Some spaces associated with multigraded rings

A thesis

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of the degree of

Doctor of Philosophy

by

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Dedicated to

My family

CERTIFICATE

Certified that the work incorporated in the thesis entitled "*Some spaces associated with multigraded rings*", submitted by *Kartik Roy* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: April 12, 2023

Dr. Vivek Mohan Mallick
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Contents

Abstract

This thesis discusses multihomogeneous spaces and their relation with T-varieties and toric varieties. Firstly, we study multihomogeneous spaces corresponding to $\mathbb{Z}^n$-graded algebras over an algebraically closed field of characteristic 0. A multihomogeneous space is a scheme associated with a graded ring where the graded group is an abelian group of finite rank. Geometrically, it is the geometric quotient of a quasi-open subscheme of the associated affine scheme by the corresponding diagonalisable group scheme. A scheme is divisorial if and only if it embeds into a multihomogeneous space. We give a criterion when a multihomogeneous space is normal. Then we mention that one could associate a sheaf with each graded module over the algebra, via a tilde construction, similar to the construction of a sheaf associated with a graded module over integer-graded rings. In doing so, we have a collection of shifted sheaves of modules associated with graded modules over algebra. As one can expect, this tilde construction is a covariant exact functor from the category of graded modules to the category of quasi-coherent sheaves of modules. We identify which shifted sheaves of modules are line bundles in terms of the graded group.

An affine T-variety is an affine scheme with an effective action of a torus. Such affine varieties can be represented by a proper polyhedral divisor over a semi projective variety. The semi projective variety is a good quotient of the action. A proper polyhedral divisor encodes a collection of ample Cartier divisors, some of which are big.

We show that for an affine T-variety, the corresponding semi projective variety and the multihomogeneous space are birational. They are generally not isomorphic due to the lack of ample divisors on the multihomogeneous space.

A toric variety is a T-variety such that the torus occurs as a dense open subscheme, and the action extends the multiplication of the torus. In toric varieties with enough invariant Cartier divisors, which includes simplicial toric varieties, points correspond to homogeneous prime ideals of a certain graded ring which Perling shows. His construction, known as tproj, reconstructs the toric variety from a graded ring where the graded group is the Picard group of the toric variety. We show that the construction of multihomogeneous space is similar to tproj; in fact, tproj, which is isomorphic to the toric variety, is an open subscheme of the multihomogeneous space associated with that graded ring. We give a criterion when a simplicial toric variety is a multihomogeneous space, and using this criterion, we classify all simplicial toric surfaces that are multihomogeneous spaces.

1

Introduction

In this chapter, we give an introduction to schemes with algebraic group actions. Toric varieties, T-varieties, and some multihomogeneous spaces are the ones we are most interested in. First, we give a short tour of the history of the above schemes and then describe our main results.

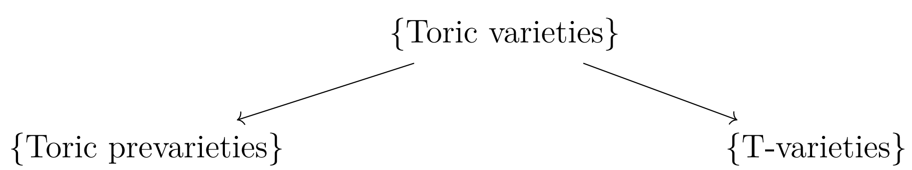
1.1 History of group actions on varieties

A normal variety with an equivariant torus embedding is called a toric prevariety. When it is separated it is called a toric variety. The category of toric varieties is a subcategory of the category of toric prevarieties. The category of toric varieties is equivalent to the category of lattice fans and the category of toric prevarieties is equivalent to the category of finite collection of compatible fans in lattice vector spaces (see [ANH]). A fan in a lattice vector space is a collection of cones in that vector space which are compatible in a certain sense. We refer [Ful], [Oda] and [CLS] for details in toric geometry.

Recall that a projective variety has a graded coordinate ring associated with it. This gives a correspondence between sheaves on the variety and graded modules on the ring. In a nice situation, coherent sheaves of modules correspond to finitely generated graded modules. Although all toric varieties are not projective they can have multiple coordinate rings attached to them. However, a priori it is not clear which one is the best coordinate ring. To remedy this Cox [Cox] came up with a coordinate ring, which is intrinsic to the toric variety, and showed that it is universal, in the sense that any other coordinate ring admits a morphism into it. In literature, this coordinate ring is known as Cox ring of the toric variety. He also showed that one can reconstruct a toric variety from its Cox ring. For example, a simplicial toric variety is a geometric quotient of a quasi-affine toric variety whose ring of global sections is the Cox ring of the original toric variety. Perling [Per] and Kajiwara [Kaj] separately showed that a toric variety with enough invariant Cartier divisors is also a geometric quotient of some quasi-affine toric variety generalising Cox's construction. Therefore we can think of the ring of global sections of the quasi-affine toric variety as a homogeneous coordinate ring of the toric variety. All these constructions are examples of a more general theory known as quotient presentation as described in [AHS1]. They classify all the quotient presentations of a toric variety up to isomorphism.

Given a category $\mathcal{C}$ there is the Yoneda embedding from $\mathcal{C}$ to $\mathrm{PSH}(\mathcal{C})$, category of presheaves of sets on $\mathcal{C}$, which sends an object $X \in \mathrm{Ob}(\mathcal{C})$ to the presheaf $\mathrm{Hom}(-, X)$. This is a fully faithful embedding and the presheaf $\mathrm{Hom}(-, X)$ uniquely determines the object X. For a toric variety X the presheaf $\mathrm{Hom}(-, X)$ was studied in [Cox], [Kaj] and [AHS1]. Eisenbud, Mustata and Stillman in [EMS] used a coordinate ring of a toric variety to compute cohomology groups of coherent sheaves.

A variety with an action of a torus is called a T-variety. A toric variety is a T-variety. There is a T-variety such that the action of the torus can not be extended to make the variety a toric variety (see [IV, example 2.10]). This makes the study of T-varieties more interesting and important. The category of T-varieties can be defined in a similar fashion as the category of toric varieties. The category of T-varieties lacks the benefits of pure combinatorial description as toric varieties due to the difference of dimensions of the variety and the torus. This difference is known as the complexity of the torus action. A toric variety is a T-variety of complexity zero. Pictorially we have the following proper subcategories



Altman and Hausen in [AH] establish a correspondence between affine T-varieties and proper polyhedral divisors (in short pp-divisors) over semi-projective varieties. A pp-divisor over a normal variety is a divisor with coefficients in the semigroup of polyhedra with a fixed tail cone. This encodes a collection of Cartier divisors with certain properties. In [AHS2] there is a way how to glue affine T-varieties to get a general T-variety and how general T-varieties correspond to divisorial fans.

Another object of interest in this thesis is the concept of a multihomogeneous space. Brenner and Schröer [BS] introduced them as a generalization of projective space where every divisorial variety embeds. These spaces correspond to multigraded rings, which are rings graded by a finitely generated abelian group. Brenner and Schröer also proved that these are simplicial torus embeddings in nice situations. Multigraded rings occur naturally in algebraic geometry, for example, as iterated blow-ups along multiple subschemes.

1.2 Main result

The results described in this thesis constitute the contents of the preprints [MR1] and [MR2].

1.2.1 T-variety and multihomogeneous space

We establish a birational morphism from a representative of a T-variety to an associated multihomogeneous space and give an isomorphism criterion. We only work with affine T-varieties over a field we describe below.

Assumption 1.2.1. *Let $D \cong \mathbb{Z}^r$ for a natural number r and suppose $A = \bigoplus_{d \in D} A_d$ be a multigraded, Noetherian, integral domain such that $A_0 = k$, where k is an algebraically closed field of characteristic* 0. $\operatorname{Proj}_{MH} A$ *is non-empty.*

$\operatorname{Proj}_{\mathrm{MH}} A$ is the multihomogeneous space corresponding to the D-grading of A (see definition 6.1.3).

We use this assumption in the following theorem.

Theorem 1.2.2 (Theorem 7.1.3)**.** *Under the assumption 1.2.1, the torus* $T = \operatorname{Spec} k[D]$

acts on $X = \operatorname{Spec} A$ *giving* X *a structure of a* T*-variety which, suppose, is represented by* $(Y, \mathfrak{D})$. *Then* Y *and* $\operatorname{Proj}_{MH} A$ *are birational.*

In the above theorem, Y is a semi-projective variety, i.e. projective over the spectrum of its ring of global sections and $\mathfrak{D}$ is a proper polyhedral(pp) divisor on Y.

Note that Y is a semi-projective variety and the multihomogeneous space $\operatorname{Proj}_{\mathrm{MH}} A$ is not a projective variety in general. Therefore, in order to have an isomorphism between them, we need the following assumption. In the following assumption ω is the weight cone of the grading in 1.2.1 and λ is a cone in the corresponding GIT fan Λ (see 5.2.7).

Assumption 1.2.3. *Suppose* $\lambda = \omega$, *i.e. the GIT fan contains only one full dimensional cone and its faces. Assume that* A *is generated by* $\bigcup_{u \in R} A_u$ *where* $R = \bigcup_{\rho \in \lambda(1)} \rho$.

Proposition 1.2.4 (Proposition 7.2.3)**.** *Assume 1.2.1 and 1.2.3. Assume that* ω *is simplicial and* A *is generated by* $\{f_\rho \,|\, \rho \in \omega(1)\}$ *such that* $\deg f_\rho \in \rho \cap D$. *Then* Y *and* $\operatorname{Proj}_{MH} A$ *are isomorphic.*

1.2.2 Toric variety and multihomogeneous space

Given a toric variety with enough invariant Cartier divisors, there exists a ring graded by the Picard group of the variety. Then one can take the multihomogeneous space corresponding to the grading. By doing so we have the following embedding.

Theorem 1.2.5 (Theorem 8.1.5)**.** *Let* X_Δ *be the toric variety with enough invariant Cartier divisors associated with the fan* Δ *and* $\operatorname{Spec}(\mathbb{C}[M])$ *its torus. Then there is a* $\operatorname{Spec}(\mathbb{C}[M])$ *equivariant open embedding* $\mu : \operatorname{tProj} A \hookrightarrow \operatorname{Proj}_{MH} A$, *where* A *is the algebra of support functions on* Δ *(defined in 4.1.10).*

The following definition is important.

Definition 1.2.6. Let Δ be a simplicial fan in $N_\mathbb{R}$ and $\Delta(1)$ be the set of rays.

1. A simplicial cone in Δ is a cone $\tau \subset N_\mathbb{R}$ generated by S, a linearly independent subset of $\Delta(1)$.

2. Δ is said to be simplicially complete if it contains every simplicial cone in Δ.

We have the following isomorphism criterion.

Theorem 1.2.7 (Theorem 8.2.5)**.** *Let X_Δ be a simplicial toric variety corresponding to the fan Δ in $N_\mathbb{R}$ satisfying 4.1.1 and A be the coordinate ring of the affine toric variety X_C (defined in 4.1.10). Then Δ is simplicially complete if and only if the morphism $\mu : \operatorname{tProj} A \to \operatorname{Proj}_{MH} A$ in 8.1.5 is an isomorphism.*

Chapter 2 is a review of combinatorial objects - cones, fans which are used in describing toric varieties and T-varieties.

Chapter 3 recalls algebraic group actions on varieties/schemes. Here we explain what an algebraic group action on a scheme means.

Chapter 4 summarises Perling's t-proj construction: This is a quotient presentation generalising Cox's construction. We also briefly talk about Kajiwara's good cone constructions.

Chapter 5 briefly describes affine T-variety in terms of proper polyhedral divisors.

Chapter 6 is devoted to multihomogeneous spaces and sheaves on them. We give a criterion for a shifted module to be a line bundle.

Chapter 7 establishes a relation between a T-variety and a multihomogeneous space associated with an affine scheme with torus action.

Chapter 8 shows how a quotient presentation of a toric variety induces a multihomogeneous space and connects it with the toric variety.

Conventions

Unless otherwise stated, we shall be working over an algebraically closed field k of characteristic 0. For us, prevarieties are integral schemes of finite type defined over k. Varieties are separated prevarieties. Toric varieties and T-varieties are assumed to be varieties in this sense. A prevariety with a torus as a dense open subscheme is a toric prevariety if the multiplication of the torus extends to an algebraic action on the prevariety.

2

Combinatorics

In the first part of this chapter, we recall cones and fans in a lattice vector space. Consult the book [CLS] for further information. Let **Ab** and $\mathbf{Vect}_{\mathbb{Q}}$ denote the categories of abelian groups and rational vector spaces respectively. There is a natural functor

$$\begin{aligned} \mathbf{Ab} &\longrightarrow \mathbf{Vect}_{\mathbb{Q}} \\ G &\longrightarrow G \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

sending an abelian group to the corresponding rational vector space.

Definition 2.0.1. A lattice N is a free abelian group of finite rank. Given a lattice N, the dual lattice of N is the group $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, which we denote by M.

For a lattice N, we denote the corresponding rational vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$ by $N_{\mathbb{Q}}$, and similarly, $M \otimes_{\mathbb{Z}} \mathbb{Q}$ by $M_{\mathbb{Q}}$. For a $\mathbb{Z}$-linear map $F : N \longrightarrow N'$ of lattices, we use the same notation $F : N_{\mathbb{Q}} \longrightarrow N'_{\mathbb{Q}}$ for the corresponding linear map of rational vector spaces. Let N be a lattice and M its dual lattice. There is a natural pairing of the lattices N and M, which we denote by

$$\begin{aligned} M \times N &\longrightarrow \mathbb{Z} \\ (u, v) &\longrightarrow \langle u, v \rangle. \end{aligned}$$

The induced pairing of rational vector spaces is denoted by

$$\begin{aligned} M_{\mathbb{Q}} \times N_{\mathbb{Q}} &\longrightarrow \mathbb{Q} \\ (u, v) &\longrightarrow \langle u, v \rangle. \end{aligned}$$

Definition 2.0.2. Let $f : N_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ be a linear functional.

1. A linear half space in $N_{\mathbb{Q}}$ is the inverse image of a linear functional f over $[0, \infty)$.
2. An affine half space in $N_{\mathbb{Q}}$ is the inverse image of a linear functional f over $[a, \infty)$ for some $a \in \mathbb{Q}$.

Linear and affine half spaces are convex and closed in $N_{\mathbb{Q}}$ equipped with Euclidean topology.

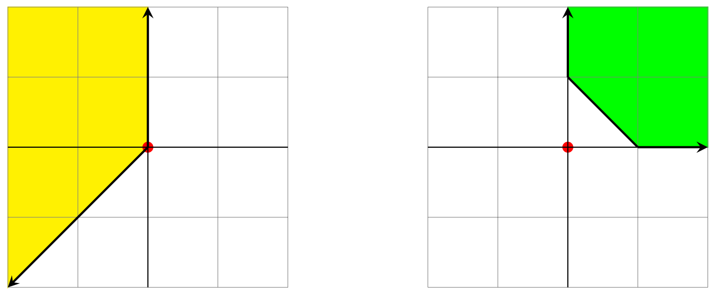
Definition 2.0.3. We define the following:

1. A polyhedral cone σ in $N_{\mathbb{Q}}$ is the intersection of finitely many linear closed half spaces. Equivalently it can be defined as the common inverse image of finitely many linear functionals f_i over $[0, \infty)$.
2. A polyhedron Δ in $N_{\mathbb{Q}}$ is the intersection of finitely many affine closed half spaces. Equivalently it can be defined as the common inverse image of finitely many linear functionals f_i over $[a_i, \infty)$.

From now on, whenever we say a cone, we mean a pointed polyhedral cone. From definition 2.0.2 it is evident that every cone is a polyhedron, and every polyhedron is convex and closed. A cone always contains the origin but a polyhedron need not contain the origin.

Definition 2.0.4. Let σ (resp. Δ) be a cone (resp. polyhedron) in $N_{\mathbb{Q}}$ and $f : N_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ be a linear functional.

1. A face of the cone σ is a subset τ such that there exists a linear functional f with $f(\sigma) \subset [0, \infty)$ and $\tau = f^{-1}(0) \cap \sigma$.
2. A face of a polyhedron Δ is a subset τ such that there exists a linear functional f with $f(\Delta) \subset [a, \infty)$ and $\tau = f^{-1}(a) \cap \sigma$.



Cone Polyhedron

Figure 2.1: Yellow represents a cone and green represents a polyhedron

3. A cone σ is pointed if 0 is a face of it.

Remark 2.0.5. *A cone is a face of itself.*

Definition 2.0.6. Let σ be a cone in $N_{\mathbb{Q}}$. The dual cone of σ is

$$\sigma^{\vee} := \{m \in M_{\mathbb{Q}} \mid \langle m, u \rangle \geq 0 \quad \forall u \in \sigma.\}$$

The dual cone $\sigma^{\vee}$ is again a polyhedral cone. Since the cone σ is pointed, the dual cone $\sigma^{\vee}$ is full dimentional. To the dual, we associate a semigroup

$$S_{\sigma} := \sigma^{\vee} \cap M.$$

It is a finitely generated saturated monoid (this follows from Gordon's lemma [CLS, proposition 1.2.17]). We denote the algebra associated with S_{σ} by A_{σ} and it has the presentation

$$A_{\sigma} := \bigoplus_{m \in S_{\sigma}} k\chi^{m}$$

with multiplication given by

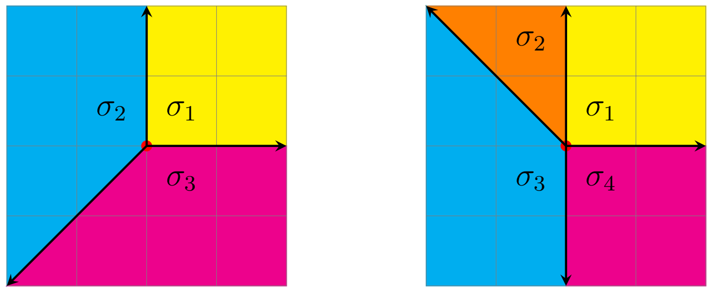
$$\chi^{m} \cdot \chi^{m'} = \chi^{m+m'}.$$

Definition 2.0.7. We define a fan as follows

1. A quasi fan Σ in $N_{\mathbb{Q}}$ is a collection of cones in $N_{\mathbb{Q}}$ such that if τ is a face of σ for some $\sigma \in \Sigma$, then $\tau \in \Sigma$ and for σ and σ' in Σ, $\sigma \cap \sigma'$ is a cone in Σ and a face of both, σ and σ'.

