Evaluation of Multi-loop Feynman integrals using Modern methods

A Thesis

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by

Prem Agarwal

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Prof. V Ravindran Co-Supervisor: Dr Naranyan Rana

Certificate

This is to certify that this dissertation entitled ”Evaluation of Multi-loop Feynman integrals using Modern methods” towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Prem Agarwal at Indian Institute of Science Education and Research under the supervision of Prof. V Ravindran, Professor at Department of Physics, Institute of Mathematical Sciences, Chennai and Dr Narayan Rana, Assistant Professor at Department of Physics, NISER Bubaneswar, during the academic year 2022-2023.

Prof. V Ravindran

Dr Narayan Rana

Committee:

Prof. V Ravindran

Dr Narayan Rana

Dr Diptimoy Ghosh

This thesis is dedicated to my mummy, papa, and my brother.

Declaration

I hereby declare that the matter embodied in the report entitled "Evaluation of Multi-loop Feynman integrals using Modern methods" are the results of the work carried out by me at the Department of Physics, Institute of Mathematical Sciences, Chennai and Indian Institute of Science Education and Research, Pune, under the supervision of Prof. V Ravindran and Dr Narayan Rana, and the same has not been submitted elsewhere for any other degree.

Prem Agarwal

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Abstract

Multi-loop Feynman integrals have been a significant roadblock in the journey of precision in collider physics and calculations in perturbative gravity. The search for New Physics amplifies the need to compute multi-loop integrals effectively. In the first half of the project, we understand the developments in analytical calculation of multi-loop integrals using modern methods such as IBP reduction and the differential equations method for the master integrals.

In the second half of the project, we study the idea of obtaining a generalized boundary condition for the differential equations by adding an auxiliary mass to the master integrals[1]. We then extend the idea by solving the differential equations of master integrals w.r.t. η using the techniques of series expansions and analytical continuation, and obtain the numerical solutions of the master integrals for one-loop and two-loop integrals, including the possibilities of having complex mass in the integrals.

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Introduction

The discovery of the Higgs boson in 2012 led to the beginning of the precision era in high-energy physics. Higgs boson, being the only non-composite scalar particle, has opened the paths to the study of various couplings with different particles, useful for the Standard Model and Beyond Standard Model corrections to various particle processes. Therefore, extensive study of the Higgs sector has extended the scope of precision collider physics. [2]

While studying the Standard Model(SM) and Beyond Standard Model(BSM) processes perturbatively, the precision calculations in collider physics play a very important role. These perturbative calculations are often done beyond leading orders to meet the precision limits detected by experiments at the LHC, which makes the computations of multi-loop integrals a key step. These perturbative precise calculations are required in order to distinguish the New Physics sector from the Standard Model sector at the high energy limit. To present why the computation of multi-loop integrals is an essential step, we try to motivate the need to calculate the Next-to-Next Leading order(NNLO), Next-to-next-to-next-to leading order(N3LO) and Higher corrections to the Quantum Chromodynamic(QCD) processes measured at the LHC.

To obtain SM and BSM physics, one of the most interesting precision studies is Higgs boson production. As gluons cannot directly couple to Higgs, QCD corrections to Higgs production is a top-mediated process, which makes it loop-induced[2]. To match the precision limits at the LHC, NNLO[3, 4, 5, 6, 7] and N3LO[8, 9, 10] corrections to Higgs production become relevant and have been calculated. Another path to Higgs production is in association with a vector boson, which gives the opportunity to probe Higgs couplings to gauge bosons. In order to obtain the contributions on the electroweak(EW) scale, Next-to-Leading Order(NLO) QCDxEW[11][12] and NNLO QCD[13][14] corrections to the Higgs production via vector boson fusion are calculated. Similarly, to match the precision of the future e^+e^- colliders, two-loop and three-loop EW and mixed QCDxEW corrections to pair production processes such as the Drell-Yan process[15, 16, 17] become important, especially to probe the Higgs-W coupling and obtain the mass of W at high precision. [18]

The bottlenecks of obtaining NNLO and N3LO corrections to the above processes are complicated multi-loop Feynman diagrams with increasing mass scales. They become significantly difficult with increasing loops, legs and mass scales of the processes targeted by the present and future colliders. To make progress, it becomes increasingly important to effectively evaluate these Feynman integrals both analytically and numerically. In the next section, we talk about the progress made to achieve the task up to the highest precision.

Multi-loop Feynman Integrals

While performing any perturbative calculation, computing scattering cross-sections becomes the most crucial step to obtain physical information about any particle process, including Standard Model Physics, Beyond Standard Model Physics and gravity. The Feynman diagrams obtained while doing a perturbative study can be a tree-level diagram, a one-loop diagram, or multi-loop diagrams. As we discussed in the last section, QCD and Electroweak corrections to Higgs physics have significant contributions from two-loop and higher Feynman diagrams. To achieve these corrections, the evaluation of multi-loop integrals becomes the key and the most challenging step in precision physics.

In the 20^{th} century, the computation of one-loop integrals of SM and BSM processes was done using various integration tricks like Schwinger and Feynman Paramatrizations, Mellin-Barnes Integrations and frame dependant integrals.[19] The onset of the 21^{st} century has seen a development in modern methods to evaluate multi-loop Feynman integrals.

The most successful method is to reduce a family of Master Integrals using Integration-by parts identities (IBP) and Lorentz Invariance (LI). Using IBP and LI identities, we obtain a set of master integrals following the Laporta algorithm [20]. The algorithm helps us to obtain master integrals, and write any integral of the family as a linear combination of master integrals. The Laporta algorithm has been publically implemented as programs like LiteRed[21], Reduze[22], FIRE[23] and KIRA[24], which enables us to perform reductions effectively up to two loops. Recent years have seen developments that suggest we perform

IBP reduction numerically over finite fields(implemented in FiniteFlow[25]) to tackle two-loop integrals with multiple mass scales.

Obtaining the solutions of master integrals has been the bottleneck in the field of precision physics. In the last two decades, analytical and numerical techniques have been developed to achieve the solution of two-loop master integrals effectively. We discuss the advantages and limitations of these techniques, and these techniques have been discussed in detail in sections [2-4].

Analytical Methods

The analytic calculations of master integrals are commonly achieved using the method of differential equations. In this method, we set up a linear differential equation for the basis of master integrals w.r.t. each kinematic variable. The differential equation is then solved via canonical transformations to ϵ-form, and we obtain the solution in terms of polylogarithms, multiple polylogarithms and elliptical polylogarithms in massive cases. These Polylogarithms can be simplified using the public packages PolyLogTools [26] and HPL[27].

In order to solve the basis of master integrals using the above method, we need a boundary condition for each master integral at a point in the physical region. This is often hard to obtain for complicated integrals and poses a limitation to this method. Another limitation of the method is that the matrix A of some basis of master integrals cannot be written into the ϵ-form, which poses a difficulty in solving the differential equation. While solving a massive Feynman integral, we obtain an elliptic integral after the canonical transformation, which results in elliptic polylogarithms, which are yet to be completely understood. This limits the use of differential equation techniques for massive Feynman integrals.

Numerical Methods

With the improving computational speed and technology, numerical evaluation of Feynman integrals can be effectively computed to match the precision required by the colliders. One of the successful algorithms for numerical integration by isolation of poles is the Sector Decomposition method[28], and has been developed into a public package *pySecDec* [29]. Another promising technique to perform numeric computation is the semi-analytic solution of the system of differential equations set up in the previous section, using series expansion. This method requires a numeric boundary condition of the master integral in any region of space, and we obtain the solutions of master integrals as a Laurent expansion in epsilon with numerical coefficients. These algorithms have been publicly released as programs *DiffExp* [30] and *SeaSyde*[31]. The package *SeaSyde* [31] also includes the possibility of evaluating Feynman diagrams containing complex masses.

One of the limitations that the series expansion method poses is that it requires us to calculate the numerical boundary condition for each master integral, which is often difficult to calculate. This limitation was overcome by developing a technique that obtains a generalized boundary condition for each master integral by adding an auxiliary mass to the propagators and taking it to a high mass limit. This method has been implemented as a Mathematica package *AMFlow* [32]. We present it in detail in section 3.

Aim of the thesis

The initial aim of the thesis was to understand and implement the modern methods of evaluating multi-loop Feynman integrals, the key aspect of precision physics. We wished to obtain the skill of performing multi-loop calculations required at the frontier of collider physics.

In the first half of the project, we focused on understanding the analytical and numerical methods stated in the above sections and getting familiar with the state-of-the-art programs such as *LiteRed*[21] to perform IBP reduction and *PolyLogTools*[26] to simplify multiple polylogarithms for some complex integrals.

In the second half of the project, we use the idea of obtaining a generalized boundary condition by adding an auxiliary mass[1] and implementing it as a Mathematica program with some modifications to the algorithm. We then wish to extend the algorithm by including the techniques to compute mixed QCD-EW corrections to particle processes, where the master integrals have complex masses due to the finite decay width of W and Z particles[17].

Chapter 1

Preliminaries

1.1 Basic perturbative QFT and scattering amplitude

Today, particle physics is understood experimentally using particle colliders, but the underlying theory to study most of the particles and fields is perturbative Quantum Field Theory. We show the representation and significance of Feynman integrals in the theory of pQFT and how it connects to the physical observables at particle colliders[33].

To understand the basic concepts, we start with a single field $\phi(x)$. The Lagrangian of the field is given by $\mathcal{L}(\phi)$, and the action functional is given by:

$$S(\phi(x)) = \int d^D x \mathcal{L}(\phi) \tag{1.1}$$

For a field ϕ with $\mathcal{L}(\phi)$, the n_{G} -point Green function in free theory is given by

$$\langle 0 \left| T\left(\phi\left(x_1\right) \ldots \phi\left(x_{n_G}\right)\right) \right| 0 \rangle = \frac{\int \mathcal{D}\phi \phi\left(x_1\right) \ldots \phi\left(x_{n_G}\right) e^{iS(\phi)}}{\int \mathcal{D}\phi e^{iS(\phi)}} \tag{1.2}$$

We wish to solve this Green's function, and it can be done in two ways: the interaction picture of $\phi(x)$ or using the path integral approach. Here, we use the path integral approach. The path integral of this field is given by:

$$U = \int D\phi e^{iS[\phi]} \tag{1.3}$$

In order to obtain a generating functional for the action, we introduce an auxiliary field $J(x)$, and integrate over all field configurations $\phi(x)$

$$U[J] = \mathcal{N} \int D\phi e^{i\left(S[\phi] + \int d^D x J(x)\phi(x)\right)} \tag{1.4}$$

Using the equation 1.4 and the functional derivatives, we can express the n_G point function in free space as:

$$\langle 0 \left| T\left(\phi\left(x_1\right) \ldots \phi\left(x_{n_G}\right)\right) \right| 0 \rangle = (-i)^{n_{\text{ext}}} \left. \frac{\delta^{n_G} U[J]}{\delta J\left(x_1\right) \ldots \delta J\left(x_{n_{\text{ext}}}\right)} \right|_{J=0} \tag{1.5}$$

To obtain the Green functions, we use the relation:

$$U[J] = e^{iV[J]} \tag{1.6}$$

and use it and equation 1.5 to write the Greens function as:

$$G_{n_{\text{ext}}}\left(x_1, \ldots, x_{n_{\text{ext}}}\right) = (-i)^{n_{\text{ext}}-1} \left. \frac{\delta^{n_{\text{ext}}} V[J]}{\delta J\left(x_1\right) \ldots \delta J\left(x_{n_{\text{ext}}}\right)} \right|_{J=0} \tag{1.7}$$

We can define the Greens function in the momentum space by a Fourier transformation given by:

$$\begin{aligned} &G_{n_G}\left(x_1, \ldots, x_{n_G}\right) = \\ &\quad \int \frac{d^D p_1}{(2\pi)^D} \ldots \frac{d^D p_{n_G}}{(2\pi)^D} (2\pi)^D \delta^D\left(p_1 + \ldots + p_{n_G}\right) \tilde{G}^p_{n_G}\left(p_1, \ldots, p_{n_G}\right) e^{-i\left(p_1 x_1 + \ldots + p_{n_G} x_{n_G}\right)} \end{aligned} \tag{1.8}$$

Using this relation, we obtain the scattering amplitude of the n_G point function of field $\phi(x)$. The scattering amplitude is given by:

$$\iota\mathcal{A}(p_1, p_2, \ldots, p_{n_G}) = \frac{\tilde{G}^p_{n_G}(p_1, \ldots, p_{n_G})}{G^p_2(p_1) G^p_2(p_2) \ldots G^p_2(p_{n_G})} \tag{1.9}$$

The scattering amplitude $\mathcal{A}$ acts as the connection between the calculations done in QFT to the physical observables measured at the colliders. Consider an observable $\mathcal{O}$ measured by experiments at the particle collider. We have the relation:

$$\mathcal{O} \propto |\mathcal{A}|^2 \tag{1.10}$$

The Greens function in equation 1.8 is expressed in the form of propagators that arise from the Lagrangian of the fields. We get the expression from equation 1.7.

The terms in a Lagrangian of fields ϕ_1 and ϕ_2 are of the form:

$$\mathcal{L} = \sum_{a,b} \phi_a D_{ab} \phi_b \tag{1.11}$$

Then the inverse of the Fourier transform of the term D_{ab} is known as the propagator for the field.

$$\tilde{P_{ab}} = \tilde{D_{ab}^{-1}} \tag{1.12}$$

The Greens function is then written as the product of the propagators. Putting it back into the momentum greens function, we get:

$$\tilde{G}^p_{n_G}(p_1, \ldots, p_{n_G} = \sum \prod_{pairs} P_{ab} \tag{1.13}$$

where we get the Greens function as a product of two or more propagators.

These propagators in the momentum space with integration over the arbitrary momentum after renormalization are what constitute a Feynman integral.

