Study on the spectral zeta function of the Jaynes-Cummings model

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by

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Certificate

This is to certify that this dissertation entitled “Study on the spectral zeta function of the Jaynes-Cummings model” towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Riddhi Manna at Indian Institute of Science Education and Research under the supervision of Prof. Masato Wakayama, Fundamental Mathematics Research Principal, NTT Institute for Fundamental Mathematics, during the academic year 2022-2023.

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This thesis is dedicated to my grandparents, parents, brother and friends

Declaration

I hereby declare that the matter embodied in the report entitled "Study on the spectral zeta function of the Jaynes-Cummings model" are the results of the work carried out by me at the NTT Institute for Fundamental Mathematics, while affiliated to the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. Masato Wakayama and the same has not been submitted elsewhere for any other degree

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Contents

Abstract

In this thesis, the spectral zeta function associated with the Jaynes-Cummings Hamiltonian is explored. The thesis first reviews the known results about the spectral properties of the JC Hamiltonian and then goes on to prove the analytic continuation of the JC spectral zeta function using summation formulas. This proof is completed in two parts, first the analytic continuation to $Re(s) > 0$ is shown followed by the analytic continuation to the entire complex plane. This proof involves analysing some hypergeometric functions arising naturally from the summation formulas used in the proof. Some possible areas where this proof might be applied in the future are discussed.

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Chapter 1

Introduction

The Jaynes-Cummings(JC) model is a quantum optics model which describes the interaction between a two-level atom and a single mode of a quantized electromagnetic field [10]. This model, being exactly solvable, finds much application in quantum optics and solid-state physics, among other applications. For example, the JC model was used to predict the existence of Rabi oscillations which was then experimentally shown to exist.

The Quantum Rabi model is an important fundamental model describing the interaction between an atom and an electromagnetic field, the fully quantised version of which was first discussed by Jaynes and Cummings in 1963. The JC model can be derived as an approximation to the Quantum Rabi model, via the rotating wave approximation when the coupling strength between the atom and the field is weak (called the JC regime of the Quantum Rabi model). The predictions of physical quantities made using the JC model have been demonstrated to be matching the experimental results for suitable values of the coupling strength[16].

The spectral zeta function of operators in quantum mechanics has interesting connections to number theory. For example, the spectral zeta function of the Quantum Harmonic Oscillator (QHO) which encodes the energy eigenvalues of the system, is the Riemann zeta multiplied by some elementary functions. The spectral zeta function of the Jaynes-Cummings model encodes the information about the energy eigenvalues of the JC Hamiltonian and is therefore interesting both mathematically and with respect to its physical applications. The spectral zeta function of the aforementioned Quantum Rabi model has been studied and its analytic continuation was shown by Sugiyama[19] and then by Reyes-Bustos and

Wakayama[18] by another method. The topic of this thesis is proving the analytic continuation of the spectral zeta function of the JC Hamiltonian and providing an explicit formula for the same. Since the eigenvalues of the JC Hamiltonian can be easily computed, which is not the case for QRM, the proof of analytic continuation of its spectral zeta function could proceed through the use of elementary summation formulas as elaborated in the thesis. The analytic continuation of the JC spectral zeta function and the derivation of an explicit formula for the same may be used to compute physical quantities related to the system.

The Jaynes-Cummings Hamiltonian is given by

$$H_{JC} = \omega a^\dagger a + \frac{g}{2}(\sigma_+ a + a^\dagger \sigma_-) + \frac{\omega_0}{2}\sigma_z,$$

Chapter 2 deals with developing the background necessary for studying this JC Hamiltonian through studying the Quantum Harmonic Oscillator. The Quantum Harmonic Oscillator is an important model of quantum mechanics which can be used to model various physical phenomena such as phonons in condensed matter or molecular vibrations. The solution to the Quantum Harmonic Oscillator will introduce the concepts of the raising and lowering operators a and $a^\dagger$ and the number operator $a^\dagger a$ which appear in the JC Hamiltonian.

The operators introduced in chapter 2 will in turn be used to find the eigenvalues of the Jaynes-Cummings Hamiltonian in chapter 3. This is possible since the JC Hamiltonian has U(1) symmetry and its action can be written in the eigenbasis of the eigenspaces of its commuting operator. Chapter 3 also includes a derivation of the JC-evolution operator as an application of the theory discussed in that chapter.

Chapter 4 introduces the concept of spectral zeta functions and discusses the spectral zeta functions of other physical models. It illustrates the interconnection between number theory and the spectral zeta functions of such models. Finally, some strategies for proving the analytic continuation of these spectral zeta functions are discussed.

Chapter 5 comprises of the main topic of this thesis- the analytic continuation of the spectral zeta function of the JC model. It discusses in detail the proof of the analytic continuation using summation formulas such as the Euler-Maclaurin summation formula. The proof involves hypergeometric functions which arise naturally from the summation formula. The concluding chapter, chapter 6 discusses the future directions the research might take.

1.1 Original contribution

This thesis consists of reviewing the literature on the spectral properties if the JC hamiltonian and then proving the analytic continuation of its spectral zeta function. The result on the analytic continuation of the JC spectral zeta function is an original result obtained during this thesis.

Chapter 2

Quantum Harmonic Oscillator

The Quantum Harmonic Oscillator (QHO) is an important model in quantum mechanics. This chapter describes in detail the Hamiltonian and spectrum of QHO to set up the necessary notation used in this thesis. This chapter consists of reviewing the literature on the QHO.

2.1 The number operator

The Quantum Harmonic Oscillator Hamiltonian is given by

$$H = \frac{P^2}{2m} + \frac{1}{2}kX^2 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 = \frac{\omega}{2}\left(\frac{P^2}{m\omega} + m\omega X^2\right), \tag{2.1}$$

where the quantities m, ω and k are positive constants. Physically m is considered as the particle's mass, and

$$\omega = \sqrt{\frac{k}{m}}$$

is considered the angular frequency of the oscillator. X and P are the position and momentum operators. The properties of these operators are discussed below.

Let $\mathscr{H}$ be a Hilbert space. The Stone-von Neumann theorem implies that if there exist two self-adjoint operators in $\mathscr{H}$ following the canonical commutation relations, these operators are unitarily equivalent to the operators X and P considered as acting on $L^2(\mathbb{R})$. The momentum and position operators X and P follow the canonical commutation relation (using the bracket notation for commutation)

$$[X,P] = iI$$

and hence can be considered as operators acting on an abstract Hilbert space $\mathscr{H}$ as well as specifically as operators acting on $L^2(\mathbb{R})$. The operators P and X can be described as acting on $L^2(\mathbb{R})$ in the following way

$$P = -i\hbar \frac{d}{dx}$$
$$X = x.$$

The theorems on momentum and position operators which are assumed (such as their self-adjoint property) are proved in [6]. Planck's constant $\hbar$ is considered as 1 in the remainder of the thesis for ease of calculation.

To find the spectrum of the QHO, an operator called the number operator can be defined as follows. Let us define

$$A := \frac{P}{\sqrt{m\omega}} \text{ and } B := \sqrt{m\omega}X.$$

The Quantum Harmonic Oscillator Hamiltonian can now be written as

$$H = \frac{1}{2}\omega(A^2 + B^2). \tag{2.2}$$

If A and B were commutative, we could easily factorize this expression as

$$H = \frac{1}{2}\omega(B + iA)(B - iA).$$

A and B do not commute, but motivated by this factorization we define

$$a = \frac{1}{\sqrt{2}}(B + iA) = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}X + i\frac{P}{\sqrt{m\omega}}\right) \tag{2.3}$$

and

$$a^\dagger = \frac{1}{\sqrt{2}}(B - iA) = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}X - i\frac{P}{\sqrt{m\omega}}\right) \tag{2.4}$$

.

Multiplying them,

$$a^\dagger a = \frac{1}{2}(B - iA)(B + iA) = \frac{1}{2}(B^2 + A^2 + i[B,A]).$$

The canonical commutation relation followed by X and P implies

$$[B,A] = [X,P] = iI.$$

Continuing the evaluation of $a^\dagger a$,

$$\begin{aligned} a^\dagger a &= \frac{1}{2}(B^2 + A^2 + i[B,A]) \\ &= \frac{1}{2}(B^2 + A^2 + i(iI)) \\ &= \frac{1}{2}(B^2 + A^2 - I). \end{aligned}$$

Hence, the operator $a^\dagger a$ can be written as

$$a^\dagger a = \frac{1}{\omega} H - \frac{1}{2} I. \tag{2.5}$$

$$\boxed{H = \omega\left(a^\dagger a + \frac{1}{2} I\right)}$$

This operator $a^\dagger a$ is called the ***number operator*** since it counts the number of particles in a system.

2.2 Relation between a and $a^\dagger$

In order to compute the spectrum of the Quantum Harmonic Oscillator Hamiltonian, it is sufficient to compute the spectrum of the number operator. To find the spectrum of the number operator, the relation between the operators a and $a^\dagger$ needs to be evaluated.

Proposition 2.2.1. *The commutator relation between a and $a^\dagger$ is given as*

$$[a, a^\dagger] = I.$$

Proof. Writing a and $a^\dagger$ in terms of A and B

$$\begin{aligned}
[a, a^\dagger] &= \frac{1}{2}[B + iA, B - iA] \\
&= \frac{1}{2}([B, B] + [B, -iA] + [iA, B] + [iA, -iA]) \\
&= \frac{1}{2}(-i[B, A] + i[A, B]).
\end{aligned}$$

Using the canonical commutation relation of A and B,

$$\begin{aligned}
&= \frac{1}{2}(-i(iI) + i(-iI)) \\
&= \frac{1}{2}(2I) = I.
\end{aligned}$$

□

Proposition 2.2.2. *The adjoint of the operator a is $a^\dagger$ (acting on $L^2(\mathbb{R})$).*

Proof. Define

$$D := \frac{1}{\sqrt{m\omega}} \quad \text{and} \quad \tilde{x} := \frac{x}{D}$$

and therefore,

$$\frac{d}{d\tilde{x}} = \frac{1}{\sqrt{m\omega}} \frac{d}{dx}.$$

Then the operators can be written in differential form in the following manner

$$\begin{aligned}
a &= \frac{1}{\sqrt{2}}(B + iA) = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}X + i\frac{P}{\sqrt{m\omega}}\right) = \frac{1}{\sqrt{2}}\left(\tilde{x} + \frac{d}{d\tilde{x}}\right) \\
a^\dagger &= \frac{1}{\sqrt{2}}(B - iA) = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}X - i\frac{P}{\sqrt{m\omega}}\right) = \frac{1}{\sqrt{2}}\left(\tilde{x} - \frac{d}{d\tilde{x}}\right).
\end{aligned}$$

Since the subspace $\mathscr{S}^2(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$ of $L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ (see [5]), it suffices to show that the adjoint of the operator a is $a^\dagger$ in $\mathscr{S}^2(\mathbb{R})$. To prove this, we need to show that given two complex functions $f(\tilde{x})$ and $g(\tilde{x}) \in \mathscr{S}^2(\mathbb{R})$, the following holds:

$$\langle f, ag \rangle = \langle a^\dagger f, g \rangle.$$

In $L^2(\mathbb{R})$ and in $\mathscr{S}^2(\mathbb{R})$, this inner product is defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \overline{f(x)} g(x)\, dx.$$

Considering the left-hand side of the equation,

$$\begin{aligned}
\langle f, ag \rangle &= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \overline{f(\tilde{x})}\; a\; g(\tilde{x})\, d\tilde{x} \\
&= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \overline{f(\tilde{x})} \left(\tilde{x} + \frac{d}{d\tilde{x}} \right) g(\tilde{x})\, d\tilde{x} \\
&= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \overline{f(\tilde{x})}\; \tilde{x}\; g(\tilde{x}) + \overline{f(\tilde{x})}\; \frac{d}{d\tilde{x}}\; g(\tilde{x})\, d\tilde{x}.
\end{aligned}$$

Using the Integration by Parts formula, this can be written as

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \overline{f(\tilde{x})}\; \tilde{x}\; g(\tilde{x}) - g(\tilde{x})\; \frac{d}{d\tilde{x}}\; \overline{f(\tilde{x})}\, d\tilde{x} + \overline{f(\tilde{x})} g(\tilde{x}) \Big|_{-\infty}^{\infty}.$$

Since $\overline{f(\tilde{x})}$ and $g(\tilde{x}) \in \mathscr{S}^2(\mathbb{R})$, the last term goes to 0 and rearranging,

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \left(\tilde{x} - \frac{d}{d\tilde{x}} \right) \overline{f(\tilde{x})}\; g(\tilde{x})\, d\tilde{x}$$

Now since $\tilde{x} \in \mathbb{R}$,

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \overline{\left(\tilde{x} - \frac{d}{d\tilde{x}} \right)}\; \overline{f(\tilde{x})}\; g(\tilde{x})\, d\tilde{x} = \langle a^\dagger f, g \rangle.$$

This proves that the adjoint of the operator a is $a^\dagger$ in $\mathscr{S}^2(\mathbb{R})$. Since $\mathscr{S}^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ the proof is complete.