2. A quasi fan Σ in $N_{\mathbb{Q}}$ is called a fan if all cones $\sigma \in \Sigma$ are pointed.

3. Support of a quasi fan Σ in $N_{\mathbb{Q}}$ is the union of all cones in Σ.



Fan of projective plane $\mathbb{P}^2$ Fan of Hirzebruch surface $\mathbb{H}_n$

Figure 2.2: Fans

Definition 2.0.8. We define the following:

1. A rational cone σ is called simplicial if it is generated by a linearly independent set of vectors in $N_{\mathbb{Q}}$.

2. A cone σ is smooth if it generated by a $\mathbb{Z}$ basis of integral vectors in $N_{\mathbb{Q}}$

3. A fan is called simplicial if each cone in it is simplicial.

4. A fan is called smooth if each cone in it is smooth.

Example 2.0.9. *The fans in figure 2.2 are simplicial. While the first fan is smooth, the second fan is not.*

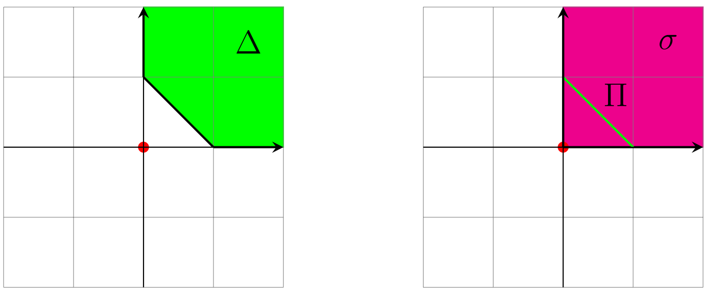
Definition 2.0.10. Let A and B be two subsets of $N_{\mathbb{Q}}$. Minkowski sum of A and B is the set $A + B := \{a + b : a \in A, b \in B\}$.

One can prove that every polyhedron Δ in $N_{\mathbb{Q}}$ can be written as a Minkowski sum decomposition $\Delta = \Pi + \sigma$ having $\Pi \subset N_{\mathbb{Q}}$, a polytope and $\sigma \subset N_{\mathbb{Q}}$, a pointed cone.

In the decomposition, σ is called the tail cone of Δ, and it is unique. In fact, given a polyhedron Δ in $N_{\mathbb{Q}}$ one computes its tail cone tail(Δ) as

$$\text{tail}(\Delta) = \{v \in N_{\mathbb{Q}}; u + av \in \Delta \text{ for all } u \in \Delta, a \in \mathbb{Q}_{\geq 0}\}$$

.



Polyhedron Minkowski decomposition

Figure 2.3: Minkowski decomposition of a polyhedron

In the Minkowski decomposition of the polyhedron Δ on the left, the polytope Π is the line segment joining $(1, 0)$ and $(0, 1)$ and the tail cone σ is the first quadrant on the right figure.

The rest of this chapter is paraphrased from [AH]. For the rest of the chapter, N denotes a lattice, M its dual. $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ denote the rational vector spaces associated with N and M respectively. σ denotes a pointed cone in $N_{\mathbb{Q}}$ and $\sigma^{\vee}$ its dual in $M_{\mathbb{Q}}$.

Definition 2.0.11.

1. A polyhedron Δ in $N_{\mathbb{Q}}$ is called a $\sigma-$polyhedron ($\sigma-$tailed polyhedron) if tail(Δ) $= \sigma$. $\text{Pol}^{+}_{\sigma}(N_{\mathbb{Q}})$ denotes the set of all $\sigma-$polyhedra in $N_{\mathbb{Q}}$.

2. A polyhedron $\Delta \in \text{Pol}^{+}_{\sigma}(N_{\mathbb{Q}})$ is called integral if in the Minkowski sum decomposition $\Delta = \Pi + \sigma$, the vertices of polytope Π consists of lattice points in N. $\text{Pol}^{+}_{\sigma}(N)$ denotes the set of all integral polyhedra in $N_{\mathbb{Q}}$.

Given $\Delta_i = \Pi_i + \sigma \in \mathrm{Pol}^+_\sigma(N_\mathbb{Q}), i = 1, 2$, their Minkowski sum is

$$\begin{aligned}\Delta_1 + \Delta_2 &= (\Pi_1 + \sigma) + (\Pi_2 + \sigma)\\ &= (\Pi_1 + \Pi_2) + \sigma\\ &= \Pi + \sigma\end{aligned}$$

where $\Pi = \Pi_1 + \Pi_2$ is again a polytope in $N_\mathbb{Q}$. Further, Δ_i are integral implying their sum is integral. In fact, $\Delta_1 + \Delta_2 = \Delta_2 + \Delta_1$ and $\Delta + \sigma = \Delta$ for $\Delta = \Pi + \sigma \in \mathrm{Pol}^+_\sigma(N_\mathbb{Q})$. Therefore, with respect to Minkowski sum $+$, $\mathrm{Pol}^+_\sigma(N_\mathbb{Q})$ and $\mathrm{Pol}^+_\sigma(N)$ are abelian monoids with identity element being σ.

Clearly the canonical map

$$\begin{aligned}\mathrm{Pol}^+_\sigma(N) &\longrightarrow \mathrm{Pol}^+_\sigma(N_\mathbb{Q})\\ \Delta &\longrightarrow \Delta\end{aligned}$$

is an injective monoid homomorphism.

Definition 2.0.12. 1. The Grothendieck group of $\mathrm{Pol}^+_\sigma(N_\mathbb{Q})$, which is generated by $\{\Delta_1 - \Delta_2; \Delta_i \in \mathrm{Pol}^+_\sigma(N_\mathbb{Q})\}$ is called the group of $\sigma-$polyhedra. Let us denote it by $\mathrm{Pol}_\sigma(N_\mathbb{Q})$.

2. The Grothendieck group of $\mathrm{Pol}^+_\sigma(N)$, which is generated by $\{\Delta_1 - \Delta_2; \Delta_i \in \mathrm{Pol}^+_\sigma(N)\}$ is called the group of integral $\sigma-$polyhedra. Let us denote it by $\mathrm{Pol}_\sigma(N)$.

The canonical map

$$\begin{aligned}\mathrm{Pol}_\sigma(N) &\longrightarrow \mathrm{Pol}_\sigma(N_\mathbb{Q})\\ \Delta &\longrightarrow \Delta\end{aligned}$$

becomes an injective group homomorphism.

Let $\mathrm{NQfan}_{\sigma^\vee}(M_\mathbb{Q})$ be the category of normal quasifans in $M_\mathbb{Q}$ with support $\sigma^\vee$. There is a functor

$$\begin{aligned}\mathrm{Pol}^+_\sigma(N_\mathbb{Q}) &\longrightarrow \mathrm{NQfan}_{\sigma^\vee}(M_\mathbb{Q})\\ \Delta &\longrightarrow \Lambda(\Delta)\end{aligned}$$

where faces $F \preceq \Delta$ correspond to cones in $\lambda(F) \in \Lambda(\Delta)$ in reverse order

$$F \longrightarrow \lambda(F) := \{u \in M_{\mathbb{Q}}; \langle u, v - v' \rangle \geq 0 \text{ for all } v \in \Delta, v' \in F\}$$

Lemma 2.0.13. *Let $\Delta \in \mathrm{Pol}^{+}_{\sigma}(N_{\mathbb{Q}})$. Then Supp$(\Lambda(\Delta)) = \sigma^{\vee}$ in $M_{\mathbb{Q}}$.*

From lemma 2.0.13 it is clear that maximal cones in $\Lambda(\Delta)$ correspond to vertices of the polyhedron Δ.

Definition 2.0.14. Let Δ be a convex set in $N_{\mathbb{Q}}$. The function

$$\begin{aligned} h_{\Delta} : M_{\mathbb{Q}} &\longrightarrow \mathbb{Q} \cup \{-\infty\} \\ u &\longrightarrow \inf_{v \in \Delta} \langle u, v \rangle \end{aligned}$$

is called the support function of Δ. The set $h_{\Delta}^{-1}(\mathbb{Q})$ in $M_{\mathbb{Q}}$ is called the domain of the support function h_{Δ}.

Lemma 2.0.15. *Let h_{Δ} be the support function associated to $\Delta \in \mathrm{Pol}^{+}_{\sigma}(N_{\mathbb{Q}})$.*

1. *The domain of h_{Δ} is $\sigma^{\vee}$, and h_{Δ} is linear on λ for all $\lambda \in \Lambda(\Delta)$.*

2. *h_{Δ} is convex: This means the function h_{Δ} satisfies the following*

$$h_{\Delta}(u) + h_{\Delta}(v) \leq h_{\Delta}(u + v)$$

for all $u, v \in \sigma^{\vee}$.
Moreover, strict inequality holds if and only if u and v lies in different maximal cones of $\Lambda(\Delta)$.

Definition 2.0.16. Let ω be a cone in $M_{\mathbb{Q}}$ and $h : M_{\mathbb{Q}} \longrightarrow \mathbb{Q} \cup \{-\infty\}$ be a function with domain ω. We call h piecewise linear if there exists a quasifan Λ in $M_{\mathbb{Q}}$ such that following hold

1. $\mathrm{Supp}(\Lambda) = \omega$ in $M_{\mathbb{Q}}$ and

2. $h|_{\lambda} : \lambda \longrightarrow \mathbb{Q}$ are linear maps for all $\lambda \in \Lambda$.

Let $\mathrm{CPL}_{\mathbb{Q}}(\omega)$ denote the set of convex piecewise linear functions on $M_{\mathbb{Q}}$ with domain ω. Then $\mathrm{CPL}_{\mathbb{Q}}(\omega)$ is an abelian monoid under pointwise addition.

Proposition 2.0.17. *The map* $\mathrm{Pol}_\sigma^+(N_\mathbb{Q}) \longrightarrow CPL_\mathbb{Q}(\sigma^\vee)$*, which sends* Δ *to* h_Δ *is an isomorphism of abelian semigroups.*

Proposition 2.0.18. *The following statements hold:*

1. *There is a commutative diagram of monoids*

$$\begin{array}{ccc} \mathrm{Pol}_\sigma^+(N) & \longrightarrow & \mathrm{Pol}_\sigma^+(N_\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Pol}_\sigma(N) & \longrightarrow & \mathrm{Pol}_\sigma(N_\mathbb{Q}) \end{array}$$

   where all arrows are canonical injective homomorphisms.

2. *For each* $\alpha \in \mathbb{Q}_{\geq 0}$ *and each* $\Delta \in \mathrm{Pol}_\sigma^+(N_\mathbb{Q})$*, define*

$$\alpha\Delta := \{\alpha v; v \in \Delta\}.$$

   Then this multiplication has unique extension to $\mathrm{Pol}_\sigma(N_\mathbb{Q})$ *and the induced scalar multiplication*

$$\mathbb{Q} \times \mathrm{Pol}_\sigma(N_\mathbb{Q}) \longrightarrow \mathrm{Pol}_\sigma(N_\mathbb{Q})$$

   makes $\mathrm{Pol}_\sigma(N_\mathbb{Q})$ *a rational vector space.*

3. *The group* $\mathrm{Pol}_\sigma(N)$ *is free abelian, and we have a canonical isomorphism of rational vector spaces*

$$\mathbb{Q} \otimes_\mathbb{Z} \mathrm{Pol}_\sigma(N) \longrightarrow \mathrm{Pol}_\sigma(N_\mathbb{Q}),$$

   induced by scalar multiplication mentioned in the statement (2).

4. *Each* $u \in \sigma^\vee$ *induces a unique linear functional, denoted by* $eval_u : \mathrm{Pol}_\sigma(N_\mathbb{Q}) \longrightarrow \mathbb{Q}$ *which satisfies the following*

$$eval_u(\Delta) = min_{v \in \Delta}\langle u, v\rangle, \text{ if } \Delta \in \mathrm{Pol}_\sigma^+(N_\mathbb{Q}).$$

5. *Two* $\sigma-$*polyhedra* $\Delta_1, \Delta_2 \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$ *are same if and only if*

$$eval_u(\Delta_1) = eval_u(\Delta_2), \text{ for all } u \in \sigma^\vee.$$

6. *A* $\sigma-$*polyhedron* $\Delta \in \mathrm{Pol}_\sigma(N_\mathbb{Q})$ *is integral if and only if all evaluations* $eval_u(\Delta)$ *are integer for all* $u \in \sigma^\vee \cap M$.

3

Algebraic group action

In this section, we define algebraic groups and their action on schemes. For a detailed exposition, see the book [Mil]. We restrict our attention to linear algebraic groups which are affine varieties with group structure given by morphisms of varieties.

Definition 3.0.1. Let G be a variety. Then G is called an algebraic group if there exist morphisms $\sigma : G \times G \to G$, $i : G \to G$, and $e : \operatorname{Spec} k \to G$, called multiplication, inversion and identity respectively, of varieties such that following diagrams are commutative:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{id \times \sigma} & G \times G \\ \downarrow{\scriptstyle \sigma \times id} & & \downarrow{\scriptstyle \sigma} \\ G \times G & \xrightarrow{\sigma} & G \end{array} \tag{3.1}$$

$$\begin{array}{ccccc} G & \xrightarrow{id \times i} & G \times G & \xleftarrow{i \times id} & G \\ \downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{e} & G & \xleftarrow{e} & \operatorname{Spec} k \end{array} \tag{3.2}$$

$$\begin{array}{ccccc} \operatorname{Spec} k \times G & \xrightarrow{e \times id} & G \times G & \xleftarrow{id \times e} & G \times \operatorname{Spec} k \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \sigma} & \swarrow{\scriptstyle \cong} & \\ & & G & & \end{array} \tag{3.3}$$

Remark 3.0.2. *Whenever way say G is an algebraic group the associated morphisms are understood.*

Definition 3.0.3. Let G be an algebraic group. Then G is a linear algebraic group if it is an affine variety.

Let G be a linear algebraic group and $A(G) := \Gamma(G, \mathscr{O}_G)$ the algebra of global sections (global regular functions). Then the groups operations σ, i, and e in 3.1, 3.2, and 3.3 correspond to algebra homomorphisms σ^*, i^* and e^* known as comultiplication, coinversion and coidentity respectively. They satisfy the following commutative diagrams

$$\begin{array}{ccc} A(G) \otimes_k A(G) \otimes_k A(G) & \xleftarrow[id\otimes\sigma^*]{} & A(G) \otimes_k A(G) \\ {\scriptstyle \sigma^*\otimes id}\uparrow & & {\scriptstyle \sigma^*}\uparrow \\ A(G) \otimes_k A(G) & \xleftarrow[\sigma^*]{} & A(G) \end{array} \tag{3.4}$$

$$\begin{array}{ccccc} A(G) & \xleftarrow[id\otimes i^*]{} & A(G) \otimes_k A(G) & \xrightarrow{i^*\otimes id} & A(G) \\ \uparrow & & {\scriptstyle \sigma^*}\uparrow & & \uparrow \\ \operatorname{Spec} k & \xleftarrow[e^*]{} & A(G) & \xrightarrow{e^*} & \operatorname{Spec} k \end{array} \tag{3.5}$$

$$\begin{array}{ccccc} k \otimes_k A(G) & \xleftarrow[e^*\otimes id]{} & A(G) \otimes_k A(G) & \xrightarrow{id\otimes e^*} & A(G) \otimes_k k \\ & {\scriptstyle \cong}\nwarrow & {\scriptstyle \sigma^*}\uparrow & \nearrow{\scriptstyle \cong} & \\ & & A(G) & & \end{array} \tag{3.6}$$

Remark 3.0.4. *$A(G)$ with σ^*, i^*, and e^* forms a Hopf algebra.*

Example 3.0.5. *Here are some simple and useful algebraic groups defined over k.*

1. *$\mathbb{G}_m(k) = k^*$ is a linear algebraic group with $A(\mathbb{G}_m) = k[T^{\pm 1}]$.*

2. *$T = (\mathbb{G}_m)^n(k)$ known as torus, is a linear algebraic group with $A(\mathbb{G}_m) = k[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.*

3. *$G = GL_n(k)$ is a linear algebraic group, where*

$$A(GL_n) = k[\{T_{i,j} | 1 \leq i, j \leq n\} \cup \{S\}]/(S \det((T_{i,j})) - 1).$$

Now we define what a morphism between algebraic groups means.

Definition 3.0.6. Let G_1 and G_2 be algebraic groups. A group homomorphism is a morphism $\pi : G_1 \to G_2$ of varieties such that the following diagram is commutative

$$\begin{CD} G_1 \times G_1 @>{\sigma_1}>> G_1 \\ @V{\pi\times\pi}VV @VV{\pi}V \\ G_2 \times G_2 @>{\sigma_2}>> G_2 \end{CD}$$

where $\sigma_i : G_i \to G_i$ are group multiplications.

Algebraic groups form a category where morphisms are group homomorphisms.

Definition 3.0.7. Let G be an algebraic group with group multiplication σ. G is called abelian if $\sigma(g, h) = \sigma(h, g)$ for all $g, h \in G$.

Definition 3.0.8. Given an algebraic group G, we define the following:

1. The set of group homomorphisms from G to $\mathbb{G}_m(k)$ forms a group. It is called the character group of G and is denoted by $\chi(G)$.

2. The set of group homomorphisms from $\mathbb{G}_m(k)$ to G also forms a group. It is called cocharacter group of G and is denoted by $\lambda(G)$.

