# Quantum Field Theory at Strong Coupling

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**ABHISHEK KUMAR MEHTA**

[**20183621**]


IISER PUNE

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**Pune - 411 008**


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# CERTIFICATE

Certified that the work incorporated in the thesis entitled

**Quantum Field Theory at Strong Coupling**

submitted by Mr.Abhishek Kumar Mehta was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.

10/05/2023

Date

Sachin Jain

Dr. Sachin Jain

Supervisor

# DECLARATION

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Abhishek Kumar Mehta
[20183621]

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*Dedicated to Maharishi Valmiki and Maharishi Vedavyasa*

# ABSTRACT

Strongly-coupled Quantum Field Theories (QFT) do not admit a perturbative expansion in the coupling and a Lagrangian formulation is not suitable for such theories. This forces us to rely on non-perturbative techniques like bootstrap, and duality where symmetries (global and broken) play an important role in computing observables like correlation functions. A class of such strongly-coupled QFTs admits conformal invariance, hence, called Conformal Field Theories ($CFT$), which completely constrains the two-point and three-point correlations. Conformal bootstrap has led to interesting developments in the study of $CFT$s and is applicable at strong or weak coupling. These studies are restricted mostly to position space and Mellin space. Recently, momentum-space $CFT$ is gaining attention due to its connection to cosmology, flat-space scattering amplitudes, and theories where perturbation theory is amenable. However, momentum-space $CFT$ has not undergone much development compared to position-space $CFT$.

In this work, I will mostly focus on 3D $CFT$ correlation functions in momentum-space and show new results even at the level three-point function that was not discovered in position space. I will discuss the existence of substructures within the three-point correlation functions which will help us demonstrate double-copy and a correspondence to flat-space scattering amplitude. In this thesis, a systematic way to compute three-point functions of arbitrary spins in momentum space is also discussed. In the cosmological correlator context, these momentum space CFT correlation functions play an important role. In particular, in this thesis, it is shown how the $\alpha$-vacua correlation function in dS space can be understood in terms of the $CFT$ correlation function in momentum space if we relax OPE consistency.

A special class of $CFT$s called the Chern-Simons Matter Theories is also discussed. This class of $CFT$s, admits a strong-weak duality and a Vasiliev dual in one higher dimension. We use analytical bootstrap tools and duality to compute the four-point functions of the scalar operator in the Supersymmetric Chern-Simons Matter theories and an all-loop conjecture for the anomalous dimension of the scalar operators in Chern-Simons theories which is not possible to compute via the Higher-Spin symmetry. We justify our calculations via various checks.

# List of Publications

[1] *S. Jain, V. Malvimat, A. Mehta, S. Prakash, N. Sudhir (2020) *All order exact result for the anomalous dimension of the scalar primary in Chern-Simons vector models*, Phys. Rev. D 101 (2020) 12, 126017, arXiv: 1906.06342 [hep-th]

[2] *K. Inbasekar, S. Jain, V. Malvimat, A. Mehta, P. Nayak (2020) *Correlation functions in $\mathcal{N} = 2$ Supersymmetric vector matter Chern-Simons theory*, JHEP 04 (2020) 207, arXiv: 1907.11722 [hep-th]

[3] *S. Jain, R. R. John, A. Mehta, A. A. Nizami, A. Suresh (2021) *Momentum space parity-odd CFT 3-point functions*, JHEP 08 (2021) 089, arXiv: 2101.11635 [hep-th]

[4] *S. Jain, R. R. John, A. Mehta, A. A. Nizami, A. Suresh (2021) *Double copy structure of parity-violating CFT correlators*, JHEP 07 (2021) 033, arXiv: 2104.12803 [hep-th]

[5] *S. Jain, R. R. John, A. Mehta, A. A. Nizami, A. Suresh (2021) *Higher spin 3-point functions in 3d CFT using spinor-helicity variables*, JHEP 09 (2021) 041, arXiv: 2106.00016 [hep-th]

[6] S. Jain, R. R. John, A. Mehta, D. K. Satyanarayanan (2022) *Constraining momentum space CFT correlators with consistent position space OPE limit and the collider bound*, JHEP 02 (2022) 084, arXiv: 2111.08024 [hep-th]

[7] *S. Jain, A. Mehta (2023) *4D flat-space scattering amplitude/$CFT_3$ correlator correspondence revisited*, Nucl. Phys. B (2023) 991 p. 116193, arXiv: 2201.07248 [hep-th]

[8] *S. Jain, N. Kundu, S. Kundu, A. Mehta, S. K. Sake (2022) *A CFT interpretation of cosmological correlation functions in $\alpha$-vacua in de-Sitter space*, arXiv: 2206.08395 [hep-th], *To be published (JHEP)*

[9] A. Mehta (2022) *Gateway-like absurdly-benign traversable wormhole solutions*, TMF 01 (2023) 214 p. 122-139, arXiv: 2211.03709 [gr-qc]

This thesis is based on the publications that are marked with the '*'.

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# INTRODUCTION

Quantum Field Theory (QFT) is one of the most successful theoretical framework that explains a wide range of physical phenomena especially the Standard Model (SM) of Particle Physics. QFT as a theoretical framework combines Classical Field Theory, Special relativity, and Quantum Mechanics. This was motivated by the need to explain various quantum mechanical phenomena like the atomic and black body spectra. This requires a self-consistent framework that combined Quantum Mechanics and Classical Field Theory. The fundamental constituents of a QFT are quantum fields which are more fundamental than particles. A QFT is characterized by a Poincare-invariant action which consists of Poincare-invariant products of these quantum fields as interactions [1]. For a small enough coupling, the interactions are represented by Feynman diagrams perturbatively. This eventually led to identifying the correct $g$-factor [2] for the electron which was useful in computing the fine structure of the hydrogen atom and derive the relativistic Compton scattering. It also led to the prediction of antiparticles. One of the most important predictions of QFT is the W and Z bosons which were confirmed by experiments at the LHC [3]. The Higgs mechanism is another prediction that was confirmed by the LHC. It has also made predictions in relation to condensed matter systems as well. These series of predictions for high-energy particles made by the QFT together form the basis of the Standard Model (SM) of particle physics. The SM unified three of the four fundamental forces - the strong, weak, and electromagnetic forces into a single theoretical framework and accounted for the existence of all the known elementary particles.

One of the fundamental tools in QFT is perturbation theory. However, this perturbation theory breaks down at strong coupling. Even at weak coupling higher-order perturbative calculations of observables always result in infinities. These divergences are sourced by Feynman diagrams having closed loops of virtual particles. Virtual particles are particles that obey momentum conservation but do not satisfy the dispersion relation. These particles are said to be off-shell. Divergences in Feynman integrals are of two kinds - Ultraviolet (UV) and Infrared (IR) [4]. UV divergences come from the high energy limit of the Feynman integrals and these divergences require a renormalization procedure to be removed. The renormalization is usually implemented by picking a fixed momenta value (or an energy scale) to define the renormalized parameter and the observables are then computed as a perturbative expansion in the renormalized parameter. Appropriate counter-terms are added to the action to cancel the divergences. This leads to a renormalization of the fields and the couplings in the theory. This procedure, however, makes the coupling run with the energy scale. For example, the effective coupling in Quantum Electrodynamics (QED) as a function of energy scale $\mu$ looks like [1, 4]

$$e_{\text{eff}}^2\left(p^2\right) = \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2}\ln\frac{p^2}{\mu^2}} \tag{1}$$

This result is obtained after the summation of 1-loop 1PI integrals. One can see that at $p = \mu$ one obtains the renormalized coupling. The renormalized coupling is something that is measured by experiments at a particular energy scale. On the other hand, the IR divergences arise due to the low momenta limit of the Feynmann integrals. There is no renormalization procedure to remove these divergences. Instead, these divergences cancel by summing various physical processes [4, 5].

QFT armed with this renormalization procedure has been confirmed with various high-energy physics experiments. The SM was also crucial in validating the renormalization procedure and reinforced the role of gauge and global symmetries in the computation of scattering amplitudes and other observables. An important quantity related to the running coupling is called the beta function that is defined as [4]

$$\beta(e_{\text{eff}}) \equiv \mu\frac{de_{\text{eff}}}{d\mu} \tag{2}$$

The beta function tells how the theory flows in the parameter space. It can also tell where the theory might break down. These points are called Landau poles. They are also important in determining fixed-points of the renormalization group (RG) flow [6] where the beta function vanishes. At these fixed-points, new symmetries arise requiring a new prescription for the theory. This new symmetry is scale invariance. This scale invariance is actually part of a larger symmetry called the conformal invariance. QFTs having these symmetries are called Conformal Field Theories ($CFT$s). QFTs with only scale invariance but no conformal invariance are rare [7, 8]. Any UV complete QFT can be thought of as an RG flow between $CFT$s, hence, studying $CFT$s helps understand the space of QFTs. These $CFT$s are especially important because, despite the tremendous success of perturbative QFT, many areas of physics are still beyond its reach. For instance, there is no renormalizable QFT for gravity. A theory is said to be non-renormalizable if it cannot be renormalized with a finite number of counter-terms. However, a perturbative prescription of quantum gravity is possible via the $AdS/CFT$ correspondence. Also, due to the perturbative nature of the renormalizable QFT, non-perturbative and strong coupling regimes of the theory remain inaccessible. As an example, one can look at the massive $\phi^4$ theory in $3D$ [9]

$$S = \int d^3x \left( \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4 \right) \tag{3}$$

In the UV, this is a free theory, hence, naturally a $CFT$. In the IR, the theory is massive. For a particular ratio of $m^2/g$, the beta function vanishes and the theory becomes a $CFT$ again. This IR $CFT$ cannot be studied by naive perturbative expansion in Feynmann diagrams. However, there are multiple QFTs or rather microscopic systems that give the same IR $CFT$. For instance, the $3D$ Ising model for a specific value of the interaction coefficient gives the same IR $CFT$ as the $\phi^4$ theory. Such theories with the same IR $CFT$ are called IR equivalent theories. They are also called dualities in the context of high-energy physics. This phenomenon of IR equivalences is called critical universality [9]. Critical universality implies that $3D$ $\phi^4$ theory can be studied via the $3D$ Ising model and vice-versa. This is a very powerful tool to simplify the study of various microscopic systems. However, one can do better by exploiting the emergent conformal symmetries through the conformal bootstrap. Eventually, through the bootstrap programme one aims to provide a fully non-perturbative formulation of QFT without a Lagrangian prescription.

### ***Conformal Bootstrap***

Through conformal bootstrap, one aims to simply focus on the $CFT$s and use the conformal symmetry to constrain or completely solve the theory. It must be noted that in some cases bootstrap is the only known strategy for understanding a theory completely as a Lagrangian description doesn't exist. There are many well-known techniques to solve these $CFT$s like the $\epsilon$-expansion where one works in $D = 4 - \epsilon$ and looks at the observables in the orders of $\epsilon$. By setting $\epsilon = 1$, one tries to recover the behvaiour of theories in $3D$. Another well-known technique is the Exact Renormalization Group (ERG) where, for example, one starts from a scalar theory and add potentially an infinite number of local operators [10]

$$(\partial\phi)^2 + \sum_i c_i \Lambda^{D-\Delta} \mathcal{O}_i \tag{4}$$

After putting the above in the RG flow equation, one looks for its fixed points. This technique helps us find the whole critical surface in the RG space and look for a UV fixed point of the theory. However, this technique is difficult to apply to gauge theories. ERG works best when a fixed point is known. The reliability of this method in finding new fixed points or finding fixed points that do not have a local description is questionable. Also, RG-based techniques to solve theories at fixed points are against the spirit of bootstrap. For the purposes of implementing the bootstrap and solving the theories at the fixed point directly, an axiomatic framework is essential.

All such fixed point theories or $CFT$s are invariant under conformal symmetry, the generators of which form the conformal algebra.

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu \\
[M_{\mu\nu}, K_\rho] &= \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu \\
[M_{\mu\nu}, M_{\rho\sigma}] &= \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\rho\mu} - \delta_{\mu\sigma} M_{\rho\nu} \\
[D, P_\mu] &= P_\mu \\
[D, K_\mu] &= -K_\mu \\
[K_\mu, P_\nu] &= 2\delta_{\mu\nu} D - 2M_{\mu\nu}
\end{aligned} \tag{5}$$

Sometimes it is convenient to think about the action of the above in terms of $\mathbb{R}^{d+1,1}$ instead of $\mathbb{R}^d$ as a $CFT_d$ algebra is isomorphic to $SO(d+1,1)$ algebra. This is called the "embedding space formalism" [11–14]. These theories are characterized by local operators called primaries that are defined by

$$[K^\kappa, \mathcal{O}(0)] = 0 \quad [D, \mathcal{O}(0)] = \Delta_{\mathcal{O}} \mathcal{O}(0) \tag{6}$$

Using the operator $\mathcal{O}$, one can define a corresponding state $|\mathcal{O}\rangle \equiv \mathcal{O}(0)|0\rangle$ which satisfy

$$K^{\kappa}|\mathcal{O}\rangle = 0 \quad D|\mathcal{O}\rangle = \Delta_{\mathcal{O}}|\mathcal{O}\rangle \tag{7}$$

which is a direct consequence of 6. This is called state-operator correspondence. This correspondence means a state of the form $|\Psi\rangle \equiv \mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle$ will have the following expansion [10, 15]

$$\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \sum_{\mathcal{O}\text{ primaries}} \lambda_{12\mathcal{O}} C_{\mathcal{O}}\left(x, \partial_y\right) \mathcal{O}(y)|_{y=0}|0\rangle \tag{8}$$

This is called the operator product expansion (OPE) and $\lambda_{12\mathcal{O}}$ is the OPE coefficient. The OPE is the most important result in this axiomatic framework as it allows one to decompose the product of two primaries at short distances into a sum of primaries with known scaling dimensions and OPE coefficients. The bootstrap program involves using the OPE to derive constraints on the scaling dimensions and OPE coefficients of the primaries. This involves looking at the four-point function of four primaries and using the OPE to expand the product of two of the operators at small distances. This leads to an expansion of the form 8 which is used to constrain the values of these variables. The four-point function can now be written in terms of the scaling dimensions and OPE coefficients, along with the conformal symmetry relating the terms in the expansion. This leads to the bootstrap equations that can be solved to constrain the values of the scaling dimensions and operator product coefficients. In short, the non-perturbative axiomatic framework that characterizes the theories at the fixed point of RG flows has the following ingredients [16]

- Spectrum of primary operators
- Conformal invariance
- State-Operator correspondence $\implies$ OPE

The aim of this framework is to solve the theory by constraining the scaling dimension of the primaries and the OPE coefficients. This is in essence the conformal bootstrap.

It is interesting to note that conformal invariance fixes the two-point and the three-point function entirely. Given the following conformal ward identities

$$0 = \left[\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} x_j^{\alpha} \frac{\partial}{\partial x_j^{\alpha}}\right] \langle \mathcal{O}_1 (\boldsymbol{x}_1) \dots \mathcal{O}_n (\boldsymbol{x}_n) \rangle \tag{9}$$

$$0 = \left[\sum_{j=1}^{n} \left(2\Delta_j x_j^{\kappa} + 2x_j^{\kappa} x_j^{\alpha} \frac{\partial}{\partial x_j^{\alpha}} - x_j^2 \frac{\partial}{\partial x_{j\kappa}}\right)\right] \langle \mathcal{O}_1 (\boldsymbol{x}_1) \dots \mathcal{O}_n (\boldsymbol{x}_n) \rangle \tag{10}$$

One can see that for $n = 2, 3$, one can find the following solutions

$$\langle \mathcal{O}_1 (x_1) \mathcal{O}_2 (x_2) \rangle = \frac{C\delta_{\Delta_1 \Delta_2}}{x_{12}^{2\Delta_1}} \tag{11}$$

$$\langle \mathcal{O}_1 (x_1) \mathcal{O}_2 (x_2) \mathcal{O}_3 (x_3) \rangle = \frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{12}$$

upto some constant coefficients [17]. For $n = 4$, we find that [9]

$$\langle \mathcal{O}_1 (x_1) \mathcal{O}_2 (x_2) \mathcal{O}_3 (x_3) \mathcal{O}_4 (x_4) \rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_2)} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_4)} \frac{G(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)} (x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}} \tag{13}$$

the solution can only be determined up to a function of conformally invariant cross-ratios $u, v$ given by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{14}$$

One can compute arbitrary $n$-point functions in terms of the $(n-1)$-point functions using the OPE as follows

$$\langle \mathcal{O}_1 (x_1) \mathcal{O}_2 (x_2) \cdots \mathcal{O}_n (x_n) \rangle = \sum_k C_{12k} (x_{12}, \partial_2) \langle \mathcal{O}_2 (x_2) \cdots \mathcal{O}_n (x_n) \rangle \tag{15}$$

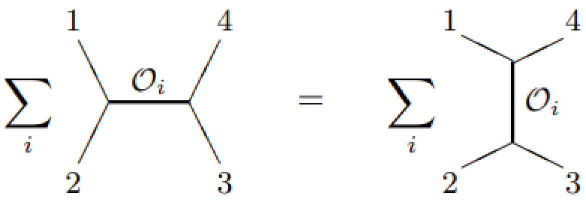
The OPE expansion allows us to compute arbitrary correlation function in terms of the conformal scaling dimensions of the external primaries $\Delta_i$ and OPE coefficients $\lambda_{ijk}$. $(\Delta_i, \lambda_{ijk})$ together forms the $CFT$ data. However, depending on how one performs the OPE, one can obtain a different expansion in terms of the $CFT$ data. These expansions must agree and this is called OPE associativity and this leads to the following crossing symmetry equation [18, 19]

$$\langle \overbracket{O_1\, O_2}\, \overbracket{O_3\, O_4} \rangle = \langle \overbracket{O_1\, \overbracket{O_2\, O_3}\, O_4} \rangle$$

where

$$\overbracket{O_i\, O_j}$$

means an OPE has been made. This is often diagrammatically represented as [9]




(16)

The crossing equation requires various numerical and analytical techniques to be solved or at least constrain the $CFT$ data.

## ***Momentum-space*** $CFT$

While $CFT$s are well-studied in position space and Mellin space, they are relatively less studied in momentum space which is surprising as momentum space $CFT$s find applications in inflationary cosmology [20–32]. Also, from the perspective of perturbative field theory which is naturally formulated in momentum space, it is of interest to study $CFT$s in the same setting. Flat-space scattering amplitudes are, via $AdS/CFT$, directly related to the flat space limit of $CFT$ correlators in momentum space [33] [1]. Studying momentum space $CFT$ correlators can therefore shed light on the structure of flat-space amplitudes. Interestingly, evidence for the double copy structure - which exists for flat space amplitudes - was seen directly in momentum space $CFT$ 3-point correlators in [39, 40]. The momentum space $CFT$ just like the position space $CFT$ is characterised by

[1]There are analogous, though somewhat less straightforward relations in Mellin space [34, 35] and position space [36–38].

the momentum space conformal ward identities [41]

$$0 = \left[ \sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right] \langle \mathcal{O}_1(\boldsymbol{p}_1) \ldots \mathcal{O}_n(\boldsymbol{p}_n) \rangle \tag{17}$$

$$0 = \left[ \sum_{j=1}^{n-1} \left( 2(\Delta_j - d) \frac{\partial}{\partial p_j^\kappa} - 2p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_j^\kappa} + (p_j)_\kappa \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_{j\alpha}} \right) \right] \langle \mathcal{O}_1(\boldsymbol{p}_1) \ldots \mathcal{O}_n(\boldsymbol{p}_n) \rangle \tag{18}$$

The solutions to these conformal ward identities for $n = 3$ can be written in the basis of triple-K integrals [41]

$$I_{\alpha\{\beta_1,\beta_2,\beta_3\}}(p_1, p_2, p_3) = \int_0^\infty dx x^\alpha \prod_{j=1}^{3} p_j^{\beta_j} K_{\beta_j}(p_j x) \tag{19}$$

where

$$\alpha = \frac{d}{2} - 1 + N, \qquad \beta_j = \Delta_j - \frac{d}{2} + k_j, \quad j = 1, 2, 3. \tag{20}$$

Unlike the position-space, however, momentum-space analogues for spinning three-point functions are quite complicated. They involve various combinations of the triple-K integrals shown above [41]. A careful treatment also requires the regularisation and renormalization of these divergences [41, 42] [2]. The triple-K integral only converges when

$$\alpha + 1 > |\beta_1| + |\beta_2| + |\beta_3| \quad p_1, p_2, p_3 > 0 \tag{21}$$

Beyond this range, the triple-K integral is defined through a unique analytical continuation. In fact, it is convenient to consider the triple-K integral to be a maximally extended analytic function that in its domain of convergence agrees with 19. In spite of this, the triple-K integral exhibits singularities for

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2n \quad n = 0, 1, 2, \cdots \tag{22}$$

Therefore, the triple-K integrals are regulated via

$$\alpha \to \tilde{\alpha} = \alpha + u\epsilon \quad \beta_j \to \tilde{\beta}_j = \beta_j + v_j\epsilon \tag{23}$$

[2] In position space one could get rid of divergences by working at non-coincident points, but in momentum-space, one cannot do this and this leads to UV divergences.

Depending on the divergences, counter terms are added to the $CFT$ action in order to remove the divergences and obtain the renormalized correlator

$$\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\, \mathcal{O}_2(\boldsymbol{p}_2)\, \mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle_{\text{ren}} = \lim_{\epsilon\to 0}\left[\langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\, \mathcal{O}_2(\boldsymbol{p}_2)\, \mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle_{\text{reg}} + \langle\langle \mathcal{O}_1(\boldsymbol{p}_1)\, \mathcal{O}_2(\boldsymbol{p}_2)\, \mathcal{O}_3(\boldsymbol{p}_3)\rangle\rangle_{\text{ct}}\right] \tag{24}$$

where 'reg' stands for regulated correlator and 'ct' stands for the contribution of the counter-term to the correlation. The story of spinning correlators is even more complicated [41, 43–45], especially the study of spinning parity-odd correlators in momentum space which is limited [43] and there is no known systematic methodology to compute correlations of arbitrary spin. In this thesis, this issue is discussed in detail and resolved by using spinor-helicity variables in chapter 1.

When it comes to the four-point functions there are multiple representations one can work with. For example, we have the simplex representation of the four-point function proposed in [46, 47]

$$\langle\langle \mathcal{O}_{\Delta_1}(\boldsymbol{p}_1)\, \mathcal{O}_{\Delta_2}(\boldsymbol{p}_2)\, \mathcal{O}_{\Delta_3}(\boldsymbol{p}_3)\, \mathcal{O}_{\Delta_4}(\boldsymbol{p}_4)\rangle\rangle = \int \frac{\mathrm{d}^d\boldsymbol{q}_1}{(2\pi)^d}\frac{\mathrm{d}^d\boldsymbol{q}_2}{(2\pi)^d}\frac{\mathrm{d}^d\boldsymbol{q}_3}{(2\pi)^d}\frac{\hat{f}(\hat{u},\hat{v})}{\mathrm{Den}_3(\boldsymbol{q}_j,\boldsymbol{p}_k)} \tag{25}$$

where

$$\mathrm{Den}_3(\boldsymbol{q}_j,\boldsymbol{p}_k) = q_3^{2\delta_{12}+d} q_2^{2\delta_{13}+d} q_1^{2\delta_{23}+d}\, |\boldsymbol{p}_1+\boldsymbol{q}_2-\boldsymbol{q}_3|^{2\delta_{14}+d}\, |\boldsymbol{p}_2+\boldsymbol{q}_3-\boldsymbol{q}_1|^{2\delta_{24}+d}\, |\boldsymbol{p}_3+\boldsymbol{q}_1-\boldsymbol{q}_2|^{2\delta_{34}+d} \tag{26}$$

The integrand is a function of momentum-dependent cross-ratios very much like the position space four-point function

$$\hat{u} = \frac{q_1^2\,|\boldsymbol{p}_1+\boldsymbol{q}_2-\boldsymbol{q}_3|^2}{q_2^2\,|\boldsymbol{p}_2+\boldsymbol{q}_3-\boldsymbol{q}_1|^2}, \quad \hat{v} = \frac{q_2^2\,|\boldsymbol{p}_2+\boldsymbol{q}_3-\boldsymbol{q}_1|^2}{q_3^2\,|\boldsymbol{p}_3+\boldsymbol{q}_1-\boldsymbol{q}_2|^2} \tag{27}$$

These cross-ratios make the crossing symmetry manifest in the momentum space representation. However, further study is required to gauge its utility in momentum-space bootstrap. Also, its generalization to higher spin and relation to scattering amplitudes is not very clear. One can also simply construct the four-point functions using the momentum-space three-point functions in a relatively straightforward manner [48]. To see that, one can use the momentum-space OPE expansion given by [49]

$$\phi_1(p_1)\phi_2(p_2)|0\rangle = \lambda_{\mathcal{O}12}\tilde{C}_{\mathcal{O}12}^{\mu_1\cdots\mu_l}(p_1, p_1+p_2)\mathcal{O}^{\mu_1\cdots\mu_l}(p_1+p_2)|0\rangle \tag{28}$$

where

$$\begin{aligned}\tilde{C}_{\mathcal{O}12}^{\mu_1\cdots\mu_l}(p,0) =& \frac{i^l 2^{d-\Delta_1-\Delta_2+\Delta_{\mathcal{O}}+1}\pi^{(d+2)/2}}{\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}+l}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}-l-d+2}{2}\right)} \\ &\times(-p^2)^{(\Delta_1+\Delta_2-\Delta_{\mathcal{O}}-l-d)/2}[p^{\mu_1}\cdots p^{\mu_l} + \text{trace terms}]\end{aligned} \tag{29}$$

to write the momentum-space four-point function as

$$\langle 0\,|[\phi_1\left(p_1\right)\phi_2\left(p_2\right)]\,[\phi_3\left(p_3\right)\phi_4\left(p_4\right)]|\,0\rangle \equiv (2\pi)^d\delta^d\left(p_1+p_2+p_3+p_4\right)G\left(p_1,p_2,p_3\right) \tag{30}$$

where

$$G\left(p_1,p_2,p_3\right) = s^{(\Delta_1+\Delta_2+\Delta_3+\Delta_4-3d)/2}\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{\mathcal{O}34}G_{\Delta,\ell}\left(p_1,p_2,p_3\right) \tag{31}$$

and $G_{\Delta,\ell}$ can be written as a product of vertex functions

$$G_{\Delta,\ell}\left(p_1,p_2,p_3\right) = \sum_{m=0}^{\ell} C_{\Delta,\ell,m}\mathcal{C}_m^{(d-3)/2}(\cos\theta)V_{\Delta,\ell,m}^{[12]}\left(\frac{p_1^2}{s},\frac{p_2^2}{s}\right)V_{\Delta,\ell,m}^{[34]}\left(\frac{p_3^2}{s},\frac{p_4^2}{s}\right) \tag{32}$$

where

$$\left\langle\left\langle[\phi_1\left(p_1\right)\phi_2\left(p_2\right)]\,\mathcal{O}^{(\ell,m)}\left(p\mid q_{12}\right)\right\rangle\right\rangle \equiv \lambda_{12\mathcal{O}}s^{(\Delta_1+\Delta_2+\Delta-2d)/2}V_{\Delta,\ell,m}^{[12]}\left(\frac{p_1^2}{s},\frac{p_2^2}{s}\right) \tag{33}$$

$$C_{\Delta,\ell,m} = \frac{2^{2\Delta-\ell+1}(m!)^2(\ell-m)!(d-2+2m)_{\ell-m}\Gamma\left(\Delta-\frac{d-2}{2}\right)\Gamma(\Delta+\ell)}{(4\pi)^{(d+2)/2}\ell!\left(\frac{d-2}{2}+m\right)_{\ell-m}\left(\frac{d-3}{2}\right)_m(\Delta-1)_m(\Delta-\ell-d+2)_{\ell-m}} \tag{34}$$

The four-point function in momentum space is again determined upto the OPE coefficients or the $CFT$ data. However, in its current form, the momentum space crossing equation is not very tractable to determine or constrain the $CFT$ data. The above has singularities and branch cuts whose physical significance is largely unknown. However, one can use unitarity and analyticity in the spirit of S-matrix bootstrap to constrain the $CFT$ data.

## *Cosmological Correlations*

One of the immediate applications of momentum-space $CFT$ is in the study of inflationary cosmology [20–32]. During the inflationary period, both quantum mechanics and gravity played an essential role where small quantum fluctuations were stretched out in a correlated fashion which we today see as the Cosmic Microwave Background (CMB). These correlations can provide insights into their origins and the conditions in which they were formed. These correlations can be traced back to the beginning of the hot Big Bang which was preceded by the inflationary epoch where the quantum fluctuations were created. Therefore, these correlations exist on the past boundary of the big bang spacetime or the future boundary of the inflationary spacetime. Now, one only has to follow the evolution of these correlations through the entirety of the spacetime to reconstruct the CMB. This makes locality, causality and unitarity completely manifest. This is the basis of the cosmological bootstrap. In the cosmological bootstrap, the correlations on the CMB are reconstructed consistently with locality, unitarity and conformal symmetry requirements. The conformal symmetry requirement arises from the observation that the inflationary epoch was very close to de Sitter ($dS_4$) space

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + \delta_{\mu\nu}dx^\mu dx^\nu) \tag{35}$$

The correlations are computed on the $\eta \to 0$ boundary of the spacetime the earliest moment in time the CMB can be traced back to from the present epoch. Since, $dS_4$ spacetime is maximally symmetric space with the following Killing vectors [50]

$$\begin{aligned} &P_i = \partial_i \quad D = -\eta\partial_\eta - x^i\partial_i \\ &J_{ij} = x_i\partial_j - x_j\partial_i \quad K_i = 2x_i\eta\partial_\eta + \left(2x^j x_i + (\eta^2 - x^2)\delta_i^j\right)\partial_j \end{aligned} \tag{36}$$

At late times, the bulk fields source operators on the $\eta \to 0$ boundary. For example, the KG field at late times behaves as

$$\phi(\mathbf{x}, \eta \to 0) = \mathcal{O}_+(\mathbf{x})\eta^{\Delta_+} + \mathcal{O}_-(\mathbf{x})\eta^{\Delta_-} \tag{37}$$

where

$$\Delta_{\pm} = \frac{d}{2} \pm i\mu \quad \mu = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}} \tag{38}$$

The correlations of the operators $\mathcal{O}_{\pm}$ are precisely the ones that are expected to reproduce the correlations on the CMB. The correlations are computed via the in-in formalism

$$\begin{aligned}&\langle \text{vac} \left| \varphi\left(\boldsymbol{k_1}\right) \varphi\left(\boldsymbol{k_2}\right) \cdots \varphi\left(\boldsymbol{k_n}\right) \right| \text{vac} \rangle (\eta) = \langle \text{vac} \left| \varphi_I\left(\eta, \boldsymbol{k_1}\right) \varphi_I\left(\eta, \boldsymbol{k_2}\right) \cdots \varphi_I\left(\eta, \boldsymbol{k_n}\right) \right| \text{vac} \rangle \\ &- i \int_{\eta_0}^{\eta} d\eta' \frac{1}{(\eta' H)^4} \int \left[ \prod_{a=1}^{n} \frac{d^3 p_a}{(2\pi)^3} \right] (2\pi)^3 \delta^3 \left( \sum_{a=1}^{n} \boldsymbol{p}_a \right) \left\langle \text{vac} \left| \left[ \varphi_I\left(\eta, \boldsymbol{k_1}\right) \varphi_I\left(\eta, \boldsymbol{k_2}\right) \cdots \varphi_I\left(\eta, \boldsymbol{k_n}\right), H^I_{int}(\eta') \right] \right| \text{vac} \right\rangle\end{aligned} \tag{39}$$

where $\eta \to 0$ and $\eta_0 \to -\infty$. The vacuum can either be Bunch-Davies or $\alpha$. Notice that in the late time limit of $\eta \to 0$ the 36 reduces to the generators of $CFT_3$ algebra. Hence, these boundary operators $\mathcal{O}_{\pm}$ form the conformal primaries and the correlations satisfy the $CFT_3$ ward identities. Due to this spacetime being maximally symmetric it doesn't have any dynamics. Hence, it is important that the $dS_4$ generators be broken. This leads to a Goldstone mode $\pi(\eta, x^i)$ with an associated perturbation $\zeta = -H\pi$ [51, 52]. The nearly scale-invariant two-point functions of $\zeta$ agree with all the current observations and there are various upper bounds on the higher-point functions. However, it is not very clear if the $dS_4$ boosts are approximate symmetries of the cosmological correlators as scale and rotational invariants fix the two-point function or the bispecturm. So, one has a choice, either one can assume that the $dS_4$ boosts are weakly broken, then one can continue to reconstruct correlators on CMB requiring full conformal symmetry or one can assume that the $dS_4$ boosts are not approximate but realized nonlinearly in certain regimes. This is the premise of boostless cosmological bootstrap [53–56]. In this bootstrap regime, $dS_4$ symmetries are strongly violated by introducing interactions of the scalar fluctuations. However, since, the symmetry is reduced the problem becomes challenging but one can use features of the singularity structure or unitarity to constrain these correlations [57].

The $dS_4$ bootstrap has the most phenomenological interest as it is applicable in several cases. In this regime, the background and dynamics is considered approximately invariant under the full $dS_4$ symmetries. Cosmological correlations are required to satisfy the full $dS_4$ symmetry. This regime of cosmological bootstrap enables computation that would otherwise be difficult. Beyond cosmology, this also provides a natural setup to study momentum space $CFT$s and provides numerous insights into the relation between $CFT$s and scattering amplitudes. This is because the flat-space limits of these cosmological correlations map onto scattering amplitudes in the flat-space limit. In chapter 3 of this thesis, however, the cosmological correlations are constrained using the $CFT$ side instead, and the consequences of OPE in constraining the cosmological observables are also discussed.

Inflationary cosmology serves as a bridge between the large and the small. To make this understanding precise, one has to understand the connection between IR observables and UV physics better. This can only be achieved via S-matrix bootstrap where the known IR physics and the unknown UV physics are related by means of dispersion relations and positivity bounds. This requires an improved understanding of unitarity for momentum space $CFT$ correlators and an insight into their analytic structures.

### *Dualities*

Another important tool to investigate physics at strong coupling is dualities. Dualities allow us to compute observables in a $CFT$, namely correlations, using a simpler field theory. These dualities are of two kinds - duality amongst field theories and strong-weak dualities. The earliest known field theory duality that exists is in electromagnetism where the Maxwell equations are invariant under [58]

$$E \to B \quad B \to -E \tag{40}$$

Also, from Dirac's quantization of electric and magnetic charge, we have [59]

$$eg = 2\pi n \tag{41}$$

where $e$ is the electric charge and $g$ is the magnetic charge and $n$ is a positive integer. Now, from the fine structure constant, we know that $e = \sqrt{4\pi\epsilon\hbar c\alpha}$, the electric charge which is also the coupling in the QED is very small. Therefore, the magnetic charge is going to be very large which will be the QED coupling in the magnetic frame. Hence, we may rewrite the Dirac's quantization rule as

$$e_{strong}e_{weak} = 2\pi n \tag{42}$$

This is the electromagnetic strong-weak duality where a theory at strong coupling is dual to a theory at weak coupling. This kind of duality also exists for $\mathcal{N} = 4$ Super-Yang-Mills theory as well. Dualities also exist in other areas of physics namely string theory, which is basically a 2D $CFT$, for instance, mirror symmetry and T-duality [60].

## *$AdS/CFT$ correspondence*

A class of dualities that help compute observables in $CFT$s is strong-weak dualities. The most famous strong-weak dualities go by the name of the $AdS/CFT$ correspondence or the Maldacena conjecture [61]. The Maldacena conjecture states that for every theory in the $AdS_{d+1}$, there exists a $CFT_d$ on its boundary. This doesn't necessarily mean that every $CFT_d$ will have an $AdS_{d+1}$ gravity dual. This correspondence provides a way to compute quantum effects in a QFT at strong coupling using classical gravitational theories. Even though the name suggests that it works for $CFT$s, this correspondence can be extended to non-conformal theories and has numerous applications in confinement, chiral symmetry breaking, non-equilibrium phenomena, and condensed matter systems. The statement of the $AdS/CFT$ correspondence is as follows [62]

$$\left\langle \exp\left(\int d^dx\mathcal{O}\phi_{(0)}\right)\right\rangle_{CFT_d} = Z_{AdS_{d+1}}\Big|_{\lim_{z\to 0}(\phi(z,x)z^{\Delta-d})=\phi_{(0)}(x)} \approx e^{-S_{classical}}\Big|_{\lim_{z\to 0}(\phi(z,x)z^{\Delta-d})=\phi_{(0)}(x)} \tag{43}$$

where $\phi$ is called the bulk field as it resides in the $AdS_{d+1}$ bulk. The bulk field sources a primary operator $\mathcal{O}$ in the boundary $CFT_d$. The boundary resides at $z = 0$ where $z$ is the spacelike coordinate for the $AdS_{d+1}$ metric given by

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \tag{44}$$

The $AdS/CFT$ correspondence provides a unique dictionary between the primary operator in the boundary $CFT_d$ and bulk field in the $AdS_{d+1}$. Unlike the duality amongst field theories, $AdS/CFT$ dualities are unique. Also, the operators and observables have a clear distinction and interpretation in this duality. The $CFT$ correlation functions can be computed using the correspondence as follows [62]

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta}{\delta \phi_0(x_1)} \cdots \frac{\delta}{\delta \phi_0(x_n)} Z_{AdS_{d+1}} \Big|_{\phi_0=0} \tag{45}$$

$AdS/CFT$, therefore, allows us to understand quantum gravity via $CFT$s. This is the essence of the holographic principle which states that the information in a space can be encoded on its boundary. This is motivated by the Bekenstein bound which states that the maximum entropy in a stored volume $V_{d+1}$ is given by $S = \frac{A_d}{4G}$. The theory of quantum gravity is defined on an $AdS \times X$ type of manifold where $X$ is some compact space. As an example, one can look a the most well-known application of the $AdS/CFT$ correspondence. It states that $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory with SU(N) gauge group and coupling $g_{YM}$ given by [61–63]

$$\begin{aligned} \mathcal{L} = \mathrm{Tr} \Bigg( & -\frac{1}{2g_{\mathrm{YM}}^2} F_{\mu\nu} F^{\mu\nu} + \frac{\vartheta}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a \\ & - \sum_i D_\mu \phi^i D^\mu \phi^i + g_{\mathrm{YM}} \sum_{a,b,i} C_i^{ab}{}_i \lambda_a \left[ \phi^i, \lambda_b \right] \\ & + g_{\mathrm{YM}} \sum_{a,b,i} \bar{C}_{iab} \bar{\lambda}^a \left[ \phi^i, \bar{\lambda}^b \right] + \frac{g_{\mathrm{YM}}^2}{2} \sum_{i,j} \left[ \phi^i, \phi^j \right]^2 \Bigg), \end{aligned} \tag{46}$$

is dynamically equivalent to a type IIB superstring theory with string length $l_s$ defined as $l_s = \sqrt{\alpha'}$ and coupling $g_s$ on $AdS_5 \times S^5$ with a radius of curvature $L$ and $N$ units of $F_{(5)}$ flux on $S^5$. The $AdS/CFT$ dictionary relates the free parameters of both theories as [62]

$$g_{YM}^2 = 2\pi g_s \quad 2g_{YM}^2 N = \frac{L^4}{\alpha'^2} \tag{47}$$

The $\phi^i$'s transform under the fundamental $SO(6)$ representation while $A^\mu$ transforms under the adjoint $SU(4)$ respresentation. The $C$'s are the Clebsh-Gordon coefficients. The $AdS/CFT$ correspondence can be established using various tests [64–66], one of them involving computing correlations of one-half BPS operators in $\mathcal{N} = 4$ SYM theory at large N [67] using both the $\mathcal{N} = 4$ SYM action and its dual supergravity action [62,67].

$AdS/CFT$ has also made some predictions concerning the properties of strongly-coupled

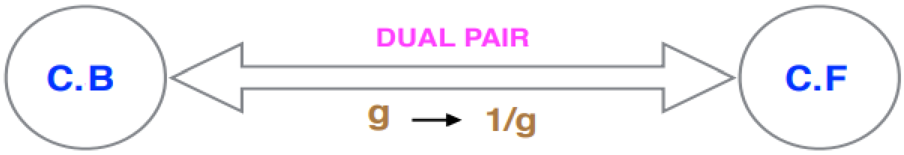



**Figure 1:** Duality between Chern-Simons Critical Fermionic (CF) and Chern-Simons Critical Bosonic (CB) theory where $g$ is the coupling. Checks: Exact partition function and exact $2 \leftrightarrow 2$ scattering.

theories that may not be easy to infer using field-theoretic means. For instance, confinement in QCD is attributed to strings in the AdS bulk. The endpoints of the strings are the quarks or gluons, so their confinement is related to the behaviour of the strings in the AdS geometry. Another prediction of $AdS/CFT$ is that the quark-gluon plasma has a black hole dual in the AdS bulk and their behaviour is directly linked to the thermodynamical properties of the bulk black hole [68, 69]. $AdS/CFT$ correspondence also maps solitons and instanton solutions in bulk to non-local operators in the $CFT$. Hence, predicting an important role for $CFT$s that violate locality.

## *Dualities in Chern-Simons matter theories*

However, some of the most intricate dualities that exist amongst $CFT$s are in Chern-Simons matter theories [70–155] and this will be one of the theories covered in this thesis. The Chern-Simon matter theories are defined by

$$S[A, \Phi] = \int d^3x [i\epsilon^{\mu\nu\rho} \frac{\kappa}{4\pi} tr(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho)] + S_m[A, \Phi] \tag{48}$$

where $A_\mu$ is a gauge field in the adjoint representation of some gauge group and $\Phi$ is some matter field (fermionic or bosonic) in the fundamental representation of the gauge group with the following actions

$$S_f\left(A_\mu, \psi\right) = -i \int d^3x \bar{\psi} \gamma^\mu \mathcal{D}_\mu \psi \quad S_b\left(A_\mu, \phi\right) = \int d^3x \mathcal{D}^\mu \bar{\phi} \mathcal{D}_\mu \phi \tag{49}$$

The gauge group is usually $SU(N)$, $U(N)$, $SO(N)$ or $O(N)$ [111, 140]. An important parameter called the t'Hooft parameter defined as $\lambda = N/\kappa$ becomes important when discussing strong coupling limit of these theories. Chern-Simons matter theories have applications in condensed matter systems [78, 101, 133, 151] and they are also important field theoretic models to study the Hall effect [104, 156, 157]. These theories also have higher-spin Vasiliev theory as a gravity dual. Some of the well-known dualities satisfied by Chern-Simons matter theories are as follows [137]

- $SU(N)$ regular fermion (RF) theories at (Yang-Mills regulated) level $k$ are dual to type 2 $U(|k|+\frac{1}{2})$ critical boson (CB) theories at level $-\text{sgn}(k)N$

- Type 2 $U(N)$ RF theories at (Yang-Mills regulated) level $k$ are dual to type 2 $SU(|k|+\frac{1}{2})$ CB theories at level $-\text{sgn}(k)N$

- Type 1 $U(N)$ RF theories at (Yang-Mills regulated) level $k$ are dual to Type 1 $U(|k|+\frac{1}{2})$ CB theories at level $-\text{sgn}(k)N$

The fermionic theories have the scalar operator $J_0^F \equiv \bar{\psi}\psi$ in their spectrum and the bosonic theories have the scalar operator $\tilde{J}_0 \equiv \bar{\phi}\phi$. These dualities satisfy various checks, for instance, all the momentum-space correlations of scalar operators in these theories are identical up to contact terms [137]

$$\begin{aligned}
&\left\langle J_0^F\left(q_1\right) J_0^F\left(q_2\right)\right\rangle_{RF}=\left\langle \tilde{J}_0\left(q_1\right) \tilde{J}_0\left(q_2\right)\right\rangle_{CB} \\
&\left\langle J_0^F\left(q_1\right) J_0^F\left(q_2\right) J_0^F\left(q_3\right)\right\rangle_{RF}=\left\langle \tilde{J}_0\left(q_1\right) \tilde{J}_0\left(q_2\right) \tilde{J}_0\left(q_3\right)\right\rangle_{CB} \\
&\left\langle J_0^F\left(q_1\right) J_0^F\left(q_2\right) J_0^F\left(q_3\right) J_0^F\left(q_4\right)\right\rangle_{RF}=\left\langle \tilde{J}_0\left(q_1\right) \tilde{J}_0\left(q_2\right) \tilde{J}_0\left(q_3\right) \tilde{J}_0\left(q_4\right)\right\rangle_{CB}
\end{aligned} \tag{50}$$

Observe how these dualities map observables from one theory to observables of the other. These dualities also lead to a map between the sourced partition functions of the theories. Since these theories exist at some fixed points in the RG flow, the mass-deformed version of these theories flow away from these fixed points. But due to the duality of the fixed point theories, the mass-deformed theories [146] are also dual to each other for a specific value of the masses used in the deformation. Also, the duality between RF and CB theories imply a relationship between their partition functions [137]

$$\int D\phi D\sigma e^{-S_{cb}(\phi,\sigma)+\int J_0(x)\zeta(x)-\frac{(2\pi)^2}{\kappa_B^2}\left(1+\mathcal{O}\left(\frac{1}{\kappa_B}\right)\right)\int \zeta^3(x)} = \int D\psi e^{-S_{rf}(\psi)+\int J_0^F(x)\zeta(x)} \tag{51}$$

One can also look at supersymmetric (SUSY) versions of Chern-Simons matter theories

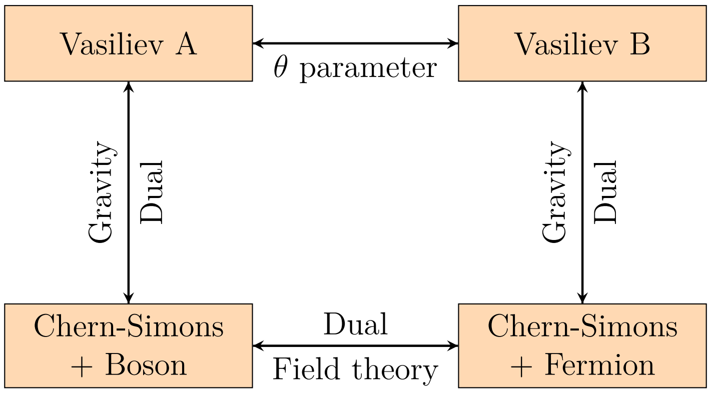



**Figure 2:** Gravity duals of Chern-Simons matter theories

($\mathcal{T}$) [76, 93, 122, 158–161]. For instance $\mathcal{N}=2$ SUSY $U(N)$ Chern-Simons theories [125]

$$\begin{aligned} S^{\mathcal{T}}_{k,N}\left(A_\mu,\varphi,\psi\right) &= \frac{ik}{4\pi}S_{\rm CS}\left(A_\mu\right)+S_b\left(A_\mu,\varphi\right)+S_f\left(A_\mu,\psi\right)+S_{bf}(\varphi,\psi) \\ S_{bf}(\varphi,\psi) &= \int d^3x\left[-\frac{4\pi i}{k}(\bar\varphi\varphi)(\bar\psi\psi)+\frac{4\pi^2}{k^2}(\bar\varphi\varphi)^3-\frac{2\pi i}{k}(\bar\psi\varphi)(\bar\varphi\psi)\right] \end{aligned} \quad (52)$$

exhibit strong-weak self-duality i.e.

$$\mathcal{T}_{k,N}\leftrightarrow\mathcal{T}_{-k,|k|-N+\frac{1}{2}} \quad k\in\mathbb{Z}+\frac{1}{2} \quad (53)$$

This self-duality of the SUSY theory leads to non-SUSY bosonization dualities which were described above [96]. In chapter 2, these dualities will be used to compute SUSY observables in terms of non-SUSY observables, in particular, the four-point function.

## *Duality between Vasiliev theories and Chern-Simons matter theories*

Free bosonic and free fermionic theories are dual to Vasiliev-type 4D higher spin gravity [162–172] and Chern-Simons matter theory interpolates between them on the field theory side while on the dual side, they are interpolated by the $\theta$-parameter, see Figure 2. Free theories have a tower of primary operators or currents defined by

$$J^b_s\equiv\sum_{r=0}^{s}(-1)^{r2s}C_{2r}\partial^r\bar\phi\partial^{s-r}\phi \quad (54)$$

$$J^f_s\equiv y^\alpha y^\beta\sum_{r=0}^{s-1}(-1)^{r+12s}C_{2r+1}\partial^r\bar\psi\partial^{s-r-1}\psi_\beta \quad (55)$$

These currents generate a higher spin symmetry. However, due to interactions in the Chern-Simons matter theories, the higher spin symmetry gets weakly broken and the higher spin currents with $s > 2$ becomes non-conserved. For example, one may have

$$\partial_\mu J^\mu{}_{---} = a_2 \left( \partial_- \tilde{j}_0 j_2 - \frac{2}{5} \tilde{j}_0 \partial_- j_2 \right) \tag{56}$$

where $a_2 \propto \lambda$. The three-point correlations of these currents have been shown to satisfy [74]

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = f^b_{s_1 s_2 s_3} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{bos} + f^f_{s_1 s_2 s_3} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{fer} + f^{odd}_{s_1 s_2 s_3} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{odd} \tag{57}$$

These currents are exactly conserved in free theories. But in interacting theories due to 56 the three-point functions of these higher-spin currents also become non-conserved. Using the the higher-spin currents one can define the following charges

$$Q_s = \int d^{3x} \partial^\mu J^s_{\mu - \cdots -} \tag{58}$$

These charges lead to the following transformation law for the currents

$$[Q_s, j_{s'}] = \sum_{s''=max[s'-s+1,0]}^{s'+s-1} \alpha_{s,s',s''} \partial^{s'+s-1-s''} j_{s''} \tag{59}$$

Both 59 and 56 can be used together to compute correlation functions for $CFT$s with slightly-broken higher-spin symmetries and compute the coefficients $f_{s_1 s_2 s_3}$ in 57. These weakly broken symmetries can also be used to compute anomalous dimensions for higher spin currents for $s > 2$. In chapter 2, a method will be presented to compute the anomalous dimension of the scalar operator without the higher spin algebra. The constraints of higher spin are strong enough to determine the correlation functions at all orders of $1/N$ provided the $AdS/CFT$ correspondence holds, i.e. provided one can show that a Vasiliev-type theory exists as a gravity dual to the Chern-Simons matter theory under consideration [74].

## ***Thesis Summary***

In this thesis, we resolve a number of unsolved issues. In chapter 1, a systematic formalism is presented to compute $CFT$ three-point correlators of arbitrary spin. We then use the results of these computations to show that these correlations satisfy double copy relations very similar to the double copy relations in three-point scattering flat-space amplitudes of gluons and gravitons. We explore this similarity between flat-space amplitudes and $CFT$ correlators in a bit more detail as well. In chapter 2, we look at the Chern-Simons matter theories where we make use of duality to compute four-point correlations in a $\mathcal{N} = 2$ SUSY Chern-Simons matter theory in terms of non-SUSY observables. We also conjecture the anomalous dimension of the scalar operator in Chern-Simons matter theory which was previously unknown. Finally, in chapter 3, we make use of momentum-space $CFT$ techniques to provide an interpretation for the $\alpha$-vacua correlations in inflationary cosmology.

# Chapter 1

# Momentum-space 3-point functions in $CFT_3$

This chapter is largely based on the following papers written with my collaborators

- S. Jain, R. R. John, A. Mehta, A. A. Nizami, A. Suresh (2021) *Momentum space parity-odd CFT 3-point functions*, JHEP 08 (2021) 089, arXiv: 2101.11635 [hep-th]
- S. Jain, R. R. John, A. Mehta, A. A. Nizami, A. Suresh (2021) *Double copy structure of parity-violating CFT correlators*, JHEP 07 (2021) 033, arXiv: 2104.12803 [hep-th]
- S. Jain, R. R. John, A. Mehta, A. A. Nizami, A. Suresh (2021) *Higher spin 3-point functions in 3d CFT using spinor-helicity variables*, JHEP 09 (2021) 041, arXiv: 2106.00016 [hep-th]
- S. Jain, A. Mehta (2023) *4D flat-space scattering amplitude/$CFT_3$ correlator correspondence revisited*, Nucl. Phys. B (2023) 991 p. 116193, arXiv: 2201.07248 [hep-th]

$CFT$s occur as endpoints of a UV-complete $QFT$. These $CFT$s are connected by the renormalization group (RG) flow. Thus, studying $CFT$s is crucial in understanding the space of $QFT$s [9]. Most of these fixed points are inaccessible to usual perturbative techniques and hence, non-perturbative techniques like duality, symmetry, and bootstrapping are essential. Various techniques have been developed over the years to compute observables in a $CFT$. However, most of these techniques are developed in position or Mellin space [9, 10, 173–175]. Considering the resemblance of momentum-space $CFT$ correlators with scattering amplitudes and its application in inflationary cosmology [21–27, 31, 50, 176–180] in the computation of non-gaussianities, the study of momentum-space $CFT$ and development of momentum-space $CFT$ bootstrap becomes essential. However, limited development is made in this direction even at the level of a three-point function given the complexities of the momentum-space ward identities [41, 150, 152, 181–188]. In this chapter, we present a systematic and standardized approach to computing any three-point correlators of arbitrary spin and parity. $CFT$ 3-point functions are determined using three complementary approaches. The first method is the more direct one and involves solving momentum-space Ward identities which were developed and used for the parity-even sector in [41]. The second approach utilizes spin-raising and weight-shifting operators. These have been used in the conformal bootstrap literature [14, 189, 190]. Parity-odd spin-raising and weight-shifting operators in momentum space is constructed and used on scalar seed correlators to generate parity-odd spinning correlators.

Despite the great utility of these approaches the finer structures of the correlations are better understood by an alternative approach. For instance, in $\langle TTT \rangle$, there is a high degree of degeneracy in the tensor structures in 3d, both in the parity-even and the parity-odd sector, which makes it difficult to choose an appropriate basis to write an ansatz for the correlator. This also means that there is no unique way to write correlators in momentum-space. The problem becomes even more complicated if we want to calculate a correlator involving higher spin conserved currents ($J_s$ with $s > 2$) both for the parity-even and parity-odd case. In this chapter, this problem is overcome using the third approach i.e. working in the spinor-helicity formalism where the degeneracy is automatically taken care of and the correlations in the spinor-helicity are represented uniquely. We solve the CWIs in these variables and then convert the results back to momentum space. Through this analysis, it is shown that the correlation function has two independent sub-structures, homogeneous and non-homogeneous. Moreover, for divergent correlation functions which require regularization and renormalization but in spinor-helicity variables they turn out to give directly the finite part without any renormalization. Some of the results are verified using weight-shifting operators.

An important feature of the momentum-space correlators is their resemblance to the scattering amplitudes. One of the interesting relationships that exist for flat space scattering amplitudes is the double-copy relation between gauge theory and gravity amplitudes, and the associated color-kinematics duality [191–193]. This means that amplitudes involving gravitons can be built out from those involving gluons. The double copy relation was first observed in Einstein gravity and pure Yang-Mills theory, and later it was extended to a whole host of theories including higher derivative conformal gravity, higher derivative gauge theories and bi-adjoint scalar theories [194–196]. The analyses in these works were for the parity-even sector in the flat-space limit. Double copy relations for the $CFT$ correlators were only studied for the parity-even sector in the flat-space limit [39]. In this chapter, we have demonstrated that the double copy relations hold for the three-point correlation functions for arbitrary spins. This close resemblance to the scattering amplitudes is very suggestive of an amplitude/correlator correspondence which will also be explored in detail in this chapter.

## 1.1 Chapter summary

The rest of this chapter is organized as follows. In Section 1.2, besides setting up the notation and terminology, we outline the three different techniques that we use in this chapter to determine parity-odd 3-point functions. We outline how to derive parity-odd correlations using both the conformal ward identity and spin-raising operators, discussed in detail in [150]. For example, the following holds

$$\langle TTO_2\rangle_{\text{odd}} = (k_1k_2)^3 P_1^{(2)} P_2^{(2)} H_{12}\widetilde{D}_{12}\langle O_1(k_1)O_2(k_2)O_2(k_3)\rangle \tag{1.1}$$

where appropriate spin-raising and dimension-raising operators have been used in the RHS. We also briefly discuss the divergences that arise, and their regularisation [150]. Finding these methods tedious for arbitrary spins due to degeneracies and Schouten identities, we present a systematic formalism to derive any correlation of arbitrary spins via spinor-helicity. In Section 1.3 the basic idea of expressing conformal correlators in terms of spinor-helicity variables is introduced and the preliminary case of 2-point functions is discussed which is based on [187]. We show that any correlator can be written as two independent sub-structures

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle = \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h}} + \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh}} \tag{1.2}$$

These structures behave differently under the Special Conformal Ward Identities [187]. Consider the example of $\langle TTT \rangle$

$$\langle T^- T^- T^- \rangle_{\text{even}} = \left( c_1 \frac{c_{123}}{E^6} + c_T \frac{E^3 - Eb_{123} - c_{123}}{c_{123}^2} \right) \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2 \tag{1.3}$$

where the leading singularity in $E = k_1 + k_2 + k_3$ is identified with the homogenous sub-structure and the sub-leading term with the non-homogenous sub-structure. Section 1.4 has the results of various 3-point correlators of spinning conserved currents and scalar operators in spinor-helicity variables. In Section 1.5, these results are translated to momentum space after carefully taking the degeneracies into account and Section 1.6 has a discussion of the renormalization of some of these correlators which have divergences. Section 1.7 contains a discussion of momentum space higher-spin conserved current correlators expressed in terms of 3-point momentum space invariants. The homogenous part of any arbitrary spinning correlator looks like

$$\begin{aligned}
\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{even}} &= k_1^{s_1-1} k_2^{s_2-1} k_3^{s_3-1} Q_{12}^{\frac{1}{2}(s_1+s_2-s_3)} Q_{23}^{\frac{1}{2}(s_2+s_3-s_1)} Q_{13}^{\frac{1}{2}(s_1+s_3-s_2)} \\
\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}} &= k_1^{s_1-1} k_2^{s_2-1} k_3^{s_3-1} S_{12} Q_{12}^{\frac{1}{2}(s_1+s_2-s_3-2)} Q_{23}^{\frac{1}{2}(s_2+s_3-s_1)} Q_{13}^{\frac{1}{2}(s_1+s_3-s_2)} \\
&\quad + \text{cyclic perm.}
\end{aligned} \tag{1.4}$$

where $Q_{ij}$ and $S_{ij}$ are the momentum space invariants. In Section 1.8 we make some important observations, including the connection between the parity-even and parity-odd parts of a correlator. We show that in spinor helicity they are only distinguished by a factor of $i$ as can be seen in the following simple example.

$$\langle J^- J^- O_4 \rangle_{\text{even}} = c_1 (k_1 k_2) I_{\frac{5}{2},\{\frac{1}{2},\frac{1}{2},\frac{5}{2}\}} \tag{1.5}$$

$$\langle J^- J^- O_4 \rangle_{\text{odd}} = i\, c_2 (k_1 k_2) I_{\frac{5}{2},\{\frac{1}{2},\frac{1}{2},\frac{5}{2}\}} \tag{1.6}$$

Section 1.9 discusses the double copy relations amongst the three-point $CFT_3$ correlators which are detailed in [187]. To establish our claim, we see that it is convenient to work with the sub-structures of the CFT correlators. In particular, we show that under double copy relations, the homogeneous part maps to a homogeneous part, and the non-homogeneous part maps to a non-homogeneous part. Let us illustrate this point by considering $\langle TTT\rangle$, the 3-point function of the stress tensor, and $\langle JJJ\rangle$, the 3-point function of the conserved spin-1 current. The correlators can be written as :

$$\begin{aligned}\langle JJJ\rangle &= \langle JJJ\rangle_{\text{homogeneous}} + \langle JJJ\rangle_{\text{non-homogeneous}}\\ \langle TTT\rangle &= \langle TTT\rangle_{\text{homogeneous}} + \langle TTT\rangle_{\text{non-homogeneous}}\end{aligned} \tag{1.7}$$

The double copy relation is then given by :

$$\begin{aligned}&\langle TTT\rangle_{\text{homogeneous}} \propto \left(\langle JJJ\rangle_{\text{homogeneous}}\right)^2\\ &\langle TTT\rangle_{\text{non-homogeneous}} \propto \left(\langle JJJ\rangle_{\text{non-homogeneous}}\right)^2\end{aligned} \tag{1.8}$$

where the proportionality factor is momentum dependent and is different for the two cases. It is given explicitly in Section 1.9.3. In Section 1.10, we discuss the correspondence between $4D$ flat-space amplitudes and $CFT_3$ correlators in detail, which is based on the discussion in [183]. We introduce the $\epsilon$-transformation that allows us to go from parity-even to parity-odd structures for covariant vertex while preserving gauge-invariance. A differential operator implements the $\epsilon$-transformation

$$[O_\epsilon]_I = \frac{1}{k_I}\epsilon(z_I k_I \frac{\partial}{\partial z_I}) \tag{1.9}$$

which acts on parity-even gauge-invariant structures to give parity-odd gauge invariant structures and vice-versa i.e

$$\begin{aligned}O_\epsilon &: \mathcal{M}_{m,e} \to \mathcal{M}_{m,o}\\ O_\epsilon &: \mathcal{M}_{nm,e} \to \mathcal{M}_{nm,o}.\end{aligned} \tag{1.10}$$

where $\mathcal{M}$ is a gauge-invariant amplitude structure. Eventually, we make use of the epsilon transformation to propose a new CFT structure that in the flat-space limit gives the extra parity-odd amplitude and discuss some examples. In the end, we have a number of appendices supplementing the main text and providing various technical details. Appendix A.4 outlines our spinor-helicity notation. In Appendix A.5 we describe in detail our terminology of homogeneous and non-homogeneous contributions to a correlator and discuss how they differ from the usual splitting of a correlation function into transverse and longitudinal pieces. Appendix A.6 has the technical details of solutions of various conformal Ward identities quoted in Section 1.4. Appendix A.7 contains useful triple-$K$ integral identities and Appendix A.8 lists the momentum space form of various 3-point correlators of conserved currents. In Appendix A.9, we highlight some parity-odd spin-raising and weight-shifting operators. In Appendix A.10, we discuss some explicit examples of flat-space amplitudes. In Appendix A.11, we discuss various identities which are useful in the main text.

## 1.2 Three approaches to determining momentum space correlators

Determining correlation functions is a significantly harder task in momentum space than in position space. For parity-odd correlators this gets even more tedious. We will now discuss three different approaches to determining momentum space correlators. We also discuss certain subtleties and limitations associated with the two approaches.

### 1.2.1 Using Conformal Ward identity : Strategy

In the first approach, following [41,197] and [42,44,45,198,199] where parity-even 3-point functions were determined, we start with an ansatz of the form $\sum_m A_m(k_i)\mathcal{T}_m$ for the correlator. Here $\mathcal{T}_m$ are all possible tensor structures that are allowed by symmetry and $A_m$ are form factors that are functions of the momenta magnitudes ($k_i$). The form factors are constrained by permutation symmetries (if any) of the correlator and by momentum space Ward identities. The latter lead to partial differential equations which can then be solved to determine the form factors, up to undetermined constants that depend on the specific theory. An excellent mathematica package that we found useful in these computations is [200].

Let us now describe the momentum space Ward identities associated with dilatation symmetry and special conformal transformations.