□

2.3 The eigenvalues of the number operator

The number operator $a^\dagger a$ acting on $L^2(\mathbb{R})$ is self-adjoint. Since a self-adjoint operator has atleast one eigenvector, let that eigenvector be ψ and the corresponding eigenvalue be λ. The eigenvalues of this operator are also non-negative because:

$$\lambda\langle\psi,\psi\rangle = \langle\psi, a^\dagger a\psi\rangle = \langle a\psi, a\psi\rangle \geq 0. \tag{2.6}$$

To compute the eigenvalues of the number operator, we first need to derive some more commutation relations.

Proposition 2.3.1. *The following commutation relations hold between the operators a and $a^\dagger$ and the number operator $a^\dagger a$*

$$[a, a^\dagger a] = a \tag{2.7}$$

$$[a^\dagger, a^\dagger a] = -a^\dagger. \tag{2.8}$$

Proof. Using Proposition 2.2.1,

$$[a, a^\dagger a] = aa^\dagger a - a^\dagger aa = (aa^\dagger - a^\dagger a)a = [a, a^\dagger]a = Ia = a.$$

Similarly using Proposition 2.2.1 again,

$$[a^\dagger, a^\dagger a] = a^\dagger a^\dagger a - a^\dagger aa^\dagger = a^\dagger(a^\dagger a - a^\dagger a) = a^\dagger(-[a, a^\dagger]) = a^\dagger(-I) = -a^\dagger.$$

□

The operator a is called the lowering operator and the operator $a^\dagger$ is called the raising operator since they lower and raise the eigenvalue of the eigenvectors of $a^\dagger a$ by 1 respectively. The following Proposition helps us compute the eigenvalue of $a^\dagger a$ and also justifies the names raising the lowering operator.

Proposition 2.3.2. *Suppose that ψ is an eigenvector for the number operator $a^\dagger a$ with eigenvalue λ. Then,*

$$a^\dagger a(a\psi) = (\lambda - 1)a\psi. \tag{2.9}$$

Either $a\psi$ is 0 or $a\psi$ is an eigenvector of $a^\dagger a$ with eigenvalue $\lambda - 1$

$$a^\dagger a(a^\dagger \psi) = (\lambda+1)a^\dagger \psi. \tag{2.10}$$

Similarly, either $a^\dagger \psi$ *is* 0 *or* $a^\dagger \psi$ *is an eigenvector of* $a^\dagger a$ *with eigenvalue* $\lambda + 1$

Proof. Using equation (2.7), we can prove the Proposition in equation (2.8) by algebraic manipulation as follows

$$\begin{aligned} a^\dagger a(a\psi) &= (aa^\dagger a - a)\psi \\ &= (a(a^\dagger a) - a)\psi \\ &= (\lambda - 1)a\psi. \end{aligned}$$

and similarly for the raising operator $a^\dagger$ using equation (2.8)

$$\begin{aligned} a^\dagger a(a^\dagger \psi) &= (a^\dagger a^\dagger a + a^\dagger)\psi \\ &= (a^\dagger(a^\dagger a) + a^\dagger)\psi \\ &= (\lambda+1)a^\dagger \psi. \end{aligned}$$

□

Since a lowers the eigenvalue of ψ, we must get zero at some point if we apply a repeatedly to ψ. This is because if $a^n\psi$ were always non-zero, for some large value of n, the eigenvalue of $a^n\psi$ would be negative, which is not possible as observed in equation 2.6.

Therefore, there exists some ψ_0 such that $a\psi_0$ equals 0. This implies that $a^\dagger a\psi_0 = 0$ and ψ_0 is an eigenvector of $a^\dagger a$ with eigenvalue 0. Now let,

$$a^n\psi = a\psi_0 = 0.$$

Since lowering the eigenvalue of ψ repeatedly n times gives the eigenvector with eigenvalue 0, the eigenvalue of the original eigenvector ψ is n.

This implies that given an eigenvector ψ of $a^\dagger a$, we can find an eigenvector ψ_0 of $a^\dagger a$ with eigenvalue 0. The other eigenvectors of $a^\dagger a$ can be derived from ψ_0 by repeatedly applying the $a^\dagger$ (raising) operator. ψ_0 can be considered as the ”ground state” of $a^\dagger a$.

2.3.1 Some identities involving a and $a^\dagger$

The action of the operators a and $a^\dagger$ on ψ_i are also important to the evaluation of the spectrum of the JC-Hamiltonian since they also occur in the JC-Hamiltonian separately from the number operator. Let ψ_0 be the ground state of the number operator as denoted earlier. Let us define

Definition 2.3.1. *The eigenvectors ψ_n of the number operator $a^\dagger a$ can be defined using a ground state of the number operator ψ_0 and the raising operator $a^\dagger$ in the following way*

$$\psi_n := (a^\dagger)^n \psi_0.$$

Proposition 2.3.3. *The vectors ψ_n have the following properties:*

1. $a^\dagger \psi_n = \psi_{n+1}$
2. $a^\dagger a \psi_n = n\psi_n$
3. $\langle \psi_n, \psi_m \rangle = n!\delta_{n,m}$
4. $a\psi_{n+1} = (n+1)\psi_n$.

Here $\delta_{n,m}$ is defined as

$$\begin{aligned} \delta_{n,m} &= 1 \quad \text{if } n = m \\ &= 0 \quad \text{if } n \neq m. \end{aligned}$$

Proof. (Proof of 1)

Using definition 2.3.1 it can be shown that

$$a^\dagger \psi_n = (a^\dagger)^{n+1} \psi_0 = \psi_{n+1}.$$

(Proof of 2)
We will show this by induction. We know that

$$a^\dagger a \psi_0 = 0.$$

Now let the induction hypothesis be

$$a^\dagger a\psi_k = k\psi_k.$$

The induction step can be written as

$$\begin{aligned}
a^\dagger a\psi_{k+1} &= a^\dagger aa^\dagger \psi_k \\
&= (a^\dagger a^\dagger a + a^\dagger)\psi_k \qquad \text{(Using equation 2.8)} \\
&= ka^\dagger \psi_k + a^\dagger \psi_k \qquad \text{(Using the induction hypothesis)} \\
&= (k+1)a^\dagger \psi_k \qquad \text{(Using equation 1)} \\
&= (k+1)\psi_{k+1}.
\end{aligned}$$

This completes the induction argument.

(Proof of 3)
If $n = m$, we prove this by induction. For n=0, $\langle \psi_0, \psi_0 \rangle = 1$ is assumed.
Next we assume

$$\langle \psi_n, \psi_n \rangle = n!$$

and compute $\langle \psi_{n+1}, \psi_{n+1} \rangle$

$$\begin{aligned}
\langle \psi_{n+1}, \psi_{n+1} \rangle &= \langle a^\dagger \psi_n, a^\dagger \psi_n \rangle \\
&= \langle \psi_n, aa^\dagger \psi_n \rangle
\end{aligned}$$

Using Proposition 2.2.1,

$$\begin{aligned}
&= \langle \psi_n, (a^\dagger a + 1)\psi_n \rangle \\
&= (n+1)\langle \psi_n, \psi_n \rangle \\
&= (n+1)!.
\end{aligned}$$

This completes the induction step.

For the case of $n \neq m$, without loss of generality, let $n > m$

$$\begin{aligned}
\langle \psi_n, \psi_m \rangle &= \langle a^{\dagger m+1} \psi_{n-m-1}, \psi_m \rangle \\
&= \langle \psi_{n-m-1}, a^{m+1} \psi_m \rangle = 0.
\end{aligned}$$

Since, $n > m$, $n - m - 1 \geq 0$.

(Proof of 4) Using Proposition 2.2.1,

$$a\psi_{n+1} = aa^\dagger \psi_n$$
$$= (a^\dagger a + 1)\psi_n$$

and using equation 1,

$$= (n+1)\psi_n.$$

□

Equation 2 demonstrates that the number operator has the non-negative integers as eigenvalues. It can also be shown that $\psi_n' s$, the eigenvectors of $a^\dagger a$ form an orthogonal basis for $L^2(\mathbb{R})$. This ensures that all the domain conditions for all the operators are met.

These identities derived in this chapter can be used to compute the spectrum of the Jaynes-Cummings Hamiltonian. For a more detailed discussion on the operators mentioned here, refer to [6]. The following section will deal with deriving the spectrum and eigenvectors of the Jaynes-Cummings Hamiltonian.

Chapter 3

The Jaynes-Cummings model

The Jaynes-Cummings model is a quantum optics model describing the interaction between a two-level atom and a quantized electromagnetic field. In this chapter, the Jaynes-Cummings Hamiltonian and its spectral properties are reviewed.

3.1 The Jaynes-Cummings Hamiltonian

The Jaynes-Cummings Hamiltonian [10] is given by

$$H_{JC} = \omega a^\dagger a + \frac{g}{2}(\sigma_+ a + a^\dagger \sigma_-) + \frac{\omega_0}{2}\sigma_z,$$

where

$$\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$$

and σ_x, σ_y and σ_z are the Pauli matrices given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Here ω, g and ω_0 are constants related to the physical system. The constants ω is the angular frequency of the electromagnetic field, ω_0 is the atom frequency and g can be interpreted as the coupling strength between the atom and the field. In the following section, we discuss the relation between the Quantum Rabi model Hamiltonian and the Jaynes-Cummings Hamiltonian.

3.1.1 QRM and derivation of H_{JC} from QRM Hamiltonian

The Quantum Rabi Model (QRM) is a fundamental model of light-matter interaction. This model was considered in its fully quantized version in 1963 by Jaynes and Cummings. The Jaynes-Cummings model Hamiltonian can be derived from the QRM Hamiltonian by ignoring some terms from the H_{int} part of the QRM Hamiltonian. This approximation is called as the rotating wave approximation (RWA). This approximation experimentally holds in certain parameter regimes, specifically when the coupling strength is weak ($\frac{g}{\omega}$ small).[4]

$$\begin{aligned} H_{Rabi} &= \omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + \frac{g}{2}\sigma_x(a^\dagger + a) \\ &= \omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + \frac{g}{2}(\sigma_+ + \sigma_-)(a^\dagger + a), \end{aligned}$$

This can be expanded as

$$= \omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + \frac{g}{2}(\sigma_+ a^\dagger + \sigma_+ a + \sigma_- a^\dagger + \sigma_- a).$$

The RWA amounts to ignoring the terms $\sigma_+ a^\dagger$ and $\sigma_- a$ which gives us the final form of the Jaynes-Cummings Hamiltonian as follows

$$H_{JC} = \omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + \frac{g}{2}(\sigma_+ a + a^\dagger \sigma_-). \tag{3.1}$$

In the following section, we establish one of the commutators of the Jaynes-Cummings Hamiltonian. This, in turn, will be used to derive the spectrum of H_{JC}.

3.2 Commuting operator to H_{JC}

The Jaynes-Cummings Hamiltonian can be separated into two commuting parts as follows

$$H_{JC} = H_I + H_{II},$$

where,

$$\begin{aligned} H_I &= \omega(a^\dagger a + \frac{\sigma_z}{2}) \\ H_{II} &= (\omega_0 - \omega)\frac{\sigma_z}{2} + \frac{g}{2}(a\sigma_+ + a^\dagger \sigma_-). \end{aligned}$$

Let $\Delta := \omega_0 - \omega$ for ease of computation in the following sections.