Remark 3.0.9. *Let G be a torus (see 3.0.5, example 2). Then the groups $\chi(G)$ and $\lambda(G)$ are finitely generated (see [Ful, section 2.3]).*

Definition 3.0.10. A k algebra R is called a Hopf algebra if it is equipped with three algebra homomorphisms σ^*, i^*, and e^* such that all diagrams in equations 3.4, 3.5, and 3.6 commutative.

Now we define group action on affine varieties.

Definition 3.0.11. Let G be an algebraic group and X a variety. An algebraic action of G on X is a morphsim $\psi : G \times X \to X$ of varieties such that the following diagram is commutative

$$\begin{CD} G \times G \times X @>{id\times\psi}>> G \times X \\ @V{\psi\times id}VV @VV{\psi}V \\ G \times X @>{\psi}>> X. \end{CD} \tag{3.7}$$

Assume G is linear and $X = \operatorname{Spec} A$ affine. Then the action ψ is a morphism of affine varieties and therefore it corresponds to an algebra homomorphism $\psi^* : A \to A(G) \otimes_k A$. The $\chi(G)$ grading on $A(G)$ induces an $\chi(G)$ grading on the algebra $A(G) \otimes_k A$ and pulling it back by ψ^*, we get an $\chi(G)$ grading on A. Therefore we have a decomposition

$$A = \bigoplus_{m \in \chi(G)} A_m$$

of $\chi(G)$ invariant k vector subspaces.

Definition 3.0.12. Let G be a linear algebraic group and X an affine variety. Assume G acts algebraically on X and $A = \Gamma(X, \mathscr{O}_X)$. For a point $x \in X$ we define the following:

1. The weight monoid $S(x)$ of x is the monoid $\{m \in \chi(G) : \exists f \in A_m \text{ such that } f(x) \neq 0\}$.
2. The weight cone $\omega(x)$ of x is the cone generated by $S(x)$ in the vector space $\chi(G)_{\mathbb{Q}}$.
3. The weight cone ω is the cone generated by $\{m \in \chi(G) : A_m \neq 0\}$ in the vector space $\chi(G)_{\mathbb{Q}}$.
4. The stabilizer group G_x of x is the subgroup $\{g \in G : g \cdot x = x\}$ of G.
5. We say the action is effective if for some $g \in G$, $g \cdot x = x$ for all $x \in X$ then $g = e$, the identity of G.

There is a relation between effective actions and the weight cone.

Fact 3.0.13. *Let G be a torus and X an affine variety. Assume G acts algebraically on X. Then the action is effective if and only if the weight cone ω has full dimension in the vector space $\chi(G)_{\mathbb{Q}}$.*

Proposition 3.0.14. *[DG, §2, Proposition 1.7] Let G be a linear algebraic group and $X = \operatorname{Spec} A$ an affine variety. Then there is a one-one correspondence between G actions on X and $\chi(G)$-graded decomposition of the k-algebra A into vector subspaces.*

4

Toric variety

A toric variety is a normal variety X with an open torus T such that the multiplication on the torus T extends to an algebraic group action on the variety X. Here, by a torus, we mean $\mathbb{G}_m^n$ (see example 3.0.5(2)). If $\iota : T \to X$ denotes the open embedding, $\sigma_T : T \times T \to T$ the group multiplication on the torus, and $\sigma : T \times X \to X$ the group action on the variety X, then we have the following commutative diagram,

$$\begin{array}{ccc} T \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle Id\times\iota}\uparrow & & \uparrow{\scriptstyle \iota} \\ T \times T & \xrightarrow{\sigma_T} & T \end{array}$$

A toric morphism between toric varieties (X_1, T_1) and (X_2, T_2) is a morphism $f : X_1 \to X_2$ of varieties which is compatible with the tori actions. That means, $f(T_1) \subset T_2$ and we denote the restriction morphism $T_1 \to T_2$ by the same f. Then we have the following commutative diagram,

$$\begin{array}{ccc} T_1 \times X_1 & \xrightarrow{\sigma_1} & X_1 \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\ T_2 \times X_2 & \xrightarrow{\sigma_2} & X_2 \end{array}$$

From now on we suppress the torus, and whenever we say a toric variety the corresponding torus is understood. For toric varieties X and Y, we denote the set of toric morphisms from X to Y by $\mathrm{Hom}_T(X, Y)$, and call it toric hom. The category of toric varieties consists of toric varieties as objects and morphisms are toric hom sets.

In this chapter, all cones are strongly convex rational polyhedral. Fix a lattice N and its dual lattice M. Let $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ be corresponding real vector spaces associated with the lattices. Let σ be a cone in $N_{\mathbb{R}}$ and $\sigma^\vee$ its dual cone in $M_{\mathbb{R}}$. Then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated saturated semigroup and the associated algebra $A_\sigma = k[S_\sigma]$ is an M-graded k-algebra of finite type. For an element $m \in S_\sigma$ we denote the corresponding element in A_σ by χ^m. Then we have the affine scheme $U_\sigma := \operatorname{Spec} A_\sigma$.

$$\sigma \rightsquigarrow \sigma^\vee \rightsquigarrow S_\sigma \rightsquigarrow A_\sigma \rightsquigarrow U_\sigma.$$

Each $m \in M_{\mathbb{R}}$ induces a linear map $\langle m, \cdot \rangle : N_{\mathbb{R}} \to \mathbb{R}$. A cone $\tau \subset N_{\mathbb{R}}$ is a face of σ if and only if there exists an $m \in \sigma^\vee$ such that $\tau = \sigma \cap \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\}$. We can take m to be integral, i.e., $m \in M$. Then we have $\tau^\vee = \sigma^\vee + m\mathbb{Z}$ and $S_\tau = S_\sigma + m\mathbb{Z}$. This gives us $A_\tau = (A_\sigma)_{\chi^m}$. By taking the prime spectrum we get $U_\tau = \operatorname{Spec} A_\tau$ is an open subscheme of $U_\sigma = \operatorname{Spec} A_\sigma$. The face 0 of σ corresponds to the semigroup $S_0 = M$ and the affine scheme $U_0 = \operatorname{Spec} k[M]$, which is a torus. By the previous argument, it is an open subscheme of U_σ. Furthermore, we have the commutative diagram of semigroups

$$\begin{array}{ccc} S_\sigma & \longrightarrow & M \oplus S_\sigma \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \oplus M \end{array} \qquad \begin{array}{ccc} m & \longrightarrow & (m, m) \\ \downarrow & & \downarrow \\ m & \longrightarrow & (m, m), \end{array}$$

satisfying certain properties. It gives rise to the following commutative diagram of k-algebras

$$\begin{array}{ccc} & & k[M \oplus S_\sigma] \\ & & \| \\ k[S_\sigma] & \longrightarrow & k[M] \otimes_k k[S_\sigma] \\ \downarrow & & \downarrow \\ k[M] & \longrightarrow & k[M] \otimes_k k[M] \\ & & \| \\ & & k[M \oplus M] \end{array} \qquad \begin{array}{ccc} \chi^m & \longrightarrow & (\chi^m \otimes \chi^m) \\ \downarrow & & \downarrow \\ \chi^m & \longrightarrow & (\chi^m \otimes \chi^m), \end{array}$$

Which satisfies the properties of a Hopf comultiplication. Therefore, the prime spectrum of the above diagram gives us an algebraic group action as follows

$$\begin{array}{ccc} T \times U_\sigma & \longrightarrow & U_\sigma \\ \uparrow & & \uparrow \\ T \times T & \longrightarrow & T \end{array}$$

where $T = U_0 = \operatorname{Spec} k[M]$ is a torus. This makes U_σ an affine toric variety.

Let Δ be a fan in $N_\mathbb{R}$. For $\sigma_1, \sigma_2 \in \Delta$ we have $\tau = \sigma_1 \cap \sigma_2 \in \Delta$. Then U_τ is an open subscheme of both U_{σ_1} and U_{σ_2}. By patching the affine toric varieties $U_\sigma, \sigma \in \Delta$ along the open subschemes given by common faces we get the toric variety X_Δ.

4.1 Perling's construction (tProj)

This section is a review of Perling's reconstruction [Per] of a toric variety X from a multigraded ring associated with X via a generalization of the Proj construction, which he denotes by tProj.

A toric variety X is a normal variety with an action of a dense open torus such that the torus action on the variety restricts to the multiplication on the torus. The category of toric varieties is equivalent to the category of pairs (N, Δ) where N is a lattice (free abelian group of finite rank) and Δ is a fan in the real vector space $N_\mathbb{R}$. The toric variety corresponding to Δ will be denoted by X_Δ. We denote the $\mathbb{Z}$-dual of N by M . See [Ful] and [CLS] for the results on toric varieties used here.

We have the following assumption on the fans associated with the toric varieties. Recall that support of a fan is the union of all its cones.

Assumption 4.1.1. *The support of Δ generates the vector space $N_\mathbb{R}$.*

Define $\Delta(1) := \{\rho \in \Delta : \dim(\rho) = 1\}$ the set of 1-dimensional cones in Δ and we denote n_ρ the primitive element of ρ. For each $\rho \in \Delta(1)$, we denote the associated T-invariant prime Weil divisor on X_Δ by D_ρ. $\mathrm{CDiv}_\mathrm{T}(X_\Delta)$ denotes the free abelian group generated by T-invariant Cartier divisors on X_Δ. We further assume the following

Assumption 4.1.2. *The Picard group* $\mathrm{Pic}(\Delta)$ *of X_Δ is free.*

The Picard group fits into an exact sequence (see [Ful, section 3.4] or [Oda]),

$$1 \to M \xrightarrow{\text{Div}} \text{CDiv}_{\text{T}}(\Delta) \xrightarrow{\deg} \text{Pic}(\Delta) \to 1.$$

Definition 4.1.3. A Δ-linear support function h is a function from the support $|\Delta|$ to $\mathbb{R}$ such that h is linear on each cone $\sigma \in \Delta$ and sends integral points in $|\Delta|$ to $\mathbb{Z}$.

We denote by $\text{SF}(\Delta)$ the free abelian group of finite rank of Δ-linear support functions. It is well known (see [Ful, section 3.4]) that such functions correspond to the Cartier divisors. Therefore $\text{CDiv}_{\text{T}}(\Delta) \cong \text{SF}(\Delta)$ is an isomorphism of abelian groups. Denote the real vector space of support functions by $\text{SF}(\Delta)_{\mathbb{R}} := \text{SF}(\Delta) \otimes_{\mathbb{Z}} \mathbb{R}$. There is a split short exact sequence of abelian groups

$$1 \to M \xrightarrow{\text{Div}} \text{SF}(\Delta) \xrightarrow{\deg} \text{Pic}(\Delta) \to 1, \tag{4.1}$$

and a corresponding exact sequence of tori

$$1 \to G \to \tilde{T} \to T \to 1,$$

where $G := \text{Spec}\,\mathbb{C}[\text{Pic}(\Delta)], \tilde{T} := \text{Spec}\,\mathbb{C}[\text{SF}(\Delta)]$ and $T := \text{Spec}\,k[M]$.

Definition 4.1.4. For a support function $h \in \text{SF}(\Delta)_{\mathbb{R}}$, we define its support as $|h| := \{\rho \in \Delta(1) : h(n_\rho) \neq 0\}$, where n_ρ is the primitive element of ρ.

For each ray $\rho \in \Delta(1)$, the spaces $H_\rho := \{h \in \text{SF}(\Delta)_{\mathbb{R}} : h(n_\rho) \geq 0\} \subset \text{SF}(\Delta)_{\mathbb{R}}$ are half spaces [Per, proposition 3.2] whose boundaries ∂H_ρ are rational hyperplanes.

Our results, which depend on Perling and Kajiwara's results, require the condition of having enough invariant Cartier divisors, which we define now.

Definition 4.1.5. [Per, proposition 3.3] A toric variety X_Δ, corresponding to a fan Δ in a lattice N, *has enough invariant Cartier divisors* if for each $\sigma \in \Delta$ there exists an effective T-invariant Cartier divisor whose support is precisely the union of D_ρ for $\rho \notin \sigma(1)$.

Remark 4.1.6. *For cone σ, if D is such a Cartier divisor and it corresponds to the support function h then $|h| = \Delta(1) \setminus \sigma(1)$. This definition agrees with that of Kajiwara [Kaj, definition 1.5] in terms of good cones.*

Example 4.1.7. *Simplicial toric varieties have enough invariant Cartier divisors (see [Cox, lemma 3.4]).*

Assumption 4.1.8. *We assume the toric variety X_Δ has enough invariant Cartier divisors.*

The set $C = \left(\bigcap_{\rho\in\Delta(1)} H_\rho\right)^\vee$ is a pointed strongly convex rational polyhedral cone in $\mathrm{SF}(\Delta)_\mathbb{R}{}^\vee$. For each $\rho \in \Delta(1)$, let l_ρ be the primitive element of the ray orthogonal to H_ρ in $\mathrm{SF}(\Delta)_\mathbb{R}{}^\vee$ which takes positive values on C. Define $\hat\sigma$ to be the cone generated by the l_ρ corresponding to the rays in σ, i.e. $\langle\{l_\rho \,|\, \rho \in \sigma(1)\}\rangle$. Then $\hat\Delta = \{\hat\sigma \,|\, \sigma \in \Delta\}$ is a subfan of C (see proof of [Per, proposition 3.6]) and the map of fans $(\mathrm{SF}(\Delta)_\mathbb{R}{}^\vee, \hat\Delta) \longrightarrow (N_\mathbb{R}, \Delta)$ is surjective. This induces the following diagram of toric morphisms

$$\begin{array}{ccc} X_{\hat\Delta} & \xrightarrow{\phi} & X_C \\ {\scriptstyle\pi}\big\downarrow & & \\ X_\Delta & & \end{array}$$

where ϕ is an open immersion and π is a quotient presentation in the sense of [AHS1].

Remark 4.1.9 (cf. remark 3.8 of [Per]). *When the fan Δ is simplicial then the quotient presentation $\pi : X_{\hat\Delta} \to X_\Delta$ is same as described by Cox [Cox]. Further $\hat\Delta$ is simplicial if and only if Δ is simplicial.*

For further properties of Cox rings consult the book [ADHL].

Notation 4.1.10. The coordinate ring $A := \mathbb{C}[C^\vee \cap \mathrm{SF}(\Delta)]$ of the affine toric variety X_C is $\mathrm{Pic}(\Delta)$-graded $\mathbb{C}$-algebra given by the homomorphism deg in 4.1. For a support function $h \in \mathrm{SF}(\Delta)$ we denote $\chi(h)$ the corresponding homogeneous element in A. Let ω be the corresponding weight cone in the real vector space $\mathrm{Pic}(\Delta)_\mathbb{R}$.

Recall that $X_{\hat\Delta} = \bigcup_{\hat\sigma\in\hat\Delta} X_{\hat\sigma}$. Let B_σ be the defining ideal for $X_C \setminus X_{\hat\sigma}$ and $B := \sum_{\sigma\in\Delta} B_\sigma \subset A$. Then, B has codimension at least 2 in A and $V(B) = X_C \setminus X_{\hat\Delta}$ proving that $\Gamma(X_{\hat\Delta}, \mathscr{O}_{X_{\hat\Delta}}) = A$. Furthermore, the dual action of G on A induces the isotypical decomposition

$$A = \bigoplus_{\alpha\in\mathrm{Pic}(\Delta)} A_\alpha.$$

Under the order reversing correspondence between faces of C and faces of $C^\vee$, $\hat{\sigma} \in \hat{\Delta}$ corresponds to $\Sigma_\sigma := C^\vee \cap \hat{\sigma}^\perp$. For each $\hat{\sigma} \in \hat{\Delta}$, we fix a support function $h_\sigma \in \operatorname{rel\,int}(\Sigma_\sigma)$. Then

$$|h_\sigma| = \Delta(1) \setminus \sigma(1).$$

For each $\sigma \in \Delta$, let $\chi(h_\sigma) \in A$ be the element corresponding to h_σ. Let $A_{(\chi(h_\sigma))}$ be the subring of degree zero elements in $\operatorname{Pic}(\Delta)$-graded localized ring $A_{\chi(h_\sigma)}$. Then we have $A^G_{\chi(h_\sigma)} \cong A_{(\chi(h_\sigma))} \cong k[\sigma_M]$ where $\sigma_M := \sigma^\vee \cap M$ for $\sigma \in \Delta$ [Per, lemma 3.10]. Thus, (X_Δ, π) is a categorical quotient of the action of G on $X_{\hat{\Delta}}$ [Per, proposition 3.11].

As a topological space,

$$\operatorname{tProj} A = \{\wp \in X_{\hat{\Delta}} \mid \wp \text{ is a homogeneous prime ideal in } A\}.$$

Let $i\colon \operatorname{tProj} A \longrightarrow X_{\hat{\Delta}}$ be the canonical embedding. One constructs a sheaf of rings $\mathscr{O}'$ on X_C which for an open set U, is the ring $\mathscr{O}'(U)$ consisting of those sections of $\mathscr{O}(U)$, which locally are fractions of homogeneous elements of degree 0. Perling [Per, definition 3.15] defines $\mathscr{O}_{\operatorname{tProj} A} = i^{-1}\mathscr{O}'$ on $\operatorname{tProj} A$. The ringed space $\operatorname{tProj} A = (\operatorname{tProj} A, \mathscr{O}_{\operatorname{tProj} A})$ is called the toric proj of A, and Perling proves that this is indeed a scheme. He shows that if $\tau < \sigma$ in Δ and h_τ and h_σ are support functions vanishing on $\tau(1)$ and $\sigma(1)$, respectively then we have the following diagram (see [Per, theorem 3.18])

$$\begin{array}{ccccccc}
\operatorname{tProj} A_{(\chi(h_\tau))} & \xrightarrow{\cong} & D_+(\chi(h_\tau)) & \xrightarrow{\cong} & \operatorname{Spec} A_{(\chi(h_\tau))} & \xrightarrow{\cong} & \operatorname{Spec} k[\tau_M] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{tProj} A_{(\chi(h_\sigma))} & \xrightarrow{\cong} & D_+(\chi(h_\sigma)) & \xrightarrow{\cong} & \operatorname{Spec} A_{(\chi(h_\sigma))} & \xrightarrow{\cong} & \operatorname{Spec} k[\sigma_M].
\end{array} \tag{4.2}$$

This establishes that $\operatorname{tProj} A$ is a scheme and is isomorphic to the toric variety. In fact, it presents X_Δ as a geometric quotient of $X_{\hat{\Delta}}$ by G [Per, theorem 3.19].