Chapter 2

Evaluation of Feynman Integrals for each Kinematic Variables

We consider the following renormalized family of integrals in the momentum space:[33]

$$I_{v_1 \ldots v_{\text{int}}}\left(D, x_1, \ldots, x_{N_B}\right) = e^{n\varepsilon\gamma_{\text{E}}}\left(\mu^2\right)^{\nu-\frac{lD}{2}} \int \prod_{r=1}^{n} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n_{\text{int}}} \frac{1}{\left(P_j\right)^{v_j}} \tag{2.1}$$

where P_j are the propagators of momentum for the particular family of integrals. Here, the kinematic variables $x_1, x_2, x_3, \ldots$ are given by: $-\frac{p_i \cdot p_j}{\mu^2}, \frac{m_i^2}{\mu^2}$

2.1 Integration by-parts relations

Integration-by-parts identities are relations that help us to express any Feynman Integral as a linear combination of basis integrals, which we call Master Integrals. They are the basic integrals for each family, which can be used to express any higher integral of the family. These identities are based on the fact that for each momenta, the integral of the derivative of the product of momenta and the integral w.r.t loop momentum is zero[33].

$$\int \prod_{r=1}^{l} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial l_i^\mu} q_{\text{IBP}}^\mu \prod_{j=1}^{n_{\text{int}}} \frac{1}{(P_j)^{v_j}} = 0 \tag{2.2}$$

This identity is derived from the fact that within dimension regularization of an integral, addition of any momentum q_{IBP}^μ to the propagators P_j does not change the integral. Consider the function:

$$\begin{gathered} g(l^\mu) = \prod_{j=1}^{n_{\text{int}}} \frac{1}{(P_j)^{v_j}} \\ \int \prod_{r=1}^{l} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} g(l^\mu) = \int \prod_{r=1}^{l} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} g(l^\mu + \lambda q^\mu) \end{gathered} \tag{2.3}$$

If we expand the RHS around $\lambda = 0$, the terms dependent on λ should go to zero for the above property to be true. Hence,

$$\begin{aligned} &\int \prod_{r=1}^{l} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial \lambda} g(l^\mu + \lambda q^\mu) = 0 \\ &= \int \prod_{r=1}^{l} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{l^\mu} [q^\mu . g(l^\mu)] = 0 \end{aligned} \tag{2.4}$$

Hence, we obtain the IBP reduction identity as shown in equation 2.2.

One-loop example

Consider the following integral:

$$\int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2)^{a_1} \left((\ell_1 + p_1)^2\right)^{a_2}} \tag{2.5}$$

The IBP identity for $q_{\text{IBP}} = l_1$ is:

$$\int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} l_1^\mu . \frac{1}{(\ell_1^2)^{a_1} \left((\ell_1 + p_1)^2\right)^{a_2}} = 0 \tag{2.6}$$

On mathematical manipulation, we obtain the following identity:

$$(D - a_1 - 2a_2)\, I_{a_1 a_2} - a_1 I_{(\mathrm{a}_1+1)(\mathrm{a}_2-1)} + \mathrm{a}_1(2 - p_1^2) I_{(\mathrm{a}_1+1)\mathrm{a}_2} + 2\mathrm{a}_2 I_{\mathrm{a}_1(\mathrm{a}_2+1)} = 0 \tag{2.7}$$

The relations 2.7 and another IBP relation obtained by setting $q_{\mathrm{IBP}} = p1$ give rise to the Master Integrals for each family. The basis of Master Integrals is also enough to calculate all higher integrals of this family.

For the one loop example, the basis of master integrals is:

$$\vec{I} = \{I_{01}, I_{02}\}$$

The algorithm to obtain master integrals and use IBP relations to reduce higher integrals of a particular family to a linear combination of master integrals is known as Laporta algorithm[20]. Using the package LiteRed[21], we perform the following reduction:

$$I_{43} = \frac{1}{12}(-12 + D)(-10 + D)(-7 + D)(-5 + D)(-3 + D) I_{11} \tag{2.8}$$

Two-loop three-point diagram

We consider the two-loop three-point integral:

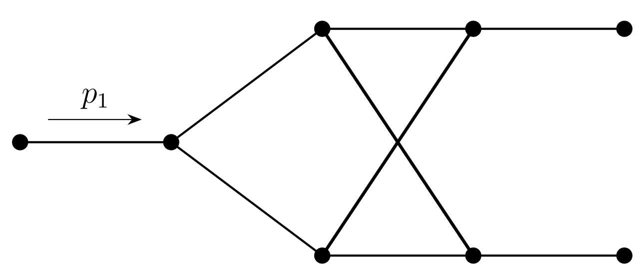

Figure 2.1: Two-loop three-point diagram

$$
I_{a_1 \dots a_7} = \int \frac{\mathrm{d}^D l_1 \ \mathrm{d}^D l_2}{\left[(l_1 - p_1)^2\right]^{a_1} \left[(l_2 - p_1)^2 - m^2\right]^{a_2} \left[(l_1 + p_2)^2\right]^{a_3} \left[(l_1 - l_2 + p_2)^2 - m^2\right]^{a_4}} \times \frac{1}{\left[(l_1 - l_2)\right]^{a_5} \left[(l_1^2)^{a_6}\right] (l_2^2)^{a_7}} \tag{2.9}
$$

We choose the kinematics as:

$$
p_1 = 0, \quad p_2 = 0
$$

$$
(p_1 + p_2)^2 = -2
$$

Using the IBP reduction package LiteRed, we obtain the Master Integrals for the above integral. The Master Integrals are:

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
2	$I_{0000101}$	J_1	–
3	$I_{0101010}$	J_2	p_1, p_2
	$I_{0110100}$	J_3	p_1, p_2
	$I_{1010001}$	J_4	p_1, p_2
4	$I_{0201010}$	J_5	p_1, p_2
	$I_{0210100}$	J_6	p_1, p_2
	$I_{0101011}$	J_7	p_1, p_2
	$I_{0101110}$	J_8	p_1, p_2
	$I_{0110101}$	J_9	p_1, p_2
	$I_{0111100}$	J_{10}	p_1, p_2
	$I_{1010101}$	J_{11}	p_1, p_2
5	$I_{2010101}$	J_{12}	p_1, p_2
	$I_{0101111}$	J_{13}	p_1, p_2
	$I_{0111101}$	J_{14}	p_1, p_2
	$I_{1011101}$	J_{15}	p_1, p_2
	$I_{1110101}$	J_{16}	p_1, p_2
6	$I_{1111101}$	J_{17}	p_1, p_2
7	$I_{2111101}$	J_{18}	p_1, p_2

Table 2.1: Master Integrals: Two-loop three-point

Two-loop four-point diagram

We consider a two-loop four-point integral with some massive legs. The topology we are considering is given by the propagators:

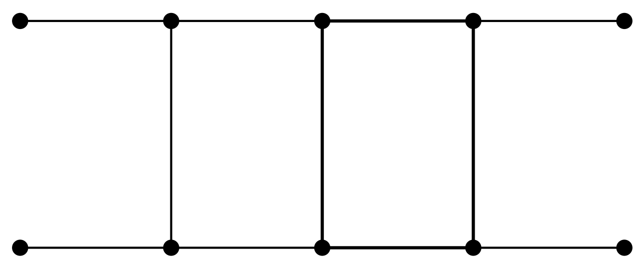

Figure 2.2: Two-loop four-point diagram (dark lines are massive)

$$\begin{aligned}&\{P_1 = l_1^2, P_2 = (p_1 - l_1)^2, P_3 = (k_l + p_2)^2, P_4 = -m_k^2 + l_2^2, P_5 = -m_k^2 + (-l_1 - l_2 + p_1)^2\\&P_6 = -m_k^2 + (l_1 + l_2 + p_2)^2, P_7 = -m_t^2 + (-l_1 - l_2 + p_1 - p_3)^2, P_8 = (l_1 + p_3)^2\\&P_9 = (l_1 + l_2)^2\}\end{aligned} \tag{2.10}$$

and the integral is of the form:

$$I = \int d^D l_1 d^D l_2 \frac{P_8^{a_8} P_9^{a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}} \tag{2.11}$$

After performing IBP reduction using LiteRed, we obtain 32 master integrals for this topology. Some of these are listed below:

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
4	$I_{000200200}$	J_1	$--$
	$I_{002020000}$	J_2	p_1, p_3
5	$I_{012000200}$	J_3	p_1, p_2
	$I_{000212000}$	J_4	p_1, p_2
	$I_{200100200}$	J_5	p_1, p_3
	$I_{001220000}$	J_6	p_1, p_2
	$I_{100200200}$	J_7	p_1, p_3
	$I_{000211100}$	J_8	p_1, p_2, p_3
	$I_{011200100}$	J_9	p_1, p_2, p_3
	$I_{100211000}$	J_{10}	p_1, p_2
	$I_{200111000}$	J_{11}	p_1, p_2
	$I_{011111000}$	J_{12}	p_1, p_2
	$I_{011101100}$	J_{13}	p_1, p_2, p_3
6	$I_{012021000}$	J_{14}	p_1, p_2
	$I_{001310100}$	J_{15}	p_1, p_2, p_3
	$I_{100301100}$	J_{16}	p_1, p_2, p_3
	$I_{012100200}$	J_{17}	p_1, p_2, p_3

Table 2.2: Master Integrals: Two-loop four-point function

2.2 Schwinger and Feynman parametrization

To set up the differential equation for each kinematic variable, the Schwinger and Feynman parametrization of integrals of the form 2.1 turn out to be helful[33] [19].

The Schwinger parametrization of 2.1 is given by:

$$I_{v_1 \ldots v_{n_{\text{int}}}}(D) = \frac{e^{l\varepsilon\gamma_{\text{E}}}}{\prod_{k=1}^{n_{\text{int}}} \Gamma(v_k)} \int_{\alpha_k \geq 0} d^{n_{\text{int}}}\alpha \left(\prod_{k=1}^{n_{\text{int}}} \alpha_k^{v_k - 1} \right) \frac{1}{\mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{U}} \tag{2.12}$$

where the $\mathcal{U}$ and $\mathcal{F}$ are Symanzik polynomials and α is the Schwinger parameter. Using the identity of the Dirac delta function given by:

$$1 = \int_{-\infty}^{\infty} dk\delta \left(k - \sum_{m=1}^{n} \alpha_m\right) = \int_{0}^{\infty} dk\delta \left(k - \sum_{m=1}^{n} \alpha_m\right) \tag{2.13}$$

and re-scaling of the Schwinger parameter by k, we can obtain the Feynman parametrization of integrals given by:

$$I_{v_1 \ldots v_{n_{\text{int}}}}(D) = P_f \int_{a_j \geq 0} d^{n_{\text{int}}} a\delta \left(1 - \sum_{m=1}^{n_{\text{legs}}} a_m\right) \left(\prod_{j=1}^{n_{\text{legs}}} a_j^{v_j - 1}\right) \frac{[\mathcal{U}(a)]^{v - \frac{(l+1)D}{2}}}{[\mathcal{F}(a)]^{v - \frac{lD}{2}}} \tag{2.14}$$

where a_j is the Feynman parameter given by $a_j = \alpha_j / k$. These two parametrizations are often useful in calculating two or higher loop vacuum integrals.

2.2.1 Calculation of $\mathcal{U}$ and $\mathcal{F}$

The calculation of Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$ can be done using various methods.[33] Here, we present one such method to perform the calculation:

Method : We use the Schwinger parametrization form given by:

$$I = \frac{e^{l\varepsilon\gamma_{\text{E}}} \left(\mu^2\right)^{v - \frac{lD}{2}}}{\prod_{j=1}^{n_{\text{int}}} \Gamma\left(v_j\right)} \int_{\alpha_j \geq 0} d^{n_{\text{int}}}\alpha \left(\prod_{j=1}^{n_{\text{int}}} \alpha_j^{v_j - 1}\right) \int \prod_{r=1}^{l} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} \exp\left(-\sum_{j=1}^{n_{\text{int}}} \alpha_j \left(-q_j^2 + m_j^2\right)\right) \tag{2.15}$$

Now we can express the exponential term in the form:

$$\sum_{j=1}^{n_{\text{int}}} \alpha_j \left(-q_j^2 + m_j^2\right) = -\sum_{x}^{n} \sum_{y}^{n} l_x W_{xy} l_y + 2 \sum_{x}^{n} l_x . \vec{T} + J \tag{2.16}$$

where J comprises terms with no loop dependence. Using this relation, we can write 2.15 as

2.12, and we can obtain the Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$ given by:

$$\mathcal{U} = det(W) \tag{2.17}$$
$$\mathcal{F} = \mathcal{U}(J + T.W^{-1}.T) \tag{2.18}$$

2.3 Obtaining the differential equation w.r.t kinematic variables

To set up the differential equation of the basis of Master Integrals w.r.t kinematic variables, we need the following operators on the Schwinger parametrization 2.12:

The dimensional shift operator [34] increases the dimension of the integral by 2. It is defined as:

$$\mathbf{D}^{\pm} I_v(D, \vec{x}) = I_v(D \pm 2, \vec{x}) \tag{2.19}$$

The raising operators $\mathbf{j}^{+}$(with $j \in \{1, \ldots, n_{\text{int}}\}$), which act on a Feynman integral as [34]

$$\mathbf{j}^{+} I_{v_1 \ldots v_j \ldots v_{n_{\text{int}}}}(D, x) = v_j \cdot I_{v_1 \ldots (v_j+1) \ldots v_{n_{\text{int}}}}(D, x) \tag{2.20}$$

We can obtain the relation between the basis of Master integrals with dimensions (D+2) and (D) given by:

$$\vec{I}(D) = \mathcal{U}^{+}(\mathbf{1}, \ldots, \mathbf{n_{int}})\, \vec{I}(D+2) \tag{2.21}$$

The $\mathcal{U}$ operator is defined with all α terms replaced with the j operator 2.20 acting on the integral basis.

To obtain the differential equation of $\vec{I}$ in terms of the above operators, we differentiate the

Schwinger parametrization w.r.t. kinematic variables. We get the relation[33]:

$$\frac{\partial}{\partial x_j} I_{v_1 \ldots v_{n_{\text{int}}}} = -\mathcal{F}'^{+}_{x_j}\left(\mathbf{1}, \ldots, \mathbf{n}_{\text{int}}\right) \mathbf{D}^{+} I_{v_1 \ldots v_{n_{\text{int}}}} \tag{2.22}$$

for $x_j \in \{x_1, \ldots, x_{N_B+1}\}$. Here $\mathcal{F}'_{x_j}$ is the derivative of the Symanzik polynomial w.r.t the kinematic variable x_j.