Also, note the fact

$$\text{the matrices } \sigma_z, \sigma_+ \text{ and } \sigma_- \text{ commute with the operators } a, a^\dagger. \qquad (3.2)$$

To show, H_I and H_{II} commute:

$$H_1 H_{11} = \left[\omega\left(a^\dagger a + \frac{\sigma_z}{2}\right)\right]\left[\Delta\frac{\sigma_z}{2} + \frac{g}{2}(a\sigma_+ + a^\dagger\sigma_-)\right]$$
$$= \omega a^\dagger a \Delta\frac{\sigma_z}{2} + \omega a^\dagger a\frac{g}{2}a\sigma_+ + \omega a^\dagger a\frac{g}{2}a^\dagger\sigma_- + \omega\Delta\frac{{\sigma_z}^2}{4} + \omega\frac{\sigma_z g}{4}a\sigma_+ + \omega\frac{\sigma_z g}{4}a^\dagger\sigma_-.$$

Using (3.2) and rearranging the terms,

$$= \omega\Delta a^\dagger a\frac{\sigma_z}{2} + \omega\frac{g}{2}\sigma_+ a^\dagger aa + \omega\frac{g}{2}\sigma_- a^\dagger aa^\dagger + \frac{\omega\Delta}{4}{\sigma_z}^2 + \frac{\omega g}{4}\sigma_z a\sigma_+ + \frac{\omega g}{4}\sigma_z\sigma_- a^\dagger.$$

$$H_{11}H_1 = \left[\Delta\frac{\sigma_z}{2} + \frac{g}{2}(a\sigma_+ + a^\dagger\sigma_-)\right]\left[\omega(a^\dagger a + \frac{\sigma_z}{2})\right]$$

$$= \Delta\frac{\sigma_z}{2}\omega a^\dagger a + \Delta\omega\frac{{\sigma_z}^2}{4} + \frac{g}{2}a\sigma_+\omega a^\dagger a + \frac{g}{2}a\sigma_+\omega\frac{\sigma_z}{2} + \frac{g}{2}a^\dagger\sigma_-\omega a^\dagger a + \frac{g}{2}a^\dagger\sigma_-\omega\frac{\sigma_z}{2}.$$

Using (3.2) and rearranging the terms,

$$H_{11}H_1 = \omega\Delta a^\dagger a\frac{\sigma_z}{2} + \omega\frac{g}{2}\sigma_+ aa^\dagger a + \omega\frac{g}{2}\sigma_- a^\dagger a^\dagger a + \frac{\omega\Delta}{4}{\sigma_z}^2 + \frac{\omega g}{4}a\sigma_+\sigma_z + \frac{\omega g}{4}a^\dagger\sigma_-\sigma_z.$$

Therefore the commutator is

$$[H_1, H_{11}] = -\omega\frac{g}{2}\sigma_+[a, a^\dagger]a + \omega\frac{g}{2}\sigma_- a^\dagger[a, a^\dagger] + \frac{\omega g}{4}a[\sigma_z, \sigma_+] + \frac{\omega g}{4}a^\dagger[\sigma_z, \sigma_-].$$

Since from Proposition 2.2.1, $[a, a^\dagger] = 1$, and also

$$[\sigma_z, \sigma_+] = 2\sigma_+ \text{ and } [\sigma_z, \sigma_-] = -2\sigma_-,$$

$$
\begin{aligned}
[H_1, H_{11}] &= -\omega\frac{g}{2}\sigma_+ a + \omega\frac{g}{2}\sigma_- a^\dagger + \frac{\omega g}{4}a(2\sigma_+) + \frac{\omega g}{4}a^\dagger(-2\sigma_-) \\
&= -\omega\frac{g}{2}\sigma_+ a - \omega\frac{g}{2}\sigma_+ a + \omega\frac{g}{2}\sigma_- a^\dagger - \omega\frac{g}{2}\sigma_- a^\dagger \\
&= 0.
\end{aligned}
$$

Therefore, H_1 can be shown to be a commuting operator to H_{JC} as follows

$$
[H_1, H_{JC}] = [H_1, H_1 + H_{11}] = [H_1, H_1] + [H_1, H_{11}] = 0. \tag{3.3}
$$

Therefore,

$$
\boxed{[H_1, H_{JC}] = 0}
$$

3.3 The spectrum of the JC-Hamiltonian

The commuting operator to H_{JC}, H_1 can be used to find the spectrum of H_{JC} by applying the following well known result

Proposition 3.3.1. *If two operators A and B commute, every eigenspace of A is B invariant. [17]*

To apply this result, first, we compute the eigenvectors and eigenvalues of H_1. We have to find ψ such that

$$
H_1\psi = \lambda\psi,
$$

that is,

$$
\omega\left(a^\dagger a + \frac{\sigma_z}{2}\right)\psi = \lambda\psi.
$$

Considering ψ_i, the eigenvectors of the number operator, this implies

$$\lambda \begin{bmatrix} \psi_n \\ \\ \psi_m \end{bmatrix} = \omega \begin{bmatrix} a^\dagger a + \frac{1}{2} & 0 \\ \\ 0 & a^\dagger a - \frac{1}{2} \end{bmatrix} \begin{bmatrix} \psi_n \\ \\ \psi_m \end{bmatrix} \tag{3.4}$$

$$= \omega \begin{bmatrix} (n+\frac{1}{2})\psi_n \\ \\ (m-\frac{1}{2})\psi_m \end{bmatrix}. \tag{3.5}$$

From (3.5), to factor out the eigenvalue λ, the following must be true

$$\omega\left(n+\frac{1}{2}\right) = \omega\left(m-\frac{1}{2}\right),$$
$$n+1 = m.$$

Therefore, the eigenvectors of H_1 are of the form

$$\begin{bmatrix} \psi_n \\ \psi_{n+1} \end{bmatrix}, \begin{bmatrix} 0 \\ \psi_{n+1} \end{bmatrix} \text{ and } \begin{bmatrix} \psi_n \\ 0 \end{bmatrix}$$

for the eigenvalue $\omega\left(n+\frac{1}{2}\right)$ for $n = 0, 1, 2...$

Additionally, the vector $\begin{bmatrix} 0 \\ \psi_0 \end{bmatrix}$ is an eigenvector of H_1 with the eigenvalue $\frac{-\omega}{2}$.

Therefore, H_1 has an eigenbasis given by

$$L^2(\mathbb{R}) \otimes \mathbb{C}^2 = V_0 \bigoplus_{n\in\mathbb{N}} V_n,$$

where

$$V_0 = \begin{bmatrix} 0 \\ \psi_0 \end{bmatrix} \text{ and } V_n = span\left\{ \begin{bmatrix} \psi_n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \psi_{n+1} \end{bmatrix} \right\}.$$

By Proposition 3.3.1, since H_{JC} and H_1 commute, H_{JC} is invariant on each of the V_i's. Therefore we can write H_{JC} in the basis of each of the subspaces V_i as follows

$$
\begin{aligned}
H_{JC}\begin{bmatrix}\psi_n\\0\end{bmatrix} &= \left(n\omega+\frac{\omega_0}{2}\right)\begin{bmatrix}\psi_n\\0\end{bmatrix}+\frac{g}{2}\begin{bmatrix}0\\\psi_{n+1}\end{bmatrix}\\
H_{JC}\begin{bmatrix}0\\\psi_{n+1}\end{bmatrix} &= (n+1)\begin{bmatrix}\psi_n\\0\end{bmatrix}+\left((n+1)\omega-\frac{\omega_0}{2}\right)\begin{bmatrix}0\\\psi_{n+1}\end{bmatrix}.
\end{aligned}
$$

Therefore, denoting the action of H_{JC} on V_n as $H^{(n)}{}_{JC}$,

$$
H_{JC}^{(n)}=\begin{bmatrix}\left(n\omega+\frac{\omega_0}{2}\right) & \frac{g(n+1)}{2}\\ \frac{g}{2} & \left((n+1)\omega-\frac{\omega_0}{2}\right)\end{bmatrix}.
$$

To find the eigenvalues of the Jaynes-Cummings Hamiltonian, we now need to solve

$$
Det(H_{JC}^{(n)}-\lambda I)=0.
$$

Expanding this,

$$
\begin{vmatrix}\left(n\omega+\frac{\omega_0}{2}\right)-\lambda & \frac{g(n+1)}{2}\\ \frac{g}{2} & \left((n+1)\omega-\frac{\omega_0}{2}\right)-\lambda\end{vmatrix}=0.
$$

Now let $\Delta=\omega_0-\omega$,

$$
\begin{vmatrix}\left(n+\frac{1}{2}\right)\omega+\frac{\Delta}{2}-\lambda & \frac{g(n+1)}{2}\\ \frac{g}{2} & \left(n+\frac{1}{2}\right)\omega-\frac{\Delta}{2}-\lambda\end{vmatrix}=0.
$$

Solving this we get,

$$\lambda = \left(n+\frac{1}{2}\right)\omega \pm \sqrt{\frac{\Delta^2}{4}+\frac{g^2(n+1)}{4}}$$
$$= \left(n+\frac{1}{2}\right)\omega \pm \sqrt{\frac{(\omega_0-\omega)^2}{4}+\frac{g^2(n+1)}{4}}.$$

The following notation for the eigenvales of H_{JC} will be followed in the remainder of this thesis.

$$E_n{}^+ = \left(n+\frac{1}{2}\right)\omega + \sqrt{\frac{(\omega_0-\omega)^2}{4}+\frac{g^2(n+1)}{4}}$$

and

$$E_n{}^- = \left(n+\frac{1}{2}\right)\omega - \sqrt{\frac{(\omega_0-\omega)^2}{4}+\frac{g^2(n+1)}{4}}.$$

The Hamiltonian of the JC model, H_{JC}, is a self-adjoint operator with a discrete spectrum of eigenvalues bounded from below. This result follows from the general theory of differential equations. In particular, a theorem in [15] implies that since H_{JC} is elliptic and has real symbol and positive principal symbol, the properties of its spectrum can be concluded.

3.4 The eigenvectors of the JC-Hamiltonian

In this section, we compute the eigenvectors of H_{JC} for completeness. To determine the eigenvectors of H_{JC}, we have to solve the following eigenvalue equation

$$H_{JC}^{(n)}\left(\alpha\begin{bmatrix}\psi_n\\0\end{bmatrix}+\beta\begin{bmatrix}0\\\psi_{n+1}\end{bmatrix}\right) = E_n{}^+\left(\alpha\begin{bmatrix}\psi_n\\0\end{bmatrix}+\beta\begin{bmatrix}0\\\psi_{n+1}\end{bmatrix}\right).$$

Replacing the matrix $H_{JC}^{(n)}$, it is of the following form

$$\begin{bmatrix} \left(n\omega + \frac{\omega_0}{2}\right) & \frac{g(n+1)}{2} \\ \frac{g}{2} & \left((n+1)\omega - \frac{\omega_0}{2}\right) \end{bmatrix} \begin{bmatrix} \alpha\psi_n \\ \beta\psi_{n+1} \end{bmatrix} = {E_n}^+ \begin{bmatrix} \alpha\psi_n \\ \beta\psi_{n+1} \end{bmatrix}.$$

The two equations obtained from this are

$$\alpha\omega n\psi_n + \frac{g}{2}\beta(n+1)\psi_n + \frac{\omega_0}{2}\alpha\psi_n = {E_n}^+\alpha\psi_n$$
$$\beta\omega(n+1)\psi_{n+1} + \frac{g}{2}\alpha\psi_{n+1} - \frac{\omega_0}{2}\beta\psi_{n+1} = {E_n}^+\beta\psi_{n+1}.$$

Equating the coefficients on both sides,

$$\alpha\omega n + \frac{g}{2}\beta(n+1) + \frac{\omega_0}{2}\alpha = {E_n}^+\alpha \tag{3.6}$$
$$\beta\omega(n+1) + \frac{g}{2}\alpha - \frac{\omega_0}{2}\beta = {E_n}^+\beta. \tag{3.7}$$

Rearranging equation (3.7),

$$\frac{g}{2}\alpha = \left({E_n}^+ - \omega(n+1) + \frac{\omega_0}{2}\right)\beta.$$

Therefore, if

$$\beta = \frac{g}{2} \text{ then, } \alpha = {E_n}^+ - \omega(n+1) + \frac{\omega_0}{2}.$$

Similarly for ${E_n}^-$,

$$\beta = \frac{g}{2} \text{ and } \alpha = {E_n}^- - \omega(n+1) + \frac{\omega_0}{2}.$$

Note 1. *Additionally, the vector* $\begin{bmatrix} 0 \\ \psi_0 \end{bmatrix}$ *is the ground state of the Jaynes-Cummings Hamiltonian with an eigenvalue of* $\frac{-\omega_0}{2}$.

This chapter concludes the discussion of the spectrum of the Jaynes-Cummings Hamiltonian. In the following chapters, the spectral zeta function associated with the JC-Hamiltonian will be discussed.

Chapter 4

Spectral zeta functions

This chapter deals with the spectral zeta function of operators, specifically the spectral zeta functions of hamiltonians of models closely related to the JC model. The chapter ends with a definition of the JC spectral zeta function and the proof of its convergence in $Re(s) > 1$. The majority of this chapter is a review with the exception of section 4.3, which is a computation done during the course of this thesis.

4.1 Definition of spectral zeta function

Let O be a self-adjoint operator on a complex Hilbert space with discrete spectrum given by

$$0 < \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_k \leq ... \to \infty$$

(eigenvalues are being counted with multiplicity). Then, the spectral zeta function of O is given by

$$\zeta_O(s) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k{}^s}, \tag{4.1}$$

for $Re(s) > 0$ sufficiently large.

