y) \tag{4.20}$$

A^{μ} is the incoming radiation that is not derived from the current . Wilson Tail for a field is given by :-

$$C(x, -\infty|t) = Pexp(-ie\int_{-\infty}^{x} dx^{\mu}A_{\mu}(x) \tag{4.21}$$

Substituting the value of field from 4.20

$$C(x,-\infty|t) = Pexp(-ie\int_{-\infty}^{x} dx^{\mu}A_{\mu}^{in}(x) + \int_{-\infty}^{x}\int dx^{\mu}d^4yG_{ret}(x-y)J_{\mu}(y) \tag{4.22}$$

where current is given by the a charged particle moving on its worldline.

$$J_{\mu}^{as}(y) = e\int d^4p'\rho(p')\int_{-\infty}^{\infty} dt'v'_{\mu}\delta^4(y-v't') \tag{4.23}$$

Plugging in the value of current and not considering the homogeneous part of field for now . Here Plugging in the Green's Function.

$$G_{ret}(x-y) = 1/2\pi\theta(x-y)\delta(x-y)^2 \tag{4.24}$$

$$C(x,-\infty|t) = \int d^4p'\rho(p')\int d^4y1/2\pi\theta(x-y)\delta(x-y)^2\int_{-\infty}^{\infty} dt'v'_{\mu}\delta^4(y-v't') \tag{4.25}$$

integrating over y gives

$$\int d^4p'\rho(p')\int_{-\infty}^{\infty} dt'v'_{\mu}\theta(x-v't')\delta(x-v't')^2 \tag{4.26}$$

The contribution comes only at time $t = t_0$

$$\int d^4p'\rho(p')\theta(x-v't'_0)\frac{v'_{\mu}}{((x.v')^2-x^2v'^2)^{1/2}} \tag{4.27}$$

Changing the integration variable from x to vt .(Integrating over worldline of particle)

$$\int d^4p'\rho(p')\frac{v.v'}{((v.v')^2-x^2v'^2)^{1/2}}\int_{-\infty}^{t_x} dt1/|t|\theta(vt-v't'_0) \tag{4.28}$$

This equals

$$i\phi = \int d^4p'\rho(p')\frac{v.v'}{((v.v')^2-v^2v'^2)^{1/2}}ln\frac{|t|}{t_0} \tag{4.29}$$

where t_0 is the IR cutoff to render the integration finite.

Now we are left to look at the contribution from the homogenous part of the field. We will parameterize the worldline of the charged particle in terms of the proper time τ. For a τ-independent four-velocity u^{μ}. Using the first order or free trajectory of particle $z^{\mu} = x^{\mu}+\tau u^{\mu}$, we obtain

$$
\begin{aligned}
ie\int_{-\infty}^{x} dz^{\mu}A_{\mu}^{in}(z) &= ie\int_{-\infty}^{0} d\tau\frac{dz^{\mu}}{d\tau}A_{\mu}^{in}(x+\tau u)\\
&= ie\int_{-\infty}^{0} d\tau u^{\mu}A_{\mu}^{in}(x+\tau u)\\
&= ie\int\widetilde{d^3k}\int_{-\infty}^{0} d\tau u^{\mu}\left[a_{\mu}(\mathbf{k})e^{ik(x+\tau u)}+a_{\mu}^{\dagger}(\mathbf{k})e^{-ik(x+\tau u)}\right],
\end{aligned}
\tag{4.30}
$$

Where we have expanded the field in mode expansion. Integration over τ has to be done taking care of the boundary terms. We find that

$$
ie\int_{-\infty}^{x} dz^{\mu}A_{\mu}^{in}(x) = -e\int\widehat{d^3k}\frac{p^{\mu}}{pk}\left[a_{\mu}^{\dagger}(\mathbf{k})e^{-ikx}-a_{\mu}(\mathbf{k})e^{ikx}\right]
$$

where we used that $\frac{u^{\mu}}{uk}=\frac{p^{\mu}}{pk}$. This is exaclty the Radiation operator derived from faddev kulish state with a few subtle difference like the charge density operator which is absent in this derivation. This discrepancy can be resolved when one keeps in mind that the cloud of photon is created for each particle . It is interesting to look at the action of these dressed state factors on the state which we will do in the coming sections.

4.2 Faddev kulish states in KMOC

In this section we will show the calculations for if we include the phase factor defined in 4.18 for the calculations of angular momentum as we did in the previous section.

The derivation follows similar steps to our earlier angular momentum calculations but with a few more complications.

Let us take the first expression in the equation but instead of using the free initial and final states we use the dress states. Our initial state is -

$$
|\psi>=e^{-i\phi(t)}|i> \tag{4.31}
$$

and the final state for amplitude calculation is the complex conjugate of the initial state, where

$$
\phi(t) = -\frac{e^2}{4\pi}\int\widetilde{d^3p}\widetilde{d^3q} : \rho(\mathbf{p})\rho(\mathbf{q}) : \frac{pq}{\sqrt{(pq)^2+m^4}}\,\mathrm{sgn}\,t\ln\frac{|t|}{t_0} \tag{4.32}
$$

(as shown earlier in 4.18) The expression for angular momentum impulse with these states is -

$$\Delta J^{\mu\nu} =< \psi|i[J^{\mu\nu}, T]|\psi > \tag{4.33}$$

Once again following the steps in .3, we will substitute the wavefunction form and open the commutator this will give the expression of form.

$$\Delta J^{\mu\nu} =< p_1p_2|e^{-i\phi(t)}J^{\mu\nu}Te^{i\phi(t)}|p_1p_2 > - < p_1p_2|e^{-i\phi(t)}TJ^{\mu\nu}e^{i\phi(t)}|p_1p_2 > \tag{4.34}$$