The advantage of this presentation is that there is an essentially surjective functor from the category of $\operatorname{Pic}(\Delta)$-graded A-modules to quasi-coherent $\mathscr{O}_{\operatorname{tProj}}$-modules. More explicitly, for a graded A-module F, the $\widetilde{(\ )}$ construction [Per, definition 3.21] gives a quasi-coherent $\mathscr{O}_{\operatorname{tProj}}$-module $\widetilde{F}$ on $\operatorname{tProj} A$. For each $\alpha \in \operatorname{Pic}(\Delta)$, we have the graded A-modules $A(\alpha)$ with decomposition $A(\alpha) := \oplus_{\beta \in \operatorname{Pic}(\Delta)} A(\alpha)_\beta$ where $\beta \in \operatorname{Pic}(\Delta)$ and $A(\alpha)_\beta := A_{\alpha+\beta}$. The associated quasi-coherent $\mathscr{O}_{\operatorname{tProj}}$-module $\mathscr{O}_{\operatorname{tProj} A}(\alpha) := \widetilde{A(\alpha)}$ is an invertible sheaf [Kaj, proposition 2.6(1)] for each $\alpha \in \operatorname{Pic}(\Delta)$. Like projective spaces, for

$\alpha, \beta \in \mathrm{Pic}(\Delta)$ we have the isomorphisms $\mathscr{O}_{\mathrm{tProj}\,A}(\alpha+\beta) \cong \mathscr{O}_{\mathrm{tProj}\,A}(\alpha) \otimes_{\mathscr{O}_{\mathrm{tProj}\,A}} \mathscr{O}_{\mathrm{tProj}\,A}(\beta)$ [Kaj, corollary 2.8] and $\widetilde{F(\alpha)} \cong \widetilde{F} \otimes_{\mathscr{O}_{\mathrm{tProj}\,A}} \mathscr{O}_{\mathrm{tProj}\,A}(\alpha)$ for a graded A-module F [Kaj, corollary 2.9].

On the other hand, for a quasi-coherent $\mathscr{O}_{\mathrm{tProj}}$-module $\mathscr{F}$, there is a graded A-module $\Gamma_*(\mathrm{tProj}\,A, \mathscr{F}) := \bigoplus_{\alpha \in \mathrm{Pic}(\Delta)} \Gamma(\mathrm{tProj}\,A, \mathscr{F}(\alpha))$ where $\mathscr{F}(\alpha) := \mathscr{F} \otimes_{\mathscr{O}_{\mathrm{tProj}\,A}} \mathscr{O}_{\mathrm{tProj}\,A}(\alpha)$ with $\widetilde{\Gamma_*(\mathrm{tProj}\,A, \mathscr{F})} \cong \mathscr{F}$. The functors $\widetilde{(\ )}$ and $\Gamma_*(\mathrm{tProj}\,A, \_)$ are adjoint functors while the former one is left adjoint, and the later one is right adjoint (see [Kaj]).

5

T-varieties

In this section, we study varieties equipped with an action of an algebraic group. The algebraic groups we mostly consider in this section and throughout this thesis are tori, denoted by $T := \mathbb{G}_m^n(k)$. The $k-$algebra of global sections of the torus T is denoted by $L := \Gamma(T, \mathscr{O}_T)$, where L means Laurent polynomials.

Now we define affine T-varieties.

Definition 5.0.1. An affine variety $X = \operatorname{Spec} A$ with an effective action of a torus T is called a T-variety.

The number $\dim(X) - \dim(T)$ is called the complexity of the torus action. T-varieties of complexity zero are toric varieties.

5.1 Varieties associated with proper polyhedral divisors

Throughout this chapter Y denotes a normal variety. We fix a lattice N and its dual lattice M. $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ denote the associated rational vector spaces. We also fix a strongly convex rational polyhedral cone σ in $N_{\mathbb{Q}}$ and its dual cone $\sigma^{\vee}$ in $M_{\mathbb{Q}}$.

Let $\operatorname{Div}(Y)$ and $\operatorname{CaDiv}(Y)$ denote the groups of Weil and Cartier divisors on Y respectively. Also, we denote the rational vector spaces of Weil and Cartier divisors by

$\mathrm{Div}_{\mathbb{Q}}(Y)$ and $\mathrm{CaDiv}_{\mathbb{Q}}(Y)$. Since Y is normal, we have the following commutative diagram of groups

$$\begin{array}{ccc} \mathrm{CaDiv}(Y) & \longrightarrow & \mathrm{Div}(Y) \\ \downarrow & & \downarrow \\ \mathrm{CaDiv}_{\mathbb{Q}}(Y) & \longrightarrow & \mathrm{Div}_{\mathbb{Q}}(Y), \end{array}$$

where all morphisms are inclusions.

Let $V \subset Y$ be an open subvariety and $D \in \mathrm{Div}_{\mathbb{Q}}(Y)$ be a rational Weil divisor on Y. Then the sheaf of sections $\mathscr{O}_D$ over V is defined as follows:

$$\Gamma(V, \mathscr{O}(D)) := \{f \in K(Y) : \mathrm{div}(f)|_V + D|_V \geq 0\} \cup \{0\} = \Gamma(V, \mathscr{O}(\lfloor D \rfloor)) \cup \{0\}.$$

If $f \in \Gamma(Y, \mathscr{O}(D))$ is a global section, then we define the vanishing locus of f as

$$Z(f) := \mathrm{Supp}(\mathrm{div}(f) + D)$$

and non-vanishing locus of f as

$$Y_f := Y \setminus Z(f).$$

We use the non-vanishing loci of global sections of Weil divisors to define the following.

Definition 5.1.1. Let $D \in \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ be a Cartier divisor on Y. Then D is called *semiample* if there exists a positive integer n such that the non-vanishing loci Y_f, $f \in \Gamma(Y, \mathscr{O}(nD))$, is an open covering of Y.

Definition 5.1.2. Let $D \in \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ be a Cartier divisor on Y. Then D is called *big* if there exists a positive integer n and a section $f \in \Gamma(Y, \mathscr{O}(nD))$ such that the non-vanishing locus Y_f is affine.

Semi-ample and big divisors are important in defining proper polyhedral divisors or pp-divisors. We define pp-divisors later.

Definition 5.1.3 ([AH], definition 2.3)**.** (i) The groups of rational polyhedral Weil divisors and rational polyhedral Cartier divisors of Y with respect to σ are

$$\mathrm{Div}_{\mathbb{Q}}(Y,\sigma) := \mathrm{Pol}_{\sigma}(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{Div}(Y)$$
$$\mathrm{CaDiv}_{\mathbb{Q}}(Y,\sigma) := \mathrm{Pol}_{\sigma}(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y)$$

respectively.

(ii) The groups of integral polyhedral Weil divisors and rational polyhedral Cartier divisors of Y with respect to σ are

$$\mathrm{Div}(Y,\sigma) := \mathrm{Pol}_{\sigma}(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{Div}(Y)$$
$$\mathrm{CaDiv}(Y,\sigma) := \mathrm{Pol}_{\sigma}(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y)$$

respectively.

Whenever we say polyhedral divisors we mean rational polyhedral divisors. Integral polyhedral divisors will be explicitly expressed. We recall the following important proposition.

Proposition 5.1.4 ([AH],proposition 2.4)**.** *(i)* $\mathrm{Div}(Y,\sigma)$ *and* $\mathrm{CaDiv}(Y,\sigma)$ *are free abelian groups, and* $\mathrm{Div}_{\mathbb{Q}}(Y,\sigma)$ *and* $\mathrm{CaDiv}_{\mathbb{Q}}(Y,\sigma)$ *are rational vector spaces.*

(ii) We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{CaDiv}(Y,\sigma) & \longrightarrow & \mathrm{Div}(Y,\sigma) \\ \downarrow & & \downarrow \\ \mathrm{CaDiv}_{\mathbb{Q}}(Y,\sigma) & \longrightarrow & \mathrm{Div}_{\mathbb{Q}}(Y,\sigma), \end{array}$$

where all morphisms are canonical inclusions. In fact, we have the isomorphisms

$$\mathrm{Div}_{\mathbb{Q}}(Y,\sigma) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Div}(Y,\sigma),$$
$$\mathrm{CaDiv}_{\mathbb{Q}}(Y,\sigma) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y,\sigma).$$

(iii) For every $u \in \sigma^\vee$, there is a linear evaluation functional

$$\begin{aligned} \mathrm{Div}_{\mathbb{Q}}(Y,\sigma) &\longrightarrow \mathrm{Div}_{\mathbb{Q}}(Y), \\ \mathfrak{D} = \sum \Delta_i \otimes D_i &\longrightarrow \mathfrak{D}(u) := \sum \mathrm{eval}_u(\Delta_i) D_i, \end{aligned}$$

induced by 2.0.18.

(iv) Two polyhedral divisors $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathrm{Div}_{\mathbb{Q}}(Y,\sigma)$ are same if and only if $\mathfrak{D}_1(u) = \mathfrak{D}_2(u)$ for all $u \in \sigma^\vee$.

(v) A polyhedral divisor $\mathfrak{D} \in \mathrm{Div}_{\mathbb{Q}}(Y,\sigma)$ is integral if and only if $\mathfrak{D}(u)$ are integral divisors for all $u \in \sigma^\vee \cap M$.

(vi) A polyhedral divisor $\mathfrak{D} \in \mathrm{Div}_{\mathbb{Q}}(Y,\sigma)$ is Cartier if and only if $\mathfrak{D}(u)$ are Cartier divisors for all $u \in \sigma^\vee$.

Definition 5.1.5 ([AH], definition 2.5)**.** (i) A polyhedral divisor $\mathfrak{D} \in \mathrm{Div}_{\mathbb{Q}}(Y,\sigma)$ is called effective if all the evaluations $\mathfrak{D}(u)$ are effective divisors on Y for all $u \in \sigma^\vee$. We write $\mathfrak{D} \geq 0$ when $\mathfrak{D} \in \mathrm{Div}_{\mathbb{Q}}(Y,\sigma)$ is effective.

(ii) A polyhedral divisor $\mathfrak{D} \in \mathrm{Div}_{\mathbb{Q}}(Y,\sigma)$ is called semiample if all the evaluations $\mathfrak{D}(u)$ are semiample divisors for all $u \in \sigma^\vee$.

Definition 5.1.6 ([AH], definition 2.7)**.** A proper polyhedral divisor (in short, pp-divisor) on Y with respect to σ is a polyhedral Cartier divisor $\mathfrak{D} \in \mathrm{CaDiv}_{\mathbb{Q}}(Y,\sigma)$ such that the following hold:

(i) $\mathfrak{D}$ is semiample,

(ii) $\mathfrak{D}$ admits a presentation $\mathfrak{D} = \sum \Delta_i \otimes D_i$ with $\Delta_i \in \mathrm{Pol}_\sigma(N_{\mathbb{Q}})$ and $D_i \in \mathrm{Div}(Y)$ effective,

(iii) for each $u \in \mathrm{rel\,int}(\sigma^\vee)$, the evaluation $\mathfrak{D}(u) \in \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ is a big Cartier divisor on Y.

The addition on $\mathrm{CaDiv}_{\mathbb{Q}}(Y,\sigma)$ induces a semigroup structure on the set of all pp-divisors.

Definition 5.1.7 ([AH], definition 2.8)**.** $\mathrm{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ denotes the semigroup of pp-divisors on Y with respect to σ.

The pp-divisors are connected to certain convex piecewise linear maps which we define now.

Definition 5.1.8 ([AH], definition 2.9)**.** Let $\omega \subset M_{\mathbb{Q}}$ be a full dimensional cone. A map $\mathfrak{h} : \omega \longrightarrow \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ is

(i) convex if $\mathfrak{h}(u) + \mathfrak{h}(u') \leq \mathfrak{h}(u + u')$ holds for any $u, u' \in \omega$,

(ii) piecewise linear if there exists a quasifan Λ in $M_{\mathbb{Q}}$ with $\mathrm{Supp}(\Lambda) = \omega$ such that $\mathfrak{h}$ is linear on each cone in Λ,

(iii) strictly semiample if $\mathfrak{h}(u)$ is semiample for all $u \in \omega$, and for all $u \in \mathrm{rel\,int}(\omega)$, $\mathfrak{h}(u)$ is big.

Definition 5.1.9. Let $\omega \subset M_{\mathbb{Q}}$ be a full dimensional cone. A convex, piecewise linear and strictly semiample map $\mathfrak{h} : \omega \longrightarrow \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ is called integral if $\mathfrak{h}(u) \in \mathrm{CaDiv}(Y)$ are integral Cartier divisors for all $u \in \omega \cap M$.

The set of all convex, piecewise linear and strictly semiample maps also form a semigroup under pointwise addition.

Definition 5.1.10 ([AH], definition 2.10)**.** Let $\omega \subset M_{\mathbb{Q}}$ be a full dimensional cone. Then $\mathrm{CPL}_{\mathbb{Q}}(Y, \omega)$ denotes the semigroup of convex, piecewise linear and strictly semiample maps $\mathfrak{h} : \omega \longrightarrow \mathrm{CaDiv}_{\mathbb{Q}}(Y)$.

We have the following correspondence.

Proposition 5.1.11 ([AH], proposition 2.11)**.** *The following map*

$$\mathrm{PPDiv}_{\mathbb{Q}}(Y, \sigma) \longrightarrow \mathrm{CPL}_{\mathbb{Q}}(Y, \sigma^{\vee}),$$
$$\mathfrak{D} \longrightarrow [u \rightarrow \mathfrak{D}(u)]$$

is a canonical isomorphism of semigroups. This isomorphism sends integral pp-divisors to integral, convex, piecewise linear and strictly semiample maps.

5.2 Construction of Y given a variety with torus action

We pay our attention to only affine T-varieties. We describe how to get the variety Y together with a pp-divisor on it.

Let A be an integral M-graded k algebra :

$$A = \bigoplus_{u \in M} A_u.$$

We assume that the weight cone

$$\omega := \langle u \in M | A_u \neq 0 \rangle$$

is full dimensional. Then by ([Gro], proposition I.4.7.3), the algebraic torus $T := \operatorname{Spec} k[M]$ acts on the affine space $X := \operatorname{Spec} A$.

Definition 5.2.1. [MFK] Let X and T be as above and $L \to X$ be a line bundle. A T-linearization of the line bundle $L \to X$ is a fiberwise T-action on L making the projection map T-equivariant.

When the line bundle $L = \mathscr{O}_X$ is trivial then the T-action on each fiber is of the form:

$$t.(x, z) = (t.x, \chi^u(t)z) \quad \text{for } (x, z) \in X \times \mathbb{A}^1_k \cong L, \tag{5.1}$$

where $\chi^u : T \to \mathbb{G}_m(k)$ is the character associated to the lattice point $u \in M$. For any positive integer n, the induced T-linearization on the line bundle $L^{\otimes n}$, which is $n-$fold tensor product is given by χ^{nu}.

To each T-linearization one associates a representation of T on the vector space of global sections:

$$t.s(x) := t.(s.(t^{-1}.x)), \tag{5.2}$$

where $x \in X$ and $s : X \to L$ is a section of the projection map $L \to X$.

Definition 5.2.2. [MFK] Let X and T be as above and $L \to X$ be the T-linearized trivial line bundle. The set of semistable points associated with the T-linearized line bundle is defined as

$$X^{ss}(L) := \{X_f \mid f \text{ is a } T\text{-invariant section of } L^{\otimes n} \text{ for some } n\}.$$

The invariant sections corresponding to $u \in M$ in 5.1 are exactly the elements $f \in A_{nu}, n \in \mathbb{N}$.

Definition 5.2.3. [AH] Let X and T be as above and $L \to X$ be the T-linearized trivial line bundle. Let $u \in M$ be a lattice point. The set of semistable points corresponding to u is defined as :

$$X^{ss}(u) := \bigcup_{f \in A_{nu}, n \in \mathbb{N}} X_f.$$

Definition 5.2.4. ([AH], definition 5.1) Let x be a point in X.

(i) The submonoid $S(x)$ containing all lattice points $u \in M$ such that there exists an element $f \in A_u$ with $f(x) \neq 0$, is called orbit monoid associated to x.

(ii) The convex cone $\omega(x)$ generated by orbit monoid $S(x)$ is called orbit cone associated to x.

(iii) The sublattice $M(x)$ generated by the orbit cone $\omega(x)$ is called the orbit lattice associated to x.

The above combinatorial data carry the following geometric information.

Proposition 5.2.5. *([AH], proposition 5.2) Let x be a point in X.*

(i) The orbit lattice $M(x)$ consists of those lattice points $u \in M$ for which there exists a homogeneous rational function $f \in \mathbb{K}(X)$ of degree u and is invertible near x.

(ii) The isotropy group $T_x \subset T$ of $x \in X$ is isomorphic to the diagonalizable group $\operatorname{Spec} k[M/M(x)]$.

(iii) The orbit closure $\overline{T.x}$ is isomorphic to the monoid algebra $\operatorname{Spec} k[S(x)]$*; there is a natural equivarint open embedding of the torus* $T/T_x = \operatorname{Spec} k[M(x)]$.

(iv) The normalization of the $\overline{T.x}$ is an affine toric variety and it corresponds to cone $\omega(x)^\vee$ in lattice $\operatorname{Hom}(M(x), \mathbb{Z})$.

The set of semistable points $X^{ss}(u)$ corresponding to lattice point $u \in M$ can be described as follows:

$$X^{ss}(u) = \{x \in X : u \in \omega(x)\}.$$

Definition 5.2.6. ([AH], definition 5.3) Let $u \in M$ be a lattice point in the weight cone ω and we define the cone

$$\lambda(u) := \bigcap_{x \in X; u \in \omega(x)} \omega(x).$$

Then $\lambda(u)$ is a polyhedral cone and we call it the GIT cone associated with u.