4.1.1 The spectral zeta function of the Quantum Harmonic Oscillator

Using point 2 from Proposition 2.3.3 and the definition of the Quantum Harmonic Oscillator (QHO) Hamiltonian given in (2.5), the spectrum of the Quantum Harmonic Oscillator Hamiltonian H acting on $L^2(\mathbb{R})$ is given by

$$Spec(H) = \{\left(n + \frac{1}{2}\right)\omega : n = 0, 1, 2, 3...\}$$

Now, for $Re(s) > 1$, the spectral zeta function of the QHO hamiltonian is given by

$$\zeta_H(s) = \frac{1}{\omega^s}\left(\frac{1}{\left(\frac{1}{2}\right)^s} + \frac{1}{\left(\frac{3}{2}\right)^s} + \frac{1}{\left(\frac{5}{2}\right)^s} + ...\right) = \frac{2^s - 1}{\omega^s}\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta function.

This illustrates that there is an intersection between the study of spectral zeta functions of physical systems and number theory. Similar to this, the spectral zeta functions of two other physical systems related to the Jaynes-Cummings model which also have a rich arithmetic structure are described in the following section.

4.2 Brief discussion of the spectral zeta functions of NcHO and QRM

The non-commutative harmonic oscillator (NcHO) was defined by M. Wakayama and A. Permeggiani [14] as a non-commutative extension to the QHO and its spectral zeta function has been shown to have rich arithmetic properties. The non-commutative harmonic oscillator (NcHO) is defined by the Hamiltonian acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ given by

$$Q := A\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) + J\left(x\frac{d}{dx} + \frac{1}{2}\right), \tag{4.2}$$

for two parameter $\alpha, \beta \in \mathbb{R}$ and where

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The analytic continuation of the spectral zeta function of the NcHO to the whole complex plane has been shown [7] and some of its special values has been explored. The special value of the NcHO spectral zeta function at $s = 2$ has been shown to have a strong connection with a solution of a singly confluent Heun's ordinary differential equation [8]. Additionally, sequences of Apery-like numbers were found in the study of these spectral zeta functions and they show interesting congruence relations [11].

The spectral zeta function of the Quantum Rabi model (QRM) described earlier in this thesis has also been studied. The meromorphic continuation of the spectral zeta function of the QRM has been shown by two different methods ([19], [18]). Interestingly, there is a relation between the NcHO and the QRM Hamiltonian through a confluence process applied to the Heun differential equation [21].

Since in the case of the QRM and the NcHO the eigenvalues cannot be explicitly computed, the analytic continuation of the spectral zeta function was shown using the following methods:

1. At first the trace of the heat kernel of the operator is considered. Then its Mellin transform is taken to obtain the integral form of the spectral zeta function which in turn helps in proving the analytic continuation of the spectral zeta function.

2. The other method to show the analytic continuation of the spectral zeta function is through the asymptotic expression of the action of the heat operator e^{-tH} (where H is the operator) on a smooth, compact, real-valued function.

The analytic continuation of the spectral zeta function of the QRM was proved through both these methods and that for the NcHO was proved through the second method. In the case of the Jaynes-Cummings Hamiltonian, since the eigenvalues are explicitly known, another more elementary way of obtaining the analytic continuation of the JC spectral zeta function is possible through the application of summation formulas.

4.3 Definition of $\zeta_{JC}(s,a)$

The spectral zeta function of the Jaynes-Cummings model is defined as:

$$\zeta_{JC}(s,a) = \sum_{n=0}^{\infty} \frac{1}{(E_n{}^{+} + a)^s} + \sum_{n=0}^{\infty} \frac{1}{(E_n{}^{-} + a)^s} + \frac{1}{(\frac{-\omega_0}{2} + a)^s} \tag{4.3}$$

where, ${E_n}^{\pm} = (n+\frac{1}{2})\omega \pm \frac{1}{2}\sqrt{g^2(n+1)+(\omega_0-\omega)^2}, a \neq -{E_n}^{\pm}$, $a, \omega > 0$ and the following condition on a must also hold.

$$a > \frac{g^2}{16\omega^2} + \frac{\Delta^2\omega}{g^2} + \frac{\omega}{2}. \tag{4.4}$$

This condition is introduced such that $\lambda + a > 0$ for all λ, where λ is an eigenvalue of the JC Hamiltonian. Introducing the constant a defines a Hurwitz-type spectral zeta function. This condition arises from the fact that the minimum possible eigenvalue of the JC Hamiltonian is given by

$$-\frac{g^2}{16\omega^2} - \frac{\Delta^2\omega}{g^2} - \frac{\omega}{2}.$$

This can be calculated by finding the minima of the function

$$f(t) = \left(t + \frac{1}{2}\right)\omega - \frac{1}{2}\sqrt{g^2(t+1)+(\omega_0-\omega)^2}.$$

This condition is necessary because say there is an eigenvalue λ which is negative, then,

$$\lambda^{-s} = e^{-slog\lambda}.$$

If λ is negative, $log\lambda$ is not defined for $\lambda \in \mathbb{R}$. To avoid this problem in the definition of the spectral zeta function, the spectral zeta function is defined using the term $(\lambda + a)^{-s}$ where a is chosen such that $\lambda + a > 0$ for all λ.

4.3.1 Convergence of $\zeta_{JC}(s,a)$ in the region $Re(s) > 1$

The domain of absolute convergence of $\zeta_{JC}(s,a)$ is discussed in this section. The proof is completed in two parts, by showing each of the two subseries involving the the ${E_n}^{+}$ and the ${E_n}^{-}$ eigenvalues converges absolutely in the region $Re(s) > 1$.

Proposition 4.3.1. *The domain of absolute convergence of the subseries*

$$\sum_{n=0}^{\infty} \frac{1}{({E_n}^{+}+a)^s} \textit{ and } \sum_{n=0}^{\infty} \frac{1}{({E_n}^{-}+a)^s}$$

is given by

$$Re(s) > 1,$$

provided $\omega > 0$.

Proof. For the subseries

$$\sum_{n=0}^{\infty} \frac{1}{(E_n{}^+ + a)^s},$$

let

$$s = \sigma + i\tau,$$

where

$$\sigma = Re(s) \text{ and } \tau = Im(s).$$

We use the comparison test for convergence. Let

$$a_n = \left| \frac{1}{(E_n{}^+ + a)^s} \right| = \frac{1}{|E_n{}^+ + a|^\sigma}$$

and

$$b_n = \frac{1}{|n\omega|^\sigma}$$

Now, since $a > 0$ and $\omega > 0$, and replacing the value of $E_n{}^+$,

$$\frac{1}{\left| \left(n + \frac{1}{2}\right)\omega + \frac{1}{2}\sqrt{g^2(n+1) + (\omega_0 - \omega)^2} + a \right|^\sigma} \leq \frac{1}{|n\omega|^\sigma}.$$

This implies

$$a_n \leq b_n.$$

By the comparison test for convergence of series, this implies if $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ will converge. The series $\sum_{n=0}^{\infty} b_n$ converges by the p-series test if $Re(s) > 1$. This implies that the series $\sum_{n=0}^{\infty} \frac{1}{(E_n{}^+ + a)^s}$ converges absolutely for $Re(s) > 1$.

For the subseries $\sum_{n=0}^{\infty} \frac{1}{(E_n{}^- + a)^s}$, the limit comparison test is used. Suppose there exists series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ such that $a_n \geq 0$ and $b_n > 0$ for all n. Then the limit comparison test states that if

$$\lim_{n \to \infty} \frac{a_n}{b_n}$$

exists and is positive then, the two series converge and diverge together.[20]

For this proof, let

$$a_n = \left| \frac{1}{(E_n{}^- + a)^s} \right| = \frac{1}{|E_n{}^- + a|^\sigma}$$

and

$$b_n = \frac{1}{|n|^\sigma}.$$

Then we have,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{\omega^\sigma}.$$

Since $\omega > 0$ this limit exists and is positive.

□

Therefore, the absolute convergence of $\zeta_{JC}(s,a)$ in the region $Re(s) > 1$ can be concluded. The following section deals with analytically continuing the spectral zeta function $\zeta_{JC}(s,a)$ defined above.

Chapter 5

Analytic continuation of $\zeta_{JC}(s,a)$

This chapter focuses on the proof of the analytic continuation of $\zeta_{JC}(s,a)$. The chapter consists of four sections the first of which deals with the proof of the analytic continuation of the Riemann zeta function using elementary summation formulas to illustrate the method used for the analytic continuation of $\zeta_{JC}(s,a)$.In section 5.2, the proof of the analytic continuation of $\zeta_{JC}(s,a)$ to $Re(s) > 0$ begins, following the first section of the proof involving $\zeta(s)$. The hypergeometric functions obtained in section 5.2 are elaborated upon in section 5.3 and the proof end here. The final section, section 5.4 contains the proof of the analytic continuation of $\zeta_{JC}(s,a)$ to the whole of $\mathbb{C}$.

The first section of this chapter is a review of the analytic continuation of the Riemann zeta function. The remaining three sections are original work carried out during this project.

5.1 Analytic continuation of $\zeta(s)$ using the Euler-Mauclaurin summation formula

The Riemann zeta function $\zeta(s)$ is defined for $Re(s) > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{5.1}$$

Lemma 5.1.1. *The series in equation (5.1) converges absolutely and uniformly in the half-plane $Re(s) \geq 1+\varepsilon$ for all $\varepsilon > 0$.[9]*

Proof. Since for $Re(s) \geq 1+\varepsilon$,

$$|n^{-s}| \leq n^{-1-\varepsilon}$$

Since the term on the right-hand side does not depend on s and the series $\sum_{n=1}^{\infty} n^{-1-\varepsilon}$ converges, this series is uniformly convergent on $Re(s) \geq 1+\varepsilon$ for all $\varepsilon > 0$. This implies that it is also absolutely convergent in the same region. □

First, the analytic continuation of $\zeta(s)$ to $Re(s) > 0$ will be demonstrated since it utilizes a simpler special case of the general summation formula used in the proof. The plan for the proof of the analytic continuation of $\zeta(s)$ to $Re(s) > 0$ is as follows:

1. Since $\zeta(s)$ converges in $Re(s) > 1$, we show that $\zeta(s)$ is equal to a function in the region $Re(s) > 1$ and that function is analytic in the region $Re(s) > 0$.
2. Conclude that $\zeta(s)$ is analytic in the region $Re(s) > 0$.

5.1.1 Euler summation formula

The Euler summation formula [2] is applied to the series $\sum_{n=1}^{\infty} n^{-s}$ to show the first part of the proof of the analytic continuation of $\zeta(s)$.

Proposition 5.1.1. *If f has a continuous derivative f' on the interval $[y,x]$, where $0 < y < x$, then*

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t])f'(t)\,dt + f(x)(x-[x]) + f(y)(y-[y]). \quad (5.2)$$

Applying this formula to the series $\sum_{n=1}^{\infty} n^{-s}$, for $x > 1$,

$$1 + \sum_{1<n\leq x} n^{-s} = 1 + \int_1^x t^{-s}\,dt + s\int_1^x ([t]-t)t^{-s-1}\,dt + (x-[x])x^{-s}.$$

This form of the series can be used to prove the following Proposition.