We see that along with the action of the angular momentum operator on the free states we will have a contribution from the action of the operator on the phase factor. Using product rule we will consider only the first term in the above equation.
We will get -

$$i[p_1^\mu \wedge \frac{\partial}{\partial p_1^\nu}] < p_1p_2|e^{-i\phi(t)}Te^{i\phi(t)}|p_1p_2 > -e^2\frac{(p_1 \wedge p_2)^{\mu\nu} \ln t/t_o}{((p_1.p_2)^2 - m^4)^{3/2}} < p_1p_2|e^{-i\phi(t)}Te^{i\phi(t)}|p_1p_2 > \tag{4.35}$$

Similarly one can calculate the second term of the above equation and combine them.
We see that the combination of the first terms are exactly the same as that of our angular momentum impulse calculation at $O(e^2)$, whereas we have an extra second term of $O(e^4)$ due to the phase factor.
After combining the second term in the expression we have -

$$\int e^2\frac{(p_1 \wedge p_2)^{\mu\nu} \ln t/t_o}{((p_1.p_2)^2 - m^4)^{3/2}} - e^2\frac{(p_1 \wedge p_2)^{\mu\nu} \ln t/t_o}{((p_1.p_2)^2 - m^4)^{3/2}} < p_1p_2|e^{-i\phi(t)}Te^{i\phi(t)}|p_1p_2 > \tag{4.36}$$

We write the transition matrix term in terms of the amplitude and using momentum transfer notation to give-

$$\int e^2\frac{(p_1 \wedge p_2)^{\mu\nu} \ln t/t_o}{((p_1.p_2)^2 - m^4)^{3/2}} - e^2\frac{(p_1 \wedge p_2)^{\mu\nu} \ln t/t_o}{((p_1.p_2)^2 - m^4)^{3/2}}[\delta(q_1 + q_2) \cdot A] \tag{4.37}$$

We will consider terms with only with leading power of the Planck's constant. After integration over dq_2 and using the delta function we get-

$$\frac{2b \wedge (p_1 + p_2)\,(p_1 \cdot p_2)\, e^4}{D^2} \ln\left(\frac{t}{t_0}\right) \tag{4.38}$$

The first term we should be dealt carefully as it might look similar to the calculation we have already done but it includes the phase factor with the transition matrix too. It follows similar steps to the Appendix C but but with some more terms .
The terms looks like -

$$\int d^4q\delta^4(q_1+q_2)\delta\left(p_2\cdot q\right)c^{-ib\cdot q}\times\frac{4\left(p_1\wedge p_2\right)}{q^2}+\int d^4q\delta(p\cdot q)e^{-(\phi'-\phi)}A\left(q\wedge\frac{\partial}{\partial q}\right)^{\mu\nu}\delta^4(q_1+q_2) \tag{4.39}$$

Simplifying this we get

$$\frac{2b^{\mu}\wedge\left(p_1^{\nu}\right)\left(p_1\cdot p_2\right)e^4}{D^2}\ln\left(\frac{t}{t_0}\right) \tag{4.40}$$

If we also take into account the impulse of particle two we shall see that we get

$$\frac{2b\wedge\left(p_1+p_2\right)\left(p_1\cdot p_2\right)e^4}{D^2}\ln\left(\frac{t}{t_0}\right) \tag{4.41}$$

We shall see that the term we obtained are nothing but the $O(e^4)$ Angular momentum and the result matches with the literature value obtained in [2].It shows that the the Faddev Kulish factor does account for interaction at a higher order in the far future and far past .It is worth working out more of these calculations using IR finite S- Matrix.

4.3 Particle trajectory from Faddev Kulish states

As we saw how the free states need to be dressed in order to give the correct asymptotic behaviour. We will show here how the Faddev kulish states give the particle trajectory [20] at asymptotic times which will be the explanation of Electromagnetic ”Scoot”. The complex phase factor is contains the information of this long range interaction of the particles under Coulomb force.
Let us consider two point charge particles with dressing.
The wavefunction will be written as -

$$|\Psi(t)>_{in}=\int dp_1dp_2e^{i\phi(t,t_0)}|p_1p_2> \tag{4.42}$$

In position space this wavefunction is given by -

$$|\Psi(t)>_{in}=\int dp_1 dp_2 e^{iF(t,x,p)}|p_1 p_2> \tag{4.43}$$

Where

$$\begin{aligned}
&F\left(t;x_1,p_1;x_2,p_2\right)\\
&=ip_1x+ip_2x_2+i\Phi\left(p_1,p_2;t,t_0\right)\\
&=\left(p_1^0+p_2^0\right)\left(t-t_0\right)-\vec{p}_1\cdot\vec{x}_1-\vec{p}_2\cdot\vec{x}_2+\frac{e_ie_j}{4\pi}\frac{-p_1\cdot p_2}{\left(\left(p_1\cdot p_2\right)^2-m_1^2m_2^2\right)^{1/2}}\log\frac{t}{t_0}
\end{aligned} \tag{4.44}$$

Under the stationary phase assumption we take the derivative of the above equation and put it to zero. We will get the trajectory of the particle.

$$\begin{aligned}
\vec{x}_1&=\frac{\vec{p}_1}{p_1^0}\left(t-t_0\right)+\frac{e_ie_j}{4\pi p_1^0}\frac{m_1^2m_2^2\left(p_2^0\vec{p}_1-p_1^0\vec{p}_2\right)}{\left(\left(p_1\cdot p_2\right)^2-m_1^2m_2^2\right)^{3/2}}\log\frac{t}{t_0}\\
&=\frac{\vec{v}_1}{v_1^0}t+\frac{e_1e_2}{4\pi m_1v_1^0}\frac{v_2^0\vec{v}_1-v_1^0\vec{v}_2}{\left(\left(v_1\cdot v_2\right)^2-1\right)^{3/2}}\ln t+\mathcal{O}\left(\frac{1}{t}\right)
\end{aligned}$$

where we have used $p_i^\mu=m_iv_i^\mu$

We see that this equation is exactly the particle trajectory upto $O(e^2)$ as was also given in [4].
We saw how it was the logarithmic behaviour of the particle trajectory at late times that eventually contributes to the angular momentum at late times and causes the phenomenon of ””Scoot””.
So it is safe to say that the phase factor is the term which accounts for ”Scoot” and since it is missing in the KMOC approach we get a non-conserved angular momentum.