### 1.2.2 Dilatation and Special Conformal Ward identities

We will denote the $n$-point Euclidean correlation function of primary operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ by $\langle \mathcal{O}_1(\boldsymbol{k}_1)\ldots\mathcal{O}_n(\boldsymbol{k}_n)\rangle$. We suppress the Lorentz indices of the operators for brevity. The correlator with the momentum conserving delta function stripped off is denoted as :

$$\langle\, \mathcal{O}_1(\boldsymbol{k}_1)\,\ldots\,\mathcal{O}_n(\boldsymbol{k}_n)\,\rangle \equiv (2\pi)^d\delta^{(3)}(\boldsymbol{k}_1+\ldots+\boldsymbol{k}_n)\langle\langle\, \mathcal{O}_1(\boldsymbol{k}_1)\,\ldots\,\mathcal{O}_n(\boldsymbol{k}_n)\,\rangle\rangle\,. \tag{1.11}$$

An $n$-point correlator with scalar or spinning operator insertions satisfies the following dilatation Ward identity [41] :

$$0 = \left[ -(n-1)d + \sum_{j=1}^{n} \Delta_j - \sum_{j=1}^{n-1} k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \right] \langle\langle\, \mathcal{O}_1(\boldsymbol{k}_1) \ldots \mathcal{O}_n(\boldsymbol{k}_n) \,\rangle\rangle \,. \tag{1.12}$$

This constrains the correlator to have the following scaling behaviour :

$$\langle\langle\, \mathcal{O}_1(\lambda\,\boldsymbol{k}_1) \ldots \mathcal{O}_n(\lambda\,\boldsymbol{k}_n) \,\rangle\rangle = \lambda^{-\left[(n-1)d - \sum_{i=1}^{n}\Delta_i\right]} \langle\langle\, \mathcal{O}_1(\boldsymbol{k}_1) \ldots \mathcal{O}_n(\boldsymbol{k}_n) \,\rangle\rangle \,. \tag{1.13}$$

The special conformal Ward identity on an $n$-point correlator with both scalar and spinning operators is [41] :

$$\begin{aligned} 0 = &\sum_{j=1}^{n-1} \left[ 2(\Delta_j - d) \frac{\partial}{\partial k_j^\kappa} - 2k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \frac{\partial}{\partial k_j^\kappa} + k_j^\kappa \frac{\partial}{\partial k_j^\alpha} \frac{\partial}{\partial k_{j\alpha}} \right] \langle\langle\, \mathcal{O}_1(\boldsymbol{k}_1) \ldots \mathcal{O}_n(\boldsymbol{k}_n) \,\rangle\rangle \\ &+ 2 \sum_{j=1}^{n-1} \sum_{k=1}^{n_j} \left( \delta^{\mu_{jk}\kappa} \frac{\partial}{\partial k_j^{\alpha_{jk}}} - \delta^\kappa_{\alpha_{jk}} \frac{\partial}{\partial k_{j_{\mu_{jk}}}} \right) \langle\langle\, \mathcal{O}_1^{\mu_{11}\ldots\mu_{1r_1}}(\boldsymbol{k}_1) \ldots \mathcal{O}_j^{\mu_{j1}\ldots\alpha_{jk}\ldots\mu_{jr_j}}(\boldsymbol{k}_j) \ldots \mathcal{O}_n^{\mu_{n1}\ldots\mu_{nr_n}}(\boldsymbol{k}_n) \,\rangle\rangle \end{aligned} \tag{1.14}$$

In the second line of the RHS of the above equation, the indices of the generator mix with the spin indices of the correlator. In principle, one can solve this equation and get the desired correlator [41]. However, for parity-odd structures in three-dimensions, the computation gets complicated and has not yet been done.

We will always be working with correlation functions with the momentum conserving delta function stripped off. From here on we will drop the double angular brackets notation to avoid clutter and use single angular brackets everywhere.

We will also use the terminology of primary and secondary conformal Ward identities [41]. A 3-point momentum space correlator can be expanded as $\sum A_i \mathcal{T}_i$ where the $A_i$ are the (scalar) form factors, whereas $\mathcal{T}_i$ give a basis for tensor structures. When one considers the action of the special conformal generator $K^\kappa$ on these correlators, it naturally results in PDEs for the form factors.

In brief, a primary Ward identity is a second-order PDE for the form factor arising from terms containing $k_1^\kappa$, $k_2^\kappa$ in the conformal Ward identity $K^\kappa\langle ...\rangle$=0. The remaining PDEs are secondary Ward identities and are first order. See section 5 of [41] for further details and properties.

#### 1.2.2.1 Divergences

Triple-$K$ integrals arise as solutions to primary conformal Ward identities which are second-order differential equations [41]. Along with the three momenta, they are expressed in terms of four other parameters :

$$I_{\alpha\{\beta_1\beta_2\beta_3\}}(k_1,k_2,k_3) \equiv \int_0^\infty dx\, x^\alpha \prod_{j=1}^{3} k_j^{\beta_j} K_{\beta_j}(k_j x) \tag{1.15}$$

where $K_{\beta_j}$ is a modified Bessel function of the second kind. While the integral is well-behaved at its upper limit, it is convergent at $x=0$ only if [41, 44] :

$$\alpha + 1 - |\beta_1| - |\beta_2| - |\beta_3| > 0 \tag{1.16}$$

When the integral is divergent one can regulate it using two parameters $u$ and $v$ [41, 44] :

$$I_{\alpha\{\beta_1\beta_2\beta_3\}} \to I_{\alpha+u\epsilon\{\beta_1+v\epsilon,\beta_2+v\epsilon,\beta_3+v\epsilon\}} \tag{1.17}$$

The regularised triple-$K$ integral is convergent except when [41, 44] :

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2n, \qquad n \in \mathbb{Z}_{\geq 0} \tag{1.18}$$

for any choice of signs. When (1.18) is satisfied, the integral is singular in the regulator $\epsilon$ and we will denote the divergence by the choice of signs $(\pm\pm\pm)$ for which (1.18) is satisfied.

Divergences of the type $(---)$ are called ultra-local and they occur when all the three operators are co-incident in position space. In momentum space, this manifests as the divergent term being analytic in all three momenta squared. Such divergences must, in general, be removed using counter-terms that are cubic in the sources, and they give rise to conformal anomalies.

Divergences of the type $(--+)$ and its permutations are called semi-local divergences.

In position space, this is a divergence that occurs when two of the operators in the correlator are at co-incident points. In momentum space, the divergence is said to be semi-local when the $O(1/\epsilon)$ term is analytic in any two of the three momenta squared. In general, these divergences must be removed by counter-terms that have two sources and an operator. Such terms lead to non-trivial beta functions.

Divergences of the kind $(+++)$ and $(++-)$ are non-local and they occur even when all three operators are at separated points in position space. In momentum space, such a divergence is analytic in at most one of the momenta squared. This is not a physical divergence and arises because the triple-$K$ integral representation of the correlator is singular. In this case no counter-term exists and the divergence is removed by imposing the condition that the constant multiplying the triple-$K$ integral vanishes as an appropriate power of $\epsilon$.

#### 1.2.2.2 Counter-terms

As we discussed above, divergences of the kind $(--+)$ and $(---)$ that correspond to ultra-local and semi-local divergences are removed using suitable counter-terms. In the case of parity-even correlators, this has been extensively studied in [41, 42, 44, 45]. We will now list a few potential counter-terms that could turn out to be useful in our study of parity-odd correlators. For ultra-local divergences, for example, we have :

$$\int d^3x\, F_3(A)\, \Box^n \phi, \quad \int d^3x\, C_{\mu\nu}\, R^{\mu\nu}\, \Box^n \phi, \quad \int d^3x\, C_{\mu\nu}\, R\, \nabla^\mu\, \nabla^\nu\, \Box^n \phi \tag{1.19}$$

and for semi-local divergences :

$$\int d^3x\, \epsilon^{\mu\nu\lambda}\, F_{\mu\nu}\, J_\lambda\, \Box^n \phi, \quad \int d^3x\, A^\mu\, J_\mu\, \Box^n \phi, \quad \int d^3x\, F^{\mu\nu}\, J_\mu D_\nu \phi, \quad \int d^3x\, C_{\mu\nu}\, T^{\mu\nu}\, \Box^n \phi \tag{1.20}$$

where $F_3(A)$ is the Chern-Simons form in three-dimensions given by,

$$F_3(A) = \epsilon_{\mu\nu\lambda} \left( A^\mu_a \partial^\nu A^\lambda_a + \frac{2}{3} f^{abc} A^\mu_a A^\nu_b A^\lambda_c \right) , \tag{1.21}$$

$C_{\mu\nu}$ is the Cotton-York tensor given by,

$$C_{\mu\nu} = \nabla^\rho \left( R^\sigma_\mu - \frac{1}{4} R\, g^\sigma_\mu \right) \epsilon_{\rho\sigma\nu} \, , \tag{1.22}$$

and $R_{\mu\nu}$ and $R$ are the Ricci tensor and the Ricci scalar respectively. In the above list of possible counter-terms (1.20) we have included certain parity-even terms such as $\int d^3x\, A^\mu\, J_\mu\, \Box^n \phi$ and $\int d^3x\, F^{\mu\nu} J_\mu D_\nu \phi$. These counter-terms could give rise to the 2-point function of currents, which has a parity-odd contribution $\langle J^\mu(p) J^\nu(-p)\rangle \propto \epsilon^{\mu\nu\rho} p_\rho$.

#### 1.2.2.3 Example: $\langle J^\mu J^\nu O\rangle_{\mathbf{odd}}$

Here we will consider the parity-odd part of the correlator $\langle J^\mu J^\nu O\rangle$. We start with the following ansatz for the correlator

$$\langle J^\mu(k_1)\, J^\nu(k_2)\, O(k_3)\rangle_{\text{odd}} = \pi^\mu_\alpha(k_1)\pi^\nu_\beta(k_2)\left[\widetilde{A}(k_1,k_2,k_3)\epsilon^{\alpha k_1 k_2} k_1^\beta + \widetilde{B}(k_1,k_2,k_3)\epsilon^{\beta k_1 k_2} k_2^\alpha\right] \tag{1.23}$$

where the orthogonal projector $\pi^\mu_\nu(p)$ is given by :

$$\pi^\nu_\mu(p) \equiv \delta^\nu_\mu - \frac{p^\nu\, p_\mu}{p^2}\,. \tag{1.24}$$

The ansatz (1.23) is chosen such that the correlator is transverse with respect to $k_1^\mu$ and $k_2^\nu$. Demanding symmetry under the exchange : $(k_1,\mu) \leftrightarrow (k_2,\nu)$ gives the following relation between the form factors :

$$\widetilde{A}(k_1,k_2,k_3) = -\widetilde{B}(k_2,k_1,k_3) \tag{1.25}$$

Using the definition of projectors (1.24), the ansatz (1.23) expands to the following :

$$\begin{aligned}\langle J^\mu(k_1)\, J^\nu(k_2)\, O(k_3)\rangle_{\text{odd}} = {} & \widetilde{A}(k_1,k_2,k_3)\epsilon^{\mu k_1 k_3}\left[\frac{(k_1^\nu + k_3^\nu)(k_1^2 + k_1\cdot k_3)}{k_2^2} - k_1^\nu\right] \\ & + \widetilde{B}(k_1,k_2,k_3)\epsilon^{\nu k_1 k_3}\left[(k_1^\mu + k_3^\mu) - \frac{k_1^\mu(k_1^2 + k_1\cdot k_3)}{k_1^2}\right]\end{aligned} \tag{1.26}$$

where we have used momentum conservation to choose $k_1$ and $k_3$ as the independent momenta. We now use Schouten identities (A.11) and (A.12) to get rid of the $\epsilon^{\mu k_1 k_2}$ tensor structure and re-express the ansatz in (1.26) as :

$$\langle J^\mu(k_1)\, J^\nu(k_2)\, O(k_3)\rangle_{\text{odd}} = -\epsilon^{\nu k_1 k_3}\left(A k_1^\mu - B k_1^\mu - B k_3^\mu\right) - \left(\epsilon^{\mu\nu k_1} + \epsilon^{\mu\nu k_3}\right)\left(A k_1^2 + B(k_1\cdot k_2)\right) \tag{1.27}$$

where the new form factors $A(k_1,k_2,k_3)$ and $B(k_1,k_2,k_3)$ are given in terms of $\widetilde{A}(k_1,k_2,k_3)$ and $\widetilde{B}(k_1,k_2,k_3)$ as follows :

$$
\begin{aligned}
A(k_1,k_2,k_3) &= \widetilde{A}(k_1,k_2,k_3) + \widetilde{B}(k_1,k_2,k_3) + \widetilde{B}(k_1,k_2,k_3)\,\frac{k_1\cdot k_3}{k_1^2} \\
B(k_1,k_2,k_3) &= \widetilde{B}(k_1,k_2,k_3) - \widetilde{A}(k_1,k_2,k_3)\frac{k_1\cdot k_2}{k_2^2}
\end{aligned} \tag{1.28}
$$

Note that the exchange symmetry (1.25) continues to hold between $A$ and $B$:

$$
A(k_1,k_2,k_3) = -B(k_2,k_1,k_3) \tag{1.29}
$$

We will now obtain the primary and secondary Ward identities that $A(k_1,k_2,k_3)$ and $B(k_1,k_2,k_3)$ satisfy, by letting the generator of special conformal transformations $K^\kappa$ (1.14) act on the ansatz (1.160) :

$$
\begin{aligned}
K^\kappa\langle J^\mu(k_1)J^\nu(k_3)O(k_2)\rangle_{\text{odd}} = \Big[ &-2\frac{\partial}{\partial k_1^\kappa} - 2k_1^\alpha\frac{\partial}{\partial k_1^\alpha}\frac{\partial}{\partial k_1^\kappa} + k_{1,\kappa}\frac{\partial}{\partial k_1^\alpha}\frac{\partial}{\partial k_{1\alpha}} \\
&+ 2(\Delta_3-3)\frac{\partial}{\partial k_3^\kappa} - 2k_3^\alpha\frac{\partial}{\partial k_3^\alpha}\frac{\partial}{\partial k_3^\kappa} + k_{3,\kappa}\frac{\partial}{\partial k_3^\alpha}\frac{\partial}{\partial k_{3\alpha}}\Big]\langle J^\mu(k_1)J^\nu(k_2)O(k_3)\rangle \\
&+ 2\left(\delta^{\mu\kappa}\frac{\partial}{\partial k_1^\alpha} - \delta^\kappa_\alpha\frac{\partial}{\partial k_{1,\mu}}\right)\langle J^\alpha(k_1)J^\nu(k_2)O(k_3)\rangle
\end{aligned} \tag{1.30}
$$

Note that by choosing $k_1$ and $k_3$ as the independent momenta, we got rid of one set of terms in the generator $K^\kappa$ that mixes with the index structure of the correlator. The primary Ward identities satisfied by $A(k_1,k_2,k_3)$ are given by :

$$
\begin{aligned}
&\frac{\partial^2 A}{\partial k_1^2} + \frac{\partial^2 A}{\partial k_3^2} + \frac{2k_1}{k_3}\frac{\partial^2 A}{\partial k_1\,\partial k_3} + \frac{2k_2}{k_3}\frac{\partial^2 A}{\partial k_2\,\partial k_3} + \frac{2}{k_1}\frac{\partial A}{\partial k_1} + \frac{8}{k_3}\frac{\partial A}{\partial k_3} = 0 \\
&\frac{\partial^2 A}{\partial k_3^2} + \frac{\partial^2 A}{\partial k_2^2} + \frac{2k_1}{k_3}\frac{\partial^2 A}{\partial k_1\,\partial k_3} + \frac{2k_2}{k_3}\frac{\partial^2 A}{\partial k_2\,\partial k_3} + \frac{8}{k_3}\frac{\partial A}{\partial k_3} = 0
\end{aligned} \tag{1.31}
$$

Similarly, the equations for $B(k_1,k_2,k_3)$ are given by :

$$
\begin{aligned}
&\frac{\partial^2 B}{\partial k_2^2} + \frac{\partial^2 B}{\partial k_3^2} + \frac{2k_1}{k_3}\frac{\partial^2 B}{\partial k_1\,\partial k_3} + \frac{2k_2}{k_3}\frac{\partial^2 B}{\partial k_2\,\partial k_3} + \frac{2}{k_2}\frac{\partial B}{\partial k_2} + \frac{8}{k_3}\frac{\partial B}{\partial k_3} = 0 \\
&\frac{\partial^2 B}{\partial k_3^2} + \frac{\partial^2 B}{\partial k_1^2} + \frac{2k_1}{k_3}\frac{\partial^2 B}{\partial k_1\,\partial k_3} + \frac{2k_2}{k_3}\frac{\partial^2 B}{\partial k_2\,\partial k_3} + \frac{8}{k_3}\frac{\partial B}{\partial k_3} = 0
\end{aligned} \tag{1.32}
$$

The general solution to both the primary Ward identities can be found in terms of triple-$K$ integrals (1.15). We solve for $\beta_1, \beta_2, \beta_3$ by substituting the triple-$K$ integral into the primary Ward identities, and obtain :

$$\begin{aligned} A &\propto I_{\alpha\{-\frac{1}{2},\frac{1}{2},\Delta_3-\frac{3}{2}\}} \\ B &\propto I_{\alpha\{\frac{1}{2},-\frac{1}{2},\Delta_3-\frac{3}{2}\}} \end{aligned} \tag{1.33}$$

The unknown $\alpha$ is determined using the dilatation Ward identity. The action of the dilatation Ward identity on the ansatz gives the degree of the form factors :

$$\begin{aligned} \deg(A) &= 1 + \Delta_3 - N_A \\ \deg(B) &= 1 + \Delta_3 - N_B \end{aligned} \tag{1.34}$$

where $N_A$ and $N_B$ are the tensorial dimensions of $A$ and $B$, defined as the number of momenta that multiply the form factor in the ansatz. We see from (1.160) and (1.23) that $N_A = N_B = 3$. Similarly, we impose the dilatation Ward identity on the triple-$K$ integral and get :

$$\deg(I_{\alpha\{\beta_j\}}) = \beta_1 + \beta_2 + \beta_3 - \alpha - 1 \tag{1.35}$$

This must equal the degree of the form factors $A$ and $B$ (1.34) giving us :

$$\alpha = 1 - \Delta_3 + \sum_{i=1}^{3} \beta_i \tag{1.36}$$

Thus we obtain :

$$\begin{aligned} A &= c_1\, I_{-\frac{1}{2}\{-\frac{1}{2},\frac{1}{2},\Delta_3-\frac{3}{2}\}} \\ B &= c_2\, I_{-\frac{1}{2}\{\frac{1}{2},-\frac{1}{2},\Delta_3-\frac{3}{2}\}} \end{aligned} \tag{1.37}$$

where $c_1$ and $c_2$ are undetermined constants. We now present the explicit expressions for the two form factors for a few values of the scaling dimension of the scalar operator $O$.

When the scalar operator has $\Delta_3 = 1$, we have,

$$\begin{aligned} A(k_1,k_2,k_3) &= c_1\sqrt{\frac{\pi^3}{8}}\frac{1}{k_1k_3\left(k_1+k_2+k_3\right)^2}, \\ B(k_1,k_2,k_3) &= c_2\sqrt{\frac{\pi^3}{8}}\frac{1}{k_2k_3\left(k_1+k_2+k_3\right)^2} \end{aligned} \tag{1.38}$$

For $\Delta_3 = 2$ :

$$\begin{aligned} A(k_1,k_2,k_3) &= c_1\sqrt{\frac{\pi^3}{8}}\frac{1}{k_1\left(k_1+k_2+k_3\right)^2}, \\ B(k_1,k_2,k_3) &= c_2\sqrt{\frac{\pi^3}{8}}\frac{1}{k_2\left(k_1+k_2+k_3\right)^2}. \end{aligned} \tag{1.39}$$

When $\Delta_3 = 3$ :

$$\begin{aligned} A(k_1,k_2,k_3) &= c_1\sqrt{\frac{\pi^3}{8}}\frac{k_1+k_2+2k_3}{k_1\left(k_1+k_2+k_3\right)^2} \\ B(k_1,k_2,k_3) &= c_2\sqrt{\frac{\pi^3}{8}}\frac{k_1+k_2+2k_3}{k_2\left(k_1+k_2+k_3\right)^2}. \end{aligned} \tag{1.40}$$

We will now look at the secondary Ward identities to fix the undetermined constants $c_1$ and $c_2$ in (1.37). There is one independent secondary Ward identity in this case which leaves just one independent, undetermined constant. The identity is given by :

$$\frac{k_1^2}{k_2}\frac{\partial A}{\partial k_2} + k_1\frac{\partial B}{\partial k_1} = 0 \tag{1.41}$$

Substituting the solutions for the form factors from (1.37) in this equation we get:

$$c_2 = -c_1 \tag{1.42}$$

which is exactly what is expected from symmetry considerations.

#### 1.2.2.3.1 Divergences and Renormalization

We saw in equations (1.38), (1.39) and (1.40) that the triple-$K$ integral is convergent for $\Delta = 1, 2, 3$. For $\Delta_3 > 3$, the integral is singular in the regulator and in some cases, we will require counter-terms to remove this divergence.

The generating functional for the theory is defined as :

$$Z = \int D\phi \, \exp\left(-\int d^3x \, (S_\phi[A_\mu, g^{\mu\nu}] + \sqrt{g} O\phi + J^\mu A_\mu)\right) \tag{1.43}$$

where $\phi$ and $A_\mu$ are sources of the scalar operator and the conserved spin-one current respectively. For certain classes of divergences, the generating functional is modified by counter-terms. We classify the values of $\Delta_3$ into two classes based on the kinds of divergences that occur. $\boldsymbol{\Delta_3 = 4 + 2n}$ **where** $n \in \mathbb{Z}_{\geq 0}$ : When $\Delta_3 = 4$, i.e. $n = 0$, (1.18) is satisfied for the choice of signs given by $(+--)$. When $n > 0$, it is satisfied for the choice of signs $(+--)$ and $(-+-)$. We choose to work in a convenient regularisation scheme where we shift $\Delta_3$ as $\Delta_3 \to \Delta_3 + \epsilon$ and keep the dimension $d$ of the space-time and the conformal dimensions $\Delta_1$ and $\Delta_2$ as in the unregulated theory. To remove this singularity, we look at the following parity-odd counter-term from (1.20)

$$S_{ct} = a(\epsilon) \int d^3x \, \mu^\epsilon \, \epsilon^{\mu\nu\lambda} \, F_{\mu\nu} \, J_\lambda \, \Box^n \phi \tag{1.44}$$

where $\mu$ is the renormalization scale. After taking suitable functional derivatives, the contribution to the correlator from this counter-term is given by

$$\begin{aligned} \langle J^\mu(x_1) J^\nu(x_2) O(x_3)\rangle_{ct} = -a(\epsilon) \Big[ & \Box^n \left(\delta^3(x_2 - x_3)\epsilon^{\rho\nu\lambda}\partial_{2\rho}\langle J^\lambda(x_1) J^\mu(x_3)\rangle\right) \\ & - \Box^n \left(\delta^3(x_1 - x_3)\epsilon^{\rho\mu\lambda}\partial_{1\rho}\langle J^\lambda(x_3) J^\nu(x_2)\rangle\right)\Big] \end{aligned} \tag{1.45}$$

A Fourier transform of the above gives :

$$\begin{aligned} \langle J^\mu(k_1) J^\nu(k_2) O(k_3)\rangle_{ct} &= -a(\epsilon)\left(k_2^{2n}\epsilon^{\nu k_2 \lambda}\pi^\mu_\lambda(k_1)\, k_1 - k_1^{2n}\epsilon^{\mu k_1 \lambda}\pi^\nu_\lambda(k_2)\, k_2\right) \\ &= -a(\epsilon)\left[k_2^{2n} k_1 \left(\epsilon^{\mu\nu k_2} + \frac{\epsilon^{\nu k_1 k_2} k_1^\mu}{k_1^2}\right) - k_1^{2n} k_2 \left(\epsilon^{\mu\nu k_1} + \frac{\epsilon^{\mu k_1 k_2} k_2^\nu}{k_2^2}\right)\right]\mu^{-\epsilon} \end{aligned} \tag{1.46}$$

where we used the following 2-point function [1] :

$$\langle J^\mu(k) J_\nu(-k)\rangle = \pi^\mu_\nu(k)\, k \tag{1.47}$$

[1] The counter-term that we used (1.211) could also contribute to the parity-even part of $\langle JJO\rangle$ since the $\langle JJ\rangle$ 2-point-function has a parity-odd contribution.

Using Schouten identities (A.11) and (A.12), the ansatz for the correlator can be written as

$$\langle J^\mu(k_1)J^\nu(k_2)O(k_3)\rangle = A_1\left(\epsilon^{\mu\nu k_2}k_1^2 + \epsilon^{\nu k_1 k_2}k_1^\mu\right) + A_2\left(\epsilon^{\mu\nu k_1}k_2^2 + \epsilon^{\mu k_1 k_2}k_2^\nu\right) \tag{1.48}$$

When $\Delta_3 = 4$ the singular part of the regularised form factors are given by

$$A_1(k_1,k_2,k_3) = \frac{1}{k_1\epsilon}, \quad A_2(k_1,k_2,k_3) = -\frac{1}{k_2\epsilon} \tag{1.49}$$

The contribution of the counter-term (1.46) to the correlator in this case ($\Delta_3 = 4$, or equivalently $n = 0$) is given by :

$$\langle J^\mu(k_1)J^\nu(k_2)O(k_3)\rangle_{ct} = -a(\epsilon)\left[k_1\left(\epsilon^{\mu\nu k_2} + \frac{\epsilon^{\nu k_1 k_2}k_1^\mu}{k_1^2}\right) - k_2\left(\epsilon^{\mu\nu k_1} + \frac{\epsilon^{\mu k_1 k_2}k_2^\nu}{k_2^2}\right)\right]\mu^{-\epsilon} \tag{1.50}$$

Comparing (1.50) and (1.48) along with (1.49) we see that choosing $a(\epsilon) = 1/\epsilon$ cancels the singular part of the correlator. After removing the divergences, the resulting form factor is given by :

$$A_1(k_1,k_2,k_3) = c_1\frac{3}{k_1}\log\left(\frac{k_1+k_2+k_3}{\mu}\right) - c_1\frac{k_3^2 + 3k_3(k_1+k_2+k_3)}{k_1(k_1+k_2+k_3)^2} \tag{1.51}$$

The second form factor is obtained by the following exchange :

$$A_2(k_1,k_2,k_3) = -A_1(k_2,k_1,k_3) \tag{1.52}$$

The anomalous dilatation Ward identity takes the form :

$$\mu\frac{\partial A_1}{\partial\mu} = -\frac{c_1}{k_1} \tag{1.53}$$

$\boldsymbol{\Delta_3 = 5 + 2n}$ where $n \in \mathbb{Z}_{\geq 0}$ : In this case, (1.18) is satisfied for the choice of signs given by $(---)$ and $(++-)$. Although we have both an ultra-local and a non-local divergence here, the term at $\mathcal{O}(1/\epsilon)$ is non-local in the momenta and therefore the divergence can be cancelled by multiplying with a constant of $\mathcal{O}(\epsilon)$ and then taking the limit $\epsilon \to 0$. In particular, when $\Delta_3 = 5$, the divergent term can be calculated to be :

$$A_1(k_1, k_2, k_3) = c_1(\epsilon)\frac{k_1 + k_2}{k_1 \epsilon} + O(\epsilon^0) \tag{1.54}$$

Choosing $c_1$ to be $O(\epsilon)$, the resulting form factor is :

$$A_1(k_1, k_2, k_3) = c_1^{(1)}\frac{k_1 + k_2}{k_1} \tag{1.55}$$

where $c_1^{(1)}$ is $\mathcal{O}(0)$ in $\epsilon$. It can be easily checked that this form factor satisfies non-anomalous Ward identities and that scale invariance is not broken.

### 1.2.3 Using Weight-shifting and Spin-raising operators

The second method of computing correlation functions in momentum space hinges on the technique of weight-shifting and spin-raising operators. In position space, this technique was initiated in [189] and extensively developed in [190]. In this approach, starting from certain seed correlators, the action of conformally covariant weight-shifting and spin-raising operators generates the desired correlator. To describe this method in some detail, let us consider a spinning correlator $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$. The first step is to count the number of independent tensor structures associated with this correlator. For parity-even correlators this number in position space is given by [14] :

$$N_{3d}^{+}(l_1, l_2, l_3) = 2l_1 l_2 + l_1 + l_2 + 1 - \frac{p(p+1)}{2} \tag{1.56}$$

where $p = \max(0, l_1 + l_2 - l_3)$. The second step is to consider a seed correlator of the form $\langle O_{\Delta_1} O_{\Delta_2} J_{s_3} \rangle$ and find out $N_{3d}^{+}(l_1, l_2, l_3)$ ways to reach $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$. This involves acting upon the seed correlator with various spin-raising and weight-shifting operators. In momentum space, we are constrained in our choice of seed correlators because correlators of the form $\langle J^{(l)} O_{\Delta_1} O_{\Delta_2} \rangle$, where $J^{(l)}$ is a spin-$l$ conserved current, are non-zero only when $\Delta_1 = \Delta_2$. A more convenient approach was recently advocated in [50, 180] to compute (parity-even) spinning cosmological correlators where instead of starting from the seed $\langle O_{\Delta_1} O_{\Delta_2} J_{s_3} \rangle$, one starts from $\langle O_{\Delta_1} O_{\Delta_2} O_{\Delta_3} \rangle$, and apply spin-raising and weight-shifting operators such that the resulting correlator satisfies the Ward-Takahashi identity. See Section 4.2.2 of [180] for an example.

#### 1.2.3.1 Subtleties with the weight-shifting and spin-raising operator approach

In momentum space, one must consider the types of divergences in the seed and target correlators. It is not always possible to reach a target correlator starting from a seed correlator although a naive application of the spin-raising and weight-shifting operators might suggest so. This is most easily understood in the case of scalar correlators. As a concrete example of such a situation, consider the following two correlators in three-dimensions :

$$\langle O_1(k_1)O_1(k_2)O_2(k_3)\rangle = \frac{1}{k_1k_2} \tag{1.57}$$

$$\langle O_2(k_1)O_2(k_2)O_2(k_3)\rangle = -\log\left(\frac{k_1+k_2+k_3}{\mu}\right) \tag{1.58}$$

where $\mu$ is the renormalisation scale. Although it might seem like we can use the weight-shifting operator $W_{12}^{++}$ (defined in (A.20)) to go from the first correlator to the second, this is clearly not possible as $\langle O_2(k_1)O_2(k_2)O_2(k_3)\rangle$ violates scale invariance whereas the seed correlator $\langle O_1(k_1)O_1(k_2)O_2(k_3)\rangle$ does not, i.e.

$$W_{12}^{++}\langle O_1(k_1)O_1(k_2)O_2(k_3)\rangle \neq \langle O_2(k_1)O_2(k_2)O_2(k_3)\rangle \tag{1.59}$$

The above example tells us that weight-shifting operators fail to reproduce the correct correlators when the divergence type changes from non-local to semi-local or ultra-local. The conditions for various types of divergences, in terms of scaling dimensions of the operator insertions, are given by :

| | |
|---|---|
| $(---)$ | $\Delta_1+\Delta_2+\Delta_3=2d+2k_1$ |
| $(--+)$ | $\Delta_1+\Delta_2-\Delta_3=d+2k_2$ |
| $(++-)$ | $-\Delta_1-\Delta_2+\Delta_3=2k_3$ |
| $(+++)$ | $\Delta_1+\Delta_2+\Delta_3=d-2k_4$ |

where $k_1, k_2, k_3, k_4 \geq 0$. We can see that the only time the divergence structure changes is when $k_i = 0$. For the non-local cases in three-dimensions, these correspond to the following for the seed correlator :

$$\Delta_3 = \Delta_1 + \Delta_2 \qquad (++-) \tag{1.60}$$

$$\Delta_3 = 3 - \Delta_1 - \Delta_2 \qquad (+++) \tag{1.61}$$

When either of these conditions is satisfied by the seed correlator, the action of $W_{12}^{++}$ does not reproduce the correct result. However, $W_{12}^{--}$ works as it can be checked that it does not change the type of divergence.

#### 1.2.3.1.1 Example: $\langle TTO_2 \rangle_{\mathbf{odd}}$

In this section we compute the odd part of $\langle TTO_2 \rangle$ using spin-raising and weight-shifting operators. We start from the renormalised scalar-seed correlator $\langle O_1(k_1)O_2(k_2)O_2(k_3)\rangle$ given by :

$$\langle O_1(k_1)O_2(k_2)O_2(k_3)\rangle = \frac{1}{k_1}\log\left[\frac{k_1+k_2+k_3}{\mu}\right] \tag{1.62}$$

where $\mu$ is the renormalization scale. We obtain $\langle TTO_2\rangle_{\text{odd}}$ from $\langle O_1 O_2 O_2\rangle$ as follows :

$$\langle TTO_2\rangle_{\text{odd}} = (k_1k_2)^3 P_1^{(2)} P_2^{(2)} H_{12} \widetilde{D}_{12} \langle O_1(k_1)O_2(k_2)O_2(k_3)\rangle \tag{1.63}$$

After making use of Schouten identities this takes the following explicit form

$$\langle TTO_2\rangle_{\text{odd}} = k_1^2 k_2^2 (k_2\cdot z_1)(k_1\cdot z_2)\frac{k_1(k_2\cdot z_1)\epsilon^{k_1k_2z_2} - k_2(k_1\cdot z_2)\epsilon^{k_1k_2z_1}}{(k_1+k_2+k_3)^4(k_1^2-2k_1k_2+k_2^2-k_3^2)^2} \tag{1.64}$$

We can easily check that the expression obtained for the correlator from the free fermion (FF) theory computation precisely matches the above expression for the correlator up to an additional contact term

$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}O_2\rangle_{\text{odd}} = \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}O_2\rangle_{\text{FF}} + 32\left\langle \frac{\delta T^{\mu_1\nu_1}}{\delta g_{\mu_2\nu_2}}(k_1,k_2)O_2(-k_3)\right\rangle \tag{1.65}$$

To arrive at (1.65) we have made repeated use of Schouten identities given in Appendix A.2. It can be easily checked going to position space that this additional term is a contact term. This difference can be accounted for through a suitable redefinition of the correlation function as was done in [44], see appendix A.3 of the same paper for more details.

## 1.3 Conformal correlators in spinor-helicity variables

In this section, we compute 3-point CFT correlators in spinor-helicity variables. It turns out that solving for CFT correlators in spinor-helicity variables is a lot simpler than doing so in momentum space. The reader may wish to refer to Appendix A.4 at this point to get familiar with our notation and convention regarding spinor-helicity variables.

We start with an ansatz for the correlator in spinor-helicity variables. To do so, we use the fact that a Lorentz transformation of the momentum $\vec{k}$ corresponds to a scale transformation of the spinors. Therefore, a Lorentz-covariant structure in spinor-helicity variables is a structure that has the correct scaling based on the helicities of the operators. An operator $O$ with helicity $h$ transforms in the following way under a scale transformation of spinors :

$$O^h(t\lambda, t^{-1}\bar{\lambda}) = t^{-2h}O(\lambda,\bar{\lambda}) \tag{1.66}$$

Therefore, the ansatz for a general correlator is given by

$$\langle O^{h_1}(k_1)O^{h_2}(k_2)O^{h_3}(k_3)\rangle = (c_1\,F_1(k_1,k_2,k_3) + i\,c_2\,F_2(k_1,k_2,k_3))$$
$$\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} \tag{1.67}$$

where $F_1(k_1,k_2,k_3)$ and $F_2(k_1,k_2,k_3)$ are form-factors that we will determine by imposing dilatation and special conformal invariance. For parity-even correlators $c_2=0$ and for parity-odd correlators $c_1=0$, and for the latter the '$i$' ensures that the correlator changes sign under conjugation since conjugation corresponds to a parity transformation for spinors.

### 1.3.1 Conformal generators

The conformal Ward identities are differential equations determined by the action of the special conformal generator on a conformal correlator. The special conformal generator in spinor-helicity variables takes the form [201] :

$$\widetilde{K}^\kappa = 2\sum_{i=1}^{n} (\sigma^\kappa)_\alpha^{\ \beta} \frac{\partial^2}{\partial\lambda_{i\alpha}\partial\bar{\lambda}_i^\beta} \tag{1.68}$$

The action of $\widetilde{K}$ on a scalar with $\Delta = 2$ is given by [180] :

$$\widetilde{K}^\kappa O_2 = -K^\kappa O_2 \tag{1.69}$$

where

$$K^\kappa = -2\partial_{k_\kappa} - 2k^\alpha\partial_{k^\alpha}\partial_{k_\kappa} + k^\kappa\partial_{k^\alpha}\partial_{k_\alpha} \tag{1.70}$$

The action of $\widetilde{K}$ on a scalar with $\Delta \neq 2$ is given by [180] :

$$\widetilde{K}^\kappa\left(\frac{O_\Delta}{k^{\Delta-2}}\right) = -\frac{1}{k^{\Delta-2}}K^\kappa O_\Delta + \frac{O_\Delta}{k^\Delta}k^\kappa(\Delta-1)(\Delta-2) \tag{1.71}$$

Similarly, the action of $\widetilde{K}$ on spin-one and spin-two conserved currents is as follows [180] :

$$\begin{aligned}\widetilde{K}^\kappa J^\pm &= \left(-z_\pm^\alpha K^\kappa + 2z_\pm^\kappa\frac{k^\alpha}{k^2}\right)J_\alpha\\ \widetilde{K}^\kappa\left(\frac{T^\pm}{k}\right) &= \left(-\frac{1}{k}z_\pm^{(\alpha}z_\pm^{\beta)}K^\kappa + 12z_\pm^\kappa\frac{z_\pm^{(\alpha}k^{\beta)}}{k^3}\right)T_{\alpha\beta}\end{aligned} \tag{1.72}$$

where $J^+ = z_\mu^+ J^\mu$ and $T^+ = z_\mu^+ z_\nu^+ T^{\mu\nu}$. In (1.70) and (1.72), $K^\kappa$ corresponds to the special conformal generator in momentum space with $\Delta = 2$. Its action on a conformally invariant correlator is zero. Therefore, the action of $\widetilde{K}^\kappa$ on a correlator in which all the operators have $\Delta = 2$ will just have a part proportional to the R.H.S. of the Ward-Takahashi identity of the correlator. When the correlator has operators with scaling dimensions other than 2, it is convenient to divide them by appropriate powers of $k$ so that the insertion has $\Delta = 2$. For a derivation, see [28].

### 1.3.2 Two-point functions

In this section we present the expressions for a few two-point correlators in spinor-helicity variables. These will later turn out to be useful when dealing with transverse Ward identities associated to spinning three-point correlators. For conserved currents of generic integer spin $s$ we have the following two-point functions:

$$\begin{aligned}
&\langle J^{s-}(k_1)J^{s-}(k_2)\rangle = \left(c_{J_s} + i\,c'_{J_s}\right)\frac{\langle 12\rangle^{2s}}{2sk_2}, \quad \langle J^{s+}(k_1)J^{s-}(k_2)\rangle = \left(c_{J_s} + i\,c'_{J_s}\right)\frac{\langle \bar{1}2\rangle^{2s}}{2sk_2}\\
&\langle J^{s+}(k_1)J^{s+}(k_2)\rangle = \left(c_{J_s} - i\,c'_{J_s}\right)\frac{\langle \bar{1}\bar{2}\rangle^{2s}}{2sk_2}, \quad \langle J^{s-}(k_1)J^{s+}(k_2)\rangle = \left(c_{J_s} - i\,c'_{J_s}\right)\frac{\langle 1\bar{2}\rangle^{2s}}{2sk_2}
\end{aligned} \tag{1.73}$$

where $c_{J_s}$ and $c'_{J_s}$ are the two-point function coefficients of the spin-$s$ current for the even and odd cases respectively.

### 1.3.3 Three-point functions: General discussion

We will now consider three-point functions with spinning operator insertions. The parity odd sector of a few correlators such as $\langle JJO\rangle$, $\langle JJJ\rangle$, and $\langle TTO\rangle$ have been studied in momentum space by solving conformal Ward identities, using spin-raising and weight-shifting operators and using higher spin equations [150,152,188]. In extending our analysis to more complicated three-point correlators we faced some difficulties as described in the beginning of this section. However, working in spinor-helicity variables, we are able to circumvent this problem and get expressions for more complicated 3-point correlators as described in detail below. We will first introduce the terminology of homogeneous and non-homogeneous solutions to conformal Ward identities which we will use throughout this paper.

#### 1.3.3.1 Homogeneous and non-homogeneous solutions

The action of the special conformal generator in spinor-helicity variables on a generic 3-point correlator takes the following form :

$$\widetilde{K}^{\kappa}\left\langle \frac{J_{s_1}}{k_1^{s_1-1}}\frac{J_{s_2}}{k_2^{s_2-1}}\frac{J_{s_3}}{k_3^{s_3-1}}\right\rangle = \text{transverse Ward identity terms} \tag{1.74}$$

where the R.H.S. contains contact-term contributions and is expressible in terms of 2-point functions. The explicit form of the generator $\widetilde{K}^\kappa$ is given in Section 1.3.1.

Being a linear differential equation, the general solution of the above is expressible as the sum of homogeneous and non-homogeneous solutions :

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h}} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{nh}} \tag{1.75}$$

where $\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h}}$ solves:

$$\widetilde{K}^\kappa \left\langle \frac{J_{s_1}}{k_1^{s_1-1}} \frac{J_{s_2}}{k_2^{s_2-1}} \frac{J_{s_3}}{k_3^{s_3-1}} \right\rangle_{\mathbf{h}} = 0 \tag{1.76}$$

and $\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{nh}}$ is a solution of :

$$\widetilde{K}^\kappa \left\langle \frac{J_{s_1}}{k_1^{s_1-1}} \frac{J_{s_2}}{k_2^{s_2-1}} \frac{J_{s_3}}{k_3^{s_3-1}} \right\rangle_{\mathbf{nh}} = \text{transverse Ward identity terms} \tag{1.77}$$

This distinction will be important to keep in mind since the homogeneous and non-homogeneous parts have different structures and properties. One way to distinguish between the two kinds of solutions in the final answer will be that the non-homogeneous solution depends on the coefficient of the two-point function. Another way is to make use of the transverse Ward identities :

$$\begin{aligned} \langle k_1 \cdot J_{s_1}(k_1) J_{s_2}(k_2) J_{s_3}(k_3) \rangle_{\mathbf{h}} &= 0 \\ \langle k_1 \cdot J_{s_1}(k_1) J_{s_2}(k_2) J_{s_3}(k_3) \rangle_{\mathbf{nh}} &= \text{WT identity terms.} \end{aligned} \tag{1.78}$$

In other words, while the homogeneous solution is completely transverse, the non-homogeneous solution gets contribution from both transverse as well as local (or longitudinal) terms.

Since the 3-point correlators can be parity-violating, it will be useful to break up the homogeneous and non-homogeneous parts further into parity-even and parity-odd contributions:

$$\begin{aligned} \langle J_{s_1} J_{s_2} J_{s_3} \rangle &= \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h}} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{nh}} \\ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h}} &= \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h},\text{even}} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h},\text{odd}} \\ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{nh}} &= \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{nh},\text{even}} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{nh},\text{odd}} \end{aligned} \tag{1.79}$$

For a detailed discussion on the homogeneous and non-homogeneous contributions to three-point correlators and their distinction from transverse and longitudinal contributions see appendix A.5.

#### 1.3.3.2 Degeneracy structure

We denote 4-dimensional Lorentzian momenta and polarisation vectors by $k_i^\mu$ and $z_i^\mu$ respectively. Here $i$ is a particle index and $\mu = 0, 1, 2, 3$ is the Lorentz index. For massless spin 2 particles the polarisation tensor can be written as an outer product $z_i^{\mu\nu} = z_i^\mu z_i^\nu$. We choose the following gauge to work with null momenta:

$$k_i^\mu = (k_i, \vec{k}_i), \quad z_i^\mu = (0, \vec{z}_i) \tag{1.80}$$

where $k_i = |\vec{k}_i|$ is the magnitude of the 3-momentum. The 3-dimensional CFT will be Euclidean and current conservation constraints translate to transversality: $k_i \cdot z_i = 0$. We will also take $z_i \cdot z_i = 0$ which in Euclidean signature implies that the components of $\vec{z}_i$ will be complex. In our computation we will find it useful to introduce the following notation for various combinations of magnitudes of momenta :

$$E = k_1 + k_2 + k_3, \quad b_{ij} = k_i k_j, \quad b_{123} = k_1 k_2 + k_2 k_3 + k_3 k_1, \quad c_{123} = k_1 k_2 k_3 \tag{1.81}$$

We also introduce the following notation :

$$J^2 = (k_1 + k_2 + k_3)(-k_1 + k_2 + k_3)(k_1 - k_2 + k_3)(k_1 + k_2 - k_3) \tag{1.82}$$

We will make use of spinor-helicity notation. The momentum vector $p_\mu$ for massless scattering in 4-dimensional flat space-time can be written as $p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ where $\lambda$ denotes a spinor-helicity variable. Since 4d amplitudes are related to 3d CFT correlators it will be useful to have a 3d version of this formalism by utilising the time-like vector $\tau^\mu = (1, 0, 0, 0)$, or $\tau^{\alpha\dot{\beta}} = \epsilon^{\alpha\dot{\beta}}$ which can be used to go from dotted to undotted indices (see appendix B of [39] and [40]). We use this to define $\bar{\lambda}^\alpha \equiv \tau^{\alpha\dot{\beta}} \tilde{\lambda}_{\dot{\beta}}$.

In three dimensions, there exist degeneracies in tensor structures which complicate the analysis of correlators. The existence of degeneracy is tied to the simple fact that not more than three vectors can be linearly independent in three dimensions. The basic problem is that the different tensor structures in the ansatz for a correlator become linearly dependent due to degeneracies. This affects the analysis of both parity-even and parity-odd correlators. For the parity-odd correlator, Schouten identities, which relate various tensor structures involving Levi-Civita tensors, are an additional source of complication. The main problem is that while solving the conformal Ward identity, one needs to identify the correct independent set of tensor structures to be able to write down differential equations for the form-factors. However, this process becomes very complicated for correlators involving spin-2 or higher spin operators. An example of such an identity in three dimensions is :

$$\epsilon^{z_1 z_2 k_1}(k_1 \cdot k_2) + \epsilon^{z_1 k_1 k_2} k_1 \cdot z_2 - \epsilon^{z_1 z_2 k_2} k_1^2 - \epsilon^{z_2 k_1 k_2} k_1 \cdot z_1 = 0, \tag{1.83}$$

where we have used the notation[2] $\epsilon^{z_2 k_1 k_2} = \epsilon_{\mu\nu\rho} z_2^\mu k_1^\nu k_2^\rho$. The structures that appear in the above equation arise in the ansatz for various parity-odd correlators such as $\langle JJO \rangle_{\text{odd}}$. The above equation then implies that a term with $\epsilon^{z_1 k_1 k_2}$ in the ansatz can be eliminated in favour of other structures [3]. This, while essential to be taken into account, makes cumbersome the correct ansatz with a minimal basis of independent structures. Other than Schouten identities, there are identities such as [41] :

$$\delta^{\mu\nu} = \frac{4}{J^2}\Big(k_i^2 k_j^\mu k_j^\nu + k_j^2 k_i^\mu k_i^\nu - \vec{k}_i.\vec{k}_j(k_i^\mu k_j^\nu + k_j^\mu k_i^\nu) + n^\mu n^\nu\Big) \tag{1.84}$$

where $n^\mu = \epsilon^{\mu\nu\rho} k_\nu k_\rho$ and $i \neq j = 1, 2, 3$. We also have [41] :

$$\Pi^{\mu\nu}_{\alpha\beta}(k_j)\, n^\alpha n^\beta = -k_j^2 \Pi^{\mu\nu}_{\alpha\beta}(k_j)\, k^\alpha_{(j+1)\text{ mod }3} k^\beta_{(j+1)\text{ mod }3} \qquad j = 1, 2, 3 \tag{1.85}$$

[2]We will often use this notation in this paper.
[3]See [188] for details of the complete momentum space analysis of $\langle JJO \rangle_{\text{odd}}$.

Another example of a degeneracy is [44] :

$$\Pi^{\alpha_1}_{\mu_1\nu_1\beta_1}(k_1)\Pi^{\alpha_2}_{\mu_2\nu_2\beta_2}(k_2)4!\delta^{\beta_1}_{[\alpha_1}\delta^{\beta_2}_{\alpha_2}k_{1\alpha_2}k_{2\alpha_4]}k_1^{\alpha_3}k_2^{\alpha_4}$$
$$= \Pi_{\mu_1\nu_1\alpha_1\beta_1}(k_1)\Pi_{\mu_2\nu_2\alpha_2\beta_2}(k_2)\bigg[k_2^{\alpha_1}k_2^{\beta_1}k_3^{\alpha_2}k_3^{\beta_2}$$
$$-(k_1^2+k_2^2-k_3^2)\delta^{\beta_1\beta_2}k_2^{\alpha_1}k_3^{\alpha_2}-\frac{J^2}{4}\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}\bigg]=0 \tag{1.86}$$

These also allow certain basis structures to be expressed in terms of others. Both parity-even and parity-odd degeneracies complicate the analysis when computing correlation functions. One of the advantages of working with spinor-helicity variables is that the degeneracies become trivial in these variables. For example, the left hand side of both (1.83) and (1.86) become identically zero in spinor-helicity variables. One can check that all the Schouten identities and other identities relating various tensor structures also become trivial in spinor-helicity variables.

## 1.4 Three-point functions : Explicit solutions in spinor-helicity variables

In this section, we focus on determining $CFT_3$ 3-point correlators in spinor-helicity variables. In particular, we compute correlators of the form $\langle J_s O_\Delta O_\Delta\rangle$, $\langle J_s J_s O_\Delta\rangle$, $\langle J_s J_s J_s\rangle$ and $\langle J_{s_1} J_s J_s\rangle$ where $J_s$ is a symmetric, traceless, spin-$s$ conserved current with scaling dimension $\Delta = s+1$, and $O_\Delta$ is a scalar operator with scaling dimension $\Delta$. In three dimensions, 3-point correlators involving only spinning operators are always finite, whereas those involving a scalar operator require renormalization for large enough values of $\Delta$.

We will observe that splitting the correlator into homogeneous and non-homogeneous parts in the sense explained in Section 1.3.3.1 is useful. As we demonstrate, whenever there exists a homogeneous parity-even solution to the conformal Ward identity in spinor-helicity variables, there also exists a homogeneous parity-odd solution and the two are identical up to some signs. Interestingly, in the case of divergent correlators, the parity-odd and the parity-even correlators continue to match even after renormalization, although the renormalization procedure for the two differs. Further, it turns out that the non-homogeneous part is always parity-even. Any parity-odd contribution to the non-homogeneous part is always a contact term. After the first example in which we present all the details, in each case we will give the correlator ansatz and then write down the form-factors as solutions of the CWIs, relegating the details to Appendix A.6.

## Notation

A spin $s$ current has various helicity components such as $J_s^{-\cdots-}, J_s^{-\cdots+-}, \cdots, J_s^{+\cdots+}$. Due to tracelessness, mixed helicity components vanish. Hence the only nontrivial helicity components are $J_s^{-\cdots-}$ and $J_s^{+\cdots+}$ which we denote by $J_s^-$ and $J_s^+$, respectively.

### 1.4.1 $\langle J_s O_\Delta O_\Delta \rangle$

In this section, we calculate correlators of the form $\langle J_s O_\Delta O_\Delta \rangle$. The Ward-Takahashi (WT) identity when the spinning operator is either a spin-one conserved current or the stress-tensor (i.e. when $s = 1$ or $s = 2$) is given by the following [41, 180]:

$$\begin{aligned} k_{1\mu}\langle J^\mu O_\Delta O_\Delta \rangle &= \langle O_\Delta(k_3)O_\Delta(-k_3)\rangle - \langle O_\Delta(k_2)O_\Delta(-k_2)\rangle \\ k_{1\mu}z_{1\nu}\langle T^{\mu\nu} O_\Delta O_\Delta \rangle &= (k_2 \cdot z_1)\left(\langle O_\Delta(k_3)O_\Delta(-k_3)\rangle - \langle O_\Delta(k_2)O_\Delta(-k_2)\rangle\right) \end{aligned} \tag{1.87}$$

where in the second equation we have contracted both sides of the WT identity with null transverse polarization vectors. It is straightforward to generalize the WT identity to arbitrary spin-$s$ conserved currents by matching the spin and scaling dimensions on both sides of the identity. This gives the following :

$$z_{1\mu_2}\cdots z_{1\mu_s}k_{1\mu_1}\langle J^{\mu_1\cdots\mu_s} O_\Delta O_\Delta \rangle = (k_2 \cdot z_1)^{s-1}(\langle O_\Delta(k_3)O_\Delta(-k_3)\rangle - \langle O_\Delta(k_2)O_\Delta(-k_2)\rangle) \tag{1.88}$$

We will see that the homogeneous part of the correlator is zero. The non-homogeneous part has the scalar two-point function on the right-hand side[4]. Consequently, the odd part of the correlator goes to zero as there is no parity-odd scalar two-point function. Thus this correlator has only a parity-even non-homogeneous part.

As noted in Section 1.3.1, when the correlator involves operators with scaling dimensions other than 2, it is convenient to divide the insertions by appropriate powers of the

[4] A correlator comprising one conserved current and two scalar operators with different scaling dimensions also vanishes, i.e.

$$\langle J_s O_{\Delta_1} O_{\Delta_2} \rangle = 0 \qquad \text{for} \quad \Delta_1 \neq \Delta_2 \tag{1.89}$$

corresponding momenta $k$ such that they have $\Delta = 2$. The correlator itself is obtained at the end by restoring the powers of $k$. Keeping this in mind, we start with the following ansatz for the correlator :

$$\left\langle \frac{J_s^-}{k_1^{s-1}} \frac{O_\Delta}{k_2^{\Delta-2}} \frac{O_\Delta}{k_3^{\Delta-2}} \right\rangle = F(k_1, k_2, k_3) \langle 12 \rangle^s \langle \bar{2}1 \rangle^s \tag{1.90}$$

The action of the generator of special conformal transformations $\widetilde{K}$ is then given by (see Section 1.3.1) :

$$\begin{aligned} \widetilde{K}^\kappa \left\langle \frac{J_s^-}{k_1^{s-1}} \frac{O_\Delta}{k_2^{\Delta-2}} \frac{O_\Delta}{k_3^{\Delta-2}} \right\rangle &= \frac{2 z_1^{-\kappa} c_O}{k_1^{2s-1} k_2^{\Delta-2} k_3^{\Delta-2}} (k_3^{2\Delta-3} - k_2^{2\Delta-3}) \\ &\quad + (\Delta-1)(\Delta-2) \left\langle J_s^- \frac{O_\Delta}{k_2^{\Delta-2}} \frac{O_\Delta}{k_3^{\Delta-2}} \right\rangle \left( \frac{k_2^\kappa}{k_2^2} - \frac{k_3^\kappa}{k_3^2} \right) \end{aligned} \tag{1.91}$$

Contracting (1.91) with $b_\kappa = (\sigma^\kappa)_\beta{}^\alpha \lambda_{1\alpha} \lambda_1^\beta$, $b_\kappa = (\sigma^\kappa)_\beta{}^\alpha (\lambda_{1\alpha} \lambda_2^\beta + \lambda_{2\alpha} \lambda_1^\beta)$ and $b_\kappa = (\sigma^\kappa)_\beta{}^\alpha \lambda_{2\alpha} \lambda_2^\beta$ gives the following :

$$\frac{\partial^2 F}{\partial k_2^2} - \frac{\partial^2 F}{\partial k_3^2} = -\frac{F}{k_2^2 k_3^2} (\Delta-1)(\Delta-2)(k_2^2 - k_3^2) \tag{1.92}$$

$$\begin{aligned} &\frac{k_1}{2} \left( \frac{\partial^2 F}{\partial k_3^2} - \frac{\partial^2 F}{\partial k_1^2} \right) + \frac{k_2}{2} \left( \frac{\partial^2 F}{\partial k_2^2} - \frac{\partial^2 F}{\partial k_3^2} \right) - s \frac{\partial F}{\partial k_1} \\ &\qquad = \frac{2(\Delta-1)(\Delta-2) F}{k_2^2 k_3^2} k_2 (k_2^2 + k_3^2 - k_1 k_2) \end{aligned} \tag{1.93}$$

$$\begin{aligned} &\frac{1}{4}(k_1 - k_2 + k_3)(-k_1 + k_2 + k_3) \left( \frac{\partial^2 F}{\partial k_1^2} - \frac{\partial^2 F}{\partial k_3^2} \right) + s^2 F + s k_2 \left( \frac{\partial F}{\partial k_1} + \frac{\partial F}{\partial k_2} \right) \\ &\qquad = c_O \frac{k_3^{2\Delta-3} - k_2^{2\Delta-3}}{k_1^3} + \frac{F}{k_3^2} (\Delta-1)(\Delta-2)(k_1 - k_2 + k_3)(-k_1 + k_2 + k_3) \end{aligned} \tag{1.94}$$

Finally, the dilatation Ward identity is given by :

$$\left( \sum_{i=1}^{3} k_i \frac{\partial F}{\partial k_i} \right) + 2sF = 0 \tag{1.95}$$

The above differential equations (1.92),(1.93), (1.94) and (1.95) can be solved to get :

$$F = c_O k_2^{-\Delta+2} k_3^{-\Delta+2} I_{\frac{1}{2}+s\{\frac{1}{2}-s, \Delta-\frac{3}{2}, \Delta-\frac{3}{2}\}} \tag{1.96}$$

where the triple-$K$ integral [41] which occurs in the RHS of this equation is defined in

(1.204). After taking the momentum factors in the denominator of the LHS of (1.90) to the RHS and using the above result for the form factor we obtain the correlator :

$$\langle J_s^- O_\Delta O_\Delta\rangle = c_O k_1^{s-1} I_{\frac{1}{2}+s\{\frac{1}{2}-s,\Delta-\frac{3}{2},\Delta-\frac{3}{2}\}}\langle 12\rangle^s\langle\bar{2}1\rangle^s \tag{1.97}$$

When $c_O = 0$, there is no non-trivial solution to the differential equations and one has :

$$\langle J_s O_\Delta O_\Delta\rangle_{\mathbf{h}} = 0. \tag{1.98}$$

### 1.4.2 $\langle J_s J_s O_\Delta\rangle$

In this section, we compute correlators of the form $\langle J_s J_s O_\Delta\rangle$ for general spin $s$. As discussed in Section 1.3.3.1, we separate out the correlator into homogeneous and non-homogeneous parts :

$$\langle J_s J_s O_\Delta\rangle = \langle J_s J_s O_\Delta\rangle_{\mathbf{h}} + \langle J_s J_s O_\Delta\rangle_{\mathbf{nh}} \tag{1.99}$$

The correlator $\langle J_s J_s O_\Delta\rangle$ is completely transverse :

$$\langle k_1\cdot J_s(k_1)J_s(k_2)O_\Delta(k_3)\rangle = \langle J_s(k_1)k_2\cdot J_s(k_2)O_\Delta(k_3)\rangle = 0 \tag{1.100}$$

where $k\cdot J_s(k) = k_{\mu_1}J^{\mu_1\mu_2\cdots\mu_s}(k)$. This implies that the non-homogeneous part of the correlator is zero :

$$\langle J_s J_s O_\Delta\rangle_{\mathbf{nh}} = 0. \tag{1.101}$$

We will now compute the explicit form of the correlators for arbitrary $\Delta$. We find that for $\Delta \geq 4$, there is a divergence and we need to regularize and renormalize to obtain finite correlators. We consider the following ansatz for the correlator : (1.67) :

$$\begin{aligned}
\left\langle \frac{J^{s-}(k_1)}{k_1^{s-1}}\frac{J^{s-}(k_2)}{k_2^{s-1}}\frac{O_\Delta(k_3)}{k_3^{\Delta-2}}\right\rangle &= (c_1\,F_1(k_1,k_2,k_3) + i\,c_2\,F_2(k_1,k_2,k_3))\,\langle 12\rangle^{2s}\\
\left\langle \frac{J^{s+}(k_1)}{k_1^{s-1}}\frac{J^{s+}(k_2)}{k_2^{s-1}}\frac{O_\Delta(k_3)}{k_3^{\Delta-2}}\right\rangle &= (c_1\,F_1(k_1,k_2,k_3) - i\,c_2\,F_2(k_1,k_2,k_3))\,\langle \bar{1}\bar{2}\rangle^{2s}\\
\left\langle \frac{J^{s-}(k_1)}{k_1^{s-1}}\frac{J^{s+}(k_2)}{k_2^{s-1}}\frac{O_\Delta(k_3)}{k_3^{\Delta-2}}\right\rangle &= (d_1\,G_1(k_1,k_2,k_3) + i\,d_2\,G_2(k_1,k_2,k_3))\,\langle 1\bar{2}\rangle^{2s}
\end{aligned} \tag{1.102}$$

It is interesting to note that the conformal Ward identity gives identical equations for the parity-odd and the parity-even parts. The details of these equations and their solution are provided in Appendix A.6.2 where we also discuss examples for special values of $\Delta$ and $s$. Here we give the final form of the solution :

$$F_1(k_1,k_2,k_3) = F_2(k_1,k_2,k_3) = k_3^{2-\Delta} I_{(\frac{1}{2}+2s)\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}$$
$$G_1(k_1,k_2,k_3) = G_2(k_1,k_2,k_3) = 0. \tag{1.103}$$

Substituting the form-factor in the ansatz (1.102) we obtain

$$\langle J_s^- J_s^- O_\Delta\rangle = \langle J_s^- J_s^- O_\Delta\rangle_{\text{even}} + \langle J_s^- J_s^- O_\Delta\rangle_{\text{odd}} = (c_1 + ic_2)\ (k_1k_2)^{s-1}\, I_{(\frac{1}{2}+2s)\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\langle 12\rangle^{2s}$$
$$\langle J_s^+ J_s^+ O_\Delta\rangle = \langle J_s^+ J_s^+ O_\Delta\rangle_{\text{even}} + \langle J_s^+ J_s^+ O_\Delta\rangle_{\text{odd}} = (c_1 - ic_2)\ (k_1k_2)^{s-1}\, I_{(\frac{1}{2}+2s)\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\langle \bar{1}\bar{2}\rangle^{2s}$$
$$\langle J_s^- J_s^+ O_\Delta\rangle = 0 \tag{1.104}$$

For $\Delta \geq 4$, the above triple-$K$ integrals and thereby the correlators are divergent. A detailed study of the renormalization of these correlators will be carried out in Section 1.6. We will see that the relationship between the parity-even and the parity-odd parts of a correlator in spinor-helicity variables continues to hold even after renormalization.

### 1.4.3 $\langle J_s J_s J_s\rangle$

In this subsection, we concentrate on the three-point function of a general spin $s$ conserved current $J_s$ [5]. Since the correlator $\langle J_s J_s J_s\rangle$ satisfies a nontrivial transverse WT identity it has both the homogeneous as well as the non-homogeneous contributions.

Let us split the correlator into the odd and even contributions :

$$\langle J_s J_s J_s\rangle = \langle J_s J_s J_s\rangle_{\text{even}} + \langle J_s J_s J_s\rangle_{\text{odd}}.$$

It will turn out that $\langle J_s J_s J_s\rangle_{\text{even}}$ has both the homogeneous and the non-homogeneous contributions whereas $\langle J_s J_s J_s\rangle_{\text{odd}}$ has a non-trivial homogeneous part but the non-homogeneous part is always a contact term.

#### 1.4.3.1 $\langle JJJ\rangle$

Let us start our analysis with the 3-point function of the spin-1 current $J_\mu$. As noted earlier, for this correlator to be non-zero, the currents have to be non-abelian. The WT

[5] If $s$ is odd then we need to consider a non-abelian current to have a non-trivial correlator

identity is given by [41, 44, 180] :

$$
\begin{aligned}
k_{1\mu}\langle J^{\mu a}(k_1)J^{\nu b}(k_2)J^{\rho c}(k_2)\rangle &= \Big(f^{adc}\langle J^{\rho d}(k_2)J^{\nu b}(-k_2)\rangle - f^{abd}\langle J^{\nu d}(k_3)J^{\rho c}(-k_3)\rangle\Big) \\
&+ \left[\left(\frac{k_2^\nu}{k_2^2} f^{abd} k_{2\alpha}\langle J^{\alpha d}(k_3)J^{\rho c}(-k_3)\rangle\right) + ((k_2,\nu)\leftrightarrow(k_3,\rho))\right]
\end{aligned}
\tag{1.105}
$$

Let us consider the following ansatz for the two helicity components of the correlator [6] :

$$
\langle J^-(k_1)J^-(k_2)J^-(k_3)\rangle = (F_1(k_1,k_2,k_3) + iF_2(k_1,k_2,k_3))\,\langle 12\rangle\langle 23\rangle\langle 31\rangle \tag{1.106}
$$

$$
\langle J^-(k_1)J^-(k_2)J^+(k_3)\rangle = (G_1(k_1,k_2,k_3) + iG_2(k_1,k_2,k_3))\,\langle 12\rangle\langle 2\bar{3}\rangle\langle \bar{3}1\rangle \tag{1.107}
$$

The solutions of the conformal Ward identity are given by (see Appendix A.6.3)

$$
F_1(k_1,k_2,k_3) = \frac{c_1}{E^3} + \frac{c_J}{k_1k_2k_3} \tag{1.108}
$$

$$
G_1(k_1,k_2,k_3) = \frac{c_2}{(k_1+k_2-k_3)^3} + c_J\frac{E-2k_3}{E(k_1k_2k_3)} \tag{1.109}
$$

$$
F_2(k_1,k_2,k_3) = \frac{c_1'}{E^3} + \frac{c_J'}{k_1k_2k_3} \tag{1.110}
$$

$$
G_2(k_1,k_2,k_3) = \frac{c_2'}{(k_1+k_2-k_3)^3} + \frac{c_J'}{k_1k_2k_3} \tag{1.111}
$$

where $c_J$ and $c_J'$ are the parity-even and parity-odd coefficients of the two-point function of conserved currents (see (1.73)). The terms are proportional to $c_1, c_1'$ and $c_2, c_2'$ are the homogeneous solutions to the differential equations and those proportional to $c_J, c_J'$ are the non-homogeneous solutions. Since $G(k_1,k_2,k_3)$ and $\tilde{G}(k_1,k_2,k_3)$ both have an un-physical pole when $k_1+k_2=k_3$, we set the coefficients of these terms to zero, i.e. $c_2 = c_2' = 0$.