Proposition 5.1.2. *$\zeta(s)$ can be analytically continued to $Re(s) > 0$.*

Proof. This proof closely follows the proof given in [2]. Using the summation formula,

$$= 1 + \int_1^x t^{-s}\,dt + s\int_1^x ([t]-t)t^{-s-1}\,dt + (x-[x])x^{-s}$$
$$= \frac{s}{s-1} - \frac{x^{1-s}}{s-1} + s\int_1^x ([t]-t)t^{-s-1}\,dt + O(x^{1-\sigma}),$$

where $\sigma = Re(s)$. If $\sigma > 1$ and $x \to \infty$, this can be written as

$$\zeta(s) = \frac{s}{s-1} + s\int_1^\infty ([t]-t)t^{-s-1}\,dt.$$

Now let A be a compact set in $Re(s) > 0$. Since $|[t]-t| \le 1$, for $\sigma > 0$,

$$\left| s\int_1^\infty ([t]-t)t^{-s-1}\,dt \right| \le 1$$

This implies that the integral is uniformly convergent in the region $Re(s) > 0$. Therefore, this expression provides the analytic continuation of $\zeta(s)$ to $Re(s) > 0$ with its only pole for $\sigma > 0$ being at s=1. □

5.1.2 The Euler-Maclaurin summation formula

The Euler-Maclaurin summation formula, a generalization of the Euler summation formula, is used to prove the analytic continuation of $\zeta(s)$ to the rest of the complex plane. The Euler-Maclaurin summation formula [2] is given by the following

$$\sum_{a\le k\le b} f(k) = \int_a^b f(t)\,dt + \frac{1}{2}(f(a)) + f(b)) + \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!}(f^{(2m-1)}(b) - f^{(2m-1)}(a)) + \int_a^b P_{2n+1}(t) f^{(2n+1)}(t)\,dt. \tag{5.3}$$

Here $n \ge 0$ is a fixed integer, $f(x) \in \mathbb{C}^{2n+1}[a,b]$, B_m is the m^{th} Bernoulli number, and P_m is the m^{th} periodic Bernoulli function defined by $P_m(x) = B_m(x-[x])$, where $B_m(x)$ is the m^{th} Bernoulli polynomial defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m(x)z^m}{m!}. \qquad (|z| < 2\pi)$$

Proposition 5.1.3. *$\zeta(s)$ is analytic in the whole complex plane except for a pole at $s = 1$.*

Proof. Applying the Euler-Maclaurin formula to the series $\sum_{n=1}^{\infty} n^{-s}$,

$$\zeta(s) = \frac{1}{1-s} + \frac{1}{2} + \sum_{m=1}^{n} \frac{B_{2m}\Gamma(s+2m-1)}{(2m)!\Gamma(s)} + \frac{\Gamma(s+2n+1)}{\Gamma(s)} \int_{1}^{\infty} P_{2n+1}(t) t^{-s-2n-1}\, dt. \tag{5.4}$$

Since $P_{2n+1}(t) = O(1)$, to bound the last integral we need,

$$\sigma + 2n > 0.$$

Since n is arbitrary, the Proposition is proved. □

5.2 Euler summation formula applied to $\zeta_{JC}(s,a)$

In this section, the analytic continuation of $\zeta_{JC}(s,a)$ to $Re(s) > 0$ will be discussed, following the same method as for $\zeta(s)$ in the previous section. The Euler summation formula is a special case of the Euler-Maclaurin summation formula which is later used to prove the analytic continuation of $\zeta_{JC}(s,a)$ to the entire complex plane. We proceed by first applying the Euler summation formula and proving the analytic continuation of $\zeta_{JC}(s,a)$ to $Re(s) > 0$ in order to introduce the various components such as the hypergeometric functions that appear in the final proof.

To recall, the spectral zeta function of the JC model is given by

$$\zeta_{JC}(s,a) = \lim_{x\to\infty} \left[\sum_{n=0}^{x} \frac{1}{(E_n{}^{+} + a)^s} + \sum_{n=0}^{x} \frac{1}{(E_n{}^{-} + a)^s} + \frac{1}{(\frac{-\omega_0}{2})^s} \right]$$

where, $E_n{}^{\pm} = (n+\frac{1}{2})\omega \pm \frac{1}{2}\sqrt{g^2(n+1) + (\omega_0 - \omega)^2}, a \neq -E_n{}^{\pm}$ and $a, \omega > 0$. the following condition on a must also hold as discussed in the previous chapter.

$$a > \frac{g^2}{16\omega^2} + \frac{\Delta^2\omega}{g^2} + \frac{\omega}{2}. \tag{5.5}$$

The three summation terms within the $\zeta_{JC}(s,a)$ will be evaluated separately using the Euler summation formula as follows.

5.2.1 The ${E_n}^+$ eigenvalue part

The expression involving the ${E_n}^+$ eigenvalues can be written using the Euler summation formula (equation 5.2) as

$$\lim_{x\to\infty}\sum_{n=1}^{x}\frac{1}{({E_n}^+ + a)^s} = \lim_{x\to\infty}\Bigg[\int_1^x\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s}dt$$
$$+s\int_1^x (t-[t])\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{\frac{-1}{2}}g^2)\,dt$$
$$+(x-[x])\left[(x+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(x+1)+\Delta^2}+a\right]^{-s}\Bigg]. \tag{5.6}$$

The first term of the subseries involving the ${E_n}^+$ eigenvalues, $\frac{1}{({E_0}^+ + a)^s}$ is not included in the analysis involving the Euler summation formula. Instead, we can argue that since it is a power function of s it is analytic in any open set in its domain which in this case is $\mathbb{C}$.

Out of the three summands in equation (5.6), the third summand can be easily estimated by considering the order of the power of x in the term. Therefore it can be written as

$$(x-[x])\left[(x+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(x+1)+\Delta^2}+a\right]^{-s}\Bigg] = O(x^{1-s}).$$

For $Re(s) > 1$, and taking $x\to\infty$, this term evaluates to 0.

Now we evaluate the second summand. Taking the limit $x\to\infty$ the integral becomes

$$s\int_1^\infty (t-[t])\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}dt.$$

Let A be a compact set in the region $Re(s) > 0$. We will try to show that the absolute value of the integral in the second summand can be bounded by a term not depending on s for $s \in A$. Now

$$\left|s\int_1^{\infty}(t-[t])\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2)\,dt\right|$$
$$\leq |s|\int_1^{\infty}\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-\sigma-1}(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2)\,dt.$$

Now, since $t \geq 1$,

$$\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2 \leq \omega+\frac{1}{4}(2g^2+\Delta^2)^{-\frac{1}{2}}g^2.$$

For simplifying calculations let,

$$k=\omega+\frac{1}{4}(2g^2+\Delta^2)^{-\frac{1}{2}}g^2.$$

The bound now becomes,

$$|s|\int_1^{\infty}\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-\sigma-1}(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2)\,dt$$
$$\leq |s|k\int_1^{\infty}\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-\sigma-1}dt.$$

To simplify the integral, we increase the bound further. Since $t \geq 1$ we have,

$$\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a \geq \frac{1}{2}\sqrt{2g^2+\Delta^2}+a.$$

This implies for $\sigma > 0$, the term inside the integral in the last bound can be bounded as

$$\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-\sigma-1} \leq \left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{2g^2+\Delta^2}+a\right]^{-\sigma-1}.$$

Therefore, the bound changes to

$$|s|k\int_0^{\infty}\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-\sigma-1}dt \leq |s|k\int_0^{\infty}\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{2g^2+\Delta^2}+a\right]^{-\sigma-1}dt.$$

Again for simplifying calculations let

$$c=\frac{\omega}{2}+\frac{1}{2}\sqrt{2g^2+\Delta^2}+a.$$

Then the integral within the final bound is

$$|s|k\int_1^\infty \left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2+\Delta^2}+a\right]^{-\sigma-1} dt = |s|k\int_1^\infty (\omega t+c)^{-\sigma-1}\,dt.$$

Therefore the integral is

$$-\frac{|s|k}{\omega}\lim_{b\to\infty}\left(\frac{1}{(\omega+b)^\sigma}-\frac{1}{(\omega+c)^\sigma}\right)=\frac{|s|k}{\omega(\omega+c)^\sigma}\leq\frac{|s^*|k}{\omega(\omega+c)^{\sigma^*}},$$

where

$$\sigma^* = \inf_{s\in A} Re(s),$$

and

$$|s^*| = \sup_{s\in A}|s|$$

σ^* and $|s^*|$ exist by the extreme value theorem for function from compact spaces to real numbers. To summarise, absolute value of the second summand can be bounded as

$$\left|s\int_1^\infty (t-[t])\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2)\,dt\right|$$
$$\leq\frac{|s|k}{(\omega+c)^\sigma}\leq\frac{|s^*|k}{(\omega+c)^{\sigma^*}}.$$

where, k, c are constants involving ω, g, Δ and a.

$\sigma = Re(s)$, $\sigma^* = \inf_{s\in A} Re(s)$, and $|s^*| = \sup_{s\in A}|s|$.

This concludes that this summand is uniformly convergent in any compact set in the region $Re(s) > 0$ and therefore it represents an analytic function in the region $Re(s) > 0$.

Now, since the third summand has been shown to evaluate to zero and the second summand is analytic in the region $Re(s) > 0$, we evaluate the first summand. The first summand evaluates to some Gaussian hypergeometric functions whose derivation and analyticity will be discussed in detail in the next section.

In the remaining part of this section we discuss along the same lines the Euler summation formula applied to the ${E_n}^-$ eigenvalue part.

5.2.2 The $E_n{}^-$ eigenvalue part

Similar to the term in $\zeta_{JC}(s,a)$ involving the $E_n{}^+$ eigenvalues, the term involving $E_n{}^-$ eigenvalues can be expanded using the Euler summation formula as follows

$$\lim_{x\to\infty}\sum_{n=1}^{x}\frac{1}{(E_n{}^-+a)^s}=\lim_{x\to\infty}\Bigg[\int_1^x\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s}dt$$
$$+s\int_1^x(t-[t])\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}(\omega-\frac{1}{4}(g^2(t+1)+\Delta^2)^{\frac{-1}{2}}g^2)\,dt$$
$$+(x-[x])\left[(x+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(x+1)+\Delta^2}+a\right]^{-s}\Bigg]. \tag{5.7}$$

Similar to the $E_n{}^+$ subseries, the term $\frac{1}{(E_0{}^-+a)^s}$ is evaluated separately and it is analytic in $\mathbb{C}$ since it is a power function of s.

The third summand in this case also evaluates to zero in the following way

$$(x-[x])\left[(x+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(x+1)+\Delta^2}+a\right]^{-s}\Bigg]=O(x^{1-s}).$$

For $Re(s)>1$, and taking $x\to\infty$, this term evaluates to 0.

Evaluating the second summand, after taking the limit, the expression is

$$s\int_1^\infty(t-[t])\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}dt$$

Let A be a compact set in the region $Re(s)>0$. Then we will show that the following integral can be bounded by a term not depending on s for $s\in A$. The integral can first be bounded by

$$\left|s\int_1^\infty(t-[t])\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}(\omega-\frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2)\,dt\right|$$
$$\leq|s|\int_1^\infty\left|(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right|^{-\sigma-1}\left|\omega-\frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2\right|dt.$$

Now if $t > 1$,

$$\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2} \geq \frac{1}{2}\sqrt{2g^2+\Delta^2}.$$

Using the above inequality and triangle inequality we can say that

$$\left|\omega - \frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2\right| \leq \omega + \frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2 \leq (\omega + \frac{1}{4}(2g^2+\Delta^2)^{-\frac{1}{2}}g^2).$$

Therefore the bound now becomes

$$|s|\int_1^\infty \left|(t+\frac{1}{2})\omega - \frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right|^{-\sigma-1}\left|\omega - \frac{1}{4}(g^2(t+1)+\Delta^2)^{-\frac{1}{2}}g^2\right| dt$$
$$\leq |s|(\omega + \frac{1}{4}(2g^2+\Delta^2)^{-\frac{1}{2}}g^2)\int_1^\infty \left|(t+\frac{1}{2})\omega - \frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right|^{-\sigma-1} dt.$$

In order to work with the absolute value on the term inside the integral, we use the reverse triangle inequality stated as follows. For real numbers x and y,

$$||x|-|y|| \leq |x-y|,$$

which implies,

$$|x|-|y| \leq |x-y|.$$

Therefore we have,

$$\left|(t+\frac{1}{2})\omega - \frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right| \geq (t+\frac{1}{2})\omega + a - \frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}.$$

The right hand side of the previous inequality is positive due to the condition on the parameter given by equation (4.4). Therefore the bound is,

$$|s|(\omega + \frac{1}{4}(2g^2+\Delta^2)^{-\frac{1}{2}}g^2)\int_1^\infty \left[\left(t+\frac{1}{2}\right)\omega - \frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-\sigma-1} dt.$$

This final bound and the first summand in $Re(s) > 0$, involves hypergeometric functions and will be discussed in the following section.