Chapter 5

Soft Theorems

Soft photons are photons emitted with a very low momentum . One can relate the emission of soft photon amplitude from an external leg with that of the amplitude without the soft photon. Using this one can derive several things simply. In papers by Weinberg he used these theorems to show charge and energy conservation in a process, derived Lorentz invariance solely using gauge invariance([32]) . We can also derive Maxwell's equation of motion using these ([33]).
In this section we will give a brief review of soft photon theorems from seminal paper [34] by Weinberg and study various results and their implications in the context of our project.

5.1 Soft photon from Feynman amplitudes

If we attach a soft-photon line with momentum q to an outgoing particle line in a Feynman diagram 5.1, we must add one extra particle propagator with momentum $p+q$ and one extra vertex term for the transition $p + q \to p$.

In the limit $q \to 0$ this factor becomes-

$$\frac{e\eta p^{\mu}}{p.q - i\eta\epsilon} \tag{5.1}$$

η is +1 for outgoing particle and -1 for incoming. This is valid for only the diagrams with

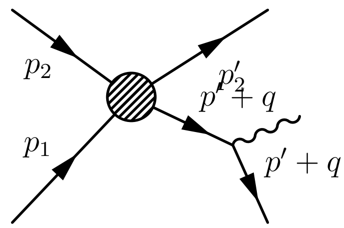

Figure 5.1: Emission of soft photon from an external leg

soft photon line attached to external legs only and not the inner vertices as it will then not have the propagator term then .
We note that if many soft photons are emitted from an outgoing charged-particle line then the charged particle propagators will contribute a factor of

$$[(p.q_1 - i\eta\epsilon)(p.(q_1 + q_2) - i\eta\epsilon)]^{-1}.... \tag{5.2}$$

While writing this we should carefully take care of the combinatoric factors.
Let us define an infrared virtual photon or as one which connects two external lines of the feynman diagram and carries energy less than Λ(5.2). In addition to this we also make a

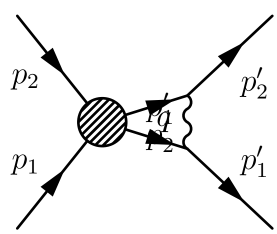

Figure 5.2: Virtual photon exchange

lower cutoff for the momentum of photon to describe the divergences .
The effect of adding N virtual infrared-photon lines is to multiply the matrix element by N pairs of the factors, each pair connected by a photon propagator with it already known to us. With the correct combinatoric factor we get the total multiplication factor to be -

$$= \frac{1}{N!}[\int_{\lambda}^{\Lambda} d^4q A(q)]^N \tag{5.3}$$

where λ is the cutoff to avoid the divergence at $q = 0$

$$e_n e_m \eta_n \eta_m A(q) = \int_{\lambda \leq |\vec{q}| \leq \Lambda} \frac{d^4q}{(2\pi)^4} \frac{-ie_n e_m \eta_n \eta_m p_n \cdot p_m}{[q^2 - i\epsilon]\,[p_n \cdot q - i\eta_n\epsilon]\,[-p_m \cdot q - i\eta_m\epsilon]}, \tag{5.4}$$

where p_n and p_m are the momenta of the two charged particles exchanging the virtual soft photons. Summing ever all the contributions from all the possible combinations of the pairs. The total amplitude becomes

$$S_{\alpha\beta} = S^0_{\alpha\beta} exp(1/2 \int_\lambda^\Lambda d^4 q A(q)) \tag{5.5}$$

The integral in the above equation becomes

$$= -\frac{1}{8\pi^2 \beta_{\mathrm{nm}}} \ln\left(\frac{1+\beta_{\mathrm{nm}}}{1-\beta_{\mathrm{nm}}}\right) \ln\left(\frac{\lambda}{\Lambda}\right) + \frac{i\delta_{\mathrm{mn}}}{4\pi\beta_{\mathrm{mm}}} \ln\left(\frac{\lambda}{\Lambda}\right) \tag{5.6}$$

The calculation for this integral is shown in the .4 β is the relative velocity factor.We will see how this result is helpful in the context of this thesis.

5.2 Sub leading soft photon theorem

. We saw in the previous sections that how that amplitude can be expanded in the powers of ω and found the contribution $O(\omega^{-1})$ We will derive the $O(\omega^0)$ contribution to the amplitude. This section will follow the notations and procedure of [16] First step is to note that the amplitude can be written as -

$$M^\mu(k, p_1...) = \frac{e\eta p^\mu}{p.k - i\eta\epsilon} M_n(p_1, p_2...) + B^\mu(k, p_1....) \tag{5.7}$$

Here the second term is the subleading term Using Ward -Takahashi identity

$$k_\mu M^\mu(k, p_1, p_2..) = 0 \tag{5.8}$$

the first equation reduces to

$$0 = M_n + k_\mu B^\mu(k, p_1....) \tag{5.9}$$

We assume that both the functions B and M are analytic around $k^\mu = 0$ so we expand around it and get,

$$0 = M_n(p_1...) + k_\mu \frac{\partial}{\partial p_\mu} M(p_1) + k_\mu B^\mu(k, p_1....) \tag{5.10}$$

using the above equation we get

$$0 = k_\mu \frac{\partial}{\partial p_\mu} M(p_1) + k_\mu B^\mu(k, p_1....) \tag{5.11}$$

From this equation we see that

$$B^\mu(0, p_1....) = -k_\mu \frac{\partial}{\partial p_\mu} M(p_1) \tag{5.12}$$

Plugging this into the expansion of amplitude.

$$M^\mu(k, p_1...) = \frac{e\eta p^\mu}{p.k - i\eta\epsilon} M_n + \frac{e\eta J^{\mu\nu}}{p.k - i\eta\epsilon} M_n \tag{5.13}$$

Where $J^{\mu\nu}$ is our friendly angular momentum operator (see 1).
It can be written in another form using Low's notation as

$$\lim_{\omega \to 0} < f|(1 + \omega\partial\omega)a_r S|i >= J_\lambda < f|S|i > \tag{5.14}$$

What is important to note here that these theorems are universal, in the sense that they are independent on the type of particle and the kind of scattering.