Theorem 5.2.7. *([AH], theorem 5.4) Let M be a lattice and A be an M-graded integral affine algebra. Then we have the following statements for the action of the torus $T := \operatorname{Spec} k[M]$ on the affine scheme $X := \operatorname{Spec} A$:*

(i) The GIT cones $\lambda(u), u \in M$, form a quasi-fan Λ in $M_{\mathbb{Q}}$.

(ii) The support $\operatorname{Supp}(\Lambda) = \omega$ in $M_{\mathbb{Q}}$.

(iii) For lattice points $u_1, u_2 \in M$, the sets of semistable points correspond to GIT cones in reverse order i.e.

$$X^{ss}(u_1) \subset X^{ss}(u_2) \Leftrightarrow \lambda(u_1) \supset \lambda(u_2).$$

Altmann and Hausen prove the following theorem. Recall that, a variety Y is semiprojective if it is projective over $\operatorname{Spec}(\Gamma(Y, \mathscr{O}_Y))$.

Theorem 5.2.8 (AH08, Theorem 3.1 and 3.4)**.** *Given a normal, semiprojective variety Y, a lattice N, the dual lattice M, a pointed cone $\sigma \subset N_{\mathbb{Q}}$, a pp-divisor $\mathfrak{D} \in \operatorname{PPDiv}_{\mathbb{Q}}(Y, \sigma)$, the affine scheme associated to $(Y, \mathfrak{D})$ is described as*

$$X = \operatorname{Spec} \Gamma \left(Y, \bigoplus_{u \in \sigma^\vee \cap M} \mathscr{O}_Y(\mathfrak{D}(u)) \right).$$

Then X is a normal T-variety where $T = \operatorname{Spec} k[M]$. Moreover given any normal affine T-variety $X = \operatorname{Spec} A$ with weight cone $\omega \in M_{\mathbb{Q}}$, there exists a normal semiprojective variety Y and a pp-divisor $\mathfrak{D} \in \operatorname{PPDiv}_{\mathbb{Q}}(Y, \omega^\vee)$ such that the T-variety associated to $(Y, \mathfrak{D})$ is X.

Remark 5.2.9. *Suppose Y and $\mathfrak{D}$ are as in the first part of the theorem 5.2.8. Let $\mathcal{A} = \bigoplus_{m \in \sigma^\vee} \mathcal{A}_m$ where $\mathcal{A}_m = \mathscr{O}_Y(\mathfrak{D}(m))$. Then, one can also consider the relative spectrum $\tilde{X} = \operatorname{Spec}_Y \mathcal{A}$. Then $A = \Gamma(Y, \mathcal{A})$. The scheme $\tilde{X}$ is a normal algebraic variety with an effective T action such that $\pi\colon \tilde{X} \longrightarrow Y$ is a good quotient. Furthermore there is a contraction morphism $r\colon \tilde{X} \longrightarrow X$ which is proper, birational and T-equivariant.*

Given a normal variety $X = \operatorname{Spec} A = \operatorname{Spec} \bigoplus_{m \in M} A_m$ with an effective torus action, theorem 5.2.8 above ensures the existence of $(Y, \mathfrak{D})$. We recall the description of Y, as it will be useful in chapter 7. According to the theory in [BH, section 2], $X_{\text{ss}}(m) = X_{\text{ss}}(m')$ for m, m' belonging to the relative interior of a GIT cone λ of the GIT fan Λ. Let $X_\lambda := X_{\text{ss}}(m)$ for some $m \in \operatorname{rel\,int} \lambda$. Then one also has that $Y_m = X_{\text{ss}}(m)//T = \operatorname{Proj} \bigoplus_{r \in \mathbb{Z}} A_{rm}$. Thus, Y_m's also depend only on the fan λ such that $m \in \operatorname{rel\,int} \lambda$, and hence are denoted by Y_λ. If $\gamma \preceq \lambda$, then one has a birational morphism $\varphi_{\gamma\lambda}\colon Y_\lambda \longrightarrow Y_\gamma$. Putting everything together compatibly one has the following diagram which also defines Y (see [AH, section 6]):

$$\begin{array}{ccccccccc} & & X' & \longrightarrow & X_\lambda & \longrightarrow & X_\gamma & \longrightarrow & X \\ & \swarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y_\lambda & \longrightarrow & Y_\gamma & & \downarrow \\ \| & & & \searrow & & \searrow & & \searrow & \\ \text{Normal}(\overline{\text{image}(X')}) & & & & \longrightarrow & & & & Y_0 \end{array} \tag{5.3}$$

where $X' = \varprojlim X_\lambda$, $Y' = \varprojlim Y_\lambda$ and $Y_0 := \operatorname{Spec}(A_0)$. It is also known that Y is a good quotient of the torus action on X.

In the previous paragraph, we constructed the Y in the pair $(Y, \mathfrak{D})$ describing affine variety X with an effective action of the torus. The construction of the pp-divisor $\mathfrak{D}$ is not relevant to this paper.

6

Multihomogeneous spaces

Projective varieties which are associated with $\mathbb{Z}$-graded rings have been studied extensively and found to be useful in studying varieties. Here we extend this class to the category of rings graded by finitely generated abelian groups. To each such graded ring, we associate a scheme, called, multihomogeneous space. It is a generalization of usual projective spaces (see [Sta, §2 27.8]). Zanchetta ([Zan]) shows that every quasi-compact and quasi-separated scheme of finite type over a ring R is a closed subscheme of a smooth open subscheme of a multihomogeneous space.

6.1 Definition and basic properties

Definition 6.1.1. [BS, §2] Let D be a finitely generated abelian group and

$$A = \bigoplus_{d \in D} A_d$$

be a D-graded ring. One says that A is *periodic* if $D' = \{d \in D \mid \exists f \in A_d \cap A^\times\}$, the subgroup of D consisting of degrees of all the homogeneous invertible elements in A, is a finite index subgroup. A homogeneous element f in a D-graded ring A is said to be *relevant* if A_f is periodic. For a relevant element f, note that the localization A_f is D-graded. We shall denote the degree 0 part as $A_{(f)}$. For a relevant element f, D_f denotes the subgroup of degrees of units in A_f.

Lemma 6.1.2. *[BS, lemma 2.1] Let D be a finitely generated abelian group and*

$$A = \bigoplus_{d \in D} A_d$$

be a D-graded periodic ring. Then the projection $\operatorname{Spec} A \longrightarrow \operatorname{Spec}(A_0)$ *is a geometric quotient in the GIT sense.*

Definition 6.1.3. [BS, definition 2.2] For D and A as in definition 6.1.1, the grading on A correspond to an action of the diagonalizable group scheme $\operatorname{Spec} A_0[D]$ on $\operatorname{Spec} A$ ([Gro], proposition I.4.7.3). Let Q be the quotient in the category of ringed spaces. Now for a relevant element f, consider the inclusion

$$D_+(f) = \operatorname{Spec} A_{(f)} \subset Q.$$

One defines

$$\operatorname{Proj}_{\mathrm{MH}} A = \bigcup_{\substack{f \in A \\ f \text{ is relevant}}} D_+(f) \subset Q.$$

Let $A_+ \subset A$ be the ideal generated by all relevant elements $f \in A$. Then we call the invariant closed subscheme $V(A_+)$, the irrelevant subscheme and the invariant open subscheme $\operatorname{Spec} A \setminus V(A_+)$, the relevant locus. Then the projection map $\operatorname{Spec} A \setminus V(A_+) \longrightarrow \operatorname{Proj}_{\mathrm{MH}} A$ is a geometric quotient for induced action of the diagonalizable group scheme $\operatorname{Spec}(A_0[D])$ (see [BS]).

Remark 6.1.4. *The points in a multihomogeneous projective space* $\operatorname{Proj}_{MH} A$ *of a D-graded ring A correspond to homogeneous ideals in A which may not be prime (see [BS, remark 2.3]). However, these ideals have the property that all the homogeneous elements in the complement form a multiplicatively closed set.*

Remark 6.1.5. *By lemma 6.1.2, the inclusion $A_{(f)} \hookrightarrow A_f$ induces map* $\operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A_{(f)}$*, which is a geometric quotient.*

By definition, The collection $\{D_+(f) \mid f \in A \text{ is homogeneous and relevant}\}$ covers $\operatorname{Proj}_{\mathrm{MH}} A$. We state the following easy fact for subsequent use.

Lemma 6.1.6. *With the notation as above, $D_+(f) \cap D_+(g) = D_+(fg) \subset \operatorname{Proj}_{MH} A$.*

Proof. This is implicit in [BS, propostion 3.1]. Note that for relevant elements f and g in A, $\operatorname{Spec} A_{fg} = \operatorname{Spec} A_f \cap \operatorname{Spec} A_g$ as subschemes of $\operatorname{Spec} A$. Now $\operatorname{Spec} A_{(fg)}$, $\operatorname{Spec} A_{(f)}$ and $\operatorname{Spec} A_{(g)}$ are geometric quotients (see remark 6.1.5) under the action of $\operatorname{Spec} A_0[D]$ and hence $\operatorname{Spec} A_{(fg)} = \operatorname{Spec} A_{(f)} \cap \operatorname{Spec} A_{(g)}$ considered as subschemes of $\operatorname{Proj}_{\mathrm{MH}} A$. □

Remark 6.1.7. *The following example clarifies why we consider only relevant elements instead of homogeneous elements. Take the $\mathbb{Z}^2$-graded ring $A = k[x, y, z, w]$ with $\deg(x) = \deg(y) = (1, 0)$ and $\deg(z) = \deg(w) = (0, 1)$. Then $A_{(x)} = k[\frac{y}{x}]$ and $A_{(xz)} = k[\frac{y}{x}, \frac{w}{z}]$. One would expect inclusion of affine spaces $\operatorname{Spec} A_{(xz)} \hookrightarrow \operatorname{Spec} A_{(x)}$. But this is not possible since the former has dimension 2 and the later has dimension 1. The reason this fails to hold is that the element x is not relevant.*

The gradation on a ring has an enormous effect on the structure of the corresponding multihomogeneous space.

Example 6.1.8. *Consider the ring $A = k[x, y, z]$*

1. *Any gradation by $\mathbb{Z}^n, n \geq 4$ on A induces $\operatorname{Proj}_{MH} A = \emptyset$ since there are no relevant elements.*

2. *For any gradation by $\mathbb{Z}^3$ on A there are two possibilities; case 1: degree of x, y and z are linearly independent, in this case $\operatorname{Proj}_{MH} A = \operatorname{Spec} k$; Case 2: degrees of x, y and z are linearly dependent, then $\operatorname{Proj}_{MH} A = \emptyset$.*

3. *Consider the $\mathbb{Z}^2$ gradation on A as follows: $\deg(x) = (1, 0)$, $\deg(y) = (0, 1)$ and $\deg(z) = (a, 0), a > 0$. Then $\operatorname{Proj}_{MH} A = \mathbb{P}^1$ for all $a > 0$. In fact, there are infinitely many structures of multihomogeneous spaces corresponding to $\mathbb{Z}^2$ gradation on A.*

4. *Set $\deg(x) = a, \deg(y) = b, \deg(z) = c$ and $a, b, c > 0, gcd(a, b, c) = 1$ as $\mathbb{Z}$ gradation on A, we have $\operatorname{Proj}_{MH} A = \mathbb{P}(a, b, c)$ weighted projective spaces.*

We give a criterion (see lemma 6.3.4) to check when the multihomogeneous space is non-empty.

Example 6.1.9. *(Proj construction) Consider a $D = \mathbb{Z}$-graded ring A which is generated by homogeneous elements $x_1, \ldots, x_n$ over A_0. Then each x_i is relevant since D_{x_i} is a* *nontrivial subgroup of* $\mathbb{Z}$*, and therefore has finite index. Then*

$$\mathrm{Proj}_{MH} A = \bigcup_{i=1}^{n} D_+(x_i) = \mathrm{Proj}\, A$$

is the usual projective variety over $\mathrm{Spec}\, A_0$*.*

Example 6.1.10. *(Non-separated multihomogeneous space) Consider the ring* $A = k[X, Y, Z]$ *and give it* $\mathbb{Z}^2$*-gradation as* $\deg(X) = (1, 1)$*,* $\deg(Y) = (1, 0)$*, and* $\deg(Z) = (0, 1)$*. Then the relevant monomials are* XY*,* XZ*, and* YZ *with* $A_{(XY)} = k[\frac{YZ}{X}]$*,* $A_{(XZ)} = k[\frac{YZ}{X}]$*,* $A_{(YZ)} = k[\frac{X}{YZ}]$*, and* $A_{(XYZ)} = k[\frac{X}{YZ}, \frac{YZ}{X}]$*. Now* $\mathrm{Spec}\, A_{(XY)}$ *and* $\mathrm{Spec}\, A_{(XZ)}$ *glue along open subscheme* $\mathrm{Spec}\, A_{(XYZ)}$ *and give us* $\mathbb{A}^1_k$ *with double origin. When we patch it with* $\mathrm{Spec}\, A_{(YZ)}$ *we get the multihomogeneous space* $\mathbb{P}^1_k$ *with double origin. It is a nonseparated scheme since* $\mathbb{A}^1_k$ *with double origin, an open subscheme, is nonseparated.*

Example 6.1.11. *(Non-separated multihomogeneous space) Consider the free abelian group* $D = \mathbb{Z}^2$ *and* D*-graded polynomial ring* $A = k[x_1, x_2, x_3, x_4]$ *with* $\deg x_1 = (4, 0), \deg x_2 = (3, 0), \deg x_3 = (0, 1)$ *and* $\deg x_4 = (12, -1)$*. The relevant elements are* x_1x_3*,* x_1x_4*,* x_2x_3*,* x_2x_4*, and* x_3x_4*. Then the multihomogeneous space is*

$$\begin{aligned}\mathrm{Proj}_{MH} A &= \bigcup_{f = x_i x_j \text{ relevant}} D_+(f) \\ &= D_+(x_3x_4) \cup D_+(x_1x_3) \cup D_+(x_2x_3) \cup D_+(x_1x_4) \cup D_+(x_2x_4)\end{aligned}$$

From example 7.2.6 we have that

$$\begin{aligned}\mathbb{P}^2 &\cong D_+(x_3x_4) \cup D_+(x_1x_3) \cup D_+(x_2x_3) \\ &\cong D_+(x_3x_4) \cup D_+(x_1x_4) \cup D_+(x_2x_4).\end{aligned}$$

Therefore the multihomogeneous space $\mathrm{Proj}_{MH} A$ *is the union of two copies of* $\mathbb{P}^2$ *glued along the affine open subscheme* $D_+(x_3x_4)$*. This is an example of a non-separated multihomogeneous space.*

Example 6.1.12. *(non-projective multihomogeneous space) [BS, example 3.9]*
Take the ring $A = k[X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Z]$ *and give it a* $\mathbb{Z}^2$ *grading as* $\deg(X_i) = (1, 0)$ *and* $\deg(Y_i) = (0, 1)$ *and* $\deg(Z) = (1, 1)$*. Now the elements* $g = X_1Y_1Z + X_2^2Y_1^2 +$

$X_1^2Y_2^2$ *and* $f = X_1Y_1Z + X_2^2Y_1^2 + X_1^2Y_2^2 + X_3X_4Y_3Y_4$ *are homogeneous of degree* $(2, 2)$. *Consider the ring* $A' = A/(f)$ *and let* $P = \text{Proj}_{MH} A, P' = \text{Proj}_{MH} A'$. *Take the open subscheme* $U = D_+(X_1Z) \cup D_+(Y_1Z)$ *and set* $U' = U \cap P'$. *Then* U' *is a separated open subscheme of* P'. *However, it is not quasi-projective since there exist points* $x \in V_+(X_1, X_2, X_3, X_4) \cap U'$ *and* $y \in V_+(Y_1, Y_2, Y_3, Y_4) \cap U'$ *that do not admit common affine neighbourhood. Suppose there exists a common affine neighbourhood* $W \subset U'$ *then the preimage* $V \subset \text{Spec}\, A'$ *is affine. Since the ring* A' *is factorial, one can write* $V = D(h)$ *where* h *is a homogeneous element in the ideal* $(X_1Z, Y_1Z) \subset A$. *Then one has* $h = pX_1Z + qY_1Z$ *with* $\deg(h) = (a, b)$ *and without loss of generality one can further assume* $a \geq b$. *Since* $\deg(Y_1Z) = (1, 2)$, q *is divisible by atleast one* X_i. *Then* $q \in (X_1, X_2, X_3, X_4)$. *This implies* $h(x) = 0$, *a contradiction. Therefore,* U' *is separated but not quasi-projective, and hence* $\text{Proj}_{MH} A'$ *is not projective.*

6.2 Known results

For later, we record two results of Brenner and Schröer regarding finiteness.

Lemma 6.2.1. *[BS, lemma 2.4] For a finitely generated abelian group* D *and a* D*-graded ring* A*, the following are equivalent:*

(i) The ring A *is Noetherian.*

(ii) A_0 *is Noetherian and* A *is an* A_0*-algebra of finite type.*

Proposition 6.2.2. *[BS, proposition 2.5] Suppose* A *is a Noetherian ring graded by a finitely generated abelian group* D. *Then the morphism* $\varphi\colon \text{Proj}_{MH} A \longrightarrow \text{Spec}\, A_0$ *is universally closed (see [Har] for definition) and of finite type.*

Lemma 6.2.3. *[BS, lemma 2.7] Let* D *be a finitely generated abelian group,* $A = \bigoplus_{d \in D} A_d$ *be a* $D-$*graded ring and let* $A_0 \longrightarrow B_0$ *be a ring map. Let*

$$\begin{array}{ccc} A_0 & \hookrightarrow & A \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B = A \otimes_{A_0} B_0 \end{array}$$

be the base change diagram. Then $\operatorname{rad}(B_+) = \operatorname{rad}((A_+)B)$*, where* $\operatorname{rad}((A_+)B)$ *is the extended ideal.*

Proposition 6.2.4. *[BS, proposition 3.1] Let D be a finitely generated abelian group and $A = \bigoplus_{d\in D} A_d$ be a $D-$graded ring. Then the diagonal embedding* $\operatorname{Proj}_{MH} A \longrightarrow \operatorname{Proj}_{MH} A \times_{\operatorname{Spec} A_0} \operatorname{Proj}_{MH} A$ *of a multihomogeneous space is affine.*

Here is a separation criterion.