For one loop two point massless case, the $\mathcal{U}$ and $\mathcal{F}$ operators will be:

$$\mathcal{U}^{+} = \mathbf{1}^{+} + \mathbf{2}^{+}$$

$$\mathcal{F}^{+} = \mathbf{1}^{+}\mathbf{2}^{+} x + (\mathbf{1}^{+} + \mathbf{2}^{+})^2$$

The $\mathcal{F}'$ is then given by:

$$\mathcal{F}'^{+}_{x} = \mathbf{1}^{+}\mathbf{2}^{+}$$

On applying the above operators, we get the linear differential equation of the form:

$$\frac{\partial}{\partial x_m} I_{\boldsymbol{v}_i} = -\sum_{k=1}^{N_{\text{master}}} A_{x_m, ik} I_{\boldsymbol{v}_k}, \quad 1 \leq i \leq N_{\text{master}}, \quad 1 \leq m \leq N_B, \tag{2.23}$$

where the coefficients $A_{x_j, ik}$ are rational functions of D and $x_1, \ldots, x_{N_B}$.

To write the differential equation in a more compact form, we define:

$$\vec{I} = \left(I_{\mathbf{v}_1}, I_{\mathbf{v}_2}, \ldots, I_{\mathbf{v}_{N_{\text{master}}}}\right)^T \tag{2.24}$$

$$dI_{v_i} = \sum_{m=1}^{N_B} \left(\frac{\partial I_{v_i}}{\partial x_m}\right) dx_m. \tag{2.25}$$

$$A = \sum_{m=1}^{N_B} A_{x_m} dx_m. \tag{2.26}$$

We obtain the compact differential equation: [33]

$$(d + A)\vec{I} = 0 \tag{2.27}$$

Here, **A** is a one-form that satisfies the integrability condition given by:

$$dA + A \wedge A = 0 \tag{2.28}$$

In the project, we have written a code to obtain the matrix A for a family of integrals. Some examples of them include:

2.3.1 One-loop two-point integral

We have the following integral:

$$I_{v_1,v_2}(D) = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2 - 1)^{v_1} ((\ell_1 + p_1)^2 - 1)^{v_2}}$$

The kinematic variable for this integral: $x_k = -p_1^2$ and the basis of master integral is:

$$\vec{I} = \{I_{01}, I_{11}\}$$

The matrix $\mathbf{A}_x$ after substituting $D \longrightarrow 4 - 2\epsilon$, is given by:

$$\mathcal{U} = \alpha_1 + \alpha_2 \tag{2.29}$$

$$\mathcal{F} = \alpha_1 \alpha_2 x + (\alpha_1 + \alpha_2)^2 \tag{2.30}$$

$$A(\epsilon, p_1)_{x_k} = \begin{pmatrix} 0 & 0 \\ \frac{2\epsilon - 2}{4p_1^2 - p_1^4} & \frac{2\epsilon p_1^2 - 4}{8p_1^2 - 2p_1^4} \end{pmatrix}$$

2.3.2 Two loop two point sunrise integral

The integral is given by:

$$I_{111}(\epsilon,\eta) = \int \frac{\mathrm{d}^D\ell_1\mathrm{d}^D\ell_2}{\mathrm{i}\pi^D} \frac{1}{(\ell_1^2 - m^2)^{a_1}\,(\ell_2^2 - m^2)^{a_2}\,((\ell_1+\ell_2-p_1)^2 - m^2)^{a_3}} \tag{2.31}$$

Taking $p_1^2 = 2m^2$, we have the kinematic variables given by:

$$x = \frac{-1}{2}$$

We obtain the basis of master integrals and the matrix $\mathbf{A}_x$ as:

$$\vec{I} = \{I_{110}, I_{111}, I_{211}\} \tag{2.32}$$

$$A(\epsilon)_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2-4\epsilon & 6 \\ \frac{-2(2\epsilon-2)^2}{17} & \frac{-21(12\epsilon^2-14\epsilon+4)}{34} & \frac{174\epsilon-104}{17} \end{pmatrix} \tag{2.33}$$

2.3.3 One loop four point box integral

The integral is given by:

$$I_{a_1a_2a_3a_4}(\epsilon,\eta) = \int \frac{\mathrm{d}^D\ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2)^{a_1}\,((\ell_1-p_1)^2)^{a_2}\,((\ell_1-p_1-p_2)^2)^{a_3}\,((\ell_1-p_1-p_2-p_3)^2)^{a_4}} \tag{2.34}$$

Taking $p_1^2 = p_2^2 = p_3^2 = 0$, we have the kinematic variables given by:

$$x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{4}$$

For the box diagram, the Symanzik polynomials are given by:

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \tag{2.35}$$

$$\mathcal{F} = \alpha_2\alpha_4 x_1 + \alpha_1\alpha_3 x_2 + \alpha_3\alpha_4 \tag{2.36}$$

Hence, we obtain the basis of master integrals and the matrix A_{x_1}andA_{x_2} as:

$$\vec{I} = \{I_{0011}, I_{0101}, I_{1010}, I_{1111}\} \tag{2.37}$$

$$A(\epsilon)_{x_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -4\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-64(2\epsilon-1)}{3} & \frac{64(2\epsilon-1)}{3} & 64(2\epsilon-1) & -2-4\epsilon \end{pmatrix} \tag{2.38}$$

$$A(\epsilon)_{x_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4\epsilon & 0 \\ \frac{-64(2\epsilon-1)}{3} & 64(2\epsilon-1) & \frac{64(2\epsilon-1)}{3} & -2-4\epsilon \end{pmatrix} \tag{2.39}$$

2.4 Canonical Transformation of A(ϵ form)

In the previous section, we have achieved a linear system of differential equations of Master Integrals, which are all coupled to each other. This system is often hard to solve analytically and is one of the bottlenecks of solving the basis of Master Integrals of higher complexity. One of the methods to simplify this system is to perform a Canonical transformation and express the matrix **A** as an ϵ form. [33]

The differential equation

$$(d + A)\vec{I} = 0$$

is in ϵ form if

$$A = \epsilon \sum_{j=1}^{N_{master}} B_j \omega_j = \epsilon \tilde{A} \tag{2.40}$$

where B_j is the $N_{master} \times N_{master}$ matrix
ω_j are the one forms of kinematic variables with a singular *dlog* pole in the matrix. They are represented as:

$$\omega_j(z_j) = \frac{dx_j}{x_j - z_j} \tag{2.41}$$

The canonical transformation to ϵ-form is useful as it helps us write $\vec{I}$ in the following way:

Firstly, we express $\vec{I}$ as a Taylor series around $\epsilon \to 0$ given as:

$$\vec{I}(\varepsilon, x_j) = \sum_{l=0}^{\infty} \vec{I}^{(l)}(x_j) \cdot \varepsilon^l \tag{2.42}$$

Substituting 2.42 and 2.40 in the 2.27, we get the relation:

$$\left(d + \varepsilon \sum_{k=1}^{N_{master}} C_k \omega_k\right) \left(\sum_{l=0}^{\infty} \vec{I}^{(l)}(x) \cdot \varepsilon^l\right) = 0 \tag{2.43}$$

Comparing each power of ϵ in this expansion, we can write:

$$\begin{aligned} d\vec{I}^{(0)}(x_j) &= 0 \\ d\vec{I}^{(l)}(x_j) &= -\sum_{k=1}^{N_L} \omega_k C_k \vec{I}^{(l-1)}(x_j), \quad j \geq 1. \end{aligned} \tag{2.44}$$

These relations can be solved recursively to obtain the basis of the Master Integrals. The general solution of the above equation can be written as:[2]

$$\vec{I}(\vec{x}, \epsilon) = \mathbb{O} \exp\left[\epsilon \int_\gamma \mathrm{d}\tilde{A}\right] \vec{I}_{boundary}(\epsilon)$$

where $\mathbb{O}$ is the path ordering operator.

Transformation matrix W

To achieve the ϵ form, we need to make a transformation to the A matrix such that it has only an explicit ϵ dependence. This is done by considering the transformation:

$$\vec{I}'(\varepsilon, x) = W(\varepsilon, x)\vec{I}(\varepsilon, x) \tag{2.45}$$

where $U(\varepsilon, x)$ is an invertible $(N_{\text{master}} \times N_{\text{master}})$-matrix, which may depend on ε and x.

We have the following relations:

$$d\vec{I} = W^{-1} d\vec{I'} + dW^{-1} \vec{I'} \tag{2.46}$$

$$d\vec{I'} = W d\vec{I} + dW \vec{I} \tag{2.47}$$

Substituting 2.46 into 2.48, we get:

$$d\vec{I'} = -(WAW^{-1} + WdW^{-1}) \vec{I'} \tag{2.48}$$

We obtain a new differential equation given by:

$$(d + A')\, \vec{I'} = 0 \tag{2.49}$$

where A' is related to A by

$$A' = WAW^{-1} + WdW^{-1} \tag{2.50}$$

Obtaining the transformation U is often a non-trivial task. The transformation is obtained via two steps, i.e., Fibre Transformation and Base Transformation. The algorithm to obtain U is lengthy and non-trivial to be discussed in the thesis but is presented in [35][33].

To achieve the ϵ-form computationally, The packages *Epsilon* [36] can be used to give a canonical transformation of matrix $\mathbf{A}$.

2.5 Polylogarithms and Multiple Polylogarithms

A Polylogarithm is a power series of complex numbers given by:

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \tag{2.51}$$

which satisfies the property:

$$Li_{s+1}(z) = \int_0^z \frac{Li_s}{t} dt \tag{2.52}$$

where the first term of this recursion is $Li_1(z) = -ln(1-z)$.

Analytical solutions of Feynman integrals often contain polylogarithms and their properties become very useful while adding many integrals to obtain the scattering amplitude.

While calculating multi-loop integrals, we obtain a multi-variable extension of polylogarithms, known as multiple polylogarithms. They satisfy the relation[33]:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t) \tag{2.53}$$

with $G(0_n, z) = \frac{1}{n!} ln^n(z)$.

Along with multiple polylogarithms, we also obtain functions known as multiple polylogarithms given by:

$$H(m, \vec{m}_k; y) \equiv \int_0^y dx f(m, x) H(\vec{m}_k; x), \quad m \in 0, \pm 1 \tag{2.54}$$

with $H(1, z) = G(1, z) = -ln(1-z)$. Here, the function f is given by:

$$f(1; y) \equiv \frac{1}{1-y}, \quad f(0; y) \equiv \frac{1}{y}, \quad f(-1; y) \equiv \frac{1}{1+y}. \tag{2.55}$$

These harmonic polylogarithms satisfy the shuffle algebra just like multiple polylogarithms. We also obtain 2d-Harmonic Polylogarithms, an extension of harmonic polylogarithms with $m \in (2, 3)$[37]. They are constructed via the functions:

$$f(2; y) \equiv f(1-z; y) \equiv \frac{1}{1-y-z}, \quad f(3; y) \equiv f(z; y) \equiv \frac{1}{y+z} \tag{2.56}$$

2.5.1 Multiple polylogarithms in master integrals

In section 2.40, we studied how canonical transformation helps us to solve the set of differential equations of each master integral [equation 2.44]. Upon solving them recursively, we obtain polylogarithms and multiple polylogarithms as functions of kinematic variables. The multiple polylogarithms obtained are of the form [33]:

$$G\left(x_1, \ldots, x_r; z\right) = G\left(\omega\left(x_1\right), \ldots, \omega\left(x_r\right); z\right) \tag{2.57}$$

where $\left(\omega\left(x_1\right), \ldots, \omega\left(x_r\right)\right)$ are the dlog one forms as discussed in section 2.40. In the next sections, we discuss certain properties and mathematical structures which are used to simplify the multiple polylogarithms present in master integrals, to achieve the cancellation of divergences and unnecessary expressions.

2.5.2 Shuffle Algebra

These multiple polylogarithms follow the properties of shuffle algebra denoted by:[38]

$$G\left(x_1, \ldots, x_{n_1}; z\right) G\left(x_{n_1+1}, \ldots, x_{n_1+n_2}; z\right) = \sum_{\sigma \in \Sigma(n_1, n_2)} G\left(x_{\sigma(1)}, \ldots, x_{\sigma(n_1+n_2)}; z\right) \tag{2.58}$$

where $\Sigma\left(n_1, n_2\right)$ is the list of all the shuffles of $n_1 + n_2$ alphabets in the multiple polylogarithms, and $x_{\sigma(n_1+n_2)} \in \left(x_1, \ldots, x_{n_1}\right) \uplus \left(x_{n_1+1}, \ldots, x_{n_1+n_2}\right)$

Example:

$$G\left(k_1, k_2; z\right) G\left(k_3, k_4; z\right) = \sum_{\sigma \in \Sigma(n_1, n_2)} G\left(x_{\sigma(1)}, \ldots, x_{\sigma(4)}; z\right)$$

where $\Sigma\left(n_1, n_2\right)$ contains the RHS of the relation:

$$\begin{aligned}\left(k_1, k_2\right) \uplus \left(k_3, k_4\right) = {} & \left(k_1, k_2, k_3, k_4\right) + \left(k_1, k_3, k_2, k_4\right) + \left(k_1, k_3, k_4, k_2\right) \\ & + \left(k_3, k_1, k_2, k_4\right) + \left(k_3, k_4, k_1, k_2\right) + \left(k_3, k_1, k_4, k_2\right)\end{aligned} \tag{2.59}$$

The shuffle product can also be used to write multiple polylogarithms of higher-degree in terms of lower-degree polylogarithms with other terms. This property helps us to simplify our expressions of master integrals.

$$G(y,0,z) = G(0,z)G(y,z) - G(0,y,z) \tag{2.60}$$

2.5.3 Hopf Algebra and Coproducts

In this section, we introduce multiple polylogarithms as a Hopf algebra, and how its maps, like coproducts, can be used to express a complex multiple polylogarithm in terms of simpler ones.