5.3 Phase factor from soft theorem

As we saw in section 5.1that the ratio of scattering matrix with virtual soft photon gives the following factor of

$$S_{\alpha\beta} = S^0_{\alpha\beta} exp(1/2) \int_\lambda^\Lambda d^4 q A(q)) \tag{5.15}$$

The real part gets cancelled with infinite soft photon emission lines (see [35] for proof)
. What we will see is that the imaginary part will be the Coloumbic phase factor or the Faddev Kulish phase factor that we derived in 4.18. Eq 5.15 can be written as

$$\frac{S_{\beta\alpha}}{S^0_{\beta\alpha}} = \exp\left\{ \frac{1}{2(2\pi)^3} \sum_{nm} e_n e_m \eta_n \eta_m (p_n \cdot p_m) J_{nm} \right\} \tag{5.16}$$

where

$$J_{nm} \equiv i \int^{\Lambda} \frac{d^4 q}{\left[q^2+\lambda^2-i\epsilon\right]\left[p_n \cdot q - i\eta_n \epsilon\right]\left[p_m \cdot q + i\eta_m \epsilon\right]} \tag{5.17}$$

Here we used photon mass λ instead of infrared cutoff. The integral is analytic except for the following value for q^0

$$\begin{aligned} q^0 &= \omega - i\epsilon, & q^0 &= -\omega + i\epsilon, \\ q^0 &= \mathbf{v}_n \cdot \mathbf{q} - i\eta_n \epsilon, & q^0 &= \mathbf{v}_m \cdot \mathbf{q} + i\eta_m \epsilon \end{aligned}$$

where $\omega = (\mathbf{q^2} + \lambda^2)^{1/2}$ If particle n is incoming and m is outgoing then we will close the contour in upper half , therefore only contribution comes from the radiation poles.
The value of J_{nm} is purely real(when one particle is incoming and the other is outgoing or vice versa) given by

$$J_{nm} = -\pi \int^{\Lambda} \frac{d^3 q}{\omega\left(\omega E_n - \mathbf{q} \cdot \mathbf{p}_n\right)\left(\omega E_m - \mathbf{q} \cdot \mathbf{p}_m\right)} \tag{5.18}$$

If both are incoming or outgoing J_{nm} is -

$$\begin{aligned} J_{nm} = -\pi \int^{\Lambda} & \frac{d^3 q}{\omega\left(\omega E_n - \mathbf{q} \cdot \mathbf{p}_n\right)\left(\omega E_m - \mathbf{q} \cdot \mathbf{p}_m\right)} \\ & + \frac{2i\pi^3}{\left[\left(p_n \cdot p_m\right)^2 - m_n^2 m_m^2\right]^{1/2}} \ln\left(\frac{\Lambda^2}{\lambda^2}+1\right) \end{aligned} \tag{5.19}$$

The imaginary divergent phase factor is

$$= \frac{2i\pi^3}{\left[\left(p_n \cdot p_m\right)^2 - m_n^2 m_m^2\right]^{1/2}} \ln\left(\frac{\Lambda^2}{\lambda^2}+1\right) \tag{5.20}$$

Since the UV cutoff is much greator than the infrared cutoff , we can write the imaginary part of the ratio of scattering matrix as

$$\frac{S_{\alpha\beta}}{S^0_{\alpha\beta}} = exp(\frac{2e^2}{\pi\left[\left(p_n \cdot p_m\right)^2 - m_n^2 m_m^2\right]^{1/2}} \ln(\frac{\Lambda}{\lambda})) \tag{5.21}$$

Like claimed,the soft theorem does give us the Faddev kulish phase factor because of the presence of virtual photon exchange. This solidifies our interpretation of the phase factor as due to the long range weak interaction between asymptotic states.

5.4 IR finite S matrix

In this section we will look at the fundamental problem of Infrared Divergences in QED which appeared in the previous section . Firstly discussing its origin and then methods to eradicate them.

We will look at the scattering of charged particles from another heavy charged particle for simplicity. The corresponding Feynman amplitude is denoted by M_0. When we are looking at case where the emitted photon has very low energy (soft photon)then, $\omega = 0$. As also derived in the previous section the new amplitude can be written in the form of old amplitude

$$M = -ieM_0[\frac{p'\epsilon}{p'k} - \frac{p\epsilon}{pk}] \tag{5.22}$$

We see that this amplitude is divergent in the soft limit . The divergence is both at the level of amplitude and cross section. In [6] Bloch and Nordsieck showed how to remove the divergence at the level of the cross section , but the problem of S-matrix divergence still prevailed .

As shown in previous section , Weinberg showed the divergences in the amplitudes and the phase factors .

The solution to this problem was first discussed in [23].We will now see how the phase factors we calculated actually cancel the divergences of the S-matrix .