## Summary of the solution

Taking into account both the parity-even and the parity-odd contributions, we obtain :

$$
\langle J^-(k_1)J^-(k_2)J^-(k_3)\rangle = \left(\frac{c_1+ic_1'}{E^3} + \frac{c_J+ic_J'}{k_1k_2k_3}\right)\langle 12\rangle\langle 23\rangle\langle 31\rangle \tag{1.112}
$$

$$
\langle J^-(k_1)J^-(k_2)J^+(k_3)\rangle = \frac{1}{k_1k_2k_3}\left((c_J+ic_J') - c_J\frac{2k_3}{E}\right)\langle 12\rangle\langle 2\bar{3}\rangle\langle \bar{3}1\rangle \tag{1.113}
$$

[6] We will suppress the color indices which amounts to suppressing an overall factor of $f^{abc}$.

In the next section, we will convert these expressions into the momentum space and see that the non-homogeneous contribution to the parity-odd correlator (term proportional to $c'_J$) becomes a contact term.

#### 1.4.3.2 $\langle TTT \rangle$

Let us now consider the correlator with three insertions of the stress-tensor operator. The transverse Ward identity satisfied by the correlator is given by [41, 44, 180]:

$$\begin{aligned}
&z_{1\mu}k_{1\nu}\langle T^{\mu\nu}(k_1)T(k_2)T(k_3)\rangle \\
&\quad= -(z_1\cdot k_2)\langle T(k_1+k_2)T(k_3)\rangle + 2(z_1\cdot z_2)k_{2\mu}z_\nu\langle T^{\mu\nu}(k_1+k_2)T(k_3)\rangle \\
&\quad\quad - (z_1\cdot k_3)\langle T(k_1+k_3)T(k_2)\rangle + 2(z_1\cdot z_3)k_{3\mu}z_{3\nu}\langle T^{\mu\nu}(k_1+k_3)T(k_2)\rangle \\
&\quad\quad + (k_1\cdot z_2)z_{1\mu}z_{2\nu}\langle T^{\mu\nu}(k_1+k_2)T(k_3)\rangle + (z_1\cdot z_2)k_{1\mu}z_{2\nu}\langle T^{\mu\nu}(k_1+k_2)T(k_3)\rangle \\
&\quad\quad + (k_1\cdot z_3)z_{1\mu}z_{3\nu}\langle T^{\mu\nu}(k_1+k_3)T(k_2)\rangle + (z_1\cdot z_3)k_{1\mu}z_{3\nu}\langle T^{\mu\nu}(k_1+k_3)T(k_2)\rangle
\end{aligned} \tag{1.114}$$

where $T(k) \equiv z_\mu z_\nu T^{\mu\nu}(k)$. Thus the correlator can have both homogeneous and non-homogeneous solutions for the parity-even and parity-odd correlation functions. The parity-even solution was already discussed in [41]. The parity-odd homogeneous contribution was computed in [26] using Feynman diagram computations in $dS_4$. Here we reproduce the same result by solving the conformal Ward identities. We also get a non-trivial non-homogeneous contribution to the parity-odd correlator which in momentum space turns out to be a contact term.

#### $\langle TTT \rangle_{\mathbf{even}}$

The parity-even contribution to the correlator $\langle TTT \rangle_{\text{even}}$ is given by [41, 180]

$$\begin{aligned}
\langle T^-T^-T^-\rangle_{\text{even}} &= \left(c_1\frac{c_{123}}{E^6} + c_T\frac{E^3 - Eb_{123} - c_{123}}{c_{123}^2}\right)\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \\
\langle T^-T^-T^+\rangle_{\text{even}} &= c_T\frac{(E-2k_3)^2(E^3 - Eb_{123} - c_{123})}{E^2c_{123}^2}\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2
\end{aligned} \tag{1.115}$$

where $b_{123} = (k_1k_2 + k_2k_3 + k_3k_1)$, $c_{123} = k_1k_2k_3$, and $c_T$ comes from the parity-even two point function of the stress tensor (1.73). The terms proportional to $c_1$ and $c_T$ are the homogeneous and the non-homogeneous contributions to the correlator respectively.

$\langle TTT \rangle_{\mathbf{odd}}$

The parity-odd part of the correlator can be solved analogously (see Appendix A.6.4 for details). The answer is given by

$$\left\langle \frac{T^-}{k_1}\frac{T^-}{k_2}\frac{T^-}{k_3}\right\rangle_{odd} = i\left(c_1'\frac{1}{E^6} + c_T'\frac{E^3 - E\,b_{123} - c_{123}}{c_{123}^3}\right)\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \tag{1.116}$$

$$\left\langle \frac{T^-}{k_1}\frac{T^-}{k_2}\frac{T^+}{k_3}\right\rangle_{odd} = i\left(c_T'\frac{(E-2k_3)^2 - (E-2k_3)(b_{123} - 2k_3\,a_{12}) + c_{123}}{c_{123}^3}\right)\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2 \tag{1.117}$$

where $a_{12} = k_1 + k_2$, $b_{123} = k_1k_2 + k_2k_3 + k_1k_3$ and $c_{123} = k_1\,k_2\,k_3$ and $c_T'$ arises in the parity-odd two point function of the stress tensor (1.73). The terms proportional to $c_1'$ and $c_T'$ are the homogeneous and the non-homogeneous contributions to the correlator respectively.

## Summary of the solution

Taking into account both the parity-even and the parity-odd contributions, we obtain :

$$\langle T^-(k_1)T^-(k_2)T^-(k_3)\rangle = \left[(c_1 + ic_1')\,\frac{c_{123}}{E^6} + (c_T + ic_T')\,\frac{E^3 - Eb_{123} - c_{123}}{c_{123}^2}\right]\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \tag{1.118}$$

$$\begin{aligned}\langle T^-(k_1)T^-(k_2)T^+(k_3)\rangle = \Big[&c_T\frac{(E-2k_3)^2(E^3 - Eb_{123} - c_{123})}{E^2c_{123}^2} + \\ &ic_T'\frac{(E-2k_3)^2 - (E-2k_3)(b_{123} - 2k_3\,a_{12}) + c_{123}}{c_{123}^3}\Big]\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2\end{aligned} \tag{1.119}$$

The other helicity components of the correlator can be obtained by complex conjugating the above results.

In the next section, we will show that the non-homogeneous contribution to the parity odd correlator (the term proportional to $c_T'$ in (1.118)) is a contact term. Thus the only non-trivial contribution to the non-homogeneous piece in the correlator comes from the parity-even part.

#### 1.4.3.3 $\langle J_s J_s J_s \rangle$ for general spin: The homogeneous part

In this subsection, we generalize the above discussion to the three-point function $\langle J_s J_s J_s \rangle$ of arbitrary spin $s$ conserved current. We first split the correlator into homogeneous and non-homogeneous pieces, and then separate them into their parity-even and parity-odd parts as indicated in (1.79). The non-homogeneous piece $\langle J_s J_s J_s \rangle_{\mathbf{nh}}$ requires us to know the WT identity which for general spin is quite complicated. However, on general grounds we can argue that the parity-odd part of this term, $\langle J_s J_s J_s \rangle_{\mathbf{nh},\text{odd}}$, is a contact term for general spin $s$. We refer the reader to Section 1.8.1 for details. In the rest of this subsection, we focus on obtaining the homogeneous part of the correlator which does not require the WT identity.

**Homogeneous solution**

For the homogeneous solution, the parity-even and the parity-odd contributions are again the same in spinor-helicity variables. We start with the following ansatz for $\langle J_s J_s J_s \rangle_{\mathbf{h}}$ :

$$\left\langle \frac{J_s^-}{k_1^{s-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^-}{k_3^{s-1}} \right\rangle_{\mathbf{h}} = F(k_1, k_2, k_3) \langle 12 \rangle^s \langle 23 \rangle^s \langle 31 \rangle^s \tag{1.120}$$

$$\left\langle \frac{J_s^-}{k_1^{s-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^+}{k_3^{s-1}} \right\rangle_{\mathbf{h}} = G(k_1, k_2, k_3) \langle 12 \rangle^s \langle 2\bar{3} \rangle^s \langle \bar{3}1 \rangle^s. \tag{1.121}$$

Since we are focusing on the homogeneous part, the action of the conformal generator is given by :

$$\widetilde{K}^\kappa \left\langle \frac{J_s^-}{k_1^{s-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^-}{k_3^{s-1}} \right\rangle_{\mathbf{h}} = 0, \quad \widetilde{K}^\kappa \left\langle \frac{J_s^-}{k_1^{s-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^+}{k_3^{s-1}} \right\rangle_{\mathbf{h}} = 0. \tag{1.122}$$

The action of $\widetilde{K}^\kappa$ on the ansatz, after dotting with $b_\kappa = (\sigma^\kappa)_\beta{}^\alpha (\lambda_{2\alpha} \lambda_3^\beta + \lambda_{3\alpha} \lambda_2^\beta)$, becomes

$$2s \left( \frac{\partial F}{\partial k_2} - \frac{\partial F}{\partial k_3} \right) + k_3 \left( \frac{\partial^2 F}{\partial k_2^2} - \frac{\partial^2 F}{\partial k_3^2} \right) - k_2 \left( \frac{\partial^2 F}{\partial k_1^2} - \frac{\partial^2 F}{\partial k_2^2} \right) = 0 \tag{1.123}$$

$$2s \left( \frac{\partial G}{\partial k_2} + \frac{\partial G}{\partial k_3} \right) - k_3 \left( \frac{\partial^2 G}{\partial k_2^2} - \frac{\partial^2 G}{\partial k_3^2} \right) - k_2 \left( \frac{\partial^2 G}{\partial k_1^2} - \frac{\partial^2 G}{\partial k_2^2} \right) = 0 \tag{1.124}$$

The dilatation Ward identity is given by :

$$\left( \sum_{i=1}^{3} k_i \frac{\partial F}{\partial k_i} \right) + 3sF = 0, \quad \left( \sum_{i=1}^{3} k_i \frac{\partial G}{\partial k_i} \right) + 3sG = 0 \tag{1.125}$$

The solutions for $F$ and $G$ are then given (see appendix A.6) by

$$F(k_1,k_2,k_3) = \frac{c_1}{E^{3s}} \tag{1.126}$$

$$G(k_1,k_2,k_3) = 0 \tag{1.127}$$

### Summary of result

Considering the parity-even and the parity-odd contributions we obtain the homogeneous part of the correlator :

$$\left\langle J_s^- J_s^- J_s^- \right\rangle_{\mathbf{h}} = (c_1 + ic_2) k_1^{s-1} k_2^{s-1} k_3^{s-1} \frac{1}{E^{3s}} \langle 12\rangle^s \langle 23\rangle^s \langle 31\rangle^s, \quad \left\langle J_s^- J_s^- J_s^+ \right\rangle_{\mathbf{h}} = 0 \tag{1.128}$$

The other helicity components can be obtained by a simple complex conjugation.

## 1.4.4 $\langle J_{s_1} J_s J_s \rangle$

In this subsection we generalize the above discussion to three-point correlators of the kind $\langle J_{s_1} J_s J_s\rangle$ for arbitrary spins $s$ and $s_1$. We again find it convenient to split the correlator into various parts as in (1.79). The WT identity for $\langle J_{s_1} J_s J_s\rangle$ for general spins $s$ and $s_1$ is quite complicated. However, as discussed in Section 1.8.1, we can argue that the parity-odd contribution to the non-homogeneous part, $\langle J_s J_s J_s\rangle_{\mathbf{nh},\mathrm{odd}}$, is a contact term. In the following we will calculate the homogeneous and the non-homogeneous contribution to the correlator $\langle TJJ\rangle$. For general spins $s$ and $s_1$, we present only the homogeneous solution.

### 1.4.4.1 $\langle TJJ\rangle$

Let us consider the correlator with a single insertion of the stress-tensor operator and two insertions of the spin-one current operator. The transverse WT identity is given by : [41, 44]

$$\begin{aligned}
k_{1\mu}\langle T^{\mu\nu}(k_1) J^\rho(k_2) J^\sigma(k_3)\rangle &= k_{3\mu}\delta^{\nu\sigma}\langle J^\mu(k_1+k_3) J^\rho(k_2)\rangle + k_{2\mu}\delta^{\nu\rho}\langle J^\mu(k_1+k_2) J^\sigma(k_3)\rangle \\
&\quad - k_{3\nu}\langle J^\sigma(k_1+k_3) J^\rho(k_2)\rangle - k_{2\nu}\langle J^\rho(k_1+k_2) J^\sigma(k_3)\rangle \\
k_{2\rho}\langle T^{\mu\nu}(k_1) J^\rho(k_2) J^\sigma(k_3)\rangle &= 0.
\end{aligned} \tag{1.129}$$

Since the WT identity is not trivial, the correlator can have both homogeneous and non-homogeneous solutions for the parity-even and the parity-odd correlation functions.

$\langle TJJ \rangle_{\mathbf{even}}$

The even part of the correlator was calculated in [41, 44] in momentum space and it is straightforward to convert that into spinor-helicity variables :

$$\begin{aligned}
\langle T^- J^- J^- \rangle &= c_1 \frac{k_1}{E^4} \langle 12 \rangle^2 \langle 13 \rangle^2 \\
\langle T^+ J^- J^- \rangle &= 0 \\
\langle T^- J^- J^+ \rangle &= c_J \frac{E + k_1}{k_1^2 E^2} \langle 12 \rangle^2 \langle 1\bar{3} \rangle^2
\end{aligned} \tag{1.130}$$

where the term proportional to $c_1$ is the homogeneous term and the term proportional to $c_J$ is the non-homogeneous term.

$\langle TJJ \rangle_{\mathbf{odd}}$

Let us now consider the parity-odd contribution to the correlator. In the parity-odd case the transverse WT identity (1.129) gives :

$$k_{1\mu} \langle T^{\mu\nu}(k_1) J^\rho(k_2) J^\sigma(k_3) \rangle_{\text{odd}} = c'_J \left( k_{2\nu} \epsilon^{\rho\sigma k_3} + k_{3\nu} \epsilon^{\sigma\rho k_2} - \delta^{\nu\rho} \epsilon^{\sigma k_3 k_2} - \delta^{\nu\sigma} \epsilon^{\rho k_2 k_3} \right) \tag{1.131}$$

where we have used $\langle J^\alpha(k_1 + k_2) J^\beta(k_3) \rangle = -c'_J \epsilon^{\alpha\beta k_3}$. Interestingly it turns out that the R.H.S. of the above equation vanishes upon using a Schouten identity. Thus, in addition to the trivial transverse WT identities w.r.t $k_2^\rho$ and $k_3^\sigma$, we have the following trivial transverse WT identity :

$$k_{1\mu} \langle T^{\mu\nu}(k_1) J^\rho(k_2) J^\sigma(k_3) \rangle_{\text{odd}} = 0 \tag{1.132}$$

This immediately implies that the parity odd part of the non-homogeneous part of the correlator is zero :

$$\langle T^{\mu\nu}(k_1) J^\rho(k_2) J^\sigma(k_3) \rangle_{\mathbf{nh},\text{odd}} = 0. \tag{1.133}$$

We now turn our attention to computing the homogeneous contribution. Let us start with the following ansatz for $\langle TJJ\rangle_{\text{odd}}$ :

$$\begin{aligned}\left\langle \frac{T^-}{k_1} J^- J^- \right\rangle_{\text{odd}} &= i\, F(k_1,k_2,k_3)\langle 12\rangle^2\langle 13\rangle^2,\\ \left\langle \frac{T^-}{k_1} J^- J^+ \right\rangle_{\text{odd}} &= i\, G(k_1,k_2,k_3)\langle 12\rangle^2\langle 1\bar{3}\rangle^2\\ \left\langle \frac{T^+}{k_1} J^- J^- \right\rangle_{\text{odd}} &= i\, H(k_1,k_2,k_3)\langle \bar{1}2\rangle^2\langle \bar{1}3\rangle^2\end{aligned} \tag{1.134}$$

The solutions to the resulting CWIs are given (see appendix A.6) by :

$$F(k_1,k_2,k_3) = \frac{c_1'}{E^4}, \quad G(k_1,k_2,k_3) = \frac{c_2'}{E^4(k_2+k_3-k_1)^2}, \quad H(k_1,k_2,k_3) = 0. \tag{1.135}$$

Since the solution for $G$ has an unphysical pole, we set its coefficient $c_2' = 0$. Substituting the above form-factors in the ansatz (1.134), we obtain the following:

$$\langle T^- J^- J^-\rangle_{\text{odd}} = i\, c_1' \frac{k_1}{E^4}\langle 12\rangle^2\langle 13\rangle^2 \tag{1.136}$$

$$\langle T^+ J^+ J^+\rangle_{\text{odd}} = -i\, c_1' \frac{k_1}{E^4}\langle \bar{1}\bar{2}\rangle^2\langle \bar{1}\bar{3}\rangle^2 \tag{1.137}$$

The other helicity components of the correlator are zero.

#### Summary of Homogeneous contribution to $\langle TJJ\rangle$

Adding up the contribution coming from the parity-even and parity-odd parts we obtain

$$\langle T^- J^- J^-\rangle_{\mathbf{h}} = (c_1 + i\, c_1')\, \frac{k_1}{E^4}\langle 12\rangle^2\langle 13\rangle^2 \tag{1.138}$$

$$\langle T^+ J^+ J^+\rangle_{\mathbf{h}} = (c_1 - i\, c_1')\, \frac{k_1}{E^4}\langle \bar{1}\bar{2}\rangle^2\langle \bar{1}\bar{3}\rangle^2 \tag{1.139}$$

with all other components being zero.

#### Summary of non-homogeneous contribution to $\langle TJJ\rangle$

As discussed above, the parity-odd contribution to the non-homogeneous part of the correlator vanishes. Thus from (1.130) we have the following for the non-homogeneous part of the correlator :

$$\langle T^- J^- J^+ \rangle_{\mathbf{nh}} = c_J \frac{E + k_1}{k_1^2 E^2} \langle 12 \rangle^2 \langle 1\bar{3} \rangle^2 \tag{1.140}$$

with all other components zero except the one obtained from complex conjugating (1.140).

#### 1.4.4.2 $\langle J_{s_1} J_s J_s \rangle$ for general spin: The homogeneous part

**Homogeneous solution**

We start with the following ansatz for $\langle J_{s_1} J_s J_s \rangle$ :

$$\begin{aligned}
\left\langle \frac{J_{s_1}^-}{k_1^{s_1-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^-}{k_3^{s-1}} \right\rangle_{\mathbf{h}} &= F(k_1, k_2, k_3) \langle 12 \rangle^{s_1} \langle 23 \rangle^{2s-s_1} \langle 31 \rangle^{s_1} \\
\left\langle \frac{J_{s_1}^-}{k_1^{s_1-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^+}{k_3^{s-1}} \right\rangle_{\mathbf{h}} &= G(k_1, k_2, k_3) \langle 12 \rangle^{s_1} \langle 2\bar{3} \rangle^{2s-s_1} \langle \bar{3}1 \rangle^{s_1} \\
\left\langle \frac{J_{s_1}^+}{k_1^{s_1-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^-}{k_3^{s-1}} \right\rangle_{\mathbf{h}} &= H(k_1, k_2, k_3) \langle 12 \rangle^{s_1} \langle 2\bar{3} \rangle^{2s-s_1} \langle \bar{3}1 \rangle^{s_1}
\end{aligned} \tag{1.141}$$

In our analysis, we assume that $2s > s_1$. The action of the conformal generator on the homogeneous part is given by :

$$\widetilde{K}^\kappa \left\langle \frac{J_{s_1}^-}{k_1^{s_1-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^-}{k_3^{s-1}} \right\rangle_{\mathbf{h}} = \widetilde{K}^\kappa \left\langle \frac{J_{s_1}^+}{k_1^{s_1-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^-}{k_3^{s-1}} \right\rangle_{\mathbf{h}} = \widetilde{K}^\kappa \left\langle \frac{J_{s_1}^-}{k_1^{s_1-1}} \frac{J_s^-}{k_2^{s-1}} \frac{J_s^+}{k_3^{s-1}} \right\rangle_{\mathbf{h}} = 0. \tag{1.142}$$

The solutions for $F$, $G$ and $H$ are given (see appendix A.6) by:

$$F(k_1, k_2, k_3) = \frac{1}{E^{2s+s_1}}, \quad G(k_1, k_2, k_3) = 0, \quad H(k_1, k_2, k_3) = 0. \tag{1.143}$$

We will now summarise the results for the homogeneous solution.

**Summary of Homogeneous contribution to $\langle J_{s_1} J_s J_s \rangle$**

$$\langle J_{s_1}^- J_s^- J_s^- \rangle_{\mathbf{h}} = (c_1 + i c_1') \frac{k_1^{s_1-1} k_2^{s-1} k_3^{s-1}}{E^{2s+s_1}} \langle 12 \rangle^{s_1} \langle 23 \rangle^{2s-s_1} \langle 31 \rangle^{s_1} \tag{1.144}$$

$$\langle J_{s_1}^+ J_s^- J_s^- \rangle_{\mathbf{h}} = 0 \tag{1.145}$$

$$\langle J_{s_1}^- J_s^- J_s^+ \rangle_{\mathbf{h}} = 0 \tag{1.146}$$

while other components can be obtained by complex conjugating the result in (1.144).

## 1.5 Conformal correlators in momentum space

In this section we present the results for higher spin $CFT_3$ correlators in momentum space. As explained in Section 1.3.3.2 a direct computation of parity-odd correlators in momentum space becomes complicated due to the large amount of degeneracy in 3d. Rather than solving the CWIs directly in momentum space, we convert our expressions for the correlators in spinor-helicity variables obtained in the previous section to momentum space. The simplest way to do this is to write down the ansatz for the correlator in momentum space and convert it to spinor-helicity variables. One can then compare it to the explicit results in spinor-helicity variables and solve for the form factors. For correlators involving higher spins, this procedure also becomes complicated, and in such cases we make use of transverse polarization tensors to represent the answers.

### 1.5.1 Two point function

Two-point functions of various conserved currents are as follows :

$$\begin{aligned}\langle J^\mu(k)J^\nu(-k)\rangle_{\text{odd}} &= c'_J\epsilon^{\mu\nu k} & \langle J^\mu(k)J^\nu(-k)\rangle_{\text{even}} &= c_J\pi^{\mu\nu}(k)k\\ \langle T^{\mu\nu}(k)T^{\rho\sigma}(-k)\rangle_{\text{odd}} &= c'_T\Delta^{\mu\nu\rho\sigma}(k)k^2 & \langle T^{\mu\nu}(k)T^{\rho\sigma}(-k)\rangle_{\text{even}} &= c_T\Pi^{\mu\nu\rho\sigma}(k)k^3\end{aligned} \tag{1.147}$$

where

$$\Delta^{\mu\nu\rho\sigma}(k) = \epsilon^{\mu\rho k}\pi^{\nu\sigma}(k) + \epsilon^{\mu\sigma k}\pi^{\nu\rho}(k) + \epsilon^{\nu\sigma k}\pi^{\mu\rho}(k) + \epsilon^{\nu\rho k}\pi^{\mu\sigma}(k) \tag{1.148}$$

$$\Pi^{\mu\nu\rho\sigma}(k) = \frac{1}{2}\left(\pi^{\mu\rho}(k)\pi^{\nu\sigma}(k) + \pi^{\mu\sigma}(k)\pi^{\nu\rho}(k) - \pi^{\mu\nu}(k)\pi^{\rho\sigma}(k)\right) \tag{1.149}$$

$$\pi^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \tag{1.150}$$

For arbitrary spin $s$, we have the following expression for the two-point function after contracting with polarization vectors :

$$\begin{aligned}\langle J_s(k)J_s(-k)\rangle_{\text{odd}} &= c'_s\epsilon^{z_1z_2k}\,(z_1\cdot z_2)^{s-1}k^{2(s-1)}\\ \langle J_s(k)J_s(-k)\rangle_{\text{even}} &= c_s(z_1\cdot z_2)^s k^{2s-1}\end{aligned} \tag{1.151}$$

From (1.151), it is clear the the parity-odd two-point function for a spin-$s$ current is a contact term as it is analytic in $k^2$.

### 1.5.2 Three point function

In this section, we convert the spinor-helicity expressions in the previous section to momentum space expressions and obtain parity-even as well as parity-odd three point correlators comprising generic spin $s$ conserved currents and scalar operators of arbitrary scaling dimensions.

#### 1.5.2.1 $\langle J_s O_\Delta O_\Delta \rangle$

In this section we give the momentum space expression for correlators of the form $\langle J_s O_\Delta O_\Delta \rangle$. As discussed earlier, the parity-odd part is zero. The parity-even part is straightforward to write down from the spinor-helicity expressions. For general spin-$s$, it is given by :

$$\langle J_s O_\Delta O_\Delta \rangle_{\text{even}} = c_1 k_1^{2s-1} I_{\frac{1}{2}+s,\{\frac{1}{2}-s,\Delta-\frac{3}{2},\Delta-\frac{3}{2}\}} (k_2 \cdot z_1)^s \tag{1.152}$$

Let us now consider the correlator for some specific values of $s$ and $\Delta$.

$\langle J_s O_2 O_2 \rangle$

For $\Delta = 2$, we have

$$\langle J_s O_2 O_2 \rangle_{\text{even}} = c_1 k_1^{2s-1} I_{\frac{1}{2}+s,\{\frac{1}{2}-s,\frac{1}{2},\frac{1}{2}\}} (k_2 \cdot z_1)^s \tag{1.153}$$

For $s = 1$, $s = 2$ and $s = 3$, we have

$$\begin{aligned}
\langle J O_2 O_2 \rangle_{\text{even}} &= c_1 \frac{1}{(k_1 + k_2 + k_3)} (k_2 \cdot z_1) \\
\langle T O_2 O_2 \rangle_{\text{even}} &= c_1 \frac{2k_1 + k_2 + k_3}{(k_1 + k_2 + k_3)^2} (k_2 \cdot z_1)^2 \\
\langle J_3 O_2 O_2 \rangle_{\text{even}} &= c_1 \frac{8k_1^2 + 9k_1(k_2 + k_3) + 3(k_2 + k_3)^2}{(k_1 + k_2 + k_3)^3} (k_2 \cdot z_1)^3
\end{aligned} \tag{1.154}$$

$\langle TO_3O_3\rangle$

For $s=2$ and $\Delta=3$, we have

$$\langle TO_3O_3\rangle_{\text{even}} = c_1\frac{k_1^3+k_2^3+k_3^3+2(k_1^2+k_2k_3)(k_2+k_3)+2k_1(k_2^2+k_2k_3+k_3^2)}{(k_1+k_2+k_3)^2}(k_2\cdot z_1)^2 \tag{1.155}$$

Let us now consider the three-point correlator with two insertions of the spin-$s$ conserved current and a scalar operator of scaling dimension $\Delta$.

#### 1.5.2.2 $\langle J_sJ_sO_\Delta\rangle$

In this section we determine the momentum space expression for correlators of the kind $\langle J_sJ_sO_\Delta\rangle$. We first discuss the $s=1$ and $s=2$ cases in detail. We then present the final result for the general spin case expressed in terms of a transverse polarization tensor.

$\langle JJO_\Delta\rangle$

The correlator is purely transverse and the even part of it takes the following form in momentum space [45]:

$$\langle J^\mu(k_1)J^\nu(k_2)O_\Delta\rangle_{\text{even}} = A_1(k_1,k_2,k_3)\pi^\mu_\alpha(k_1)\pi^\nu_\beta(k_2)\left[k_2^\alpha k_3^\beta-\chi\delta^{\alpha\beta}\right] \tag{1.156}$$

where the form factor $A(k_1,k_2,k_3)$ is given by :

$$A_1(k_1,k_2,k_3) = I_{\frac{5}{2},\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}} \tag{1.157}$$

and

$$\chi = \frac{1}{2}(k_1+k_2+k_3)(k_1+k_2-k_3) \tag{1.158}$$

As an example let us consider the case when the scaling dimension of the scalar operator is $\Delta=2$. In this case, after evaluating the integral (1.157) we obtain the form factor in the ansatz (1.156) to take the form :

$$A_1(k_1,k_2,k_3) = \frac{1}{(k_1+k_2+k_3)^2} \tag{1.159}$$

Let us now consider the parity-odd sector. In [187] we computed $\langle JJO_\Delta\rangle_{\text{odd}}$ by imposing conformal invariance and obtained :

$$\langle J^\mu(k_1)\, J^\nu(k_2)\, O(k_3)\rangle_{\text{odd}} = \pi^\mu_\alpha(k_1)\pi^\nu_\beta(k_2)\left[A(k_1,k_2,k_3)\epsilon^{\alpha k_1 k_2}k_1^\beta + B(k_1,k_2,k_3)\epsilon^{\beta k_1 k_2}k_2^\alpha\right] \tag{1.160}$$

where

$$\begin{aligned} A(k_1,k_2,k_3) &= \frac{k_2^2(I_1k_1^2+I_2k_1\cdot k_2)}{k_1^2k_2^2-(k_1\cdot k_2)^2} \\ B(k_1,k_2,k_3) &= \frac{k_1^2(I_2k_2^2+I_1k_1\cdot k_2)}{k_1^2k_2^2-(k_1\cdot k_2)^2} \end{aligned} \tag{1.161}$$

where $I_1$ and $I_2$ are the following two triple-$K$ integrals respectively :

$$I_1 = c_1\, I_{\frac{5}{2}\{-\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}} \tag{1.162}$$

$$I_2 = -c_1\, I_{\frac{5}{2}\{\frac{1}{2},-\frac{1}{2},\Delta-\frac{3}{2}\}} \tag{1.163}$$

We further verified our results by computing the correlator for $\Delta = 1,\ldots,5$ using weight-shifting operators and matching with the results obtained above.

Again, when $\Delta = 2$, after evaluating the above triple-$K$ integrals we obtain the form factors in (1.160) to be :

$$\begin{aligned} A(k_1,k_2,k_3) &= -\frac{k_2}{(k_1+k_2+k_3)^2\,((k_1-k_2)^2-k_3^2)} \\ B(k_1,k_2,k_3) &= \frac{k_1}{(k_1+k_2+k_3)^2\,((k_1-k_2)^2-k_3^2)} \end{aligned} \tag{1.164}$$

Note that, as expected there is a total energy singularity when $E = k_1+k_2+k_3 \to 0$. It seems from the above expression that there is also an apparent collinear divergence when any two momentum vectors are proportional to each other. In this case, momentum conservation implies that all 3 momenta are along a line and it is easy to check that the

$((k_1 - k_2)^2 - k_3^2)$ factor in the denominator above vanishes. However, in this case the numerator of the full correlator also vanishes appropriately, leaving the correlator finite. Hence as expected the correlator has no singularities other than the $E \to 0$ singularity [7].

$\langle TTO_\Delta \rangle$

The transverse and traceless part of the correlator in three dimensions is given by [45]:

$$\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} O_\Delta \rangle_{\text{even}} = -2k_1^2 k_2^2 A_1 \Pi_{\mu_1\nu_1\alpha\beta_1}(k_1)\Pi_{\mu_2\nu_2\alpha\beta_2}(k_2)(k_2^{\beta_1}k_3^{\beta_2} - \chi\,\delta^{\beta_1\beta_2}) \tag{1.165}$$

where

$$A_1(k_1, k_2, k_3) = c_1\, I_{\frac{9}{2},\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}} \tag{1.166}$$

and $\chi$ is defined in (1.158). For the case when $\Delta = 2$ one obtains the following :

$$A_1(k_1, k_2, k_3) = \frac{1}{(k_1 + k_2 + k_3)^4} \tag{1.167}$$

Let us now consider the parity odd contribution to the correlator. In [188] we obtained the momentum space expressions for $\langle TTO_1 \rangle_{\text{odd}}$ and $\langle TTO_2 \rangle_{\text{odd}}$ using spin-raising and weight-shifting operators. We will now use the expressions we obtained in spinor-helicity variables for $\langle TTO_\Delta \rangle$, for a generic $\Delta$, to obtain a momentum space expression for the same. We start with the following ansatz in momentum space :

$$\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} O_\Delta \rangle_{\text{odd}} = \Pi^{\alpha_1\beta_1}_{\mu_1\nu_1}(k_1)\Pi^{\alpha_2\beta_2}_{\mu_2\nu_2}\left(A_1 \epsilon^{\mu_1\mu_2 k_1}\delta^{\nu_1\nu_2} + A_2 \epsilon^{\mu_1\mu_2 k_2}\delta^{\nu_1\nu_2}\right) \tag{1.168}$$

Symmetry considerations tell us that :

$$A_2 = -A_1(k_1 \leftrightarrow k_2) \tag{1.169}$$

Dotting with transverse, null polarization vectors, we get

$$\langle TTO_\Delta \rangle_{\text{odd}} = A_1 \epsilon^{k_1 z_1 z_2} z_1 \cdot z_2 - A_1(k_1 \leftrightarrow k_2)\epsilon^{k_2 z_1 z_2} z_1 \cdot z_2 \tag{1.170}$$

[7] There is also an alternative form of this correlator (see eq. 5.16 of [188]) using which it is easy to see that there are no collinear divergences.

Converting into spinor-helicity variables we get :

$$\begin{aligned}\langle T^-T^+O_\Delta\rangle_{\text{odd}} &= \frac{A_1k_1 - A_1(k_1\leftrightarrow k_2)k_2}{4k_1^2k_2^2}\langle 1\bar{2}\rangle^4\\ \langle T^-T^-O_\Delta\rangle_{\text{odd}} &= \frac{A_1k_1 + A_1(k_1\leftrightarrow k_2)k_2}{4k_1^2k_2^2}\langle 12\rangle^4\end{aligned} \tag{1.171}$$

Comparing (1.171) and (A.60), we get the following for the form factor :

$$A_1 = 2c_1k_1^2k_2^3 I_{\frac{9}{2}\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}} \tag{1.172}$$

which matches the form factor (1.166) that appears in the even part of the same correlator. Let us now come to the case where the spinning operator in the correlator is a generic spin $s$ conserved current.

$\langle J_sJ_sO_\Delta\rangle$

As mentioned in the introduction to this section, for correlators involving higher spin operators, it is convenient to introduce transverse polarization tensors. It is straightforward to write down the momentum space expression for these correlators from their expression in spinor-helicity variables. This can be done upon observing that

$$\begin{aligned}\left[k_2\,\epsilon^{k_1z_1z_2} - k_1\,\epsilon^{k_2z_1z_2}\right] &\mapsto i\langle 12\rangle^2\\ \left[(\vec{z}_1\cdot\vec{k}_2)(\vec{z}_2\cdot\vec{k}_1) + \frac{1}{2}E(E-2k_3)\vec{z}_1\cdot\vec{z}_2\right] &\mapsto \langle 12\rangle^2\end{aligned} \tag{1.173}$$

We then have

$$\langle J_sJ_sO_\Delta\rangle_{\text{even}} = (k_1k_2)^{s-1}I_{\frac{1}{2}+2s\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\left[2(\vec{z}_1\cdot\vec{k}_2)(\vec{z}_2\cdot\vec{k}_1) + E(E-2k_3)\vec{z}_1\cdot\vec{z}_2\right]^s \tag{1.174}$$

$$\begin{aligned}\langle J_sJ_sO_\Delta\rangle_{\text{odd}} &= (k_1k_2)^{s-1}I_{\frac{1}{2}+2s\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\left[k_2\,\epsilon^{k_1z_1z_2} - k_1\,\epsilon^{k_2z_1z_2}\right]\\ &\quad\times\left[2(\vec{z}_1\cdot\vec{k}_2)(\vec{z}_2\cdot\vec{k}_1) + E(E-2k_3)\vec{z}_1\cdot\vec{z}_2\right]^{s-1}\end{aligned} \tag{1.175}$$

Let us now look at this correlator for specific values of the scaling dimension of the scalar operator.

$\langle J_s J_s O_1 \rangle$

For $\Delta = 1$, the correlator is given by :

$$\begin{aligned}\langle J_s J_s O_1 \rangle_{\text{odd}} &= (k_1 k_2)^{s-1} \frac{1}{k_3 E^{2s}} \left[ k_2\, \epsilon^{k_1 z_1 z_2} - k_1\, \epsilon^{k_2 z_1 z_2} \right] \\ &\times \left[ 2(\vec{z}_1 \cdot \vec{k}_2)(\vec{z}_2 \cdot \vec{k}_1) + E(E - 2k_3)\vec{z}_1 \cdot \vec{z}_2 \right]^{s-1} \end{aligned} \tag{1.176}$$

$\langle J_s J_s O_2 \rangle$

For $\Delta = 2$, the correlator is given by :

$$\begin{aligned}\langle J_s J_s O_2 \rangle_{\text{odd}} &= (k_1 k_2)^{s-1} \frac{1}{E^{2s}} \left[ k_2\, \epsilon^{k_1 z_1 z_2} - k_1\, \epsilon^{k_2 z_1 z_2} \right] \\ &\times \left[ 2(\vec{z}_1 \cdot \vec{k}_2)(\vec{z}_2 \cdot \vec{k}_1) + E(E - 2k_3)\vec{z}_1 \cdot \vec{z}_2 \right]^{s-1} \end{aligned} \tag{1.177}$$

$\langle J_s J_s O_3 \rangle$

For $\Delta = 3$, the correlator is given by :

$$\begin{aligned}\langle J_s J_s O_3 \rangle_{\text{odd}} &= (k_1 k_2)^{s-1} \frac{(E + (2s-1)k_3)}{E^{2s}} \left[ k_2\, \epsilon^{k_1 z_1 z_2} - k_1\, \epsilon^{k_2 z_1 z_2} \right] \\ &\times \left[ 2(\vec{z}_1 \cdot \vec{k}_2)(\vec{z}_2 \cdot \vec{k}_1) + E(E - 2k_3)\vec{z}_1 \cdot \vec{z}_2 \right]^{s-1} \end{aligned} \tag{1.178}$$

#### 1.5.2.3 $\langle J_s J_s J_s \rangle$

We will now determine the correlator $\langle J_s J_s J_s \rangle$ in momentum space. We first discuss the $s = 1$ and $s = 2$ cases in detail. For the case of general spin, we present only the final result expressed in terms of transverse polarization tensor. For this, we restrict our attention to the homogeneous part. For the parity-odd part of the correlator, the non-homogeneous contribution is always a contact term.

$\langle JJJ \rangle$

Unlike $\langle J_s J_s O_\Delta \rangle$ the correlator $\langle JJJ \rangle$ is not purely transverse and it has a local term given by [44]:

$$\langle J^{\mu a} J^{\nu b} J^{\rho c} \rangle_{\text{local}} = \Bigg[ \frac{k_1^\mu}{k_1^2} \left( f^{adc} \langle J^{\rho d}(k_2) J^{\nu b}(-k_2) \rangle - f^{abd} \langle J^{\nu d}(k_3) J^{\rho c}(-k_3) \rangle \right)$$
$$+ ((k_1, \mu) \leftrightarrow (k_2, \nu)) + ((k_1, \mu) \leftrightarrow (k_3, \rho)) \Bigg] + \Bigg[ \left( \frac{k_1^\mu k_2^\nu}{k_1^2 k_2^2} f^{abd} k_{2\alpha} \langle J^{\alpha d}(k_3) J^{\rho c}(-k_3) \rangle \right)$$
$$+ ((k_1, \mu) \leftrightarrow (k_3, \rho)) + ((k_2, \nu) \leftrightarrow (k_3, \rho)) \Bigg] \tag{1.179}$$

The transverse part of the even part of the correlator denoted by $\langle j^{\mu_1 a_1} j^{\mu_2 a_2} j^{\mu_3 a_3} \rangle_{\text{even}}$ was computed in [44] :

$$\langle j^{\mu_1 a_1} j^{\mu_2 a_2} j^{\mu_3 a_3} \rangle_{\text{even}} = \pi^{\mu_1}_{\alpha_1}(k_1) \pi^{\mu_2}_{\alpha_2}(k_2) \pi^{\mu_3}_{\alpha_3}(k_3) [A_1 k_2^{\alpha_1} k_3^{\alpha_2} k_1^{\alpha_3} + A_2 \delta^{\alpha_1 \alpha_2} k_1^{\alpha_3}$$
$$+ A_2(k_3, k_1, k_2) \delta^{\alpha_1 \alpha_3} k_3^{\alpha_2} + A_2(k_2, k_3, k_1) \delta^{\alpha_2 \alpha_3} k_2^{\alpha_1}] \tag{1.180}$$

where the form factors are given by :

$$A_1 = \frac{2c_1}{(k_1 + k_2 + k_3)^3}$$
$$A_2 = c_1 \frac{k_3}{(k_1 + k_2 + k_3)^2} - \frac{2c_J}{(k_1 + k_2 + k_3)} \tag{1.181}$$

Here and in the following we suppress the color indices for brevity. After dotting with transverse, null polarization vectors, the correlator can be separated into homogeneous and non-homogeneous parts as follows :

$$\langle JJJ \rangle_{\text{even},\mathbf{h}} = \frac{c_1}{(k_1 + k_2 + k_3)^2} \left[ \frac{2(k_2 \cdot z_1)(k_3 \cdot z_2)(k_1 \cdot z_3)}{(k_1 + k_2 + k_3)} + (k_3 (z_1 \cdot z_2)(k_1 \cdot z_3) + \text{cyclic perm.}) \right]$$
$$\langle JJJ \rangle_{\text{even},\mathbf{nh}} = -\frac{2c_J}{(k_1 + k_2 + k_3)} ((z_1 \cdot z_2)(k_1 \cdot z_3) + \text{cyclic perm.}) \tag{1.182}$$

In [188] we computed the odd part of $\langle JJJ \rangle$ using the action of spin-raising and weight-shifting operators on a scalar seed correlator. The correlator is given by the sum of its local terms (1.179) and transverse parts. The latter is given by :

$$\langle j^{\mu a} j^{\nu b} j^{\rho c}\rangle_{\text{odd}} = \pi^{\mu}_{\alpha}(k_1)\pi^{\nu}_{\beta}(k_2)\pi^{\rho}_{\gamma}(k_3) X^{\alpha\beta\gamma} \tag{1.183}$$

where

$$X^{\alpha\beta\gamma} = A_1 \epsilon^{k_1 k_2 \alpha} k_1^{\gamma} k_3^{\beta} + A_2 \epsilon^{k_1 k_2 \alpha} \delta^{\beta\gamma} + A_3 \epsilon^{k_1 \alpha\beta} k_1^{\gamma} + A_4 \epsilon^{k_1 \alpha\gamma} k_3^{\beta} + \text{cyclic perm.} \tag{1.184}$$

where

$$\begin{aligned} A_1 &= -\frac{2}{k_1(k_1+k_2+k_3)^3}, \qquad & A_2 &= -\frac{1}{(k_1+k_2+k_3)^2} \\ A_3 &= \frac{k_1+k_2+2\,k_3}{k_1(k_1+k_2+k_3)^2}, \qquad & A_4 &= \frac{k_1+2\,k_2+k_3}{k_1(k_1+k_2+k_3)^2} \end{aligned} \tag{1.185}$$

After dotting with transverse polarization vectors, the correlator can be rewritten in the following form using Schouten identities :

$$\begin{aligned} \langle JJJ\rangle_{\text{odd},\mathbf{h}} &= \frac{c_1'}{E^3}\Big[\Big\{(\vec{k}_1\cdot\vec{z}_3)\left(\epsilon^{k_3 z_1 z_2} k_1 - \epsilon^{k_1 z_1 z_2} k_3\right) + (\vec{k}_3\cdot\vec{z}_2)\left(\epsilon^{k_1 z_1 z_3} k_2 - \epsilon^{k_2 z_1 z_3} k_1\right) \\ &\qquad -(\vec{z}_2\cdot\vec{z}_3)\epsilon^{k_1 k_2 z_1} E + \frac{k_1}{2}\epsilon^{z_1 z_2 z_3} E(E-2k_1)\Big\} + \text{cyclic perm}\Big] \\ \langle JJJ\rangle_{\text{odd},\mathbf{nh}} &= c_J' \epsilon^{z_1 z_2 z_3} \end{aligned} \tag{1.186}$$

We see that the non-homogeneous part of the parity-odd part of the correlator is a contact term. This term can be explained from the $dS_4$ perspective by considering the three-point tree-level amplitude arising from the interaction term $F\widetilde{F}$.

In the rest of this section we obtain the momentum space expressions for the correlators $\langle TJJ\rangle$ and $\langle TTT\rangle$. As described in Section 1.3.3.2, a direct computation of these correlators by solving the conformal Ward identities in momentum space is quite difficult.

$\langle TTT\rangle$

The momentum space expression for the even part of the correlator $\langle TTT\rangle$ was obtained in [44] and it was shown to have two structures. We will now obtain the momentum space expression for the odd part of the correlator.

In (1.116) we obtained the following result for the parity odd part of the correlator

$\langle TTT\rangle$ in spinor-helicity variables :

$$\begin{aligned}\langle T^-T^-T^-\rangle_{odd} &= \left(c_1'\frac{c_{123}}{E^6}+c_T'\frac{E^3-Eb_{123}-c_{123}}{c_{123}^2}\right)\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2\\ \langle T^-T^-T^+\rangle_{odd} &= c_T'\frac{(E-2k_3)^3-(E-2k_3)(b_{123}-2k_3a_{12})+c_{123}}{c_{123}^2}\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2\end{aligned} \tag{1.187}$$

Let us consider the following ansatz for the transverse part of the correlator :

$$\begin{aligned}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle_{\text{odd}} = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(k_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(k_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(k_3)\Big(&A_1\epsilon^{k_1k_2\alpha_1}\epsilon^{k_1k_2\alpha_2}\epsilon^{k_1k_2\alpha_3}k_1^{\beta_3}k_2^{\beta_1}k_3^{\beta_2}\\ &+A_2\epsilon^{k_1k_2\alpha_3}k_1^{\beta_3}k_3^{\alpha_2}k_3^{\beta_2}k_2^{\alpha_1}k_2^{\beta_1}+A_2(k_2\leftrightarrow k_3)\epsilon^{k_2k_3\alpha_2}k_3^{\beta_2}k_1^{\alpha_3}k_1^{\beta_3}k_2^{\alpha_1}k_2^{\beta_1}\\ &+A_2(k_1\leftrightarrow k_3)\epsilon^{k_1k_2\alpha_1}k_2^{\beta_1}k_3^{\alpha_2}k_3^{\beta_2}k_1^{\alpha_3}k_1^{\beta_3}\Big)\end{aligned} \tag{1.188}$$

One could have started with a more general ansatz with many more tensor structures than exhibited by (1.188). However, it turns out that there are several Schouten identities that relate those tensor structures and upon using them one ends up with the minimal (and complete) ansatz in (1.188) [8].

Contracting with null and transverse polarization vectors, we get

$$\begin{aligned}\langle TTT\rangle_{\text{odd}} = &A_1\epsilon^{k_3k_1z_1}\epsilon^{k_1k_2z_2}\epsilon^{k_2k_3z_3}(k_2\cdot z_1)(k_3\cdot z_2)(k_1\cdot z_3)+A_2\epsilon^{k_2k_3z_3}(k_1\cdot z_3)(k_3\cdot z_2)^2(k_2\cdot z_1)^2\\ &+A_2(k_2\leftrightarrow k_3)\epsilon^{k_1k_2z_2}(k_1\cdot z_3)^2(k_3\cdot z_2)(k_2\cdot z_1)^2+A_2(k_1\leftrightarrow k_3)\epsilon^{k_3k_1z_1}(k_1\cdot z_3)^2(k_3\cdot z_2)^2(k_2\cdot z_1)\end{aligned} \tag{1.189}$$

Converting this into spinor-helicity variables, we get

$$\begin{aligned}\langle T^-T^-T^+\rangle &= \langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2\frac{J^4\left(A_1c_{123}+A_2k_3-A_2(k_2\leftrightarrow k_3)k_2-A_2(k_1\leftrightarrow k_3)k_1\right)}{(E-2k_3)^2c_{123}^2}\\ \langle T^-T^-T^-\rangle &= -\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2\frac{J^4\left(A_1c_{123}+A_2k_3+A_2(k_2\leftrightarrow k_3)k_2+A_2(k_1\leftrightarrow k_3)k_1\right)}{E^2c_{123}^2}\end{aligned} \tag{1.190}$$

Comparing (1.187) and (1.190) and solving for the momentum space form factors we get

[8]Schouten identities that turn out to be useful are given in Appendix C of [188].

:

$$A_1 = c'_1 \frac{c_{123}^2}{2J^4E^4} - c'_T \frac{12(k_1^2+k_2^2+k_3^2)}{J^4} \tag{1.191}$$

$$A_2 = -c'_1 \frac{b_{12}c_{123}^2}{2J^4E^4} + c'_T \frac{(k_3^4+7k_3^2(k_1^2+k_2^2)+4(k_1^4+4k_1^2k_2^2+k_2^4))}{J^4} \tag{1.192}$$

Naively it might look like there are two contributions to the parity-odd correlation function. However, as will be shown below the term proportional to $c'_T$ is a contact term.

## Contact term in parity-odd $\langle TTT\rangle$

The fact that the term proportional to $c_T$ is a contact term can be seen more explicitly by switching to a basis where the factor of $J^4$ in the denominator disappears. One such basis is given by[9]:

$$\langle TTT\rangle = \Big[B_1\epsilon^{k_1z_1z_2}(z_1\cdot z_2)(k_1\cdot z_3)^2 - B_1(k_1\leftrightarrow k_2)\epsilon^{k_2z_1z_2}(z_1\cdot z_2)(k_1\cdot z_3)^2$$
$$+ B_2\epsilon^{k_1z_1z_2}(z_1\cdot z_3)(z_2\cdot z_3) - B_2(k_1\leftrightarrow k_2)\epsilon^{k_2z_1z_2}(z_1\cdot z_3)(z_2\cdot z_3)\Big] + \text{cyclic perm.} \tag{1.193}$$

The ansatz in (1.193) is related to (1.189) by Schouten identities. Converting (1.193) into spinor-helicity variables and comparing with (1.190), we can solve for $B_1$ and $B_2$. We get the solutions for the terms proportional to $c'_T$ to be :

$$B_1 = c'_T\frac{1}{24}, \qquad\qquad B_2 = c'_T\frac{1}{12}\left(k_1^2+\frac{7}{4}k_2^2+\frac{7}{4}k_3^2\right) \tag{1.194}$$

The fact that $B_1$ is a constant and $B_2$ is dependent on $k^2$ implies that if we evaluate $\langle TTT\rangle$ in the basis (1.193) and convert it to position space, we will get delta functions or derivatives on delta functions which are nothing but contact terms. Since the odd non-homogeneous part is a contact term, the full $\langle TTT\rangle$ correlator has only 3 non-trivial contributions, 2 parity-even and 1 parity-odd conformally invariant structures. This

[9]This choice of basis is not unique. One can find several other bases in which un-physical poles do not appear. To do this, we start with the most general ansatz for $\langle TTT\rangle$ containing all possible tensor structures. We then solve for the form-factors in this most general ansatz by relating it to the known answer for the correlator. Not all the form-factors in the ansatz are fixed this way and those that are not fixed can be set to zero. Which of them are set to zero is a choice made while solving and different choices lead to different bases.

agrees with the analysis of [73].

From the $dS_4$ perspective this contribution to the correlator can be understood to arise from the $W\widetilde{W}$ interaction. The $W\widetilde{W}$ interaction also reproduces the parity-odd two-point function of the stress tensor.

$\langle J_s J_s J_s \rangle$

For general spin $s$ it is easy to write down the homogeneous part of $\langle J_s J_s J_s \rangle$ in momentum space using the transverse polarization. The answer is given by :

$$\begin{aligned}
\langle J_s J_s J_s \rangle_{\text{even},\mathbf{h}} &= (k_1 k_2 k_3)^{s-1} \left[\frac{1}{E^3}\Big\{2\,(\vec{z}_1\cdot\vec{k}_2)\,(\vec{z}_2\cdot\vec{k}_3)\,(\vec{z}_3\cdot\vec{k}_1) + E\{k_3\,(\vec{z}_1\cdot\vec{z}_2)\,(\vec{z}_3\cdot\vec{k}_1) + \text{cyclic}\}\Big\}\right]^s \\
\langle J_s J_s J_s \rangle_{\text{odd},\mathbf{h}} &= (k_1 k_2 k_3)^{s-1}\frac{1}{E^3}\Big[\Big\{(\vec{k}_1\cdot\vec{z}_3)\left(\epsilon^{k_3 z_1 z_2}k_1 - \epsilon^{k_1 z_1 z_2}k_3\right) + (\vec{k}_3\cdot\vec{z}_2)\left(\epsilon^{k_1 z_1 z_3}k_2 - \epsilon^{k_2 z_1 z_3}k_1\right) \\
&\qquad -(\vec{z}_2\cdot\vec{z}_3)\epsilon^{k_1 k_2 z_1}E + \frac{k_1}{2}\epsilon^{z_1 z_2 z_3}E(E-2k_1)\Big\} + \text{cyclic perm}\Big] \\
&\qquad \times\left[\frac{1}{E^3}\Big\{2\,(\vec{z}_1\cdot\vec{k}_2)\,(\vec{z}_2\cdot\vec{k}_3)\,(\vec{z}_3\cdot\vec{k}_1) + E\{k_3\,(\vec{z}_1\cdot\vec{z}_2)\,(\vec{z}_3\cdot\vec{k}_1) + \text{cyclic}\}\Big\}\right]^{s-1}
\end{aligned} \tag{1.195}$$

The parity-odd contribution to the non-homogeneous piece is just a contact term.

#### 1.5.2.4 $\langle J_{2s} J_s J_s \rangle$

We will now look at correlators of the form $\langle J_{2s} J_s J_s \rangle$. We focus the discussion on the $\langle TJJ \rangle$ correlator and also give the results for the $\langle J_4 TT \rangle$ correlator.

$\langle TJJ \rangle$

We saw in (1.129) and (1.132) that the odd part of the correlator $\langle TJJ \rangle$ satisfies trivial transverse Ward identities [10]. We also note that in three-dimensions the trace Ward identity obeyed by this correlator is trivial. Taking these into account we write down the

[10] As a result the non-homogeneous part of this parity-odd correlator is zero. One can understand this from the $dS_4$ perspective by the following argument. The only interaction term that could possibly have contributed to this correlator is $F\widetilde{F}$. However, since this term is independent of the metric the contribution from it to $\langle TJJ \rangle$ is zero. In fact, there is no interaction term that one can have from the gravity side that contributes to the non-homogeneous parity-odd part of $\langle TJJ \rangle$.

following ansatz for the correlator in momentum space :

$$\langle T^{\mu_1\nu_1}(k_1)J^{\mu_2}(k_2)J^{\mu_3}(k_3)\rangle_{\text{odd}} = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(k_1)\pi^{\mu_2}_{\alpha_2}(k_2)\pi^{\mu_3}_{\alpha_3}(k_3)\Big(A_1 k_2^{\alpha_1}k_3^{\alpha_2}\epsilon^{\beta_1\alpha_3 k_1} + A_2 k_2^{\alpha_1}k_3^{\alpha_2}\epsilon^{\beta_1\alpha_3 k_3} + A_3\delta^{\alpha_1\alpha_2}\epsilon^{\beta_1\alpha_3 k_1} + A_4\delta^{\alpha_1\alpha_2}\epsilon^{\beta_1\alpha_3 k_3}\Big) \tag{1.196}$$

Let us now contract this with polarization vectors and this gives :

$$\langle TJJ\rangle_{\text{odd}} = A_1(k_2\cdot z_1)(k_3\cdot z_2)\epsilon^{k_1 z_1 z_3} + A_2(k_2\cdot z_1)(k_3\cdot z_2)\epsilon^{k_3 z_1 z_3} + A_3(z_1\cdot z_2)\epsilon^{k_1 z_1 z_3} + A_4(z_1\cdot z_2)\epsilon^{k_3 z_1 z_3} \tag{1.197}$$

We now convert this expression into spinor-helicity variables to obtain :

$$\begin{aligned}
\left\langle T^- J^- J^-\right\rangle_{\text{odd}} &= \frac{\langle 12\rangle^2\langle 13\rangle^2}{8k_1^2k_2k_3}\left(2A_3k_1 - 2A_4k_3 + (k_1-k_2-k_3)(k_1-k_2+k_3)(A_1k_1 - A_2k_3)\right)\\
\left\langle T^+ J^- J^-\right\rangle_{\text{odd}} &= \frac{\langle \bar{1}2\rangle^2\langle \bar{1}3\rangle^2}{8k_1^2k_2k_3}\left(2A_3k_1 + 2A_4k_3 + (k_1+k_2+k_3)(k_1+k_2-k_3)(A_1k_1 + A_2k_3)\right)\\
\left\langle T^- J^- J^+\right\rangle_{\text{odd}} &= \frac{\langle 12\rangle^2\langle 1\bar{3}\rangle^2}{8k_1^2k_2k_3}\left(2A_3k_1 + 2A_4k_3 + (k_1-k_2-k_3)(k_1-k_2+k_3)(A_1k_1 + A_2k_3)\right)\\
\left\langle T^- J^+ J^-\right\rangle_{\text{odd}} &= \frac{\langle 1\bar{2}\rangle^2\langle 13\rangle^2}{8k_1^2k_2k_3}\left(2A_3k_1 - 2A_4k_3 + (k_1+k_2-k_3)(k_1+k_2+k_3)(A_1k_1 - A_2k_3)\right)
\end{aligned} \tag{1.198}$$

We obtained the following explicit results for these correlators in (1.136) :

$$\begin{aligned}
&\langle T^- J^- J^-\rangle_{\text{odd}} = c_1'\frac{k_1}{E^4}\langle 12\rangle^2\langle 13\rangle^2 \qquad &&\langle T^+ J^- J^-\rangle_{\text{odd}} = 0\\
&\langle T^- J^- J^+\rangle_{\text{odd}} = 0 &&\langle T^- J^+ J^-\rangle_{\text{odd}} = 0
\end{aligned} \tag{1.199}$$

Comparing (1.199) and (1.198), we get the following solutions for the form factors :

$$\begin{aligned}
A_1 &= -c_1'\frac{k_1k_3}{E^4} \qquad & A_3 &= c_1'\frac{k_1k_3(k_1+k_2-k_3)}{2E^3}\\
A_2 &= c_1'\frac{k_1^2}{E^4} & A_4 &= -c_1'\frac{k_1^2(k_1+k_2-k_3)}{2E^3}
\end{aligned} \tag{1.200}$$

Plugging back the solution (1.200) in (1.197) we obtain :

$$\langle TJJ\rangle_{\text{odd}} = \frac{k_1}{4E^4}\left(-2(k_3\cdot z_2)(k_2\cdot z_1)+E(E-2k_3)(z_1\cdot z_2)\right)\left(k_1\epsilon^{z_1z_3k_3}-k_3\epsilon^{k_1z_1z_3}\right) \tag{1.201}$$

which matches the result in [187] obtained by computing a tree level $dS_4$ amplitude.

The solution (1.201) is not manifestly symmetric under a $(2\leftrightarrow 3)$ exchange in this basis. However, we can use Schouten identities to convert the ansatz (1.197) to the following form where it is manifestly symmetric under a $(2\leftrightarrow 3)$ exchange :

$$\begin{aligned}\langle TJJ\rangle_{\text{odd}} = {} & B_1\epsilon^{k_1k_2z_1}(k_1\cdot z_1)(k_1\cdot z_3)(k_3\cdot z_2)+B_2\epsilon^{k_1k_2z_1}(k_2\cdot z_1)^2(k_1\cdot z_3)\\ & +B_3\epsilon^{k_1k_2z_3}(k_2\cdot z_1)^2(k_3\cdot z_2)+B_4\epsilon^{k_1k_2z_1}(k_2\cdot z_1)(z_2\cdot z_3)\end{aligned} \tag{1.202}$$

The relation between the form-factors in the two bases (1.197) and (1.202) is given by :

$$\begin{aligned}B_1 = {} & \frac{16}{J^4}\Big(4A_4((k_1^2-k_2^2)^2+2(k_1^2+k_2^2)k_3^2-3k_3^4)-2(A_1(k_1^2-k_2^2)+(A_1-2A_2)k_3^2)\\ & \times((k_1^2-k_2^2)^2-2(k_1^2+k_2^2)k_3^2+k_3^4)+4A_3(-3k_1^4+(k_2^2-k_3^2)^2+2k_1^2(k_2^2+k_3^2)\Big)\\ B_2 = {} & -\frac{128}{J^4}\Big(k_1^2(-2A_4k_3^2+A_3(k_1^2+k_3^2-k_2^2)\Big)\\ B_3 = {} & -\frac{16}{J^4}\Big(-8A_3k_1^2(k_1^2+k_2^2-k_3^2)+4A_4(k_1^4-2(k_2^2-k_3^2)^2)-8(2A_1k_1^2-A_2(k_1^2-k_2^2+k_3^2))J^2\Big)\\ B_4 = {} & \frac{8}{J^2}\Big(-2A_3k_1^2+A_4(k_1^2-k_2^2+k_3^2)\Big)\end{aligned}$$

For the case $s_1=4$ and $s=2$, the momentum space expression that we get after converting the answer in spinor-helicity variables given in Section 1.4.4.2 is the following :

$$\begin{aligned}\langle J_4TT\rangle_{\text{odd}} = {} & c_1'\frac{k_1^3k_2k_3}{E^8}\Big[(2(k_2\cdot z_1)(k_3\cdot z_2)-(z_1\cdot z_2)(E-2k_3)E)\left(k_1\epsilon^{z_1z_3k_3}-k_3\epsilon^{z_1z_3k_1}\right)\Big]\\ & \times\Big[\Big((k_3\cdot z_2)(k_2\cdot z_1)-\frac{1}{2}E(E-2k_3)(z_1\cdot z_2)\Big)\Big((k_1\cdot z_3)(k_2\cdot z_1)-\frac{1}{2}E(E-2k_2)(z_1\cdot z_3)\Big)\Big]\end{aligned} \tag{1.203}$$

The parity-odd contribution to the non-homogeneous part is again a contact term.

## 1.6 Renormalisation

In Sections 1.4 and 1.5 we saw that CFT correlators in spinor-helicity variables and momentum space are given by triple-$K$ integrals of the kind :

$$I_{\alpha\{\beta_1,\beta_2,\beta_3\}}(k_1,k_2,k_3) = \int_0^\infty dx\, x^\alpha \prod_{j=1}^{3} k_j^{\beta_j}\, K_{\beta_j}(k_j\, x) \tag{1.204}$$

where $K_\nu$ is a modified Bessel function of the second kind and $\alpha$ and $\beta_i$ are discrete parameters that depend on the dimension of space and the conformal dimensions of the operators. The integral is convergent except when the following equality is satisfied for any (or many) choice of signs [41, 42, 44, 45] :

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2n, \qquad n \in \mathbb{Z}_{\geq 0} \tag{1.205}$$

When the integral is divergent, one regulates it by shifting the dimension of the space and the conformal dimensions [41, 42, 44, 45] :

$$\begin{aligned} d &\to \widetilde{d} = d + 2u\epsilon \\ \Delta_i &\to \widetilde{\Delta}_i = \Delta_i + (u + v_i)\epsilon \end{aligned} \tag{1.206}$$

where $u$ and $v_i$ are four real parameters. This results in a shift in the discrete parameters of the triple-$K$ integral indicated as follows :

$$I_{\alpha\{\beta_1,\beta_2,\beta_3\}} \to I_{\widetilde{\alpha}\{\widetilde{\beta}_1,\widetilde{\beta}_2,\widetilde{\beta}_3\}} = I_{\alpha+u\epsilon\{\beta_1+v_1\epsilon,\beta_2+v_2\epsilon,\beta_3+v_3\epsilon\}} \tag{1.207}$$

Here we note that when one deals with parity odd contributions to a correlator one has to set $u = 0$ as the Levi-Civita tensors are defined in the original dimensions and not in the shifted dimensions and hence one cannot use dimensional regularisation.