Definition 2.5.1. *A tensor algebra on a vector space V is a vector space $\mathcal{A}$ with the map:*

$$\begin{aligned} m : \mathcal{A} \times \mathcal{A} &\to \mathcal{A} \\ (a,b) &\mapsto m(a,b) \equiv a \cdot b \end{aligned}$$

that is associative and has a unit vector ε. In this case the map m is a tensor product given by: a unique linear map $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that

$$a \cdot b = m(a,b) = \mu(a \otimes b)$$

Definition 2.5.2. *If $\mathcal{A}$ is an algebra with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and unit e, we define the coalgebra as the dual of an algebra, i.e. $\mathcal{C} = \mathcal{A}^*$, which is equipped with the coproduct linear map:*

$$\Delta = \mu^\dagger : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$

and the unit covectors $e^{'} = e^\dagger$.

Definition 2.5.3. *The Hopf Algebra $\mathcal{H}$ is a bialgebra i.e. an algebra on a vector space with both the product map μ and the coproduct map Δ. It has an additional structure of antipode, a map $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ given by:*

$$S(x \cdot y) = S(x) \cdot S(y) \tag{2.61}$$

$$\mu(\mathrm{id} \otimes S)\Delta = \mu(S \otimes \mathrm{id})\Delta = 0 \tag{2.62}$$

Hopf Algebra in Multiple Polylogarithms

In order to use the properties of Hopf Algebra to simplify polylogarithms, one needs to show that multiple polylogarithms follow Hopf Algebra. The proof is motivated as follows:

In the previous sections, we showed that multiple polylogarithms follow the properties of shuffle algebra. One can show that shuffle algebra is a non-commutative Hopf algebra[39]. Hence, we can use the coproduct map Δ to express higher-weight polylogarithms into polylogarithms of lower weight.

For a Hopf algebra $\mathcal{H}$, the coproduct map has the iterative property given by[38]:

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \mathrm{id}} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \, \mathrm{id} \, \otimes \, \mathrm{id}} \dots \tag{2.63}$$

We know that, as the shuffle product preserves the weight of the MPLs, MPLs are given by a graded algebra. This gives out additional properties for the coproduct. If a graded Hopf algebra $\mathcal{H}$ is decomposed via a coproduct, it's weight n is preserved:

$$\mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{a+b=k} \mathcal{H}_a \otimes \mathcal{H}_b. \tag{2.64}$$

The action of Δ on $\mathcal{H}_n$ can be expressed as:

$$\Delta = \sum_{a+b=k} \Delta_{a,b} \tag{2.65}$$

Using these relations, coproducts of logarithms and basic polylogarithms have been obtained[38]:

$$\begin{aligned} \Delta(\ln z) &= 1 \otimes \ln z + \ln z \otimes 1, \\ \Delta\left(\mathrm{Li}_n(z)\right) &= 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!} \end{aligned} \tag{2.66}$$

$$\Delta(\zeta_{2k}) = \zeta_{2k} \otimes 1$$
$$\Delta(\pi) = \pi \otimes 1 \tag{2.67}$$

Here, the terms $1 \otimes k + k \otimes 1$ are primitive terms, so we can define the reduced coproduct given by:

$$\Delta(k) = 1 \otimes k + k \otimes 1 + \Delta^{'}(k) \tag{2.68}$$

Hence the reduced coproducts are:

$$\Delta^{'}(\ln z) = 0 = \Delta^{'}(\zeta_{2n}) = \Delta^{'}(\pi)$$
$$\Delta^{'}\left(\mathrm{Li}_n(z)\right) = \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!} \tag{2.69}$$

The main conjecture of using Hopf algebra in the context of multiple polylogarithms is as follows:

If we have an expression F_w containing multiple polylogarithms and we wish to express it in the form of a simpler function G_w, the expression G_w should be such that if:

$$\Delta'(F_w) = \Delta'(G_w)$$

then we can imply that

$$F_w = G_w + \sum_i c_i k_{w,i} \tag{2.70}$$

where $k_{w,i}$ are the powers of functions, such as π and ζ, which have their coproduct as zero but can be part of the function F_w.

Examples

The relation in equation 2.70 can be used to simplify polylogarithms with the alphabets of the form $\left(1 - \frac{1}{f}\right)$.

$$Li_k\left(1 - \frac{1}{f}, z\right) = c_i Li_k(f, z) + \text{Other terms}$$

This has been done for a few examples[38]:

$$\mathrm{Li}_3\left(\frac{1}{x}\right) = \mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x - \frac{\pi^2}{3}\ln x \tag{2.71}$$

$$\mathrm{Li}_4\left(1 - ze^{i\delta\pi}\right) = -\mathrm{Li}_4\left(\frac{1}{1+z}\right) - \frac{1}{24}\ln^4(1+z) - \frac{i\delta\pi}{6}\ln^3(1+z) + \frac{\pi^2}{6}\ln^2(1+z) + \frac{\pi^4}{45} \tag{2.72}$$

Using coproducts to simplify multiple polylogarithms becomes more complex, and examples were not worked out for the same.

2.5.4 Fibration Basis

A general multiple polylogarithm can be written in several representations due to its functional properties. In the many variable cases, it's important to express the multiple polylogarithms as a basis. In this section, we show how polylogarithms with many algebraic variables (kinematic variables in the case of Feynman integrals) can be written in a simpler form using its fibration basis. [33] [26]

Consider a multiple polylogarithm with $x_1, \ldots, x_n$ kinematic variables and rational functions $(f_1, \ldots, f_n)$. In the language of shuffle algebra, consider these f_n as letters and a word defined by $w = f_1 f_2 .... f_n$. Consider a multiple polylogarithm $G(w; 1) = G(f_1, f_2, \ldots, f_n)$. We also know that the letters f_i are of the form:

$$f_i(x_1, \ldots, x_n) = c_i(x_1, \ldots, x_n)x_i + d(x_1, \ldots, x_n)$$

Then we can express the multiple polylogarithms above as:

$$G(f_1, f_2, \ldots, f_n) = \sum_k a_k G(w_{1,k}; x_1) G(w_{2,k}; x_2) \ldots G(w_{n,k}; x_n) \tag{2.73}$$

where $w_{i,k}$ are words containing only c_m letters and are algebraic functions of variables $x_{i+1} \ldots x_n$. The method of fibration basis is very useful to simplify polylogarithms containing functions of different variables and express them as functions of a single variable, which is useful for the cancellation of terms while simplifying scattering amplitudes.

Examples

The multiple polylogarithm $G(0, 1, 1+x, 1-y)$ can be expressed in its fibration basis as:

$$\begin{aligned} G(0,1,1+x,1-z) = & \frac{\pi^2 G(-1,x)}{6} + G(0,z)G(0,-1,x) + G(0,-1,-1,x) - \\ & G(-1,x)G(1,0,z) - G(0,-1,-z,x) + G(1,0,0,z) + \zeta(3) \end{aligned} \tag{2.74}$$

The fibration basis of $G(2, 1-x)$ in x is:

$$G(2,1-x) = -G(2,x) + ln(x)G(1,x) + \zeta(2) \tag{2.75}$$

The fibration basis and coproducts of multiple polylogarithms can be obtained using the Mathematica package *PolyLogTools*[26].

Chapter 3

Numerical solutions of Master Integrals via addition of an auxiliary mass

As the complexity of solving Master Integrals analytically increases with increasing loops, legs and mass scales, numerical methods of the complex integrals become more feasible than analytical methods. In the next few sections, we'll discuss one technique of performing numerical computations of Master Integrals and its advantages over heavy analytical results.

3.1 Obtaining a generalized boundary condition

The methods of setting up a differential equation of the basis of MIs w.r.t kinematic variables have been successful in calculating many physical processes. However, even after successful IBP reduction and canonical transformation to ϵ-form, the limitation or the bottleneck of this method is to know a boundary condition of the Master Integrals at one point in space.

This section presents a method to obtain a generalized boundary condition of a master integral at an unphysical point in space by adding an auxiliary mass to the Master integral.

This is done by introducing an auxiliary mass for each propagator in the integral:[1]

$$\frac{1}{D_p} \longrightarrow \frac{1}{D_p + \iota\eta} \tag{3.1}$$

where D_p is the propagator and η is the auxiliary mass.

The advantage of introducing η is that, by taking $\eta \to \infty, I(\vec{x_j}, \epsilon, \eta)$ can be reduced to linear combinations of simpler integrals. For tree propagators, the denominator can be easily expanded for $\eta \to \infty$.

$$\frac{1}{\mathcal{D}_{tree} + \eta} \overset{\eta\to\infty}{\longrightarrow} \frac{1}{\eta} \sum_{j=0}^{+\infty} \left(\frac{-\mathcal{D}_{tree}}{\eta} \right)^j \tag{3.2}$$

As the loop momenta can be any large value, one cannot perform the expansion of the loop propagators in the same way as tree propagators. However, one can do the following auxiliary mass expansion: [1]

$$\frac{1}{\mathcal{D}_{loop} + \mathrm{i}0^+} = \begin{cases} \frac{1}{\mathcal{D}_\eta + \mathrm{i}0^+} \sum_{j=0}^{+\infty} \left(\frac{-K_\alpha}{\mathcal{D}_\eta + \mathrm{i}0^+} \right)^j & \text{if } \mathcal{D}_\eta \neq 0, \\ \frac{1}{\mathcal{D} + \mathrm{i}0^+} & \text{if } \mathcal{D}_\eta = 0, \end{cases} \tag{3.3}$$

where we decompose $\mathcal{D}_{loop} = \mathcal{D}_\eta + K_\alpha$, with $\mathcal{D}_\eta$ including only the part at the order of $|\eta|$, while K_α contains the finite terms that are smaller than $|\eta|$.

We find that, as $\eta \to \infty, I(\vec{x_j}, \epsilon, \eta)$ is simplified to a linear combination of integrals with less inverse propagators in the denominator or multiplied by single-scale vacuum integrals. This is true as all the terms dependent on external momenta are a part of the K_α term, due to which the propagators of the expansion are independent of external momenta.

The one loop vacuum integrals can be solved easily by using the relation already in the literature[19]:

$$\int \frac{d^D\bar{l}}{(2\pi)^D} \frac{\left(\bar{l}^2\right)^x}{\left(\bar{l}^2 + K\right)^y} = \frac{\Gamma\left(y - x - \frac{1}{2}D\right)\Gamma\left(x + \frac{1}{2}D\right)}{(4\pi)^{D/2}\Gamma(y)\Gamma\left(\frac{1}{2}D\right)} K^{-(y-x-D/2)} \tag{3.4}$$

For two loops vacuum integrals or higher, calculations using Feynman parametrizations and other techniques is possible, which has been done in the literature.[40][41]

Hence, using this method, we know the calculations of Master Integrals $I(\vec{x_j}, \epsilon, \eta)$ in an unphysical region, i.e. at $\eta \to \infty$, which can further act as a boundary condition for obtaining the Master Integral at $\eta = 0$. Furthermore, the next section discusses the certain strategies we need to achieve the above results.

3.1.1 IBP reduction with the auxiliary mass

In order to use the auxiliary mass expansion, we need to add η to the propagators before performing the IBP reduction in order to obtain the correct set of master integrals. As adding an auxiliary mass to the propagator is equivalent to having a massive propagator while doing IBP reduction, adding the mass to all the propagators would increase the number of Master Integrals by a great amount. This would highly complicate the computation of a diagram like two-loop three-point massless integral.

Consider a two-loop three-point integral with on-shell propagators, as shown in figure 2.1. For the case when we don't add any auxiliary mass η to the propagators, we get just 4 master integrals. If we add η to all the propagators, we get 37 master integrals. Hence, adding η to only four of the propagators with loop momentum l_2 gives us 17 master integrals.

As the addition of η to all propagators increases the Master Integrals by more than $9X$, we cannot casually add η to the propagators in order to optimize this method. Therefore, we need an optimized strategy to perform IBP reduction with an auxiliary mass[42]. Some possible strategies are:

- If there are massive propagators present, add η to the mass. In this case, no increase in the number of Master Integrals would be seen.
- Adding η only to the propagators with no external momentum can give an optimized basis of Master Integrals
- Adding η to only a few propagators such that all differential equations are dependant on η

3.1.2 Differential equation w.r.t. auxiliary mass

The advantage of adding an auxiliary mass to the Integrals becomes prominent when we obtain a differential equation of the basis of master integrals in the following manner:[43]

$$\frac{\partial}{\partial \eta}\vec{I}(\vec{x},\epsilon,\eta) = M(\vec{x},\epsilon,\eta)\vec{I}(\vec{x},\epsilon,\eta) \tag{3.5}$$

On differentiating, the master integrals become higher-order integrals, which we write as a linear combination of master integrals using IBP reduction. This gives rise to a linear set of differential equations as written in the equation 3.5.

To solve this differential equation, we require a boundary condition which has already been discussed in section 3.1. As our primary aim is to obtain the master integrals at $\eta = 0$, we use the boundary condition i.e. the master integral at $\eta \to \infty$, and perform an analytical continuation to obtain the desired integral. This is discussed in detail in the next section.

3.2 Numerical Solution of MIs using auxiliary mass method

Upon obtaining the basis of Master Integrals after adding an auxiliary mass η, we obtain a differential equation of the Master Integrals w.r.t η as discussed in section 3.1.2. The equation obtained is of the form:

$$\frac{\partial}{\partial \eta}\vec{I}(\vec{x},\eta,\epsilon) = M(\vec{x},\eta,\epsilon)\vec{I}(\vec{x},\eta,\epsilon) \tag{3.6}$$

The solution of this set of differential equations at $\eta = 0$ gives us the numerical solution of the desired basis of MIs. To obtain the solution, we use the generalized boundary condition of each master integral in the unphysical region i.e. $\eta \to \infty$ and use the methods of analytical continuation to obtain the solution of MIs in the desired physical region i.e. $\eta = 0$.