The kind of divergences that appear are -

- Soft photon emission
- Virtual soft photon exchange

. Let us look try to work out how the FK states actually make the S matrix IR-finite. IR finite S matrix elements are given by -

$$< f|e^{R(t)}e^{i\phi(t)}Se^{-R(t)}e^{i-\phi(t)}|i > \tag{5.23}$$

We have already calculated R(t) previously in 4.18. Let us introduce the following shorthand notation:

$$S_\mu(p,k) \equiv e f_\mu(\mathbf{p},\mathbf{k})$$
$$P_\mu(p,k) \equiv e\frac{p_\mu}{p\cdot k}$$

There are some subtleties regarding these factors satisfying the Gupta-Beluer conditions , which we will not discuss but can be looked at from [13].
Let us see action of this factor on the incoming and outgoing states firstly , the incoming state is modified to -

$$|\mathrm{i}\rangle = e^{R_f(p_i)} c^\dagger(\mathbf{p}_i)|0\rangle = \exp\left(\int \widetilde{d^3k}\left[S_i^\mu a_\mu^\dagger(\mathbf{k}) - S_i^\mu a_\mu(\mathbf{k})\right]\right) c^\dagger(\mathbf{p}_i)\,. \tag{5.24}$$

using commutation relation for photon fields discussed in the Preliminaries 1.

$$\left[\left(\int \widetilde{d^3k} S_i^\mu a_\mu^\dagger\right), \left(-\int \widetilde{d^3k} S_i^\nu a_\nu\right)\right] = \int \widetilde{d^3k} S_i^\mu \eta_{\mu\nu} S_i^\nu. \tag{5.25}$$

Using BCH formula we can write it as

$$\exp\left(\int \widetilde{d^3k}\left[S_i^\mu a_\mu^\dagger(\mathbf{k}) - S_i^\mu a_\mu(\mathbf{k})\right]\right) = \exp\left(\int \widetilde{d^3k} S_i^\mu a_\mu^\dagger(\mathbf{k})\right)\exp\left(-\int \widetilde{d^3k} S_i^\mu a_\mu(\mathbf{k})\right)\times \exp\left(-\frac{1}{2}\int \widetilde{d^3k} S_i^\mu \eta_{\mu\nu} S_i^\nu\right) \tag{5.26}$$

To the lowest order in the photon creation operators this is

$$|\mathrm{i}\rangle = \left(1 - \frac{1}{2}\int \widetilde{d^3k} S_i^\mu \eta_{\mu\nu} S_i^\nu + \int \widetilde{d^3k} S_i^\mu a_\mu^\dagger(\mathbf{k})\right) c^\dagger(\mathbf{p}_i)|0\rangle. \tag{5.27}$$

Similarly, the final state may be written as,

$$< f| = \langle 0|c(p_f)\left(1 - \frac{1}{2}\int \widetilde{d^3k} S_f^\mu \eta_{\mu\nu} S_f^\nu + \int \widetilde{d^3k} S_f^\mu a_\mu(\mathbf{k})\right). \tag{5.28}$$

from

$$\langle \mathrm{f}| = \langle 0|c(p_f)\, e^{-R_f(p_f)}$$

Let us look at cancelling of IR divergence due to soft photon emission .
Let us assume we have a soft photon emission then the final state is -

$$< K, r| =< 0|a_\mu(k) \tag{5.29}$$

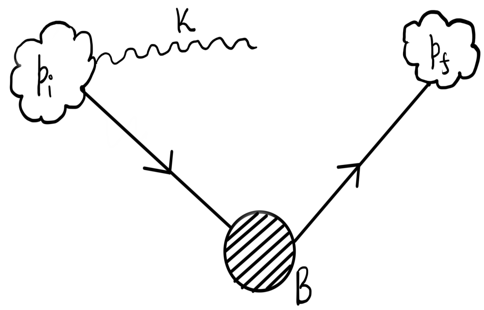

Figure 5.3: Soft photon emission from the photon cloud

r denotes the polarization of the photon. Action of phase will give

$$< K, r|e^{-R_f(p)} S_D e^{-R_f(p)}|i > = < 0|e^{-R_f(p)}[a_\mu + [a_\mu, e^{R_f}]]S_D e^{R_f(p)}|i > \tag{5.30}$$

Here the first term will be the soft photon factor from the external leg 5.1 we already know about.

The second term will also give the same factor but with a negative sign once we open up the commutator and act it on the state 5.3(Notice that in the figure we have use 1-1 scattering , it is clear that the cancellation takes place for 2-2 scattering as well , it is just easier to show in this case). It accounts for the soft emission from the cloud of photons. Hence this first kind of divergences will cancel.

Now the second kind of divergence produces the factor given in 5.15.
These terms will be cancelled if we take the exchange of soft photon between the clouds and the external legs and exchange between the clouds themselves clearly as they are higher order terms in e or charge.These process are given in the form of the diagram as - The contribution from the exchange between cloud and external leg is given by the third term in 5.27
These contribution to the S matrix elements are order e^4 given by,

$$M^a_{\text{int}} = \int \widetilde{d^3k} S^\mu (p_i, k)\, \eta_{\mu\nu} \frac{i}{-2p_i \cdot k} (2iep^\nu_1) = \int \widetilde{d^3k} S^\mu (p_i, k)\, \eta_{\mu\nu} e \frac{p^\nu_i}{p_i \cdot k} \tag{5.31}$$

Similarly the contribution from other figures in the diagram are given by ,

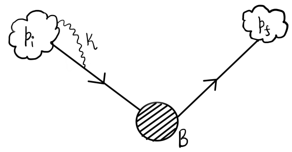

Figure 5.4: exchange diagram (a)

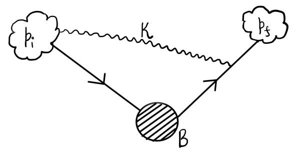

Figure 5.5: exchange diagram(b)

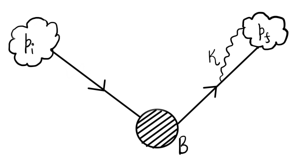

Figure 5.6: exchange diagram(c)

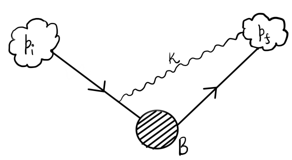

Figure 5.7: exchange diagram(d)

For b)

$$M_{\text{int}}^{b} = -\int \widetilde{d^3k} S^{\mu}\left(p_i, k\right) \eta_{\mu\nu} P^{\nu}\left(p_f, k\right) \tag{5.32}$$