For cases where the divergence condition (1.205) is satisfied for the choice of signs $(---)$ or $(--+)$ or its permutations, one gets rid of the singularities in the regularised correlator by adding suitable counter-terms to the CFT action. For cases where the equality (1.205) is satisfied for the choice of signs $(-++)$ and its permutations or $(+++)$ there are no appropriate counter-terms that one can add to the action. These correspond to cases where it is the triple-$K$ representation of the correlator itself that is singular

[42, 44, 45].

In the following, we will show using the example of the $\langle JJO_\Delta\rangle$ correlator that in the spinor-helicity variables the relation between the parity-even and the parity-odd parts continues to hold even after renormalization. To renormalize the correlators, we first convert our answers in the spinor-helicity variables to momentum space expressions, cure the divergences and obtain finite answers in momentum space. We then convert the resulting correlators back to spinor-helicity variables.

### 1.6.1 $\langle JJO_\Delta\rangle$

The momentum space expression for the even part of the correlator was given in (1.156). It can be checked that the discrete parameters in the triple-$K$ integral satisfy the divergence condition (1.205) when $\Delta \geq 4$. When $\Delta = 4$, the divergence condition (1.205) is satisfied for the choice of signs given by $(- - -)$. A convenient scheme of regularisation is choosing $v_3 \neq 0$ and $u = v_1 = v_2 = 0$ [45]. The divergence terms are removed by adding the counter-term with three sources [45]:

$$S_{ct} = a(\epsilon)\int d^3x\, \mu^{v_3\epsilon}\, F_{\mu\nu}F^{\mu\nu}\phi \tag{1.208}$$

where $\mu$ is the renormalization scale. After removing the divergence by choosing an appropriate $a(\epsilon)$ such that the singular term in the regularised correlator is canceled, we get the following finite renormalized form factor :

$$A_1^{\text{reno}}(k_1,k_2,k_3) = 3\,c_1\log\left(\frac{k_1+k_2+k_3}{\mu}\right) - c_1\frac{k_3^2+3k_3(k_1+k_2+k_3)}{(k_1+k_2+k_3)^2} \tag{1.209}$$

In spinor-helicity variables, the renormalized correlator takes the following form :

$$\langle J^-J^-O_4\rangle_{\text{even}} = A_1^{\text{reno}}(k_1,k_2,k_3)\langle 12\rangle^4 \tag{1.210}$$

Let us now discuss the parity odd part of $\langle JJO_\Delta\rangle$. The momentum space expression for the correlator takes the form in (1.160). For $\Delta = 4$, the two triple-$K$ integrals are singular for the choice of signs $(+ - -)$ and $(- + -)$ respectively. To remove the singularity, we add the following parity-odd counter-term with two sources and one operator :

$$S_{ct} = a(\epsilon)\int d^3x\, \mu^{-\epsilon}\, \epsilon^{\mu\nu\lambda}\, F_{\mu\nu}\, J_\lambda\, \phi \tag{1.211}$$

After removing the divergences, the resulting form factor is given by :

$$B_1^{\text{reno}}(k_1,k_2,k_3) = c_1 \frac{3}{k_1} \log\left(\frac{k_1+k_2+k_3}{\mu}\right) - c_1 \frac{k_3^2+3k_3(k_1+k_2+k_3)}{k_1(k_1+k_2+k_3)^2} \tag{1.212}$$

Note that the form factor $B_1^{\text{reno}}(k_1,k_2,k_3)$ is related to the one in the even case $A_1^{\text{reno}}(k_1,k_2,k_3)$ by the following simple relation :

$$B_1^{\text{reno}}(k_1,k_2,k_3) = \frac{1}{k_1} A_1^{\text{reno}}(k_1,k_2,k_3) \tag{1.213}$$

In spinor-helicity variables, the correlator again takes the same form as in the parity even case :

$$\langle J^- J^- O_4\rangle_{\text{odd}} = i\, A_1^{\text{reno}}(k_1,k_2,k_3)\langle 12\rangle^4 \tag{1.214}$$

Thus we have illustrated following the case of $\langle JJO_4\rangle$ that the parity-even and the parity-odd parts of the correlator are given by the same form factor even after renormalization.

### 1.6.2 $\langle TTO_\Delta\rangle$

Let us now consider the $\langle TTO_\Delta\rangle$ correlator. The transverse and traceless part of the even part of the correlator is given by [45] :

$$\begin{aligned}&\langle T_{\mu_1\nu_1}(k_1)T_{\mu_2\nu_2}(k_2)O(k_3)\rangle_{\text{even}}\\ &\quad= \Pi_{\mu_1\nu_1\alpha_1\beta_1}(k_1)\Pi_{\mu_2\nu_2\alpha_2\beta_2}(k_2)\left[A_1\, k_2^{\alpha_1}k_2^{\beta_1}k_3^{\alpha_2}k_3^{\beta_2} + A_2\delta^{\alpha_1\alpha_2}k_2^{\beta_1}k_3^{\beta_2} + A_3\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}\right]\end{aligned} \tag{1.215}$$

In $d=3$, the solutions of the primary Ward identities were obtained to be [45] :

$$\begin{aligned}A_1 &= c_1 I_{\frac{9}{2}\{\frac{3}{2},\frac{3}{2},\Delta-\frac{3}{2}\}}\\ A_2 &= 4c_1 I_{\frac{7}{2}\{\frac{3}{2},\frac{3}{2},\Delta-\frac{1}{2}\}} + c_2 I_{\frac{5}{2}\{\frac{3}{2},\frac{3}{2},\Delta-\frac{3}{2}\}}\\ A_3 &= 2c_1 I_{\frac{5}{2}\{\frac{3}{2},\frac{3}{2},\Delta+\frac{1}{2}\}} + c_2 I_{\frac{3}{2}\{\frac{3}{2},\frac{3}{2},\Delta-\frac{1}{2}\}} + c_3 I_{\frac{1}{2}\{\frac{3}{2},\frac{3}{2},\Delta-\frac{3}{2}\}}\end{aligned} \tag{1.216}$$

One can easily check that for $\Delta=1$, $\Delta=2$, $\Delta=3$ a subset of the triple-$K$ integrals that appear in the form-factors above are divergent. A convenient regularisation scheme to work with is $u=v_1=v_2$ and $v_3\neq u$. For $\Delta=1,2,3$ one does not have counter-terms to remove the divergences. It turns out that the divergences that appear in these cases are exactly cancelled by the primary constants determined by the secondary Ward identities [45].

#### 1.6.2.1 $\langle TTO_4\rangle$

$\langle TTO_4\rangle$ deserves special discussion as this is the first case where a type-$A$ anomaly could occur [45]. It was noticed in [45] that in the regularised correlator the pole in the regulator $\epsilon$ multiplies a degenerate combination of form factors in the numerator and hence the divergent form factors amount to a finite anomalous contribution to the correlator. Thus counter-terms are not essential to remove such divergences. It was also shown in [45] that the divergences in the regularised correlator and the anomaly can be removed using an appropriate counter-term with suitable coefficients. The form-factors of the renormalized correlator takes the following form (we present only the scheme-independent part) :

$$\begin{aligned} A_1 &= \frac{c_1}{E^4}\mathcal{E}_1 \\ A_2 &= \frac{c_1}{E^3}(-\mathcal{E}_1(k_1+k_2-k_3)+2\mathcal{E}_2k_1k_2) \\ A_3 &= \frac{c_1(k_1+k_2-k_3)}{4E^2}(\mathcal{E}_1(k_1+k_2-k_3)-4\mathcal{E}_2k_1k_2) \end{aligned} \tag{1.217}$$

where

$$\begin{aligned} \mathcal{E}_1 &= (k_1+k_2)^2((k_1+k_2)^2+12k_1k_2)+16(k_1+k_2)((k_1+k_2)^2+3k_1k_2)k_3 \\ &\qquad +6(7(k_1+k_2)^2+10k_1k_2)k_3^2+32(k_1+k_2)k_3^3+5k_3^4 \\ \mathcal{E}_2 &= (k_1+k_2)^3+15(k_1+k_2)^2k_3+27(k_1+k_2)k_3^2+5k_3^3 \end{aligned} \tag{1.218}$$

We now convert this result in momentum space to the spinor-helicity variables and obtain :

$$\langle T^-T^-O_4\rangle = k_1\,k_2\frac{(k_1+k_2)^2+4(k_1+k_2)k_3+5k_3^2}{(k_1+k_2+k_3)^4}\langle 12\rangle^4 \tag{1.219}$$

This precisely matches the result that we obtained for the correlator by directly solving the conformal Ward identities in spinor-helicity variables (1.104). For $\Delta=4$ (or more generally $\Delta\le 5$) the triple-$K$ integral in (1.104) is convergent and we get finite results for the correlator without any renormalization. For $\Delta\ge 6$ the above triple-$K$ integral is singular and one needs to regularise and renormalize appropriately.

## 1.7 CFT correlators in terms of momentum space invariants

The aim of this section is to write down CFT correlators derived in previous sections in terms of a few conformal invariant momentum space structures. Let us define

$$Q_{12} = \frac{1}{E^2}\left[2\left(\vec{z}_1\cdot\vec{k}_2\right)\left(\vec{z}_2\cdot\vec{k}_1\right) + E\left(E-2k_3\right)\vec{z}_1\cdot\vec{z}_2\right] \tag{1.220}$$
$$S_{12} = \frac{1}{E^2}\left[k_2\epsilon^{k_1 z_1 z_2} - k_1\epsilon^{k_2 z_1 z_2}\right] \tag{1.221}$$
$$P_{123} = \frac{1}{E^3}\left[2\left(\vec{z}_1\cdot\vec{k}_2\right)\left(\vec{z}_2\cdot\vec{k}_3\right)\left(\vec{z}_3\cdot\vec{k}_1\right) + E\left(k_3\left(\vec{z}_1\cdot\vec{z}_2\right)\left(\vec{z}_3\cdot\vec{k}_1\right) + \text{ cyclic }\right)\right] \tag{1.222}$$
$$\begin{aligned} R_{123} = \frac{1}{E^3}\Big[\Big\{&(\vec{k}_1\cdot\vec{z}_3)\left(\epsilon^{k_3 z_1 z_2}k_1 - \epsilon^{k_1 z_1 z_2}k_3\right) + (\vec{k}_3\cdot\vec{z}_2)\left(\epsilon^{k_1 z_1 z_3}k_2 - \epsilon^{k_2 z_1 z_3}k_1\right) \\ &-(\vec{z}_2\cdot\vec{z}_3)\epsilon^{k_1 k_2 z_1}E + \frac{k_1}{2}\epsilon^{z_1 z_2 z_3}E(E-2k_1)\Big\} + \text{cyclic perm}\Big] \end{aligned} \tag{1.223}$$

These can be used as building blocks for writing down momentum space 3-point conserved correlators since they arise naturally in the expressions for such correlators [11]. There are some interesting relations among the above defined quantities. For example

$$\begin{aligned} &S_{ij}^2 = Q_{ij}^2, \quad R_{ijk}^2 = P_{ijk}^2, \quad P_{123}^2 = Q_{12}Q_{23}Q_{31}, \quad S_{ij}S_{jk} = Q_{ij}Q_{jk} \\ &P_{123}R_{123} = S_{12}Q_{23}Q_{31} + \text{cyclic perm.} \end{aligned} \tag{1.224}$$

up-to degeneracies.

## Homogeneous contribution

From the summary in A.109, we may now write the momentum space three-point functions in a compact manner using the above invariants. Let us note that we are concerned only with correlators which satisfy triangle inequality. To do this we divide the correlator into two different classes.

[11] That they are conformal invariants follows from (1.226) - (1.229). If we put $s = 1$, in each case a structure is equal to a particular correlator which is of course, by definition, conformally invariant.

### $s_1 + s_2 + s_3 = 2n \ \ (n \in \mathbb{Z})$

For this class of correlators we only require $Q_{ij}$ and $S_{ij}$. Consider $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ such that $s_1 \geq s_2 \geq s_3$, $s_1 \leq s_2 + s_3$. Then, we have

$$\begin{aligned} \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{even}} &= k_1^{s_1-1} k_2^{s_2-1} k_3^{s_3-1} Q_{12}^{\frac{1}{2}(s_1+s_2-s_3)} Q_{23}^{\frac{1}{2}(s_2+s_3-s_1)} Q_{13}^{\frac{1}{2}(s_1+s_3-s_2)} \\ \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}} &= k_1^{s_1-1} k_2^{s_2-1} k_3^{s_3-1} S_{12} Q_{12}^{\frac{1}{2}(s_1+s_2-s_3-2)} Q_{23}^{\frac{1}{2}(s_2+s_3-s_1)} Q_{13}^{\frac{1}{2}(s_1+s_3-s_2)} \\ &\quad + \text{cyclic perm.} \end{aligned} \tag{1.225}$$

Correlators involving scalar operators can also be written this way,

$$\langle J_s J_s O_2 \rangle_{\text{even},\mathbf{h}} = b_{12}^{s-1} Q_{12}^{s} \tag{1.226}$$

$$\langle J_s J_s O_2 \rangle_{\text{odd},\mathbf{h}} = b_{12}^{s-1} S_{12} Q_{12}^{s-1} \tag{1.227}$$

where $b_{ij} = k_i k_j$ and $c_{123} = k_1 k_2 k_3$. The 3-point function involving $\Delta = 1$ is obtained by shadow transforming (1.226). One can also write the correlator involving a generic scalar operator dimension $\delta$ but we do not reproduce this here.

### $s_1 + s_2 + s_3 = 2n + 1 \ \ (n \in \mathbb{Z})$

We require $P_{123}$ and $R_{123}$ as well when the sum of the spins is odd. For example, when $s$ is odd, we have

$$\langle J_s J_s J_s \rangle_{\text{even},\mathbf{h}} = c_{123}^{s-1} P_{123}^{s} \tag{1.228}$$

$$\langle J_s J_s J_s \rangle_{\text{odd},\mathbf{h}} = c_{123}^{s-1} R_{123} P_{123}^{s-1} \tag{1.229}$$

When $s$ is odd, these are the only structures using which $\langle J_s J_s J_s \rangle$ can be written. One can use (1.224) to substitute even powers of $P_{123}$ in terms of $Q_{ij}$'s. Other correlators with $s_1 + s_2 + s_3 = \text{odd}$ can also be considered similarly.

## Non-Homogeneous contribution

We have discussed homogeneous contributions so far. The story for the non-homogeneous contribution is more complicated. We do not have a generic form of the WT identity to evaluate three-point functions involving operators of arbitrary spin. For example, if we consider the solutions for $\langle J_s O_2 O_2 \rangle$ as given in (1.154), there is no discernible underlying structure to these expressions. The numerator becomes increasingly complicated as we consider higher values of $s$ and one cannot write these as the power of some simple structure. We can similarly identify some structures based on the answers for $\langle JJJ \rangle$, $\langle JJT \rangle$ and $\langle TTT \rangle$ however those relations are not illuminating as in the case of the homogeneous part, (see (1.226)) and we do not present them here.

## 1.8 Some interesting observations

In this section we collect a few interesting observations about the correlators discussed so far. For the purposes of this discussion, it will be useful to write the correlators as in (1.79).

### 1.8.1 Contact terms

To properly understand correlators in momentum space it is very important to understand the contact terms which arise in both parity-odd and parity-even cases. For example, $\langle JJJ \rangle$ correlation function has a contact term which is parity odd and is given by (1.186). Fourier transforming this to position space will give us a term of the form

$$\langle J^a_\mu J^b_\nu J^c_\rho \rangle_{\text{contact}} \propto c'_J f^{abc} \epsilon_{\mu\nu\rho} \delta^3(x_1 - x_2)\delta^3(x_2 - x_3). \tag{1.230}$$

Another example of a correlation function where both parity even and parity odd part has contact term, let us consider $\langle TTT \rangle$. The parity even contact term is given by [39]

$$\langle TTT \rangle_{\text{even}} \propto c_T \left(k_1^3 + k_2^3 + k_3^3\right) z_1 \cdot z_2 z_2 \cdot z_3 z_3 \cdot z_1 \tag{1.231}$$

which when converted to position space gives contact term of the form

$$\langle TTT \rangle_{\text{contact}} \propto c_T \left( f(x_1)\delta^3(x_2 - x_3) + f(x_2)\delta^3(x_3 - x_1) + f(x_3)\delta^3(x_1 - x_2) \right). \tag{1.232}$$

Parity odd contact term is given[12] by (1.193) which becomes

$$\langle TTT\rangle_{\text{contact}} \propto c_T' \epsilon_{z_1 z_2 z_3} \delta^3(x_1 - x_2)\delta^3(x_2 - x_3) + \cdots \tag{1.233}$$

where $z$ are polarization tensors. Once again we have not mentioned the exact form of the contact term. Interestingly, for both parity even and parity odd parts, the contact term arises in the non-homogeneous contribution. One way to understand parity odd case is to look at (1.77). The right-hand side of this equation for parity odd case is always a contact term for the correlator we have considered. For example, the transverse Ward identity for $\langle JJJ\rangle$ takes the form (1.105)

$$\begin{aligned} k_{1\mu}\langle J^{\mu a}(k_1)J^{\nu b}(k_2)J^{\rho c}(k_2)\rangle &= f^{adc}\langle J^{\rho d}(k_2)J^{\nu b}(-k_2)\rangle - f^{abd}\langle J^{\nu d}(k_3)J^{\rho c}(-k_3)\rangle \\ &= f^{abc}\epsilon^{\nu\rho k_1} \end{aligned}$$

which is a contact term. One can check the same explicitly for $\langle TTT\rangle$ as well as other correlators computed in previous sections. We expect on general grounds that $\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\text{odd},\mathbf{nh}}$ is a contact term. To conclude, we observe that

**A.** Contribution to the contact term comes from the non-homogeneous part of both parity-even and parity-odd correlator. For parity-even it was observed in $\langle TTT\rangle$ only.

**B.** Parity-odd non-homogeneous piece of the CFT correlator is always a contact term.

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh},\text{odd}} = \text{contact term}. \tag{1.234}$$

However parity-even non-homogeneous piece can be nontrivial as is discussed in previous sections. The reader is referred to appendix 1.8.1 for a detailed discussion regarding whether the contact parity-odd terms can be set to zero by field redefinitions.

### 1.8.2 Relation between parity-even and parity-odd solutions

If we look at the correlator in momentum space, see section 1.5, there seems to exist no clear relationship between the parity odd and parity even part of the correlator. However,

[12] Let us note that, we have neglected the precise functional dependence. We have just indicated the form of the delta function that arises.

as is seen in section 1.3 and 1.4, there exists a remarkable relationship between them in spinor-helicity variables, namely

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h},\text{odd}} = \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\mathbf{h},\text{even}} \qquad \text{(In spinor helicity variables)} \tag{1.235}$$

up-to some signs and factors of $i$. Let us explain this in terms of some concrete equations. To start, let us consider the ansatz

$$\begin{aligned} \langle J^-_{s_1} J^-_{s_2} J^-_{s_3} \rangle &= (F_1(k_1,k_2,k_3) + iF_2(k_1,k_2,k_3)) \langle 12 \rangle^{s_1+s_2-s_3} \langle 23 \rangle^{-s_1+s_2+s_3} \langle 31 \rangle^{s_1-s_2+s_3} \\ \langle J^-_{s_1} J^-_{s_2} J^+_{s_3} \rangle &= (G_1(k_1,k_2,k_3) + iG_2(k_1,k_2,k_3)) \langle 12 \rangle^{s_1+s_2-s_3} \langle 23 \rangle^{-s_1+s_2+s_3} \langle 31 \rangle^{s_1-s_2+s_3} \end{aligned} \tag{1.236}$$

where $F_1, G_1$ and $F_2, G_2$ are form-factors for the parity-even and parity-odd parts of the correlator. Both $F_1$ and $F_2$ satisfy the same non-homogeneous equation, see for example (A.74), (A.83). However, the form factors $G_1$ and $G_2$ satisfy a different non-homogeneous equation, see for example (A.75), (A.84) in appendix A.6. This difference is coming due to the different contribution of WT identity to parity-even and parity-odd parts for $--+$ helicity component[13]. This implies that non-homogeneous contributions to parity-even and odd cases generally differ, whereas the homogeneous solution is always the same.

This relation becomes even more nontrivial in cases where there is a divergence in the correlator. For example, for $\langle JJO_4 \rangle$ the solution of the conformal Ward identity is given by

$$\langle J^- J^- O_4 \rangle_{\text{even}} = c_1 (k_1 k_2) I_{\frac{5}{2},\{\frac{1}{2},\frac{1}{2},\frac{5}{2}\}} \tag{1.237}$$

$$\langle J^- J^- O_4 \rangle_{\text{odd}} = i\, c_2 (k_1 k_2) I_{\frac{5}{2},\{\frac{1}{2},\frac{1}{2},\frac{5}{2}\}} \tag{1.238}$$

However, the triple-$K$ integral is divergent and one needs to regularise and renormalize the correlator. To do so we go to momentum space (see section 1.6). The renormalization procedure for even and odd parts is also completely different and we required quite different kinds of counter-terms. However, converting back the renormalized results in spinor-helicity variables, we remarkably obtained the same result again. This happens to all other correlators having divergences and it would be interesting to have a better

[13]For $---$ helicity the WT identity contributes the same for parity even and odd case.

understanding of this observation.

### 1.8.3 Manifest locality test

In the context of the cosmological bootstrap, a manifestly local test (MLT) was derived for wavefunctions of scalars of dimension 3 and gravitons in any manifestly local, unitary theory [57]. MLT imposes the following condition on such wavefunctions [57] :

$$\lim_{k_c \to 0} \frac{\partial}{\partial k_c} \psi_n(k_1, \ldots, k_c, \ldots, k_n) = 0 \tag{1.239}$$

In the following we discuss how this analysis can be used for calculating CFT correlator $\langle T O_3 O_3 \rangle$. Based on the symmetries of the correlator we write down the following ansatz for the correlator :

$$\langle T_{\mu\nu}(k_1) O_3(k_2) O_3(k_3) \rangle = \Pi_{\mu\nu\alpha\beta}(k_1)\, A_1(k_1, k_2, k_3)\, k_2^\alpha k_2^\beta \tag{1.240}$$

where we take the following ansatz for the form factor :

$$\begin{aligned} A_1(k_1, k_2, k_3) = \frac{1}{(k_1 + k_2 + k_3)^2} \Big[ & c_1 k_1^3 + c_2 k_1^2 (k_2 + k_3) + c_3 k_1 k_2 k_3 \\ & + c_4 k_1 (k_2 + k_3)^2 + c_5 (k_2 + k_3) k_2 k_3 + c_6 (k_2 + k_3)^3 \Big] \end{aligned} \tag{1.241}$$

where the pole in $k_1 + k_2 + k_3 = 0$ can be argued on general grounds and the power of the pole is fixed by dilatation Ward identity. We will now fix the coefficients that appear in the ansatz by imposing manifest locality. With respect to one of the scalar operators we have :

$$\lim_{k_2 \to 0} \frac{\partial}{\partial k_2} \langle T(k_1) O_3(k_2) O_3(k_3) \rangle = 0 \tag{1.242}$$

This gives the following relation between the coefficients :

$$c_2 = 2c_1, \quad c_4 = \frac{2c_1 - c_3}{2}, \quad c_6 = -c_5, \quad c_3 = 2c_5 \tag{1.243}$$

We can easily check that with these conditions, MLT with respect to the second scalar operator is also satisfied, i.e. :

$$\lim_{k_3\to 0}\frac{\partial}{\partial k_3}\langle T(k_1)O_3(k_2)O_3(k_3)\rangle = 0. \tag{1.244}$$

Let us now impose manifest locality with respect to the stress-tensor operator :

$$\lim_{k_1\to 0}\frac{\partial}{\partial k_1}\langle T(k_1)O_3(k_2)O_3(k_3)\rangle = 0 \tag{1.245}$$

This gives the following constraint $c_5 = -c_1$. We now substitute the coefficients back into the ansatz to get :

$$A(k_1,k_2,k_3) = \frac{c_1}{(k_1+k_2+k_3)^2}\left[k_1^3+k_2^3+k_3^3+2(k_1^2+k_2k_3)(k_2+k_3)+2k_1(k_2^2+k_2k_3+k_3^2)\right] \tag{1.246}$$

Notice that the form factor in (1.246) matches explicitly with the form factor presented in (1.155). We hope to come back to this in the future for a better understanding of other 3-point functions.

### 1.8.4 A comparison between position and momentum space invariants

It is interesting to compare momentum space invariants discussed in section 1.7 and position space invariants introduced in [14, 202]. To illustrate this, let us consider $\langle JJT\rangle$ even part. This is given by

$$\langle T(x_1)J(x_2)J(x_3)\rangle_{\text{even}} = \frac{1}{|x_{12}||x_{23}||x_{31}|}\left(a_1P_1^2Q_1^2+a_2P_2^2P_3^2+a_3Q_1^2Q_2Q_3+a_4P_1P_2P_3Q_1\right), \tag{1.247}$$

We refer the reader to [202] for details about the notation. We see that there are 4 structures. Demanding conservation equation for currents, we get two relation $a_2 = -4a_1,\ a_3 = -\frac{5}{2}a_1$ which leaves two independent structures

$$\langle T(x_1)J(x_2)J(x_3)\rangle_{\text{even}} = \frac{1}{|x_{12}||x_{23}||x_{31}|}\left[a_1\left(P_1^2Q_1^2-4P_2^2P_3^2-\frac{5}{2}Q_1^2Q_2Q_3\right)+a_4P_1P_2P_3Q_1\right] \tag{1.248}$$

Furthermore, using WT identity we get a relation between $a_4, a_1$ and the two-point function coefficient $c_j$. Eliminating $a_4$ we obtain

$$\langle T(x_1)J(x_2)J(x_3)\rangle_{\text{even}} = \frac{1}{|x_{12}||x_{23}||x_{31}|}\Big[a_1\left(P_1^2Q_1^2 - 4P_2^2P_3^2 - \frac{5}{2}Q_1^2Q_2Q_3 - 2P_1P_2P_3Q_1\right) + \frac{3}{8}c_jP_1P_2P_3Q_1\Big] \tag{1.249}$$

where $c_j$ appears in two point function of $J_\mu$. Let us emphasize that, (1.247) is built out of conformal invariant structures whereas (1.249) is built out of conformally invariant *conserved* structures[14]. In (1.249), we can identify the term proportional to $a_1$ as homogeneous and the term proportional to $c_j$ as the non-homogeneous contribution. Let us note that, for a generic correlator involving arbitrary spin-$s$ currents, in general, it is quite complicated to arrive at the analogue of (1.249) starting from more the readily obtainable expression (1.247). Moreover, finding the non-homogeneous term in position space is equally complicated. However, in momentum-space, we naturally obtain the analogue of (1.249) directly. In other words, in momentum space we naturally divide the answer into homogeneous and non-homogeneous contributions, and the conformal invariant conserved structure is naturally built in.

## 1.9 Double copy structure of CFT correlators

In this section, we discuss the double copy structure of CFT 3-point correlation functions in momentum space. For the homogeneous piece, we have in the spinor-helicity variables

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h,e}} \propto \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h,o}}. \tag{1.250}$$

Using such properties it is easy to establish in momentum space [187]

$$\begin{aligned}\langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{h,e}} &\propto \left(\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{even,\mathbf{h}}\right)^2 \propto \left(\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h,o}}\right)^2\\ \langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{h,o}} &\propto \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h,e}}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h,o}}\end{aligned} \tag{1.251}$$

[14]It is quite difficult to build conformally invariant conserved structures directly without first writing conformal invariants and then demanding conservation. Free theory generating functions defined in [202] might be of help, however, it will be difficult to separate out the homogeneous and non-homogeneous contributions. However, in momentum-space, we directly get the conformal invariant conserved structures, and getting conformal invariant structures without the WT identity constraint would be a more challenging task.

using which we get

$$\langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{h,o}} + \langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{o,\mathbf{h}} \propto \left(\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h,e}} + \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{h,o}}\right)^2. \tag{1.252}$$

It was also shown that non-homogeneous structures separately satisfy double copy relation

$$\langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{nh,e}} \propto \left(\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,e}}\right)^2 \tag{1.253}$$

As discussed in the main text, even for non-homogeneous parity-odd and even pieces defined in (1.291) we have analogue of (1.250) in spinor helicity variables

$$\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,e}} \propto \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,o}}. \tag{1.254}$$

This again implies [187]

$$\begin{aligned}
&\langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{nh,e}} \propto \left(\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,e}}\right)^2 \propto \left(\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,o}}\right)^2\\
&\langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{nh,o}} \propto \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,e}}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,o}}
\end{aligned} \tag{1.255}$$

using which we get

$$\langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{nh,e}} + \langle J_{2s_1}J_{2s_2}J_{2s_3}\rangle_{\mathbf{nh,o}} \propto \left(\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,e}} + \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{\mathbf{nh,o}}\right)^2. \tag{1.256}$$

We will now look at some specific examples.

### 1.9.1 Example: $s_3 = 0$

The following double copy structure of $\langle TTO_3\rangle_{\text{even}}$ was established in [39] :

$$\langle TTO_3\rangle_{\text{even},\mathbf{h}} = \frac{(E+3k_3)k_1k_2}{(E+k_3)^2}\langle JJO_3\rangle_{\text{even},\mathbf{h}}\langle JJO_3\rangle_{\text{even},\mathbf{h}} \tag{1.257}$$

From the explicit expressions for the correlators, we notice that the double copy relations extend to the parity-odd sector :

$$\langle TTO_3\rangle_{\text{odd},\mathbf{h}} = \frac{(E+3k_3)k_1k_2}{(E+k_3)^2}\langle JJO_3\rangle_{\text{odd},\mathbf{h}}\langle JJO_3\rangle_{\text{even},\mathbf{h}} \tag{1.258}$$

Remarkably, $\langle TTO_3\rangle_{\text{even}}$ is also given by the square of $\langle JJO_3\rangle_{\text{odd}}$

$$\langle TTO_3\rangle_{\text{even},\mathbf{h}} = \frac{(E+3k_3)k_1k_2}{(E+k_3)^2}\langle JJO_3\rangle_{\text{odd},\mathbf{h}}\langle JJO_3\rangle_{\text{odd},\mathbf{h}} \tag{1.259}$$

The above double copy relations for $\langle TTO_3\rangle_{\text{even}}$ and $\langle TTO_3\rangle_{\text{odd}}$ immediately imply the following double copy structure for the complete correlator :

$$\begin{aligned}\langle TTO_3\rangle_{\text{even},\mathbf{h}} + \langle TTO_3\rangle_{\text{odd},\mathbf{h}} &= \frac{(E+3k_3)k_1k_2}{(E+k_3)^2}\left(\langle JJO_3\rangle_{\text{even},\mathbf{h}} + \langle JJO_3\rangle_{\text{odd},\mathbf{h}}\right)^2\\ \implies \langle TTO_3\rangle_{\mathbf{h}} &= \frac{(E+3k_3)k_1k_2}{(E+k_3)^2}\langle JJO_3\rangle^2_{\mathbf{h}}\end{aligned} \tag{1.260}$$

In writing the above double copy relation it is crucial that we have the following relation between $\langle JJO_3\rangle_{\text{even}}$ and $\langle JJO_3\rangle_{\text{odd}}$ :

$$\langle JJO_3\rangle^2_{\text{even},\mathbf{h}} = c\langle JJO_3\rangle^2_{\text{odd},\mathbf{h}} \tag{1.261}$$

where $c$ is some constant. For correlators such as $\langle TTO_3\rangle$ and $\langle JJO_3\rangle$ the conformal Ward identity (in spinor helicity variables) does not have a non-homogeneous term. Hence the double copy structure that we obtained above is purely for correlators that satisfy the homogeneous conformal Ward identity.

### 1.9.2 Example: $s_1, s_2 > 2$

We will now extend our analysis of the double copy structure of $\langle TTO_3\rangle$ to higher spin correlators of the form $\langle J_sJ_sO_3\rangle$. One can show that the higher spin correlators take the following form :

$$\begin{aligned}\langle J_4J_4O_3\rangle_{\text{even},\mathbf{h}} &= k_1k_2\frac{(E+7k_3)}{(E+3k_3)^2}\langle J_2J_2O_3\rangle^2_{\text{even}}\\ \langle J_3J_3O_3\rangle_{\text{even},\mathbf{h}} &= k_1k_2\frac{(E+5k_3)}{(E+k_3)(E+3k_3)}\langle J_2J_2O_3\rangle_{\text{even}}\langle J_1J_1O_3\rangle_{\text{even}}.\end{aligned} \tag{1.262}$$

Calculating the parity-odd contribution to these three point functions is difficult due to the high amount of degeneracy [188]. However, in spinor helicity variables the computation becomes easier. In these variables one has the following remarkable relation between the parity-even and parity-odd contributions [?] :

$$\langle J_s{}^- J_s{}^- O_3\rangle_{\text{even},\mathbf{h}} = i\langle J_s{}^- J_s{}^- O_3\rangle_{\text{odd},\mathbf{h}}, \quad \langle J_s{}^+ J_s{}^+ O_3\rangle_{\text{even},\mathbf{h}} = -i\langle J_s{}^+ J_s{}^+ O_3\rangle_{\text{odd},\mathbf{h}} \tag{1.263}$$

for any $s$ and all the other spinor helicity components are zero. Using this one can generalize (1.262) to include the parity-odd sector. The double copy relation (1.262) then becomes

$$\begin{aligned}
\langle J_4 J_4 O_3\rangle_{\text{even},\mathbf{h}} + \langle J_4 J_4 O_3\rangle_{\text{odd},\mathbf{h}} &= \frac{k_1 k_2 (E+7k_3)}{(E+3k_3)^2}\left(\langle J_2 J_2 O_3\rangle_{\text{even},\mathbf{h}} + \langle J_2 J_2 O_3\rangle_{\text{odd},\mathbf{h}}\right)^2 \\
\implies \langle J_4 J_4 O_3\rangle_{\mathbf{h}} &= \frac{k_1 k_2 (E+7k_3)}{(E+3k_3)^2}\langle J_2 J_2 O_3\rangle_{\mathbf{h}}^2 \\
\langle J_3 J_3 O_3\rangle_{\text{even},\mathbf{h}} + \langle J_3 J_3 O_3\rangle_{\text{odd},\mathbf{h}} &= \frac{k_1 k_2 (E+5k_3)}{(E+k_3)(E+3k_3)}\left(\langle J_2 J_2 O_3\rangle_{\text{even},\mathbf{h}} + \langle J_2 J_2 O_3\rangle_{\text{odd},\mathbf{h}}\right) \\
&\quad \times \left(\langle J_1 J_1 O_3\rangle_{\text{even},\mathbf{h}} + \langle J_1 J_1 O_3\rangle_{\text{odd},\mathbf{h}}\right) \\
\implies \langle J_3 J_3 O_3\rangle_{\mathbf{h}} &= \frac{k_1 k_2 (E+5k_3)}{(E+k_3)(E+3k_3)}\langle J_2 J_2 O_3\rangle_{\mathbf{h}}\langle J_1 J_1 O_3\rangle_{\mathbf{h}}
\end{aligned}$$

To write down the above double copy relations it is crucial that we have the following relation between the parity-odd and parity-even parts of the correlators :

$$\begin{aligned}
\langle J_2 J_2 O_3\rangle_{\text{even},\mathbf{h}}^2 &= \langle J_2 J_2 O_3\rangle_{\text{odd},\mathbf{h}}^2 \\
\langle J_2 J_2 O_3\rangle_{\text{even},\mathbf{h}}\langle J_1 J_1 O_3\rangle_{\text{even},\mathbf{h}} &= \langle J_2 J_2 O_3\rangle_{\text{odd},\mathbf{h}}\langle J_1 J_1 O_3\rangle_{\text{odd},\mathbf{h}}
\end{aligned} \tag{1.264}$$

As we noted in the case of the double copy structure of $\langle TTO_3\rangle$ in terms of $\langle JJO_3\rangle$, the conformal Ward identity for correlators of the form $\langle J_s J_s O_3\rangle$ does not have a non-homogeneous term. Hence the double copy relations that we arrived at here are purely for the homogeneous terms.

### 1.9.3 Example: $s_1 = s_2 = s_3$

The double copy relation between $\langle JJJ\rangle$ and $\langle TTT\rangle$ is more subtle than those for correlators with a scalar operator insertion. Unlike $\langle TTO_3\rangle$ or $\langle JJO_3\rangle$ these correlators have a non-homogeneous term as well and we will see that the double copy structures map homogeneous terms to homogeneous terms and non-homogeneous terms get mapped to non-homogeneous terms.

### Homogeneous terms

The following double copy structure was noticed in [39] for the homogeneous term in the even part of $\langle TTT \rangle$ :

$$\langle TTT \rangle_{\text{even},\mathbf{h}} = k_1 k_2 k_3 \langle JJJ \rangle_{\text{even},\mathbf{h}} \langle JJJ \rangle_{\text{even},\mathbf{h}} \tag{1.265}$$

From the explicit expressions for the correlators, we notice :

$$\langle TTT \rangle_{\text{odd},\mathbf{h}} = k_1 k_2 k_3 \langle JJJ \rangle_{\text{odd},\mathbf{h}} \langle JJJ \rangle_{\text{even},\mathbf{h}} \tag{1.266}$$

We also have the remarkable relationship that the parity-even part of the homogeneous term is given by the square of the odd part of the homogeneous term in $\langle JJJ \rangle$ :

$$\langle TTT \rangle_{\text{even},\mathbf{h}} = k_1 k_2 k_3 \langle JJJ \rangle_{\text{odd},\mathbf{h}} \langle JJJ \rangle_{\text{odd},\mathbf{h}} \tag{1.267}$$

Combining these relations we obtain the following double copy relation for the complete homogeneous term of the $\langle TTT \rangle$ correlator :

$$\begin{aligned} \langle TTT \rangle_{\text{even},\mathbf{h}} + \langle TTT \rangle_{\text{odd},\mathbf{h}} &= k_1 k_2 k_3 (\langle JJJ \rangle_{\text{even},\mathbf{h}} + \langle JJJ \rangle_{\text{odd},\mathbf{h}})^2 \\ \implies \langle TTT \rangle_{\mathbf{h}} &= k_1 k_2 k_3 \, \langle JJJ \rangle_{\mathbf{h}}^2 \end{aligned} \tag{1.268}$$

### Non-homogeneous terms

$\langle JJJ \rangle_{\text{even}}$ and $\langle TTT \rangle_{\text{even}}$ also have non-trivial non-homogeneous parts between which there exists the following double copy relation :

$$\langle TTT \rangle_{\text{even},\mathbf{nh}} = (E^3 - E(k_1 k_2 + k_2 k_3 + k_1 k_3) - k_1 k_2 k_3) \langle JJJ \rangle_{\text{even},\mathbf{nh}}^2 \tag{1.269}$$

This relation is independent of the double copy of the homogeneous part as the pre-factor is different. The non-homogeneous parts of $\langle TTT \rangle_{\text{odd}}$ and $\langle JJJ \rangle_{\text{odd}}$ are trivial as they are contact terms.

### 1.9.4 Spin $s$ current correlator as $s$ copies of the spin one current correlator

In this sub-section, we note that we can write correlators of the form $\langle J_s J_s O_3\rangle$ and $\langle J_s J_s J_s\rangle$ as $s$ copies of correlators of the spin-one current. Using the double copy relations recursively we notice that :

$$\langle J_s J_s O\rangle_{\mathbf{h}} = (k_1 k_2)^{s-1}\,\frac{E+(2s-1)k_3}{(E+k_3)^s}\,(\langle JJO\rangle_{\mathbf{h}})^s \tag{1.270}$$

Similarly, using the double copy relations recursively we notice that :

$$\langle J_s J_s J_s\rangle_{\mathbf{h}} = (k_1 k_2 k_3)^{s-1}\,(\langle JJJ\rangle_{\mathbf{h}})^s \tag{1.271}$$

## 1.10 4D Flat-space scattering amplitude $/CFT_3$ correlator correspondence revisited

In this section, we look at the correspondence between flat-space amplitudes and CFT correlators. Momentum-space correlators not only resemble scattering amplitudes due to nontrivial double-copy relations but they also reproduce flat-space scattering amplitudes in appropriate limits. A remarkable feature of the momentum space CFT correlation function is its connection to flat-space amplitude [33,34]. Any $d$-dimensional three-point CFT correlator can be shown to give rise to $d+1$-dimensional three-point flat-space amplitude in the flat-space limit. In the previous chapter, this was shown to hold true for $d=3$.

Any three-point CFT correlator of conserved current has a maximum of three structures, two parity-even and one parity-odd structure [74,202]. Two parity-even structures can be obtained from free bosonic or free fermionic theory whereas the parity-odd structures can also be obtained from local Lagrangian such as Chern-Simons matter theories [71,72]. On the other hand, there are two parity-even amplitude structures and two parity-odd covariant amplitude structures [203,204]. This is summarised in the following table.

The table immediately makes it clear that there is a gross mismatch in number of independent structures of amplitude in flat space and CFT correlator. As can be seen

| Three-point | |
|---|---|
| **4D Flat-space Amplitudes** | **3D CFT correlator** |
| two parity-even<br>two parity-odd | two parity-even<br>one parity-odd<br>1 missing |

**Table 1.1:** The amplitudes are of massless, arbitrary spin $s$ gauge fields and the $CFT$ correlations are for conserved, arbitrary spin $s$ currents.

in the table 1.1, the number of CFT correlators is less than the number of flat space covariant amplitudes or the flat space spinor helicity amplitude. This immediately raises the question on validity of the flat space amplitude/ CFT correlator correspondence.

Let us concentrate on the case of[15] $s_3 \leq s_1 + s_2$. In this case as is well known, see [14, 202], three are total three structures for CFT correlators of conserved currents two parity even and one parity odd [16]. This implies, in this case, we see there is one less parity odd structure in CFT as compared to the covariant vertex. Interestingly, we point out a CFT structure that in the flat space limit reproduces the missing flat space amplitude. To do this we first show that parity odd flat space covariant amplitude can be constructed starting from parity even flat space covariant amplitude by what we call an epsilon transformation. We show that by using the same epsilon transform on some combination of parity even free fermion and free boson CFT correlator of conserved currents, we can construct a new parity odd CFT correlator which in the flat space limit reproduces the correct flat space covariant parity odd vertex. Even though this new parity odd CFT correlator is constructed out of epsilon transform of correlation of conserved current, the resulting correlator is not that of conserved current, as will be discussed.

[15]For outside the triangle $s_3 > s_1 + s_2$, see table 1.1 for counting. In this case, if we consider slightly broken HS currents, then one gets one parity odd structure, which is still a mismatch with flat space amplitude.

[16]However, if we allow for non-conserved currents then there are more allowed structures [14,205,206]. The point of interest of this paper is three-point function of conserved currents for which how the counting works for flat space amplitude and CFT correlators is not discussed in these papers.

### 1.10.1 Brief review of flat-space covariant vertex

In this section, we review known results for flat-space amplitude in four dimensions for massless particles. We follow closely the notation and discussion in [204].

#### 1.10.1.1 Covariant vertex

Consider a generic action of the form

$$S^{(3)} = \int d^4x C(\partial_{x_I}, \partial_{u_I})\phi^1(x_1, u_1)\phi^2(x_2, u_2)\phi^3(x_3, u_3)\Big|_{u_I=0, x_I=x} \tag{1.272}$$

where

$$\phi(x, u) = \phi_{\mu_1 \cdots \mu_s}(x) u^{\mu_1} \cdots u^{\mu_s} \tag{1.273}$$

are higher spin fields and $u_I$ are some some auxiliary variables and $C$ is some operator which makes the higher-spin fields contract with each other with various derivatives forming a cubic interaction. Under higher spin symmetry and gauge invariance, $C$ can be made up of only certain parity-even and parity-odd structures namely

| Flat-space amplitudes Building blocks | | |
|---|---|---|
| **Parity** | **Position space** | **Momentum space** |
| Even | $Y_I = \partial_{u_I}.\partial_{x_{I+1}}$<br>$Z_I = \partial_{u_{I+1}}.\partial_{u_{I-1}}$ | $Y_I = z_I.p_{I+1}$<br>$Z_I = z_{I+1}.z_{I-1}$ |
| Odd | $V_I = \epsilon^{\mu\nu\rho\sigma}\partial_{u^{\mu}_{I+1}}\partial_{x^{\nu}_{I+1}}\partial_{u^{\rho}_{I-1}}\partial_{x^{\sigma}_{I-1}}$<br>$W_I = \epsilon^{\mu\nu\rho\sigma}\partial_{u^{\mu}_1}\partial_{u^{\nu}_2}\partial_{u^{\rho}_3}\partial_{x^{\sigma}_I}$ | $V_I = \epsilon(z_{I+1}p_{I+1}z_{I-1}p_{I-1})$<br>$W_I = \epsilon(z_1 z_2 z_3 p_I)$ |

**Table 1.2:** Building blocks for flat-space amplitude in 4d. $\epsilon$ is 4D totally antisymmetric tensor and the notation $\epsilon(xyzw) = \epsilon_{\mu\nu\rho\sigma}x^{\mu}y^{\nu}z^{\rho}w^{\sigma}$.

The gauge symmetry and higher spin symmetry constrain the possible amplitudes to be

with $G = Y_1Z_1 + Y_2Z_2 + Y_3Z_3$. We have also assumed that $s_3 > s_2, s_1$ with out loss of generality. The minimal and non-minimal distinction comes from the number of

| Flat-space amplitudes | | |
|---|---|---|
| **Parity** | **Minimal** | **Non-Minimal** |
| parity-even | $g_{m,e}G^{s_1}Y_2^{s_2-s_1}Y_3^{s_3-s_1}$ | $g_{nm,e}Y_1^{s_1}Y_2^{s_2}Y_3^{s_3}$ |
| parity-odd | $g_{m,o}V_1G_1^{s_1}Y_2^{s_2-s_1-1}Y_3^{s_3-s_1-1}$ | $g_{nm,o}V_1Y_1^{s_1}Y_2^{s_2-1}Y_3^{s_3-1}$ |

**Table 1.3:** Amplitudes in 4d. The subscript "m" means minimal, "nm" means non-minimal, "e" means parity even, "o" means parity odd.

derivatives or the number of momentum factors present in the amplitudes. Notice that for the odd case, there is an issue when spins are coincident i.e. $s_i = s_j$, one obtains negative powers of derivatives which is not possible due to locality. For the case of $s_1 = s_2 < s_3$, one may use the identity

$$V_1G^{s_1}Y_2^{-1}Y_3^{s_3-s_1-1} \approx \frac{1}{2}\left[V_1Z_2 - V_2Z_1 + (W_2 - W_1)\,Y_3\right]G^{s_1-1}Y_3^{s_3-s_1-1} \tag{1.274}$$

to remove the negative powers. Such identities are derived using Schouten identities and momentum conservation. For the case of coincidence of $s_1 = s_2 < s_3$, notice that (1.274) is anti-symmetric under the $1 \leftrightarrow 2$ exchange. This implies for parity odd minimal amplitude to be non-zero for this case, we do require the Chan-Paton factor which is antisymmetric in exchange of $1 \leftrightarrow 2$ indices. Together with the Chan-Paton factor and (1.274), the amplitude becomes symmetric. For the case $s_1 = s_2 = s_3$, it can be shown that the negative powers for parity odd minimal amplitude cannot be removed [204]. In this case, the cubic vertices with negative powers are simply dropped. The final results for special cases are summarized below in the table. Let us note that, the discussion involving scalar is simpler and can be obtained from results in Table 1.3. Some explicit examples are worked in detail in the appendix A.10. In the next section, we show an interesting relation between the parity even and parity odd part of the covariant vertex discussed till now.

#### 1.10.1.2 $\epsilon$-transformation

In this section, we introduce what we call $\epsilon$-transform which maps parity-even amplitude to parity-odd amplitude and vice-versa. In 4D one may work with the following choice

| parity-odd flat-space amplitudes: special cases | | |
|---|---|---|
| **Configuration** | **Minimal** | **Non-Minimal** |
| $s_1 = s_2 < s_3$ | $g_{m,o}\left(V_1 Z_2 - V_2 Z_1 + (W_2 - W_1)\, Y_3\right) G^{s_1-1} Y_3^{s_3-s_1-1}$ | $g_{nm,o} V_1 Y_1^{s_1} Y_2^{s_1-1} Y_3^{s_3-1}$ |
| $s_1 = s_2 = s_3$ | $\times$ | $g_{nm,o} V_1 Y_1^{s_1} Y_2^{s_1-1} Y_3^{s_1-1}$ |

**Table 1.4:** parity-odd special cases of amplitudes in 4d. The parity-even part of the amplitude is as given in Table 1.3.

of momenta and polarization

$$p^\mu = (k, k^i) \quad z^\mu = (0, z^i) \tag{1.275}$$

with this choice the momentum space expressions in Table 1.2 take the form

$$\begin{aligned} V_I &= \epsilon(z_{I+1} z_{I-1} k_{I-1}) k_{I+1} - \epsilon(z_{I+1} z_{I-1} k_{I+1}) k_{I-1}, \quad Y_I = z_I . k_{I+1} \\ Z_I &= z_{I+1} . z_{I-1}, \quad p_I . p_J = -k_I k_J + k_I . k_J = 0 \end{aligned} \tag{1.276}$$

where we have used three-dimensional epsilon in the last expression, more precisely $\epsilon^{0\mu\nu\rho} = \epsilon^{\mu\nu\rho}$ where index 0 is time direction. Let us define an $\epsilon$-transformation [185] which is given by

$$z^i \to \frac{\epsilon^{zki}}{k} \tag{1.277}$$

The $\epsilon$-transformation can also be implemented by a differential operator

$$[O_\epsilon]_I = \frac{1}{k_I} \epsilon(z_I k_I \frac{\partial}{\partial z_I}) \tag{1.278}$$

which acts on parity-even gauge-invariant structures to give parity-odd gauge invariant structures i.e

$$\begin{aligned} O_\epsilon &: \mathcal{M}_{m,e} \to \mathcal{M}_{m,o} \\ O_\epsilon &: \mathcal{M}_{nm,e} \to \mathcal{M}_{nm,o}. \end{aligned} \tag{1.279}$$

We show this explicitly below. Consider the $\epsilon$-transformation of the even minimal amplitude

$$[O_\epsilon]_2\mathcal{M}_{m,e} = s_1 G^{s_1-1}Y_2^{s_2-s_1}Y_3^{s_3-s_1}[O_\epsilon]_2 G + (s_2-s_1)G^{s_1}Y_2^{s_2-s_1-1}Y_3^{s_3-s_1}[O_\epsilon]_2 Y_2 \quad (1.280)$$

Now, it can be shown [17] that

$$Y_2 Y_3[O_\epsilon]_2 G = -GV_1 \quad (1.281)$$

$$Y_3[O_\epsilon]_2 Y_2 = -V_1 \quad (1.282)$$

which when used in (1.280) gives

$$[O_\epsilon]_2\mathcal{M}_{m,e} = -s_2 V_1 G^{s_1}Y_2^{s_2-s_1-1}Y_3^{s_3-s_1-1} = \mathcal{M}_{m,o} \quad (1.283)$$

which is precisely the odd minimal amplitude. Similarly, for the even non-minimal amplitude, we have

$$[O_\epsilon]_2\mathcal{M}_{nm,e} = s_2 Y_1^{s_1}Y_2^{s_2-1}Y_3^{s_3}[O_\epsilon]_2 Y_2 = -s_2 V_1 Y_1^{s_1}Y_2^{s_2-1}Y_3^{s_3-1} = \mathcal{M}_{nm,o} \quad (1.284)$$

where in the second equality (1.282) was used and we have precisely obtained the odd non-minimal amplitude given in 1.3. Let us now consider some special examples to illustrate this.

### All equal spin: $s_1 = s_2 = s_3$

Notice when $s_1 = s_2 = s_3 = s$ for minimal even amplitude, from (1.283) we get

$$[O_\epsilon]_2\mathcal{M}_{m,e}^{sss} = -sG^{s-1}Y_2^{-1}Y_3^{-1} \quad (1.285)$$

These negative powers cannot be removed by any Schouten identities or degeneracies, therefore, we conclude that parity odd minimal gauge-invariant vertex does not exist.

[17] Refer to Appendix C for its derivation.

This is consistent with table 1.4. For the case of $s_1 = s_2 = s_3 = 2$, we have

$$\begin{aligned}\mathcal{M}_m^{222} &= g_{m,e} G^2 \\ \mathcal{M}_{nm}^{222} &= g_{nm,e} Y_1^2 Y_2^2 Y_3^2 + g_{nm,o} V_1 Y_1^2 Y_2 Y_3\end{aligned} \tag{1.286}$$

It is easy to show that non-minimal parity even and parity odd terms are related by $\epsilon$-transform. However, if we look at the $\epsilon$-transform of the minimal even amplitude we get

$$[\mathcal{M}']^{222} = ([O_\epsilon]_1 + [O_\epsilon]_2 + [O_\epsilon]_3) G^2 = -\frac{G}{k_1 k_2 k_3}(k_2 k_3 \epsilon(z_1 k_1 k_2) z_2.z_3 + \text{cyclic terms}) \tag{1.287}$$

which cannot be rewritten as a $4D$ Lorentz invariant structure. Hence it is not a valid covariant amplitude. This is consistent with the fact that there can not exist any parity odd minimal covariant vertex for an equal spin case.

### Two equal spin: $s_1 = s_2 \neq s_3$

Another coincidence point of concern is $s_1 = s_2 = s$, where we obtain

$$[O_\epsilon]_2 \mathcal{M}_{m,e}^{sss_3} = -s V_1 G Y_2^{-1} Y_3^{s_3-s-1} \tag{1.288}$$

These negative powers can be removed by using the Schouten identities [188] and re-write them as

$$[O_\epsilon]_2 \mathcal{M}_{m,e}^{sss_3} = -s\frac{1}{2}\left[V_1 Z_2 - V_2 Z_1 + \left(W_2 - W_1\right) Y_3\right] G^{s-1} Y_3^{s_3-s-1} \tag{1.289}$$

Due to the nature of the Schouten identities in 4D momentum space, an anti-symmetrization in $\epsilon$-transforms at $1, 2$ was not necessary to derive the above, but one can in principle still use anti-symmetrization and obtain the same result using (A.134)

$$([O_\epsilon]_1 - [O_\epsilon]_2) \mathcal{M}_{m,e}^{sss_3} = s\left[V_1 Z_2 - V_2 Z_1 + \left(W_2 - W_1\right) Y_3\right] G^{s-1} Y_3^{s_3-s-1} \tag{1.290}$$

However, as we will see later, this anti-symmetrization of the $\epsilon$-transforms is necessary at the level of the CFT correlator because not all Schouten identities in 4D momentum space carry on to the 3D momentum space. The $\epsilon$-transform can be done at any operator in the correlator, however, for the case of $s_1 = s_2$, some care needs to be taken. Notice that (1.289) is anti-symmetric under $1 \leftrightarrow 2$ while its minimal even counterpart is symmetric under the exchange. This is due to the presence of Chan-Paton factors for the case of $s_1 = s_2$. Therefore, an $\epsilon$-transform of minimal even for $s_3$ will give zero. Hence, when Chan-Paton factors are involved, the $\epsilon$-transform of $s_3$ must be avoided.

### 1.10.2 CFT Correlator/ covariant vertex correspondence: A new parity-odd CFT correlation function

In this section, we construct a parity-odd CFT correlation function which in the flat-space limit goes over to parity-odd minimal amplitude listed in section 1.10.1. In the case of flat-space amplitude, we explicitly showed in section 1.10.1.2 that the parity-odd minimal amplitude is obtained from parity-even minimal amplitude by doing what is called $\epsilon$-transform. Our proposal for the CFT correlation function is exactly the analogue of the amplitude case. We propose

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o} = [\mathcal{O}_\epsilon] \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,e} = [\mathcal{O}_\epsilon] \frac{\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FB} + \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{FF}}{2} \tag{1.291}$$

which by construction produces the correct flat-space minimal parity-odd amplitude[18]. Since this is non-homogeneous, this CFT correlator satisfies Ward-Takahashi (WT) identity and is given by

$$WT[\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o}] = [\mathcal{O}_\epsilon] \frac{WT_{FB} + WT_{FF}}{2}. \tag{1.292}$$

When the spins satisfy triangle inequality $s_i \leq s_j + s_k$, we have $WT_{FB} = WT_{FF}$ [184] which gives

$$WT[\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{nh,o}] = [\mathcal{O}_\epsilon] WT_{FB}. \tag{1.293}$$

For correlators with spin that violate the triangle inequality, we have $WT_{FB} \neq WT_{FF}$, and hence we have to use (1.291). The explicit form of WT identity for general spins can be very complicated. Below we work out a few simple examples of this correlator (1.291)

[18] In spinor helicity variables, the parity even non-homogeneous term used in (1.291) and parity odd term defined in the same equation (1.291) are identical upto imaginary factor of $i$.

and their flat-space limit. We also discuss the WT identity (1.292).

#### 1.10.2.1 Example: $\langle JJT \rangle$

Let us consider the simplest example of two spin-1 and one spin-2 operators. The parity even part of the CFT correlator is given by [41, 186]

$$\begin{aligned}
&\langle J(z_1,k_1)J(z_2,k_2)T(z_3,k_3)\rangle_e = \langle J(z_1,k_1)J(z_2,k_2)T(z_3,k_3)\rangle_{h,e} + \langle J(z_1,k_1)J(z_2,k_2)T(z_3,k_3)\rangle_{nh,e}\\
&= \frac{2(3k_3+E)}{E^4}(z_3.k_2)^2(z_2.k_3)(z_1.k_3) + \left[\frac{2k_3^2}{E^3} - c_J\frac{2(k_3+E)}{E^2}\right](z_3.k_2)^2(z_1.z_2)\\
&+ \left[\frac{1}{E^3}(-2k_3^2 - k_2^2 + k_1^2 - 3k_3k_2 + 3k_3k_1) - c_J\frac{2(k_3+E)}{E^2}\right](z_3.z_2)(z_3.k_2)(z_1.k_3)\\
&+ \left[\frac{1}{E^3}(-2k_3^2 - k_1^2 + k_2^2 - 3k_3k_1 + 3k_3k_2) - c_J\frac{2(k_3+E)}{E^2}\right](z_1.z_3)(z_3.k_2)(z_2.k_1)\\
&+ \left[\frac{(k_3+E)(k_3^2 - (k_2+k_1)^2 + 4k_2k_1)}{2E^2} + c_J(\frac{2k_3^2}{E} - k_2 - k_1)\right](z_1.z_3)(z_3.z_2) \qquad (1.294)
\end{aligned}$$

where term proportional to $c_J$ is non-homogeneous parity even part and rest is parity even homogeneous contribution. In the flat-space limit we get

$$\begin{aligned}
&\lim_{E\to 0}\langle J(z_1,k_1)J(z_2,k_2)T(z_3,k_3)\rangle = \frac{6k_3}{E^4}(z_3.k_2)^2(z_2.k_3)(z_1.k_2) + O(\frac{1}{E^3})\\
&+ c_J\frac{2k_3^2}{E^3}(z_1.z_2z_3.k_1 + z_2.z_3z_1.k_2 + z_3.z_1z_2.k_3)(z_3.k_2) + c_J\ O(\frac{1}{E^2}) \qquad (1.295)
\end{aligned}$$

which perfectly match with parity even minimal and non-minimal vertices

$$\mathcal{M}_{nm,e} = (z_3.k_1)^2(z_2.k_3)(z_1.k_2) \quad \mathcal{M}_{m,e} = (z_1.z_2z_3.k_1 + z_2.z_3z_1.k_2 + z_3.z_1z_2.k_3)(z_3.k_2). \qquad (1.296)$$

The non-minimal parity odd amplitude is obtained by taking the flat space limit of parity odd homogeneous contribution. The parity odd CFT correlator can be found in [186]. It is easy to show that in the flat space limit,

$$\lim_{E\to 0}\langle JJT\rangle_{h,o} \to \mathcal{M}_{nm,o}. \qquad (1.297)$$

However as it is clear, there is no analogue of the CFT correlator which in the flat space limit reproduces correct flat space minimal parity odd amplitude. We now show using definition (1.291) we get a CFT correlator which in the flat space reproduces correct parity odd minimal amplitude. As was discussed below (1.289), in this case also we need to introduce Chan-Paton factors and anti-symmetric with respect to two spin-1 currents to get parity-odd non-homogeneous results. Using the proposal (1.291), we see that for $\langle TJJ\rangle_{nh,odd}$, we get

$$\begin{aligned}\langle JJT\rangle_{nh,o} &= ([O_\epsilon]_2 - [O_\epsilon]_1)\langle JJT\rangle_{nh,e}\\ &= ([O_\epsilon]_2 - [O_\epsilon]_1)\frac{\langle JJT\rangle_{FB} + \langle JJT\rangle_{FF}}{2}\\ &= A(k_1,k_2,k_3)z_3.k_1 z_3.z_2\epsilon(z_1k_1k_3) + B(k_1,k_2,k_3)z_3.z_2\epsilon(z_1k_1z_3)\\ &+ C(k_1,k_2,k_3)z_3.k_1 z_2.k_3\epsilon(z_1k_1z_3) + D(k_1,k_2,k_3)(z_3.k_1)^2\epsilon(z_1k_1z_2)\\ &- (2\leftrightarrow 1)\end{aligned} \tag{1.298}$$

where

$$D = C = -A = \frac{E+k_3}{k_2E^2} \quad B = -\frac{1}{2k_2}\left(-k_2 - k_1 + \frac{2k_3^2}{E}\right). \tag{1.299}$$

This new parity odd CFT correlator has a pole only in the total energy $E$. The ward identity can be obtained using the proposal (1.292)

$$\begin{aligned}\langle JJk_3.T\rangle_{nh,o} &= ([O_\epsilon]_2 - [O_\epsilon]_1)\langle JJk_3.T\rangle_{nh,e}\\ &= -z_1.z_3\epsilon(z_2k_2k_3) + \frac{k_1}{k_2}[(z_1.k_2)\epsilon(z_2k_2z_3) - (z_3.k_2)\epsilon(z_2k_2z_1)] - z_3.k_1\epsilon(z_2k_2z_1)\\ &- (2\leftrightarrow 1)\end{aligned} \tag{1.300}$$

One can confirm that (1.298) and (1.300) are consistent with each other by going to Spinor-Helicity variables and checking conformal ward identity.

In the flat-space limit, we obtain

$$\lim_{E\to 0}\langle JJT\rangle_{nh,o} \sim \frac{1}{E^2}\left(-\left(\epsilon\left(z_2z_3k_2\right)k_3 - \epsilon\left(z_2z_3k_3\right)k_2\right)\left(z_3\cdot z_1\right) + \epsilon\left(z_1z_2z_3\right)k_2\left(z_3\cdot k_1\right)\right) + \mathcal{O}(\frac{1}{E}) \tag{1.301}$$

which matches with minimal odd amplitude in (A.119) and converting it in 4D notation we obtain amplitude given in (A.117). Even though the correlator that has been constructed has a nice behaviour that it has only total energy singularity, it also has some other unusual properties which become clear in position space.