In this section, we discuss the methodology for solving the above differential equation. For every value of η, we use the Frobenius method of series expansion to solve the n^{th} order differential equations and obtain the solution of MIs as a Laurent expansion in ϵ with numerical coefficients. We also discuss the path of analytical continuation of MIs from $\eta \to \infty$ to $\eta = 0$, including the method to avoid branch cuts while performing analytical continuation.

3.2.1 Analytical continuation

Analytic Continuation is a powerful statement that provides a way of extending the complex domain over which we have a defined function. It is formally defined as:

Definition 3.2.1. *If g_1 and g_2 are analytic functions on complex domains X_1 and X_2, respectively, and the intersection $X_1 \cap X_2 \neq \phi$ and $g_1 = g_2$ on $X_1 \cap X_2$. Then g_2 is called an analytic continuation of g_1 to the domain X_2.*

The most useful way to perform analytical continuation is using a power series of the form:

$$p(z) = \sum_n a_n (z - z_o)^n \tag{3.7}$$

As this power series is analytical in its radius of convergence, we can perform analytical continuation of the series from z_o to any other point in its analytical domain defined via its radius of convergence.

Analytical continuation of Master Integrals

In order to obtain Master Integrals at $\eta = 0$, our aim is to perform an analytical continuation from $\eta \to \infty$ to $\eta = 0$. To achieve this, the best way is to express the Master integrals as a power series around the points near ∞ and extend their convergence domain to points near 0. This motivates us to use the Frobenius method to obtain a numerical solution to the differential equations.

3.2.2 Frobenius Method

We use the most common semi-analytical method to solve our first and second-order differential equations i.e. the Frobenius method. Consider the following differential equation:

$$f'(x) + a(x)f(x) = b(x) \tag{3.8}$$

The general solution of the differential equation is given by: $f_{total} = K(x)f_{homg}$. So we need to first solve the homogeneous part of the differential equation.

Homogeneous part

Consider the following differential equation:

$$(2x + 1)f'(x) + 3f(x) = 2x + 3$$

In the above example, we have the homogeneous part of the form:

$$f'(x) + \frac{3}{2x + 1}f(x) = 0 \tag{3.9}$$

In order to solve this first-order homogeneous differential equation around a point p, we take the ansatz as the expansion of $f(x)$ around the point p:

$$f_{homg}(x) = (x - p)^k \sum_{n=0}^{N} a_n (x - p)^n \tag{3.10}$$

To obtain the value of k, we solve the indical equation obtained by comparing the terms with the least power i.e. the coefficient of x^{k-1}. We get:

$$k = 0$$

After putting in the solution of the indical equation, we obtain a set of N equations that relate the coefficients a_n with each other. On solving the set of equations recursively, we obtain all the coefficients $a_1......a_n$ in terms of an unknown coefficient a_0. Here, we get the :

$$f_{homg} = a_0 + 3a_0x + \frac{3}{2}a_0x^2 - \frac{1}{2}a_0x^3 + \frac{3}{8}a_0x^4 + ...$$

Inhomogeneous part

To find the solution to the equation, including the inhomogeneous terms, we obtain the inhomogeneous solution of the differential equation using the variation of the constant method. The total solution is given by:

$$f_{total} = K(x)f_{homg} \tag{3.11}$$

$$f_{total}^{'} = K(x)f_{homg}^{'} + K^{'}(x)f_{homg} \tag{3.12}$$

On substituting 3.11 and 3.12 into 3.8, we get the following relation:

$$K(x)f_{homg}^{'} + K^{'}(x)f_{homg} + a(x)K(x)f_{homg} = b(x) \tag{3.13}$$

As f_{homg} satisfies the following equation:

$$f'(x) + a(x)f(x) = 0$$

we substitute it back into 3.13 to get:

$$K^{'}(x)f_{homg} = b(x)$$

Using this relation and integrating the equation w.r.t x, we get:

$$K(x) = \int \frac{b(x)}{f_{homg}} + C$$

Hence, the total solution of any first-order differential equation at a point p using the Frobenius method is given by:

$$f(x) = f_{homg}.\int \frac{b(x)}{f_{homg}} + Cf_{homg} \tag{3.14}$$

3.2.3 MIs as a Laurent expansion in ϵ

In order to make the algorithm of solving the differential equations of MIs simpler and faster, we express our MIs as a Laurent expansion in terms of ϵ.

$$\vec{I}(\vec{x},\eta,\epsilon)=\sum_{j=1}^{2L}\frac{1}{\epsilon^j}\vec{I}(\vec{x},\eta,\frac{1}{\epsilon^j})+\epsilon^0\vec{I}(\vec{x},\eta,\epsilon^0)+\epsilon\vec{I}(\vec{x},\eta,\epsilon^1)+\mathcal{O}(\epsilon^2) \tag{3.15}$$

Inserting the Laurent expansion of MIs into the differential equation 3.5, we get a separate differential equation for each term $\vec{I}(\vec{x},\eta,\epsilon^{-j})$.The idea of solving each coefficient of the master integral is advantageous in the following ways:

- As we have a differential equation for each coefficient of ϵ, the number of differential equations to solve increases by (2L+3) times. Even though the number of differential equations increases, this reduces the complexity of each differential equation in terms of η, leading to a faster computation of the Frobenius solution.
- In many cases, such as two-loop two-point function, the coupled nature of the differential equations is also simplified using the Laurent expansion of the master integrals. As we go to higher coupled systems, this method reduces the coupled equations to simpler coupled ones, hence simplifying our problem to a great deal.

3.2.4 Path of Analytical Continuation

In the sections 3.2.1 and 3.2.2, we discussed the method to obtain the solution at a point in space and the concept of extending the analytical domain of the solution using continuation. In this section, we talk about the path chosen to perform analytical continuation in the context of Feynman integrals.

We need to choose such a path such that we do not extend the domain of the power series solution of the master integrals(of the type shown in equation 3.10) beyond its radius of convergence. This depends on the simple poles of η that are present in the matrix M (equation 3.5). The poles of η can either be all real poles or might contain some complex poles. Also, while obtaining the differential equation of each term of the ϵ-expansion of master integral(discussed in section 3.2.3) w.r.t η, we obtain logarithmic terms of η. As logarithmic functions are multi-valued functions, we need to be extra careful of logarithmic singularities leading to branch cuts. To perform analytic continuation of master integrals, we need to successfully avoid the branch cuts in order to achieve the correct results. Keeping all these intricacies in mind, we discuss the two cases in detail:

Real poles

While computing the integrals of the particle processes with scalars and fermions, we only have real poles of η as the propagators contain only real-valued terms. In this case, the logarithmic branch cut is present on the real line. To avoid the branch cut and stay in the analytical domain, we can access all the points on the imaginary line to perform analytical continuation.

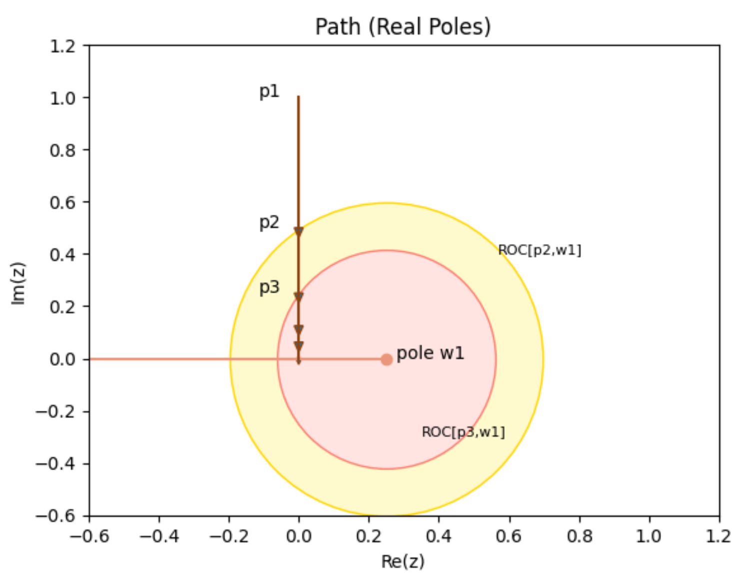

Figure 3.1: The path for pole $w_1 = \frac{1}{4}$. The figure shows the circle of convergence of point p_2 and p_3 where the next point of the path is inside the circle.

Complex Poles

The particle processes involving the production of $W^{\pm}$ and Z particles have complex mass terms in their propagators .[17] [16] In such cases, we often get complex poles of η. As discussed above, this complex pole gives rise to a logarithmic branch cut. Avoiding the branch cut, in this case, is non-trivial because it divides the complex plane into two different Riemann surfaces. In order to avoid these branch cuts, an algorithm was written to find the path with the least number of points of analytical continuation from η going from ∞ to 0. The rules followed by the algorithm are:

- After evaluating the integral at $\eta \to \infty$, the starting point of the path is taken to be a random real number between [2×Im[Max[poles]],Im[Max[poles]]].
- If the master integral is evaluated at a point, it can only be continued within its circle of convergence i.e. the circle enclosed by the radius of convergence of the closest pole.
- We move away left or right from the branch cut when the distance to the branch cut is less than the radius of convergence w.r.t the closest pole, and we move towards the origin after avoiding the branch cut.

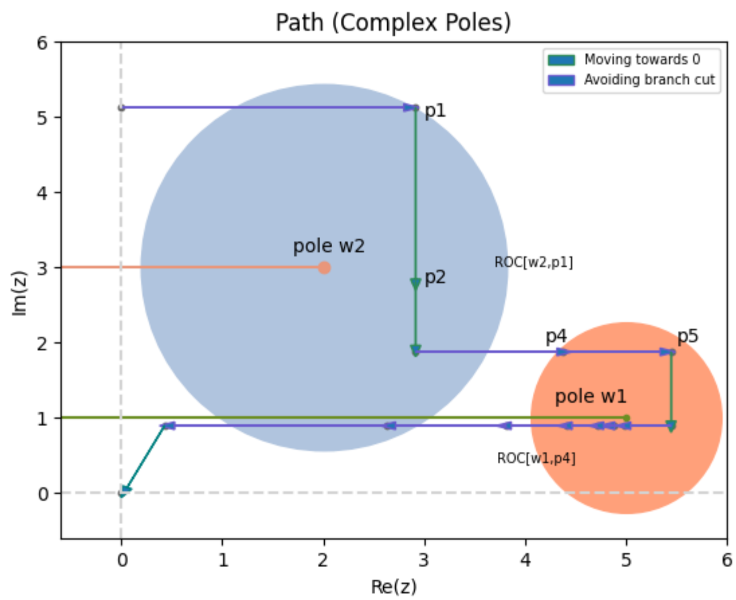

Figure 3.2: The figure shows the path with complex poles: $w_1 = 5 + \iota$ and $w_2 = 2 + 3\iota$.

Chapter 4

Methodology of Numerical Evaluation of Master Integrals

All computations are done via Wolfram Mathematica.

In this chapter, we discuss the methodology of implementing the Mathematica code to numerically compute the master integrals of the one-loop and two-loop Feynman diagrams, using the techniques discussed in section 3.

4.1 General construct

The algorithm to solve the master integrals could be expressed into two parts: 1) Setting up the differential equation and 2) Solving the differential equation by semi-analytical methods.

In order to set up the differential equation of the basis if master integrals $\vec{I}$ w.r.t. η, we perform the following operations:

- In order to obtain the basis of master integrals, we perform an IBP Reduction of the family of Feynman integrals, with numerical kinematic values, upon adding an auxiliary mass η to the propagators, as mentioned in section 3.1.1. The packages LiteRed[21] and Mint[44] have been used to perform the IBP reduction.

- Upon obtaining the basis of master integrals for each family, we differentiate the master integrals w.r.t η to obtain the differential equation 3.5 and the matrix M. Using this matrix M, we obtain the differential equation of each coefficient of ϵ for a master integral i.e. $\vec{I}(\vec{x}, \eta, \frac{1}{\epsilon^j})$, as discussed in section 3.2.3.

After obtaining the set of differential equations for the basis of master integrals, we use the series expansion method i.e. The Frobenius method, to solve the differential equation of each master integral. The algorithm used is as follows:

- The power series of the master integral at $\eta \to \infty$ can be obtained using the vacuum integrals obtained after performing an expansion at $\eta \to \infty$ (Appendix A) and obtaining a general form of the power series. We then use the equation 3.5 to obtain a recursive relation of the coefficients of the power series, which is then solved using the boundary condition i.e. evaluation of the first integral of the expansion. This step is discussed for each case in the next section.
- Consider the path of analytical continuation given by $p = \{p_0, p_1, \ldots, 0\}$. We use the power series of master integral at $\eta \to \infty$ and evaluate it at p_0 i.e. the first point of the path of analytical continuation. This acts as a boundary condition for the next step i.e. solving the differential equation for each coefficient of ϵ at p_0.
- Obtaining the boundary condition, we can solve the differential equation for $\vec{I}(\vec{x}, \eta, \frac{1}{\epsilon^j})$ using the Frobenius method in section 3.2.2. The solution obtained by us for $\eta = p_0$ is then evaluated at $\eta = p_1$ and the same step is repeated along the path p until we obtain the solution of $\vec{I}(\vec{x}, \eta, \frac{1}{\epsilon^j})$ at $\eta = 0$.

4.2 One-loop computations

In this section, we focus on the intricacies of solving one-loop diagrams using the above general construct.

Upon setting up a differential equation, we obtain the set of differential equations where the basic one-propagator and two-propagator master integrals can be solved independently, and they can be used to solve the dependant master integrals such as master integrals with

3-propagators and higher. To understand this, we consider the matrix M of a one-loop three-point function: $I = \{I_{001}, I_{011}, I_{101}, I_{110}, I_{111}\}$

$$M(\epsilon,\eta) = \begin{pmatrix} \frac{1-\epsilon}{\eta} & 0 & 0 & 0 & 0 \\ \frac{-1+\epsilon}{2\eta^2+\eta} & \frac{1-2\epsilon}{2\eta+1} & 0 & 0 & 0 \\ \frac{-2+2\epsilon}{4\eta^2+\eta} & 0 & \frac{2-4\epsilon}{4\eta+1} & 0 & 0 \\ \frac{-2+2\epsilon}{4\eta^2+\eta} & 0 & 0 & \frac{2-4\epsilon}{4\eta+1} & 0 \\ \frac{2-2\epsilon}{\eta+6\eta^2+8\eta^3} & 0 & \frac{-1+2\epsilon}{1+8\eta^2+6\eta} & \frac{-1+2\epsilon}{1+8\eta^2+6\eta} & \frac{-2\epsilon}{1+2\eta} \end{pmatrix} \tag{4.1}$$

The first master integral I_{001} can be solved independently. Following that, we solve the master integrals I_{011}, I_{101}andI_{110} using the solution I_{001}. Similarly, we use the other four master integrals to solve the master integral I_{111}.