5.3 The hypergeometric functions

This section discusses the Gaussian hypergeometric functions which arise while evaluating the integral in the first summand of the Euler summation formula. The Gaussian hypergeometric function is the solution of a homogeneous second-order linear differential equation with some special properties (refer to [3] for details). For $|z| < 1, z \in \mathbb{C}$, and c in positive integers and the Gaussian hypergeometric function is defined as[3]

$$ {}_2F_1 = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where $(a)_n$ is the pochammer symbol defined by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \text{ and } n = 1, 2, 3... $$

The following section elaborates on the derivation of the hypergeometric functions from the first summand. Some computation using Wolfram Alpha was initially done to guess the solution to the integral in the first term of the right side of equation (5.6) hence the derivation proceeds in a way so that this integral can be modified into one of the integral forms of the Gaussian hypergeometric function given as follows

$$ {}_2F_1(a,b;c;1-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\infty} s^{b-1}(1+s)^{a-c}(1+sz)^{-a}\, ds. \tag{5.8} $$

This holds true for $Re(c) > Re(b)$.[3]

5.3.1 The $E_n{}^+$ eigenvalue part

The first summand can be integrated as follows

$$ \lim_{x\to\infty} \int_1^x \left[(t+\frac{1}{2})\omega + \frac{1}{2}\sqrt{g^2(t+1)+\Delta^2} + a \right]^{-s} dt. $$

We use the following change of variable. Let

$$ q = \sqrt{g^2(t+1)+\Delta^2}, $$

and so,

$$dt = \frac{2q}{g^2}dq.$$

Replacing this in the integral,

$$= \frac{2}{g^2} \lim_{x\to\infty} \int_{\sqrt{2g^2+\Delta^2}}^{\sqrt{g^2(x+1)+\Delta^2}} q\left[\frac{q^2\omega}{g^2} + \frac{q}{2} - \frac{\Delta^2\omega}{g^2} - \frac{\omega}{2} + a\right]^{-s} dq.$$

The term within the integral can be factorised into linear factors of q as follows, let for ease of calculations,

$$m = \sqrt{\frac{g^2}{4} - 4\omega a + \frac{4\Delta^2\omega^2}{g^2} + 2\omega^2}.$$

Then the integral can be written as

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty} \int_{\sqrt{2g^2+\Delta^2}}^{\sqrt{g^2(x+1)+\Delta^2}} q\left(q + \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s}\left(q + \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s} dq$$

In order to modify this integral into one of the integral forms of the Gaussian hypergeometric function, the lower limit of the integral should be 0. To achieve this we do the following change of variable

$$r = q - \sqrt{2g^2+\Delta^2}.$$

Therefore the integral becomes

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty} \int_{0}^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} (r + \sqrt{2g^2+\Delta^2})\left(r + \sqrt{2g^2+\Delta^2} + \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s}$$
$$\left(r + \sqrt{2g^2+\Delta^2} + \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s} dr.$$

This integral then needs to be split into two integrals so that the powers on the factors are in the required form

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty}\int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} r\left(r+\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s}$$
$$\left(r+\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s} dr$$
$$+\frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{2g^2+\Delta^2}\lim_{x\to\infty}\int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} \left(r+\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s}$$
$$\left(r+\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s} dr.$$

For ease of calculation the constants would be grouped together and named as follows. Let,

$$k_1 = \sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}+\frac{mg}{2\omega} \text{ and } n_1 = \sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}-\frac{mg}{2\omega}$$

Replacing these constants,

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty}\int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} r(r+k_1)^{-s}(r+n_1)^{-s}\,dr$$
$$+\frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{2g^2+\Delta^2}\lim_{x\to\infty}\int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} (r+k_1)^{-s}(r+n_1)^{-s}\,dr.$$

Now the terms inside the integral will be divided and multiplied by k_1 and n_1 respectively in order to make them in the form of $(1+\text{variable})$

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} k_1{}^{-s} n_1{}^{-s} \lim_{x\to\infty}\int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} r\left(1+\frac{r}{k_1}\right)^{-s}\left(1+\frac{r}{n_1}\right)^{-s} dr$$
$$+\frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{2g^2+\Delta^2}\,k_1{}^{-s} n_1{}^{-s}\lim_{x\to\infty}\int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} \left(1+\frac{r}{k_1}\right)^{-s}(1+\frac{r}{n_1}\Big)^{-s} dr.$$

Now let,

$$p = \frac{r}{k_1}.$$

Taking the limit and changing the variable,

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} {k_1}^{-s+2}{n_1}^{-s}\int_0^\infty p(1+p)^{-s}\left(1+\frac{pk_1}{n_1}\right)^{-s} dp$$
$$+\frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{2g^2+\Delta^2}\, {k_1}^{-s+1}{n_1}^{-s}\int_0^\infty (1+p)^{-s}\left(1+\frac{pk_1}{n_1}\right)^{-s} dp.$$

Comparing these two integrals with the integral form of the gaussian hypergeometric function in equation (5.8) the integrals are equal to, for $Re(s) > 1$ (since $Re(c) > Re(b)$)

$$= \frac{{c_1}^+}{2(s-1)(2s-1)}{}_2F_1(s,2;2s;1-\frac{k_1}{n_1}) + \frac{{c_2}^+}{2s-1}{}_2F_1(s,1;2s;1-\frac{k_1}{n_1}), \tag{5.9}$$

where,

$${c_1}^+ = \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}{k_1}^{-s+2}{n_1}^{-s}$$
$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\left(\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s+2}\left(\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s},$$

$${c_2}^+ = \frac{2}{g^2}\sqrt{2g^2+\Delta^2}\left(\frac{\omega}{g^2}\right)^{-s}{k_1}^{-s+1}{n_1}^{-s}$$
$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{2g^2+\Delta^2}\left(\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s+1}\left(\sqrt{2g^2+\Delta^2}+\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s},$$

and,

$$m = \sqrt{\frac{g^2}{4} - 4\omega a + \frac{4\Delta^2\omega^2}{g^2} + 2\omega^2}.$$

5.3.2 The ${E_n}^-$ eigenvalue part

To recall, the first summand for the ${E_n}^-$ eigenvalue part is given as follows

$$\lim_{x\to\infty}\int_1^x \left[(t+\frac{1}{2})\omega - \frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}\right]^{-s} dt.$$

Similar to the $E_n{}^+$ part, we do a change of variable. Let

$$q = \sqrt{g^2(t+1)+\Delta^2}, \text{ and } dt = \frac{2q}{g^2}dq.$$

Therefore, the integral can now be written as

$$= \frac{2}{g^2} \lim_{x\to\infty} \int_{\sqrt{2g^2+\Delta^2}}^{\sqrt{g^2(x+1)+\Delta^2}} q\left[\frac{q^2\omega}{g^2} - \frac{q}{2} - \frac{\Delta^2\omega}{g^2} - \frac{\omega}{2} + a\right]^{-s} dq.$$

Again, to factorise this expression, let

$$m = \sqrt{\frac{g^2}{4} - 4\omega a + \frac{4\Delta^2\omega^2}{g^2} + 2\omega^2}.$$

Note that this m is the same as that in the $E_n{}^+$ eigenvalue part. Therefore upon factorising the integral is

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty} \int_{\sqrt{2g^2+\Delta^2}}^{\sqrt{g^2(x+1)+\Delta^2}} q\left(q - \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s}\left(q - \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s} dq.$$

The next change of variable is done to make the lower bound of the integral 0. Let

$$r = q - \sqrt{2g^2+\Delta^2}.$$

The integral becomes

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty} \int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{g^2+\Delta^2}} (r+\sqrt{2g^2+\Delta^2})\left(r+\sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s}$$
$$\left(r+\sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s} dr.$$

Splitting the integral in a way similar to the last section,

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty} \int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} r\left(r+\sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s}$$
$$\left(r+\sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s} dr$$
$$+ \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \sqrt{2g^2+\Delta^2} \lim_{x\to\infty} \int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} \left(r+\sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s}$$
$$\left(r+\sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s} dr.$$

The constants are grouped together and named as follows. Let,

$$k_2 = \sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} + \frac{mg}{2\omega} \text{ and } n_2 = \sqrt{2g^2+\Delta^2} - \frac{g^2}{4\omega} - \frac{mg}{2\omega}$$

Replacing these constants we get,

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \lim_{x\to\infty} \int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} r(r+n_2)^{-s}(r+k_2)^{-s}\,dr$$
$$+ \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \sqrt{2g^2+\Delta^2} \lim_{x\to\infty} \int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} (r+n_2)^{-s}(r+k_2)^{-s}\,dr.$$

Now the terms inside the integral will be divided and multiplied by n_2 and k_2 respectively in order to make them in the form of $(1+\text{variable})$

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} k_2{}^{-s} n_2{}^{-s} \lim_{x\to\infty} \int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} r\left(1+\frac{r}{n_2}\right)^{-s}\left(1+\frac{r}{k_2}\right)^{-s} dr$$
$$+ \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \sqrt{2g^2+\Delta^2}\; k_2{}^{-s} n_2{}^{-s} \lim_{x\to\infty} \int_0^{\sqrt{g^2(x+1)+\Delta^2}-\sqrt{2g^2+\Delta^2}} \left(1+\frac{r}{n_2}\right)^{-s}(1+\frac{r}{k_2}\bigg)^{-s} dr.$$

Now let,

$$p = \frac{r}{k_2}.$$

Taking the limit and changing the variable,

$$= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} k_2{}^{-s+2} n_2{}^{-s} \int_0^\infty p(1+p)^{-s}\left(1+\frac{pk_2}{n_2}\right)^{-s} dp$$
$$+ \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} \sqrt{2g^2+\Delta^2}\; k_2{}^{-s+1} n_2{}^{-s} \int_0^\infty (1+p)^{-s}\left(1+\frac{pk_2}{n_2}\right)^{-s} dp.$$

As before, we compare these two integrals with the integral form of the gaussian hypergeometric function in equation (5.8). Therefore, the integrals are equal to, for $Re(s) > 1$ (since $Re(c) > Re(b)$)

$$= \frac{c_1{}^-}{2(s-1)(2s-1)}{}_2F_1(s,2;2s;1-\frac{k_2}{n_2}) + \frac{c_2{}^-}{2s-1}{}_2F_1(s,1;2s;1-\frac{k_2}{n_2}), \tag{5.10}$$

where,

$$
\begin{aligned}
{c_1}^- &= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s} {k_2}^{-s+2}{n_2}^{-s} \\
&= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\left(\sqrt{2g^2+\Delta^2}-\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s+2}\left(\sqrt{2g^2+\Delta^2}-\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s},
\end{aligned}
$$

$$
\begin{aligned}
{c_2}^- &= \frac{2}{g^2}\sqrt{2g^2+\Delta^2}\left(\frac{\omega}{g^2}\right)^{-s} {k_2}^{-s+1}{n_2}^{-s} \\
&= \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{g^2+\Delta^2}\left(\sqrt{2g^2+\Delta^2}-\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s+1}\left(\sqrt{2g^2+\Delta^2}-\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s},
\end{aligned}
$$

and,

$$
m = \sqrt{\frac{g^2}{4}-4\omega a+\frac{4\Delta^2\omega^2}{g^2}+2\omega^2}.
$$

5.3.3 Analyticity of the hypergeometric functions

To show the analyticity of the first summand of both the ${E_n}^+$ and ${E_n}^-$ parts we will use the known results on the analyticity of the hypergeometric function as a function of its parameters. To do this we first use the following transformation formula to transform the hypergeometric functions so that only one of the parameters contains the variable s

$$
F(a,b;2b;z) = \left(1-\frac{1}{2}z\right)^{-a} F\left(\frac{a}{2},\frac{1}{2}+\frac{a}{2};b+\frac{1}{2};[z/(2-z)]^2\right).
$$

For the hypergeometric function in the ${E_n}^+$ case (equation 5.9), it transforms as

$$
\begin{aligned}
{}_2F_1\left(s,2;2s;1-\frac{k_1}{n_1}\right) &= {}_2F_1\left(2,s;2s;1-\frac{k_1}{n_1}\right) \\
&= \frac{4{n_1}^2}{(n_1+k_1)^2}\, {}_2F_1\left(1,\frac{3}{2};s+\frac{1}{2};\frac{(n_1-k_1)^2}{(n_1+k_1)^2}\right).
\end{aligned}
$$

Now, we refer to the following Proposition from [3].

Proposition 5.3.1. ${}_2F_1(a,b;c;z_0)/\Gamma(c)$ *is an entire analytic function of* a,b,c, *if* z_0 *is fixed and* $|z_0|<1$.

According to Proposition 5.3.1, we need to check the condition on $|z_0|$. This is given by

$$\frac{(n_1-k_1)^2}{(n_1+k_1)^2}<1.$$

Replacing the value of n_1 and k_1 in terms of g,ω,Δ and m, this is equal to

$$a>-\frac{\sqrt{2g^2+\Delta^2}}{2}-\frac{3\omega}{2}.$$

This condition holds by the condition on a given by equation (4.4). This is because this is equivalent to

$$E_1{}^{+}+a>0.$$

For the hypergeometric function involving the $E_n{}^{-}$ eigenvalues, the hypergeometric function transforms as

$$\begin{aligned} {}_2F_1\left(s,2;2s;1-\frac{k_2}{n_2}\right) &= {}_2F_1\left(2,s;2s;1-\frac{k_2}{n_2}\right) \\ &= \frac{4{n_1}^2}{(n_2+k_2)^2}\,{}_2F_1\left(1,\frac{3}{2};s+\frac{1}{2};\frac{(n_2-k_2)^2}{(n_2+k_2)^2}\right). \end{aligned} \tag{5.11}$$

Then the analyticity condition is given by

$$a>\frac{\sqrt{2g^2+\Delta^2}}{2}-\frac{3\omega}{2}.$$

This condition also holds by equation (4.4) since this condition is equivalent to

$$E_1{}^{-}+a>0.$$

The $\Gamma(c)$ mentioned in Proposition 5.3.1 is also included in the expressions 5.9 and 5.10. Now, since we have the analyticity of the hypergeometric functions, we need to complete the argument for the second summand in equation 5.7. The second summand could be bounded by

$$\sigma(\omega+\frac{1}{4}(g^2+\Delta^2)^{-\frac{1}{2}}g^2)\int_1^{\infty}\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-\sigma-1}dt.$$

This integral was derived to be the hypergeometric function given by 5.11. Therefore, for a compact set A in the region $Re(s)>0$, and the hypergeometric function is defined for any s in that region since it was shown to be analytic for all s in $\mathbb{C}$. We take $\sigma^* = s$ for $s \in A$ such that the value of the hypergeometric function given by 5.11 is the maximum. The value of the hypergeometric function at $\sigma*$ is the bound for this function.