Proposition 6.2.5. *[BS, proposition 3.2] With the above hypothesis, if every pair of points $x, y \in \operatorname{Proj}_{MH} A$ admits a common affine neighbourhood $D_+(f)$ for some relevant element $f \in A$, then $\operatorname{Proj}_{MH} A$ is separated.*

Let $f \in A$ be a homogeneous element and H_f be the set of homogeneous divisors $g|f^n, n \geq 0$. Let $C_f \subset D \otimes \mathbb{R}$ be the cone generated by $\deg(g), g \in H_f$. Then we have f is relevant if and only if C_f has non-empty interior (see [BS]).

Proposition 6.2.6. *[BS, proposition 3.3] Assume the above hypothesis. Let F be a collection of relevant elements such that for every pair $f_i, f_j \in F$, the set $C_{f_i} \cap C_{f_j} \subset D \otimes \mathbb{R}$ admits non-empty interior. Then $\bigcup_{f_i\in F} D_+(f_i) \subset \operatorname{Proj}_{MH} A$ is a separated open subscheme.*

Definition 6.2.7. [BS, page 10] Let R be a ring, M be a free abelian group of finite rank, and $N := \operatorname{Hom}(M, \mathbb{Z})$ be the dual of M. Let X be an $R-$scheme and $T := \operatorname{Spec} R[M]$ be the torus. A simplicial torus embedding of torus T is $T-$equivariant open map $T \hookrightarrow X$ locally given by semigroup algebra homomorphisms $R[\sigma^\vee \cap M] \to R[M]$, where σ is a strongly convex, simplicial cone in N.

Remark 6.2.8. *If X is a simplicial toric variety with torus T, then X is a simplicial torus embedding of the torus T. There are other schemes which are simplicial torus embeddings of some torus. The homogeneous spectrum of multigraded polynomial algebras are examples of this type.*

Let D be an abelian group of finite rank and $A = k[x_1, \dots, x_n]$ be a $D-$graded polynomial $k-$algebra. Suppose the grading is given by a linear map $P : \mathbb{Z}^n \to D$ with finite co-kernel. Then we have the following sequence of abelian groups

$$0 \to M \to \mathbb{Z}^n \to D,$$

where M is the kernel of P.

Proposition 6.2.9. *[BS, proposition 3.4] Assume the above setting. Then* $\mathrm{Proj}_{MH} A$ *is a simplicial torus embedding of the torus* $\mathrm{Spec}\, k[M]$*.*

Remark 6.2.10. *[BS, remark 3.7] Again we assume the above setting. Let* $I = \{1, \dots, n\}$ *be an index set and* $N := \mathrm{Hom}(M, \mathbb{Z})$ *the dual of* M*. Let* $pr_i : \mathbb{Z}^n \to \mathbb{Z}, i \in I$ *be projections. We associate each subset* $J \subset I$ *to cone* $\sigma_J \subset N_{\mathbb{R}}$ *generated by* $pr_i|_M, i \in J$*. Then we have a correspondence between subsets* J *of* I *with* $\prod_{i \in J} x_i$ *relevant, and strongly convex, simplicial cones* $\sigma_{I \setminus J} \subset N_{\mathbb{R}}$*.*

6.3 Some results about Multihomogenous spaces

Proposition 6.3.1. *Suppose* D *is a free finitely generated* $\mathbb{Z}$*-module and suppose* $A = \bigoplus_{d \in D} A_d$ *be a* D*-graded ring. Assume that we have a collection of relevant elements* F *such that*

$$\mathrm{Proj}_{MH} A = \bigcup_{f \in F} \mathrm{Spec}\, A_{(f)}$$

and for each $f \in F$, $\left\{ d \in D \,\middle|\, d = \deg g \text{ for some } g \in A_f^\times \right\} = D$*. Then every point* $p \in \mathrm{Proj}_{MH} A$ *corresponds to a homogeneous prime in* A*.*

Proof. Suppose $p \in \mathrm{Spec}\, A_{(f)}$ for some relevant element $f \in A$. Then A_f is periodic and

$$D' = \left\{ d \in D \,\middle|\, d = \deg g \text{ for some } g \in A_f^\times \right\}$$

is a free subgroup of D of finite index. Define

$$A'_f = \bigoplus_{d \in D'} (A_f)_d.$$

It is easy to see that in this case, $A'_f = A_{(f)}[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$ where $r = \mathrm{rank}\, D'$.

Note the primes $P \in A_{(f)}$ correspond to the primes $P[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \subset A'_f$. Now consider the diagram

$$\begin{array}{ccccc} & & & & A \\ & & & & \downarrow \\ A_{(f)} & \hookrightarrow & A'_f & \hookrightarrow & A_f \end{array}$$

It is easy to see that if $A'_f = A_f$, then the primes in $A_{(f)}$ would correspond to homogeneous primes in A which do not contain f. The condition $A'_f = A_f$ holds whenever the hypothesis of the proposition is satisfied. □

Corollary 6.3.2. *Under the hypothesis of proposition 6.3.1, the points in $D_+(f) \subset \operatorname{Proj}_{MH} A$ correspond to all homogeneous primes in A which do not contain f.*

Proof. This was mentioned in the proof of proposition 6.3.1 after the diagram. □

Corollary 6.3.3. *The hypothesis of the proposition 6.3.1 holds in the following cases and hence in these cases points in the multihomogeneous space will correspond to prime ideals in the graded ring.*

- Proj *of $\mathbb{Z}$-graded rings.*
- $\operatorname{Proj}_{MH} A$ *where D is a free abelian group and A is a D-graded algebra generated over A_0 by a set $\{ a_1, \dots, a_n \}$ of homogeneous elements such that $\{ \deg a_1, \dots, \deg a_n \}$ contain a basis for D.*

A criterion for non-emptiness of multihomogeneous space.

Lemma 6.3.4. *Suppose D is a finitely generated abelian group and $A = \bigoplus_{d \in D} A_d$ be a D-graded ring which is finitely generated by homogeneous elements $x_1, \dots, x_r \in A$ over the ring A_0. Then $\operatorname{Proj}_{MH} A$ is non-empty if and only if $\{\deg x_i \mid 1 \le i \le r\}$ generates a finite index subgroup of D.*

The functor $\operatorname{Proj}_{\mathrm{MH}}$ commutes with the product.

Theorem 6.3.5. *Let $X_i = \operatorname{Proj}_{MH} A_i, i = 1, \dots, n$ be a finite set of multihomogeneous spaces over k. Then the product $\prod_{i=1}^n X_i$ is the multihomogeneous space $\operatorname{Proj}_{MH} \left(\otimes_{i=1}^n A_i \right)$.*

Proof. Let the diagonalizable groups $\operatorname{Spec} k[D_i]$ act on the affine spaces $\operatorname{Spec} A_i$ inducing X_i as quotients. Then the diagonalizable group $\prod_{i=1}^n \operatorname{Spec} k[D_i]$ which is $\operatorname{Spec} \left(\otimes_{i=1}^n k[D_i] \right)$

acts on the affine space $\operatorname{Spec}\left(\otimes_{i=1}^{n} A_i\right)$ component wise. Then the quotient is the product of the quotient of each component. And we have $\prod_{i=1}^{n} X_i = \operatorname{Proj}_{\mathrm{MH}}\left(\otimes_{i=1}^{n} A_i\right)$. $\square$

One can see that whenever A is integral the multihomogeneous space $\operatorname{Proj}_{\mathrm{MH}} A$ is integral. In fact, We can say the same thing for normality also.

Proposition 6.3.6. *Suppose A is a Noetherian normal ring satisfying the above hypothesis. Then $\operatorname{Proj}_{MH} A$ is a normal scheme.*

Proof. It is enough to check normality over an affine cover $\{D_+(f_i)\}$ of $\operatorname{Proj}_{\mathrm{MH}} A$. Let f be a relevant element of A, H be the set of nonzero homogeneous elements in A_f, $k(A_f)$ and $k(A_{(f)})$ be function fields of A_f and $A_{(f)}$ respectively. Then $H^{-1}(A_f)$ is a graded ring and $A_f \hookrightarrow H^{-1}(A_f)$ is a graded monomorphism. Furthermore, we have the inclusion of the 0-th component of the grading: $k(A_{(f)}) \subset \left(H^{-1}(A_f)\right)_0$.

$$\begin{array}{ccc}
 & \nearrow & k(A_f) \\
 & & \uparrow \\
A_f & \hookrightarrow & (A_f)_H \\
\uparrow & & \uparrow \\
A_{(f)} & \hookrightarrow & k(A_{(f)})
\end{array}$$

Since A is normal, we have A_f is normal in $k(A_f)$ and therefore integrally closed in $(A_f)_H$. We want to show that $A_{(f)}$ is normal. Let $a \in k(A_{(f)})$ be integral over $A_{(f)}$. Then a is integral over A_f and hence $a \in (A_f)_0 = A_{(f)}$ since $a \in \left(H^{-1}(A_f)\right)_0$. $\square$

6.4 Sheaves associated with multigraded modules

Consider a finitely generated abelian group D and let A be a D-graded ring. Suppose $M = \bigoplus_{d \in D} M_d$ is a D-graded A-module. Just as in the case of quasi-coherent modules over Proj of a $\mathbb{N}$-graded ring [Har, definition before proposition 5.11, page 116], we can construct $\widetilde{M}$. We sketch some details to fix the notation and show the similarities between the two set-ups.

In the construction of Proj of a $\mathbb{Z}$-graded module over a $\mathbb{N}$-graded ring, by the definition of $\operatorname{Proj} A$ the points corresponding to homogeneous prime ideals in the defining graded ring which do not contain the whole of the irrelevant ideal. However, this is no longer true for a multihomogeneous space and a point P may correspond to an ideal which is not a prime.

One can associate a sheaf $\widetilde{M}$ on $\operatorname{Proj} A$, on an $\mathbb{N}$-graded ring A, for example as done in [Har, definition preceding proposition 5.11, pg 116]. Now to describe sheaves of multihomogeneous spaces, we mimic this construction. Let A be a D-graded ring and M be a D-graded coherent sheaf on A with the usual condition that $A_d M_{d'} \subset M_{d+d'}$. Since the points p in $\operatorname{Proj}_{\mathrm{MH}} A$ correspond to graded ideals I_p in A such that the homogeneous elements in the complement $A \setminus I_p$ form a multiplicatiely closed set. It is still true that the stalk of the structure sheaf at p is given by $A_{(I_p)}$ (see remark 6.1.4). One can now define $\widetilde{M}$ in the same way by associating to $U \subset \operatorname{Proj}_{\mathrm{MH}} A$, the $\mathscr{O}_{\operatorname{Proj}_{\mathrm{MH}} A}(U)$-module of sections $s\colon U \to \coprod_{p \in U} M_{(I_p)}$ satisfying the usual condition that locally such s should be defined by a single element of the form m/a with $m \in M$ and $a \in A$ but not in any of the ideals I_p. These modules are coherent under some mild conditions, as we state below. Note that, given a D-graded A-module $M = \bigoplus_{d \in D} M_d$ and an $e \in D$, one can define a graded module $M(e)$ whereas A-modules $M(e) = M$, but $M(e)_d = M_{d+e}$ $\forall d \in D$.

Lemma 6.4.1. *Suppose D is a finitely generated abelian group and A be a D-graded integral Noetherian ring. Then for $X = \operatorname{Proj}_{MH} A$, the following hold*

(a) $\widetilde{A} = \mathscr{O}_X$. This allows us to define

$$\mathscr{O}_X(d) := \widetilde{A(d)}.$$

$\mathscr{O}_X(d)$ is a coherent sheaf.

(b) For a D-graded A-module M, $\widetilde{M}$ is quasi-coherent and $\widetilde{M}\Big|_{D_+(f)} \cong \widetilde{M_{(f)}}$ for any relevant element $f \in A$, where $\widetilde{M_{(f)}}$ is the sheaf of modules over $\operatorname{Spec} A_{(f)}$ corresponding to the module $M_{(f)}$, the degree zero elements in M_f. Moreover, $\widetilde{M}$ is coherent whenever M is finitely generated.

(c) The functor $M \to \widetilde{M}$ is a covariant additive exact functor from the category of D-graded A-modules to the category of quasi-coherent $\mathscr{O}_X$-modules, and commutes with direct limits and direct sums.

The proof is similar to the proof of [Har, proposition 5.11]. Note that, for an open subscheme $U \subset D_+(f)$, where $f \in A$ is relevant, the sections of $\widetilde{M}|_{D_+(f)}(U)$ and $\widetilde{M_f}(U)$ agree. This proves *(b)* and hence *(a)* and *(c)*.

The proof of the next lemma is also evident. We will see later the functor $\widetilde{(\ )}$ is essentially surjective.

Lemma 6.4.2. *Suppose D is a finitely generated abelian group and A be a D-graded algebra such that $A = A_0[x_1, \ldots, x_r]$, where $x_i \in A_{d_i}$ are homogeneous. Then $\{d \in D \,|\, A_d \neq 0\}$ generate a finite index subgroup of D if and only if $\{d_i \,|\, 1 \leq i \leq r\}$ does.*

This lemma provides a way to ensure one of the points of the hypothesis in the theorem below.

Theorem 6.4.3. *Suppose D is a* free *finitely generated abelian group and $A = \bigoplus_{d \in D} A_d$ is a D-graded* integral domain *which is finitely generated by homogeneous elements $x_1, \ldots, x_r \in A$ over the ring A_0. Also assume that for all k, $1 \leq k \leq r$, the set $\{\deg x_i \,|\, 1 \leq i \leq r, i \neq k\}$ generates a finite index subgroup of D. Let $X = \mathrm{Proj}_{MH} A$. Then $\Gamma(X, \mathscr{O}_X(d)) \cong A_d$. Furthermore, $\mathscr{O}_X(d)$ is a reflexive sheaf.*

Before proving the theorem we observe a fact.

Lemma 6.4.4. *Under the hypothesis of theorem 6.4.3,*

$$X = \mathrm{Proj}_{MH} A = \bigcup_{\substack{f:\ \text{is relevant and} \\ \text{is a monomial in } x_1, \ldots, x_r}} D_+(f).$$

Proof. We shall prove this for each $D_+(f)$ for every relevant f and the lemma will follow. Suppose $f = m_1 + \cdots + m_t$ is the homogeneous decomposition, where each m_i is a monomial. This means $\deg f$ is the same as $\deg m_i$ for each $1 \leq i \leq t$. Let $\{\deg g_j, 1 \leq j \leq l\}$ generates D_f (see definition 6.1.1). Then for some positive integer N, each g_j divides f^N and hence g_{j_k} divides m_i^N where g_{j_k} appears in the homogeneous decomposition of g_j. Now the fact that $\deg g_j$ is the same as $\deg g_{j_k}$ implies m_i is relevant.

Under the given hypothesis, corollary 6.3.2 implies that for any relevant g, $D_+(g)$ corresponds to all homogeneous prime ideals in A which do not contain g. In particular, if the set of homogeneous primes in A is denoted by H,

$$D_+(f) = \{P \in H \mid f \notin P\} \subset \bigcup_{i=1}^{t} \{P \in H \mid m_i \notin P\} = \bigcup_{i=1}^{t} D_+(m_i)$$

as was to be proved. □

Now we return to the proof of the theorem.

Proof of theorem 6.4.3. Giving an element $t \in \Gamma(X, \mathscr{O}_X(d))$ is the same as giving a collection $t_f \in D_+(f) = \operatorname{Spec} A_{(f)}$ for each relevant monomial f such that they agree on the pairwise intersections: $D_+(f) \cap D_+(g) = D_+(fg)$ (see lemma 6.1.6).

Suppose $t \in \Gamma(X, \mathscr{O}_X(d))$. For each relevant monomial $f \in A$ (which are enough to consider by lemma 6.4.4),

$$t|_{D_+(f)} \in \mathscr{O}_X(d)\Big(D_+(f)\Big) = \Big(\widetilde{A_f(d)}\Big)_{(0)}(D_+(f)) = (A_f)_d,$$

the d-th component of the D-graded ring A_f. Thus, for each such f write

$$t|_{D_+(f)} = \frac{p_f}{f^{k_f}}$$

where $\deg p_f - k_f \deg f = d$. Now since A is a domain, each $A_f \subset A_{x_1 \cdots x_r}$, and since the expression of t match over the intersections, t is of the form $x_1^{\alpha_1} \cdots x_r^{\alpha_r} f'$ with $f' \in A$. Since for each i, $x_1 \cdots \hat{x}_i \cdots x_r$ is relevant, $x_1^{\alpha_1} \cdots x_r^{\alpha_r} f' \in A_{x_1 \cdots \hat{x}_i \cdots x_r}$ implies that $\alpha_i \geq 0$. This proves that $t \in A$ and therefore, $t \in A_d$.