4.2.1 Solving the power expansion at $\eta \to \infty$

The power expansion at $\eta \to \infty$ can be obtained by solving the vacuum integrals obtained after the integral expansion around η(See Appendix A.1). The first integral is usually of the form:

$$\begin{aligned} I(\text{term1}) &= \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2-\eta)^k} \\ &= -\eta^{(2-k-\epsilon)}\Gamma(\epsilon+k-2) \end{aligned} \tag{4.2}$$

Hence, we can write the power expansion at $\eta \to \infty$ as:

$$I_{inf} = \sum_{n=0} c_n \eta^{(2-k-\epsilon-n)} \tag{4.3}$$

Now, we use the equation 3.5 to obtain the recursive relations which give us $c_1, \ldots, c_n$ in terms of c_0, which is given by equation 4.2 for each k. Hence, we obtain the power series of the master integral at $\eta \to \infty$.

4.2.2 Solving the power series along the path

To solve the differential equation of $\vec{I}(\vec{x}, \eta, \frac{1}{\epsilon^j})$ at the next point in the path i.e. $\eta_o = p_0$, we first obtain the boundary value by evaluating the master integral I_{inf} at η_o and obtaining the coefficient of the required power of ϵ.

The power series at η_o is taken to be:

$$I_{\eta_o} = (\eta - \eta_o)^r \sum_{n=0} a_n (\eta - \eta_o)^n \tag{4.4}$$

We use the Frobenius method to obtain a_n in terms of a_0, where a_0 is again the obtained boundary condition at η_o. This can be done for all points in the path p to obtain $\vec{I}(\vec{x}, \eta, \frac{1}{\epsilon^j})$ at $\eta = 0$. On solving the lowest coefficient of ϵ, we solve the subsequent dependant coefficients and obtain the master integrals $I(0, \epsilon)$.

4.3 Two-loop computations

We discuss the complications that arise while obtaining the numerical solution of the master integrals of two-loop diagrams, and the strategies adopted to solve them.

4.3.1 Two-loop vacuum integrals

In order to obtain the power series at $\eta \to \infty$, we need to solve two-loop vacuum integrals as the boundary condition. Unlike one-loop vacuum integrals, two-loop integrals do not have any general solution. To solve them, Feynman parametrizations and Mellin-Barnes integrations prove to be useful strategies. The vacuum integral of the two-loop sunrise diagram has been calculated in the Appendix section A.2.1. In the literature, vacuum integrals of up to five loops have been solved and can be used to solve complicated diagrams.[40][41]

4.3.2 Coupled Differential equations

Upon performing the IBP reduction for a two-loop diagram and setting up the differential equation, we obtain the matrix M, where a few of the master integrals cannot be calculated independently as they are coupled to other master integrals. This complicates the solution of the set of differential equations and they have to be tackled using certain strategies. One such strategy is to decouple these equations by rewriting the equation as a higher-order equation. We show this for two coupled equations:

Consider the two differential equations:

$$\begin{pmatrix} \frac{\partial}{\partial\eta}x \\ \frac{\partial}{\partial\eta}y \end{pmatrix} = \begin{pmatrix} a(\eta)x + b(\eta)y \\ c(\eta)x + d(\eta)y \end{pmatrix} \tag{4.5}$$

We can use the first equation of 4.5 to express:

$$y = \frac{1}{b(\eta)}\left(\frac{\partial}{\partial\eta}x - a(\eta)x\right) \tag{4.6}$$

We substitute 4.6 into the second equation of 4.5 to get a second order differential equation of variable x:

$$\frac{1}{b(\eta)}\left(\frac{\partial^2}{\partial^2\eta}x\right) - \left(\frac{a(\eta)+d(\eta)}{b(\eta)}\right)\frac{\partial x}{\partial\eta} + \frac{a(\eta)d(\eta)-b(\eta)c(\eta)}{b(\eta)}x = 0 \tag{4.7}$$

We can do the same to get a second-order differential equation in y. These two decoupled 2^{nd}-order equations can now be solved using the Frobenius method.

We use the same strategy with differential equations of master integrals, and we can do perform the same method for more than two coupled differential equations. This strategy was used to solve the two-loop sunrise integral which contains only two coupled differential equations of master integrals.

As the matrix M contains uncoupled equations and coupled equations of different orders, an algorithm was developed to identify the uncoupled and coupled equations from the matrix and solve them accordingly. This is being used to solve two-loop kite integrals (another topology of two-loop two-point diagrams) and two-loop three-point integrals.

Chapter 5

Results

All numerical values are obtained by computation via Wolfram Mathematica. The reference values are obtained using *AMFlow*[32].

In this section, we present the numerical results of the master integrals for various one-loop and two-loop Feynman diagrams with defined kinematics, and compare them with the master integrals obtained using *AMFlow*[32]. We then talk about their improving precision with the number of terms in the series.

5.0.1 One loop two-point bubble diagrams

One loop two point diagrams are the basic one loop diagrams and hence our first calculated result. We present two cases of the bubble diagrams: massive internal lines and massless internal lines.

Massless internal propagators

We consider the following one-loop two-point Feynman diagram. The propagators P_k for the above diagram are given by:

$$P_1 = l_1^2, \quad P_2 = (l_1 - p_1)^2$$

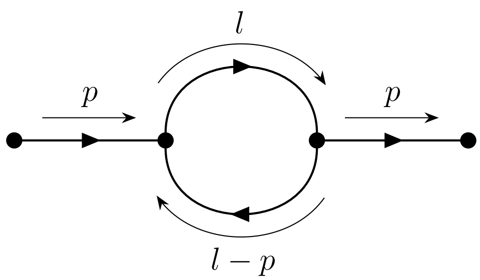

Figure 5.1: One-loop two-point diagram

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
1	I_{01}	J_1	-
2	I_{11}	J_2	p_1

Table 5.1: Master Integrals of two-loop two-point integral

Upon IBP Reduction using LiteRed, we get the following two Master Integrals:

On adding auxiliary mass η to each propagator, we get the differential equation matrix $M(\epsilon, \eta)$ given by:

$$M(\epsilon,\eta) = \begin{pmatrix} \frac{1-\epsilon}{\eta} & 0 \\ \frac{2\epsilon-2}{4\eta^2+\eta} & \frac{2-4\epsilon}{4\eta+1} \end{pmatrix} \tag{5.1}$$

The master integrals with auxiliary masses are given by:

$$I_{10}^{\text{aux}}(\epsilon,\eta) = \int \frac{\mathrm{d}^D\ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2-\eta)^1}$$

$$I_{11}^{\text{aux}}(1,1,\epsilon,\eta) = \int \frac{\mathrm{d}^D\ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2-\eta)^1\left((\ell_1+p_1)^2-\eta\right)^1}$$

We use the equation 3.4 to obtain the exact solution to the vacuum integral I_{10}:

$$I_{10}^{\text{aux}} = -\eta^{(1-\epsilon)}\Gamma(\epsilon-1) \tag{5.2}$$

To solve the integral I_{11}^{aux}, we calculate it's value at $\eta \longrightarrow \infty$. To achieve that, the expansion around $\eta \longrightarrow \infty$ is given by:

$$I_{11}^{aux} = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell_1^2 - \eta\right)^1 \left(\ell_1^2 - \eta\right)^1} \sum_{j=0}^{\infty} \left(\frac{2\ell p - p^2}{\ell_1^2 - \eta}\right)^j \tag{5.3}$$

This expansion can be calculated using the boundary condition i.e. the first term of the expansion. The first term of the expansion is a vacuum integral given by:

$$\begin{aligned} I_{11}^{aux}(\text{first term}) &= \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell_1^2 - \eta\right)^2} \\ &= \eta^{-\epsilon}\Gamma(\epsilon) \end{aligned} \tag{5.4}$$

Hence, the boundary conditions for a one loop massless bubble diagram at $\eta \to \infty$ are (see Appendix A.1):

$$I_{10}^{\text{aux}} = -\eta^{(1-\epsilon)}\Gamma(\epsilon - 1)$$

$$I_{11}^{\text{aux}} = \eta^{-\epsilon}\Gamma(\epsilon)$$

We obtain the values of the above Master Integrals at $\eta = 0$ using the above boundary conditions. The values of the master integrals for various precision terms are:

J_i	precision= 20	precision= 30	precision= 40
J_1	0	0	0
J_2	$1.4228043604 + \frac{1.0000000000}{\epsilon}$	$1.4228202324 + \frac{1.0000000000}{\epsilon}$	$1.4227970875 + \frac{1.0000000000}{\epsilon}$

Table 5.2: Numerical Values of master integrals

The reference value using AMFlow is given by $J_2 = 1.4227843350 + \frac{1.0000000000}{\epsilon}$.

Massive case

The propagators P_k for the massive case are given by:

$$P_1 = l_1^2 - m^2, \quad P_2 = (l_1 - p_1)^2 - m^2$$

The master integrals and the boundary conditions are the same as the massless cases, as the auxiliary mass was anyway added before the IBP reduction. The kinematics of this configuration is given by:

$$p_1^2 = -1 \quad m^2 = 2$$

The matrix $M(\epsilon, \eta)$ is given by:

$$M(\epsilon,\eta) = \begin{pmatrix} \frac{1-\epsilon}{2+\eta} & 0 \\ \frac{2\epsilon-2}{4\eta^2+17\eta+18} & \frac{2-4\epsilon}{4\eta+9} \end{pmatrix} \tag{5.5}$$

The two master integrals are given for precision terms=40:

J_i	precision= 40	Reference Value
J_1	-3.8540401040 + $\frac{4}{\epsilon}$	-3.8540401040 + $\frac{4}{\epsilon}$
J_2	$-2.0041707455 + \frac{0.9999999999}{\epsilon}$	$-2.0041707455 + \frac{0.9999999999}{\epsilon}$

Table 5.3: Numerical values of Master integrals: One-loop two-point massive

5.0.2 One loop three-point diagram

Here, we present the numerical solution to the basis of Master Integrals for a one-loop three-point diagram with massless internal propagators.

We consider the following Feynman diagram:

The propagators P_k for the above diagram are given by:

$$P_1 = l_1^2, \quad P_2 = (l_1 - p_1)^2, \quad P_3 = (l_1 + p_2)^2$$

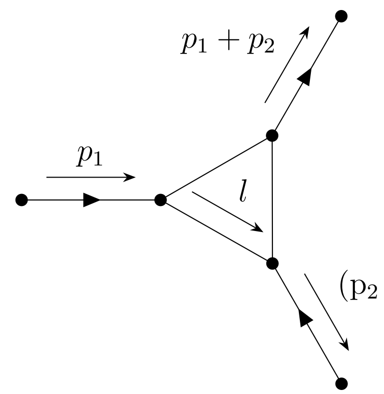

Figure 5.2: One-loop three point function

Off-Shell Case

For the kinematics with $p_1^2 \neq 0$ and $p_2^2 \neq 0$ and $p_1 = p_2$, we get the following Master Integrals via LiteRed:

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
1	I_{001}	J_1	-
2	I_{011}	J_2	p_1, p_2
	I_{110}	J_3	p_1
	I_{101}	J_4	p_2
3	I_{111}	J_5	p_1, p_2

Table 5.4: Master integrals: One-loop three-point

The matrix $M(\epsilon, \eta)$ for this diagram is given by:

$$M(\epsilon,\eta) = \begin{pmatrix} \frac{1-\epsilon}{\eta} & 0 & 0 & 0 & 0 \\ \frac{-1+\epsilon}{2\eta^2+\eta} & \frac{1-2\epsilon}{2\eta+1} & 0 & 0 & 0 \\ \frac{-2+2\epsilon}{4\eta^2+\eta} & 0 & \frac{2-4\epsilon}{4\eta+1} & 0 & 0 \\ \frac{-2+2\epsilon}{4\eta^2+\eta} & 0 & 0 & \frac{2-4\epsilon}{4\eta+1} & 0 \\ \frac{2-2\epsilon}{\eta+6\eta^2+8\eta^3} & 0 & \frac{-1+2\epsilon}{1+8\eta^2+6\eta} & \frac{-1+2\epsilon}{1+8\eta^2+6\eta} & \frac{-2\epsilon}{1+2\eta} \end{pmatrix} \tag{5.6}$$

The boundary values for each of the five Master Integrals at $\eta \longrightarrow \infty$ have been calculated (see Appendix A.1). They are given by:

$$I_{001}^{\text{aux}} = -\eta^{(1-\epsilon)}\Gamma(\epsilon - 1)$$

$$I_{011}^{\text{aux}} = I_{101}^{\text{aux}} = I_{110}^{\text{aux}} = \eta^{-\epsilon}\Gamma(\epsilon)$$

$$I_{111}^{\text{aux}} = -\eta^{(-1-\epsilon)}\frac{\Gamma(\epsilon+1)}{2}$$

We obtain the values of the above 5 Master Integrals for precision=40:

Kinematics 1

$$p_1^2 = 1 \quad p_2^2 = 1 \quad (p_1 + p_2)^2 = -4$$

J_i	precision= 40	Reference Value
J_1	0	0
J_2	$0.03595315519 + \frac{1.0000000000}{\epsilon}$	$0.03595315519 + \frac{1.0000000000}{\epsilon}$
J_3	$1.4227970875 + \frac{1.0000000000}{\epsilon}$	$1.4227843350 + \frac{1.0000000000}{\epsilon}$
J_4	$1.4227970875 + \frac{1.0000000000}{\epsilon}$	$1.4227843350 + \frac{1.0000000000}{\epsilon}$
J_5	0.4868757134	0.4868757134