#### 1.10.2.1.1 In position space

In position space, the $\epsilon$-transform is given by

$$[O_\epsilon]_I : \langle O_1(x_1)\cdots O^{\mu_1\cdots\mu_i\cdots\mu_I}(x_I)\cdots O_n(x_n)\rangle \to \epsilon^{\mu_i}{}_{\sigma\alpha}\int \frac{d^3y_I}{|x_I-y_I|^2}\partial^\sigma_{x_I}\langle O_1(x_1)\cdots O^{\mu_1\cdots\alpha\cdots\mu_I}(y_I)\cdots O(x_n)\rangle \tag{1.302}$$

From the above, it is clear that the transformation is not so straightforward in position space, unlike the momentum space. Let us look into the simplest example $\langle TJJ\rangle$. To start with, let us consider the $\epsilon$-transform of ward identity. The ward identity for $\langle TJJ\rangle$ in position space is given by

$$\begin{aligned}\partial_3^\mu\langle J_\rho(x_1)J_\sigma(x_2)T_{\mu\nu}(x_3)\rangle &= \partial_{3\nu}\delta^{(3)}(x_3-x_2)\langle J_\sigma(x_3)J_\rho(x_1)\rangle - \partial_{3\mu}\delta^{(3)}(x_3-x_2)\delta_{\nu\sigma}\langle J_\mu(x_3)J_\rho(x_1)\rangle\\ &+\partial_\nu\delta^{(3)}(x_1-x_3)\langle J_\sigma(x_2)J_\rho(x_3)\rangle - \partial_\mu\delta^{(3)}(x_1-x_3)\delta_{\nu\rho}\langle J_\sigma(x_2)J_\mu(x_1)\rangle\end{aligned} \tag{1.303}$$

Using (1.302) in the above equation, we get

$$\begin{aligned}&\partial_3^\mu\langle J_\rho(x_1)J_\sigma(x_2)T_{\mu\nu}(x_3)\rangle_{nh,o} = ([O_\epsilon]_2-[O_\epsilon]_1)\partial_3^\mu\langle J_\rho(x_1)J_\sigma(x_2)T_{\mu\nu}(x_3)\rangle\\ &= \epsilon_{\sigma\alpha\tau}\partial_{2\nu}\partial_2^\alpha\frac{1}{|x_2-x_3|^2}\langle J^\tau(x_3)J^\rho(x_1)\rangle - \epsilon_{\sigma\alpha\nu}\partial_{2\mu}\partial_2^\alpha\frac{1}{|x_2-x_3|^2}\langle J^\mu(x_3)J^\rho(x_1)\rangle\\ &+\partial_{3\nu}\delta^{(3)}(x_1-x_3)\langle J^\sigma(x_2)J^\rho(x_3)\rangle_o - \partial_{1\mu}\delta^{(3)}(x_1-x_3)\delta_{\nu\rho}\langle J^\sigma(x_2)J^\mu(x_3)\rangle_o\\ &-[(2,\sigma)\leftrightarrow(1,\rho)]\end{aligned} \tag{1.304}$$

In the first line, we made use of the properties of the delta function and we also have made use of

$$\langle J_\mu(x)J_\nu(y)\rangle_{\rm o} = \epsilon_{\mu\sigma\alpha}\int\frac{d^3x_1}{|x-x_1|^2}\partial^\sigma_{x_1}\langle J^\alpha(x_1)\,J_\nu(y)\rangle_{\rm e} \tag{1.305}$$

in the second line. The RHS of (1.304) gives

$$
\begin{aligned}
&\partial_3^\mu \langle J_\rho(x_1) J_\sigma(x_2) T_{\mu\nu}(x_3)\rangle_{nh,o} \\
&= \epsilon_{\sigma\alpha\tau}\left(-\frac{2\delta_\nu^\alpha}{x_{32}^4}+\frac{8x_{32}^\alpha x_{\nu 32}}{x_{32}^6}\right)\left(\frac{\delta_\rho^\tau}{x_{13}^4}-\frac{2x_{13}^\tau x_{13\rho}}{x_{13}^6}\right)-\epsilon_{\sigma\alpha\nu}\left(-\frac{2\delta_\mu^\alpha}{x_{32}^4}+\frac{8x_{32}^\alpha x_{\mu 32}}{x_{32}^6}\right)\left(\frac{\delta_\rho^\mu}{x_{13}^4}-\frac{2x_{13}^\mu x_{13\rho}}{x_{13}^6}\right) \\
&+\partial_{1\nu}\delta^{(3)}(x_1-x_3)\epsilon_{\sigma\rho\tau}\partial_2^\tau\delta^{(3)}(x_2-x_3)-\partial_{1\mu}\delta^{(3)}(x_1-x_3)\delta_{\nu\rho}\epsilon_{\sigma\mu\tau}\partial_2^\tau\delta^{(3)}(x_2-x_3) \\
&-[(2,\sigma)\leftrightarrow(1,\rho)] \\
&\equiv \epsilon_{\sigma\alpha\tau}\left(-\frac{2\delta_\nu^\alpha}{x_{32}^4}+\frac{8x_{32}^\alpha x_{\nu 32}}{x_{32}^6}\right)\left(\frac{\delta_\rho^\tau}{x_{13}^4}-\frac{2x_{13}^\tau x_{13\rho}}{x_{13}^6}\right)-\epsilon_{\sigma\alpha\nu}\left(-\frac{2\delta_\mu^\alpha}{x_{32}^4}+\frac{8x_{32}^\alpha x_{\mu 32}}{x_{32}^6}\right)\left(\frac{\delta_\rho^\mu}{x_{13}^4}-\frac{2x_{13}^\mu x_{13\rho}}{x_{13}^6}\right) \\
&-[(2,\sigma)\leftrightarrow(1,\rho)]
\end{aligned}
\tag{1.306}
$$

where in the last line we have removed contact terms. These contact terms will give rise to a contact term in correlation function, see [186] for similar discussion. However, it is important to note that, unlike in (1.303), no delta function appears in (1.306). This implies that spin-2 current is not conserved even away from coincident points. This implies we can't identify this spin-2 current as a stress tensor. This is as expected as for exactly conserved current we can't have more than three structures.

## WT identity for $\langle JTJ_3\rangle$

The fact that Ward-Takahashi identity is non zero even away from contact points, is a universal fact for non-homogeneous parity-odd terms, which can be checked easily. Consider now the ward identity for $\langle J_1 J_2 J_3\rangle$

$$
\begin{aligned}
&\partial_3^\gamma\langle J_{1\mu}(x_1)T_{\nu\rho}(x_2)J_{3\alpha\beta\gamma}(x_3)\rangle \sim \partial_{3\mu}\delta^{(3)}(x_3-x_1)\langle T_{\alpha\beta}(x_3)T_{\nu\rho}(x_2)\rangle+\partial_{3(\alpha}\delta^{(3)}(x_3-x_1)\langle T_{\beta)\mu}(x_3)T_{\nu\rho}(x_2)\rangle \\
&+(3\partial_{3\alpha}\partial_{3\beta}-\delta_{\alpha\beta}\Box_3)\partial_{3(\nu}\delta^{(3)}(x_3-x_2)\langle J_{\rho)}(x_3)J_{1\mu}(x_1)\rangle+(3\partial_{3\nu}\partial_{3\rho}-\delta_{\nu\rho}\Box_3)\partial_{3(\alpha}\delta^{(3)}(x_3-x_2)\langle J_{\beta)}(x_3)J_{1\mu}(x
\end{aligned}
\tag{1.307}
$$

After an $\epsilon$-transform we get

$$
\begin{aligned}
&[O_\epsilon]_1\partial_3^\gamma\langle J_{1\mu}(x_1)T_{\nu\rho}(x_2)J_{3\alpha\beta\gamma}(x_3)\rangle \sim \partial_{1\mu}\frac{1}{|x_{13}|^2}\langle T_{\alpha\beta}(x_3)T_{\nu\rho}(x_2)\rangle+\partial_{1(\alpha}\frac{1}{|x_{13}|^2}\langle T_{\beta)\mu}(x_3)T_{\nu\rho}(x_2)\rangle \\
&+(3\partial_{3\alpha}\partial_{3\beta}-\delta_{\alpha\beta}\Box_3)\partial_{3(\nu}\delta^{(3)}(x_3-x_2)\langle J_{\rho)}(x_3)J_{1\mu}(x_1)\rangle_o+(3\partial_{3\nu}\partial_{3\rho}-\delta_{\nu\rho}\Box_3)\partial_{3(\alpha}\delta^{(3)}(x_3-x_2)\langle J_{\beta)}(x_3)J_{1\mu}(
\end{aligned}
\tag{1.308}
$$

where

$$\langle J_{\alpha)}(x_1)J_{1\beta}(x_2)\rangle_o = \epsilon_{\alpha\beta\lambda}\partial_1^{\lambda}\delta^{(3)}(x_1 - x_2) \tag{1.309}$$

Just like $\langle JJT\rangle$, we see that an $\epsilon$-transform at $x_1$ gives rise to contact terms in the second line of (1.308) and is therefore dropped. We also get terms that do not vanish at non-coincident points. Hence, we see that the $\epsilon$-transform of conserved current correlations gives rise to correlations that are not conserved. Let us consider another example, all equal spin $\langle TTT\rangle$.

#### 1.10.2.1.2 Example: $\langle TTT\rangle$

Consider now the $\langle TTT\rangle_e$ correlator in momentum space

$$\begin{aligned}\langle T(z_1,k_1)T(z_2,k_2)T(z_3,k_3)\rangle_e &= c_1\frac{k_1k_2k_3}{E^6}[2z_1.k_2z_2.k_3z_3.k_1 + E(k_3z_1.z_2z_3.k_1 + k_1z_2.z_3z_1.k_2 + k_2z_3.z_1z_2.k\\ &+ c_T\left(\frac{k_1k_2k_3}{E^2} + \frac{k_1k_2+k_2k_3+k_3k_1}{E} - E\right)(z_1.z_2z_3.k_1 + z_2.z_3z_1.k_2 + z_3.z_1z_2.k_3)^2\end{aligned} \tag{1.310}$$

which in the flat-space limit gives

$$\begin{aligned}\lim_{E\to 0}\langle T(z_1,k_1)T(z_2,k_2)T(z_3,k_3)\rangle_e &= c_1\frac{k_1k_2k_3}{E^6}[2z_1.k_2z_2.k_3z_3.k_1] + O(\frac{1}{E^5})\\ &+ c_T\frac{k_1k_2k_3}{E^2}(z_1.z_2z_3.k_1 + z_2.z_3z_1.k_2 + z_3.z_1z_2.k_3)^2 + c_TO(\frac{1}{E})\end{aligned} \tag{1.311}$$

which perfectly matches the minimal and non-minimal parity even vertices in (1.286). For parity-odd case [19], we have

$$\begin{aligned}
&\langle T(z_1,k_1)T(z_2,k_2)T(z_3,k_3)\rangle_{h,o} \\
&= (k_1k_2k_3)\,\frac{1}{E^3}\Big[\Big\{\left(\vec{k}_1\cdot\vec{z}_3\right)\left(\epsilon^{k_3z_1z_2}k_1-\epsilon^{k_1z_1z_2}k_3\right)+\left(\vec{k}_3\cdot\vec{z}_2\right)\left(\epsilon^{k_1z_1z_3}k_2-\epsilon^{k_2z_1z_3}k_1\right) \\
&\quad -\left(\vec{z}_2\cdot\vec{z}_3\right)\epsilon^{k_1k_2z_1}E+\frac{k_1}{2}\epsilon^{z_1z_2z_3}E\left(E-2k_1\right)\Big\}+\text{ cyclic perm }\Big] \\
&\times\left[\frac{1}{E^3}\left\{2\left(\vec{z}_1\cdot\vec{k}_2\right)\left(\vec{z}_2\cdot\vec{k}_3\right)\left(\vec{z}_3\cdot\vec{k}_1\right)+E\left\{k_3\left(\vec{z}_1\cdot\vec{z}_2\right)\left(\vec{z}_3\cdot\vec{k}_1\right)+\text{ cyclic }\right\}\right\}\right] \qquad (1.313)
\end{aligned}$$

In the flat-space limit, which becomes

$$\lim_{E\to 0}\langle TTT\rangle_{h,o}=\frac{k_1k_2k_3}{E^6}\left[\left(\vec{k}_1\cdot\vec{z}_3\right)\left(\epsilon^{k_3z_1z_2}k_1-\epsilon^{k_1z_1z_2}k_3\right)+\text{ cyclic perm }\right]\left(\vec{z}_1\cdot\vec{k}_2\right)\left(\vec{z}_2\cdot\vec{k}_3\right)\left(\vec{z}_3\cdot\vec{k}_1\right) \qquad (1.314)$$

which is precisely the non-minimal parity-odd cubic vertex mentioned in (1.286). Since the three-point function of conserved currents at maximum can only have three structures, we see that just like cubic vertex we do not have any analogue of parity odd-minimal amplitude at the level CFT correlation function. However, let us define another parity-odd structure namely

$$\begin{aligned}
\langle TTT\rangle'_o &= ([O_\epsilon]_1+[O_\epsilon]_2+[O_\epsilon]_3)\langle TTT\rangle_{nh,e} \\
&= \frac{E^3-E\left(k_1k_2+k_2k_3+k_3k_1\right)-k_1k_2k_3}{E^2k_1k_2k_3}(z_1.k_2z_2.z_3+z_1.z_2z_3.k_1+z_2.k_3z_3.z_1) \\
&[k_2k_3z_2.z_3\epsilon(z_1k_1k_2)+\text{cyclic terms}-k_1k_2k_3E\epsilon(z_1z_2z_3)] \qquad (1.315)
\end{aligned}$$

which in the flat-space limit gives

$$\lim_{E\to 0}\langle TTT\rangle'_o\sim-\frac{c_{123}}{E^2}[V']^{222}+\mathcal{O}(\frac{1}{E}) \qquad (1.316)$$

[19] Following [186], one can write parity-odd non-homogeneous piece as follows

$$\begin{aligned}
\langle TTT\rangle_{nh,o} &= \frac{1}{24}[\epsilon(z_1z_2k_1)(z_1.z_2)(z_3.k_1)^2-\epsilon(z_1z_2k_2)(z_1.z_3)(z_2.z_3)] \\
&+\frac{1}{12}[(z_1.z_3)(z_2.z_3)\epsilon(z_1z_2k_1)(k_1^2+\frac{7}{4}k_2^2+\frac{7}{4}k_3^2)-(z_1.z_2)(z_3.k_1)^2\epsilon(z_1z_2k_2)(k_2^2+\frac{7}{4}k_1^2+\frac{7}{4}k_3^2)]+\text{cyclic terms} \qquad (1.312)
\end{aligned}$$

However, one can see that $\langle TTT\rangle_{nh,o}$ is just a contact term. This does not correspond to any cubic vertex.

which is precisely what we computed in (1.287). As mentioned before, this flat-space limit cannot be recast as a $4D$ flat-space amplitude. It is easy to show that, spin two current that appears in (1.315) is not conserved. To show this we work in position space.

#### 1.10.2.1.3 $\epsilon$-transform WT identity of $\langle TTT\rangle$ in position space

Consider the $\langle TTT\rangle$ ward identity in position space

$$\partial^\mu\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}\rangle = \partial_\nu\delta^{(3)}(x-y)\langle T_{\sigma\rho}(x)T_{\alpha\beta}(z)\rangle + \Big\{\partial_\sigma(\delta^{(3)}(x-y)\langle T_{\rho\nu}(x)T_{\alpha\beta}(z)\rangle) + \sigma\leftrightarrow\rho\Big\}$$
$$+\,\partial_\nu\delta^{(3)}(x-z)\langle T_{\sigma\rho}(y)T_{\alpha\beta}(x)\rangle + \Big\{\partial_\alpha(\delta^{(3)}(x-z)\langle T_{\beta\nu}(x)T_{\sigma\rho}(y)\rangle) + \alpha\leftrightarrow\beta\Big\} \tag{1.317}$$

where we now perform an $\epsilon$-transform and just like for the case of $\langle JJT\rangle$ we find that

$$\begin{aligned}
&[O_\epsilon]_y\partial^\mu\langle T_{\mu\nu}(x)T_{\sigma\rho}(y)T_{\alpha\beta}\rangle\\
&\sim \epsilon_{\sigma\eta\zeta}\partial^\zeta\partial_\nu\frac{1}{|x-y|^2}\langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle + \partial_\rho(\epsilon_{\sigma\eta\zeta}\partial^\zeta\frac{1}{|x-y|^2}\langle T_{\eta\nu}(x)T_{\alpha\beta}(z)\rangle)\\
&+\epsilon_{\sigma\eta\zeta}\partial^\zeta\frac{1}{|x-y|^2}\partial_\eta\langle T_{\rho\nu}(x)T_{\alpha\beta}(z)\rangle + +\epsilon_{\sigma\eta\zeta}\partial^\zeta\frac{1}{|x-y|^2}\partial_\rho\langle T_{\eta\nu}(x)T_{\alpha\beta}(z)\rangle\\
&+\partial_\nu\delta^{(3)}(x-z)\langle T_{\sigma\rho}(y)T_{\alpha\beta}(x)\rangle_{odd} + \Big\{\partial_\alpha(\delta^{(3)}(x-z)\langle T_{\beta\nu}(x)T_{\sigma\rho}(y)\rangle_{odd}) + \alpha\leftrightarrow\beta\Big\}\\
&\sim \epsilon_{\sigma\eta\zeta}\partial^\zeta\partial_\nu\frac{1}{|x-y|^2}\langle T_{\eta\rho}(x)T_{\alpha\beta}(z)\rangle + \partial_\rho(\epsilon_{\sigma\eta\zeta}\partial^\zeta\frac{1}{|x-y|^2}\langle T_{\eta\nu}(x)T_{\alpha\beta}(z)\rangle)\\
&+\epsilon_{\sigma\eta\zeta}\partial^\zeta\frac{1}{|x-y|^2}\partial_\eta\langle T_{\rho\nu}(x)T_{\alpha\beta}(z)\rangle + \epsilon_{\sigma\eta\zeta}\partial^\zeta\frac{1}{|x-y|^2}\partial_\rho\langle T_{\eta\nu}(x)T_{\alpha\beta}(z)\rangle
\end{aligned} \tag{1.318}$$

Again, the Ward identity has terms that survive at non-coincident points and therefore, show that the $\epsilon$-transform leads to a non-conserved spin-2 current.

## 1.11 Summary and Discussion

To summarize, we have systematically solved for 3-point CFT correlators involving higher spin conserved currents and scalar operators in three dimensions. Spinor-helicity formalism simplifies considerably the CWI-based analysis of correlators. It solves the problems associated with degeneracy which makes direct computation in momentum space difficult. In these variables, we found that the homogeneous part of the correlator gets an identical contribution from the parity-even and parity-odd parts. We were also able to write down momentum space correlators in terms of conserved conformally invariant structures. For some correlators which are divergent in momentum space, a careful renormalization analysis is required. However, in spinor-helicity variables, we observed that it turns out we directly get the finite part of the correlator which does not require any renormalization. We also verified some of the results using weight-shifting operators. We also established various double copy relations for parity-violating $CFT_3$ momentum space 3-point correlators. The double copy structure is a very special property of CFT correlators in momentum space, the analogue of which does not exist in position space. To understand this structure, we divided the momentum space CFT correlation function into two parts, which we called homogeneous and non-homogeneous pieces. It was crucial for our analysis that the homogeneous part consists of two pieces of conformally invariant structures, namely one parity-even and one parity-odd structure whereas the non-homogeneous part has only one parity-even conformally invariant piece - all other contributions are contact terms. Squaring the homogeneous piece could in principle generate three structures. However, interestingly it turns out that squaring the parity-odd and even part produces exactly the same structure, whereas the cross-term which is generated by multiplying the parity-odd and parity-even part gives rise to the needed parity-odd structure. It was also shown that in the flat-space limit, the three-point correlators reproduce the flat-space three-point amplitudes in $4D$. The flat-space limits of the double copy relations also reproduce the double copy relations known for flat-space three-point amplitudes in $4D$. This is very suggestive of a correspondence that may exist between correlators and amplitudes. We focused in particular on the connection between the three-point $CFT_3$ correlation function of conserved currents with flat space three-point amplitude of massless gauge fields. In particular, we show the following map

$$\begin{aligned}\langle J_{s_1}J_{s_2}J_{s_3}\rangle_{nh,e} &\rightarrow M^{s_1s_2s_3}_{m,e}\\ \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{h,e} &\rightarrow M^{s_1s_2s_3}_{nm,e}\\ \langle J_{s_1}J_{s_2}J_{s_3}\rangle_{h,o} &\rightarrow M^{s_1s_2s_3}_{nm,o}.\end{aligned} \tag{1.319}$$

This map indicates that the number of $CFT$ correlation functions of conserved currents are always less than the number of allowed structures for flat space amplitude. To generate the missing parity-odd $CFT_3$ structure, we demonstrated the existence of the $\epsilon$-transform that exists in the flat-space amplitudes. This $\epsilon$-transform relates parity-even and parity-odd amplitudes in momentum space. We simply generated the extra parity-odd $CFT$

structure using this transform. However, we showed that this extra parity odd $CFT$ correlator can not be constructed out of conserved currents. Interestingly, this extra $CFT$ correlator is consistent with the position space OPE limit. One of the obvious future directions one can pursue is solving the momentum space four-point conformal ward identities. It is already well-known that conformal blocks for scalar four-point functions are given by the product of momentum space three-point functions as in . However, a simpler expression in terms of momentum space structures is desired to demonstrate double copy for the spinning four-point correlations. This will be the first step toward establishing an amplitude/correlator correspondence for the four-point function. Such a correspondence can shed some light on the parallels between the S-matrix and the $CFT$ bootstrap.

# Chapter 2

# Chern-Simons Matter Theories

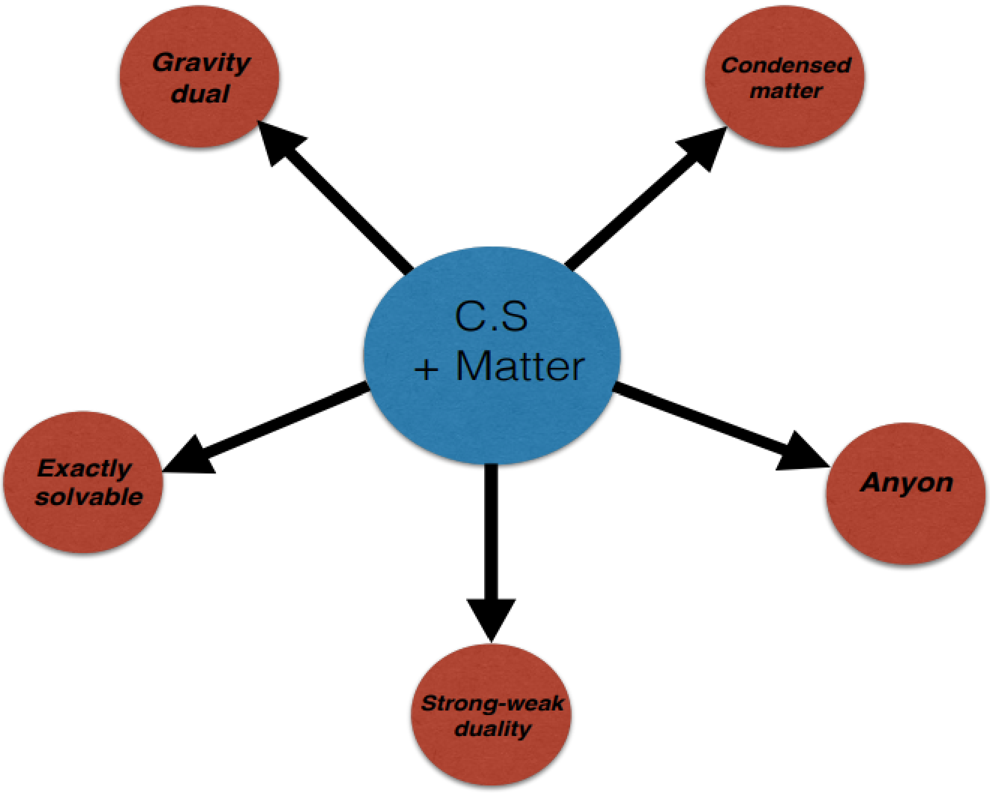
This chapter is largely based on the following papers written with my collaborators

- S. Jain, V. Malvimat, A. Mehta, S. Prakash, N. Sudhir (2020) *All order exact result for the anomalous dimension of the scalar primary in Chern-Simons vector models*, Phys. Rev. D 101 (2020) 12, 126017, arXiv: 1906.06342 [hep-th]

- K. Inbasekar, S. Jain, V. Malvimat, A. Mehta, P. Nayak (2020) *Correlation functions in $\mathcal{N}=2$ Supersymmetric vector matter Chern-Simons theory*, JHEP 04 (2020) 207, arXiv: 1907.11722 [hep-th]

Three-dimensional free fermion and free boson theories are some of the simplest CFTs. One of the examples of the interacting non-supersymmetric theory is boson or fermion coupled to the Chern-Simons gauge field. If the matter is in fundamental representation, these theories are called Chern-Simons matter theories [70–155]. The supersymmetric extensions of these theories exist [76,93,122,158–161]. In this chapter, we will be making use of some techniques of conformal bootstrap, namely the Inversion Formula [207] to compute the four-point function of scalar multiplets in $\mathcal{N}=2$ supersymmetric Chern-Simons theories. This was first done for quasi-fermionic and quasi-bosonic theories in [208]. One of the most well-known examples of Chern-Simons matter theory in $\mathcal{N}=6$ ABJM theory where the matter is adjoint [158]. In this chapter, we will focus on the matter in fundamental representation. Chern-Simons matter theories are interesting for various reasons, see Figure 2.1. Chern-Simons matter theories are one of the simplest CFTs that are exactly solvable in the large-$N$ limit and there exists a lot of dualities amongst various Chern-Simons theories [71–74, 77]. They also provide the simplest example for

**Figure 2.1:** Properties of Chern-Simons matter theories




non-SUSY gauge/gravity duality and field theory dualities [72]. They have a higher-spin gravity dual called Vasiliev-type theories and have applications in condensed matter physics where they are useful in understanding the Quantum Hall Effect [104, 156, 157]. Another important property of these theories is that they display anyonic behaviour [209]. Free bosonic and fermionic field theories along with the conformal symmetry enjoy higher-spin symmetry which gets slightly broken when coupled to the Chern-Simons gauge field. This also causes the operators in these theories to acquire anomalous dimensions which can be easily computed using the slightly-broken higher-spin symmetry for $s > 2$ [109]. However, the anomalous dimension for the scalar operator cannot be ascertained using this method and hence, remains unknown. In this chapter, we conjecture an all-loop anomalous dimension of the scalar operator and justify it using loop calculations and other checks. There are five types of Chern-Simons matter theories that are of interest in this chapter. They are explicitly as follows

- The $\mathcal{N} = 2$ supersymmetric (S) Chern-Simons-matter theory with a single chiral multiplet in the fundamental representation:

$$\mathcal{S}^{S}_{\mathcal{N}=2}(\phi,\psi) = \int d^3x \left[ i\varepsilon^{\mu\nu\rho}\frac{\kappa}{4\pi}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + D_\mu\bar{\phi}D^\mu\phi + \bar{\psi}\gamma^\mu D_\mu\psi \right.$$
$$\left. + \frac{4\pi^2}{\kappa^2}(\bar{\phi}\phi)^3 + \frac{4\pi}{\kappa}(\bar{\psi}\psi)(\bar{\phi}\phi) + \frac{2\pi}{\kappa}(\bar{\psi}\phi)(\bar{\phi}\psi)\right]. \tag{2.1}$$

- The critical bosonic (CB) theory

$$S_{CB}(\phi,\sigma_B) = \int d^3x \left[ i\varepsilon^{\mu\nu\rho}\frac{\kappa_B}{4\pi}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + D_\mu\bar{\phi}D^\mu\phi + \sigma_B\bar{\phi}\phi\right] \tag{2.2}$$

- The regular fermion (RF) theory

$$S_{RF}(\psi) = \int d^3x \left[ i\varepsilon^{\mu\nu\rho}\frac{\kappa_F}{4\pi}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + \bar{\psi}\gamma_\mu D^\mu\psi\right] \tag{2.3}$$

- The regular boson (RB) theory

$$S_{RB}(\phi) = \int d^3x \left[ i\varepsilon^{\mu\nu\rho}\frac{\kappa_B}{4\pi}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + D_\mu\bar{\phi}D^\mu\phi + \frac{(2\pi)^2}{\kappa_B^2}\left(x_6^B+1\right)(\bar{\phi}\phi)^3\right] \tag{2.4}$$

- The critical fermion (CF) theory

$$S_{CF}(\psi,\sigma_F) = \int d^3x \left[ i\varepsilon^{\mu\nu\rho}\frac{\kappa_F}{4\pi}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + \bar{\psi}\gamma_\mu D^\mu\psi - \frac{4\pi}{\kappa_F}\zeta\bar{\psi}\psi + \frac{(2\pi)^2}{\kappa_F^2}x_6^F\zeta^3\right] \tag{2.5}$$

The RF and CB theories are together called 'quasi-fermionic' theories while the RB and CF theories were referred to as 'quasi-bosonic' theories. We will employ this nomenclature in the rest of this chapter. Let us carefully review the theories under study and their relations via RG flow and bosonization duality. The quasi-bosonic family of theories flows to the quasi-fermionic family of theories under RG flow. In [210], the quasi-bosonic family is described by three parameters[1] $\tilde{\lambda}_{QB}$, $\tilde{N}_{QB}$ and $\tilde{\lambda}_{6,QB}$; and the quasi-fermionic

[1]The analysis of [210] is valid for theories with only even spins, e.g. $O(N)$ vector models. For the $U(N)$ vector models we study here, the analysis of [210] has not been carried out, and there may be an

family is described by two parameters $\tilde{\lambda}_{QF}$ and $\tilde{N}_{QF}$. The parameter $\tilde{N}$ is defined via the two-point function of the stress-energy tensor, and is a measure of the number of degrees of freedom of each theory – we will only be interested in the large $\tilde{N}$ limit and the first non-trivial $1/\tilde{N}$ corrections. In this limit, the spectrum is independent of the parameter $\tilde{\lambda}_{6,QB}$ so we will ignore it in the discussion that follows.

The celebrated bosonization duality states that each family of theories has two very different-looking descriptions. The quasi-bosonic family can be described as a theory of $N_b$ complex bosons transforming in the fundamental representation of $U(N_b)$, coupled to a level-$\kappa_b$ Chern-Simons gauge field. It can also be described as a theory of $N_f$ Dirac "critical" fermions, in the fundamental representation of $U(N_f)$ coupled to a level $\kappa_f$ Chern-Simons gauge field. The quasi-fermionic family can be described as a theory of $N_b$ critical complex bosons transforming in the fundamental representation of $U(N_b)$, coupled to a level-$\kappa_b$ Chern-Simons gauge field. It can also be described as a theory of $N_f$ Dirac fermions, in the fundamental representation of $U(N_f)$ coupled to a level $\kappa_f$ Chern-Simons gauge field. This duality is well-tested in the large $N_{b/f}$ limit, with $\lambda_{b/f} \equiv \frac{N_{b/f}}{k_{b/f}}$ held fixed. In this limit we have the following relation between the parameters:

$$\begin{aligned}
\tilde{N}_{QB} &= 2N_b \frac{\sin(\pi\lambda_b)}{\pi\lambda_b} = 2N_f \frac{\sin(\pi\lambda_f)}{\pi\lambda_f} \\
\tilde{N}_{QF} &= 2N_b \frac{\sin(\pi\lambda_b)}{\pi\lambda_b} = 2N_f \frac{\sin(\pi\lambda_f)}{\pi\lambda_f} \\
\tilde{\lambda}_{QB} &= \tan\left(\frac{\pi\lambda_b}{2}\right) = \cot\left(\frac{\pi\lambda_f}{2}\right) \\
\tilde{\lambda}_{QF} &= \cot\left(\frac{\pi\lambda_b}{2}\right) = \tan\left(\frac{\pi\lambda_f}{2}\right)
\end{aligned} \tag{2.6}$$

Because $N_{b/f}$ and $\kappa_{b/f}$ are integers (or half-integers), the parameters $\lambda_{b/f}$ and $N_{b/f}$ do not run under RG flow from quasi-bosonic theory to quasi-fermionic theory. Under RG flow, the quasi-bosonic theory defined by $\tilde{\lambda}_{QB}$ and $\tilde{N}_{QB}$ flows to the quasi-fermionic theory described by:

$$\tilde{\lambda}_{QF} = \frac{1}{\tilde{\lambda}_{QB}} \tag{2.7}$$

$$\tilde{N}_{QF} = \tilde{N}_{QB}. \tag{2.8}$$

additional parameter, corresponding to the strength of an additional Chern-Simons $U(1)$ Chern-Simons field that could couple to the spin-1 conserved current, which we assume is turned off here.

We henceforth use $\tilde{N}$ without any subscript.

## 2.1 Chapter summary

This chapter is structured as follows. In section 2.2, the problem with computing the anomalous dimension of scalar operator is explained along with our two-loop correction computed for the critical boson and regular boson theories in 2.30 and 2.33. The details of the Feynman integrals are also given. In section 2.3, the all loop conjecture is presented for quasi-fermion and quasi-boson, respectively

$$\tilde{\gamma}_0 = -\frac{32}{3\pi^2}\frac{\tilde{\lambda}_{QF}^2}{1+\tilde{\lambda}_{QF}^2}\frac{1}{\tilde{N}} \quad \gamma_0 = -\frac{32}{3\pi^2}\frac{\tilde{\lambda}_{QB}^2}{1+\tilde{\lambda}_{QB}^2}\frac{1}{\tilde{N}}. \tag{2.9}$$

The conjecture is consistent with the two-loop computations and other all loops check which are discussed in successive sections. This discussion is taken from [144]. In section 2.5, we review the $\mathcal{N}=2$ theory and its operator spectrum in some detail. In section 2.6, we determine the scalar multiplet 2 and 3-point functions via a direct computation. In section 2.7, we determine the 4-point function of the bosonic and the fermionic scalar operators in this theory using the double discontinuity technique developed in [208]. One may refer to [145] for more details. We show that

$$\langle J_0^b(x_1)J_0^b(x_2)J_0^b(x_3)J_0^b(x_4)\rangle = disc + \tfrac{1}{x_{13}^2x_{24}^2}F(u,v) \tag{2.10}$$

$$\langle J_0^f(x_1)J_0^f(x_2)J_0^f(x_3)J_0^f(x_4)\rangle = disc + \tfrac{1}{x_{13}^4x_{24}^4}\mathcal{G}(u,v) \tag{2.11}$$

where

$$F(u,v) = \frac{1+\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2}f_{fb}(u,v) - \frac{8}{\tilde{N}}\frac{2\tilde{\lambda}^2}{\pi^{5/2}(1+\tilde{\lambda}^2)^2}\Big[\bar{D}_{11\frac{1}{2}\frac{1}{2}}(u,v) + \bar{D}_{11\frac{1}{2}\frac{1}{2}}(v,u) + \frac{1}{u}\bar{D}_{11\frac{1}{2}\frac{1}{2}}(\frac{1}{u},\frac{v}{u})\Big]$$
$$+ a_1\bar{D}_{1111}(u,v) + \tilde{c}_1G^{AdS}_{\phi^4} + c_2G^{AdS}_{(\partial\phi)^4} + c_3G^{AdS}_{\phi^2(\partial^3\phi)^2} \tag{2.12}$$

$$\mathcal{G}(u,v) = \frac{1+\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2}f_{ff}(u,v) + \bar{c}_1G^{AdS}_{\phi^4} + \bar{c}_2G^{AdS}_{(\partial\phi)^4} + \bar{c}_3G^{AdS}_{\phi^2(\partial^3\phi)^2} \tag{2.13}$$

Finally, in section 2.8, we summarize our results and outline related open questions and future directions. In various appendices, we collect our notation and conventions, some technical details of the results in the main text of the paper and briefly summarize our attempt at the direct computation of the 4-point function.

## 2.2 Anomalous dimension of operators with spin $s > 0$

Due to interactions in the Chern-Simons matter theories, the higher-spin symmetry is broken and the infinite tower of higher-spin conserved currents becomes non-conserved. These currents then obey a non-conservation equation of the form

$$\partial \cdot J_s = \mathcal{K}_{s-1} \tag{2.14}$$

where $\mathcal{K}_{s-1}$ is a multitrace primary operator. Assuming that the scaling dimension of $J_s$ is given by $s + 1 + \gamma_s$, one can then look at

$$\langle \partial \cdot J_s(x, z_1) \partial \cdot J_s(0, z_2) \rangle = \langle \mathcal{K}_{s-1}(x, z_1) \mathcal{K}_{s-1}(0, z_2) \rangle \tag{2.15}$$

By equating the divergence both sides one obtains the following formula for the anomalous dimension

$$\gamma_s = -\frac{1}{s^2 \left(s^2 - \frac{1}{4}\right)} \frac{(z \cdot x)^2 \langle \mathcal{K}_{s-1}(x, z) \mathcal{K}_{s-1}(0, z) \rangle_0}{\langle j_s(x, z) j_s(0, z) \rangle_0} \tag{2.16}$$

where the subscript '0' stands for the free theory correlation. This formula eventually leads to the expressions in 2.37 taken from [109]. Notice how this method cannot be used for computing anomalous dimension of scalar operators as a non-conversation equation for $s = 0$ doesn't make sense. Therefore, in the coming sections, we will try to conjecture the anomalous dimension of scalar operators and justify it via loop calculations and other checks. Let us denote the scaling dimension of the scalar primary $j_0$ in the quasi-bosonic theory as $\Delta_0$, and the scaling dimension of $\tilde{j}_0$ in the QF theory as $\tilde{\Delta}_0$. We define the anomalous dimension as:

$$\Delta_0 = 1 + \gamma_0, \quad \tilde{\Delta}_0 = 2 + \tilde{\gamma}_0 \tag{2.17}$$

### 2.2.1 Anomalous Dimension of $j_0$ at Two Loops

In this subsection, we calculate the anomalous dimension of the operator $j_0$ at two loops. The diagrams which we need to evaluate are given in figure 2.2,2.3,2.4. The Feynmann rules for these diagrams are listed in Appendix B.6 The logarithmic divergences arising

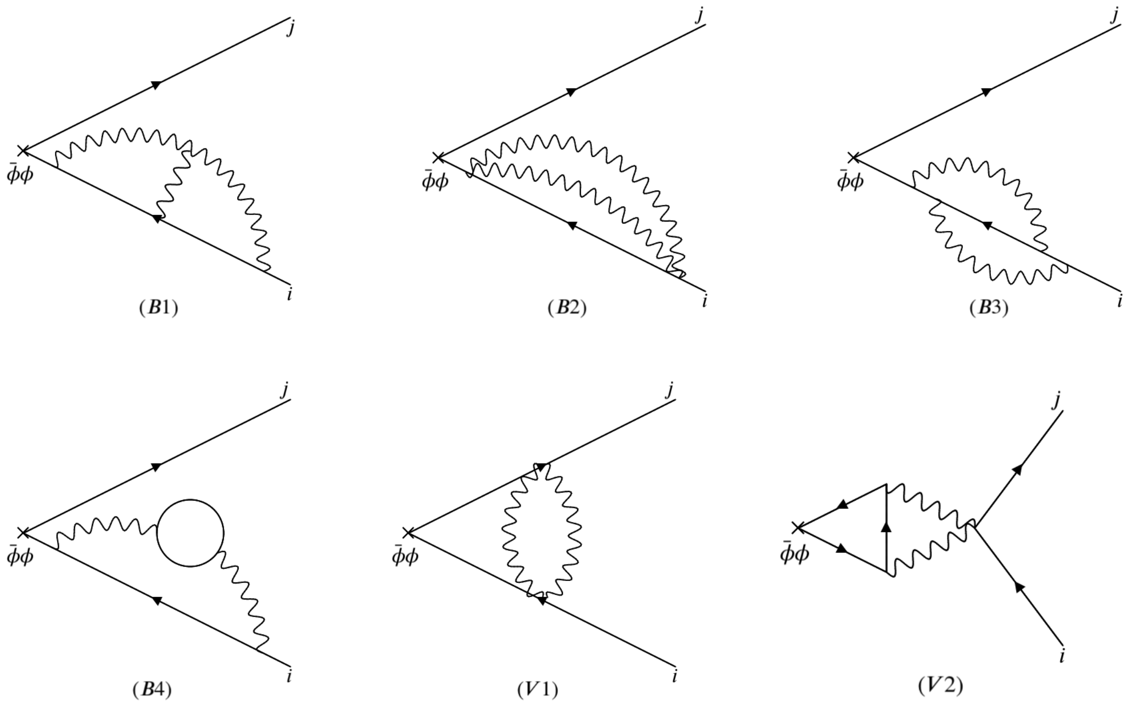



**Figure 2.2:** Diagrams $(B1)$-$(B4)$ depict loop corrections to the propagator. Diagrams (V1) and (V2) are loop corrections to the vertex.

due to the loop correction of the propagators depicted in the diagrams $(B1)$-$(B4)$ of the figure 2.2 are given by

$$(B1) = \frac{2}{3k^2} C_2 C_3 \ \log[\Lambda] = \frac{1-N^2}{3k^2} \ \log[\Lambda] \tag{2.18}$$

$$(B2) = \frac{2}{3k^2}(C_3^2 + \frac{C_2 C_3}{4}) \ \log[\Lambda] = \frac{N^4 - 3N^2 + 2}{12k^2 N^2} \log[\Lambda] \tag{2.19}$$

$$(B3) = \frac{8}{3k^2}(C_3^2 + \frac{C_2 C_3}{2}) \log[\Lambda] = \frac{2}{3k^2}(\frac{1}{N^2} - 1) \log[\Lambda] \tag{2.20}$$

$$(B4) = \frac{4}{3k^2} C_1 C_3 \log[\Lambda] = \frac{1}{3k^2}\left(N - \frac{1}{N}\right) \log[\Lambda] \tag{2.21}$$

The logarithmic divergences arising from the corrections to the vertex depicted in figure 2.2 are given by

$$(V1) = \frac{4}{k^2}(C_3^2 + \frac{C_2 C_3}{4}) \ \log[\Lambda] = \frac{N^4 - 3N^2 + 2}{2 \ k^2 N^2} \log[\Lambda] \tag{2.22}$$

$$(V2) = \frac{8}{k^2} C_1 C_3 \ \log[\Lambda] = \frac{2}{k^2}\left(N - \frac{1}{N}\right) \log[\Lambda]. \tag{2.23}$$

Following [86], we use these results, to compute the $\mathcal{O}(\frac{1}{N})$ logarithmic divergence of the two-point function $\langle j_0 j_0 \rangle$ to be:

$$2(B1 + B2 + B3 + B4) + V1 + V2 = \frac{8}{3}\frac{\lambda^2}{N}\log[\Lambda] + O(\frac{1}{N^2}), \tag{2.24}$$

where we have used re-expressed the result in terms of $\lambda \equiv \frac{N}{k}$. Note that the $U(N)$ result is just the larg-$N$ limit of the $SU(N)$ result. Note that the loop corrections to the propagator should be taken on each of the two legs of the vertex diagrams ($B1$)-($B4$) depicted in figure 2.2 and hence they contribute twice to the two-point function. The two-point correlation function of the scalars in a $d$-dimensional CFT in momentum space is given by

$$\langle j_0(p) j_0(0) \rangle = \frac{c}{p^{2\Delta - d}}. \tag{2.25}$$

Now we briefly describe how to obtain the anomalous dimension of operator $j_0$ from two point function of the same operator.The two-point correlation function of the scalars in a $d$-dimensional CFT in momentum space is given by

$$\langle j_0(p) j_0(0) \rangle = \frac{c}{p^{2\Delta - d}}. \tag{2.26}$$

The scaling dimension $\Delta$ can be expressed in $\frac{1}{N}$ expansion as

$$\Delta = \Delta_0 + \frac{\gamma_0}{N} + \mathcal{O}(\frac{1}{N^2}) \tag{2.27}$$

where $\Delta_0$ is classical scaling dimension, $\gamma_0$ is anomalous dimension to order $\frac{1}{N}$. Plugging (2.27) in (2.26) and expanding ot leading order we obtain

$$\langle j_0(p) j_0(0) \rangle \quad = \quad \frac{1}{p^{2\Delta_0 - d}}(1 - 2\frac{\gamma_0}{N}\log p)] \tag{2.28}$$

Hence the anomalous dimension is given by $-1/2$ times the logarithmic divergence we obtained earlier. Keeping corrections in the anomalous dimension upto $\mathcal{O}[\frac{1}{N}]$, this leads us to the following expression for the anomalous dimension at $\mathcal{O}[\lambda^2]$

$$\gamma_0 = -\frac{4}{3}\lambda^2. \tag{2.29}$$

Note that expression obtained for anomalous dimension in Eq.(2.29) is same as that we obtain from perturbative expansion of our conjectured answer in (2.49).

## UV Finite diagrams

Apart from the diagrams depicted in fig.2.2 there are other two-loop diagrams that do not contribute to the anomalous dimension at order $1/N$. They are depicted in Figure 2.3.

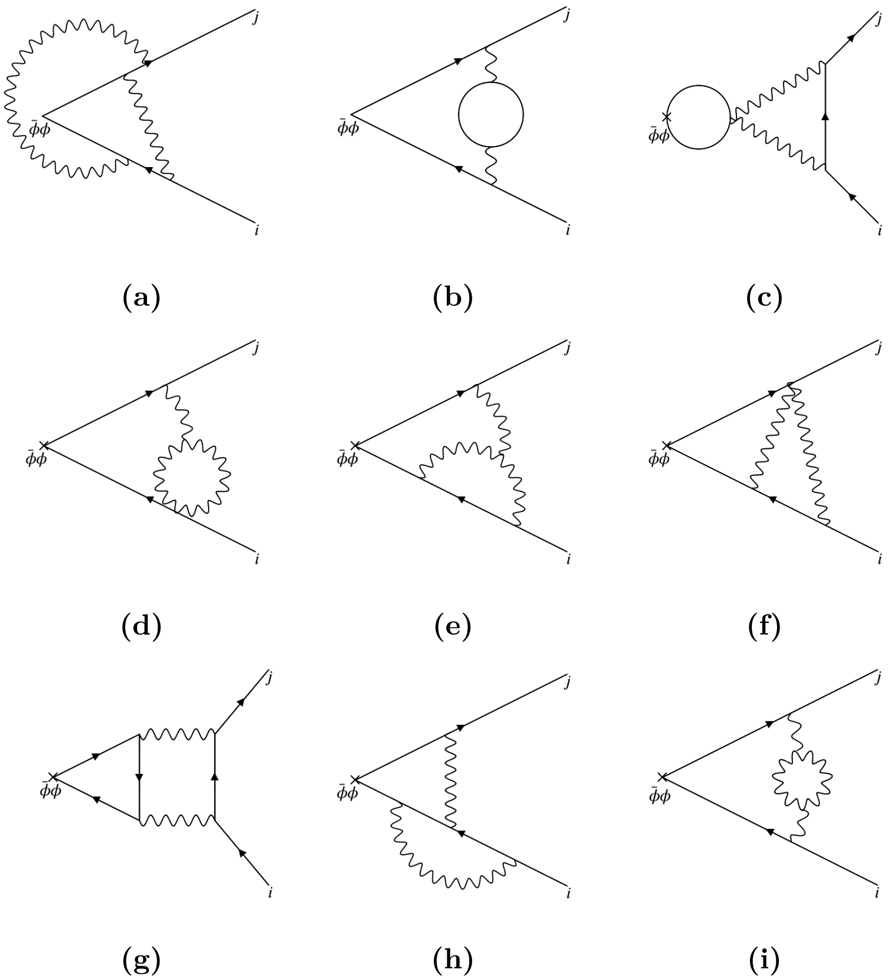


**Figure 2.3:** All the diagrams appearing in this figure do not contribute to the anomalous dimension at $1/N$.

### 2.2.2 Quasi-Fermionic Theory

The leading order $1/N$ anomalous dimension for the critical bosonic theory appears in [211] (see also [212, 213]). From the calculation of the order-$\lambda_b^2$ correction carried out above, we get

$$\tilde{\gamma}_0 = \frac{1}{N_b}\left(-\frac{16}{3\pi^2} + \frac{4}{9}\lambda_b^2 + O(\lambda_b^4)\right). \tag{2.30}$$

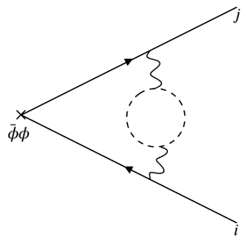


**Figure 2.4:** Diagram with a ghost loop which cancels diagram $(i)$.

To order $\lambda_f^2$, the anomalous dimension in the regular fermionic theory appears in [86,109]:

$$\tilde{\gamma}_0 = -\frac{4}{3}\lambda_f^2 \frac{1}{N_f} + O(\lambda_f^4). \tag{2.31}$$

### 2.2.3 Quasi-Bosonic Theory

The order $\lambda_f^2$ correction anomalous dimension in the critical fermionic theory can be calculated following [109, 214] to be:

$$\gamma_0 = \frac{1}{N_f}\left(-\frac{16}{3\pi^2} + \frac{4}{9}\lambda_f^2 + O(\lambda_f^4)\right). \tag{2.32}$$

To order $\lambda_b^2$, from the calculation carried out above the anomalous dimension in the regular bosonic theory is

$$\gamma_0 = -\frac{4}{3}\lambda_b^2 \frac{1}{N_b} + O(\lambda_b^4). \tag{2.33}$$

The contributing two-loop Feynmann diagrams are given in 2.2.

### 2.2.4 Relation between the critical and non-critical theories

Equation (3.24) of [137], derives the following relation between $\tilde{\gamma}_0$ and $\gamma_0$:

$$\tilde{\gamma}_0 + \gamma_0 = -\frac{16\lambda_b}{3\pi \sin \pi\lambda_b}\frac{1}{N_b} = -\frac{16\lambda_f}{3\pi \sin \pi\lambda_f}\frac{1}{N_f} = -\frac{32}{3\pi^2}\frac{1}{\tilde{N}}, \tag{2.34}$$

Equation (2.34) is explicitly satisfied by the two-loop results above. However, it is a non-trivial constraint that must be satisfied to all orders in $\lambda$ by our conjecture below.

### 2.2.5 Higher-Spin Results

The $1/N$ higher-spin spectrum for the quasi-fermionic theory, which is known to all orders in $\tilde{\lambda}_{QF}$, is given by [109] :

$$\gamma_s^{QF} = \frac{1}{\tilde{N}}\left(a_s^{QF}\frac{\tilde{\lambda}_{QF}^2}{1+\tilde{\lambda}_{QF}^2} + b_s^{\text{QB}}\frac{\tilde{\lambda}_{QF}^2}{(1+\tilde{\lambda}_{QF}^2)^2}\right) + O(\frac{1}{N^2})\,. \tag{2.35}$$

Here $\gamma_s^{QF} = \Delta_s - (s+1)$ is the anomalous dimension of the spin-$s$ primary. A similar expression holds for the quasi-bosonic theory. The expressions for the spin-dependent constants turn out to be identical[2] for both the quasi-bosonic and quasi-fermionic theories and is:

$$a_s = \begin{cases} \frac{16}{3\pi^2}\frac{s-2}{2s-1}\,, & \text{for even } s\,, \\ \frac{32}{3\pi^2}\frac{s^2-1}{4s^2-1}\,, & \text{for odd } s\,, \end{cases} \tag{2.36}$$

$$b_s = \begin{cases} \frac{2}{3\pi^2}\left(3\sum_{n=1}^{s}\frac{1}{n-1/2} + \frac{-38s^4+24s^3+34s^2-24s-32}{4s^4-5s^2+1}\right)\,, & \text{for even } s\,, \\ \frac{2}{3\pi^2}\left(3\sum_{n=1}^{s}\frac{1}{n-1/2} + \frac{20-38s^2}{4s^2-1}\right)\,, & \text{for odd } s\,. \end{cases} \tag{2.37}$$

While this result does not apply to the case of spin-0, it serves as an inspiration for our conjecture. One might be tempted to "analytically continue" the expressions for $a_s$ and $b_s$ in [109], to $s = 0$, using

$$\sum_{n=1}^{s}\frac{1}{n-1/2} = \gamma - \psi(s) + 2\psi(2s) = H_{s-1/2} + 2\ln 2 \tag{2.38}$$

resulting in

$$a_0^{AC} \to \frac{32}{3\pi^2},\quad b_0^{AC} \to -\frac{64}{3\pi^2}. \tag{2.39}$$

While this so-called "analytic continuation" gives the correct answer for the value of $\gamma_0$ obtained from the two-loop calculation in the regular fermionic (bosonic) theory, it leads to an incorrect prediction for $\gamma_0$ and $\tilde{\gamma}_0$ in the critical bosonic (fermionic) theories at $\tilde{\lambda} \to \infty$.

[2] This is an unexplained coincidence at present, and is not true at order $1/N^2$, as can be seen from [215, 216].

## 2.3 Our conjecture

Here, we conjecture that the anomalous dimension of the scalar still takes the form given by equation (2.35) for $s = 0$, and we attempt to determine constants $a_0$ and $b_0$ that satisfy the results listed in section 2.2.

### 2.3.1 Conjecture in Quasi-Fermionic theory: The $\tilde{\gamma}_0$

We can determine $a_0$ and $b_0$ in the quasi-fermionic theory, by first expanding around $\tilde{\lambda}_{QF} = \infty$:

$$\tilde{\gamma}_0 = \frac{1}{\tilde{N}}\left(a_0^{QF} + \frac{b_0^{QF} - a_0^{QF}}{\tilde{\lambda}_{QF}^2} + O(\frac{1}{\tilde{\lambda}_{QF}^4})\right). \tag{2.40}$$

We can now compare this to the two-loop result from the critical bosonic theory (2.30). This yields:

$$a_0^{QF} = -\frac{32}{3\pi^2} \tag{2.41}$$

$$b_0^{QF} = 0. \tag{2.42}$$

We thus obtain the following expression for $\tilde{\gamma}_0$:

$$\tilde{\gamma}_0 = -\frac{32}{3\pi^2}\frac{\tilde{\lambda}_{QF}^2}{1+\tilde{\lambda}_{QF}^2}\frac{1}{\tilde{N}} \tag{2.43}$$

Making a perturbative expansion around $\tilde{\lambda} = 0$, we find

$$\tilde{\gamma}_0 = -\frac{32}{3\pi^2}\tilde{\lambda}_{QF}^2\frac{1}{\tilde{N}} \tag{2.44}$$

which precisely reproduces the two-loop result in the regular fermionic theory (2.31), thus providing us a non-trivial test of our conjecture.

### 2.3.2 Conjecture in Quasi-Bosonic theory: The $\gamma_0$

Repeating this procedure in the quasi-bosonic theory, we again find that

$$a_0^{QB} = -\frac{32}{3\pi^2} \tag{2.45}$$

$$b_0^{QB} = 0. \tag{2.46}$$

so

$$\gamma_0 = -\frac{32}{3\pi^2}\frac{\tilde{\lambda}_{QB}^2}{1+\tilde{\lambda}_{QB}^2}\frac{1}{\tilde{N}}. \tag{2.47}$$

### 2.3.3 All loop check of our conjecture

As a final non-trivial check of our conjecture we note that, using our expressions for $\gamma_0$ and $\tilde{\gamma}_0$, we obtain

$$\tilde{\gamma}_0 + \gamma_0 = -\frac{16\lambda_b}{3\pi\sin\pi\lambda_b}\frac{1}{N_b} = -\frac{16\lambda_f}{3\pi\sin\pi\lambda_f}\frac{1}{N_f} = -\frac{32}{3\pi^2}\frac{1}{\tilde{N}}, \tag{2.48}$$

which is exactly equation (2.34) and is satisfied to all orders in $\tilde{\lambda}$.

### 2.3.4 Anomalus dimension interms of $\lambda_b$ and $\lambda_f$ variables

Let us conclude by presenting the expression for $\tilde{\gamma}_0$ and $\gamma_0$ in terms of variables $\lambda_b$ and $\lambda_f$ variables. Using (2.6), we have:

$$\tilde{\gamma}_0 = -\frac{8\lambda_b}{3\pi N_b}\cot\left(\frac{\pi\lambda_b}{2}\right) = -\frac{8\lambda_f}{3\pi N_f}\tan\left(\frac{\pi\lambda_f}{2}\right) \tag{2.49}$$

and

$$\gamma_0 = -\frac{8\lambda_b}{3\pi N_b}\tan\left(\frac{\pi\lambda_b}{2}\right) = -\frac{8\lambda_f}{3\pi N_f}\cot\left(\frac{\pi\lambda_f}{2}\right). \tag{2.50}$$

Note that our conjecture also reproduces the all the known results reported in section 2.2.

## 2.4 Two-sided Padé approximation

Let us also observe that our conjecture can be thought of as a two-sided Padé approximation. In this sense, even if our conjecture turns out to be incorrect, it provides a good estimate for the anomalous dimension of the scalar primary that takes into account all known weak-coupling and strong-coupling calculations. Consider making an $(m,n)$-Padé approximation of $\gamma_0$ as follows:

$$\gamma_0^{(m,n)} = \frac{A_0 + A_2\tilde{\lambda}_{QB}^2 + \ldots + A_m\tilde{\lambda}_{QB}^m}{1 + B_2\tilde{\lambda}_{QB}^2 + \ldots + B_n\tilde{\lambda}^n}. \tag{2.51}$$

We only include even powers of $\tilde{\lambda}$ as the anomalous dimension must be parity-invariant.

The $(2,2)$ Padé approximation has three unknowns. We have four perturbative data to constrain it:

- The fact that $\gamma_0$ vanishes when $\tilde{\lambda}_{QB} = 0$.
- A two-loop calculation $\gamma_0$ in the regular-bosonic theory.
- The value of $\gamma_0$ in the critical fermionic theory at $\lambda_b = 0$.
- A two-loop (order $\lambda_b^2$) calculation of $\gamma_0$ in the critical fermionic theory.

Hence the Padé-approximation is overconstrained. Nevertheless, it is possible to fit all four results with following choice of three coefficients.

$$A_0 = 0, \quad A_2 = -\frac{32}{3\pi^2}, \quad B_2 = 1. \tag{2.52}$$

Repeating the calculation to obtain a $(2,2)$ Padé approximation for the quasi-fermionic theory, we obtain the same coefficients. However, we also have to impose the extra constraint of equation (2.34), which turns out to be automatically satisfied. Hence, the simplest Padé approximation to the perturbative data we have seems to work very well. Of course, it is possible to obtain higher-order Padé approximations that satisfy all these constraints, so our answer is not uniquely determined by this procedure. But, it is an interesting observation that, for a variety of physical quantities, such as planar three-point functions [210], planar four-point function of the scalar primary [208], and the $1/N$ higher-spin spectrum [109], a relatively simple Padé approximation defined using the variables $\tilde{\lambda}$ and $\tilde{N}$, happens to coincide with the exact answer.

## 2.5 $\mathcal{N} = 2$ theory and its Operator Spectrum

In this section, we are interested in $\mathcal{N} = 2$ $U(N)$ Chern-Simons theory coupled to single chiral multiplet, $\Phi \equiv (\phi, \psi)$, in the fundamental representation of the gauge group. The position space Lagrangian for the theory is given in 2.2. The theory has two parameters : the rank of the gauge group, $N$, and the Chern-Simons level, $\kappa$, which is quantized to take only integer values [217]. $\kappa^{-1}$ controls the strength of gauge interactions and the theory is perturbative for large values of $\kappa$ at any finite $N$. This theory is conjectured to be self-dual under a strong-weak type duality, [70]. In the 't Hooft like large $N$ limit

$$\kappa \to \infty, N \to \infty \quad \text{with} \quad \lambda = \frac{N}{\kappa} \quad \text{fixed} \tag{2.53}$$

of interest in this paper, the duality transformation is

$$\kappa \to -\kappa, \quad \lambda \to \lambda - \text{sgn}(\lambda). \tag{2.54}$$

Apart from the matching of many of the supersymmetric observables which can be computed at finite $N$ and $\kappa$ using supersymmetric localization techniques, recent exact computation of many non-supersymmetric observables, e.g. the thermal partition function, in the large $N$ limit [71, 72, 77, 79, 83, 85] has provided ample evidence for this conjectured duality.

The theory is quantum mechanically (super) conformal for all values of $\kappa$ and $N$. In the 't Hooft limit, one can focus on the single trace superconformal primary operator spectrum of the theory. Though our theory has $\mathcal{N}=2$ superconformal symmetry, in this paper we will work in the $\mathcal{N}=1$ superspace formulation to allow us to use the relevant results of [93] for our computations. In the $\mathcal{N}=1$ language, the operator spectrum of the theory consists of a set of supercurrent operators [218]

$$J^{(s)} = \sum_{r=0}^{2s}(-1)^{\frac{r(r+1)}{2}} \begin{pmatrix} 2s \\ r \end{pmatrix} \nabla^r \bar{\Phi} \nabla^{2s-r} \Phi \ , \tag{2.55}$$

which are written in terms of the superfields,

$$\Phi = \phi + \theta\psi - \theta^2 F, \quad \bar{\Phi} = \bar{\phi} + \theta\bar{\psi} - \theta^2 \bar{F} \ .$$

and the superscript $s$ in(2.55) takes values in $\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. Here, we have also defined

(2.56)using the auxiliary commuting polarisation spinors, $\lambda^{\alpha_i}$, which keep track of the spin; and $\nabla_{\alpha_i}$ are the standard supersymmetry invariant gauge-covariant derivatives.

Their action on the matter superfields of our theory is given by,

$$\begin{aligned}\nabla_\alpha \Phi =& D_\alpha \Phi - i\Gamma_\alpha \Phi \\ \nabla_\alpha \bar{\Phi} =& D_\alpha \bar{\Phi} + i\Gamma_\alpha \bar{\Phi}\end{aligned} \tag{2.57}$$

The explicit expressions for the spin 0 operator and the first few spin-$s$ currents are,

$$\begin{aligned} J_0 &= \bar{\Phi}\Phi \\ J_\alpha &= \bar{\Phi}\nabla_\alpha\Phi - \nabla_\alpha\bar{\Phi}\Phi = \bar{\Phi}D_\alpha\Phi - D_\alpha\bar{\Phi}\Phi - 2i\bar{\Phi}\Gamma_\alpha\Phi \\ J_{\alpha\beta} &= \bar{\Phi}\nabla_\alpha\nabla_\beta\Phi - 2\nabla_\alpha\bar{\Phi}\nabla_\beta\Phi + \nabla_\alpha\nabla_\beta\bar{\Phi}\Phi \\ J_{\alpha\beta\gamma} &= \bar{\Phi}\nabla_\alpha\nabla_\beta\nabla_\gamma\Phi - 3\nabla_\alpha\bar{\Phi}\nabla_\beta\nabla_\gamma\Phi - 3\nabla_\alpha\nabla_\beta\bar{\Phi}\nabla_\gamma\Phi + \nabla_\alpha\nabla_\beta\nabla_\gamma\bar{\Phi}\Phi \end{aligned} \tag{2.58}$$

In the free limit of the theory i.e. $\lambda \to 0$, each of these supercurrents, $J^{(s)}$ with $s \neq 0$, satisfies the conservation equation

$$\mathcal{D}^\alpha \left( \frac{\partial}{\partial \lambda^\alpha} J^{(s)} \right) = 0 \tag{2.59}$$

and constitutes two component conserved current operators $\{J^{(s)}, J^{(s+\frac{1}{2})}\}$ in its $\theta$ expansion [218]. At finite $\lambda$, the conservation equation (2.59) is violated at order $\frac{1}{N}$ by double trace operators for $s \geq 2$ [71, 218]. Here, we are interested in the scalar operator $J_0(\theta, x)$. There is no conservation equation associated with this operator and it constitutes 2 scalar and 1 spin half operator as follows

$$J^{(0)}(\theta, x) = J_0^b(x) + \theta^\alpha \Psi_\alpha(x) - \theta^2 J_0^f(x) \tag{2.60}$$

where

$$J_0^b(x) = \bar{\phi}\phi(x), \quad \Psi_\alpha(x) = (\bar{\phi}\psi_\alpha + \bar{\psi}_\alpha\phi)(x), \quad J_0^f(x) = \bar{\psi}\psi(x). \tag{2.61}$$

In the subsequent sections, we compute the 2 and 3-point functions of the $J^{(0)}$ operator and two component of the 4-point function.