For c)

$$M_{\text{int}}^{c} = +\int \widetilde{d^3k} S^{\mu}\left(p_f, k\right) \eta_{\mu\nu} P^{\nu}\left(p_f, k\right) \tag{5.33}$$

For d)

$$M_{\text{int}}^{d} = -\int \widetilde{d^3k} S^{\mu}\left(p_f, k\right) \eta_{\mu\nu} P^{\nu}\left(p_i, k\right) \tag{5.34}$$

For the figure involving emission and re-absorption of soft photon by the same cloud comes from the 2nd term in 5.27, calculated to give,

$$\begin{aligned} M^{i} &= -\frac{1}{2} \int \widetilde{d^3k} S^{\mu}\left(p_i, k\right) S^{\nu}\left(p_i, k\right) \left\langle 0 \left| a_{\mu}(\mathbf{k}) a_{\nu}^{\dagger}(\mathbf{k}) \right| 0 \right\rangle = \int \widetilde{d^3k} S_i \cdot S_i. \\ M^{f} &= -\frac{1}{2} \int \widetilde{d^3k} S^{\mu}\left(p_f, k\right) S^{\nu}\left(p_f, k\right) \left\langle 0 \left| a_{\mu}(\mathbf{k}) a_{\nu}^{\dagger}(\mathbf{k}) \right| 0 \right\rangle = \int \widetilde{d^3k} S_f \cdot S_f. \end{aligned} \tag{5.35}$$

Here we have done it for both clouds associated with initial and final states.
The exchange between the two different clouds comes as -

$$\int \widetilde{d^3k d^3k'} S^{\mu}\left(p_f, k'\right) S^{\nu}\left(p_i, k\right) \left\langle 0 \left| a_{\mu}(\mathbf{k}) a_{\nu}^{\dagger}\left(\mathbf{k}'\right) \right| 0 \right\rangle . \tag{5.36}$$

this equals to

$$M_{\text{ctc}}^{\text{dis}} = \int \widetilde{d^3k} S^{\mu}\left(p_f, k\right) \eta_{\mu\nu} S^{\nu}\left(p_i, k\right) = \int \widetilde{d^3k} S_i \cdot S_f \tag{5.37}$$

The total contribution is calculated by adding all of the terms we found and it equals to -

$$M = \frac{1}{2} \int \widehat{d^3k} \left[P_i \cdot P_i + P_f \cdot P_f - 2P_i \cdot P_f\right] \tag{5.38}$$

Slight modifications by going back to our original notation of particle momentum p_i we see that this is exactly the lowest power expansion of the soft factor given in 5.15
We can hence see that the IR divergences are cancelled , This cancellation can be proven to occur at all orders refer to [9].

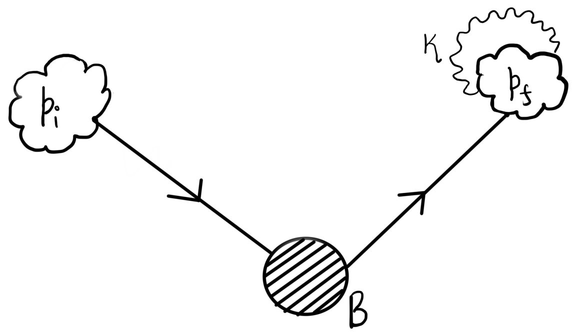

Figure 5.8: Cloud to cloud disconnected diagram)(a)

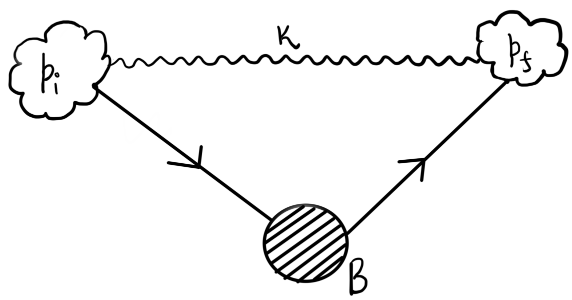

Figure 5.9: Cloud to cloud disconnected diagram (b)

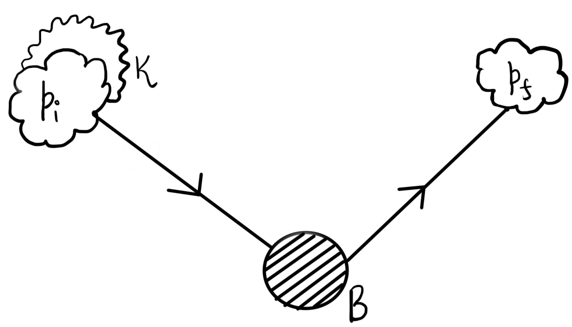

Figure 5.10: Cloud to cloud connected diagram

Chapter 6

Summary and Outlook

In this thesis we presented a review of formalism that calculates classical obervable from feynman amplitude methods.We used that formalism to calculate angular momentum impulse in a 2 body scattering problem. . Some new results regarding Angular momentum impulse were obtained which were non intuitive. We then looked at the classical calculations which also produced similar results as that of our KMOC result .The classical counter part of the Angular momentum Impulse which was called ""Scoot"" ,was understood to be the main reason for this phenomenon.
We alo tried to find a solution to this problem of ""Scoot"" which lead us to review Faddev Kulish States in the process.These states were shown to inculcate the information of the particle in the far past and thus explained the phenomenon of ""Scoot"".Faddev Kulish state introduced us to the problem of IR divergence of the QED theory. We studied similar concepts of the infrared divergence problems using soft theorems. We saw how the Faddev Kulish states cancel the divergences in the amplitude and the cross section due to emission of soft photons.
With this basic clarity about various concepts it becomes possible to tackle many open problems in this field. One can extend this work by using the dressed states instead of asymptotic states to calculate observables using KMOC. The finite S marix elements have been formulated but never used somehow. The concept of FK states relates to the existence of coherent states of photon . These coherent states are readily used to calculates observables via Thompson Scattering. "Scoot" can be anaylsed through this approach of coherent incoming state od photons in amplitude as well then. The similar calculations can

be performed for gravitons and can help develop the idea of Memory effects via KMOC. This project wide plethora of questions and conceptual understanding of heavily used but less understood concepts in QFT .