5.3.4 Poles and residues

From the expression involving the hypergeometric functions, it seems the poles of $\zeta_{JC}(s,a)$ are at $s = \frac{1}{2}$ and $s = 1$. To verify this, we first calculate the residue of $\zeta_{JC}(s,a)$ at $s = \frac{1}{2}$. Putting $s = \frac{1}{2}$ in equation (5.9) gives the following term in the numerator of $2s-1$

$$\frac{{c_1}^+}{2} {}_2F_1\left(\frac{1}{2}, 2; 1; 1 - \frac{k_1}{n_1}\right) + {c_2}^+ {}_2F_1\left(\frac{1}{2}, 1; 1; 1 - \frac{k_1}{n_1}\right). \tag{5.12}$$

Similarly, putting $s = \frac{1}{2}$ in equation (5.10) gives the following term in the numerator of $2s-1$

$$\frac{{c_1}^-}{2} {}_2F_1\left(\frac{1}{2}, 2; 1; 1 - \frac{k_1}{n_1}\right) + {c_2}^- {}_2F_1\left(\frac{1}{2}, 1; 1; 1 - \frac{k_1}{n_1}\right). \tag{5.13}$$

The hypergeometric functions at specific values of the parameters were computed from [1] and [22] and are given by

$${}_2F_1\left(\frac{1}{2}, 2; 1; 1 - a\right) = \frac{a+1}{2a^{\frac{2}{3}}},$$

and

$${}_2F_1\left(\frac{1}{2}, 1; 1; 1 - a\right) = \frac{1}{\sqrt{a}}.$$

Therefore the term in equation (5.12) becomes (after replacing k_1 and n_1),

$$\frac{-1}{g\sqrt{\omega}}\left(2\sqrt{2g^2 + \Delta^2} + \frac{g^2}{2\omega}\right) + \frac{2}{g\sqrt{\omega}}\sqrt{2g^2 + \Delta^2}, \tag{5.14}$$

and the term in equation (5.13) becomes

$$\frac{-1}{g\sqrt{\omega}}\left(2\sqrt{2g^2 + \Delta^2} - \frac{g^2}{2\omega}\right) + \frac{2}{g\sqrt{\omega}}\sqrt{2g^2 + \Delta^2}. \tag{5.15}$$

Adding equation (5.14) and (5.15) gives the residue at $s = \frac{1}{2}$ as 0. This implies that $\zeta_{JC}(s,a)$ does not have a pole at $s = \frac{1}{2}$.

The residue of $\zeta_{JC}(s,a)$ at $s = 1$ can also be computed using the following value of the hypergeometric function

$$ {}_2F_1\left(1,2;2;1-a\right) = \frac{1}{a}. $$

Therefore the residue is

$$ \frac{{c_1}^+}{2}\left(\frac{n_1}{k_1}\right) + \frac{{c_1}^-}{2}\left(\frac{n_2}{k_2}\right). $$

This is equal to

$$ \frac{1}{2}\left(\frac{2k_1}{\omega n_1}\right)\left(\frac{n_1}{k_1}\right) + \frac{1}{2}\left(\frac{2k_2}{\omega n_2}\right)\left(\frac{n_2}{k_2}\right) = \frac{2}{\omega}. $$

Therefore $\zeta_{JC}(s,a)$ has a pole at $s = 1$ with a residue of $\frac{2}{\omega}$.

5.3.5 Analytic continuation of $\zeta_{JC}(s,a)$ to $Re(s) > 0$

In this subsection, the proof for the analytic continuation of $\zeta_{JC}(s,a)$ to $Re(s) > 0$ will be discussed by summarizing the techniques discussed in the previous subsection of chapter 5.

Proposition 5.3.2. *The spectral zeta function of the JC-Hamiltonian, $\zeta_{JC}(s,a)$ can be analytically continued to $Re(s) > 0$ except for a simple pole at $s = 1$.*

Proof. We know

$$ \zeta_{JC}(s,a) = \lim_{x\to\infty}\left[\sum_{n=0}^{x}\frac{1}{({E_n}^+ + a)^s} + \sum_{n=0}^{x}\frac{1}{({E_n}^- + a)^s} + \frac{1}{(\frac{-\omega_0}{2})^s}\right] $$

where, ${E_n}^{\pm} = (n+\frac{1}{2})\omega \pm \frac{1}{2}\sqrt{g^2(n+1)+(\omega_0-\omega)^2}$ and assuming the necessary conditions on a and ω. The terms $(\frac{-\omega_0}{2})^{-s}$, $({E_0}^+ + a)^{-s}$ and $({E_0}^- + a)^{-s}$ are considered separately as mentioned in subsection 5.2.1.

Now, the Euler summation formula while applied to the subseries involving the ${E_n}^+$ eigenvalues (except for the first term $({E_0}^+ + a)^{-s}$) gives the following terms

$$\lim_{x\to\infty}\sum_{n=1}^{x}\frac{1}{({E_n}^+ + a)^s} = \lim_{x\to\infty}\left[\int_1^x\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s}dt\right.$$
$$+s\int_1^x(t-[t])\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}\left(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{\frac{-1}{2}}g^2\right)dt$$
$$\left.+(x-[x])\left[(x+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(x+1)+\Delta^2}+a\right]^{-s}\right].$$

Among these terms, the first term evaluates to the hypergeometric functions which are derived in subsection 5.3.1. These functions are analytic in the whole complex plane as shown in subsection 5.3.3.

The second integral term can be shown to be uniformly convergent and hence analytic in the region $Re(s) > 0$ as elaborated in subsection 5.2.1. The third term evaluates to 0 upon taking the limit $x \to \infty$.

Similarly, applying the Euler summation formula to the subseries involving the ${E_n}^-$ eigenvalues (except for the first term $({E_0}^- + a)^{-s}$) gives the following terms

$$\lim_{x\to\infty}\sum_{n=1}^{x}\frac{1}{({E_n}^- + a)^s} = \lim_{x\to\infty}\left[\int_1^x\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s}dt\right.$$
$$+s\int_1^x(t-[t])\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s-1}\left(\omega-\frac{1}{4}(g^2(t+1)+\Delta^2)^{\frac{-1}{2}}g^2\right)dt$$
$$\left.+(x-[x])\left[(x+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(x+1)+\Delta^2}+a\right]^{-s}\right].$$

In this case too the first term evaluates to hypergeometric functions detailed in subsection 5.3.2. The analyticity of these hypergeometric functions to the entire complex plane is discussed in subsection 5.3.3.

The second integral can be shown to be uniformly convergent and hence analytic in the region $Re(s) > 0$. This is shown in two parts, the first part of which is showing that the absolute value of the integral can be bounded by a term as shown in subsection 5.2.2. This bound is then shown to be equal to a hypergeometric function which does not depend on s for s belonging to a compact set in the region $Re(s) > 0$, as given in subsection 5.3.3. The remaining third term evaluates to 0 after taking the limit $x \to \infty$.

This concludes the proof of the analytic continuation of $\zeta_{JC}(s,a)$ to $Re(s) > 0$. The hypergeometric functions in subsections 5.3.1 and 5.3.2 indicate that $\zeta_{JC}(s,a)$ might contain poles at $s = 1$ and $s = \frac{1}{2}$. Upon calculating the residues (discussed in subsection5.3.4), we can conclude that $\zeta_{JC}(s,a)$ has a simple pole at $s = 1$. □

5.4 Analytic continuation of $\zeta_{JC}(s,a)$ to whole of $\mathbb{C}$

5.4.1 The Euler-Maclaurin summation formula applied to $\zeta_{JC}(s,a)$

To show the analytic continuation of $\zeta_{JC}(s,a)$ to the rest of the complex plane, the Euler-Maclaurin summation formula is used which is given as follows [9]

$$\sum_{a \leq k \leq b} f(k) = \int_a^b f(t)\,dt + \frac{1}{2}(f(a)) + f(b))$$
$$+ \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!}(f^{(2m-1)}(b) - f^{(2m-1)}(a)) + \int_a^b P_{2n+1}(t) f^{(2n+1)}(t)\,dt.$$

Here $n \geq 0$ is a fixed integer, $f(x) \in \mathbb{C}^{2n+1}[a,b]$, B_m is the m^{th} Bernoulli number, and P_m is the m^{th} periodic Bernoulli function defined by $P_m(x) = B_m(x - [x])$, where $B_m(x)$ is the m^{th} Bernoulli polynomial defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m(x) z^m}{m!} (|z| < 2\pi).$$

Applying the Euler-Maclaurin summation formula to the subseries

$$\sum_{n=1}^{\infty} \frac{1}{(E_n{}^+ + a)^s}$$

we have,

$$f(t) = \left[(t + \frac{1}{2})\omega + \frac{1}{2}\sqrt{g^2(t+1) + \Delta^2} + a\right]^{-s}.$$

and,

$$\sum_{k=1}^{\infty}\frac{1}{(E_k{}^{+}+a)^s}=\frac{1}{2}\left(\frac{3\omega}{2}+\frac{1}{2}\sqrt{2g^2+\Delta^2}+a\right)^{-s}+\int_1^{\infty}\left[(t+\frac{1}{2})\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s}dt$$
$$+\sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}(\lim_{b\to\infty}f^{(2m-1)}(b))-f^{(2m-1)}(1)+\int_1^{\infty}P_{2n+1}(t)f^{(2n+1)}(t)\,dt. \tag{5.16}$$

The term $(E_0{}^{+}+a)^{-s}$ will be added separately to the sum after this evaluation. The first term in this sum is a power function of s and is hence analytic for all s in its domain that is all $s\in\mathbb{C}$. The second term gives the hypergeometric function and is analytic for all $s\in\mathbb{C}$.
Now, applying the Euler-Maclaurin summation formula to the subseries

$$\sum_{k=0}^{\infty}\frac{1}{(E_k{}^{-}+a)^s}$$

We have,

$$\sum_{k=1}^{\infty}\frac{1}{(E_k{}^{-}+a)^s}=\frac{1}{2}\left(\frac{3\omega}{2}-\frac{1}{2}\sqrt{g^2+\Delta^2}+a\right)^{-s}+\int_1^{\infty}\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s}dt$$
$$+\sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}(\lim_{b\to\infty}f^{(2m-1)}(b))-f^{(2m-1)}(1)+\int_1^{\infty}P_{2n+1}(t)f^{(2n+1)}(t)\,dt.$$

Similarly to the $E_n{}^{+}$ eigenvalues, the first term is an elementary function of s so it is analytic for all s in $\mathbb{C}$. Here too, the term $(E_0{}^{-}+a)^{-s}$ is added separately later to the final formula. The second term is the hypergeometric function whose analyticity for all $s\in\mathbb{C}$ was shown in section 5.3.3. We discuss the remaining two terms in the next section.

5.4.2 The Bernoulli number term and the residual term

For the other two terms in the case of $E_n{}^{+}$, $f^{(n)}(t)$ needs to be estimated. By induction arguments, the following Proposition can be shown that

Proposition 5.4.1. *The n^{th} derivative of $f(t)$ where*

$$f(t)=\left[\left(t+\frac{1}{2}\right)\omega+\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s},$$

is given by

$$f^{(n)}(t) = \frac{h(u(t))}{(g(t))^{s+n}}.$$

Here,

$$u(t) = (g^2(t+1) + \Delta^2)^{-\frac{1}{2}},$$

and, $h(u(t))$ is a polynomial in $u(t)$ with the highest power of $u(t)$ in $h(u(t))$ as $2n-1$. The polynomial in the denominator is given by

$$g(t) = \left(t + \frac{1}{2}\right)\omega + \frac{1}{2}\sqrt{g^2(t+1) + \Delta^2} + a.$$

The polynomial $h(t)$ also has $g(t)$ as coefficients with the highest power of $g(t)$ as $n-1$.