Fix a relevant element $f \in A$ and an element $d \in D$. Then $D_+(f) = \operatorname{Spec} A_{(f)}$. Note that $A(d)$ as an A-module is generated by $1 \in A(d)$ which has degree $-d \in D$. Therefore, the map $\operatorname{Hom}_A(A(d), A) \to A(-d)$, which sends $\phi \to \phi(1)$, gives a graded isomorphism of graded A-modules, where $e \in D$ graded part, $\operatorname{Hom}_A(A(d), A)_e$ consists of all $\phi \in \operatorname{Hom}_A(A(d), A)$ such that $\phi(1) \in A_{(e-d)}$. This in turn gives a graded isomorphism $\operatorname{Hom}_{A_f}(A(d)_f, A_f) \cong A(-d)_f$ of graded A_f-modules. Now taking the invariant part, we get $\operatorname{Hom}_{A_{(f)}}(A(d)_{(f)}, A_{(f)}) \cong A(-d)_{(f)}$ as $A_{(f)}$-modules or $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_X(d), \mathscr{O}_X)|_{D_+(f)} \cong \mathscr{O}_X(-d)|_{D_+(f)}$. The compatibility of these local isomorphisms gives an isomorphism $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_X(d), \mathscr{O}_X) \cong \mathscr{O}_X(-d)$ for all $d \in D$. This proves the reflexivity of $\mathscr{O}_X(d)$. □

Theorem 6.4.5. *Assume the hypothesis as in theorem 6.4.3. If M is a D-graded A-module then $\Gamma(X, \widetilde{M(d)}) \cong M_d$ for each $d \in D$.*

Proof. In the proof of theorem 6.4.3, replace $\mathscr{O}_X(d), A_f$ and $f' \in A$ with $\widetilde{M(d)}, M_f$ and $f' \in M$ respectively. □

Example 6.4.6. *The hypothesis of the above theorem 6.4.3 is necessary. For example, consider the ring $A = \mathbb{C}[X, Y, Z]$ with $\mathbb{Z}^2$-grading given by*

$$\deg X = (0, 1) \qquad\qquad \deg Y = (1, 0) = \deg Z$$

The scheme $\operatorname{Proj}_{MH} A$ is covered by two affines $D_+(XY)$ and $D_+(XZ)$. Now consider the module $M = A(2, -1)$. Consider the section YZ/X which is defined over both $\widetilde{M}(D_+(XY))$ and $\widetilde{M}(D_+(XZ))$. Therefore, $YZ/X \in \Gamma(\operatorname{Proj}_{MH} A, \widetilde{M})$, whereas $A_{(2,-1)} = 0$.

6.5 Line bundles on Multihomogeneous spaces

The reflexive coherent sheaves $\mathscr{O}_X(d)$ will not be line bundles for every $d \in D$. We give a criterion for these to be line bundles generalizing the well-known similar results for weighted projective spaces. Before that, we prove a short lemma.

Lemma 6.5.1. *Suppose A is a D-graded ring for a finitely generated free abelian group D, generated as an A_0-algebra by homogeneous elements $x_1, \ldots, x_r$. Assume that f is a relevant monomial in A. Suppose $d \in D_f$, where D_f is the sublattice of D generated by*

$$\left\{ \deg a \,\middle|\, a \text{ divides } f^N \text{ for some } N > 0 \right\}.$$

Then there is a monomial m in $x_1, \ldots, x_r$ and $k \in \mathbb{N} \cup \{0\}$ such that $\deg(m/f^k) = d$ and $m \mid f^N$ for some $N > 0$.

Proof. Suppose $f = x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ and that $d_{i_k} = \deg(x_{i_k})$ for $1 \leq k \leq s$. Then D_f is generated by $\{ d_{i_1}, \ldots, d_{i_s} \}$. Then for $d \in D_f$, there exists integers $a_1, \ldots, a_s$ such that $d = \sum_{j=1}^{s} a_j d_{i_j}$. Consider the element $a = x_{i_1}^{a_1} \cdots x_{i_s}^{a_s}$. Let $I = \{k \,|\, 1 \leq k \leq s, a_k < 0\}$.

Note that $\prod_{j\in I} x_{i_j}^{-a_j} | f^M$ for some $M > 0$. Let $b \in A$ be such that

$$\prod_{j\in I} x_{i_j}^{-a_j} b = f^M$$

Then

$$a = \frac{\prod_{j\notin I} x_{i_j}^{a_j} b}{f^M}.$$

This completes the proof by taking $m = \prod_{j\notin I} x_{i_j}^{a_j} b$, $N = M$. □

Theorem 6.5.2. *Suppose $X = \mathrm{Proj}_{MH} A$ is a multihomogeneous space defined for a D-graded integral domain $A = \bigoplus_{d\in D} A_d$ generated by homogeneous elements $x_1, \ldots, x_r$ such that $\{d_i = \deg x_i \,|\, 1 \le i \le r\}$ generates D. Moreover, assume that A_0 is a field. Let $d \in D$ be such that*

$$d \in D_f = \left\{ \deg a \,\middle|\, a \in A_f^\times \right\}$$

for every relevant element $f \in A$. Then $\mathscr{O}_X(d)$ is a line bundle.

Proof. By lemma 6.4.4, we can consider an open cover of X given by relevant monomials. Fix a d such that $d \in D_f$ for all relevant f. And fix an f which is a relevant monomial. On $D_+(f)$,

$$\mathscr{O}_X(d)|_{D_+(f)} = \widetilde{A_f(d)_{(0)}} = \widetilde{\left(A(d)\right)_{(f)}}.$$

by lemma 6.4.1(*b*). We claim that $A(d)_{(f)} \cong A_{(f)}$. Note that $1 \in A(d)$ has degree $-d$, which belongs to D_f by hypothesis. Thus by lemma 6.5.1, we can find an m such that $m \mid f^N$ for some N and $\deg(m/f^k) = -d$ for some k. This implies m/f^k is invertible in A_f and $\deg f^k/m = d$. Now it is evident that for any element of the form $\prod_{i=1}^r x_i^{a_i}/f^\nu$ in $A(d)_{(f)}$,

$$\deg_{A(d)} \frac{\prod_{i=1}^r x_i^{a_i}}{f^\nu} = 0 \iff \deg_A \frac{\prod_{i=1}^r x_i^{a_1}}{f^\nu} = d \iff \deg_A \frac{\prod_{i=1}^r x_i^{a_1}}{f^\nu} \frac{m}{f^k} = 0$$

and thus $\frac{\prod_{i=1}^r x_i^{a_1}}{f^\nu} \frac{m}{f^k} \in A_{(f)}$. Since m/f^k is invertible in A_f, this gives an isomorphism of $A_{(f)}$-modules. This proves that $\mathscr{O}_X(d)$ is a line bundle. □

Example 6.5.3. *In case of a weighted projective space, $P = \mathrm{Proj}\, \mathbb{C}[x_0, \ldots, x_n]$ with $\deg x_i = d_i$, theorem 6.5.2 reduces to saying $\mathscr{O}_P(d)$ is a line bundle iff d is divisible* *by each of the d_i's. This is well known (see [Del, remark 1.8]).*

Definition 6.5.4. Let $X = \mathrm{Proj}_{\mathrm{MH}} A$ be the multihomogeneous space associated with a D-graded ring A. For a quasi-coherent $\mathscr{O}_X$-module $\mathcal{F}$, we define a D-graded A-module

$$\Gamma_*(X, \mathcal{F}) := \bigoplus_{d \in D} \Gamma(X, \mathcal{F}(d)),$$

where $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(d), d \in D$.

Remark 6.5.5. *With the hypothesis as in theorem 6.4.3 one has $A \cong \Gamma_*(X, \mathscr{O}_X)$.*

$\Gamma_*(X, \_)$ is a covariant additive functor from the category of quasi-coherent $\mathscr{O}_X$-modules to the category of D-graded A-modules. However, it is not exact.

Proposition 6.5.6. *Assume X as in theorem 6.4.3. Let $\mathcal{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the homomorphism $\mu : \widetilde{\Gamma_*(X, \mathcal{F})} \to \mathcal{F}$ is an isomorphism. In fact, every quasi-coherent $\mathscr{O}_X$-module is of the form $\widetilde{F}$ for some D-graded A-module F.*

Proof. The proof is similar to the proof of [Cox, theorem 3.2] which itself is a generalization of [Har, proposition II.5.15]. □

Proposition 6.5.7. *Let D be a finitely generated free abelian group and A a D-graded ring. Let $X = \mathrm{Proj}_{MH} A$ be the corresponding multihomogeneous space. Then the functor $\widetilde{(\ )}$ from the category of D-graded A-modules to the category of quasi-coherent sheaves of $\mathscr{O}_X$-modules is essentially surjective.*

Proof. It follows from the previous proposition. □

7

Relation between multihomogeneous space and T-variety

7.1 Birationality

To study the relationship, we need a couple of assumptions. We shall explore them one by one.

Assumption 7.1.1. *Let $D \cong \mathbb{Z}^r$ for a natural number r and suppose $A = \bigoplus_{d\in D} A_d$ be a multigraded, Noetherian, normal domain such that $A_0 = k$, where k is an algebraically closed field of characteristic 0. $\operatorname{Proj}_{MH} A$ is non-empty.*

Lemma 7.1.2. *Suppose Λ is the GIT fan (see 5.2.7) associated to the $T = \operatorname{Spec} k[D]$ action on $X = \operatorname{Spec} A$ induced by the D-grading. Suppose λ is a full-dimensional cone in the quasi-fan Λ. Then there exists $u \in \operatorname{rel\,int} \lambda$ such that A_u contains a relevant element.*

Proof. By the definition of a quasi-fan, each of the rays $\rho \in \lambda(1)$ is also an orbit cone and hence there exists an $u_\rho \in \rho \cap D$ such that $A_{u_\rho} \neq \{ 0 \}$.

Since λ is full dimensional, $|\lambda(1)| \geq \dim \lambda$ and hence (λ be a strongly convex polyhedral cone) $\{u_\rho \,|\, \rho \in \lambda(1)\}$ is a spanning set of D over $\mathbb{Q}$. Choose a homogeneous $f_\rho \in A_{u_\rho}$ for each ρ and consider $f = \prod_{\rho\in\lambda(1)} f_\rho$.

We claim that f is relevant. This follows as once f is inverted, the degrees of units in A_f contains $\{\pm u_\rho \,|\, \rho \in \lambda(1)\}$ and hence $[D : D_f] < \infty$, where D_f is defined in the statement of theorem 6.5.2. □

Theorem 7.1.3. *Under the assumption 7.1.1, the torus $T = \operatorname{Spec} k[D]$ acts on $X = \operatorname{Spec} A$ giving X a structure of a T-variety which, suppose, is represented by $(Y, \mathfrak{D})$. Then Y and $\operatorname{Proj}_{MH} A$ are birational.*

Proof. Let Λ be the GIT fan and λ be a cone of maximal dimension. Choose a relevant f using lemma 7.1.2 such that $\deg f \in \operatorname{rel\,int} \lambda$. Suppose $u = \deg f$. Note that $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$ is a T-equivariant embedding. On the other hand, consider $X_{\mathrm{ss}}(u) \subset \operatorname{Spec} A$. Being open irreducible subsets of $\operatorname{Spec} A$, $X_{\mathrm{ss}}(u)$ and $\operatorname{Spec} A_f$ are birational. Now the result follows from the following commutative diagram:

$$\begin{array}{ccccccc} & & X_{\mathrm{ss}}(u) & \leftarrow\!\text{-}\,\text{-}\,\text{-}\!\rightarrow & \operatorname{Spec} A_f & \hookrightarrow & X = \operatorname{Spec} A \\ & & \downarrow & & \downarrow & & \\ Y & \longrightarrow & Y_\lambda & \leftarrow\!\text{-}\,\text{-}\,\text{-}\,\text{-}\,\text{-}\!\rightarrow & U & \hookrightarrow & \operatorname{Proj}_{\mathrm{MH}} A \end{array}$$

where the two vertical maps are geometric quotients (by the remark 6.1.5) under the action of T. The birational map $X_{\mathrm{ss}}(u) \dashrightarrow \operatorname{Spec} A_f$ is T-equivariant. Hence, Y_λ and U, the geometric quotients of $X_{\mathrm{ss}}(u)$ and $\operatorname{Spec} A_f$ by T, are also birational. By construction of multihomogeneous spaces, U is an open subset of $\operatorname{Proj}_{\mathrm{MH}} A$. The fact that $Y \longrightarrow Y_\lambda$ is birational follows from [AH, lemma 6.1]. This proves that Y and $\operatorname{Proj}_{\mathrm{MH}} A$ are birational. □

7.2 Conditions for isomorphism

In the rest of this section, we shall explore the conditions under which they become isomorphic.

Assumption 7.2.1. *Suppose $\lambda = \omega$, i.e. the GIT fan contains only one full dimensional cone and its faces. Assume that A is generated by $\bigcup_{u \in R} A_u$ where $R = \bigcup_{\rho \in \lambda(1)} \rho$.*

Proposition 7.2.2. *Assume 7.1.1 and 7.2.1. Assume that ω is simplicial and A is gen-*

erated by $\{f_\rho \,|\, \rho \in \omega(1)\}$ *such that* $\deg f_\rho \in \rho \cap D$. *Then* $\mathrm{Proj}_{MH} A$ *is projective.*

Proof. One can see that each relevant f is of the form $\prod_{\rho\in\omega(1)} f_\rho$. Then cone C_f associated to f is ω for all relevant f. Therefore by 6.2.2 and 6.2.6, $\mathrm{Proj}_{\mathrm{MH}} A$ is projective. □

Proposition 7.2.3. *Assume 7.1.1 and 7.2.1. Assume that* ω *is simplicial and* A *is generated by* $\{f_\rho \,|\, \rho \in \omega(1)\}$ *such that* $\deg f_\rho \in \rho \cap D$. *Then* Y *and* $\mathrm{Proj}_{MH} A$ *are isomorphic.*

Proof. Under the given conditions, there exists a collection of relevant monomials $\prod_{\rho\in\omega(1)} f_\rho$ which have degree $u = nu'$ where $u' = \sum_\rho u_\rho$, $n \in \mathbb{N}$ and

$$\mathrm{Proj}_{\mathrm{MH}} A = \bigcup D_+ \left(\prod_{\rho\in\omega(1)} f_\rho \right)$$

Consider $A_{(u)} = \bigoplus_{n\geq 0} A_{nu}$. It is generated by $A_u = (A_{(u)})_1$. Therefore, $\mathrm{Proj}_{\mathrm{MH}} A = \mathrm{Proj}\, A_{(u)} \cong Y$ (see [AH, 6.1]). □

Remark 7.2.4. *In the special case when* $A = k[X_1, \ldots, X_n]$ *with* $\deg X_i \in \mathbb{Z}^d$, *the affine space becomes a* T*-variety with the action of a d-dimensional torus. Assume that this action is effective. Then we know that the* Y *one gets from the description of the* T*-variety is normal and projective. It is difficult to characterize these further.*

Example 7.2.5. *The hypothesis in proposition 7.2.3 is satisfied for products of projective and weighted projective spaces. This is because, in the case of projective spaces and weighted projective spaces, the weight cone is the only full dimensional cone in the GIT fan. Also, if* X *and* Y *are varieties where the weight cones are the only full dimensional cones in their GIT fans, then the same is true for* $X \times Y$.

We can not weaken the hypothesis of the above proposition. Here is an example of an affine toric variety X and a subtorus T such that corresponding varieties Y and $\mathrm{Proj}_{\mathrm{MH}} A$, where A is the algebra of global sections of X, are not isomorphic.

Example 7.2.6 ([AH], example 11.1)**.** *Take the affine toric variety* $X = k^4$ *associated to the canonical cone* $\delta := (\mathbb{Z}_{\geq 0})^4$ *in* $N_X = \mathbb{Z}^4$ *and consider the subtorus* $T := k^{*2}$ *action on* X *given in standard coordinates by the embedding* $t = (t_1, t_2) \to (t_1^4, t_1^3, t_2, t_1^{12} t_2^{-1})$. *Then we have the following short exact sequence of lattices:*

$$0 \to N_T \xrightarrow{F} N_X \xrightarrow{P} N_Y \to 0,$$

where N_T *is the lattice of one parameter subgroups of* T *and* $N_Y := N_X/N_T$ *is the quotient lattice. The linear maps* F *and* P *are given by*

$$\begin{bmatrix} 4 & 0 \\ 3 & 0 \\ 0 & 1 \\ 12 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 & -1 & -1 \\ 0 & 4 & -1 & -1 \end{bmatrix}$$

Let Σ_Y *be the coarsest fan in* $(N_Y)_{\mathbb{Q}}$ *generated by* $P(\delta_0)$ *where* δ_0 *are faces of* δ*. The maximal cones of* Σ_Y *are given by*

$$\sigma_1 = \langle (1,0),(0,1)\rangle, \;\; \sigma_2 = \langle (0,1),(-1,-1)\rangle \text{ and } \;\sigma_3 = \langle (-1,-1),(1,0)\rangle.$$

Then the toric variety Y *is* $\mathbb{P}^2$ *and there exists a pp-divisor* $\mathfrak{D}$ *over* $Y = \mathbb{P}^2$ *such that the t-variety* (X,T) *is represented by the pair* $(Y,\mathfrak{D})$*.*

Now the algebra of global sections $A = k[x_1,x_2,x_3,x_4]$ *of* X *has a gradation by* $M_T = \mathbb{Z}^2$ *given by the deg map in the following short exact sequence*

$$0 \to M_Y \xrightarrow{\tilde{P}} M_X \xrightarrow{\tilde{F}} M_T \to 0$$

The linear maps $\tilde{P}$ *and* $\tilde{F}$ *are given by*

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \\ -1 & -1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 3 & 0 & 12 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Let $I = \{1,2,3,4\}$ *be an index set. Then* $\deg x_1 = (4,0), \deg x_2 = (3,0), \deg x_3 = (0,1)$ *and* $\deg x_4 = (12,-1)$ *in* M_T*. Let* $pr_i : M_X \to \mathbb{Z}, i \in I$ *be the projections and* $\rho_i := pr_i|_{M_Y} \in N_Y$*. Then we have four rays* $\rho_1 = (1,0)\mathbb{R}$*,* $\rho_2 = (0,1)\mathbb{R}$ *and* $\rho_3 = \rho_4 = (-1,-1)\mathbb{R}$ *generated by primitive vectors. Then by remark 6.2.10, a monomial* $f = x_i x_j \in A$ *where* $i,j \in I$ *is relevant if and only if the cone* $\sigma_f = \langle \rho_i : i \in I \text{ and } x_i \nmid f\rangle$ *is simplicial.*

Therefore, One can compute that

$$\begin{aligned}\mathrm{Proj}_{MH} A &= \bigcup_{f=x_ix_j \ relevant} D_+(f)\\ &= D_+(x_3x_4) \cup D_+(x_1x_3) \cup D_+(x_2x_3) \cup D_+(x_1x_4) \cup D_+(x_2x_4)\end{aligned}$$

Note that

$$\begin{aligned}Y = \mathbb{P}^2 &= D_+(x_3x_4) \cup D_+(x_1x_3) \cup D_+(x_2x_3)\\ &= D_+(x_3x_4) \cup D_+(x_1x_4) \cup D_+(x_2x_4)\end{aligned}$$

Therefore the multihomogeneous space $\mathrm{Proj}_{MH} A$ *is a union of two copies of* $\mathbb{P}^2$ *glued along open subscheme* $D_+(x_3x_4)$*. However the canonical map in 7.1.3 identifies* Y *with either* $D_+(x_3x_4) \cup D_+(x_1x_3) \cup D_+(x_2x_3)$ *or* $D_+(x_3x_4) \cup D_+(x_1x_4) \cup D_+(x_2x_4)$ *in* $\mathrm{Proj}_{MH} A$*. And hence the map in 7.1.3 is not an isomorphism. The weight cone* ω*, generated by* $(0,1)$ *and* $(12,-1)$*, is simplicial. The isomorphism fails to hold because the cone* ω *is not a GIT cone.*

8

Relation between multihomogeneous space and toric variety

8.1 An open embedding

Let X_Δ be a toric variety associated to the fan Δ in $N_\mathbb{R}$ satisfying assumptions 4.1.1 and 4.1.2, i.e. the fan does not lie in a lower dimensional subspace and the Picard group $\mathrm{Pic}(\Delta)$ of X is free. Further, assume that X_Δ has enough invariant Cartier divisors (see 4.1.5).