Table 5.5: Numerical values of Master integrals: One-loop three-point kin1

Kinematics 2

$$p_1^2 = -1 \quad p_2^2 = -1 \quad (p_1 + p_2)^2 = 0$$

J_i	precision= 40	Reference Value
J_1	0	0
J_2	$0.72963715453 + \frac{1.0000000000}{\epsilon}$	$0.72963715453 + \frac{1.0000000000}{\epsilon}$
J_3	$1.4227843350 + \frac{1.0000000000}{\epsilon}$	$1.4227843350 + \frac{1.0000000000}{\epsilon}$
J_4	$1.4227843350 + \frac{1.0000000000}{\epsilon}$	$1.4227843350 + \frac{1.0000000000}{\epsilon}$
J_5	-1.8319311884	-1.8319311884

Table 5.6: Numerical values of Master integrals: One-loop three-point kin2

On-shell case

For the kinematics with $p_1^2 = 0$ and $p_2^2 = 0$ and $p_1 = p_2$, we get the following Master Integrals via LiteRed:

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
1	I_{001}	J_1	-
2	I_{011}	J_2	p_1, p_2
3	I_{111}	J_3	p_1, p_2

Table 5.7: Master integrals: One-loop three-point on-shell

The kinematics of the following integral is:

$$u = (p_1 + p_2)^2 = -2$$

The matrix $M(\epsilon, \eta)$ for this integral and kinematic configurations is given by:

$$M(\epsilon, \eta) = \begin{pmatrix} \frac{1-\epsilon}{\eta} & 0 & 0 \\ \frac{-1+\epsilon}{2\eta^2+\eta} & \frac{1-2\epsilon}{2\eta+1} & 0 \\ \frac{1-\epsilon}{2\eta^2+4\eta^3} & \frac{-1+2\epsilon}{4\eta^2+2\eta} & \frac{-\epsilon}{\eta} \end{pmatrix}$$

The boundary values for each of the five Master Integrals at $\eta \longrightarrow \infty$ have been calculated (see Appendix A.1). They are given by:

$$I_{001}^{\text{aux}} = -\eta^{(1-\epsilon)}\Gamma(\epsilon - 1)$$

$$I_{011}^{\text{aux}} = \eta^{-\epsilon}\Gamma(\epsilon)$$

$$I_{111}^{\text{aux}} = -\eta^{(-1-\epsilon)}\Gamma(\epsilon + 1)$$

The values of the above 3 Master Integrals for various precision terms are:

J_i	precision= 40	Reference Value
J_1	0	0
J_2	0.0077671357 - $\frac{0.500000000}{\epsilon^2} + \frac{0.63517892341}{\epsilon}$	$0.0077780769 - \frac{0.4999999999}{\epsilon^2} + \frac{0.63518142273}{\epsilon}$
J_3	$0.7296421621 + \frac{1.000000000}{\epsilon}$	$0.7296371545 + \frac{0.9999999999}{\epsilon}$

Table 5.8: Numerical values of Master integrals: One-loop three-point

5.0.3 One loop 4 point function

We calculated the numerical solutions to the Master Integrals for a one-loop four-point function with 4 on-shell external legs and massless internal propagators. The Feynman diagram for the same is given below.

Considering all incoming particles, the propagators P_k for the above diagram are given by:

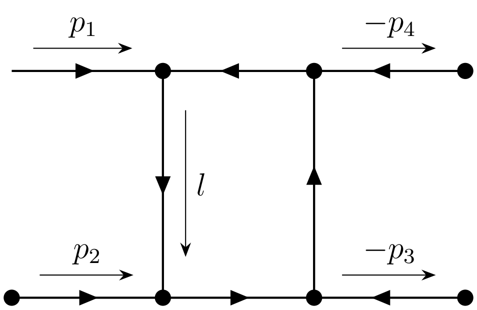

Figure 5.3: One-loop four-point diagram

$$P_1 = l_1^2, \quad P_2 = (l_1 + p_1)^2, \quad P_3 = (l_1 + p_1 + p_2)^2, \quad P_4 = (l_1 + p_1 + p_2 + p_3)^2$$

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
1	I_{0001}	J_1	p_1, p_2, p_3
2	I_{0101}	J_2	p_1, p_2, p_3
	I_{1001}	J_3	p_1, p_2, p_3
	I_{1010}	J_4	p_1, p_2
3	I_{0111}	J_5	p_1, p_2, p_3
	I_{1011}	J_6	p_1, p_2, p_3
	I_{1101}	J_7	p_1, p_2, p_3
	I_{1110}	J_8	p_1, p_2
4	I_{1111}	J_9	p_1, p_2, p_3

Table 5.9: Master integrals: One-loop four-point

For $p_1^2 = 0$, $p_2^2 = 0$ and $p_3^2 = 0$, we get the following Master Integrals via LiteRed:

The kinematics of the following integral are:

$$u = (p_1 + p_2)^2 = -4$$

$$v = (p_1 + p_3)^2 = -4$$

$$w = (p_2 + p_3)^2 = -4$$

$M(\epsilon, \eta) =$

$$\begin{pmatrix}
\frac{1-\epsilon}{\eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1+\epsilon}{2\eta^2+2\eta} & \frac{1-2\epsilon}{2\eta+2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1+\epsilon}{2\eta^2+6\eta} & 0 & \frac{1-2\epsilon}{2\eta+6} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1+\epsilon}{2\eta^2+2\eta} & 0 & 0 & \frac{1-2\epsilon}{2\eta+2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1-\epsilon}{4\eta^2(1+\eta)} & \frac{-1+2\epsilon}{4\eta(1+\eta)} & 0 & 0 & -\frac{\epsilon}{\eta} & 0 & 0 & 0 & 0 \\
\frac{1-\epsilon}{4\eta(1+\eta)(3+\eta)} & 0 & \frac{-3+6\epsilon}{8\eta(3+\eta)} & \frac{1-2\epsilon}{8\eta+8\eta^2} & 0 & -\frac{\epsilon}{\eta} & 0 & 0 & 0 \\
\frac{1-\epsilon}{4\eta(1+\eta)(3+\eta)} & \frac{1-2\epsilon}{8\eta+8\eta^2} & \frac{-3+6\epsilon}{8\eta(3+\eta)} & 0 & 0 & 0 & -\frac{\epsilon}{\eta} & 0 & 0 \\
\frac{1-\epsilon}{4\eta^2(1+\eta)} & 0 & 0 & \frac{-1+2\epsilon}{4\eta+4\eta^2} & 0 & 0 & 0 & -\frac{\epsilon}{\eta} & 0 \\
\frac{(-1+\epsilon)(-3+\eta)}{8(-1+\eta)\eta^2(1+\eta)(3+\eta)} & \frac{1-2\epsilon}{8\eta-8\eta^3} & \frac{3-6\epsilon}{8(-1+\eta)\eta(3+\eta)} & \frac{1-2\epsilon}{8\eta-8\eta^3} & \frac{\epsilon}{4\eta-4\eta^2} & \frac{\epsilon}{-2\eta+2\eta^2} & \frac{\epsilon}{-2\eta+2\eta^2} & \frac{\epsilon}{4\eta-4\eta^2} & \frac{1+2\epsilon}{2-2\eta}
\end{pmatrix} \tag{5.7}$$

The boundary values for each of the three Master Integrals at $\eta \longrightarrow \infty$ have been calculated (see Appendix A.1). They are given by:

$$I^{\text{aux}}_{0001} = -\eta^{(1-\epsilon)}\Gamma(\epsilon-1)$$
$$I^{\text{aux}}_{0101} = I^{\text{aux}}_{1001} = I^{\text{aux}}_{1010} = \eta^{-\epsilon}\Gamma(\epsilon)$$
$$I^{\text{aux}}_{0111} = I^{\text{aux}}_{1011} = I^{\text{aux}}_{1101} = I^{\text{aux}}_{1110} = \eta^{(-1-\epsilon)}\frac{\Gamma(\epsilon+1)}{2}$$
$$I^{\text{aux}}_{1111} = \eta^{(-2-\epsilon)}\frac{\Gamma(\epsilon+2)}{6}$$

The values of the above 9 Master Integrals for 40 precision terms are:

J_i	precision= 40	Reference Value
J_1	0	0
J_2	$-0.6566572065 + \frac{1}{\epsilon}$	$-0.6566572065 + \frac{1}{\epsilon}$
J_3	0	0
J_4	$0.03648997397. + \frac{1}{\epsilon}$	$0.03648997397. + \frac{1}{\epsilon}$
J_5	$-0.2763046944 - \frac{0.25}{\epsilon^2} + \frac{0.4908775065}{\epsilon}$	$-0.2763046944 - \frac{0.25}{\epsilon^2} + \frac{0.4908775065}{\epsilon}$
J_6	$-0.2785444346 + \frac{0.125}{\epsilon^2} - \frac{0.3320821508}{\epsilon}$	$-0.2785444346 + \frac{0.125}{\epsilon^2} - \frac{0.3320821508}{\epsilon}$
J_7	$-0.2785444346 + \frac{0.125}{\epsilon^2} - \frac{0.3320821508}{\epsilon}$	$-0.2785444346 + \frac{0.125}{\epsilon^2} - \frac{0.3320821508}{\epsilon}$
J_8	$-0.2763046944 - \frac{0.25}{\epsilon^2} + \frac{0.4908775065}{\epsilon}$	$-0.2763046944 - \frac{0.25}{\epsilon^2} + \frac{0.4908775065}{\epsilon}$
J_9	$0.0852102004 - \frac{0.125}{\epsilon^2} + \frac{0.2887604520}{\epsilon}$	$0.0852102004 - \frac{0.125}{\epsilon^2} + \frac{0.2887604520}{\epsilon}$

Table 5.10: Numerical values of Master integrals: One-loop four-point

5.0.4 2 loop sunrise integral

After calculating the basic diagrams with one loop momenta and four external legs, we calculate the two-loop two-point sunrise integral for uniform masses. We consider the following Feynman diagram shown in figure 5.4.

The propagators P_k for the above diagram are given by:

$$P_1 = l_1^2, \quad P_2 = (l_2)^2, \quad P_3 = (-l_1 - l_2 + p_1)^2$$

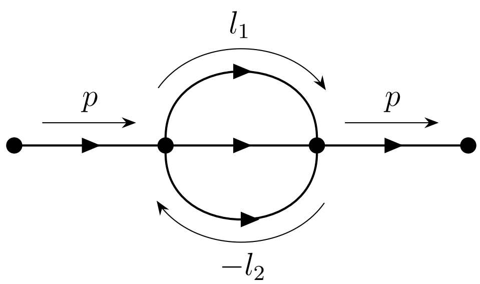

Figure 5.4: Two loop two point sunrise diagram

For the kinematics with $p_1^2 \neq 0$, we get the following Master Integrals via LiteRed:

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
3	I_{111}	J_1	p_1
3	I_{211}	J_2	p_1

Table 5.11: Master integrals: Two-loop two-point

We obtain the boundary conditions for these propagators(see Appendix A.2):

$$I_{111} = \left(\mathrm{i}\pi^D\right) \frac{\Gamma(2\varepsilon - 1)}{(\eta)^{2\varepsilon-1}} \frac{\Gamma(\epsilon)^2\Gamma(1-\epsilon)}{\Gamma(2\epsilon)\Gamma(2-\epsilon)} \tag{5.8}$$

$$I_{211} = \left(\mathrm{i}\pi^D\right) \frac{\Gamma(2\varepsilon)}{(\eta)^{2\varepsilon-1}} \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2\epsilon+1)\Gamma(2-\epsilon)} \tag{5.9}$$

The kinematics of the following integral is:

$$u = (p_1)^2 = -1$$

The matrix $M(\epsilon, \eta)$ for this integral and kinematic configurations is given by:

$$M(\epsilon, \eta) = \begin{pmatrix} \frac{2-3\epsilon}{2\eta} & \frac{\epsilon+\epsilon\eta}{2\eta-4\epsilon\eta} \\ \frac{2-7\epsilon+6\epsilon^2}{2\eta+2\eta^2} & \frac{\epsilon+2\eta-7\epsilon\eta}{2\eta+2\eta^2} \end{pmatrix} \tag{5.10}$$

The values of the above 2 Master Integrals for various precision terms are:

J_i	precision= 40	Reference Value
J_1	$1.33639216754 + \frac{0.2500000000}{\epsilon}$	$1.33639216754 + \frac{0.2500000000}{\epsilon}$
J_2	-2.2790638956 - $\frac{0.50000000000}{\epsilon^2} - \frac{0.9227843350}{\epsilon}$	-2.2790638956 - $\frac{0.50000000000}{\epsilon^2} - \frac{0.9227843350}{\epsilon}$

Table 5.12: Numerical values of Master integrals: Two-loop two-point

5.0.5 2 loop kite integral

We calculate the two-loop two-point kite integral for uniform masses. We consider the following Feynman diagram shown in figure 5.5.