.1 Appendix A

Here we will show how to calculate certain integrals used in 2.23

The integrals we are working with have the form :-

$$I = iq_1q_2 \int \frac{d^4q}{(2\pi)^4}\hat{\delta}\left(2p_1 \cdot l\right)\hat{\delta}\left(2p_2 \cdot l\right)e^{-ibq}\frac{F(p_1,p_2)}{q^2}q^{\mu} \tag{1}$$

We notice that we have two delta functions inside the integral. We proceed with integration over q^0 and use the $i\epsilon$ prescription to bypass the poles.

$$\frac{1}{q^2+i\epsilon} = \frac{1}{(q^0+|\boldsymbol{q}|+i\epsilon)(q^0-|\boldsymbol{q}|-i\epsilon)} \tag{2}$$

But since we have delta function $\delta(p_2.q)$ we can choose to evaluate the integral in the rest frame of p_2. The delta function would then become $\delta(q^0)$ This delta function will pick up the pole at $q^0 = 0$ and the value of propagator would be.

$$\frac{1}{|\boldsymbol{q}|+i\epsilon)(-|\boldsymbol{q}|-i\epsilon)} \tag{3}$$

Hence we can say that if we have atleast one delta function in the numerator along with a propagator we need not consider the contribution of poles from the propagator .

We can solve this integral by choosing a particular frame , but we will look at general methods to solve it . We will decompose q into components along the velocity and transverse to it .

$$q_1 = \alpha_1 p_1 + \alpha_2 p_2 + q_{\perp} \tag{4}$$

with

$$\begin{aligned}\alpha_1 &= \frac{(q\cdot p_1)m_2^2-(q\cdot p_2)(p_2\cdot p_1)}{D}\\ \alpha_2 &= \frac{(q\cdot p_2)m_1^2-(q\cdot p_1)(p_2\cdot p_1)}{D}\end{aligned} \tag{5}$$

Where

$$D = (p_1.p_2)^2-(m_1m_2)^2 \tag{6}$$

With change of variables the integration measure becomes

$$d^4q = \frac{1}{\sqrt{D}} d^2q_\perp dx_1 dx_2 \tag{7}$$

Integration over x can be done easily , all we need to do is to substitute the new varible to 0 in the denominator of the propagator.
We will be left with the following integral

$$I = iq_1q_2 \int \frac{d^2q}{(2\pi)^2} e^{-ibq_\perp} q_\perp^\mu q_\perp^2 (8)$$

Converting the $q_\perp^\mu$ integral in polar coordinates.

$$I = \frac{q_1q_2}{\sqrt{D}} \partial_{b^\mu} lim_{\mu\to 0} \int_\mu^\infty \frac{dq_\perp}{q_\perp} \int_0^{2\pi} d\theta e^{-ibq_\perp} \tag{9}$$

$$I = \frac{q_1q_2}{\sqrt{D}} \partial_{b^\mu} lim_{\mu\to 0} \int_\mu^\infty \frac{dq_{\perp J_0(q_\perp b_\perp)}}{q_\perp} \tag{10}$$

Where J_0 is the zeroth order Bessels function , its integration is known.

$$I = \frac{q_1q_2}{\sqrt{D}} \partial_{b^\mu} lim_{\mu\to 0} \log(-b^2\mu^2) \tag{11}$$

Now evaluating the limit after differentiation gives.

$$\Delta p_1^\mu = -k^2 \frac{2(p_1.p_2)^2 - m_1^2 m_2^2 b^\mu}{\sqrt{D} b^2} \tag{12}$$

.2 Appendix B

Feynman rules for scalar graviton coupling (for derivation refer [12]).
The vertex term is given by -

$$i\frac{k}{2}[(p_\mu p'_\nu + p'_\mu p_\nu) - \eta_{\mu\nu}(p.p' - m^2)]$$

The graviton propagator is given by -

$$i\frac{\eta^{\alpha\gamma}\eta^{\beta\delta}+\eta^{\alpha\delta}\eta^{\beta\gamma}-\eta^{\alpha\beta}\eta^{\gamma\delta}}{2q^2}$$

.3 Appendix C

The expression we start with is similar to momentum impulse expression-

$$\Delta J^{\mu\nu} =< \psi|i[J^{\mu\nu},T]|\psi > \tag{13}$$

expanding the the commutator we get :-

$$\Delta J^{\mu\nu} =< \psi|iJ^{\mu\nu},T|\psi > - < \psi|iTJ^{\mu\nu}|\psi > \tag{14}$$

The angular momentum operator action in the ket state is easy but on the action of bra state one has to insert a complete set of sets in between . The expression we get after doing that and integrating is-

$$\Delta J^{\mu\nu} = i[p_1^{\mu}\wedge\frac{\partial}{\partial p_1^{\nu}} + p_1^{'\mu}\wedge\frac{\partial}{\partial p_1^{'\nu}}] < \psi|T|\psi > \tag{15}$$

Inserting the form the wavefunction we get-

$$\Delta J^{\mu\nu} = \int d^4q_1 d^4q_2 \hat{\delta}\left(q_1\cdot p_1\right)\hat{\delta}\left(q_2\cdot p_2\right)e^{-ib\cdot q}[p_1^{\mu}\wedge\frac{\partial}{\partial p_1^{\nu}}+p_1^{'\mu}\wedge\frac{\partial}{\partial p_1^{'\nu}}](\delta^4\left(q_1+q_2\right)\frac{(p_1+p_1^{'}).(p_2+p_2^{'})}{q^2}) \tag{16}$$