Lemma 8.1.1. *The weight cone ω (described in 4.1.10) is a full dimensional pointed strictly convex rational polyhedral cone in* $\mathrm{Pic}(\Delta)_\mathbb{R}$.

Proof. It is clear from the fact that (4.1) is split exact sequence and $C^\vee$ surjects onto ω. □

Definition 8.1.2 (cf. definition 1.3(2) in [Kaj])**.** For each $\sigma \in \Delta$, $\mathrm{SF}(\check{\sigma})$ denotes the group of integral support functions on Δ with support precisely $\check{\sigma} := \Delta(1) \setminus \sigma(1)$.

Lemma 8.1.3 (cf. lemma 1.7(2) in [Kaj])**.** *Let* $\sigma \in \Delta$*. Then* $\deg(\mathrm{SF}(\check{\sigma})) = \mathrm{Pic}(\Delta)$.

Proof. The proof follows from computing the rank of the following short exact sequence

$$1 \to M \cap \sigma^\perp \xrightarrow{\mathrm{Div}} \mathrm{SF}(\check{\sigma}) \xrightarrow{\deg} \mathrm{Pic}(\Delta) \to 1,$$

induced from 4.1. □

Proposition 8.1.4. *Let X_Δ be the toric variety associated with the fan Δ in $N_\mathbb{R}$. Then $\chi(h_\sigma)$ is a relevant element in A, the algebra of support functions (defined in §2), with respect to* $\mathrm{Pic}(\Delta)$*-grading for all $\sigma \in \Delta$.*

Proof. For the $\mathrm{Pic}(\Delta)$-graded $\mathbb{C}$-algebra $A_{\chi(h_\sigma)}$, the subgroup generated by degrees of units $D' = \deg(\mathrm{SF}(\check{\sigma})) = \mathrm{Pic}(\Delta)$ by lemma 8.1.3. Therefore $\chi(h_\sigma)$ is a relevant element in A by definition 6.1.1 for all cones $\sigma \in \Delta$. □

Theorem 8.1.5. *Let X_Δ be the toric variety with enough invariant Cartier divisors associated with the fan Δ and* $\mathrm{Spec}(\mathbb{C}[M])$ *its torus. Then there is a* $\mathrm{Spec}(\mathbb{C}[M])$ *equivariant open embedding $\mu : \mathrm{tProj}\, A \hookrightarrow \mathrm{Proj}_{MH} A$, where A is the algebra of support functions on Δ (defined in 4.1.10).*

Proof. Recall that $\mathrm{Proj}_{\mathrm{MH}} A := \bigcup_{f:\, f \text{ is relevant in } A} \mathrm{Spec}\, A_{(f)}$ and $\chi(h_\sigma)$ is relevant for all $\sigma \in \Delta$. For each $\sigma \in \Delta$, we have $U_\sigma \cong \mathrm{Spec}\, k[\sigma_M] \cong \mathrm{Spec}(A_{(\chi(h_\sigma))})$ by [Per, lemma 3.10] and for $\tau \preceq \sigma$, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{tProj}(A_{(\chi(h_\tau))}) & \xrightarrow{\cong} & \mathrm{Spec}(k[\tau_M]) & \xrightarrow{\cong} & \mathrm{Spec}(A_{(\chi(h_\tau))}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{tProj}(A_{(\chi(h_\sigma))}) & \xrightarrow{\cong} & \mathrm{Spec}(k[\sigma_M]) & \xrightarrow{\cong} & \mathrm{Spec}(A_{(\chi(h_\sigma))}). \end{array} \tag{8.1}$$

It follows from the above diagram that X_Δ is isomorphic to $\mathrm{tProj}\, A$. Moreover, the morphism μ, which is the composition of the following morphisms,

$$\mathrm{tProj}\, A \cong \bigcup_{\sigma\in\Delta} \mathrm{tProj}(A_{(\chi(h_\sigma))}) \cong \bigcup_{\sigma\in\Delta} \mathrm{Spec}(A_{(\chi(h_\sigma))}) \hookrightarrow \mathrm{Proj}_{\mathrm{MH}} A \tag{8.2}$$

is an open embedding. Note that the morphisms in 8.1 are $\mathrm{Spec}(\mathbb{C}[M])$ equivariant. This makes μ a $\mathrm{Spec}(\mathbb{C}[M])$ equivariant open embedding. □

8.2 Criterion for isomorphism

In this section, we give a criterion for the open embedding to be an isomorphism.

Let X_Δ be a simplicial toric variety associated to the fan Δ in $N_\mathbb{R}$ satisfying assumptions 4.1.1 and 4.1.2, i.e. the fan does not lie in a lower dimensional subspace and the Picard group $\mathrm{Pic}(\Delta)$ of X is free. Then we have the short exact sequence 4.1.

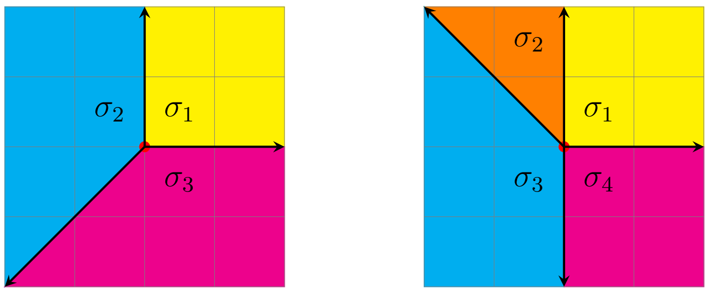
Keeping in mind remark 6.2.10, we define the following:

Definition 8.2.1. Let Δ be a simplicial fan in $N_\mathbb{R}$ and $\Delta(1)$ be the set of rays.

1. A simplicial cone in Δ is a cone $\tau \subset N_\mathbb{R}$ generated by S, a linearly independent subset of $\Delta(1)$.

2. Δ is said to be simplicially complete if it contains every simplicial cone in Δ.

Example 8.2.2. *The fans of projective and weighted projective spaces are simplicially complete but the fans of Hirzebruch surface $H_r, r \geq 1$ are not.*



Fan of $\mathbb{P}^2$ Fan of $\mathbb{H}_n$

Figure 8.1: Examples of simplicially complete and simplicially incomplete fans

Let Δ be a simplicial fan in $N_\mathbb{R}$. For each ray $\rho \in \Delta(1)$, we define an integral support function which takes the following values on the rays

$$K_\rho : |\Delta| \to \mathbb{R}, \qquad K_\rho(n_{\rho'}) = \begin{cases} \lambda_\rho, & \text{if } \rho' = \rho \\ 0, & \text{if } \rho' \neq \rho. \end{cases} \tag{8.3}$$

We choose $\lambda_\rho \in \mathbb{Z}_{\geq 0}$ in a way such that K_ρ are primitive element in the lattice $\mathrm{SF}(\Delta)$. Furthermore, K_ρ are linearly independent and are contained in $C^\vee \cap \mathrm{SF}(\Delta)$. For a fan Δ (not necessarily simplicial) in $N_\mathbb{R}$ we define the map $\Phi : \mathrm{SF}(\Delta)_\mathbb{R} \to \mathbb{R}^{\Delta(1)}$ by sending a support function h to $(h(n_\rho))_{\rho\in\Delta(1)}$. Then Φ is an injective linear map and the following diagram commutes

$$\begin{array}{ccc} \mathrm{SF}(\Delta) & \xrightarrow{\Phi} & \mathbb{Z}^{\Delta(1)} \\ \downarrow & & \downarrow \\ \mathrm{SF}(\Delta)_{\mathbb{R}} & \xrightarrow{\Phi} & \mathbb{R}^{\Delta(1)}. \end{array} \tag{8.4}$$

Φ has finite cokernel whenever Δ is simplicial.

Lemma 8.2.3. *Let X_Δ be a toric variety corresponding to the fan Δ in $N_{\mathbb{R}}$ satisfying 4.1.1 and A be the coordinate ring of the affine toric variety X_C (defined in 4.1.10). Then Δ is simplicial if and only if A is a polynomial $\mathbb{C}$-algebra of Krull dimension* $\dim(\mathrm{SF}(\Delta)_{\mathbb{R}})$.

Proof. Assume Δ is simplicial. Note that $\{K_\rho\}_{\rho\in\Delta(1)}$ forms a basis of $\mathrm{SF}(\Delta)$. If $h \in \mathrm{SF}(\Delta)$ and $\Phi(h) = (c_\rho)_{\rho\in\Delta(1)} \in \Phi(\mathrm{SF}(\Delta)) \cap (\mathbb{Z}_{\geq 0})^{\Delta(1)}$, then $c_\rho = h(n_\rho) = a_\rho\lambda_\rho$ implying $h = \sum_{\rho\in\Delta(1)} a_\rho K_\rho$. Therefore $\langle K_\rho : \rho \in \Delta(1)\rangle = C^\vee \cap \mathrm{SF}(\Delta) = \Phi(\mathrm{SF}(\Delta)) \cap (\mathbb{Z}_{\geq 0})^{\Delta(1)}$ which implies $A = \mathbb{C}[C^\vee \cap \mathrm{SF}(\Delta)]$ is a polynomial $\mathbb{C}$-algebra. It is clear that Krull dimension of A is $\dim(\mathrm{SF}(\Delta)_{\mathbb{R}})$.

On the other hand, assume that A is a polynomial $\mathbb{C}$-algebra of Krull dimension $\dim(\mathrm{SF}(\Delta)_{\mathbb{R}})$. It is then clear that there exists a basis for the semigroup $C^\vee \cap \mathrm{SF}(\Delta)$ which makes $C^\vee$ a simplicial cone. Therefore the cone C and its faces $\hat{\sigma}$, where $\sigma \in \Delta$, are simplicial. Finally remark 4.1.9 implies the cones $\sigma \in \Delta$ are simplicial. □

Remark 8.2.4. *When Δ is simplicial the $\mathbb{C}$-algebra $A = \mathbb{C}[\chi(K_\rho) : \rho \in \Delta(1)]$. For each $\sigma \in \Delta$ we can take $h_\sigma = \sum_{\rho\in\sigma(1)} K_\rho$ and therefore $\chi(h_\sigma) = \prod_{\rho\in\sigma(1)} \chi(K_\rho)$.*

Theorem 8.2.5. *Let X_Δ be a simplicial toric variety corresponding to the fan Δ in $N_{\mathbb{R}}$ satisfying 4.1.1 and A be the coordinate ring of the affine toric variety X_C (defined in 4.1.10). Then Δ is simplicially complete if and only if the morphism $\mu : \mathrm{tProj}\, A \to \mathrm{Proj}_{MH} A$ in 8.1.5 is an isomorphism.*

Proof. Note that $\mathcal{B} := \{K_\rho\}_{\rho\in\Delta(1)}$ is a basis of $\mathrm{SF}(\Delta)$ and for each ρ, $\{K_{\rho'}\}_{\rho\neq\rho'\in\Delta(1)}$ is a basis of the boundary ∂H_ρ. With respect to $\mathcal{B}$, the projections in 6.2.10 are given by $pr_\rho = r_\rho K_\rho^* = l_\rho \in \mathrm{SF}(\Delta)^\vee, \rho \in \Delta(1)$ where $r_\rho \in \mathbb{R}_{>0}$ and $K_\rho^* \in {\mathrm{SF}(\Delta)_{\mathbb{R}}}^\vee$ with $K_\rho^*(K_\rho) = 1$. Then the restrictions $pr_\rho|_M = \mathrm{image}(l_\rho) = n_\rho \in N$. Let S be a set of relevant monomials such that $\operatorname{Proj}_{\mathrm{MH}} A = \cup_{f \in S} D_+(f)$. Then by remark 6.2.10 each $f \in S$ corresponds to the simplicial cone

$$\begin{aligned} \sigma_f &= \langle pr_\rho : \rho \in \Delta(1) \setminus |f| \rangle \\ &= \langle n_\rho : \rho \in \Delta(1) \setminus |f| \rangle \end{aligned}$$

over Δ, and since Δ is simplicially complete, we have $\sigma_f = \sigma$ for some $\sigma \in \Delta$. Therefore $f = \chi(h_\sigma)$ for each $f \in S$ and hence $\operatorname{tProj} A = \cup_{\sigma \in \Delta} D_+(\chi(h_\sigma)) = \operatorname{Proj}_{\mathrm{MH}} A$.

Now assume μ is an isomorphism. Then we have $\operatorname{tProj} A \cong \cup_{\sigma \in \Delta} D_+(\chi(h_\sigma)) \cong \operatorname{Proj}_{\mathrm{MH}} A$. Suppose Δ is not simplicially complete and σ is a simplicial cone over Δ, not contained in Δ. Then σ corresponds to the relevant monomial $f = \prod_{\rho \in \Delta(1) \setminus \sigma(1)} \chi(K_\rho)$. The homogeneous prime ideal $\mathscr{P} = (\chi(K_\rho) : \rho \in \sigma(1))$ is contained in $\operatorname{Spec} A_{(f)}$. For any $\tau \in \Delta$ the corresponding relevant monomial $g = \prod_{\rho \in \Delta(1) \setminus \tau(1)} \chi(K_\rho)$ takes zero at the point $\mathscr{P}$ and therefore $\mathscr{P} \notin \operatorname{Spec} A_{(g)}$. Hence we get $\mathscr{P} \in \operatorname{Spec} A_{(f)} \subset \operatorname{Proj}_{\mathrm{MH}} A$ and $\mathscr{P} \notin \cup_{\sigma \in \Delta} D_+(\chi(h_\sigma)) = \operatorname{Proj}_{\mathrm{MH}} A$, a contradiction. Therefore Δ is simplicially complete. □

The following are some easy implications of the preceding theorem 8.2.5.

Corollary 8.2.6. *Let X be a simplicial toric surface associated to Δ in $N_{\mathbb{R}}$ satisfying 4.1.1. Then the morphism $\mu : X \to \operatorname{Proj}_{MH} A$ in 8.1.5 is not an isomorphism if and only if there exists a simplicial cone τ over Δ, not contained in Δ, with $\sigma \subset \tau$ for some $\sigma \in \Delta$.*

Proof. μ not being an isomorphism means Δ is not simplicially complete. In the forward direction, we can take σ to be a one-dimensional cone. The opposite direction is obvious. □

Corollary 8.2.7. *Let X be a simplicial toric surface associated to Δ in $N_{\mathbb{R}}$ satisfying 4.1.1. If rank$(\operatorname{Pic}(\Delta)) \geq 3$ then the morphism $\mu : X \to \operatorname{Proj}_{MH} A$ in theorem 8.1.5 is not an isomorphism.*

Proof. The hypothesis implies cardinality of $\Delta(1)$ is at least 5. Note that each cone in Δ is generated by at most 2 one-dimensional cones. There are 3 one-dimensional cones in less than 180°. This says Δ is not simplicially complete, in other words, μ is not an isomorphism. □

Remark 8.2.8. *Let X be a simplicial toric surface associated to Δ in $N_{\mathbb{R}}$ satisfying 4.1.1. One can show that the condition rank$(\mathrm{Pic}(\Delta)) \geq 3$ is equivalent to the condition that the fan Δ has at least 5 one-dimensional cones.*

Recall that, all complete nonsingular toric surfaces are gotten by successive blow-ups of either $\mathbb{P}^2$ or Hirzebruch surfaces $H_r, r \geq 0$.

Corollary 8.2.9. *Let X be a complete nonsingular toric surface corresponding to fan Δ, A be the $\mathbb{C}$-algebra of support functions on Δ and $\mathrm{Proj}_{MH} A$ the associated multihomogeneous space.*

(i) *$X = \mathbb{P}^2$: Theorem 8.2.5 implies $\mu : X \to \mathrm{Proj}_{MH} A$ is an isomorphism since the fan is simplicially complete.*

(ii) *$X = H_0 = \mathbb{P}^1 \times \mathbb{P}^1$: Same argument as in (i).*

(iii) *$X = H_r, r \geq 1$: It is obvious to see that the fan of X is not simplicially complete, so by theorem 8.2.5 $\mu : X \to \mathrm{Proj}_{MH} A$ is not an isomorphism.*

(iv) *X is a (successive) blowup of $\mathbb{P}^2$ or $H_r, r \geq 0$: Recall that blowup at a torus invariant subset corresponds to a refinement of the fan. Therefore, by statement (iii), the refinement is also simplicially incomplete and then theorem 8.2.5 says $\mu : X \to \mathrm{Proj}_{MH} A$ is not an isomorphism.*

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