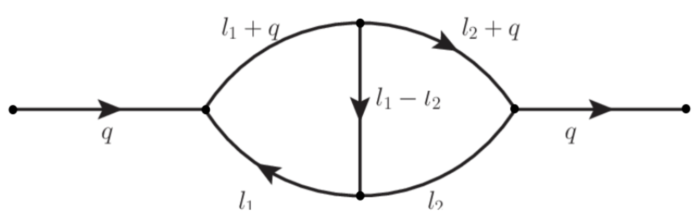

Figure 5.5: Two loop two point kite diagram

The propagators P_k for the above diagram are given by:

$$P_1 = l_1^2, \quad P_2 = l_2^2, \quad P_3 = (-l_1 - l_2 + p_1)^2, \quad P_4 = (-l_1 + p_1)^2, \quad P_5 = (-l_2 + p_1)^2$$

For the kinematics with $p_1^2 - 1$, we get the following Master Integrals via LiteRed:

The kinematics of the following integral is:

$$u = (p_1)^2 = -1$$

Number of Propagators	Master Integrals $\vec{I}$	Master Integral number $\vec{J}$	Kinematic Dependence
2	I_{01100}	J_1	p_1
3	I_{01011}	J_2	p_1
3	I_{01110}	J_3	p_1
3	I_{11100}	J_4	p_1
3	I_{02011}	J_5	p_1
3	I_{21100}	J_4	p_1
4	I_{10111}	J_7	p_1
5	I_{11111}	J_8	p_1

Table 5.13: Master integrals: Two-loop two-point

The matrix $M(\epsilon, \eta)$ for this integral and kinematic configurations is given by:

$$\begin{pmatrix}
\frac{2-2\epsilon}{\eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{-1+\epsilon}{\eta^2+\eta} & 0 & \frac{1-\epsilon+2\eta-3\epsilon\eta}{\eta^2+\eta} & 0 & 0 & 0 & 0 & 0 \\
\frac{-1+\epsilon}{2\eta^2+2\eta} & 0 & 0 & \frac{1-2\epsilon}{2\eta+2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & \frac{-2+7\epsilon-6\epsilon^2}{\eta+\eta^2} & 0 & 0 & \frac{-\epsilon+2\eta-5\eta\epsilon}{\eta+\eta^2} & 0 & 0 & 0 \\
\frac{-2(1-\epsilon)^2}{9\eta^2+10\eta^3+\eta^4} & 0 & 0 & \frac{(-2+7\epsilon-6\epsilon^2)}{9\eta^2+10\eta^3+\eta^4} & 0 & \frac{10\eta+18\eta^2}{9\eta^2+10\eta^3+\eta^4} & 0 & 0 \\
0 & 0 & \frac{-2(1-\epsilon)}{\eta(1+\eta)} & 0 & 0 & 0 & \frac{2-4\epsilon}{\eta+1} & 0 \\
\frac{2(1-\epsilon)^2}{(1-2\eta)\eta^2(1+\eta)^2} & \frac{-2+3\epsilon}{\eta(1+\eta)^2} & \frac{2-2\epsilon}{\eta^2(1+\eta)^2} & \frac{-2+3\epsilon}{\eta(1+\eta)^2} & \frac{-1+\eta}{\eta(1+\eta)^2} & \frac{-1+3\eta}{\eta(1+\eta)^2} & \frac{-1+2\epsilon}{\eta(1+\eta)^2} & \frac{-\epsilon+3\epsilon\eta}{\eta(1+\eta)}
\end{pmatrix}$$

The values of the above 2 Master Integrals for various precision terms are:

J_i	precision terms= 40	Reference Value
J_1	0	0
J_2	$1.3363921675 + \frac{0.25000000000}{\epsilon}$	$1.3363921675 + \frac{0.25000000000}{\epsilon}$
J_3	$1.3363921675 + \frac{0.25000000000}{\epsilon}$	$1.3363921675 + \frac{0.25000000000}{\epsilon}$
J_4	0	0
J_5	$-2.2790638956 - \frac{0.4999999999}{\epsilon^2} - \frac{0.9227843350}{\epsilon}$	$-2.2790638956 - \frac{0.4999999999}{\epsilon^2} - \frac{0.9227843350}{\epsilon}$
J_6	$-2.2790638956 - \frac{0.4999999999}{\epsilon^2} - \frac{0.9227843350}{\epsilon}$	$-2.2790638956 - \frac{0.4999999999}{\epsilon^2} - \frac{0.9227843350}{\epsilon}$
J_7	$6.4036964615 + \frac{0.9999999999}{\epsilon^2} + \frac{2.8455686701}{\epsilon}$	$6.4036964615 + \frac{0.9999999999}{\epsilon^2} + \frac{2.8455686701}{\epsilon}$
J_8	-7.2123414189	-7.2123414189

Table 5.14: Numerical values of Master integrals: Two-loop two-point

Chapter 6

Conclusion and Future works

In the first half of the thesis, we focused on understanding the modern methods of computing multi-loop integrals, specifically by setting up a differential equation of the basis of master integrals. Some of the computations performed while understanding the techniques are included as examples for each technique. These techniques and their examples have been presented in Chapter 2. The idea to simplify some known scattering amplitudes containing 2DHPLs using PolyLogTools was attempted but could not be completed with a successful result.

In the second half of the thesis, we studied the method of obtaining generalized boundary conditions by adding an auxiliary mass η(chapter 3). Using the power series method of solving the differential equation and analytical continuation, we wrote a Mathematica program to obtain the numerical solution of the master integrals for one-loop and two-loop diagrams. The most complex diagram to be successfully evaluated is a two-loop two-point function, and the computation of two-loop three-point function is being programmed using the decoupling method (section 4.3.2).

In Chapter 5, we presented the results of the master integrals obtained using the Mathematica program. The master integrals are expressed as a Laurent expansion in ϵ with numerical real coefficients and are presented up to the ϵ^0 order. These results have been compared to the values obtained by evaluating the same diagrams using AMFlow[32].

On a careful study of the results, we observe that improving precision by increasing the number of terms by 10 improves the accuracy of the master integrals by one decimal place. We also observe that all the master integrals with 40 terms of the series expansion are accurate up to the 12^{th} decimal place in comparison with the 40th decimal precision by AMFlow. The limitations in the precision of our numerical values of master integrals are:

- The Mathematica built-in function *Solve* is computationally slow and inaccurate beyond twenty terms in the matrix. When precision=40 is considered, the row reduction of the function gives inaccurate results at higher decimal places.
- The precision of the value is limited by the other built-in functions used in the Mathematica program.
- The precision of the computed master integrals can be improved by including more terms in the power series solution, but the computational time increases drastically with an increasing number of terms.

Future works and prospective

The Mathematica program is still in its developing stages, and many interesting prospects can be implemented in the same:

- The program needs to be developed to obtain two-loop three-point and two-loop four-point integrals for single mass scales.
- The decoupling method used in the program is effective only when the n^{th} order decoupled differential equation is factorizable. In order to solve systems where decoupling isn't possible, we wish to use algorithms that can solve any coupled system more effectively.
- In order to increase the speed and precision of the master integrals obtained using the program, effective ways of solving the recursive relations of the Frobenius solution would be necessary to produce significant results.
- The Mathematica program can also be improved to include the computation of phase-space integrals using the reverse unitarity condition[3] of IBP reduction.

Appendix A

Vacuum integrals

A.1 Infinity expansion of one-loop integrals

All the one-loop vacuum integrals are of the form:

$$
\begin{aligned}
I &= \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell^2 - \eta\right)^k} \\
&= \frac{\Gamma\left(k - \frac{1}{2}D\right)\Gamma\left(\frac{1}{2}D\right)}{(4\pi)^{D/2}\Gamma(k)\Gamma\left(\frac{1}{2}D\right)}\eta^{-(k-D/2)}
\end{aligned}
\tag{A.1}
$$

We use this relation to calculate the boundary conditions containing one-loop vacuum integrals in section 5.

A.1.1 One-loop two-point integral

We have the following one-loop two point integral:

$$
I_{11}^{\mathrm{aux}}(1,1,\epsilon,\eta) = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell_1^2 - \eta\right)^1 \left((\ell_1 + p_1)^2 - \eta\right)^1}
$$

The expansion of the above integral at $\eta \to \infty$ is given by:

$$
\begin{aligned}
I_{11} &= \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell_1^2 - \eta\right)^1 \left(\ell_1^2 - \eta\right)^1} \sum_{j=0}^{\infty} \left(\frac{2\ell p - p^2}{\ell_1^2 - \eta}\right)^j \\
&= \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell^2 - \eta\right)^2} + \text{Higher powers}
\end{aligned}
\tag{A.2}
$$

Using the equation A.1, we get:

$$
I_{11} = \eta^{-\epsilon} \Gamma(\epsilon) \tag{A.3}
$$

A.1.2 One-loop three-point integral

We have the following one-loop three-point integral:

$$
I_{111}(\epsilon, \eta) = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell_1^2 - \eta\right)\left((\ell_1 - p_1)^2 - \eta\right)\left((\ell_1 + p_2)^2 - \eta\right)} \tag{A.4}
$$

The expansion of the above integral at $\eta \to \infty$ is given by:

$$
\begin{aligned}
I_{111} &= \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell_1^2 - \eta\right)^3} \sum_{j=0}^{\infty} \left(\frac{2\ell p_1 - p_1^2}{\ell_1^2 - \eta}\right)^j \sum_{k=0}^{\infty} \left(\frac{2\ell p_2 + p_2^2}{\ell_1^2 - \eta}\right)^k \\
&= \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell^2 - \eta\right)^3} + \text{Higher powers}
\end{aligned}
\tag{A.5}
$$

Using the equation A.1, we get:

$$
I_{111} = \eta^{(-1-\epsilon)} \frac{\Gamma(\epsilon + 1)}{2} \tag{A.6}
$$

A.1.3 One-loop four-point integral

We have the following one-loop four-point integral:

$$
I_{1111}(\epsilon, \eta) = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{\left(\ell_1^2 - \eta\right)\left((\ell_1 - p_1)^2 - \eta\right)\left((\ell_1 - p_1 - p_2)^2 - \eta\right)\left((\ell_1 - p_1 - p_2 - p_3)^2 - \eta\right)} \tag{A.7}
$$

The expansion of the above integral at $\eta \to \infty$ is given by:

$$I_{1111} = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2 - \eta)^4} \sum_{j=0}^{\infty} \left(\frac{2\ell p_1 - p_1^2}{\ell_1^2 - \eta} \right)^j \sum_{k=0}^{\infty} \left(\frac{2\ell(p_2 + p_1) - p_1^2 - p_2^2}{\ell_1^2 - \eta} \right)^k \times \tag{A.8}$$

$$\sum_{l=0}^{\infty} \left(\frac{2\ell(p_2 + p_1 + p_3) - p_1^2 - p_2^2 - p_3^2}{\ell_1^2 - \eta} \right)^l \tag{A.9}$$

$$= \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - \eta)^4} + \text{Higher powers} \tag{A.10}$$

Using the equation A.1, we get:

$$I_{1111} = \eta^{(-2-\epsilon)} \frac{\Gamma(\epsilon + 2)}{6} \tag{A.11}$$

A.2 Infinity expansion of two-loop integrals

A.2.1 Two-loop two point sunrise diagram

$$I_{111}(\epsilon, \eta) = \int \frac{\mathrm{d}^D \ell_1 \mathrm{d}^D \ell_2}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2 - \eta)(\ell_2^2 - \eta)((\ell_1 + \ell_2 - p_1)^2 - \eta)} \tag{A.12}$$

The expansion of the above integral at $\eta \to \infty$ is given by:

$$\begin{aligned} I_{111} &= \int \frac{\mathrm{d}^D \ell_1 \mathrm{d}^D \ell_2}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2 - \eta)(\ell_2^2 - \eta)((\ell_1 + \ell_2)^2 - \eta)} \sum_{j=0}^{\infty} \left(\frac{2(\ell_1 + \ell_2)p_1 - p_1^2}{((\ell_1 + \ell_2)^2 - \eta)} \right)^j \\ &= \int \frac{\mathrm{d}^D \ell_1 \mathrm{d}^D \ell_2}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2 - \eta)(\ell_2^2 - \eta)((\ell_1 + \ell_2)^2 - \eta)} + \text{Higher powers} \end{aligned} \tag{A.13}$$

Calculation of the first term

The first term is a massive vacuum two-loop sunrise diagram. This is easier to solve using the Feynman parametrization of integrals. Consider the integral:

$$\int \frac{\mathrm{d}^D\ell_1 \mathrm{d}^D\ell_2}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell_1^2-\eta)(\ell_2^2-\eta)((\ell_1+\ell_2)^2)} \tag{A.14}$$

We first calculate the Symanzik polynomials:

$$\mathcal{U} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 \tag{A.15}$$

$$\mathcal{F} = (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3)(\alpha_1+\alpha_2)x \tag{A.16}$$

The Feynman representation using equation 2.14 is given by:

$$\begin{aligned} I = \left(\mathrm{i}\pi^D\right) \frac{\Gamma(+2\varepsilon-1)}{(\eta)^{2\varepsilon-1}} \int_0^\infty \int_0^\infty \int_0^\infty (\, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3) \\ \times \delta\left(\sum_l \alpha_l - 1\right) \frac{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)^{\varepsilon-2}}{(\alpha_1+\alpha_2)^{2\varepsilon-1}}. \end{aligned} \tag{A.17}$$

Now, we use the Cheng-Wu identity to write:

$$\delta\left(\sum_l \alpha_l - 1\right) = \delta\left(\alpha_1+\alpha_2-1\right) \tag{A.18}$$

We redefine the variables $\alpha_1 = \beta, \quad \alpha_2 = 1-\beta$ and simplify the integral as:

$$I = \left(\mathrm{i}\pi^D\right) \frac{\Gamma(+2\varepsilon-1)}{(\eta)^{2\varepsilon-1}} \int_0^1 \int_0^\infty \, \mathrm{d}\beta \, \mathrm{d}\alpha_3 \frac{(\beta(1-\beta)+\alpha_3)^{\varepsilon-2}}{1^{2\varepsilon-1}}. \tag{A.19}$$

We integrate over α_3 to obtain:

$$I = \left(\mathrm{i}\pi^D\right) \frac{\Gamma(+2\varepsilon-1)}{(\eta)^{2\varepsilon-1}} \int_0^1 \, \mathrm{d}\beta \frac{1}{\epsilon-1} \frac{1}{(\beta(1-\beta))^{(1-\epsilon)}} \tag{A.20}$$

On integrating, we get the solution:

$$I_{111} = \left(\mathrm{i}\pi^{D}\right) \frac{\Gamma(2\varepsilon - 1)}{(\eta)^{2\varepsilon-1}} \frac{\Gamma(\epsilon)^2\Gamma(1-\epsilon)}{\Gamma(2\epsilon)\Gamma(2-\epsilon)} \tag{A.21}$$

$$I_{211} = \left(\mathrm{i}\pi^{D}\right) \frac{\Gamma(2\varepsilon)}{(\eta)^{2\varepsilon-1}} \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2\epsilon+1)\Gamma(2-\epsilon)} \tag{A.22}$$

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