Now we will use product rule , the first term will be when the operator acts on the stripped amplitude and the second where the operator acts on the delta function. The expression for the first term is easy but should be done carefully considering what quatitites out of p,p' and q are taken to be independent. We get -

$$= \delta^4\left(q_1+q_2\right)\frac{4p_1\wedge p_2}{q^2}(h^{-1}) \tag{17}$$

We will work with the second term from the product rule

$$\int d^4q_1 d^4q_2 \hat{\delta}\left(q_1 \cdot p_1\right) \hat{\delta}\left(q_2 \cdot p_2\right) e^{-ib\cdot q} \frac{(p_1 + p_1^{'}).(p_2 + p_2^{'})}{q^2} [p_1^{\mu} \wedge \frac{\partial}{\partial p_1^{\nu}} + p_1^{'\mu} \wedge \frac{\partial}{\partial p_1^{'\nu}}] \delta^4\left(q_1 + q_2\right) \quad (18)$$

$$= \int d^4q_1 d^4q_2 \hat{\delta}\left(q_1 \cdot p_1\right) \hat{\delta}\left(q_2 \cdot p_2\right) e^{-ib\cdot q} \frac{(p_1 + p_1^{'}).(p_2 + p_2^{'})}{q^2}$$

$\overline{q_1^{\mu} \wedge \frac{\partial}{\partial q_1^{\nu}} \delta^4(q_1+q_2)(19)}$ Applying by parts:-

$$-\int d^4q_2 \delta^4\left(q_1 + q_2\right) \delta\left(2q_2 \cdot p_2 + q^2\right) q^{\mu} \wedge \frac{\partial}{\partial q^{\nu}} [\delta\left(2q_1 \cdot p_1 + q^2\right) e^{-ib\cdot q} \frac{(p_1 + p_1^{'}).(p_2 + p_2^{'})}{q^2}] \quad (20)$$

$$= -\int d^4q_2 \delta^4\left(q_1 + q_2\right) \delta\left(2q_2 \cdot p_2 + q^2\right) q^{\mu} \wedge (b^{\nu} e^{-ib\cdot q} + e^{-ib\cdot q} \frac{\partial}{\partial q^{\nu}}) [\delta\left(2q_1 \cdot p_1 + q^2\right) \frac{(p_1 + p_1^{'}).(p_2 + p_2^{'})}{q^2}] \quad (21)$$

$$= -\int d^4q_2 \delta^4\left(q_1 + q_2\right) \delta(p_2.q) [A q_1 \wedge p_1 \delta^{'}\left(2p_1.q_1\right)] \quad (22)$$

Other terms of form $q \wedge b$ is 0 as δp^{μ} is in the direction of b^{μ} and other term of higher order in h

$$\int d^4q_1 \delta(p_2.q) A (q_1 \wedge p_1) \frac{\partial}{\partial(p_1.q_1)} \delta(2p_1.q_1)] \quad (23)$$

Applying by parts again

$$\int d^4q_1 \delta(p_2.q) [A \frac{\partial}{\partial(p_1.q_1)} (q_1 \wedge p_1) + (q_1 \wedge p_1) \frac{\partial}{\partial(p_1.q_1)} A] \quad (24)$$

We get $O(h^{-1})$ terms only from the first term in the above expression

We shall expand q^{μ} in terms of the momentum as we did previously and substitute it. The final expression we get from the second term hence is

$$\int d^4q_1 \delta(p_2.q) [A (p_1 \wedge p_2) p_1.p_2 / D \quad (25)$$

All that is left is to combine these to terms. We shall do similar calculation in other section therefore one must look at this appendix for reference for those calculations as well.

.4 Appendix D

We shall look at the integrals of form 5.6. The integral expression is -

$$e_n e_m \eta_n \eta_m A(q) = \int_{\lambda \leq |\vec{q}| \leq \Lambda} \frac{d^4 q}{(2\pi)^4} \frac{-i e_n e_m \eta_n \eta_m p_n \cdot p_m}{[q^2 - i\epsilon] [p_n \cdot q - i\eta_n \epsilon] [-p_m \cdot q - i\eta_m \epsilon]}, \tag{26}$$

The integral over l^0 can be evaluated vi residues. first closing the contour in the upper half-plane lends to

$$J_{nm} = i\pi \int_{|\vec{r}| < \Delta} \frac{\vec{i}}{(2E_n|q| + 2\vec{p}\hat{q})(2E_m|q| - 2\vec{p})|q|} \tag{27}$$

and, going to polar cordinates $\vec{q} - \omega\hat{\imath}$ with $|\hat{\hat{n}}| = 1$.

$$I_{-} = \frac{i\pi}{4} \int_0^{\lambda} \frac{d\omega}{\omega^{2+2}} \int \frac{d\Omega(\tilde{n})}{(E + \vec{p}\hat{n})(E - \vec{p}\hat{n})}$$

For small ϵ

$$\int_0^{\lambda} \frac{dy}{\omega^{1+2\epsilon}} - \left[-\frac{\omega^{-2\epsilon}}{2x} \right]_0^3 = \frac{\Lambda^{-2}}{2\epsilon} = \frac{1}{2\epsilon} + O\left(e^2\right)$$

Evaluating the two-dimensional angular integral over the sphere gives

$$I_{-} = -\frac{i\pi^2}{4\epsilon} \int_{-1}^{+1} \frac{dx}{(E_n + |\vec{p}|x)(E_m - |\vec{p}|x)} = \frac{-i\pi^2}{4\epsilon \{E_n + E')|\vec{p}|} \log \frac{(E + |\vec{p}|)(E_m + |\vec{p}|)}{(E - |\vec{p}|](E_m - |\vec{p}|)}.$$

This result can be cast in the invariant form

$$I_{-}(p, p') = K(p, p') = -\frac{1}{\epsilon 2mm'} \frac{arc \cosh \sigma_{pp'}}{\sqrt{\sigma^2} - 1}.$$

where

$$\sigma_{pp'} = -\frac{pp'}{mm^r}, \quad p^2 = m^2, \quad p^2 = m^2$$

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