## 2.6 Correlation functions

In this section, we compute the two and three-point correlation function of the $J_0(\theta, p)$ operator in momemtum space. Two of the main ingredients for these computations are the exact propagator (2.63) and the renormalized four point vertex for the fundamental superfield $\Phi(\theta, p)$ ($\nu_4$ in 2.64). These were computed in [93] for a more general class of theories with $\mathcal{N} = 1$ supersymmetry which can be thought of as one parameter[3] deformation of the $\mathcal{N} = 2$ theory of interest in this paper. Below, we list these results for our $\mathcal{N} = 2$ theory, conveniently stated in term of the exact quantum effective action

$$\begin{aligned}
&S = S_2 + S_4 \ , \\
&S_2 = \int \frac{d^3p}{(2\pi)^3} d^2\theta_1 d^2\theta_2 \left[\bar{\Phi}(\theta_1, -p) e^{-\theta_1^\alpha p_{\alpha\beta} \theta_2^\beta} \Phi(\theta_2, p)\right] , \\
&S_4 = \frac{1}{2} \int \frac{d^2p}{(2\pi)^3} \frac{d^2q}{(2\pi)^3} \frac{d^2k}{(2\pi)^3} d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\theta_4 \\
&\qquad \left[\nu_4(\theta_1, \theta_2, \theta_3, \theta_4; p, q, k) \Phi_i(\theta_1, -(p+q)) \bar{\Phi}^i(\theta_2, p) \bar{\Phi}^j(\theta_3, k+q) \Phi_j(\theta_4, -k)\right]
\end{aligned} \tag{2.62}$$

The quadratic part of the effective action receives no quantum corrections at large $N$ in the $\mathcal{N} = 2$ theory. The propagator is thus tree-level exact and given by

$$\begin{aligned}
\langle \bar{\Phi}(\theta_1, p_1) \Phi(\theta_2, p_2) \rangle &= (2\pi)^3 \delta^3(p_1 + p_2) \mathcal{P}(\theta_1, \theta_2; p_1) \\
&= (2\pi)^3 \delta^3(p_1 + p_2) \frac{e^{-\theta_1^\alpha \theta_2^\beta (p_1)_{\alpha\beta}}}{p_1^2}.
\end{aligned} \tag{2.63}$$

The quartic superspace vertex, $\nu_4$, does receive quantum corrections and takes the following form

$$\begin{aligned}
\nu_4(\theta_1, \theta_2, \theta_3, \theta_4; p, q, k) &= e^{\frac{1}{4} X.(p.\theta_{12} + q.\theta_{13} + k.\theta_{43})} F_4(\theta_{12}, \theta_{13}, \theta_{43}; p, q, k), \\
\text{with} \quad F_4 &= \theta_{12}^+ \theta_{43}^+ \Big[ A(p,q,k) \theta_{12}^- \theta_{43}^- \theta_{13}^+ \theta_{13}^- + C(p,q,k) \theta_{12}^- \theta_{13}^+ \\
&\qquad + D(p,q,k) \theta_{13}^+ \theta_{43}^- \Big]
\end{aligned} \tag{2.64}$$

Here, we have used the following notation for the sum and the difference of Grassmann variables to avoid clutter,

$$X^\alpha = \sum_{i=1}^{n} \theta_i^\alpha \quad , \quad \theta_{in}^\alpha = \theta_i^\alpha - \theta_n^\alpha. \tag{2.65}$$

[3]Quartic superpotential term : $-\frac{\pi(\omega-1)}{\kappa} \int d^3x \, d^2\theta \, (\bar{\Phi}\Phi)^2$. $\omega = 1$ is the $\mathcal{N} = 2$ point.

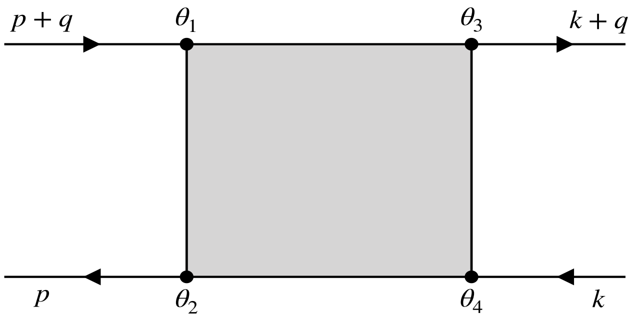



**Figure 2.5:** Diagrammatic representation of the exact four point vertex, $\nu_4$ in (2.64).

The overall exponential factor is determined by supersymmetric Ward identity (2.69), while the coefficient functions A, C, and D require explicit computation and are given by [93]

$$\begin{aligned} A(p,q,k) &= -\frac{2\pi i}{\kappa} e^{2i\lambda\left[\tan^{-1}\left(\frac{2k_s}{q_3}\right) - \tan^{-1}\left(\frac{2p_s}{q_3}\right)\right]}, \\ C(p,q,k) &= D(p,q,k) = \frac{2A(p,q,k)}{(k-p)_-}. \end{aligned} \tag{2.66}$$

Note that the vertex $\nu_4$ was computed in a special momentum configuration, namely

$$q_+ = q_- = 0. \tag{2.67}$$

while the momenta $p$ and $k$ are arbitrary [4]. For this reason, our computation of correlation functions will also be a restricted configuration in which the momentum of $J_0$ operators are restricted to lie only in the 3-direction. Diagrammatically, the exact four-point vertex will be represented as in Figure 2.5.

### 2.6.1 Constraints on correlation functions from supersymmetry

To begin with, let us study the constraints on an arbitrary correlation function due to supersymmetry. As stated earlier, although our theory has $\mathcal{N} = 2$ supersymmetry, we will be working in $\mathcal{N} = 1$ superspace following [93]. A general n-point correlation function of $\mathcal{N} = 1$ scalar superfield is constrained by supersymmetry and translation invariance

[4] We refer the reader to appendix B.1 for conventions for labeling momenta.

to take the following form [93]

$$\langle \mathcal{O}_1(\theta_1, p_1) \dots \mathcal{O}_n(\theta_n, p_n) \rangle$$
$$= (2\pi)^3 \delta^3 \left( \sum_{i=1}^{n} p_i \right) \exp \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \theta_i \right) . \left( \sum_{i=1}^{n} p_i . \theta_i \right) \right] F_n(\{\theta_{in}\}; \{p_i\}). \tag{2.68}$$

The $\delta^3(\sum_i p_i)$ follows from translation invariance while the overall Grassmann exponential factor follows from invariance under $\mathcal{N} = 1$ supersymmetry. Note that the function $F_n$ above only depend on the differences of the Grassmann coordinates. Following [93], the form is easily derived as follows

$$\begin{aligned} 0 &= \left[ \sum_{i=1}^{n} \mathcal{Q}_\alpha^{(i)} \right] \langle \mathcal{O}_1(\theta_1, p_1) \dots \mathcal{O}_n(\theta_n, p_n) \rangle \\ &= \left[ \sum_{i=1}^{n} \left( \frac{\partial}{\partial \theta_i^\alpha} - (p_i)_{\alpha\beta} \theta_i^\beta \right) \right] \langle \mathcal{O}_1(\theta_1, p_1) \dots \mathcal{O}_n(\theta_n, p_n) \rangle \\ &= \left( n \frac{\partial}{\partial X^\alpha} - \sum_{i=1}^{n-1} (p_i)_{\alpha\beta} \theta_{in}^\beta \right) \langle \mathcal{O}_1(\theta_1, p_1) \dots \mathcal{O}_n(\theta_n, p_n) \rangle. \end{aligned} \tag{2.69}$$

In the last line above, we used momentum conservation to replace $p_n$ with $\sum_{i=1}^{n-1}(-p_i)$. The factorized form in (2.68) follows as the solution to the last equation in (2.69).

### 2.6.2 $J_0$ vertex

Before proceeding to the computation of correlation functions, it would be useful to compute an intermediate quantity, the $J_0$-vertex. It is defined by stripping of the propagators from $\langle J_0 \Phi \bar{\Phi} \rangle$ as follows

$$\langle J_0(\theta_1, p_1) \Phi(\theta_2, p_2) \bar{\Phi}(\theta_3, p_3) \rangle =$$
$$\int \frac{d^3 p_2'}{(2\pi)^3} \frac{d^3 p_3'}{(2\pi)^3} d^2\theta_2' d^2\theta_3' \left[ \langle J_0(\theta_1, p_1) \Phi(\theta_2', p_2') \bar{\Phi}(\theta_3', p_3') \rangle_{ver} \mathcal{P}(\theta_2', \theta_2; -p_2) \mathcal{P}(\theta_3', \theta_3; p_3) \right] \tag{2.70}$$

and satisfies the same Ward identity as a three-point function (2.68). The vertex receives contribution both from the free propagation of the fundamental field as well as from the interaction vertices in the theory. The free part vertex is simply proportional to the momentum and the Grassmannian $\delta$-functions while the interacting part of the vertex can be computed from the exact $\nu_4$ vertex. Figure (2.6) shows the relevant diagrams.

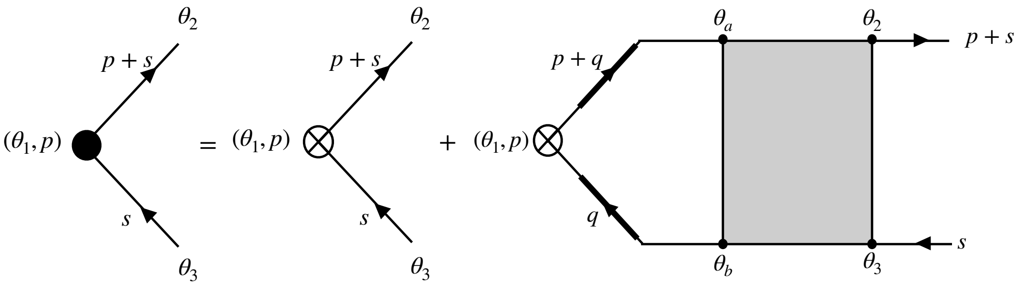



**Figure 2.6:** Solid circle on the LHS represents the full exact $J_0$ vertex and the first diagram on RHS is the free vertex. The second diagram on RHS includes all the interactions which are accounted by insertion of exact 4 point vertex (2.64) connected to the external $\mathcal{J}^{(0)}$ operator using the exact propagator.

$$\langle J_0(\theta_1, p)\Phi(\theta_2, r)\bar{\Phi}(\theta_3, s)\rangle_{ver} = \langle J_0(\theta_1, p)\Phi(\theta_2, r)\bar{\Phi}(\theta_3, s)\rangle_{ver,free} + \langle J_0(\theta_1, p)\Phi(\theta_2, r)\bar{\Phi}(\theta_3, s)\rangle_{ver,int}$$

$$\begin{aligned}
\text{where} \quad \langle J_0(\theta_1, p)\Phi(\theta_2, r)\bar{\Phi}(\theta_3, s)\rangle_{ver,free} &= (2\pi)^3\delta^3(p+r+s) \quad \nu_{3,free}(\theta_{12}, \theta_{32}; p, s) \\
&= (2\pi)^3\delta^3(p+r+s)\theta^+_{32}\theta^-_{32}\theta^+_{12}\theta^-_{12} \\
\text{and} \quad \langle J_0(\theta_1, p)\Phi(\theta_2, r)\bar{\Phi}(\theta_3, s)\rangle_{ver,int} & \\
= (2\pi)^3\delta^3(p+r+s)\left[\int \frac{d^3q}{(2\pi^3)} d^2\theta_a d^2\theta_b \ \mathcal{P}(\theta_1, \theta_a; q+p)\mathcal{P}(\theta_b, \theta_1; q)\nu_4(\theta_a, \theta_b, \theta_2, \theta_3; q, p, s)\right] & \\
= (2\pi)^3\delta^3(p+r+s)e^{\frac{1}{3}\theta_{123}.(p.\theta_{12}+s.\theta_{32})}\nu_{3,int}(\theta_{12}, \theta_{32}, p, s) &
\end{aligned} \tag{2.71}$$

Explicit computation of the above integral, with constraint $p_+ = p_- = 0$ following from the (2.67), leads to the following result for the full $J_0$-vertex factor

$$\begin{aligned}
\nu_3 &= (\nu_{3,free} + \nu_{3,int})(\theta_{12}, \theta_{32}, p, s) \\
&= \frac{1}{2s^+}\left[1 - e^{2i\lambda \tan^{-1}(\frac{2s_s}{p_3})}\right]\theta^+_{32}\theta^+_{12} + \frac{1}{2p_3}\left(e^{2i\lambda\tan^{-1}(\frac{2s_s}{p_3}) - i\pi\lambda sgn(p_3)} - 1\right)\theta^+_{32}\theta^-_{32} \\
&\quad + \left(1 + \frac{1}{6}(-4 + e^{2i\lambda\tan^{-1}(\frac{2s_s}{p_3})} + 3e^{2i\lambda\tan^{-1}(\frac{2s_s}{p_3}) - i\pi\lambda sgn(p_3)})\right)\theta^+_{32}\theta^-_{32}\theta^+_{12}\theta^-_{12}
\end{aligned} \tag{2.72}$$

The $J_0$-vertex computed above will be useful in further computations of 2 and 3-point functions of the $J_0$ operator.

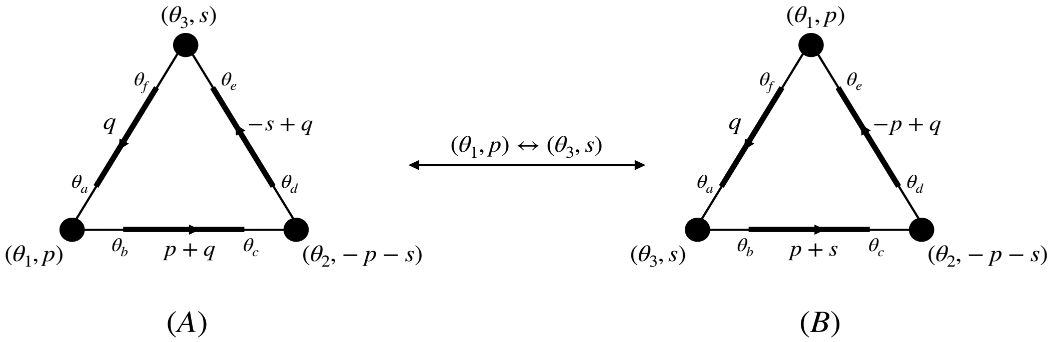



**Figure 2.7:** The full $J_0$ 3 point function is obtained by connecting three exact $J_0$-vertices with exact propagators. There are two such diagrams, as shown above, which turn out to be equal.

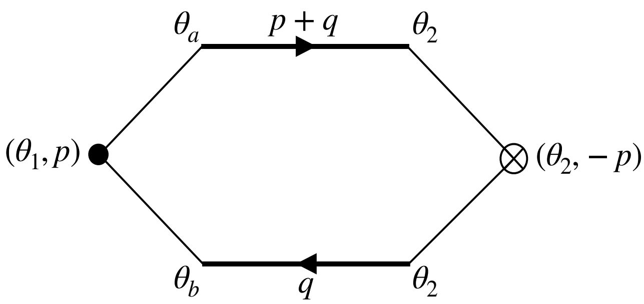



**Figure 2.8:** The full $J_0$ 2 point function is obtained by connecting the exact $J_0$-vertex (solid circle) to the free vertex (cross) with exact propagators (thick line).

### 2.6.3 $\langle J_0 J_0 \rangle$ correlation function

The two-point function can be straightforwardly computed from the $J_0$-vertex determined in the previous section by combining the exact vertex on one side with the free vertex on the other side. Figure (2.8) shows the relevant diagram which when computed leads to the following non-vanishing component of the correlators

$$\begin{aligned}
\langle J_0^b(p) J_0^b(-p) \rangle &= \frac{N}{8|p|} \frac{\sin(\pi\lambda)}{\pi\lambda} \\
\langle J_0^f(p) J_0^f(-p) \rangle &= -\frac{N|p|}{8} \frac{\sin(\pi\lambda)}{\pi\lambda} \\
\langle \Psi_\alpha(p) \Psi_\beta(-p) \rangle &= \frac{N}{8} \left( \frac{p_{\alpha\beta}}{|p|} \frac{\sin(\pi\lambda)}{\pi\lambda} + C_{\alpha\beta} \frac{1-\cos(\pi\lambda)}{\pi\lambda} \right) \\
\langle J_0^b(p) J_0^f(-p) \rangle &= -\frac{N}{8} \frac{(1-\cos(\pi\lambda))}{\pi\lambda}
\end{aligned} \tag{2.73}$$

Let us compare the above two-point functions with the corresponding two-point functions in the regular fermionic and regular bosonic theories studied in [77] and [79] respectively. Note that as opposed to the regular bosonic and regular fermionic theories studied in [77] and [79], the $\lambda$ dependence of the two-point function of $J_0^b$ and $J_0^f$ operators is the same as that of the higher spin currents in the non-supersymmetric cases. Further, using the double trace factorization argument of [96] relating the two-point function of current operators in the supersymmetric and the above-mentioned non-supersymmetric theories, we know that the two-point function of all the current operators in our supersymmetric theory is exactly the same as those of the corresponding regular boson/fermion theory. Thus, we see that in our theory the two-point function of scalar operators is the same as that for the higher spin current operators. The reason for this is supersymmetry. Though we are working in $\mathcal{N}=1$ superspace language, our theory has underlying $\mathcal{N}=2$ supersymmetry under which the scalar operators $J_0^b, J_0^f$ belong to the same supersymmetry multiplet as the spin 1 conserved current and thus the two-point function of the two are thus related by supersymmetry.

### 2.6.4 $\langle J_0 J_0 J_0 \rangle$ correlation function

The full 3-point function can be constructed by combining three $J_0$ vertices with exact propagators. There are two such diagrams shown in figure (2.7). Each of these two diagrams can easily be shown to be cyclically symmetric and related to each other by pair-exchange of any two $J_0$ insertions. An explicit computation of the diagram shows that each of the diagrams is completely symmetric (cyclic as well as under pair-exchange) by itself and the two diagrams are equal. The full 3 point function is then just twice the contribution of the first diagram from which the non-zero components of the three-point function can be extracted

$$\begin{aligned}
\langle J_0^b(p_3)J_0^b(s_3)J_0^b(-p_3-s_3)\rangle &= \frac{\sin(2\pi\lambda)}{2\pi\lambda}\frac{N}{8|p_3s_3(p_3+s_3)|}\\
\langle J_0^f(p_3)J_0^f(s_3)J_0^f(-p_3-s_3)\rangle &= \frac{-iN}{8}\frac{(\sin(\pi\lambda))^2}{\pi\lambda}\\
\langle J_0^b(p_3)J_0^b(s_3)J_0^f(-p_3-s_3)\rangle &= \frac{(\sin(\pi\lambda))^2}{\pi\lambda}\frac{(-iN)}{8|p_3s_3|}\\
\langle J_0^f(p_3)J_0^f(s_3)J_0^b(-p_3-s_3)\rangle &= \frac{\sin(2\pi\lambda)}{2\pi\lambda}\frac{N}{16|p_3+s_3|}\\
\langle \Psi_+(p_3)\Psi_-(s_3)J_0^b(-p_3-s_3)\rangle &= \frac{N}{16p_ss_3(p_3+s_3)}\Bigg(\frac{\sin(2\pi\lambda)}{2\pi\lambda}\Big(|p_3|-|s_3|-(p_3-s_3)\mathrm{sgn}(p_3+s_3)\Big)\\
&\qquad - i\frac{(\sin(\pi\lambda))^2}{\pi\lambda}\mathrm{sgn}(p_3+s_3)(|p_3+s_3|-|p_3|+|s_3|)\Bigg)\\
\langle \Psi_+(p_3)\Psi_-(s_3)J_0^f(-p_3-s_3)\rangle &= \frac{N}{16p_ss_3}\Bigg(\frac{\sin(2\pi\lambda)}{2\pi\lambda}\Big(|p_3+s_3|-|p_3|-|s_3|\Big)\\
&\qquad + i\frac{(\sin(\pi\lambda))^2}{\pi\lambda}\mathrm{sgn}(p_3+s_3)\Big((p_3-s_3)|p_3+s_3|-|p_3|+|s_3|\Big)\Bigg)
\end{aligned} \tag{2.74}$$

Notice that in the above result for 3 point functions, two different functional forms of $\lambda$ dependences appear, namely $\frac{\sin(2\pi\lambda)}{2\pi\lambda}$ and $\frac{\sin^2\pi\lambda}{\pi\lambda}$. The two of them differ in a crucial way. The first one has a finite $\lambda \to 0$ limit and is invariant under parity under which $\lambda$ is odd. The second is odd under parity and vanishes in $\lambda \to 0$ limit. This result thus provides some support for the conjecture made in [218] that the three-point functions in $\mathcal{N} = 1$ superconformal theories with higher spin symmetry have exactly one parity even and one parity odd structure.

## 2.7 Four point functions

In the previous section, we evaluated the 3-point functions involving the $\mathcal{J}_0$ operator in the $\mathcal{N} = 2$ supersymmetric theory by computing the required vertex. However, the direct computation of the four-point function of $J_0$ operator following the same technique has proven to be intractable in our attempt till now. We describe our attempt to evaluate this four-point function in momentum space through the required vertices in the Appendix (B.4).

In this section, we determine the four-point correlators of the $J_0^b$ and $J_0^f$ operators using a novel method developed in [208], which we briefly review below. Note that we will be evaluating the 4-point correlation function in the position space as in [208].

Consider the position space four-point correlator of the identical external operators with conformal dimensions $\Delta$. The function $\mathcal{A}$ which is known as the reduced correlator is defined as follows

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}}\frac{1}{x_{34}^{2\Delta}}\mathcal{A}(u,v) = \frac{1}{x_{13}^{2\Delta}}\frac{1}{x_{24}^{2\Delta}}\frac{\mathcal{A}(u,v)}{u^{\Delta}} \,. \tag{2.75}$$

Here, $u, v$ are the standard cross-ratios:

$$u = \left(\frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}\right)^2 , \quad v = \left(\frac{|x_{14}||x_{23}|}{|x_{13}||x_{24}|}\right)^2 .$$

The conformal block expansion expressed in terms of the reduced correlator $\mathcal{A}(u,v)$ is given as

$$\frac{\mathcal{A}(u,v)}{u^{\Delta}} = \frac{1}{u^{\Delta}}\sum_k C^2_{\mathcal{O}\mathcal{O}\mathcal{O}_k} G_{\Delta_k,J_k}(u,v) \tag{2.76}$$

where $G_{\Delta_k,J_k}(u,v)$ is known as the conformal block corresponding to the operator $\mathcal{O}_k$ with scaling dimension $\Delta_k$ and spin $J_k$.

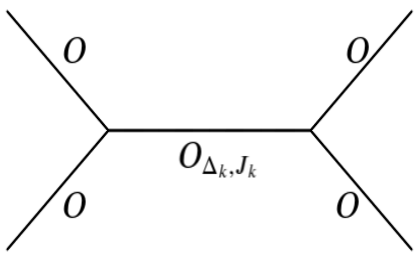



**Figure 2.9:** Schematic for the conformal block expansion

In the supersymmetric four point functions of $J_0$ operators, the relevant exchanges are schematically shown below

### 2.7.1 Review of the double discontinuity technique

In [208], the authors determine the four-point correlation functions of the scalar operator in the non-supersymmetric scalar/fermion coupled to Chern Simons gauge field i.e. quasi-bosonic and quasi-fermionic theory respectively. In order to obtain the required four-point

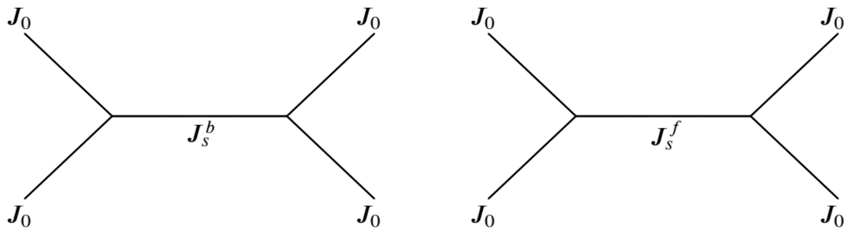



**Figure 2.10:** Schematic for the exchanges relevant in the supersymmetric scalar correlators

functions, the authors utilize the inversion formula which relates the double discontinuity to the OPE coefficients [207]. The authors first prove an interesting theorem that in the large-N limit of a $CFT_d$, the double discontinuity constrains the four-point correlator up to three contact terms in $AdS_{d+1}$. Suppose there are two solutions $G_1$ and $G_2$ to the crossing equation with the same double discontinuity then they are related by the contact interactions in the AdS as follows

$$G_1 = G_2 + c_1 G^{AdS}_{\phi^4} + c_2 G^{AdS}_{(\partial\phi)^4} + c_2 G^{AdS}_{\phi^2(\partial^3\phi)^2} \tag{2.77}$$

Furthermore, the authors showed[5] that for the four-point function of single trace scalar operator in Chern-Simons coupled fundamental scalar/fermion theories these $AdS_4$ contact terms do not contribute and hence the double discontinuity completely determines the four-point functions.

Consider the normalized three-point functions of the operators $\mathcal{O}_i (i = 1, 2, 3)$.[6] In [73, 74, 208], it was noticed that the square of these normalized coefficients in the quasi-fermionic theories ($C^2_{s,qf}$) is related to that of a single free Majorana fermion ($C^2_{s,ff}$) as follows

$$C^2_{s,qf} \;\; = \;\; \frac{1}{\tilde{N}} C^2_{s,ff} \tag{2.78}$$

where $\tilde{N}$ is related to the the rank of the gauge group $N$ and coupling $\lambda_{qf}$ by,

$$\tilde{N} = 2N \frac{\sin(\pi\lambda_{qf})}{\pi\lambda_{qf}}. \tag{2.79}$$

[5]via explicit numerical computation

[6]For the conventions of normalization correlation functions please refer to appendix B.3.

Note that the normalized coefficients of quasi-fermionic theory and free fermionic theory are proportional to each other as given in (2.78). Hence, the double discontinuity of the scalar four-point function in the free fermionic theory is the same as that of the quasi-fermionic theories up to an overall factor that depends only on $N$ and $\lambda_{qf}$.

On the other hand, the square of the normalized coefficients of the quasi-bosonic theories ($C^2_{s,qb}$) is related to the theory of a free real boson ($C^2_{s,fb}$) as follows

$$C^2_{s,qb} = \frac{1}{\tilde{N}} C^2_{s,fb} \quad s > 0, \tag{2.80}$$

$$C^2_{0,qb} = \frac{1}{\tilde{N}} \frac{1}{(1+\tilde{\lambda}^2_{qb})} C^2_{0,fb} = \frac{1}{\tilde{N}} C^2_{0,fb} - \frac{1}{\tilde{N}} \frac{\tilde{\lambda}^2_{qb}}{(1+\tilde{\lambda}^2_{qb})} C^2_{0,fb}. \tag{2.81}$$

where $\tilde{N}$ and $\tilde{\lambda}$ are related to $N$ and coupling $\lambda_{qb}$ as

$$\tilde{N} = 2N \frac{\sin(\pi\lambda_{qb})}{\pi\lambda_{qb}}, \tag{2.82}$$

$$\tilde{\lambda}_{qb} = \tan\left(\frac{\pi\lambda_{qb}}{2}\right). \tag{2.83}$$

Note that, unlike the normalized coefficients of the quasi-fermionic theories, in the quasi-bosonic theories, the spin $s = 0$ and $s \neq 0$ coefficients given above have different factors in front of their free bosonic counterparts. In order to account for the second term on the RHS of (2.81) one needs to add a conformal partial wave with spin-0 exchange which is given by the well-known $\bar{D}$-function with the correct pre-factor [208]. We now proceed to employ this technique for the supersymmetric case.

### 2.7.2 Double discontinuity and the supersymmetric correlators

Here, we utilize the technique described above to compute the four-point correlators for spin-0 operators $J^b_0$ and $J^f_0$ in our supersymmetric theory. Since we are considering correlators of identical external operators[7], only even spin operators will contribute to the block expansion.

[7]Although we have all the three-point correlators required, we do not compute mixed correlators such as $\langle J^b_0 J^b_0 J^f_0 J^f_0 \rangle$ here, currently a free theory analogue for such correlators is not clear. We reserve this issue for future investigations.

#### 2.7.2.1 $\langle J_0^b(x_1)J_0^b(x_2)J_0^b(x_3)J_0^b(x_4)\rangle$

The four-point function of the $J_0^b$ operators is expressed as follows[8]

$$\langle J_0^b(x_1)J_0^b(x_2)J_0^b(x_3)J_0^b(x_4)\rangle = disc + \frac{1}{x_{13}^2 x_{24}^2}F(u,v). \tag{2.84}$$

Here, disc corresponds to the disconnected part given by

$$disc = \frac{1}{x_{12}^2 x_{34}^2} + \frac{1}{x_{13}^2 x_{24}^2} + \frac{1}{x_{14}^2 x_{23}^2} \tag{2.85}$$

while $F(u,v)$ is given by

$$F(u,v) = \frac{1}{u}\sum_k C^2_{\mathcal{O}\mathcal{O}O_k} G_{\Delta_k,J_k}(u,v) \tag{2.86}$$

In order to determine the double discontinuity and hence the 4-point functions in the supersymmetric case using the method described above, we need the normalized 3-point function coefficients for the operators running in OPE of two $J_0^b$ operators. For the case of spin 0 operators, i.e. $J_0^b, J_0^f$, these normalized coefficients can directly be obtained from our explicit computations for the 2 and 3-point functions in (2.73),(2.74). For the contribution of higher spin operators $(J_s^b, J_s^f)$, these coefficients can be computed by relating to them to the regular boson (fermion) theories using the large $N$ *double trace factorization* (see e.g. [96]) of correlation functions. We relegate the computation of these to appendix (B.3) and only collect the final result here.

For scalar operators $J_0^{b,f}$, we have

$$\begin{aligned} C^{2(BBB)}_{0,susy} &= \frac{1}{\tilde{N}}\frac{(1-\tilde{\lambda}^2)^2}{(1+\tilde{\lambda}^2)^2}\, C^2_{0,fb,} \\ C^{2(BBF)}_{0,susy} &= \frac{8}{\pi^2}\frac{\tilde{\lambda}^2}{\tilde{N}(1+\tilde{\lambda}^2)^2} C^2_{0,fb}. \end{aligned}$$

For the higher spin operators, $J_s^{b,f}$ $(s \in (2,4,6,\ldots))$, we get

$$\begin{aligned} C^{2(BBB)}_{s,susy} &= \frac{1}{\tilde{N}(1+\tilde{\lambda}^2)^2} C^2_{s,fb} \qquad s>0, \\ C^{2(BBF)}_{s,susy} &= \frac{\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2} C^2_{s,fb} \qquad s>0. \end{aligned} \tag{2.87}$$

[8] Note that, it is useful to redefine operators such that the normalization is fixed to be $\langle J_0 J_0\rangle = x^{-2\Delta}$ [208]. We work with this normalization in this section.

Note that we may re-express the spin 0 coefficient $C^{2(BBB)}_{0,susy}$ above as follows

$$C^{2(BBB)}_{0,susy} = \frac{1}{\tilde{N}(1+\tilde{\lambda}^2)^2}C^2_{0,fb} + \frac{\tilde{\lambda}^4 - 2\tilde{\lambda}^2}{\tilde{N}(1+\tilde{\lambda}^2)^2}C^2_{0,fb},$$

(2.88)

Observe that $C^{2(BBB)}_{s,susy}$ in (2.87) and the first term of $C^{2(BBB)}_{0,susy}$ in (2.88) have the same pre-factor. This is similar to the case of the quasibosnic case given in (2.80) and (2.81) reviewed earlier. Consider, now, the double discontinuity of the conformal blocks

$$\text{dDisc}[G_{\Delta,J}(1-z,1-\bar{z})] = \sin^2(\frac{\pi}{2}(\Delta - J - 2\Delta_\phi))G_{\Delta,J}(1-z,1-\bar{z}) \tag{2.89}$$

where $\Delta_\phi$ being the conformal dimension of the external operator. Notice that for $\Delta = 2\Delta_\phi + J + 2m$, the double-discontinuity vanishes. Therefore, for the double-trace exchange, the double-discontinuity vanishes. That is why the OPE of single-trace operators are sufficient to construct a function that has a double-discontinutiy equal to the four-point correlator. However, notice that the single-trace exchange $J^{FF}_0$ with quantum numbers $(\Delta, J) = (2,0)$ also vanish. Coincidently, the double-trace operator $[J^b_0, J^b_0]_{0,0}$ also has the same quantum numbers.[9] By inspection, we can see that the function below has the right double-discontinuity

$$F(u,v) = \frac{1+\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2}f_{fb}(u,v) - \frac{8}{\tilde{N}}\frac{2\tilde{\lambda}^2}{\pi^{5/2}(1+\tilde{\lambda}^2)^2}\left[\bar{D}_{11\frac{1}{2}\frac{1}{2}}(u,v) + \bar{D}_{11\frac{1}{2}\frac{1}{2}}(v,u) + \frac{1}{u}\bar{D}_{11\frac{1}{2}\frac{1}{2}}(\frac{1}{u},\frac{v}{u})\right]$$
$$+ c_1 G^{AdS}_{\phi^4} + c_2 G^{AdS}_{(\partial\phi)^4} + c_3 G^{AdS}_{\phi^2(\partial^3\phi)^2} \tag{2.90}$$

where, the function $f_{fb}(u,v)$ is the free bosonic part given by.[10]

$$f_{fb}(u,v) = 4\frac{1+u^{1/2}+v^{1/2}}{u^{1/2}v^{1/2}} \tag{2.91}$$

[9] $[\mathcal{O},\mathcal{O}]_{n,l} = \mathcal{O}\Box^n\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_l}\mathcal{O}$ − traces where $\mathcal{O}$ is a single-trace operator.

[10] Note that we may have used two separate tree-level $\phi^3$ exchange Witten diagrams corresponding to $\Delta = 1$ and $\Delta = 2$ bulk exchange with arbitrary coefficients instead [219]. But Witten diagrams themselves admitting an expansion in contact terms would compound the problem. The $\bar{D}$-functions, therefore, represent the choice with the least number of contact terms and the right double-discontinuity.

The contact terms are explicitly provided in B.46. Note that $c_1$ contains contributions from both single-trace and double-trace operators which we have separated in the following equation as $a_1$ and $\tilde{c}_1$

$$F(u,v) = \frac{1+\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2} f_{fb}(u,v) - \frac{8}{\tilde{N}} \frac{2\tilde{\lambda}^2}{\pi^{5/2}(1+\tilde{\lambda}^2)^2} \left[\bar{D}_{11\frac{1}{2}\frac{1}{2}}(u,v) + \bar{D}_{11\frac{1}{2}\frac{1}{2}}(v,u) + \frac{1}{u}\bar{D}_{11\frac{1}{2}\frac{1}{2}}(\frac{1}{u},\frac{v}{u})\right] + a_1 \bar{D}_{1111}(u,v) + \tilde{c}_1 G^{AdS}_{\phi^4} + c_2 G^{AdS}_{(\partial\phi)^4} + c_3 G^{AdS}_{\phi^2(\partial^3\phi)^2} \tag{2.92}$$

To determine $a_1$ we take the OPE limit. In the OPE limit[11] the conformal blocks behave as follow [220]

$$G_{\Delta,J}(u,v) \approx \frac{J!}{2^J(h-1)_J} u^{\Delta/2} C_J^{h-1}\left(\frac{v-1}{2\sqrt{u}}\right) \qquad \left(\text{here } h = \frac{d}{2} = \frac{3}{2}\right) \tag{2.93}$$

For $(\Delta, J) = (2,0)$ i.e. for $J_0^f$ exchange, we have $G_{2,0}(u,v) \approx u$ in the OPE limit. Since we are interested in the single-trace operator $J_0^f$, hence, we have

$$F(u,v) \approx C^{2(BBF)}_{0,susy} \tag{2.94}$$

In the OPE limit, we have for $\phi^4$ contact term

$$\bar{D}_{1111}(u,v) \approx 2 \tag{2.95}$$

By only looking at the single-trace contributions we obtain

$$a_1 = \frac{C^{2(BBF)}_{0,susy}}{2} \tag{2.96}$$

Now, we focus our attention on double-trace operators. Coefficient $\tilde{c}_1$ can now be determined by looking at the double-trace trace operator $[J_0^b J_0^b]_{0,0}$. Since, $(\Delta, J) = (2,0)$ for the double-trace is the same as that of the single-trace operator $J_0^f$, we use the same method to obtain $\tilde{c}_1$. [12]

[11] OPE limit: $u \to 0,\ v \to 1,$ with $(v-1)/u^{1/2}$ fixed

[12] The procedure above thus determines the coefficient $\tilde{c}_1$ of the first $AdS_4$ contact Witten diagram in term of the contribution of operators $J_0^f$ and $[J_0^b J_0^b]_{0,0}$. We collect the formal relation below and leave the explicit computation of these OPE coefficients for future work.

$$\tilde{c}_1 = \frac{1}{2}\left([C^{2(BBB)}_{0,susy}]_{(J_0^b)^2} - \frac{1}{\tilde{N}} \frac{4\tilde{\lambda}^2}{(1+\tilde{\lambda}^2)^2\pi^2} C^2_{0,fb} - \frac{1+\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2}[C^2_{0,fb}]_{(J_0^b)^2}\right) \tag{2.97}$$

#### 2.7.2.2 $\langle J_0^f(x_1)J_0^f(x_2)J_0^f(x_3)J_0^f(x_4)\rangle$

The four-point function of $J_0^f$ is given by the following expression

$$\langle J_0^f(x_1)J_0^f(x_2)J_0^f(x_3)J_0^f(x_4)\rangle = disc + \frac{1}{x_{13}^4 x_{24}^4}\mathcal{G}(u,v) \tag{2.98}$$

where *disc* denotes the disconnected piece given by

$$disc = \frac{1}{x_{12}^4 x_{34}^4} + \frac{1}{x_{13}^4 x_{24}^4} + \frac{1}{x_{14}^4 x_{23}^4} \tag{2.99}$$

while $F(u,v)$ is given by

$$\mathcal{G}(u,v) = \frac{1}{u}\sum_k C^2_{\mathcal{O}\mathcal{O}O_k} G_{\Delta_k,J_k}(u,v) \tag{2.100}$$

We now proceed to determine the four-point function $J_0^f$ using the same technique as above. The relevant normalized 3-point function coefficient squared are collected below[13] (see appendix (B.3) for details)

$$C^{2(FFF)}_{s,susy} = \frac{1}{\tilde{N}(1+\tilde{\lambda}^2)^2} C^2_{s,ff}, \tag{2.101}$$

$$C^{2(FFB)}_{s,susy} = \frac{\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2} C^2_{s,ff}, \tag{2.102}$$

where $C^2_{s,ff}$ is the normalized three-point functions for free fermionic theory. Note that the 3-point functions of the spin-0 exchanges given by $C^{2(FFF)}_{0,susy}$ and $C^{2(FFB)}_{0,susy}$ are contact terms in this case which, therefore, may be set to zero. This implies that the above relation is trivially satisfied for the spin $s = 0$ case as the free fermionic coeffcient $C^2_{0,ff} = 0$. Hence, both the $s = 0$ and $s \neq 0$ coefficients in this case come with the same pre-factor. This implies that the function which has the correct double discontinuity is given by

$$\mathcal{G}(u,v) = \frac{1+\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2} f_{ff}(u,v) + \bar{c}_1 G^{AdS}_{\phi^4} + \bar{c}_2 G^{AdS}_{(\partial\phi)^4} + \bar{c}_3 G^{AdS}_{\phi^2(\partial^3\phi)^2}, \tag{2.103}$$

Note that computationally $[\tilde{C}^{2(BBB)}_{0,susy}]_{(J_0^b)^2} = \frac{\langle J_0^b J_0^b (J_0^b)^2\rangle}{\langle J_0^b J_0^b\rangle\sqrt{\langle (J_0^b)^2 (J_0^b)^2\rangle}}$, the OPE coefficient involving double trace operator $(J_0^b)^2$ is as difficult as computing 4-point function. However, it may be of use to write contact term coefficients in terms of these OPE coefficients.

[13]Note that $C^{FFF}_{s,susy} = \frac{\langle J_0^f J_0^f J_s^f\rangle}{\langle J_0^f J_0^f\rangle\sqrt{\langle J_s^f J_s^f\rangle}}$ and $C^{FFB}_{s,susy} = \frac{\langle J_0^f J_0^f J_s^b\rangle}{\langle J_0^f J_0^f\rangle\sqrt{\langle J_s^b J_s^b\rangle}}$.

where $f_{ff}(u,v)$ is the free fermionic part given by

$$f_{ff}(u,v) = \frac{1 + u^{5/2} + v^{5/2} - u^{3/2}(1+v) - v^{3/2}(1+u) - u - v}{u^{3/2}v^{3/2}} \tag{2.104}$$

The coefficients $\bar{c}_i$ can be related to OPE coefficients involving double trace operator as discussed in the previous section. We leave this for future work[14].

## 2.8 Summary and Discussion

To summarize, we have proposed a conjecture for the leading $1/N$ anomalous dimension of the scalar primary operator in $U(N)_k$ Chern-Simons theories coupled to a single fundamental field, to all orders in $\lambda = N/k$. We demonstrated that our conjecture is consistent with all the existing two-loop perturbative results. We also performed a two-loop calculation of the anomalous dimension of the scalar primary $j_0$ in the bosonic theory, which provides an additional test of our conjecture. Furthermore, we showed that our conjectured expression for the leading $1/N$ anomalous dimension for the quasi-bosonic and quasi-fermionic theories satisfies an all-loop relation that was previously derived in the literature. This non-trivial consistency check gives further evidence for our proposal.

We also focused our attention on the $\mathcal{N}=2$ $U(N)$ Chern Simons theory coupled with a single fundamental chiral multiplet in the 't Hooft large $N$ limit and presented the computations for the exact 2 and 3-point functions for the scalar supermultiplet. The result are invariant under duality transformation (2.54) and can be seen as an independent confirmation of the duality. For the case of 4-point function, though we are not able to perform the direct computation for the full scalar supermultiplet, we are able to use a combination of techniques from conformal bootstrap, factorization of 3-point functions via double trace interactions along with the self-duality of our theory to determine two of the component 4 point function, namely $\langle J_0^b J_0^b J_0^b J_0^b\rangle$ and $\langle J_0^f J_0^f J_0^f J_0^f\rangle$, up to 3 undetermined coefficients. These undetermined coefficients can be fixed in terms of the OPE coefficients involving specific double trace operators. The approach used is following [208], to compute the $J_0^b$ and $J_0^f$ 4-point functions relies crucially on the fact that the double discontinuity of

[14]The coefficient $\bar{c}_1$ can be evaluated easily and is given by

$$\bar{c}_1 = \frac{3\pi^{1/2}}{8P_1^{(2)}(0,0)}\left([\tilde{C}^{2(FFF)}_{0,susy}]_{(J_0^f)^2} - \frac{1+\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2}[\tilde{C}^2_{0,ff}]_{(J_0^f)^2}\right) \tag{2.105}$$

where $P_1^{(2)}(0,0)$ is defined in appendix B.5.2

the 4-point function in the interacting theory is almost the same as that of the free theory. We could thus write down the full interacting 4-point function in terms of the free 4-point function. An obvious future direction one can pursue is to compute mixed scalar four-point functions or perhaps higher spin four-point functions in the SUSY Chern-Simons model. An interesting direction, however, would be to generalize these analytical tools like double-discontinuity and the inversion formula for momentum space $CFT$s.

# Chapter 3

# A CFT interpretation of cosmological correlation functions in $\alpha$-vacua in de-Sitter space

This chapter is largely based on the following paper written with my collaborators

- S. Jain, N. Kundu, S. Kundu, A. Mehta, S. K. Sake (2022) *A CFT interpretation of cosmological correlation functions in $\alpha$-vacua in de-Sitter space*, arXiv: 2206.08395 [hep-th], *To be published (JHEP)*

Inflation is the most elegant solution proposed to solve the issues in the standard Big Bang, for instance, the flatness and the horizon problem. These problems are easily remedied by adding an inflationary phase preceding the Big Bang characterized by the de-Sitter spacetime. The future late-time boundary of the de-Sitter signifies the end of inflation and the beginning of the standard Big Bang. In fact, the Cosmic Microwave Background (CMB) is characterized by the spatial correlations on this future boundary of the de-Sitter metric. Since the isometries of the future boundary of the de-Sitter metric are generated by $CFT_3$ algebra, these spatial correlations should also satisfy the conformal symmetry making the cosmological correlations amenable to the tools of $CFT$ bootstrap. This provides an efficient way to calculate the cosmological correlation functions [21–32, 176, 221, 222], for recent exciting development see [9, 50, 55–57, 177–180, 223–225].

The correlations in the CMB were created prior to the Big Bang and currently only the two-point correlations on the CMB have been measured and they are shown to be

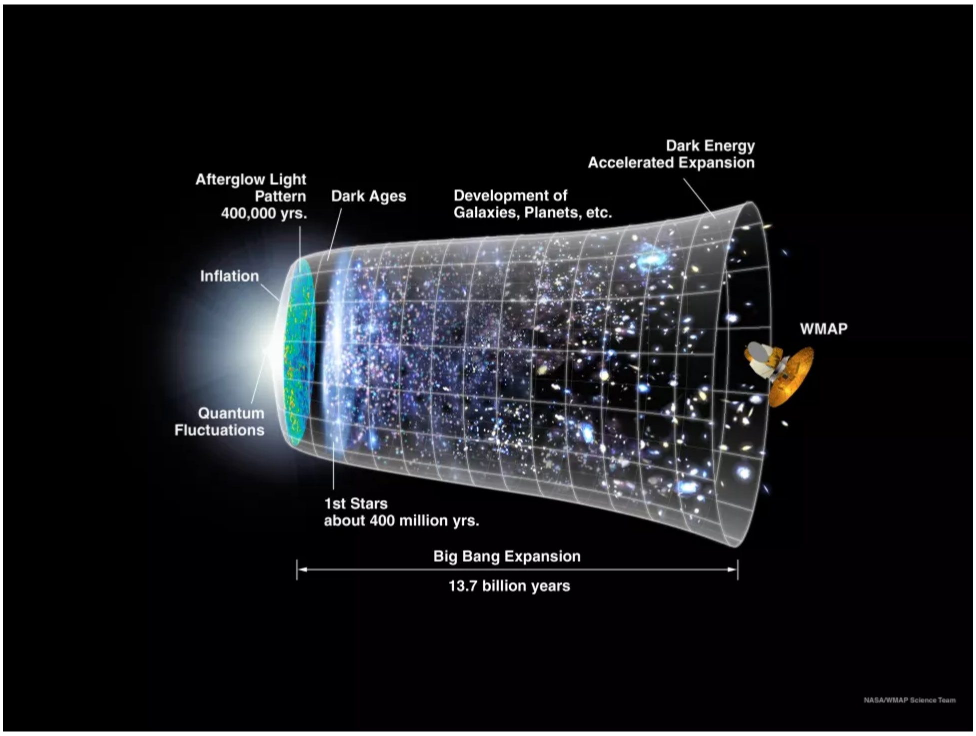



**Figure 3.1:** Inflationary cosmology *(Image credit: NASA)*

approximately scale-invariant. However, more precise measurements can reveal higher point correlations in the CMB which is known as non-gaussianity [20, 26, 226]. Any small deviation in the initial conditions is of great interest and will lead to cosmological non-gaussianity. These higher-point correlations must satisfy additional symmetries beyond scale invariance. The best possible extension of scale invariance is conformal invariance, as mentioned previously, a de Sitter expansion preceding the standard Big Bang can solve the Horizon problem etc. As the late-time symmetries of the de Sitter expansion are conformal symmetries, therefore, it is the best possible extension of the scale invariance in order to compute the non-gaussianities. These correlation functions are mostly computed in Bunch-Davies (BD) vacuum. dS space in general allows for a more general class of vacua [227, 228] which are related to BD vacuum by Bogoliubov transformation. Parametrized by a single parameter $\alpha$, they are, therefore, called the $\alpha$-vacua. Their phenomenological significance has been discussed and they are very relevant from an observational perspective [229–231]. In this chapter, we show that the momentum space CFT correlation function plays a very important role in understanding the correlations in $\alpha-$vacua. We compute these correlation functions at the late time slice of a nearly $dS_4$ background in the slow-roll approximation. We demonstrate that the correlation functions in $\alpha$-vacua can also be understood in terms of $CFT$ correlation functions with relaxed OPE consistency and we further show that the correlations in $\alpha$-vacua are related to the correlations in the Bunch-Davies vacuum by some simple sign changes of the momentum scalar.

## 3.1 Chapter summary

The rest of the chapter is organized as follows and is based on [232]. In section 3.2 we review basic facts about $\alpha-$vacua and then calculate the scalar and spinning three-point function using in-in formalism. Unlike in flat-space QFT, where the transition amplitudes are specified by the inner product between states defined on a far-past time slice and a far-future time slice, in cosmology it is not possible to define or distinguish states like that. Also, a single observer cannot access an entire time slice in de Sitter, so a transition amplitude has no meaning. Therefore, the best way to compute observables is via the in-in formalism as follows

$$\begin{aligned}&\langle\phi^{int}(x_1,t)\phi^{int}(x_2,t)\cdots\phi^{int}(x_n,t)\rangle\\&=\frac{\langle\bar{\mathcal{T}}\left(\exp\left(i\int_{-\infty(1+i\epsilon)}^{t}dt'H_I^{int}\right)\right)\prod_{i=1}^{n}\phi^{int}(x_i,t)\,\mathcal{T}\left(\exp\left(-i\int_{-\infty(1-i\epsilon)}^{t}dt'H_I^{int}\right)\right)\rangle}{\langle\bar{\mathcal{T}}\left(\exp\left(i\int_{-\infty(1+i\epsilon)}^{t}dt'H_I^{int}\right)\right)\mathcal{T}\left(\exp\left(-i\int_{-\infty(1-i\epsilon)}^{t}dt'H_I^{int}\right)\right)\rangle}\end{aligned} \tag{3.1}$$

where

$$\prod_{i=1}^{n}\phi^{int}(x_i,t)=\phi^{int}(x_1,t)\phi^{int}(x_2,t)\cdots\phi^{int}(x_n,t)\,. \tag{3.2}$$

In section 3.3 we demonstrate how the results obtained in $\alpha-$vacua can be obtained from the CFT perspective. In section 3.4 we show that conformal ward identity allows for a more general class of solutions. We also discuss their relation to correlation functions in generalized $\alpha-$vacua. In section 3.5 we discuss how to obtain parity odd correlation function contribution for $\alpha-$vacua using CFT results only. Next, in section 3.6 we show that correlation functions in $\alpha-$vacua can be expressed in terms of the same answer in BD vacuum. In other words, we will show that given the structure in BD vacuum, we can obtain the corresponding expression for the correlator in $\alpha-$vacuum. In section 3.7 we summarise the findings. In Appendices, we collect some useful background details which are helpful in the main draft.

## 3.2 Cosmological correlation function in $\alpha-$vacua

The aim of this section is to use the in-in formalism to calculate cosmological correlation functions. We first start with a brief review of $\alpha-$vacua. We then use in-in formalism to calculate scalar and spinning three-point functions. We work in Poincare coordinate in dS. The metric is given by

$$ds^2 = \frac{1}{H^2\eta^2}\left(-d\eta^2 + dx^i dx_i\right) \tag{3.3}$$

with $-\infty < \eta \leq 0$ where $H$ is the Hubble parameter, which we set to unity in this paper. Let us consider free scalar field theory

$$S = -\frac{1}{2}\int d^3x d\eta \sqrt{-g} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi. \tag{3.4}$$

Mode expansion of the scalar field in the Bunch-Davies vacuum is given by

$$\varphi(x,\eta) = \int \frac{d^3k}{(2\pi)^3}\left(a_k v_k(\eta) + a^\dagger_{-k} v^*_k(\eta)\right) e^{ik.x} \tag{3.5}$$

where $v_k(\eta) = \frac{1}{\sqrt{2k^3}}\left(1 + ik\eta\right) e^{-ik\eta}$. The Bunch-Davies vacuum is defined by

$$a_k|0\rangle = 0 \;\forall\; k \tag{3.6}$$

For dS space, one can define two real parameter sets of general vacuum $|\alpha,\beta\rangle$ [228]. The mode expansion of massless scalar field (3.4) $\varphi$ is given by

$$\varphi(\eta,\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\left[b_{\boldsymbol{k}} u_k(\eta) + b^\dagger_{-\boldsymbol{k}} u^*_k(\eta)\right] \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{3.7}$$

and

$$u_k(\eta) = \frac{1}{\sqrt{2k^3}}\left\{A(1-ik\eta)\mathrm{e}^{ik\eta} + B(1+ik\eta)\mathrm{e}^{-ik\eta}\right\} \tag{3.8}$$

where $A, B$ are arbitrary complex numbers that satisfy

$$|A|^2 - |B|^2 = 1. \tag{3.9}$$

One choice of parameters is

$$A = \cosh(\alpha), \qquad B = -ie^{i\beta}\sinh(\alpha). \tag{3.10}$$

The $|\alpha, \beta\rangle$ vacuum is defined by

$$b_k|\alpha, \beta\rangle = 0 \quad \forall\ k \tag{3.11}$$

Let us note that BD vacuum is a special case $\alpha = 0$ of this general vacuum. One can write down the following relation

$$|\alpha, \beta\rangle = \prod_k \frac{1}{\sqrt{|B|}} \exp\left(\frac{A^*}{2B^*} a_k^\dagger a_{-k}^\dagger\right) |0\rangle. \tag{3.12}$$

## Mode expansion for metric fluctuations

We take the mode expansion in $\alpha-$vacua for the metric fluctuation to be given by

$$\gamma^{\mu\nu}(x,\eta) = \int \frac{d^3k}{(2\pi)^3} \left(a_k u_k^{\mu\nu}(\eta) + a_{-k}^\dagger u_k^{*\mu\nu}(\eta)\right) e^{ik.x} \tag{3.13}$$

with

$$u^{\mu\nu}(k,\eta) = \frac{z^\mu z^\nu}{\sqrt{2k^3}}[e^{ik\eta}(1-ik\eta)C + e^{-ik\eta}(1+ik\eta)D] \tag{3.14}$$

where $z^\mu, z^\nu$ are polarization tensors satisfying the conditions

$$z^\mu = (0, \vec{z}), \quad \text{where} \quad z^2 = 0,\ z\cdot k = 0, \tag{3.15}$$

and $C, D$ are some complex numbers satisfying $|C|^2 - |D|^2 = 1$

$$C = \cosh(\tilde{\alpha}), \qquad D = -ie^{i\tilde{\beta}}\sinh(\tilde{\alpha}). \tag{3.16}$$

The $|\tilde{\alpha}, \tilde{\beta}\rangle$ vacuum is defined by

$$a_k|\tilde{\alpha}, \tilde{\beta}\rangle = 0. \tag{3.17}$$

We now use these mode expansions to calculate the correlation function in $\alpha-$vacua.

To calculate the cosmological correlation function we are going to use in-in formalism which gives the correlation function

$$\begin{aligned}&\langle\phi^{int}(x_1,t)\phi^{int}(x_2,t)\cdots\phi^{int}(x_n,t)\rangle\\&=\frac{\langle\bar{\mathcal{T}}\left(\exp\left(i\int_{-\infty(1+i\epsilon)}^{t}dt'H_I^{int}\right)\right)\prod_{i=1}^{n}\phi^{int}(x_i,t)\,\mathcal{T}\left(\exp\left(-i\int_{-\infty(1-i\epsilon)}^{t}dt'H_I^{int}\right)\right)\rangle}{\langle\bar{\mathcal{T}}\left(\exp\left(i\int_{-\infty(1+i\epsilon)}^{t}dt'H_I^{int}\right)\right)\mathcal{T}\left(\exp\left(-i\int_{-\infty(1-i\epsilon)}^{t}dt'H_I^{int}\right)\right)\rangle}\end{aligned} \tag{3.18}$$

where

$$\prod_{i=1}^{n}\phi^{int}(x_i,t)=\phi^{int}(x_1,t)\phi^{int}(x_2,t)\cdots\phi^{int}(x_n,t)\,. \tag{3.19}$$

In (3.18) $\phi^{int}(x_i,t)$ are the fields written in the interaction picture, and $H_I^{int}$ is the interacting Hamiltonian in the interaction picture[1].

In the following sub-sections, we will use (3.18) to compute two and three-point functions involving scalar and tensor fields. For that, we will transform the time coordinate to conformal time variable $\eta$. Once we know the explicit form of $H_I^{int}$ relevant for the particular correlation function, the RHS of (3.18) will be determined perturbatively by bringing down factors of $H_I^{int}$ from the exponential. Next, one needs to substitute mode expansions for the fields given in (3.7) and (3.13). Finally, one needs to use the desired vacuum given in (3.17). Once all these are taken care of, ultimately, we will be left with an integration over the conformal time $\eta$ along a chosen contour from $\eta=-\infty$ to $\eta=0$ (see [26]). In the following sections, we will look into various cases and will perform this $\eta$-integration.

These correlation functions can be interpreted as correlation functions of some appropriate operators in three-dimensional Euclidean CFT, for details see [26]. For example, a conformally coupled scalar field (denoted by $\varphi$) with mass $m^2=2H^2$ in $dS_4$, will correspond to an operator $O_\Delta$ with scaling dimension $\Delta=2$ in $CFT_3$. Similarly, for a massless scalar field $\phi$ in $dS_4$, we will have $O_\Delta$ with $\Delta=3$. Also, for a massless vector field $A_\mu$ in $dS_4$, we will get a conserved spin one current $J_i$ (with $\Delta=2$), and a spin two graviton $\gamma_{\mu\nu}$ in $dS_4$ will correspond to a conserved spin two current $T_{ij}$ (with $\Delta=3$) in $CFT_3$ [2]. Performing the calculation on the RHS of (3.18), we will obtain correlation functions

[1]We will drop the subscripts and the superscripts on the fields in the following discussions to avoid the clutter of notations.

[2]It should be emphasized that we are not assuming any kind of dS/CFT correspondence here. These statements follow just from the fact that the asymptotic isometry of $dS_4$ (as $\eta\to 0$) is the same as the symmetry group of 3-dimensional Euclidean CFT.

involving late time values of the fields $\varphi$, $A_\mu$, $\gamma_{\mu\nu}$. Next, following the steps outlined in appendix C.2 we can translate them to correlators involving the operators $O$, $J$, $T$ in CFT.

### 3.2.1 Two-point function

In this section, we calculate the two-point function of scalar and graviton field in rigid dS background using in-in formalism.

#### 3.2.1.1 $\langle\varphi\varphi\rangle$

In this case, the two-point function is given by

$$\langle\varphi(k_2)\varphi(-k_2)\rangle = \frac{(A+B)(\bar{A}+\bar{B})}{2k_2^3} \tag{3.20}$$

where the mode expansion eq.(3.7) is used in obtaining the above result. After the shadow transform, discussed in appendix C.2 , gives the following correlator

$$\langle O(k_2)O(-k_2)\rangle = \frac{2k_2^3}{(A+B)(\bar{A}+\bar{B})} \tag{3.21}$$

where in the last line we have used $|A|^2 - |B|^2 = 1$. Now using (3.10) we have

$$\langle O(k_2)O(-k_2)\rangle = \frac{k_2^3}{(\cosh 2\alpha + \sin\beta\sinh 2\alpha)} \tag{3.22}$$

and when $\beta = 0$ we get

$$\langle O(k_2)O(-k_2)\rangle = \frac{1}{\cosh 2\alpha}k_2^3. \tag{3.23}$$

We now compute a two-point function of the graviton.

#### 3.2.1.2 $\langle TT\rangle$

We recalculate $\langle TT\rangle$ in a general vacuum using the mode expansion for the graviton given by (3.14). The $\langle\gamma\gamma\rangle$ can be computed to be

$$\langle\gamma(k_1)\gamma(k_2)\rangle = (z_1\cdot z_2)^2\frac{(C+D)(\bar{C}+\bar{D})}{2k_1^3}. \tag{3.24}$$

After the shadow transform, we have

$$\langle T(z_1,k_1)T(z_2,k_2)\rangle = (z_1\cdot z_2)^2\frac{2k_1^3}{(C+D)(\bar{C}+\bar{D})}. \tag{3.25}$$

Now using (3.16) we get

$$\langle T(z_1,k_1)T(z_2,k_2)\rangle = (z_1\cdot z_2)^2\frac{1}{(\cosh 2\tilde{\alpha}+\sin\tilde{\beta}\sinh 2\tilde{\alpha})}k_2^3 \tag{3.26}$$

and when $\tilde{\beta}=0$ we get

$$\langle T(z_1,k_1)T(z_2,k_2)\rangle = (z_1\cdot z_2)^2\frac{1}{\cosh 2\tilde{\alpha}}k_2^3. \tag{3.27}$$

### 3.2.2 Three-point function

We now turn our attention to calculating the three-point functions. We start with the simplest case of scalars and then move on to spinning fields.

#### 3.2.2.1 $\langle OOO\rangle$

We recalculate $\langle OOO\rangle$ in a general vacua characterized by the mode expansion (3.7). Using the mode expansion in (3.18), with

$$H_{int} = \int d^4x\sqrt{-g}\;\phi^3 \tag{3.28}$$

we obtain the following time integral

$$\begin{aligned}6\,\mathrm{Im}\Bigg[\int_{-\infty}^{0}\frac{d\eta}{\eta^4}\Bigg(&[-f_{k_3}(A,B)\bar{u}_{k_3}(\eta)+\bar{f}_{k_3}(A,B)u_{k_3}(\eta)]f_{k_1}(A,B)f_{k_2}(A,B)\bar{u}_{k_1}(\eta)\bar{u}_{k_2}(\eta)\\ &+[-f_{k_2}(A,B)\bar{u}_{k_2}(\eta)+\bar{f}_{k_2}(A,B)u_{k_2}(\eta)f_{k_1}(A,B)\bar{f}_{k_3}(A,B)\bar{u}_{k_1}(\eta)u_{k_3}(\eta)+[-f_{k_1}(A,B)\bar{u}_{k_1}(\eta)\\ &+\bar{f}_{k_1}(A,B)u_{k_1}(\eta)]\bar{f}_{k_2}(A,B)\bar{f}_{k_3}(A,B)u_{k_2}(\eta)u_{k_3}(\eta)\Bigg)\Bigg]\end{aligned} \tag{3.29}$$

where $f_k(A,B)$ and $u_k(\eta)$ have been defined previously in (3.8) and eq.(C.7). The integral once computed, after the shadow transform, leads to

$$\langle OOO\rangle = aR_1 + b\,(R_2+R_3+R_4) \tag{3.30}$$

where

$$
\begin{aligned}
a &= \frac{1}{2\mathcal{N}^3(A,B)}[(2A^2+3AB+3B^2)(\bar{A}^2+\bar{B}^2+\bar{A}\bar{B})+(A-B)\bar{B}(|A|^2+|B|^2+A\bar{B})] \\
b &= \frac{1}{2\mathcal{N}^3(A,B)}[(A^2+6AB+B^2)\bar{A}\bar{B}+A\bar{B}^2(-A+B)+(A-B)B\bar{A}^2] \\
\mathcal{N}(A,B) &= (A+B)\left(\bar{A}+\bar{B}\right)
\end{aligned}
\tag{3.31}
$$

and

$$
R_1 = R(k_1,k_2,k_3) \quad R_2 = R(k_1,k_2,-k_3) \quad R_3 = R(k_1,-k_2,k_3) \quad R_4 = R(-k_1,k_2,k_3) \tag{3.32}
$$

with

$$
R(k_1,k_2,k_3) = -\frac{4}{9}\sum_a k_a^3 - \frac{1}{3}\sum_{a\neq b} k_a^2 k_b + \frac{1}{3}k_1k_2k_3 + 3\sum_a k_a^3 \log E \tag{3.33}
$$

where $E = k_1 + k_2 + k_3$. Notice the bulk computation automatically forces exchange symmetry. Also, note that $a, b$ is real i.e. under $A \leftrightarrow \bar{A}, B \leftrightarrow \bar{B}$ the coefficients $a, b$ are invariant meaning $a^* = a, b^* = b$. Using the parameterization as in (3.10) we have

$$
\langle OOO \rangle = aR_1 + b\left(R_2 + R_3 + R_4\right) \tag{3.34}
$$

where

$$
\begin{aligned}
a &= \frac{5+3\cosh 4\alpha - 6\cos 2\beta \sinh^2 2\alpha + 6\sin\beta\sinh 4\alpha}{8(\cosh 2\alpha + \sin\beta\sinh 2\alpha)^3} \\
b &= \frac{[\cosh 2\alpha \sin\beta + (3+\cos 2\beta)\cosh\alpha\sinh\alpha]\sinh 2\alpha}{2(\cosh 2\alpha + \sin\beta\sinh 2\alpha)^3}
\end{aligned}
\tag{3.35}
$$

For the special case of $\beta = 0$ we have

$$
\begin{aligned}
a &= \frac{1}{\cosh^3 2\alpha} \\
b &= \frac{\sinh^2\alpha}{\cosh^3 2\alpha}
\end{aligned}
\tag{3.36}
$$

which gives

$$\langle OOO\rangle = \frac{1}{\cosh^3 2\alpha}[R_1 + \sinh^2 2\alpha\,(R_2 + R_3 + R_4)]. \tag{3.37}$$

The results in (3.37) are the same as those that appears in [233].