Proof. Let the induction hypothesis be the same as the Proposition.
For n=1,

$$\begin{aligned} f^{(1)}(t) &= \frac{d}{dt}\left[\left(t + \frac{1}{2}\right)\omega + \frac{1}{2}\sqrt{g^2(t+1) + \Delta^2} + a\right]^{-s} \\ &= -s\left[\left(t + \frac{1}{2}\right)\omega + \frac{1}{2}\sqrt{g^2(t+1) + \Delta^2} + a\right]^{-s-1}(\omega + \frac{1}{4}(g^2(t+1) + \Delta^2)^{\frac{-1}{2}} g^2). \end{aligned}$$

This satisfies the induction hypothesis since this is in the form of

$$\frac{h(u(t))}{(g(t))^{s+1}},$$

where $h(u(t))$ is a polynomial in $(g^2(t+1) + \Delta^2)^{\frac{-1}{2}}$ with its highest power as 1.
Now for $n = k$ let,

$$f^{(k)}(t) = \frac{h(u(t))}{(g(t))^{s+k}}.$$

The polynomial $h(u(t))$ is of degree $(2k-1)$ has $g(t)$ as coefficients with the highest power of $g(t)$ being $k-1$.
Now for $n = k+1$,

$$f^{(k+1)}(t) = \frac{(-s-k)h(u(t))}{(g(t))^{s+k+1}} + \frac{g(t)h^{(1)}(u(t))}{(g(t))^{s+k+1}}. \tag{5.17}$$

We need to check the highest power of $u(t)$ and $g(t)$ in $h^{(1)}(u(t))$ in order to complete the induction argument. For this, it is enough to consider the term in $h(u(t))$ with the highest possible power of $u(t)$ and $g(t)$. Now, the term with the highest possible power of $g(t)$ and $u(t)$ in $h(u(t))$ is,

$$g(t)^{(k-1)}u(t)^{(2k-1)} = g(t)^{(k-1)}(g^2(t+1)+\Delta^2)^{-\frac{(2k-1)}{2}}.$$

Therefore after computing $h^{(1)}(u(t))$ this term will be,

$$g(t)^{k-2}(g^2(t+1)+\Delta^2)^{-\frac{(2k-1)}{2}}\left(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{\frac{-1}{2}}g^2\right)$$
$$-\frac{(2k-1)}{2}g(t)^{k-1}(g^2(t+1)+\Delta^2)^{-\frac{(2(k+1)-1)}{2}}.$$

Multiplying this by $g(t)$ as in the second term of equation (5.17), this becomes

$$g(t)^{k-1}(g^2(t+1)+\Delta^2)^{-\frac{(2k-1)}{2}}\left(\omega+\frac{1}{4}(g^2(t+1)+\Delta^2)^{\frac{-1}{2}}g^2\right)$$
$$-\frac{(2k-1)}{2}g(t)^{k}(g^2(t+1)+\Delta^2)^{-\frac{(2(k+1)-1)}{2}}.$$

Therefore, $f^{(k+1)}$ satisfies the induction hypothesis. This completes the induction argument.

□

Now for the second last term in equation (5.16), $\lim_{b\to\infty} f^{(2m-1)}(b)$ can be estimated by finding the order of t in $f^{(2m-1)}(t)$ using Proposition 5.4.1.

The highest power of t in the numerator of $f^{(2m-1)}(t)$ is

$$-\frac{((2m-1)-1)}{2}+(2m-1-1) = m-3,$$

and

$$(\text{lowest power of } t \text{ in the denominator}) - s = 2m-1.$$

Therefore,

$$f^{(2m-1)}(t) = \frac{c_1 t^{m-3}+f_1(t)}{c_2 t^{s+2m-1}+f_2(t)}.$$

Here $f_1(t)$ and $f_2(t)$ are polynomials and the degree of $f_1(t)$ is less than $m-3$ while $\deg(f_2)-s$ is higher than $2m-1$. Let $\deg(f_2)-s-2m+1=k$ and $k>0$ by the previous assertion.
Now, multiplying the numerator and denominator by $t^{-s-2m+1}$,

$$f^{(2m-1)}(t)=\frac{c_3t^{-s-m-2}+f_3(t)}{c_{k+4}t^k+\ldots+c_4t+c_5}.$$

Here, $f_3(t)$ is a polynomial and $\deg(f_3)+s$ is less than $-m-2$. Therefore, for the second last term in (5.16),

$$\lim_{b\to\infty}f^{(2m-1)}(b)=\lim_{b\to\infty}\frac{c_3}{c_2}b^{-s-m-2}+\lim_{b\to\infty}f_3(b)\lim_{b\to\infty}\left(\frac{1}{c_{k+4}t^k+\ldots+c_5t+c_4}\right).$$

Since the leading term goes to zero for $Re(s)>-m-2$, and the $\deg(f_3)+s$ is less than $-m-2$,

$$\lim_{b\to\infty}f_3(b)=0$$

The term multiplied to this second term is,

$$\lim_{b\to\infty}\frac{1}{c_{k+4}b^k+\ldots+c_5b+c_4}=\frac{1}{c_4}.$$

The above is true since $k>0$. Therefore,

$$\lim_{b\to\infty}f^{(2m-1)}(b)=0.$$

Evaluating $f^{(2m-1)}(t)$ at 1,

$$f^{(2m-1)}(1)=\frac{h(u(1))}{(g(1))^{s+2m-1}}.$$

The only case where this term would not be finite is if

$$g(1)=\frac{3\omega}{2}+a+\frac{\sqrt{2g^2+\Delta^2}}{2}=0$$

This is not possible due to the condition assumed on a in equation (4.4). Therefore, $f^{(2m-1)}(1)$ is a power function of s that is analytic in $\mathbb{C}$.

Proposition 5.4.2. *The integral $\int_1^\infty P_{2n+1}(t)f^{(2n+1)}(t)\,dt$ converges uniformly and is analytic for $Re(s)>-n$.*

Proof. The polynomial

$$P_{2n+1}(t) = O(1),$$

The term $f^{(2n+1)}(t)$ can be approximated using Proposition 5.4.1 as

$$f^{(2n+1)}(t) = \frac{c_1 t^{-s-n-1} + f_2(t)}{c_{k+2}t^k + \ldots + c_3 t + c_2}$$

Where $f_2(t)$ is a polynomial with $\deg(f_2) + s$ less than $(-n-1)$. Substituting the value of $f^{(2n+1)}(t)$ in the integral gives

$$\int_1^\infty P_{2n+1}(t) f^{(2n+1)}(t)\, dt = \int_1^\infty P_{2n+1}(t) \left(\frac{c_1 t^{-s-n-1} + f_2(t)}{c_{k+2}t^k + \ldots + c_3 t + c_2} \right) dt.$$

Now since $P_{2n+1}(t) = O(1)$,

$$\left| \int_1^\infty P_{2n+1}(t) \left(\frac{c_1 t^{-s-n-1} + f_2(t)}{c_{k+2}t^k + \ldots + c_3 t + c_2} \right) dt \right| \leq \frac{c_1}{c_2} \int_1^\infty t^{-\sigma-n-1}\, dt + \frac{1}{c_2} \int_1^\infty |f_2(t)|\, dt \tag{5.18}$$

Therefore for uniform convergence of the first integral in the bound of (5.18) we need,

$$\sigma > -n$$

where $Re(s) = \sigma$. For the other integral in the bound in equation (5.18), since $\deg(f_2) + s$ is less than $(-n-1)$, the uniform convergence of that integral can also be concluded by the same condition on σ. □

Therefore for the last term in the Euler-Maclaurin summation of the subseries given by (5.16), using Proposition 5.4.2, a suitable n is chosen such that

$$Re(s) > -n.$$

This concludes the analyticity of equation (5.16) for such suitable n.
For the $E_n{}^-$ eigenvalues, the Proposition 5.4.1 stays the same and only the definition of $f(t)$ becomes

$$f(t) = \left[(t + \frac{1}{2})\omega - \frac{1}{2}\sqrt{g^2(t+1) + \Delta^2} + a \right]^{-s}.$$

The rest of the analysis involving the powers of t is the same and the analytic continuation of $\zeta_{JC}(s,a)$ follows.

5.4.3 The analytic formula for $\zeta_{JC}(s,a)$

Proposition 5.4.3. *The spectral zeta function of the Jaynes-Cummings model can be given by (using suitable n as in Proposition 5.4.2 and except for the pole at $s=1$)*

$$\begin{aligned}
\zeta_{JC}(s,a) = &\frac{1}{2}\left(\frac{\omega}{2}+\frac{1}{2}\sqrt{g^2+\Delta^2}+a\right)^{-s}+\frac{1}{2}\left(\frac{3\omega}{2}+\frac{1}{2}\sqrt{2g^2+\Delta^2}+a\right)^{-s}\\
&+\sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}f^{(2m-1)}(1)+\int_1^\infty P_{2n+1}(t)f^{(2n+1)}(t)\,dt\\
&+\frac{{c_1}^+}{2(s-1)(2s-1)}{}_2F_1(s,2;2s;1-\frac{k_1}{n_1})+\frac{{c_2}^+}{2s-1}{}_2F_1(s,1;2s;1-\frac{k_1}{n_1})\\
&+\frac{1}{2}\left(\frac{\omega}{2}-\frac{1}{2}\sqrt{g^2+\Delta^2}+a\right)^{-s}+\frac{1}{2}\left(\frac{3\omega}{2}-\frac{1}{2}\sqrt{2g^2+\Delta^2}+a\right)^{-s}\\
&+\sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}g^{(2m-1)}(1)+\int_1^\infty P_{2n+1}(t)g^{(2n+1)}(t)\,dt\\
&+\frac{{c_1}^-}{2(s-1)(2s-1)}{}_2F_1(s,2;2s;1-\frac{k_2}{n_2})+\frac{{c_2}^-}{2s-1}{}_2F_1(s,1;2s;1-\frac{k_2}{n_2}).
\end{aligned} \tag{5.19}$$

where

$$f(t)=\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s},$$

and,

$$g(t)=\left[(t+\frac{1}{2})\omega-\frac{1}{2}\sqrt{g^2(t+1)+\Delta^2}+a\right]^{-s}.$$

The constants ${c_1}^+,{c_2}^+,k_1,n_1,{c_1}^-,{c_2}^-,k_2$ and n_2 are given by

1.

$${c_1}^+=\frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\left(\sqrt{g^2+\Delta^2}+\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s+2}\left(\sqrt{g^2+\Delta^2}+\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s},$$

$${c_2}^+=\frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{g^2+\Delta^2}\left(\sqrt{g^2+\Delta^2}+\frac{g^2}{4\omega}+\frac{mg}{2\omega}\right)^{-s+1}\left(\sqrt{g^2+\Delta^2}+\frac{g^2}{4\omega}-\frac{mg}{2\omega}\right)^{-s},$$

2.

$$k_1 = \sqrt{g^2+\Delta^2} + \frac{g^2}{4\omega} + \frac{mg}{2\omega},$$
$$n_1 = \sqrt{g^2+\Delta^2} + \frac{g^2}{4\omega} - \frac{mg}{2\omega},$$

3.

$$c_1{}^{-} = \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\left(\sqrt{g^2+\Delta^2} - \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s+2}\left(\sqrt{g^2+\Delta^2} - \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s},$$
$$c_2{}^{-} = \frac{2}{g^2}\left(\frac{\omega}{g^2}\right)^{-s}\sqrt{g^2+\Delta^2}\left(\sqrt{g^2+\Delta^2} - \frac{g^2}{4\omega} + \frac{mg}{2\omega}\right)^{-s+2}\left(\sqrt{g^2+\Delta^2} - \frac{g^2}{4\omega} - \frac{mg}{2\omega}\right)^{-s},$$

4.

$$k_2 = \sqrt{g^2+\Delta^2} - \frac{g^2}{4\omega} + \frac{mg}{2\omega},$$
$$n_2 = \sqrt{g^2+\Delta^2} - \frac{g^2}{4\omega} - \frac{mg}{2\omega},$$

5.

$$m = \sqrt{\frac{g^2}{4} - 4\omega a + \frac{4\Delta^2\omega^2}{g^2} + 2\omega^2}.$$

Chapter 6

Conclusions

The previous chapters conclude the proof of the analytic continuation of the spectral zeta function of the Jaynes-Cummings model. The following topics may be pursued in the future in this direction:

1. Finding the negative integer values of $\zeta_{JC}(s,a)$ in a way similar to that done for the Riemann zeta function using the Euler-Maclaurin summation formula.

2. The partition function which is given by the inverse Mellin transform of $\zeta_{JC}(s,a)$ may be explicitly computed.

Incidentally, the recent work of Marcello Malagutti [13] also proves the analytic continuation of the Jaynes-Cummings spectral zeta function through the development and application of another method similar to the method of Ichinose-Wakayama[7] and Sugiyama[19].

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