## Spinning correlator

For this case we need to define the vacuum suitably. We have characterized the vacuum for scalar mode in (3.11) and for tensor mode in (3.17). In general when we consider correlation function involving both graviton and scalar we can consider most general vacuum defined of the form $|\alpha, \beta, \tilde{\alpha}, \tilde{\beta}\rangle$. However to start with let us take a simpler situation when we have $A = C, B = D$ which implies $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$.

### 3.2.2.2 $\langle TOO\rangle$

To compute two scalars and one graviton amplitude we need to consider

$$H_{int} = \int d^4x\sqrt{-g}\; g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{3.38}$$

Here we just provide the final results, the computational details of which are provided in Appendix C.3.1. The correlation function is given by

$$\langle TO_3O_3\rangle = a\langle TO_3O_3\rangle_{R_1} + b\,(\langle TO_3O_3\rangle_{R_2} + \langle TO_3O_3\rangle_{R_3} + \langle TO_3O_3\rangle_{R_4}) \tag{3.39}$$

where the form of $a, b$ is same as that appears in (3.31) and

$$\begin{aligned}
\langle TO_3O_3\rangle_{R_1} &= \left[k_1 + k_2 + k_3 - \frac{k_1k_2 + k_2k_3 + k_3k_1}{k_1 + k_2 + k_3} - \frac{k_1k_2k_3}{(k_1 + k_2 + k_3)^2}\right](z_1.k_2)^2\\
\langle TO_3O_3\rangle_{R_2} &= \left[-k_1 + k_2 + k_3 - \frac{-k_1k_2 + k_2k_3 - k_3k_1}{-k_1 + k_2 + k_3} + \frac{k_1k_2k_3}{(-k_1 + k_2 + k_3)^2}\right](z_1.k_2)^2\\
\langle TO_3O_3\rangle_{R_3} &= \left[k_1 - k_2 + k_3 - \frac{-k_1k_2 - k_2k_3 + k_3k_1}{k_1 - k_2 + k_3} + \frac{k_1k_2k_3}{(k_1 - k_2 + k_3)^2}\right](z_1.k_2)^2\\
\langle TO_3O_3\rangle_{R_4} &= \left[k_1 + k_2 - k_3 - \frac{k_1k_2 - k_2k_3 - k_3k_1}{k_1 + k_2 - k_3} + \frac{k_1k_2k_3}{(k_1 + k_2 - k_3)^2}\right](z_1.k_2)^2
\end{aligned} \tag{3.40}$$

#### 3.2.2.3 $\langle TTO_3 \rangle$

To calculate two gravitons and one scalar amplitude we consider the following interaction term

$$H_{int} = \int d^4x \, \sqrt{-g}\varphi \mathcal{W}_{\rho\sigma\alpha\beta}\mathcal{W}_{\rho\sigma\alpha\beta} \tag{3.41}$$

Here we write down the final result, details of the computation are provided in appendix C.3.2. The correlation function is given by

$$\langle TTO_3 \rangle_\alpha = a\langle TTO_3 \rangle_{R_1} + b\left(\langle TTO_3 \rangle_{R_4} + \langle TTO_3 \rangle_{R_2} + \langle TTO_3 \rangle_{R_3}\right). \tag{3.42}$$

where the form of $a, b$ is same as that appears in (3.31). The explicit expressions for $\langle TTO_3 \rangle_{R_i}$ are complicated. They are best written in spinor helicity variables. We give their explicit forms in section 3.3.3.

#### 3.2.2.4 $\langle TTT \rangle$

The three graviton amplitude can get contributions from two different sources, the Einstein Hilbert term, and the $Weyl^3$ term. Let us start with the Weyl tensor contribution.

#### $Weyl^3$ contribution

To calculate the three graviton amplitude we need to consider the following interaction

$$S^{(3)}_{\gamma,\mathcal{W}^3} = \int d^4x\sqrt{-g}\; \mathcal{W}^3. \tag{3.43}$$

where $\mathcal{W}_{abcd}$ is the Weyl tensor. We provide computational details in appendix C.3.3. The correlation functions are given by

$$\langle TTT \rangle_{\mathcal{W}^3,\alpha} = a\langle TTT \rangle_{\mathcal{W}^3,1} + b(\langle TTT \rangle_{\mathcal{W}^3,2} + \langle TTT \rangle_{\mathcal{W}^3,3} + \langle TTT \rangle_{\mathcal{W}^3,4}) \tag{3.44}$$

where $a, b$ are same as that appears in (3.31). We also have

$$\langle TTT \rangle_{\mathcal{W}^3,i} = \langle TTT \rangle_{h,R_i} \tag{3.45}$$

where $\langle TTT \rangle_{h,R_i}$ appear in (3.119), (3.120).

**Two-derivative interaction the Einstein-Hilbert contribution**

Consider now the interaction [3]

$$S^{(3)}_{\gamma,EG} = \int d^4x\sqrt{-g}\ R \tag{3.46}$$

Again details are provided in Appendix C.3.3. The correlation function in this case is given by

$$\langle TTT\rangle_{EG,\alpha} = a\langle TTT\rangle_{EG,1} + b(\langle TTT\rangle_{EG,2} + \langle TTT\rangle_{EG,3} + \langle TTT\rangle_{EG,4}) \tag{3.47}$$

where $a, b$ are precisely given again by (3.31) and

$$\langle TTT\rangle_{EG,i} = \langle TTT\rangle_{nh,R_i} \quad i = 1,2,3,4 \tag{3.48}$$

where $\langle TTT\rangle_{nh,R_i}$'s appear in (3.123).

### 3.2.3 Spinning correlator in general vacuum

In this subsection we consider more spinning correlator of the form $\langle TOO\rangle$ and $\langle TTO\rangle$. In earlier section we consider this correlation function for the special case when $A = C, B = D$ which implies $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$. We now consider the case when $A \neq C, B \neq D$ which implies $\alpha \neq \tilde{\alpha}$ and $\beta \neq \tilde{\beta}$. For this case the general vacuum is denoted by $|\alpha, \beta, \tilde{\alpha}, \tilde{\beta}\rangle$. For this case as well calculation is straightforward and is given in Appendix C.3.1 and C.3.2. The $\langle TOO\rangle$ is then obtained to be

$$\langle TO_3O_3\rangle = c_1\langle TO_3O_3\rangle_{R_1} + c_2\langle TO_3O_3\rangle_{R_2} + c_3\left(\langle TO_3O_3\rangle_{R_3} + \langle TO_3O_3\rangle_{R_4}\right) \tag{3.49}$$

where $c_i$ are given in (C.10). Let us note that as compared to (3.39) where the correlator was parameterized by two parameters, here we have an extra parameter. The Ward-Takahashi(WT) identity is given by

$$k_1^\mu z^\nu\langle T_{\mu\nu}O_3O_3\rangle = (c_1 + c_2)(z_1.k_2)(k_2^3 - k_3^3). \tag{3.50}$$

For the case of $\langle TTO_3\rangle$ we have

$$\langle TTO_3\rangle_\alpha = d_1\langle TTO_3\rangle_{R_1} + d_2\langle TTO_3\rangle_{R_4} + d_3(\langle TTO_3\rangle_{R_2} + \langle TTO_3\rangle_{R_3}) \tag{3.51}$$

[3]Before we put our paper on arXiv, [231] appeared which computes this correlation function and discusses its phenomenological implications.

where $d_i$ are given in (C.24). Again, let us note that as compared to (3.42) where the correlator was parameterized by two parameters, here we have an extra parameter and is thus parameterized by three parameters.

## 3.3 Spinning CFT correlators using weight shifting and spin raising operator

In this section, we discuss how to understand the results obtained in the previous section from a CFT perspective. Let us start our discussion with a two-point function.

We shall be using the notation $k \equiv |\vec{k}|$ in the following. We also choose to work with null and transverse polarization tensors such that $z_i.k_i = 0, z_i^2 = 0$.

### 3.3.1 Two-point function

Consider the following two-point functions of stress tensor and scalar operator $O_3$

$$\langle T(z_1, k_1)T(z_2, k_2)\rangle = C_T(z_1.z_2)^2 k_1^3 \tag{3.52}$$

$$\langle O_3(k_1)O_3(k_2)\rangle = C_O k_3^3 \tag{3.53}$$

Comparing this result with general $\alpha-$vacuum answer (3.25) and (C.17) one can obtain $C_O, C_T$.

### 3.3.2 Three-point function

We now turn our attention to computing the three-point function. We start with the scalar three-point function. The momentum space result for the three-point function is much richer than the position space counterpart. For scalar three-point function, there are four different solutions to conformal ward identity [41]. If we impose consistency with OPE limit, only one particular combination of four solutions survives, which matches with the cosmological correlation function computed in the Bunch-Davies (BD) vacuum and is also can be thought of as a Fourier transform of position space answer. As will be shown in this section, the cosmological correlation function in general $\alpha-$vacua requires us to consider even the solutions which are not consistent with OPE limit. As will be shown below, this is true even for correlation functions with spinning operators.

#### 3.3.2.1 $\langle O_3 O_3 O_3 \rangle$

For the case of a scalar three-point function, we have four solutions to conformal ward identity. These are given by [41]

$$R_1 = A(k_1, k_2, k_3) \quad R_2 = A(k_1, k_2, -k_3) \quad R_3 = A(k_1, -k_2, k_3) \quad R_4 = A(-k_1, k_2, k_3) \tag{3.54}$$

where

$$A(k_1, k_2, k_3) = -\frac{4}{9}\sum_a k_a^3 - \frac{1}{3}\sum_{a\neq b} k_a^2 k_b + \frac{1}{3}k_1 k_2 k_3 + 3\sum_a k_a^3 \log E \tag{3.55}$$

where $E = k_1 + k_2 + k_3$. The most general answer to the three-point function is given by

$$\langle O_3(k_1) O_3(k_2) O_3(k_3)\rangle = c_1 R_1 + c_2 R_2 + c_3 R_3 + c_4 R_4. \tag{3.56}$$

Consistency with OPE limit sets $c_2 = c_3 = c_3 = 0$ and gives the Bunch-Davies (BD) vacuum answer

$$\langle O_3(k_1) O_3(k_2) O_3(k_3)\rangle_{BD} = -\frac{4}{9}\sum_a k_a^3 - \frac{1}{3}\sum_{a\neq b} k_a^2 k_b + \frac{1}{3}k_1 k_2 k_3 + 3\sum_a k_a^3 \log E. \tag{3.57}$$

However, if we don't demand consistency with OPE, we obtain the most general answer given in (3.56). The exchange symmetry $1 \leftrightarrow 2 \leftrightarrow 3$ requires $c_2 = c_3 = c_4$ which gives

$$\langle O_3(k_1) O_3(k_2) O_3(k_3)\rangle = c_1 R_1 + c_2 (R_2 + R_3 + R_4). \tag{3.58}$$

Let us note that, $c_1$ and $c_2$ are real numbers because the correlator is real. Comparing with general $\alpha-$vacua result (3.30),(3.31) we obtain values of parameter $c_1, c_2$

$$c_1 = a, c_2 = b \tag{3.59}$$

Having discussed the scalar three-point function case, let us consider the spinning correlator. The story here is much richer than the scalar case. Let us start with the simplest of the spinning correlation function $\langle T O_3 O_3\rangle$.

#### 3.3.2.2 $\langle T O_3 O_3 \rangle$

The easiest way to get to the result is to use the spin and dimension raising operator starting from $\langle O_3 O_3 O_3\rangle$. The spin and dimension raising operator is reviewed in the appendix **??**. The correlator $\langle TO_3O_3\rangle$ can be obtained by

$$\langle TO_3O_3\rangle = D_{13}D_{12}\mathcal{W}_{23}^{++}\langle O_3O_3O_3\rangle \tag{3.60}$$

which in the momentum space is given by

$$\langle TO_3O_3\rangle_\alpha = a\langle TO_3O_3\rangle_{R_1} + b(\langle TO_3O_3\rangle_{R_2} + \langle TO_3O_3\rangle_{R_3} + \langle TO_3O_3\rangle_{R_4}) \tag{3.61}$$

where $a, b$ are precisely as given in (3.31) and $\langle TO_3O_3\rangle_{R_i}$ is same as that appears in (3.40). This also satisfies Ward identity given by

$$\begin{aligned} k_1^\mu z^\nu\langle T_{\mu\nu}O_3O_3\rangle &= (a+b)(z_1.k_2)(k_2^3 - k_3^3) \\ &= (z_1.k_2)(\langle O_3(k_2)O_3(-k_2)\rangle - \langle O_2(k_3)O_2(-k_3)\rangle) \end{aligned} \tag{3.62}$$

where we have identified $a + b = C_{\Delta=3}$. We would like to understand the results from the perspective of solutions of conformal ward identity as we did for the case of the scalar correlator in the previous subsection. We do this in the next section.

#### 3.3.2.3 $\langle JJO_3\rangle$

Let us now consider $\langle JJO_3\rangle$. The correlator $\langle JJO_3\rangle$ may be computed from $\langle O_3O_3O_3\rangle$ using spin raising and weight shifting operators as follows

$$\langle JJO_3\rangle = k_1k_2H_{12}\mathcal{W}_{12}^{--}\langle O_3O_3O_3\rangle \tag{3.63}$$

which in momentum space is given by

$$\langle JJO_3\rangle = a\langle JJO_3\rangle_{R_1} + b(\langle JJO_3\rangle_{R_2} + \langle JJO_3\rangle_{R_3} + \langle JJO_3\rangle_{R_4}) \tag{3.64}$$

where

$$\begin{aligned} \langle JJO_3\rangle_{R_1} &= \frac{12(k_1+k_2+2k_3)}{(k_1+k_2+k_3)^2}[(k_1+k_2+k_3)(k_1+k_2-k_3)(z_1.z_2) + 2z_1.k_2z_2.k_1] \\ \langle JJO_3\rangle_{R_2} &= \frac{12(-k_1+k_2+2k_3)}{(k_2+k_3-k_1)^2}[(k_1-k_2-k_3)(k_1+k_3-k_2)(z_1.z_2) + 2z_1.k_2z_2.k_1] \\ \langle JJO_3\rangle_{R_3} &= \frac{12(k_1-k_2+2k_3)}{(k_1-k_2+k_3)^2}[(k_1-k_2+k_3)(k_1-k_3-k_2)(z_1.z_2) + 2z_1.k_2z_2.k_1] \\ \langle JJO_3\rangle_{R_4} &= \frac{12(k_1+k_2-2k_3)}{(k_1+k_2-k_3)^2}[(k_1+k_2+k_3)(k_1+k_2-k_3)(z_1.z_2) + 2z_1.k_2z_2.k_1] \end{aligned} \tag{3.65}$$

Again demanding consistency with OPE limit would give

$$\langle JJO_3\rangle = \langle JJO_3\rangle_{R_1} \tag{3.66}$$

which is precisely the correlation function obtained in the Bunch-Davies vacuum. In spinor helicity variables, results look much simpler and take the following form

$$\langle J^-J^-O_3\rangle_{R_1} = \frac{12(k_1+k_2+2k_3)}{(k_1+k_2+k_3)^2}\langle 12\rangle^2 \quad \langle J^-J^-O_3\rangle_{R_4} = \frac{12(k_1+k_2-2k_3)}{(k_1+k_2-k_3)^2}\langle 12\rangle^2 \tag{3.67}$$

$$\langle J^-J^-O_3\rangle_{R_2} = \langle J^-J^-O_3\rangle_{R_3} = 0 \tag{3.68}$$

$$\begin{aligned}&\langle J^-J^+O_3\rangle_{R_1} = \langle J^-J^+O_3\rangle_{R_4} = 0\\ &\langle J^-J^+O_3\rangle_{R_2} = \frac{12(-k_1+k_2+2k_3)}{E^2(k_2+k_3-k_1)^2}\langle 31\rangle^2\langle\bar{2}\bar{3}\rangle^2 \quad \langle J^-J^+O_3\rangle_{R_3} = \frac{12(k_1-k_2+2k_3)}{E^2(k_1-k_2+k_3)^2}\langle 31\rangle^2\langle\bar{2}\bar{3}\rangle^2\end{aligned} \tag{3.69}$$

and its complex conjugates. One can check that this result is consistent with the calculation in dS space if we consider

$$\int \varphi F_{\mu\nu}F^{\mu\nu}. \tag{3.70}$$

The relation between the correlators in general $\alpha-$vacua and BD vacuum can be written as

$$\langle JJO_3\rangle_\alpha - (a+b)\langle JJO_3\rangle_{BD} = -b[\langle JJO_3\rangle_{R_1} - \langle JJO_3\rangle_{R_2} - \langle JJO_3\rangle_{R_3} - \langle JJO_3\rangle_{R_3}]. \tag{3.71}$$

### 3.3.3 $\langle TTO_3\rangle$

Similarly, correlator $\langle TTO_3\rangle$ may be computed from $\langle O_3O_3O_3\rangle$ using

$$\langle TTO_3\rangle = k_1^3k_2^3H_{12}^2\mathcal{W}_{12}^{--}\langle O_3O_3O_3\rangle \tag{3.72}$$

which again in momentum space becomes

$$\langle TTO_3\rangle = a\langle TTO_3\rangle_{R_1} + b(\langle TTO_3\rangle_{R_2} + \langle TTO_3\rangle_{R_3} + \langle TTO_3\rangle_{R_4}) \tag{3.73}$$

In momentum space $\langle TTO_3\rangle$ is complicated, however, in spinor helicity variables they take a simple form and are given by

$$\langle T^-T^-O_3\rangle_{R_1} = \frac{48k_1k_2(k_1+k_2+4k_3)}{(k_1+k_2+k_3)^4}\langle 12\rangle^4 \quad \langle T^-T^-O_3\rangle_{R_4} = \frac{48k_1k_2(k_1+k_2-4k_3)}{(k_1+k_2-k_3)^4}\langle 12\rangle^4$$
$$\langle T^-T^-O_3\rangle_{R_2} = \langle T^-T^-O_3\rangle_{R_3} = 0. \tag{3.74}$$

For the mixed helicity, we have

$$\langle T^-T^+O_3\rangle_{R_1} = \langle T^-T^+O_3\rangle_{R_4} = 0$$
$$\langle T^-T^+O_3\rangle_{R_2} = \frac{48k_1k_2(-k_1+k_2+4k_3)}{E^4(k_2+k_3-k_1)^4}\langle 31\rangle^4\langle\bar{2}\bar{3}\rangle^4 \quad \langle T^-T^+O_3\rangle_{R_3} = \frac{48k_1k_2(k_1-k_2+4k_3)}{E^4(k_1-k_2+k_3)^4}\langle 31\rangle^4\langle\bar{2}\bar{3}\rangle^4. \tag{3.75}$$

In general vacua, we have

$$\langle TTO_3\rangle_\alpha - (a+b)\langle TTO_3\rangle_{BD} = -b[\langle TTO_3\rangle_{R_1} - \langle TTO_3\rangle_{R_2} - \langle TTO_3\rangle_{R_3} - \langle TTO_3\rangle_{R_4}] \tag{3.76}$$

In spinor-helicity variables, the independent components are

$$\langle T^-T^-O_3\rangle_\alpha - (a+b)\langle T^-T^-O_3\rangle_{BD} = -b(\langle T^-T^-O_3\rangle_{R_1} - \langle T^-T^-O_3\rangle_{R_4}) \tag{3.77}$$
$$\langle T^-T^+O_3\rangle_\alpha - (a+b)\langle T^-T^+O_3\rangle_{BD} = -b(\langle T^-T^+O_3\rangle_{R_2} + \langle T^-T^+O_3\rangle_{R_3}) \tag{3.78}$$

### 3.3.4 $\langle TTT\rangle$

Let us now turn our attention to $\langle TTT\rangle$. The Ward-Takahashi identity is given by

$$\begin{aligned} k_1^\mu z_1^\nu\langle T_{\mu\nu}T(z_2,k_2)T(z_3,k_3)\rangle =& -(z_1.k_2)\, z_2^i z_2^j\langle T^{ij}_{\vec{k}_2+\vec{k}_1}T_{\vec{k}_3}\rangle + 2\,(z_1.z_2)\, z_2^j k_2^i\langle T^{ij}_{\vec{k}_2+\vec{k}_1}T_{\vec{k}_3}\rangle \\ &-(z_1\cdot k_3)\, z_3^i z_3^j\langle T_{\vec{k}_2}T^{ij}_{\vec{k}_3+\vec{k}_1}\rangle + 2(z_1.z_3) z_3^j k_3^i\langle T_{\vec{k}_2}T^{ij}_{\vec{k}_3+\vec{k}_1}\rangle \\ &+(z_2.k_1) z_1^i z_2^j\langle T^{ij}_{\vec{k}_2+\vec{k}_1}T_{\vec{k}_3}\rangle + (z_1.z_2)k_1^i z_2^j\langle T^{ij}_{\vec{k}_2+\vec{k}_1}T_{\vec{k}_3}\rangle \\ &+(z_3.k_1) z_1^i z_3^j\langle T_{\vec{k}_2}T^{ij}_{\vec{k}_1+\vec{k}_3}\rangle + (z_1.z_3)k_1^i z_3^j\langle T_{\vec{k}_2}T^{ij}_{\vec{k}_1+\vec{k}_3}\rangle. \end{aligned} \tag{3.79}$$

So the solution for $\langle TTT\rangle$ is given by [186]

$$\langle TTT\rangle = \langle TTT\rangle_h + \langle TTT\rangle_{nh}. \tag{3.80}$$

#### 3.3.4.1 Homogenous part of $\langle TTT\rangle$

The homogenous part of $\langle TTT\rangle$ may be computed from $\langle O_3O_3O_3\rangle$ using spin-raising and weight shifting operators as follows [50, 180],

$$\langle TTT\rangle = k_3^3k_2^3(\mathcal{W}_{23}^{--})^2(S_{23}^{++})^2D_{13}D_{12}\langle O_3O_3O_3\rangle \tag{3.81}$$

which gives

$$\langle TTT\rangle = a\langle TTT\rangle_{h,R_1} + b(\langle TTT\rangle_{h,R_2} + \langle TTT\rangle_{h,R_3} + \langle TTT\rangle_{h,R_4}) \tag{3.82}$$

In the spinor helicity variables, we have

$$\langle T^-T^-T^-\rangle_{h,R_1} = \frac{k_1k_2k_3}{(k_1+k_2+k_3)^6}\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \tag{3.83}$$

$$\langle T^-T^-T^-\rangle_{h,R_2} = \langle T^-T^-T^-\rangle_{h,R_3} = \langle T^-T^-T^-\rangle_{h,R_4} = 0 \tag{3.84}$$

$$\langle T^-T^-T^+\rangle_{h,R_4} = \frac{k_1k_2k_3}{(k_1+k_2-k_3)^6}\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2 \tag{3.85}$$

$$\langle T^-T^-T^+\rangle_{h,R_2} = \langle T^-T^-T^+\rangle_{h,R_3} = \langle T^-T^-T^+\rangle_{h,R_1} = 0 \tag{3.86}$$

$$\langle T^+T^-T^-\rangle_{h,R_2} = \frac{k_1k_2k_3}{(k_2+k_3-k_1)^6}\langle \bar{1}2\rangle^2\langle 23\rangle^2\langle 3\bar{1}\rangle^2 \tag{3.87}$$

$$\langle T^+T^-T^-\rangle_{h,R_4} = \langle T^+T^-T^-\rangle_{h,R_3} = \langle T^+T^-T^-\rangle_{h,R_1} = 0 \tag{3.88}$$

$$\langle T^-T^+T^-\rangle_{h,R_3} = \frac{k_1k_2k_3}{(k_1+k_3-k_2)^6}\langle 1\bar{2}\rangle^2\langle \bar{2}3\rangle^2\langle 31\rangle^2 \tag{3.89}$$

$$\langle T^-T^+T^-\rangle_{h,R_2} = \langle T^-T^+T^-\rangle_{h,R_4} = \langle T^-T^+T^-\rangle_{h,R_1} = 0 \tag{3.90}$$

The results $\langle TTT\rangle_{h,R_i}$ are precisely same as that appear in (C.37). This implies that the homogeneous part is same as that obtained from Weyl term (C.37) $W^3$ in dS space [187]. Again the following interesting relation between general $\alpha-$vacua and Bunch-Davies vacuum holds,

$$\langle TTT\rangle_{h,\alpha} - (a+b)\langle TTT\rangle_{h,BD} = -b(\langle TTT\rangle_{h,R_1} - \langle TTT\rangle_{h,R_2} - \langle TTT\rangle_{h,R_3} - \langle TTT\rangle_{h,R_4}). \tag{3.91}$$

#### 3.3.4.2 Non-homogeneous solution

The non-homogeneous part of $\langle TTT\rangle$ is computed as follows using the weight shifting and spin raising operators [50, 180]

$$\langle TTT\rangle_{A_i} = \mathcal{S}_{31}^{++}\mathcal{S}_{23}^{++}\mathcal{S}_{12}^{++}R_i(k_1,k_2,k_3) \tag{3.92}$$
$$\langle TTT\rangle_{B_i} = \left(\mathcal{S}_{23}^{++}\right)^2\mathcal{W}_{23}^{++}D_{13}D_{12}R_i(k_1,k_2,k_3) + \text{ perms.} \tag{3.93}$$
$$\langle TTT\rangle_{C_i} = D_{13}D_{12}\left(\mathcal{S}_{23}^{++}\right)^2\mathcal{W}_{23}^{++}R_i(k_1,k_2,k_3) + \text{ perms.} \tag{3.94}$$

There is also a contact term that is added

$$\langle TTT\rangle_{D_i} = (s_{1i}k_1^3 + s_{2i}k_2^3 + s_{3i}k_3^3)z_1.z_2z_2.z_3z_3.z_1 \tag{3.95}$$

where $s_i = \pm$ depending on the basis we are working with. The full $\langle TTT\rangle$ is given by

$$\langle TTT\rangle_{nh,R_i} = \frac{1}{4}\langle TTT\rangle_{A_i} - \frac{7}{108}\langle TTT\rangle_{B_i} - \frac{1}{135}\langle TTT\rangle_{C_i} - \frac{36}{5}\langle TTT\rangle_{D_i} \tag{3.96}$$

where $\langle TTT\rangle_{nh,R_i}$ are given by

$$\langle TTT\rangle_{nh,R_1} = A_1(k_1,k_2,k_3)(z_3.k_1z_1.z_2 + z_1.k_2z_2.z_3 + z_2.k_3z_3.z_1)^2 \tag{3.97}$$
$$\sum_{i=2}^{4}\langle TTT\rangle_{nh,R_i} = A_2(k_1,k_2,k_3)(z_3.k_1z_1.z_2 + z_1.k_2z_2.z_3 + z_2.k_3z_3.z_1)^2 \tag{3.98}$$

where

$$A_2(k_1,k_2,k_3) = A_1(-k_1,k_2,k_3) + A_1(k_1,-k_2,k_3) + A_1(k_1,k_2,-k_3) \tag{3.99}$$
$$A_1(k_1,k_2,k_3) = \frac{E^3 + (k_1^2+k_2^2+k_3^2)E - 2k_1k_2k_3}{2E^2} \tag{3.100}$$

The full non-homogeneous answer is given by

$$\langle TTT\rangle_{nh} = a\langle TTT\rangle_{nh,R_1} + b(\langle TTT\rangle_{nh,R_2} + \langle TTT\rangle_{nh,R_3} + \langle TTT\rangle_{nh,R_4}). \tag{3.101}$$

Let us note that this is precisely same as that obtained from Einstein-term in dS space, see (3.47). For the general vacua, we have

$$\langle TTT\rangle_{nh,\alpha} - (a+b)\langle TTT\rangle_{nh,BD} = -b(\langle TTT\rangle_{nh,R_1} - \langle TTT\rangle_{nh,R_2} - \langle TTT\rangle_{nh,R_3} - \langle TTT\rangle_{nh,R_4}) \tag{3.102}$$

To summarize this section, we have shown that the scalar correlation function in $|\alpha,\beta\rangle$ vacuum can be obtained in three-dimensional CFT by taking a linear combination of four solutions of conformal ward identity and imposing permutation invariance. For the spinning correlator, we have shown that by acting with spin and dimension-raising operators on this scalar correlator we can obtain the correlation function that was obtained in $\alpha-$vacua. However, it is important to note that, spin and dimension-raising operators do not reproduce the correlation functions in the generalized vacuum discussed in section 3.2.3.

## 3.4 Spinning correlator in $\alpha-$vacua by solving conformal ward identity

The aim of this section is two-fold. First, we want to understand the results previous section about spinning correlator starting from a solution of conformal ward identity, and second, we would like to understand the correlation function in a more generalized vacuum $|\alpha,\beta,\tilde{\alpha},\tilde{\beta}\rangle$ that appeared in section 3.2.3. Let us start our discussion with the simplest example of $\langle TO_3O_3\rangle$.

### 3.4.1 $\langle TO_3O_3\rangle$

Ward identity for $\langle TO_3O_3\rangle$ is given by

$$\langle k_{1,\mu}T^{\mu\nu}(k_1)O(k_2)O(k_3)\rangle \propto k_{2,\nu}\left(\langle O(k_2)O(-k_2)\rangle - \langle O(k_3)O(-k_3)\rangle\right). \tag{3.103}$$

We can now solve conformal ward identity in spinor-helicity variables. Let us split the solution into homogeneous and non-homogeneous parts as follows

$$\langle TO_3O_3\rangle = \langle TO_3O_3\rangle_h + \langle TO_3O_3\rangle_{nh}. \tag{3.104}$$

One can show that there are two homogeneous solutions and are given by

$$\begin{aligned}\langle TO_3O_3\rangle_{h_1} &= \frac{k_1(k_1^2-k_2^2-4k_2k_3-k_3^2)}{E^2}\left(\frac{\langle 12\rangle\langle 13\rangle}{\langle 23\rangle}\right)^2\\ \langle TO_3O_3\rangle_{h_2} &= \frac{k_1(E-2k_1)^2(-k_1^2+k_2^2-4k_2k_3+k_3^2)}{(E-2k_3)^2(E-2k_2)^2}\left(\frac{\langle 12\rangle\langle 13\rangle}{\langle 23\rangle}\right)^2\end{aligned} \tag{3.105}$$

and the non-homogeneous solution is given by

$$\langle TO_3O_3\rangle_{nh} = \frac{(k_1+k_3-k_1)^2}{k_1^2}\left[k_1+k_2+k_3-\frac{k_1k_2+k_2k_3+k_3k_1}{k_1+k_2+k_3}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right]\left(\frac{\langle 12\rangle\langle 13\rangle}{\langle 23\rangle}\right)^2 \tag{3.106}$$

The most general solution is given by

$$\langle TO_3O_3\rangle_\alpha = a_1\langle TO_3O_3\rangle_{nh} + a_2\langle TO_3O_3\rangle_{h_1} + a_3\langle TO_3O_3\rangle_{h_2}. \tag{3.107}$$

We would like compare this with (3.49). Consistency with WT identity [4] of (3.49) and (3.107) implies $a_1 = c_1+c_2$. However the coefficient $a_2, a_3$ in (3.107) remains undetermined. Let us note here that, if we demand consistency with OPE limit we need to set $a_2 = a_3 = 0$, see [184] for details. This is consistent with the fact that in the BD vacuum, we only have a non-homogeneous solution. However, if we relax the consistency with OPE limit we see that both $a_2$ and $a_3$ are non-zero. One can determine both $a_2, a_3$ by comparing (3.107) with (3.49), (C.10). One can also identify [5]

$$\begin{aligned}\langle TO_3O_3\rangle_{h_1} &= 2\left(\langle TO_3O_3\rangle_{R_1} - \langle TO_3O_3\rangle_{R_2}\right)\\ \langle TO_3O_3\rangle_{h_2} &= -2\left(\langle TO_3O_3\rangle_{R_3} + \langle TO_3O_3\rangle_{R_4}\right).\end{aligned} \tag{3.109}$$

which gives $a_1 = c_1 + c_2,\ a_2 = -\frac{c_2}{2},\ a_3 = -\frac{c_3}{2}$.

### 3.4.2 $\langle JJO_3\rangle$

The WT identity for $\langle JJO_3\rangle$ is given by

$$k_{1,\mu}\langle J^\mu(k_1)J^\nu(k_2)O_3(k_3)\rangle = 0. \tag{3.110}$$

This implies that only homogeneous solutions exist for conformal ward identity.

[4]

$$k_1^\mu z^\nu\langle T_{\mu\nu}O_3O_3\rangle = (c_1+c_2)(z_1.k_2)(k_2^3-k_3^3). \tag{3.108}$$

[5] It is interesting to note that the homogeneous solutions (3.109) satisfies MLT test [57] even though they are disallowed by OPE consistency condition.

The solution to the homogeneous spinor-helicity ward identities is given by

$$\langle J^-J^-O_3\rangle = \frac{12(k_1+k_2+2k_3)}{(k_1+k_2+k_3)^2}\langle 12\rangle^2 \quad \langle J^-J^-O_3\rangle = \frac{12(k_1+k_2-2k_3)}{(k_1+k_2-k_3)^2}\langle 12\rangle^2 \tag{3.111}$$

$$\langle J^-J^+O_3\rangle = \frac{12(-k_1+k_2+2k_3)}{E^2(k_2+k_3-k_1)^2}\langle 31\rangle^2\langle\bar{2}\bar{3}\rangle^2 \quad \langle J^-J^+O_3\rangle = \frac{12(k_1-k_2+2k_3)}{E^2(k_1-k_2+k_3)^2}\langle 31\rangle^2\langle\bar{2}\bar{3}\rangle^2 \tag{3.112}$$

Notice the homogeneous solutions and their complex conjugates are reproduced in various components of (3.65). This shows that each of (3.65) satisfies the homogeneous ward identity. This implies that we have four homogeneous solutions which are given by

$$\langle JJO_3\rangle_{h_i} = \langle JJO_3\rangle_{R_i} \tag{3.113}$$

with $i = 1$ to 4 and $\langle JJO_3\rangle_{R_i}$ are given in (3.65).

The most general solution is then given by

$$\langle JJO_3\rangle = \sum_{i=1}^{4} a_i\langle JJO_3\rangle_{h_i}. \tag{3.114}$$

By demanding permutation symmetry between 1 and 2 we only get $a_2 = a_3$ that is

$$\langle JJO_3\rangle = a_1\langle JJO_3\rangle_{h_1} + a_4\langle JJO_3\rangle_{h_4} + a_2\left(\langle JJO_3\rangle_{h_2} + \langle JJO_3\rangle_{h_3}\right) \tag{3.115}$$

Let us note that, this is a more general result than (3.64). To obtain this from dS computation one needs to do similar computation as was done in section 3.2.3.

### 3.4.3 $\langle TTO_3\rangle$

The discussion for $\langle TTO_3\rangle$ is precisely the same as for $\langle JJO_3\rangle$. Again one can show that for $\langle TTO_3\rangle$ we get four homogeneous solutions. By demanding $1 \leftrightarrow 2$ exchange symmetry we obtain

$$\langle TTO_3\rangle = a_1\langle TTO_3\rangle_{h_1} + a_4\langle TTO_3\rangle_{h_4} + a_2\left(\langle TTO_3\rangle_{h_2} + \langle TTO_3\rangle_{h_3}\right). \tag{3.116}$$

By comparing (3.116) and (3.51) we obtain values of $a_1, a_2, a_4$ in terms of $\alpha-$ vacua parameter. Let us note that as compared to results in section 3.3.3 we have more general results. In the special case, results in this section reduce to the results of the section 3.3.3.

### 3.4.4 $\langle TTT\rangle$

Let us now turn our attention to the three-point function of the stress tensor. The WT identity is given by (3.79). The three-point function can be written as

$$\langle TTT\rangle = \langle TTT\rangle_h + \langle TTT\rangle_{nh}. \tag{3.117}$$

In spinor helicity variables the homogeneous solution for $\langle TTT\rangle$ is given by

$$\langle T^{h_1}T^{h_2}T^{h_3}\rangle_h = f_{h_1,h_2,h_3}\left(k_1,k_2,k_3\right)\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} \tag{3.118}$$

with

$$\begin{aligned}
&f^{(1)}_{h_1,h_2,h_3}\left(k_1,k_2,k_3\right) = \frac{k_1k_2k_3}{E^{h_1s_1+h_2s_2+h_3s_3}}\Bigg|_{s_1=s_2=s_3=2}\\
&f^{(2)}_{h_1,h_2,h_3}\left(k_1,k_2,k_3\right)\\
&= k_1k_2k_3\left(k_1+k_2-k_3\right)^{h_1s_1+h_2s_2-h_3s_3}\left(k_2+k_3-k_1\right)^{h_3s_3+h_2s_2-h_1s_1}\left(k_1+k_3-k_2\right)^{h_1s_1+h_3s_3-h_2s_2}\Bigg|_{s_1=s_2=s_3=2}
\end{aligned} \tag{3.119}$$

By choosing $h_i = \pm$ one can obtain various helicity components.

### 3.4.5 Homogeneous solution: Matching CFT answer with $\mathcal{W}^3$ contribution

In this subsection we show how to obtain correlation function coming from $\mathcal{W}^3$ term (C.37). It is interesting to note that, not all the homogeneous solution in (3.119) is required to reproduce (3.82). The solutions that are required by the $\alpha-$vacua correlator (3.82) are as follows

$$\begin{aligned}
\langle T^-T^-T^-\rangle_{h,R_1} &= f^{(1)}_{---}\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2\\
\langle T^-T^-T^+\rangle_{h,R_4} &= f^{(2)}_{--+}\frac{\langle 12\rangle^6}{\langle 31\rangle^2\langle 23\rangle^2}\\
\langle T^-T^+T^-\rangle_{h,R_3} &= f^{(2)}_{-+-}\frac{\langle 31\rangle^6}{\langle 23\rangle^2\langle 12\rangle^2}\\
\langle T^+T^-T^-\rangle_{h,R_2} &= f^{(2)}_{+--}\frac{\langle 23\rangle^6}{\langle 12\rangle^2\langle 13\rangle^2}
\end{aligned} \tag{3.120}$$

By demanding permutation invariance we get [6]

$$\langle TTT\rangle_h = a\langle TTT\rangle_{h,R_1} + b\left(\langle TTT\rangle_{h,R_2} + \langle TTT\rangle_{h,R_3} + \langle TTT\rangle_{h,R_4}\right) \tag{3.122}$$

Again we would like to point out, the $\alpha-$vacua solution is just a special combination of the most general solutions that are allowed. In spinor helicity variables, it may look like we have a total of eight independent homogeneous solutions in (3.119), however all of them do not have a good corresponding momentum space representation which is permutation invariant.

#### 3.4.5.1 Non-homogeneous solutions: Matching contribution due to Einstein term

The non-homogeneous solution to conformal ward identity has to satisfy the Ward-Takahashi identity given in (3.79), it can only differ from the BD vacuum by some homogeneous solution. This is simply because the BD vacuum answer saturates the WT identity by itself. Since the $\alpha-$vacua answer contains the BD answer as well as some additional pieces, these additional pieces must satisfy homogeneous ward identity. Let us check this expectation explicitly.

In the previous subsection, the homogeneous solutions which did not play any role, in general, $\alpha-$vacua are given by

$$\begin{aligned}
\langle T^-T^-T^-\rangle_{G_1} &= \frac{k_1k_2k_3}{(k_1+k_2-k_3)^2(k_1-k_2+k_3)^2(-k_1+k_2+k_3)^2}\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 = f^{(2)}_{---}\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2\\
\langle T^-T^-T^+\rangle_{G_4} &= \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2} = f^{(1)}_{--+}\frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\\
\langle T^+T^-T^-\rangle_{G_2} &= \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\frac{\langle 23\rangle^6}{\langle 13\rangle^2\langle 21\rangle^2} = f^{(1)}_{+--}\frac{\langle 23\rangle^6}{\langle 13\rangle^2\langle 21\rangle^2}\\
\langle T^-T^+T^-\rangle_{G_3} &= \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\frac{\langle 13\rangle^6}{\langle 23\rangle^2\langle 21\rangle^2} = f^{(1)}_{-+-}\frac{\langle 13\rangle^6}{\langle 23\rangle^2\langle 21\rangle^2}
\end{aligned} \tag{3.123}$$

We now show that these solutions play an important role in determining a non-homogeneous solution for the stress tensor three-point function.

[6] The BD answer is given by

$$\langle TTT\rangle_{h,R_1} = \langle TTT\rangle_{h,BD} = F(k_1,k_2,k_3).$$

The alpha vacuum answer should contain all BD vacuum answers as well as other solutions as follows

$$F(-k_1,k_2,k_3), F(k_1,-k_2,k_3), F(k_1,k_2,-k_3). \tag{3.121}$$

The permutation invariance then fixes the correlator to take the form in (3.122).

For the general vacuum, we have (3.102), (3.47)

$$\langle TTT\rangle_{nh,\alpha} - (c_1+c_2)\langle TTT\rangle_{nh,BD} = -c_2(\langle TTT\rangle_{nh,R_1} - \langle TTT\rangle_{nh,R_2} - \langle TTT\rangle_{nh,R_3} - \langle TTT\rangle_{nh,R_4}) \tag{3.124}$$

where $\langle TTT\rangle_{nh,R_i}$ appears explicitly in (3.96). In the spinor helicity variables, we can check it gives

$$\begin{aligned}
\langle T^-T^-T^-\rangle_\alpha - (c_1+c_2)\langle T^-T^-T^-\rangle_{BD} &= -96c_2\langle T^-T^-T^-\rangle_{G_1}\\
\langle T^-T^-T^+\rangle_\alpha - (c_1+c_2)\langle T^-T^-T^+\rangle_{BD} &= -96c_2\langle T^-T^-T^+\rangle_{G_4}\\
\langle T^+T^-T^-\rangle_\alpha - (c_1+c_2)\langle T^+T^-T^-\rangle_{BD} &= -96c_2\langle T^+T^-T^-\rangle_{G_2}\\
\langle T^-T^+T^-\rangle_\alpha - (c_1+c_2)\langle T^-T^+T^-\rangle_{BD} &= -96c_2\langle T^-T^+T^-\rangle_{G_3}
\end{aligned} \tag{3.125}$$

which are precisely the homogeneous solutions that were left out in (3.119) and are summarised in (3.123). We conclude that (3.125) precisely matches our expectations.

## 3.5 Parity odd contribution to cosmological correlation function in $\alpha-$ vacuum

Parity odd contribution to non-gaussianity might also play an important role. In general, one can use the in-in formalism to calculate them just like the parity even case [26, 187]. For CFT one can also use spin raising and dimension raising operators to calculate them [188]. However, very recently it was understood that parity odd CFT three-point function can be obtained by doing epsilon transformation [185, 234] starting from parity even CFT correlation function. Let us illustrate this in more detail with examples.

### $\langle JJO_3\rangle$

To calculate the parity odd part of $\langle JJO_3\rangle$ we need to consider

$$\int \varphi F_{\mu\nu}F_{\rho\sigma}dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma \tag{3.126}$$

in $dS_4$ space. However, as was shown in [185] the result is given by

$$\langle J_\mu(k_1)J_\nu(k_2)O_3(k_3)\rangle_{odd} = \frac{1}{k_1}\epsilon_{\mu\alpha\beta}k_1^\beta\langle J^\alpha J_\nu O_3\rangle_{even}. \tag{3.127}$$

To obtain the parity odd contribution for $\alpha-$vacua, all we have to do is just plug $\langle J^\alpha J_\nu O_3\rangle_{even}$ that appears in (3.64).

### $\langle TTO_3 \rangle$

For $\langle TTO_3 \rangle$ the story is the same. To calculate the parity odd part of $\langle TTO_3 \rangle$ we need to consider

$$\int \varphi \mathcal{W}_{\mu\nu}^{\alpha\beta} \mathcal{W}_{\rho\sigma\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{3.128}$$

in $dS_4$ space. However, as was shown in [185] the result is given by

$$\langle T_{\mu\nu}(k_1) T_{\rho\sigma}(k_2) O_3(k_3) \rangle_{odd} = \frac{1}{k_1} \epsilon_{(\mu\alpha\beta} k_1^\beta \langle T^\alpha_{\nu)}(k_1) T_{\rho\sigma}(k_2) O_3(k_3) \rangle_{even}. \tag{3.129}$$

where we have symmetrized appropriately.

### $\langle TTT \rangle$

To calculate the parity odd part of $\langle TTT \rangle$ we need to consider

$$\int \mathcal{W}^2 \tilde{W} \tag{3.130}$$

in $dS_4$ space where $W$ is the Weyl tensor, $\tilde{W}$ is the Hodge-dual of the Weyl tensor . However as was shown in [185] the result is given by

$$\langle T_{\mu\nu}(k_1) T_{\rho\sigma}(k_2) T_{\gamma\delta}(k_3) \rangle_{odd} = \frac{1}{k_1} \epsilon_{(\mu\alpha\beta} k_1^\beta \langle T^\alpha_{\nu)}(k_1) T_{\rho\sigma}(k_2) T_{\gamma\delta}(k_3) \rangle_{even,h}. \tag{3.131}$$

## General discussion on parity odd contribution in $\alpha-$ vacuum

From the CFT perspective, it is very interesting to understand the general structure of parity odd correlation functions in $\alpha-$vacua. In [202], it was shown via some examples that for correlation function of conserved currents, when the triangle inequality is violated $s_i + s_j < s_k$ for $i, j, k$ taking value $1, 2, 3$, the parity odd contribution is zero. In [109] a more detailed proof of this statement was provided in position space. For example, $\langle TOO \rangle$ can not have parity odd contribution as it violates the triangle inequality. In [184] a much simpler and intuitive proof was provided and it was shown using momentum space analysis that consistency with OPE forbids parity odd contribution outside the triangle. However, as explained the $\alpha-$vacua correlators need not be consistent with OPE expansion. This implies one can allow for parity odd contribution even for correlation functions outside the triangle such as $\langle TOO \rangle$. However, these correlators are not present as there are no suitable interactions from the $dS_4$ perspective.

## 3.6 $\alpha-$vacua correlator in terms of BD vacuum correlator

In this section, we express results in $\alpha-$vacua, in terms of BD vacuum answers. Let us start with the simplest of cases namely the correlation function in $|\alpha,\beta\rangle$ vacua. For this case, the spinning as well as scalar fields of the modes in $\alpha-$vacua are defined by the same $\alpha,\beta$ parameters. See section 3.2.2 and 3.3 for explicit results. Let us start our discussion with the case of the scalar three-point function. In this case it is easy to see that

$$\begin{aligned}\langle O(k_1)O(k_2)O(k_3)\rangle_{BD} &= F(k_1,k_2,k_3)\\ \langle O(k_1)O(k_2)O(k_3)\rangle_{\alpha} &= a\ F(k_1,k_2,k_3)+b\,(F(-k_1,k_2,k_3)+F(k_1,-k_2,k_3)+F(k_1,k_2,-k_3))\end{aligned} \tag{3.132}$$

where $a,b$ are given in (3.35), (3.36). For any spinning correlator, the relation is the same. The relation can be summarised as follows. Let us define the correlator in BD vacuum to be given by

$$\sum_i (\text{ distinct tensor structure })_i\, A_i(k_1,k_2,k_3) \tag{3.133}$$

then the answer in $\alpha-$vacua takes the following form

$$\sum_i (\text{ distinct tensor structure })_i\ [a\ A_i(k_1,k_2,k_3)+b\,(A_i(-k_1,k_2,k_3)+A_i(k_1,-k_2,k_3)+A_i(k_1,k_2,-k_3))] \tag{3.134}$$

where $A_i(k_1,k_2,k_3)$ are the form factors. This can be checked to be true for all the correlators discussed in 3.2.2 and 3.3 which includes $\langle TOO\rangle,\langle TTO\rangle,\langle TTT\rangle$.

Now let us consider more general case when we have $\alpha-$vacua defined as $|\alpha,\beta,\tilde{\alpha},\tilde{\beta}\rangle$, see section 3.2.3 for results. For simplicity, we focus on $\langle TOO\rangle$ and $\langle TTO\rangle$. Any other correlators can also be discussed similarly. For $\langle TOO\rangle$ we have

$$\langle TOO\rangle_{BD} = (z_1.k_2)^2 A_T(k_1,k_2,k_3) \tag{3.135}$$

where the form factor $A_T$ can be found in (3.40). For general $\alpha-$vacua we have

$$\langle TOO\rangle_{\alpha} = (z_1.k_2)^2\,(c_1A_T(k_1,k_2,k_3)+c_2A_T(-k_1,k_2,k_3)+c_3\,(A_T(k_1,-k_2,k_3)+A_T(k_1,k_2,-k_3))) \tag{3.136}$$

where $c_i$ are as same as in (3.49). Let us note that (3.136) is consistent with $2\leftrightarrow 3$ exchange symmetry. Now let us consider the case of $\langle TTO\rangle$. For the BD vacuum we have

$$\langle TTO\rangle_{BD} = \sum_i (\text{ distinct tensor structure })_i \, B_i(k_1, k_2, k_3) \tag{3.137}$$

where $B_i$ are form factors. The correlator in $\alpha-$vacua in terms of BD vacua is given by

$$\begin{aligned}\langle TTO\rangle_\alpha = \sum_i (\text{ distinct tensor structure })_i \Big[ & d_1 \; B_i(k_1, k_2, k_3) + d_2 B_i(k_1, k_2, -k_3) \\ & + d_3 \left(B_i(-k_1, k_2, k_3) + B_i(k_1, -k_2, k_3)\right)\Big]\end{aligned} \tag{3.138}$$

which is consistent with (3.51). Let us note that (3.138) is consistent with $1 \leftrightarrow 2$ exchange symmetry.

To conclude, we have shown that given answers in BD vacuum, it is straightforward to obtain correlation function in $\alpha-$vacua.

## 3.7 Summary and Discussion

In this chapter, we have discussed the cosmological correlation function in general $\alpha$-vacua in rigid dS space. One of the main purposes of this paper is to understand how to construct these cosmological correlation functions from a CFT perspective. We showed that for this purpose we need to consider CFT correlators that are not consistent with OPE limit. Interestingly, conformal ward identity in momentum space allows for such solutions. For example, solving conformal ward identity for the three-point function of the scalar operator gives in general four different solutions. Out of these four different solutions, only one of them is consistent with OPE and coincides with the correlation function in the Bunch-Davies vacuum as well as consistent with the position space correlation function. However, for $\alpha-$vacua all four solutions are important and the most general solution is a linear combination of all these four solutions consistent with permutation symmetry. For the spinning correlator, we then used spin and dimension raising operators as well as a solution of conformal ward identity in momentum/spinor helicity variables. Based on our computation we summarise the result as follows

- BD vacuum answers $\implies$ Imposing consistency with OPE limit and permutation invariance on a solution to conformal ward identity in momentum space.
- $\alpha-$vacua answer $\implies$ Relaxing consistency with OPE limit however keeping consistency with permutation invariance on the solution to conformal ward identity in momentum space.

There are several important aspects of these analyses that we plan to explore in the future.

## Inflationary correlation function

We have mostly focused on the calculation of the three-point function in rigid dS space. It would be interesting to calculate the correlation function for the inflationary case. For the rigid de-Sitter case we have shown how to obtain correlation function in $\alpha-$vacua given correlation function BD vacuum, see 3.6. It will be interesting to check if the same relation continues to hold in the inflationary scenario. This will also provide an easy way to obtain answers in $\alpha-$vacua given the plethora of results already known in the BD vacuum.

## Observational significance

We have not studied any phenomenological implication of the $\alpha-$vacua. It is well known that signal for the $\alpha-$vacua can be significantly enhanced as compared to BD vacuum, see [231] for a recent discussion on this issue. We have also not studied the issue of consistency of $\alpha-$vacua. One of the problems with $\alpha-$vacua is that the stress tensor expectation value of the probe scalar field is divergent and as a result, its back-reaction can invalidate the rigid dS approximation. To renormalize one needs non-local and $\alpha-$dependent counter-term. This seems to be problematic, however, it was argued in [235] that $\alpha-$parameter-dependent counter-terms are fine. We would like to come back to this issue in the future.

# CONCLUSION

In this thesis, we have looked at various aspects of physics at strong coupling. Perturbation theory has already made significant contributions to physics, but to understand physics at strong coupling one needs to go beyond perturbation theory. In this study, we have looked into various possibilities such as the use of symmetry, bootstrap, and duality. In duality, we predominantly worked with field theory-field theory duality, and more specifically, we explored the aspects of non-SUSY and SUSY duality. Not all theories have these dualities and Chern-Simons matter theories are the simplest theories that have them. Apart from dualities among field theories, Chern-Simons matter theories also have an $AdS$ higher-spin gravity dual. $AdS/CFT$ is a special kind of duality that allows us to study $CFT$s non-perturbatively via weakly coupled gravitational theories and vice-versa i.e. to study perturbative quantum corrections in gravity via $CFT$. However, dualities such as these are theory-dependent. Therefore, it would be interesting to find a general prescription for dualities in $CFT$s. In this study, we also looked into momentum-space $CFT$s due to their applications in cosmology and connection to S-matrix bootstrap. We were able to derive interesting results even at the level of three-point functions. Namely, we were able to establish a relation between parity-even and parity-odd correlations and show double-copy relations for the correlators. We also developed a standardized methodology to compute three-point correlations of arbitrary spin and saw a nice connection to the S-matrix. It would be interesting to explore the connection between $CFT$ correlations at the level of four and higher-point functions. Establishing such a connection would allow us to formulate a $CFT$ analogue of S-matrix BCFW recurrence relations and unitarity conditions. Finally, we used the momentum-space $CFT$ techniques we developed in cosmology to compute $CMB$ non-gaussianity. We demonstrated the role of OPE-consistency of $CFT$ correlators in the selection of the vacuum. This is an important application of $CFT$ and given the observational significance of these results, it will be interesting to look forward to what the future has to offer.

Strong coupling is an interesting area of physics to understand and we believe this will lead to interesting developments in the future. An important aspect is duality and its further investigation will play an important role in understanding physics at strong coupling. Looking at ways to understand these dualities and how they may help us better understand $CFT$s and to find ways to derive these dualities are some of the aspects that need to be looked into. It would be interesting to look for a duality-invariant formalism for $CFT$s. Connecting in momentum-space $CFT$ bootstrap with the S-matrix bootstrap will help us understand the correspondence between amplitudes and $CFT$ correlators better. I hope to explore these issues in the future.

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# Appendix A

## A.1 Parity-odd two-point functions

As is well known, scale invariance completely fixes CFT two-point functions. parity-odd structures can exist for two-point functions of spinning operators.

### A.1.1 Four and Higher dimensions

In four or higher dimensions, it is not possible to have any parity-odd two-point function of either spin-one or any other spinning symmetric spinning correlator. This is because a parity-odd correlator must necessarily involve the $\epsilon$ tensor and it is simple to show that it is impossible to have any parity-odd 2-point function of a spin-1 or any symmetric tensor operator.

### A.1.2 Three-dimensions

In three-dimensions parity-odd two-point functions exist. These come from purely contact terms [1]. We will look at the parity-odd 2-point functions of spin-one and spin two conserved currents.

#### $\langle J^\mu J^\nu \rangle_{\textbf{odd}}$

The general ansatz for the correlator is given by

$$\langle J^\mu(k) J^\nu(-k)\rangle_{\text{odd}} = A(k)\epsilon^{\mu\nu k} \tag{A.1}$$

The ansatz guarantees that the correlator is transverse to the momentum. Imposing scale invariance gives the following differential equation for the form factor $A(k)$ :

$$k\frac{\partial}{\partial k}A(k) = 0 \tag{A.2}$$

This implies that the form factor is just a constant in this case and we have :

$$\langle J^\mu(k) J^\nu(-k)\rangle_{\text{odd}} = c_J\epsilon^{\mu\nu k} \tag{A.3}$$

We will now consider the parity-odd 2-point function of the stress-tensor.

#### $\langle T^{\mu\nu} T^{\rho\sigma} \rangle_{\textbf{odd}}$

We consider the following ansatz for this correlator :

$$\langle T^{\mu\nu}(k) T^{\rho\sigma}(-k)\rangle_{\text{odd}} = B(k)\Delta^{\mu\nu\rho\sigma}(k) \tag{A.4}$$

where $\Delta^{\mu\nu\rho\sigma}(k)$ is a parity-odd, transverse-traceless projector given by :

$$\Delta^{\mu\nu\rho\sigma}(k) = \epsilon^{\mu\rho k}\pi^{\nu\sigma}(k) + \epsilon^{\mu\sigma k}\pi^{\nu\rho}(k) + \epsilon^{\nu\sigma k}\pi^{\mu\rho}(k) + \epsilon^{\nu\rho k}\pi^{\mu\sigma}(k) \tag{A.5}$$

[1] In this case, the corresponding position space correlator with separated points vanishes.

where $\pi^{\mu}_{\nu}(k)$ is the same projector used in previous sections. The ansatz guarantees that the correlator is transverse and traceless.

The dilatation Ward identity gives the following equation for the form factor $B(k)$ :

$$\left(k\frac{\partial}{\partial k}-2\right)B(k)=0 \tag{A.6}$$

This can be easily solved to get

$$B(k)=c_T k^2 \tag{A.7}$$

Therefore, the correlator is given by

$$\langle T^{\mu\nu}(k)T^{\rho\sigma}(-k)\rangle_{\text{odd}}=c_T\Delta^{\mu\nu\rho\sigma}(k)k^2 \tag{A.8}$$

## A.2 Schouten Identities

Here, we list the Schouten identities used in our calculations in the main text. The most general form of a Schouten identity in $d$-dimensions is

$$\epsilon^{[\mu_1\mu_2\cdots\mu_d}\,\delta^{\nu]\rho}=0 \tag{A.9}$$

In three-dimensions, this translates to

$$\epsilon^{\mu_1\mu_2\mu_3}\delta^{\nu\rho}-\epsilon^{\mu_2\mu_3\nu}\delta^{\mu_1\rho}+\epsilon^{\mu_3\nu\mu_1}\delta^{\mu_2\rho}-\epsilon^{\nu\mu_1\mu_2}\delta^{\mu_3\rho}=0 \tag{A.10}$$

Dotting the indices in (A.10) with momenta lets us relate different epsilon structures that occur in the correlation functions calculated earlier. Dotting with $k_{1\mu_3}$, $k_{1\rho}$ and $k_{2\nu}$ gives

$$\epsilon^{\mu_1\mu_2k_1}(k_1\cdot k_2)+\epsilon^{\mu_1k_1k_2}k_1^{\mu_2}=\epsilon^{\mu_1\mu_2k_2}k_1^2+\epsilon^{\mu_2k_1k_2}k_1^{\mu_1} \tag{A.11}$$

Similarly, dotting with $k_{1\mu_3}$, $k_{2\rho}$ and $k_{2\nu}$

$$\epsilon^{\mu_1\mu_2k_2}(k_1\cdot k_2)+\epsilon^{\mu_2k_1k_2}k_2^{\mu_1}=\epsilon^{\mu_1\mu_2k_1}k_2^2+\epsilon^{\mu_1k_1k_2}k_2^{\mu_2} \tag{A.12}$$

(A.11) and (A.12) were useful in rewriting the $\langle JJO\rangle$ ansatz. One can also derive these two identities by considering the contraction of three Levi-Civita tensors.

$$\begin{aligned}\epsilon^{\mu_1\alpha k_1}\epsilon^{\beta k_2\mu_2}\epsilon_{\beta\rho\alpha}&=\epsilon^{\mu_1\mu_2k_1}k_{2\rho}+\epsilon^{\mu_1k_1k_2}\delta^{\mu_2}_{\rho}\\ \epsilon^{\beta k_2\mu_2}\epsilon^{\alpha k_1\mu_1}\epsilon_{\alpha\beta\rho}&=\epsilon^{\mu_1\mu_2k_2}k_{1\rho}+\epsilon^{\mu_2k_1k_2}\delta^{\mu_1}_{\rho}\end{aligned} \tag{A.13}$$

Equating the RHS of the two equations after dotting them with $k_1^{\rho}$ and $k_2^{\rho}$ respectively, we get back (A.11) and (A.12). Similarly, while checking the transverse identity for $\langle JJJ\rangle$, we used the following Schouten identities

$$\epsilon^{k_1k_2\mu_3}k_2^{\mu_2}=\epsilon^{k_2\mu_3\mu_2}(k_1\cdot k_2)+\epsilon^{\mu_2\mu_3k_1}k_2^2+\epsilon^{k_1k_2\mu_2}k_2^{\mu_3} \tag{A.14}$$

$$\epsilon^{k_1k_2\mu_3}k_3^{\mu_2}=\epsilon^{k_2\mu_3\mu_2}(k_1\cdot k_3)+\epsilon^{\mu_2\mu_3k_1}(k_2\cdot k_3)+\epsilon^{\mu_3k_1k_2}k_3^{\mu_3} \tag{A.15}$$

In four-dimensions, we use the following identities to rewrite the ansatz for $\langle JJJ\rangle$.

$$\begin{aligned}&(k_1\cdot k_2)\epsilon_{\mu_1\mu_2\mu_3k_1}-k_{1\mu_1}\epsilon_{k_2\mu_2\mu_3k_1}-k_{1\mu_2}\epsilon_{\mu_1k_2\mu_3k_1}-k_{1\mu_3}\epsilon_{\mu_1\mu_2k_2k_1}-k_1^2\epsilon_{\mu_1\mu_2\mu_3k_2}=0\\ &(k_1\cdot k_2)\epsilon_{\mu_1\mu_2\mu_3k_2}-k_{2\mu_1}\epsilon_{k_2\mu_2\mu_3k_2}-k_{2\mu_2}\epsilon_{\mu_1k_1\mu_3k_2}-k_{2\mu_3}\epsilon_{\mu_1\mu_2k_1k_2}-k_2^2\epsilon_{\mu_1\mu_2\mu_3k_1}=0\end{aligned} \tag{A.16}$$

## A.3 Parity-even spin-raising and weight-shifting operators

In this section we list out all the parity-even weight-shifting operators used in the main text of the paper [50, 180].

The operator that decreases the scaling dimension of operators at points 1 and 2 is :

$$W_{12}^{--} = \frac{1}{2}\vec{K}_{12}^{-} \cdot \vec{K}_{12}^{-} \tag{A.17}$$

where

$$K_{12}^{-\mu} = \partial_{k_{1\mu}} - \partial_{k_{2\mu}} \tag{A.18}$$

We also use

$$K_{12}^{+\mu} = \partial_{k_{1\mu}} + \partial_{k_{2\mu}} \tag{A.19}$$

We can also define an operator that increases the scaling dimension at 2-points. Although this has a very complicated expression, it simplifies when acting on scalar operators and is given by :

$$\begin{aligned} W_{12}^{++} =& (k_1 k_2)^2\, W_{12}^{--} - (d-2\,\Delta_1)(d-2\,\Delta_2) k_1 \cdot k_2 \\ &+ \Big(k_2^2 (d-2\,\Delta_1)(d-1-\Delta_1 + k_1 \cdot K_{12}) + (1 \leftrightarrow 2)\Big) \end{aligned} \tag{A.20}$$

$D_{11}$ raises the spin of the operator at point 1 and simultaneously lowers its weight. This was used in the construction of both $\langle TTO \rangle$ and $\langle JJJ \rangle$ :

$$D_{11} = (\Delta_2 - 3 + \vec{k}_2 \cdot \vec{K}_{12}) \vec{z}_1 \cdot \vec{K}_{12} - (\vec{k}_2 \cdot \vec{z}_1) W_{12}^{--} - (\vec{z}_2 \cdot \vec{K}_{12})\,(\vec{z}_1 \cdot \partial_{\vec{z}_2}) + (\vec{z}_1 \cdot \vec{z}_2) \partial_{\vec{z}_2} \cdot \vec{K}_{12} \tag{A.21}$$

We can similarly define $D_{22}$ and $D_{33}$ by doing cyclic permutations of the momenta and polarization vectors in (A.21). For example,

$$D_{22}\left((k_1, z_1), (k_2, z_2), (k_3, z_3)\right) = D_{11}\left((k_3, z_3), (k_1, z_1), (k_2, z_2)\right) \tag{A.22}$$

$S_{12}^{++}$ raises the spin at points 1 and 2 :

$$\begin{aligned} S_{12}^{++} =& (s_1 + \Delta_1 - 1)(s_2 + \Delta_2 - 1) z_1 \cdot z_2 - (z_1 \cdot k_1)(z_2 \cdot k_2) W_{12}^{--} \\ &+ [(s_1 + \Delta_1 - 1)(k_2 \cdot z_2)(z_1 \cdot K_{12}) + (1 \leftrightarrow 2)] \end{aligned} \tag{A.23}$$

$S_{23}^{++}$ and $S_{13}^{++}$ are once again defined by cyclic permutations of (A.23).

The operator $H_{12}$ which raises the spin at points 1 and 2 and also lowers the weight at both the points is given by :

$$H_{12} = 2\,(z_1 \cdot K_{12})(z_2 \cdot K_{12}) - 2\,(z_1 \cdot z_2) W_{12}^{--} \tag{A.24}$$

The operator that raises the spin at point 1 and simulataneously lowers the weight at point 2 is given by :

$$D_{12} = (\Delta_1 + s_1 - 1) z_1 \cdot K_{12} - (z_1 \cdot k_1) W_{12}^{--} \tag{A.25}$$

A $(1 \leftrightarrow 2)$ exchange in this operator gives $D_{21}$. Both of these were used in the construction of $\langle TTO \rangle$.

## A.4 Spinor-helicity notation

In this appendix we will quickly summarise the spinor-helicity variables for 3d CFTs. For more details see [28, 39] . We first embed the Euclidean 3-momentum $\vec{k}$ into a null momentum vector $k_\mu$ in 3+1 dimensions :

$$k_\mu = (k, \vec{k}) \tag{A.26}$$

such that $k = |\vec{k}|$. Given the 4-momentum we express it in spinor notation as :

$$k_{\alpha\dot{\alpha}} = k_\mu \sigma^\mu_{\dot{\alpha}\alpha} = \lambda_\alpha \widetilde{\lambda}_{\dot{\alpha}} \tag{A.27}$$

where $\alpha$ and $\dot{\alpha}$ are $SL(2,\mathbb{C})$ transform under inequivalent (conjugate) representations of $SL(2,\mathbb{C})$. However, in 3 dimensions one has an identification between the dotted and undotted indices. To see this let us consider the vector $\tau^\mu = (1, 0, 0, 0)$. In spinor-helicity variables :

$$\tau_{\alpha\dot{\alpha}} = \tau_\mu (\sigma^\mu)_{\alpha\dot{\alpha}} = -\mathbb{I}_{\alpha\dot{\alpha}} \tag{A.28}$$

We can now convert dotted indices to undotted indices using the following tensor :

$$\tau^{\dot{\alpha}}_\alpha = -\epsilon^{\dot{\alpha}\dot{\beta}} \mathbb{I}_{\dot{\beta}\alpha} \tag{A.29}$$

We also introduce the barred spinors as follows :

$$\bar{\lambda}_\alpha \equiv \widetilde{\lambda}_{\dot{\alpha}} \tau^{\dot{\alpha}}_\alpha \tag{A.30}$$

We then have the following relations between the 3-momentum and the spinors.

$$\lambda_\alpha \bar{\lambda}_\beta = k_i \left(\hat{\sigma}^i\right)_{\alpha\beta} + k\epsilon_{\alpha\beta} \tag{A.31}$$

$$k^i = \frac{1}{2} \left(\sigma^i\right)^\alpha_\beta \lambda_\alpha \bar{\lambda}^\beta \tag{A.32}$$

Since $\epsilon_{\alpha\beta}$ is an $SL(2,\mathbb{C})$ invariant, we can use it to define dot products between spinors.

$$\begin{aligned}
\langle ij \rangle &= \epsilon^{\alpha\beta} \lambda^i_\alpha \lambda^j_\beta \\
\langle \bar{i}\bar{j} \rangle &= \epsilon^{\alpha\beta} \bar{\lambda}^i_\alpha \bar{\lambda}^j_\beta \\
\langle i\bar{j} \rangle &= \epsilon^{\alpha\beta} \lambda^i_\alpha \bar{\lambda}^j_\beta
\end{aligned} \tag{A.33}$$

It can be also be used to raise and lower indices on the spinors for which we will use the following convention.

$$\lambda_\beta = \epsilon_{\alpha\beta} \lambda^\alpha \tag{A.34}$$

The reader is referred to appendix B of [39] or appendix C in [180] which contains a set of useful relations between spinor brackets that will be used throughout the main text. Finally, we also define the following polarization vectors which when dotted with the momentum space expression of a correlator, gives its expression in spinor-helicity variables.

$$z^-_{\alpha\beta} = \frac{\lambda_\alpha \lambda_\beta}{2k} \qquad\qquad z^+_{\alpha\beta} = \frac{\bar{\lambda}_\alpha \bar{\lambda}_\beta}{2k} \tag{A.35}$$

## A.5 Homogeneous & non-homogeneous vs transverse & longitudinal contributions

While computing momentum space correlation functions one often splits the correlator into its transverse and longitudinal parts [41]. In this paper we find it more useful to split correlators into their homogeneous and non-homogeneous parts as defined in sec. 1.3.3.1. In this appendix we emphasise and illustrate through examples that the transverse and homogeneous parts of a correlator are not identical, and also that the longitudinal and non-homogeneous parts are not identical. In particular, we will show that while the homogeneous part of a correlator is always transverse, the non-homogeneous part in general contains both transverse and longitudinal contributions and is proportional to 2-point function coefficients.

As an example consider $\langle TOO\rangle$. The correlator is given by [41]

$$\langle TOO\rangle = \langle TOO\rangle_{\text{transverse}} + \langle TOO\rangle_{\text{longitudinal}} \tag{A.36}$$

where the transverse part is given by

$$\langle TOO\rangle_{\text{transverse}} = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(k_1)A_1 k_2^{\alpha_1}k_2^{\beta_1} \tag{A.37}$$

For example when the scalar operator $O$ has scaling dimension $\Delta = 1$ the form factor is given by [41]

$$A_1 = c_O \frac{2k_1 + k_2 + k_3}{k_2 k_3 (k_1 + k_2 + k_3)^2}. \tag{A.38}$$

The form-factor is proportional to the coefficient of the scalar two-point function $c_O$

$$\langle O(k)O(-k)\rangle_{\Delta=1} = c_O \frac{1}{k}.$$

The longitudinal part of the correlator for $\Delta = 1$ is

$$\langle TOO\rangle_{\text{longitudinal}} = \left[k_2^{\alpha}\mathcal{I}^{\mu_1\nu_1}_{\alpha}(k_1) - \frac{1}{2}\pi^{\mu_1\nu_1}(k_1)\right] c_O \frac{1}{k_2} + k_2 \leftrightarrow k_3 \tag{A.39}$$

where

$$\mathcal{I}^{\mu\nu}_{\alpha}(k) = \frac{1}{k^2}\left[2p^{(\mu}\delta^{\nu)}_{\alpha} - \frac{k_\alpha}{2}\left(\delta^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right)\right]. \tag{A.40}$$

We see that the full correlator is proportional to the two-point function coefficient $c_O$. Thus in our terminology the full answer is non-homogeneous and there is no homogeneous contribution to it. To summarize we have

$$\begin{aligned}\langle TOO\rangle &= \langle TOO\rangle_{\text{transverse}} + \langle TOO\rangle_{\text{longitudinal}} \\ &= \langle TOO\rangle_{\textbf{nh}}\end{aligned} \tag{A.41}$$

Let us now consider the case of $\langle TTT\rangle$. The full answer in the terminology of [41] is given by

$$\langle TTT\rangle = \langle TTT\rangle_{\text{transverse}} + \langle TTT\rangle_{\text{longitudinal}} \tag{A.42}$$

which can as well be split into homogeneous and non-homogeneous pieces as follows

$$\begin{aligned}\langle TTT\rangle &= \langle TTT\rangle_{\text{transverse}} + \langle TTT\rangle_{\text{longitudinal}}\\ &= \langle TTT\rangle_{\text{transverse},\mathbf{h}} + \langle TTT\rangle_{\text{transverse},\mathbf{nh}} + \langle TTT\rangle_{\text{longitudinal}}\\ &= \langle TTT\rangle_{\mathbf{h}} + \langle TTT\rangle_{\mathbf{nh}}\end{aligned} \tag{A.43}$$

where we made the following identification

$$\begin{aligned}\langle TTT\rangle_{\mathbf{h}} &= \langle TTT\rangle_{\text{transverse},\mathbf{h}}\\ \langle TTT\rangle_{\mathbf{nh}} &= \langle TTT\rangle_{\text{transverse},\mathbf{nh}} + \langle TTT\rangle_{\text{longitudinal}}\end{aligned} \tag{A.44}$$

We now give explicit identification of the homogeneous and non-homogeneous contribution. To simplify the discussion, we make use of transverse, null polarization vectors that are contracted with the free indices of the correlator. The longitudinal term drops out and what remains are the transverse pieces. For convenience we reproduce it here [39, 187]

$$\langle TTT\rangle_{\text{even}} = \frac{C_1 c_{123}}{E^6}\mathcal{M}_{W^3} + 2C_{TT}\left(\frac{c_{123}}{E^2} + \frac{b_{123}}{E} - E\right)\mathcal{M}_{EG} \tag{A.45}$$

where $C_{TT}$ is defined by the two-point function

$$\langle T(k)T(-k)\rangle = C_{TT}(z_1\cdot z_2)^2 k^3 \tag{A.46}$$

In the transverse correlator (A.45), the term proportional to $C_{TT}$ is non-homogeneous and the rest of it (the term proportional to $C_1$) is homogeneous. To summarize, the term that is dependent on the two-point function coefficient (fixed by secondary conformal Ward identity in the language of [41]) is the non-homogeneous contribution. From the $dS_4$ perspective the interpretation is that the term getting contribution from $W^3$ (term proportional to $C_1$) is homogeneous and the term getting contribution from Einstein-gravity $\sqrt{g}R$ (term proportional to $C_{TT}$) is non-homogeneous.

To conclude, the non-homogeneous part of the correlator can contain both transverse as well as longitudinal parts. From the $dS_4$ perspective as well, the origins of the homogeneous and non-homogeneous contributions are distinct.

## A.6 Details of solutions of CWIs for various correlators

In this appendix we provide details of the calculations related to solving conformal Ward identities (CWIs) in spinor-helicity variables.

### A.6.1 $\langle J_s O_\Delta O_\Delta\rangle$

The details of the conformal Ward identities for this case were already given in Section 1.4.1. Here we consider a few examples. The $s = 1$ and the $s = 2$ cases have already been computed in [41].

### Example - Spin one current: $\langle J_\mu O_\Delta O_\Delta\rangle$

Setting $s = 1$ in (1.97) we obtain :

$$\begin{aligned}\langle J^- O_\Delta O_\Delta\rangle &= c_O I_{\frac{3}{2}\{-\frac{1}{2},\Delta-\frac{3}{2},\Delta-\frac{3}{2}\}}\langle 12\rangle\langle \bar{2}1\rangle\\ &= c_O I_{\frac{3}{2}\{-\frac{1}{2},\Delta-\frac{3}{2},\Delta-\frac{3}{2}\}}\frac{\langle 12\rangle\langle 13\rangle}{\langle 23\rangle}(k_2+k_3-k_1)\end{aligned} \tag{A.47}$$

We see that the correlator gets a minus sign under a $(2 \leftrightarrow 3)$ exchange. Therefore, this correlator is non-zero only when all the three operators have non-abelian indices. The non-abelian indices add an extra factor of $f^{abc}$ to the correlator which results in a plus sign under a $(2 \leftrightarrow 3)$ exchange. This result holds for any $\langle J_s O_\Delta O_\Delta \rangle$ whenever $s$ is odd. For the specific case of $\Delta = 2$, the correlator is given by

$$\langle J^- O_2 O_2 \rangle = c_O \frac{1}{k_1 E} \langle 12 \rangle \langle \bar{2}1 \rangle \tag{A.48}$$

The correlator is divergent for $\Delta \geq 3$ and needs to be renormalized for higher scaling dimensions.

### Example - Spin two current: $\langle T_{\mu\nu} O_\Delta O_\Delta \rangle$

Setting $s = 2$ and $\Delta = 2$ in (1.97) we obtain :

$$\langle T^- O_2 O_2 \rangle = c_O \frac{E + k_1}{k_1^2 E^2} \langle 12 \rangle^2 \langle \bar{2}1 \rangle^2 \tag{A.49}$$

Setting $s = 2$ and $\Delta = 3$ in (1.97) we obtain :

$$\langle T^- O_3 O_3 \rangle = c_O \frac{k_1^2(E + k_2 + k_3) + (E + k_1)(k_2^2 + k_2 k_3 + k_3^2)}{k_1^2 E^2} \langle 12 \rangle^2 \langle \bar{2}1 \rangle^2 \tag{A.50}$$

For $\Delta > 3$, the correlator is divergent and needs to renormalized.

## A.6.2 $\langle J_s J_s O_\Delta \rangle$

From the action of the special conformal generator on the scalar operator and conserved spin-$s$ currents (1.71) and (1.72), we get the following :

$$\widetilde{K}^\kappa \left\langle \frac{J^{s-}}{k_1^{s-1}} \frac{J^{s-}}{k_2^{s-1}} \frac{O_\Delta}{k_3^{\Delta-2}} \right\rangle = 2 \Big[ \frac{z_1^{-\kappa}}{k_1^{s+1} k_2^{s-1} k_3^{\Delta-2}} \langle k_1 \cdot J_s(k_1) J_s^-(k_2) O(k_3) \rangle$$
$$+ \frac{z_2^{-\kappa}}{k_1^{s-1} k_2^{s+1} k_3^{\Delta-2}} \langle J_s^-(k_1) k_2 \cdot J_s(k_2) O(k_3) \rangle + \frac{k_3^\kappa (\Delta - 2)(\Delta - 1)}{k_1^{s-1} k_2^{s-1} k_3^\Delta} \langle J_s^- J_s^- O_\Delta \rangle \Big] \tag{A.51}$$

Making use of the trivial transverse Ward identity (1.100), the first and the second terms on the RHS of the above equation drop out and we obtain :

$$\widetilde{K}^\kappa \left\langle \frac{J^{s-}}{k_1^{s-1}} \frac{J^{s-}}{k_2^{s-1}} \frac{O_\Delta}{k_3^{\Delta-2}} \right\rangle = \frac{k_3^\kappa}{k_1^{s-1} k_2^{s-1} k_3^\Delta} (\Delta - 2)(\Delta - 1) \langle J_s^- J_s^- O_\Delta \rangle \tag{A.52}$$

Contracting (A.52) with $k_1 z_1^{-\kappa}$ and with $k_2 z_2^{-\kappa}$ we get the following equations for the parity even part of the correlator (1.102) [2] :

$$\left(\frac{\partial^2 F_1}{\partial k_2^2}-\frac{\partial^2 F_1}{\partial k_3^2}\right)=-\frac{F_1}{k_3^2}(\Delta-1)(\Delta-2)$$
$$\left(\frac{\partial^2 F_1}{\partial k_3^2}-\frac{\partial^2 F_1}{\partial k_1^2}\right)=-\frac{F_1}{k_3^2}(\Delta-1)(\Delta-2) \tag{A.53}$$

$$\frac{(k_2+k_3-k_1)}{4}\left(\frac{\partial^2 G_1}{\partial k_2^2}-\frac{\partial^2 G_1}{\partial k_3^2}\right)+s\frac{\partial G_1}{\partial k_2}=-\frac{G_1}{k_3^2}(\Delta-1)(\Delta-2)(k_2+k_3-k_1)$$
$$\frac{(k_1+k_3-k_2)}{4}\left(\frac{\partial^2 G_1}{\partial k_3^2}-\frac{\partial^2 G_1}{\partial k_1^2}\right)-s\frac{\partial G_1}{\partial k_1}=-\frac{G_1}{k_3^2}(\Delta-1)(\Delta-2)(k_1+k_3-k_2) \tag{A.54}$$

From the form of the ansatz for the correlator in (1.102) and since the conformal Ward identity takes the form in (A.52), the equations satisfied by the odd parts $F_2$ and $G_2$ of the correlator (1.102) are identical to those for the even parts $F_1$ and $G_1$ respectively.

We note that the equation for $F_1$ (and $F_2$) is independent of the spin $s$. The dependence on the spin comes through the dilation Ward identity and is given by :

$$\left(\sum_{i=1}^{3}k_i\frac{\partial F_1}{\partial k_i}\right)-(\Delta-2(s+1))F_1=0 \tag{A.55}$$

The same equation is satisfied by $F_2$ as well. The equations (A.54) for $G_1$ (and $G_2$) do not have a non-trivial solution. Solving (A.53) and (A.55) we obtain the result in (1.103).

## Examples

In the following we consider a few examples of the correlator $\langle J_s J_s O_\Delta\rangle$ for specific values of $s$ and $\Delta$.

## Spin one current: $\langle J_\mu J_\nu O_\Delta\rangle$

Setting $s=1$ in the expression for the generic correlator (1.104) we obtain :

$$\begin{aligned}&\langle J^- J^+ O_\Delta\rangle=0\\&\langle J^- J^- O_\Delta\rangle=\langle J^- J^- O_\Delta\rangle_{\text{even}}+\langle J^- J^- O_\Delta\rangle_{\text{odd}}=(c_1+ic_2)\ I_{\frac{5}{2}\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\langle 12\rangle^2\\&\langle J^+ J^+ O_\Delta\rangle=\langle J^+ J^+ O_\Delta\rangle_{\text{even}}+\langle J^+ J^+ O_\Delta\rangle_{\text{odd}}=(c_1-ic_2)\ I_{\frac{5}{2}\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\langle\bar{1}\bar{2}\rangle^2\end{aligned} \tag{A.56}$$

### Example: $\Delta=1$

When $\Delta=1$ we have :

$$\begin{aligned}&\langle J^- J^- O_1\rangle_{\text{even}}=c_1\frac{1}{k_3(k_1+k_2+k_3)^2}\langle 12\rangle^2 &&\langle J^- J^+ O_1\rangle_{\text{even}}=0\\&\langle J^- J^- O_1\rangle_{\text{odd}}=ic_1'\frac{1}{k_3(k_1+k_2+k_3)^2}\langle 12\rangle^2 &&\langle J^- J^+ O_1\rangle_{\text{odd}}=0\end{aligned} \tag{A.57}$$

[2] $z_1^{-\kappa}=\frac{(\sigma^\kappa)^{\alpha\beta}\lambda_{1\alpha}\lambda_{1\beta}}{2k_1}$

**Example: $\Delta = 2$**

When $\Delta = 2$ we have :

$$\begin{aligned}
\langle J^- J^- O_2\rangle_{\text{even}} &= c_1 \frac{1}{(k_1+k_2+k_3)^2}\langle 12\rangle^2 & \langle J^- J^+ O_2\rangle_{\text{even}} &= 0 \\
\langle J^- J^- O_2\rangle_{\text{odd}} &= i c_1' \frac{1}{(k_1+k_2+k_3)^2}\langle 12\rangle^2 & \langle J^- J^+ O_2\rangle_{\text{odd}} &= 0
\end{aligned} \tag{A.58}$$

**Example: $\Delta = 3$**

When $\Delta = 3$ we have :

$$\begin{aligned}
\langle J^- J^- O_3\rangle_{\text{even}} &= c_1 \frac{k_1+k_2+2k_3}{(k_1+k_2+k_3)^2}\langle 12\rangle^2 & \langle J^- J^+ O_3\rangle_{\text{even}} &= 0 \\
\langle J^- J^- O_3\rangle_{\text{odd}} &= i c_1' \frac{k_1+k_2+2k_3}{(k_1+k_2+k_3)^2}\langle 12\rangle^2 & \langle J^- J^+ O_3\rangle_{\text{odd}} &= 0
\end{aligned} \tag{A.59}$$

We see that the solution for $\Delta = 1$ is just the shadow transform of the $\Delta = 2$ solution. In Section 1.5 we convert this answer to momentum space and check that it matches the known answer previously computed in [188].

## Spin Two current : $\langle TTO_\Delta\rangle$

Setting $s = 2$ in the expression for the generic correlator (1.104) we obtain :

$$\begin{aligned}
&\langle T^- T^+ O_\Delta\rangle = 0 \\
&\langle T^- T^- O_\Delta\rangle = \langle T^- T^- O_\Delta\rangle_{\text{even}} + \langle T^- T^- O_\Delta\rangle_{\text{odd}} = (c_1 + i c_2)\ k_1 k_2 I_{\frac{9}{2}\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\langle 12\rangle^4 \\
&\langle T^+ T^+ O_\Delta\rangle = \langle T^+ T^+ O_\Delta\rangle_{\text{even}} + \langle T^+ T^+ O_\Delta\rangle_{\text{odd}} = (c_1 - i c_2)\ k_1 k_2 I_{\frac{9}{2}\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}\}}\langle \bar{1}\bar{2}\rangle^4
\end{aligned} \tag{A.60}$$

## Example: $\Delta = 1$

When $\Delta = 1$ we have :

$$\begin{aligned}
\langle T^- T^- O_1\rangle_{\text{even}} &= c_1 k_1 k_2 \frac{1}{k_3(k_1+k_2+k_3)^4}\langle 12\rangle^4 & \langle T^- T^+ O_1\rangle_{\text{even}} &= 0 \\
\langle T^- T^- O_1\rangle_{\text{odd}} &= i c_1' k_1 k_2 \frac{1}{k_3(k_1+k_2+k_3)^4}\langle 12\rangle^4 & \langle T^- T^+ O_1\rangle_{\text{odd}} &= 0
\end{aligned} \tag{A.61}$$

## Example: $\Delta = 2$

When $\Delta = 2$ we have :

$$\begin{aligned}
\langle T^- T^- O_2\rangle_{\text{even}} &= c_1 k_1 k_2 \frac{1}{(k_1+k_2+k_3)^4}\langle 12\rangle^4 & \langle T^- T^+ O_2\rangle_{\text{even}} &= 0 \\
\langle T^- T^- O_2\rangle_{\text{odd}} &= i c_1' k_1 k_2 \frac{1}{(k_1+k_2+k_3)^4}\langle 12\rangle^4 & \langle T^- T^+ O_2\rangle_{\text{odd}} &= 0
\end{aligned} \tag{A.62}$$

## Example: $\Delta = 3$

When $\Delta = 3$ we have :

$$\begin{aligned}
\langle T^- T^- O_3 \rangle_{\text{even}} &= c_1 k_1 k_2 \frac{k_1 + k_2 + 4k_3}{(k_1 + k_2 + k_3)^4} \langle 12 \rangle^4 & \langle T^- T^+ O_3 \rangle_{\text{even}} &= 0 \\
\langle T^- T^- O_3 \rangle_{\text{odd}} &= i c_1' k_1 k_2 \frac{k_1 + k_2 + 4k_3}{(k_1 + k_2 + k_3)^4} \langle 12 \rangle^4 & \langle T^- T^+ O_3 \rangle_{\text{odd}} &= 0
\end{aligned} \tag{A.63}$$

Again, we see that the $\Delta = 1$ solution and the $\Delta = 2$ solution are just shadow transforms of each other. For $\Delta \geq 6$, the triple-$K$ integrals show a divergence and the correlators need to be renormalized.

## Higher spin example

Let us now discuss a few correlators involving higher spin conserved currents. When the scalar operator $O_\Delta$ has scaling dimension $\Delta = 3$ and the conserved current operator $J_s$ has spin $s = 3$, we have from (1.104) :

$$\begin{aligned}
\langle J^{3-} J^{3-} O_3 \rangle &= (c_1 + i\, c_2)(k_1 k_2)^2 I_{\frac{13}{2}\{\frac{1}{2},\frac{1}{2},\frac{3}{2}\}} \\
&= (c_1 + i\, c_2)(k_1 k_2)^2 \frac{E + 5k_3}{E^6} \langle 12 \rangle^6
\end{aligned} \tag{A.64}$$

When the scalar operator $O_\Delta$ has scaling dimension $\Delta = 3$ and the conserved current operator $J_s$ has spin $s = 4$, we have from (1.104) :

$$\begin{aligned}
\langle J^{4-} J^{4-} O_3 \rangle &= (c_1 + i\, c_2)(k_1 k_2)^4 I_{\frac{17}{2}\{\frac{1}{2},\frac{1}{2},\frac{3}{2}\}} \langle 12 \rangle^8 \\
&= (c_1 + i\, c_2)(k_1 k_2)^4 \frac{E + 7k_3}{E^8} \langle 12 \rangle^8
\end{aligned} \tag{A.65}$$

We can also get the parity even part of the above two results using weight-shifting and spin-raising operators in momentum space [50, 188] and then converting the answer into spinor-helicity variables :

$$\langle J^3 J^3 O_3 \rangle = (k_1 k_2)^2 P_1^{(3)} P_2^{(3)} H_{12}^3 \langle O_2 O_2 O_3 \rangle \tag{A.66}$$

$$\langle J^4 J^4 O_3 \rangle = (k_1 k_2)^3 P_1^{(4)} P_2^{(4)} H_{12}^4 \langle O_2 O_2 O_3 \rangle \tag{A.67}$$

where $P_i^{(s)}$ are spin-$s$ projectors transverse to $k_i$ and $H_{12}$ is a bilocal operator that raises the spin of the operators at insertions 1 and 2. It can be verified that the answers obtained this way match the answers in (A.64) and (A.65).

### A.6.3 $\langle JJJ \rangle$

The ansatz for the correlator is given in (1.106). We will analyze the parity-odd and the parity-even parts separately here as they have different WT identities.

$\langle JJJ \rangle_{\textbf{even}}$

$$\langle J^-(k_1) J^-(k_2) J^-(k_3) \rangle_{\text{even}} = F_1(k_1, k_2, k_3) \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \tag{A.68}$$

$$\langle J^-(k_1) J^-(k_2) J^+(k_3) \rangle_{\text{even}} = G_1(k_1, k_2, k_3) \langle 12 \rangle \langle 2\bar{3} \rangle \langle \bar{3}1 \rangle \tag{A.69}$$

The action of the conformal generator is given by :

$$\widetilde{K}^\kappa\langle J^-J^-J^-\rangle = 2\left(z_1^{-\kappa}\frac{k_{1\mu}}{k_1^2}\langle J^\mu J^-J^-\rangle + z_2^{-\kappa}\frac{k_{2\mu}}{k_2^2}\langle J^-J^\mu J^-\rangle + z_3^{-\kappa}\frac{k_{3\mu}}{k_3^2}\langle J^-J^-J^\mu\rangle\right)$$
$$\widetilde{K}^\kappa\langle J^-J^-J^+\rangle = 2\left(z_1^{-\kappa}\frac{k_{1\mu}}{k_1^2}\langle J^\mu J^-J^+\rangle + z_2^{-\kappa}\frac{k_{2\mu}}{k_2^2}\langle J^-J^\mu J^+\rangle + z_3^{+\kappa}\frac{k_{3\mu}}{k_3^2}\langle J^-J^-J^\mu\rangle\right) \quad (A.70)$$

The transverse Ward identities of $\langle JJJ\rangle$ [44] are non-trivial :

$$\frac{k_{1\mu}}{k_1^2}\langle J^\mu J^-J^-\rangle_{\text{even}} = c_J\frac{1}{k_1^2k_2k_3}\langle 23\rangle^2(k_3-k_2)$$
$$\frac{k_{1\mu}}{k_1^2}\langle J^\mu J^-J^+\rangle_{\text{even}} = c_J\frac{1}{k_1^2k_2k_3}\langle 2\bar{3}\rangle^2(k_3-k_2) \quad (A.71)$$

Using (A.71) in the R.H.S. of (A.70) we obtain :

$$\widetilde{K}^\kappa\langle J^-J^-J^-\rangle_{\text{even}} = z_1^{-\kappa}c_J\frac{\langle 23\rangle^2}{k_1^2k_2k_3}(k_2-k_3) + \text{cyclic perm.} \quad (A.72)$$

$$\widetilde{K}^\kappa\langle J^-J^-J^+\rangle_{\text{even}} = z_1^{-\kappa}c_J\frac{\langle 2\bar{3}\rangle^2}{k_1^2k_2k_3}(k_2-k_3) + \text{cyclic perm.} \quad (A.73)$$

Expanding out the left hand side and dotting with $(\sigma^\kappa)^\beta_\alpha(\lambda_2^\alpha\lambda_{3\beta}+\lambda_{2\beta}\lambda_3^\alpha)$ gives us the following equations for the form factors :

$$2\left(\frac{\partial F_1}{\partial k_2}-\frac{\partial F_1}{\partial k_3}\right)+k_2\left(\frac{\partial^2F_1}{\partial k_2^2}-\frac{\partial^2F_1}{\partial k_1^2}\right)+k_3\left(\frac{\partial^2F_1}{\partial k_1^2}-\frac{\partial^2F_1}{\partial k_3^2}\right) = 2c_J\frac{(k_3-k_2)}{k_1^3k_2k_3} \quad (A.74)$$

$$2\left(\frac{\partial G_1}{\partial k_2}+\frac{\partial G_1}{\partial k_3}\right)+k_2\left(\frac{\partial^2G_1}{\partial k_2^2}-\frac{\partial^2G_1}{\partial k_1^2}\right)-k_3\left(\frac{\partial^2G_1}{\partial k_1^2}-\frac{\partial^2G_1}{\partial k_3^2}\right) = 2c_J\frac{(k_3-k_2)}{k_1^3k_2k_3} \quad (A.75)$$

Similarly, dotting with $(\sigma^\kappa)^\beta_\alpha(\lambda_1^\alpha\lambda_{3\beta}+\lambda_{1\beta}\lambda_3^\alpha)$ gives :

$$2\left(\frac{\partial F_1}{\partial k_1}-\frac{\partial F_1}{\partial k_3}\right)-k_1\left(\frac{\partial^2F_1}{\partial k_2^2}-\frac{\partial^2F_1}{\partial k_1^2}\right)+k_3\left(\frac{\partial^2F_1}{\partial k_2^2}-\frac{\partial^2F_1}{\partial k_3^2}\right) = 2c_J\frac{(k_3-k_1)}{k_1k_2^3k_3} \quad (A.76)$$

$$2\left(\frac{\partial G_1}{\partial k_1}+\frac{\partial G_1}{\partial k_3}\right)-k_1\left(\frac{\partial^2G_1}{\partial k_2^2}-\frac{\partial^2G_1}{\partial k_1^2}\right)-k_3\left(\frac{\partial^2G_1}{\partial k_2^2}-\frac{\partial^2G_1}{\partial k_3^2}\right) = 2c_J\frac{(k_3-k_1)}{k_1k_2^3k_3} \quad (A.77)$$

The dilatation Ward identity is given by

$$\left(\sum_{i=1}^{3}k_i\frac{\partial F_1}{\partial k_i}\right)+3F_1=0, \qquad \left(\sum_{i=1}^{3}k_i\frac{\partial G_1}{\partial k_i}\right)+3G_1=0 \quad (A.78)$$

Solving these equations we obtain $F_1(k_1,k_2,k_3)$ and $G_1(k_1,k_2,k_3)$ in (1.111).

### $\langle JJJ\rangle_{\text{odd}}$

We now turn our attention to the odd part of the correlator. The ansatz is given by

$$\langle J^-(k_1)J^-(k_2)J^-(k_3)\rangle_{\text{odd}} = iF_2(k_1,k_2,k_3)\langle 12\rangle\langle 23\rangle\langle 31\rangle \quad (A.79)$$
$$\langle J^-(k_1)J^-(k_2)J^+(k_3)\rangle_{\text{odd}} = iG_2(k_1,k_2,k_3)\langle 12\rangle\langle 2\bar{3}\rangle\langle\bar{3}1\rangle \quad (A.80)$$

The transverse WT identity in this case is given by :

$$\frac{k_{1\mu}}{k_1^2}\langle J^\mu J^- J^-\rangle_{\rm odd} = c_J'\frac{1}{k_1^2k_2k_3}\langle 23\rangle^2(k_3-k_2)$$
$$\frac{k_{1\mu}}{k_1^2}\langle J^\mu J^- J^+\rangle_{\rm odd} = c_J'\frac{1}{k_1^2k_2k_3}\langle 2\bar{3}\rangle^2(k_3+k_2) \tag{A.81}$$

Substituting (A.81) into the right hand side of the conformal identity (A.70), we get :

$$\begin{aligned}\widetilde{K}^\kappa\langle J^- J^- J^-\rangle_{\rm odd} &= z_1^{-\kappa} i\, c_J'\frac{\langle 23\rangle^2}{k_1^2k_2k_3}(k_2-k_3) + \text{cyclic perm.}\\ \widetilde{K}^\kappa\langle J^- J^- J^+\rangle_{\rm odd} &= z_1^{-\kappa} i\, c_J'\frac{\langle 2\bar{3}\rangle^2}{k_1^2k_2k_3}(k_2+k_3) + \text{cyclic perm.}\end{aligned} \tag{A.82}$$

Following the same procedure as in the parity-even case, we get :

$$2\left(\frac{\partial F_2}{\partial k_2}-\frac{\partial F_2}{\partial k_3}\right)+k_2\left(\frac{\partial^2 F_2}{\partial k_2^2}-\frac{\partial^2 F_2}{\partial k_1^2}\right)+k_3\left(\frac{\partial^2 F_2}{\partial k_1^2}-\frac{\partial^2 F_2}{\partial k_3^2}\right)=2c_J'\frac{(k_3-k_2)}{k_1^3k_2k_3} \tag{A.83}$$
$$2\left(\frac{\partial G_2}{\partial k_2}+\frac{\partial G_2}{\partial k_3}\right)+k_2\left(\frac{\partial^2 G_2}{\partial k_2^2}-\frac{\partial^2 G_2}{\partial k_1^2}\right)-k_3\left(\frac{\partial^2 G_2}{\partial k_1^2}-\frac{\partial^2 G_2}{\partial k_3^2}\right)=2c_J'\frac{(k_3+k_2)}{k_1^3k_2k_3} \tag{A.84}$$

and

$$2\left(\frac{\partial F_2}{\partial k_1}-\frac{\partial F_2}{\partial k_3}\right)-k_1\left(\frac{\partial^2 F_2}{\partial k_2^2}-\frac{\partial^2 F_2}{\partial k_1^2}\right)+k_3\left(\frac{\partial^2 F_2}{\partial k_2^2}-\frac{\partial^2 F_2}{\partial k_3^2}\right)=2c_J'\frac{(k_3-k_1)}{k_1k_2^3k_3} \tag{A.85}$$
$$2\left(\frac{\partial G_2}{\partial k_1}+\frac{\partial G_2}{\partial k_3}\right)-k_1\left(\frac{\partial^2 G_2}{\partial k_2^2}-\frac{\partial^2 G_2}{\partial k_1^2}\right)-k_3\left(\frac{\partial^2 G_2}{\partial k_2^2}-\frac{\partial^2 G_2}{\partial k_3^2}\right)=2c_J'\frac{(k_3+k_1)}{k_1k_2^3k_3} \tag{A.86}$$

Let us note that (A.83), (A.85) are exactly identical to (A.74), (A.76), whereas comparing (A.84), (A.86) with (A.75), (A.77), we see that the r.h.s. of the equations are different. Solving these equations we obtain $F_2(k_1,k_2,k_3)$ and $G_2(k_1,k_2,k_3)$ in (1.111).

### A.6.4 $\langle TTT\rangle$

The even part of this correlator was obtained earlier in [41,180]. We focus on obtaining the odd part.

#### $\langle TTT\rangle_{\rm odd}$

We start with the following ansatz for $\langle TTT\rangle_{\rm odd}$ :

$$\left\langle\frac{T^-}{k_1}\frac{T^-}{k_2}\frac{T^-}{k_3}\right\rangle_{\rm odd} = i\,F(k_1,k_2,k_3)\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \tag{A.87}$$
$$\left\langle\frac{T^-}{k_1}\frac{T^-}{k_2}\frac{T^+}{k_3}\right\rangle_{\rm odd} = i\,G(k_1,k_2,k_3)\langle 12\rangle^2\langle 2\bar{3}\rangle^2\langle \bar{3}1\rangle^2 \tag{A.88}$$

The action of the conformal generator is given by :

$$\widetilde{K}^\kappa \left\langle \frac{T^-}{k_1}\frac{T^-}{k_2}\frac{T^-}{k_3}\right\rangle = 12 z^-_{1\kappa}\frac{k_{(1\mu}z^-_{1\nu)}}{k_1^3}\left\langle T^{\mu\nu}\frac{T^-}{k_2}\frac{T^-}{k_3}\right\rangle + 12 z^-_{2\kappa}\frac{k_{(2\mu}z^-_{2\nu)}}{k_2^3}\left\langle \frac{T^-}{k_1}T^{\mu\nu}\frac{T^-}{k_3}\right\rangle$$
$$+ 12 z^-_{3\kappa}\frac{k_{(3\mu}z^-_{3\nu)}}{k_3^3}\left\langle \frac{T^-}{k_1}\frac{T^-}{k_2}T^{\mu\nu}\right\rangle$$
$$\widetilde{K}^\kappa \left\langle \frac{T^-}{k_1}\frac{T^-}{k_2}\frac{T^+}{k_3}\right\rangle = 12 z^-_{1\kappa}\frac{k_{(1\mu}z^-_{1\nu)}}{k_1^3}\left\langle T^{\mu\nu}\frac{T^-}{k_2}\frac{T^+}{k_3}\right\rangle + 12 z^-_{2\kappa}\frac{k_{(2\mu}z^-_{2\nu)}}{k_2^3}\left\langle \frac{T^-}{k_1}T^{\mu\nu}\frac{T^+}{k_3}\right\rangle$$
$$+ 12 z^+_{3\kappa}\frac{k_{(3\mu}z^+_{3\nu)}}{k_3^3}\left\langle \frac{T^-}{k_1}\frac{T^-}{k_2}T^{\mu\nu}\right\rangle \tag{A.89}$$

Using (1.114) we find for parity odd contribution

$$\frac{k_{(1\mu}z^-_{1\nu)}}{k_1^3}\left\langle T^{\mu\nu}\frac{T^-}{k_2}\frac{T^-}{k_3}\right\rangle = E\frac{\langle 12\rangle\langle 23\rangle^3\langle 31\rangle}{k_1^4k_2^3k_3^3}(k_3^3-k_2^3)$$
$$\frac{k_{(1\mu}z^-_{1\nu)}}{k_1^3}\left\langle T^{\mu\nu}\frac{T^-}{k_2}\frac{T^+}{k_3}\right\rangle = (E-2k_3)\frac{\langle 12\rangle\langle 2\bar{3}\rangle^3\langle \bar{3}1\rangle}{k_1^4k_2^3k_3^3}(k_3^3+k_2^3) \tag{A.90}$$

The action of $\widetilde{K}^\kappa$ on the ansatz, after dotting with $b_\kappa = (\sigma_\kappa)^\beta_\alpha(\lambda_2^\alpha\lambda_{3\beta}+\lambda_{2\beta}\lambda_3^\alpha)$, becomes

$$4\left(\frac{\partial F}{\partial k_2}-\frac{\partial F}{\partial k_3}\right)+k_3\left(\frac{\partial^2 F}{\partial k_1^2}-\frac{\partial^2 F}{\partial k_3^2}\right)-k_2\left(\frac{\partial^2 F}{\partial k_1^2}-\frac{\partial^2 F}{\partial k_2^2}\right)=c_T'\frac{E(k_2^3-k_3^3)}{k_1^2(k_1k_2k_3)^3} \tag{A.91}$$
$$4\left(\frac{\partial G}{\partial k_2}+\frac{\partial G}{\partial k_3}\right)-k_3\left(\frac{\partial^2 G}{\partial k_1^2}-\frac{\partial^2 G}{\partial k_3^2}\right)-k_2\left(\frac{\partial^2 G}{\partial k_1^2}-\frac{\partial^2 G}{\partial k_2^2}\right)=c_T'\frac{(E-2k_3)(k_2^3+k_3^3)}{k_1^2(k_1k_2k_3)^3} \tag{A.92}$$

The dilatation Ward identity is given by

$$\left(\sum_{i=1}^3 k_i\frac{\partial F}{\partial k_i}\right)+6F=0,\qquad \left(\sum_{i=1}^3 k_i\frac{\partial G}{\partial k_i}\right)+6G=0 \tag{A.93}$$

The solutions for $F$ and $G$ are then given by :

$$F(k_1,k_2,k_3)=\frac{c_1'}{E^6}+c_T'\frac{E^3-E\,b_{123}-c_{123}}{c_{123}^3} \tag{A.94}$$
$$G(k_1,k_2,k_3)=c_T'\frac{(E-2k_3)^3-(E-2k_3)(b_{123}-2k_3\,a_{12})+c_{123}}{c_{123}^3} \tag{A.95}$$

where $a_{12}=k_1+k_2$, $b_{123}=k_1k_2+k_2k_3+k_1k_3$ and $c_{123}=k_1\,k_2\,k_3$.

### A.6.5 $\langle TJJ\rangle$

We once again focus on only the odd part of the correlator. Since we have shown that the transverse WT identities are trivial in (1.129)and (1.132), the action of $\widetilde{K}^\kappa$ on the ansatz

(1.134) becomes :

$$\widetilde{K}^{\kappa}\left\langle \frac{T^-}{k_1} J^- J^- \right\rangle_{\text{odd}} = 0$$
$$\widetilde{K}^{\kappa}\left\langle \frac{T^-}{k_1} J^- J^+ \right\rangle_{\text{odd}} = 0. \tag{A.96}$$

Expanding out the l.h.s. and dotting with an appropriate $b_\kappa = (\sigma_\kappa)^\beta_\alpha(\lambda_2^\alpha\lambda_{3\beta} + \lambda_{2\beta}\lambda_3^\alpha)$, we get

$$k_3\left(\frac{\partial^2 F}{\partial k_1^2} - \frac{\partial^2 F}{\partial k_3^2}\right) - k_2\left(\frac{\partial^2 F}{\partial k_1^2} - \frac{\partial^2 F}{\partial k_2^2}\right) + 2\left(\frac{\partial F}{\partial k_2} - \frac{\partial F}{\partial k_3}\right) = 0 \tag{A.97}$$

$$k_3\left(\frac{\partial^2 G}{\partial k_1^2} - \frac{\partial^2 G}{\partial k_3^2}\right) - k_2\left(\frac{\partial^2 G}{\partial k_1^2} - \frac{\partial^2 G}{\partial k_2^2}\right) + 2\left(\frac{\partial G}{\partial k_2} + \frac{\partial G}{\partial k_3} - 2\frac{\partial G}{\partial k_1}\right) = 0 \tag{A.98}$$

The solutions to these are given by (1.135).

### A.6.6 $\langle J_{s_1} J_s J_s \rangle$

Dotting (1.142) with $b_\kappa = (\sigma^\kappa)\lambda_{1\alpha}{\lambda_1}^\beta$, we get :

$$(-k_1 + k_2 + k_3)\left(\frac{\partial^2 F}{\partial k_2^2} - \frac{\partial^2 F}{\partial k_3^2}\right) + 2(2s - s_1)\left(\frac{\partial F}{\partial k_2} - \frac{\partial F}{\partial k_3}\right) = 0$$

$$(-k_1 + k_2 - k_3)\left(\frac{\partial^2 H}{\partial k_2^2} - \frac{\partial^2 H}{\partial k_3^2}\right) + 2(2s - s_1)\left(\frac{\partial H}{\partial k_2} + \frac{\partial H}{\partial k_3}\right) = 0 \tag{A.99}$$

Similarly, dotting (1.142) with $b_\kappa = (\sigma^\kappa)\lambda_{2\alpha}{\lambda_2}^\beta$, we get :

$$(k_1 - k_2 + k_3)\left(\frac{\partial^2 F}{\partial k_3^2} - \frac{\partial^2 F}{\partial k_1^2}\right) + 2s_1\left(\frac{\partial F}{\partial k_3} - \frac{\partial F}{\partial k_1}\right) = 0$$

$$(k_3 - k_2 - k_1)\left(\frac{\partial^2 G}{\partial k_3^2} - \frac{\partial^2 G}{\partial k_1^2}\right) + 2s_1\left(\frac{\partial G}{\partial k_3} + \frac{\partial G}{\partial k_1}\right) = 0 \tag{A.100}$$

The dilatation Ward identity is given by

$$\left(\sum_{i=1}^{3} k_i \frac{\partial F}{\partial k_i}\right) + (2s + s_1)F = 0$$

$$\left(\sum_{i=1}^{3} k_i \frac{\partial G}{\partial k_i}\right) + (2s + s_1)G = 0$$

$$\left(\sum_{i=1}^{3} k_i \frac{\partial H}{\partial k_i}\right) + (2s + s_1)H = 0 \tag{A.101}$$

We have considered only one equation for $G$ and $H$ as these by themselves imply that there is no homogeneous solution for the two form factors. The solutions for $F$, $G$ and $H$ are then given by (1.144).

## A.7 Identities involving Triple-$K$ integrals

In this section we obtain non-trivial identities involving triple-$K$ integrals by matching our results obtained for the correlator in spinor-helicity variables to the results obtained for the same in momentum space after converting to spinor-helicity variables.

Let us first consider the correlator $\langle JJO_\Delta\rangle$. We will work in a convenient regularisation scheme in which we set $u = v_1 = v_2 = 0$ and $v_3 \neq 0$. The momentum space expression for the correlator after converting to spinor-helicity variables takes the following form :

$$\langle J^- J^- O\rangle = -\frac{2A_2 + A_1\left[(k_1-k_2)^2 - k_3^2\right]}{4k_1k_2}\langle 12\rangle^2 \tag{A.102}$$

where [45] :

$$\begin{aligned} A_1 &= c_1 I_{\frac{5}{2},\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}+v_3\epsilon\}} \\ A_2 &= c_1 I_{\frac{3}{2},\{\frac{1}{2},\frac{1}{2},\Delta-\frac{1}{2}+v_3\epsilon\}} + c_1\frac{\Delta}{2}(1-\Delta)\, I_{\frac{1}{2},\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}+v_3\epsilon\}} \end{aligned} \tag{A.103}$$

Comparing with our results for the same correlator obtained by solving the conformal Ward identities directly in spinor-helicity variables (A.56) we get the following identity involving triple$-K$ integrals which we have verified to $O(1)$ in the regulator the following relation :

$$-\frac{2A_2 + A_1\left[(k_1-k_2)^2 - k_3^2\right]}{4k_1k_2} = c_1 I_{\frac{5}{2},\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}+v_3\epsilon\}} \tag{A.104}$$

Let us now consider the correlator $\langle TTO_\Delta\rangle$. The momentum space expression for the correlator after converting to spinor-helicity variables takes the following form :

$$\langle T^- T^- O\rangle = \frac{4A_3 + \left[(k_1-k_2)^2 - k_3^2\right]\left[2A_2 + A_1((k_1-k_2)^2 - k_3^2)\right]}{16k_1^2k_2^2}\langle 12\rangle^4 \tag{A.105}$$

We will continue to work in the scheme where $u = v_1 = v_2 = 0$ and only $v_3$ is non-zero and in this scheme the form factors are given by [45] :

$$\begin{aligned} A_1 &= c_1\, I_{\frac{9}{2},\{\frac{3}{2},\frac{3}{2},\Delta-\frac{3}{2}+v_3\epsilon\}} \\ A_2 &= 4c_1\, I_{\frac{7}{2},\{\frac{3}{2},\frac{3}{2},\Delta-\frac{1}{2}+v_3\epsilon\}} + c_2\, I_{\frac{5}{2},\{\frac{3}{2},\frac{3}{2},\Delta-\frac{3}{2}+v_3\epsilon\}} \\ A_3 &= 2c_1\, I_{\frac{5}{2},\{\frac{3}{2},\frac{3}{2},\Delta+\frac{1}{2}+v_3\epsilon\}} + c_2\, I_{\frac{3}{2},\{\frac{3}{2},\frac{3}{2},\Delta-\frac{1}{2}+v_3\epsilon\}} + c_3\, I_{\frac{1}{2},\{\frac{3}{2},\frac{3}{2},\Delta-\frac{3}{2}+v_3\epsilon\}} \end{aligned} \tag{A.106}$$

where

$$\begin{aligned} c_2 &= c_1(1-\Delta-v_3\epsilon)(\Delta+2+v_3\epsilon) \\ c_3 &= \frac{c_1}{4}(\Delta-3+v_3\epsilon)(\Delta-1+v_3\epsilon)(\Delta+v_3\epsilon)(\Delta+2+v_3\epsilon) \end{aligned} \tag{A.107}$$

Matching with our answers obtained by solving conformal Ward identities in spinor-helicity variables (A.60) we obtain the following identity for triple-$K$ integrals we have verified to $O(1)$ in the regulator :

$$4A_3 + \left[(k_1-k_2)^2 - k_3^2\right]\left[2A_2 + A_1((k_1-k_2)^2 - k_3^2)\right] = 16c_1k_1^3\,k_2^3\, I_{\frac{9}{2},\{\frac{1}{2},\frac{1}{2},\Delta-\frac{3}{2}+v_3\epsilon\}} \tag{A.108}$$

## A.8 Higher-spin momentum space correlators

In this section we summarise the momentum space expression for the parity-even and parity-odd homogeneous parts of higher spin correlators using the results of section 1.5, see also Appendix D of [187].

For $\langle J_s J_s O_2\rangle$ we have

$$\langle J_s J_s O_2\rangle_{\text{even},\mathbf{h}} = (k_1k_2)^{s-1}\left[\frac{1}{E^2}\left\{2(\vec{z}_1\cdot\vec{k}_2)(\vec{z}_2\cdot\vec{k}_1)+E(E-2k_3)\vec{z}_1\cdot\vec{z}_2\right\}\right]^s$$
$$\langle J_s J_s O_2\rangle_{\text{odd},\mathbf{h}} = (k_1k_2)^{s-1}\frac{1}{E^{2s}}\left[k_2\,\epsilon^{k_1z_1z_2}-k_1\,\epsilon^{k_2z_1z_2}\right]$$
$$\times\left[2(\vec{z}_1\cdot\vec{k}_2)(\vec{z}_2\cdot\vec{k}_1)+E(E-2k_3)\vec{z}_1\cdot\vec{z}_2\right]^{s-1} \tag{A.109}$$

while for $\langle J_s J_s O_3\rangle$ we get

$$\langle J_s J_s O_3\rangle_{\text{even},\mathbf{h}} = (k_1k_2)^{s-1}(E+(2s-1)k_3)\left[\frac{1}{E^2}\left\{2(\vec{z}_1\cdot\vec{k}_2)(\vec{z}_2\cdot\vec{k}_1)+E(E-2k_3)\vec{z}_1\cdot\vec{z}_2\right\}\right]^s$$
$$\langle J_s J_s O_3\rangle_{\text{odd},\mathbf{h}} = (k_1k_2)^{s-1}\frac{(E+(2s-1)k_3)}{E^{2s}}\left[k_2\,\epsilon^{k_1z_1z_2}-k_1\,\epsilon^{k_2z_1z_2}\right]$$
$$\times\left[2(\vec{z}_1\cdot\vec{k}_2)(\vec{z}_2\cdot\vec{k}_1)+E(E-2k_3)\vec{z}_1\cdot\vec{z}_2\right]^{s-1} \tag{A.110}$$

The homogeneous part of the $J_s$ 3-point correlator is

$$\langle J_sJ_sJ_s\rangle_{\text{even},\mathbf{h}} = (k_1k_2k_3)^{s-1}\left[\frac{1}{E^3}\left\{2\,(\vec{z}_1\cdot\vec{k}_2)\,(\vec{z}_2\cdot\vec{k}_3)\,(\vec{z}_3\cdot\vec{k}_1)+E\{k_3\,(\vec{z}_1\cdot\vec{z}_2)\,(\vec{z}_3\cdot\vec{k}_1)+\text{cyclic}\}\right\}\right]^s$$
$$\langle J_sJ_sJ_s\rangle_{\text{odd},\mathbf{h}} = (k_1k_2k_3)^{s-1}\frac{1}{E^3}\left[\left\{(\vec{k}_1\cdot\vec{z}_3)\left(\epsilon^{k_3z_1z_2}k_1-\epsilon^{k_1z_1z_2}k_3\right)+(\vec{k}_3\cdot\vec{z}_2)\left(\epsilon^{k_1z_1z_3}k_2-\epsilon^{k_2z_1z_3}k_1\right)\right.\right.$$
$$\left.\left.-(\vec{z}_2\cdot\vec{z}_3)\epsilon^{k_1k_2z_1}E+\frac{k_1}{2}\epsilon^{z_1z_2z_3}E(E-2k_1)\right\}+\text{cyclic perm}\right]$$
$$\times\left[\frac{1}{E^3}\left\{2\,(\vec{z}_1\cdot\vec{k}_2)\,(\vec{z}_2\cdot\vec{k}_3)\,(\vec{z}_3\cdot\vec{k}_1)+E\{k_3\,(\vec{z}_1\cdot\vec{z}_2)\,(\vec{z}_3\cdot\vec{k}_1)+\text{cyclic}\}\right\}\right]^{s-1} \tag{A.111}$$

whereas for $\langle J_{2s}J_sJ_s\rangle$ we have

$$\langle J_{2s}J_sJ_s\rangle_{\text{odd},\mathbf{h}} = \frac{k_1^{2s-1}(k_2k_3)^{s-1}}{E^{4s}}\left[\left((k_3\cdot z_2)(k_2\cdot z_1)-\frac{1}{2}E(E-2k_3)(z_1\cdot z_2)\right)\left(k_1\epsilon^{z_1z_3k_3}-k_3\epsilon^{z_1z_3k_1}\right)\right]$$
$$\times\left[\left((k_3\cdot z_2)(k_2\cdot z_1)-\frac{1}{2}E(E-2k_3)(z_1\cdot z_2)\right)\left((k_1\cdot z_3)(k_2\cdot z_1)-\frac{1}{2}E(E-2k_2)(z_1\cdot z_3)\right)\right]^{s-1}$$
$$\langle J_{2s}J_sJ_s\rangle_{\text{even},\mathbf{h}} = \frac{k_1^{2s-1}(k_2k_3)^{s-1}}{E^{4s}}\left[\left((k_3\cdot z_2)(k_2\cdot z_1)-\frac{1}{2}E(E-2k_3)(z_1\cdot z_2)\right)\right.$$
$$\left.\times\left((k_1\cdot z_3)(k_2\cdot z_1)-\frac{1}{2}E(E-2k_2)(z_1\cdot z_3)\right)\right]^s \tag{A.112}$$

## A.9 Parity-odd weight-shifting operators

We need the following spin and dimension raising operators [50, 180, 188],

$$
\begin{aligned}
H_{12} =& 2\left(z_1 \cdot K_{12} z_2 \cdot K_{12} - 2 z_1 \cdot z_2 W_{12}^{--}\right), \\
\tilde{D}_{12} =& -\frac{1}{2}\Big[\epsilon(z_1 z_2 K_{12}^-)(\Delta_1 - d - k_1 \cdot \frac{\partial}{\partial k_1}) + \frac{K_{12}^- K_{12}^+}{2}\epsilon(k_1 z_1 z_2) + \epsilon(k_1 K_{12}^- z_1)(z_2 \cdot \frac{\partial}{\partial k_2}) \\
& + \epsilon(k_1 z_2 K_{12}^-)(z_1 \cdot \frac{\partial}{\partial k_1})\Big].
\end{aligned} \tag{A.113}
$$

where expressions for $K_{12}, W_{12}^{--}$ can be found in the above mentioned references. The following sequence of operators reproduces $\langle TJJ\rangle_{\rm odd}$

$$
\langle T(k_1)J(k_2)J(k_3)\rangle_{\rm odd} = P_1^{(2)} P_2^{(1)} P_3^{(1)} H_{13} \widetilde{D}_{12} \langle O_1(k_1) O_2(k_2) O_2(k_3)\rangle + (2 \leftrightarrow 3) \tag{A.114}
$$

where $P_i^{(s)}$ is a spin$-s$ projector. The explicit momentum space expression for the correlator is given by

$$
\begin{aligned}
\langle TJJ\rangle_{\rm odd} = \Big[& A_1 \epsilon^{k_1 k_2 z_1}(k_2 \cdot z_1)(k_3 \cdot z_2)(k_1 \cdot z_3) + A_2 \epsilon^{k_1 k_2 z_1}(z_2 \cdot z_3)(k_2 \cdot z_1) \\
& + A_3 \epsilon^{k_1 z_1 z_2}(k_2 \cdot z_1)(k_1 \cdot z_3) + A_4 \epsilon^{k_2 z_1 z_2}(k_2 \cdot z_1)(k_1 \cdot z_3) \\
& + A_5 \epsilon^{k_1 z_1 z_2}(z_1 \cdot z_3) + A_6 \epsilon^{k_2 z_1 z_2}(z_1 \cdot z_3) \\
& + A_7 \epsilon^{k_1 k_2 z_1}(z_1 \cdot z_2)(k_1 \cdot z_3) + A_8 \epsilon^{z_1 z_2 z_3}(k_2 \cdot z_1)\Big] + (2 \leftrightarrow 3)
\end{aligned} \tag{A.115}
$$

where the form factors are given by

$$
\begin{aligned}
A_1 &= 12 \frac{5k_1^2 + 4k_1(k_2+k_3) + (k_2+k_3)^2}{k_1^2 (k_1+k_2+k_3)^4} \\
A_2 &= 4 \frac{k_1 + k_2 + 3k_3}{(k_1+k_2+k_3)^3} \\
A_3 &= \frac{15k_1^3 + 13k_1^2(k_2+k_3) + 9k_1(k_2+k_3)^2 + 3(k_2+k_3)^3}{k_1^2(k_1+k_2+k_3)^3} \\
A_4 &= \frac{k_1 + k_2 + 3k_3}{(k_1+k_2+k_3)^3} \\
A_5 &= \frac{-3k_1^4 + 2k_1^3(5k_2 - 3k_3) + 4k_1^2 k_2(2k_2 - k_3) + 6k_1(k_2-k_3)^2(k_2+k_3) + 3(k_2^2 - k_3^2)^2}{2k_1^2(k_1+k_2+k_3)^2} \\
A_6 &= 4 \frac{k_2(k_1 + k_2 + 2k_3}{(k_1+k_2+k_3)^2} \\
A_7 &= \frac{-3k_1^3 - 3(k_2 - 3k_3)(k_2+k_3)^2 + k_1^2(-9k_2 + 23k_3) - 9k_1(k_2^2 - 2k_2k_3 - 3k_3^2))}{k_1^2(k_1+k_2+k_3)^3} \\
A_8 &= -2 \frac{3k_1^2 + 2k_1(k_2+k_3) + (k_2+k_3)^2}{(k_1+k_2+k_3)^2}
\end{aligned} \tag{A.116}
$$

Although this expression looks very different from the expression obtained earlier in (1.197), they are actually the same up to some Schouten identities. This can easily be seen by converting both of them to spinor-helicity variables where they match exactly.

## A.10 Flat-space amplitudes: Examples

In this section, we give some simple examples of flat-space $4D$ scattering amplitude. We define two sets of amplitudes, one that satisfies $s_i \leq s_j + s_k$ is called inside the triangle and one that violates this is called outside the triangle. This distinction becomes very important for momentum space CFT correlators as was discussed in [184].

### A.10.1 Inside the triangle inequality

We take a simple example of two photon and graviton scattering. The results are given by four structures, two parity-even and one parity-odd

$$\mathcal{M}^{112}_{even} = g_{m,e}(z_1.p_2 z_2.z_3 + z_2.p_3 z_3.z_1 + z_3.p_1 z_1.z_2)(z_3.p_1) + g_{nm,e}(z_1.p_2)(z_2.p_3)(z_3.p_1)^2 \quad \text{(A.117)}$$

$$\begin{aligned}\mathcal{M}^{112}_{odd} =& g_{m,o}[\epsilon(z_2p_2z_3p_3)(z_3.z_1) + \epsilon(z_3p_3z_1p_1)(z_2.z_3) + (\epsilon(z_1z_2z_3p_2) - \epsilon(z_1z_2z_3p_1))(z_3.k_1)] \\ &+ g_{nm,o}\epsilon(z_2p_2z_3p_3)(z_1.p_2)(z_3.p_1)\end{aligned} \quad \text{(A.118)}$$

Both the minimal and non-minimal amplitudes are present for both parity-even and parity-odd cases. Notice that the odd minimal amplitude is antisymmetric under $1 \leftrightarrow 2$ exchange. Therefore, one needs to introduce Chan-Paton factors for that amplitude. In fact, it turns out Chan-Paton factors must be introduced for amplitudes with $s_1 = s_2 < s_3$ for even $s_3$. We rewrite the odd amplitude in $3D$ momentum space variables (1.275)

$$\begin{aligned}\mathcal{M}^{112}_{odd} =& g_{m,o}[-(\epsilon(z_2z_3k_2)k_3 - \epsilon(z_2z_3k_3)k_2)(z_1\cdot z_3) + \epsilon(z_1z_2z_3)k_2(z_3\cdot k_1)] \\ &+ g_{nm,o}[-(\epsilon(z_2z_3k_2)k_3 - \epsilon(z_2z_3k_3)k_2)](z_1.k_2)(z_3.k_1)\end{aligned} \quad \text{(A.119)}$$

It is easy to show under epsilon transform that the parity-even amplitude in (A.117) maps to parity-odd amplitude (A.119). Let us for completeness show this below explicitly. Consider

$$\begin{aligned}[O_\epsilon]_2\mathcal{M}^{112}_e =& g_{m,e}(z_1.k_2\frac{\epsilon(z_2k_2z_3)}{k_2} + z_3.z_1\frac{\epsilon(z_2k_2k_3)}{k_2} + z_3.k_1\frac{\epsilon(z_2k_2z_1)}{k_2})(z_3.k_1) \\ &+ g_{nm,e}(z_1.k_2)\frac{\epsilon(z_2k_2k_3)}{k_2}(z_3.k_1)^2\end{aligned} \quad \text{(A.120)}$$

Using the Schouten identities

$$(z_1.k_2)\epsilon(z_2k_2z_3) = k_2^2\epsilon(z_2z_1z_3) - z_3.k_1\epsilon(z_2k_2z_1) \quad \text{(A.121)}$$

$$(z_3.k_1)\epsilon(z_2k_2k_3) = -k_2^2\epsilon(z_2z_3k_3) - k_2k_3\epsilon(z_2k_2z_3) \quad \text{(A.122)}$$

in the above we exactly get (A.119). We have also made use of $k_Ik_J = k_I.k_J$ in the above. A more abstract derivation of the same is given in Section 1.10.1.2. From here on-wards, we write all the flat-space amplitudes in $3D$ momentum space variables.

## Three spin-s amplitude

For general three spin-s amplitude we have

$$\begin{aligned}\mathcal{M}^{sss}_e &= g_{m,e}(z_1.k_2z_2.z_3 + z_2.k_3z_3.z_1 + z_3.k_1z_1.z_2)^s + g_{nm,e}(z_1.k_2)^s(z_2.k_3)^s(z_3.k_1)^s \\ \mathcal{M}^{sss}_o &= g_{nm,o}[-\epsilon(z_2z_3k_2)k_3 + \epsilon(z_2z_3k_3)k_2](z_1.k_2)^s(z_2.k_1)^{s-1}(z_3.k_1)^{s-1}\end{aligned} \quad \text{(A.123)}$$

Notice no minimal amplitude present for the parity-odd case. This is as mentioned before in Table 1.4. The minimal term was dropped as the negative powers appearing due to $s_1 = s_2 = s_3$.

For example, three photon amplitude is given by

$$\mathcal{M}_e^{111} = g_{m,e}(z_1.k_2 z_2.z_3 + z_2.k_3 z_3.z_1 + z_3.k_1 z_1.z_2) + g_{nm,e}(z_1.k_2)(z_2.k_3)(z_3.k_1)$$
$$\mathcal{M}_o^{111} = g_{nm,o}[-\epsilon(z_2 z_3 k_2)k_3 + \epsilon(z_2 z_3 k_3)k_2](z_1.k_2) \tag{A.124}$$

Another simple example is of three graviton scattering where we have

$$\mathcal{M}_e^{222} = g_{m,e}(z_1.k_2 z_2.z_3 + z_2.k_3 z_3.z_1 + z_3.k_1 z_1.z_2)^2 + g_{nm,e}(z_1.k_2)^2(z_2.k_3)^2(z_3.k_1)^2$$
$$\mathcal{M}_o^{222} = g_{nm,o}[-\epsilon(z_2 z_3 k_2)k_3 + \epsilon(z_2 z_3 k_3)k_2](z_1.k_2)^2(z_2.k_1)(z_3.k_1) \tag{A.125}$$

**One scalar two spin-s amplitude**

For two photon and one scalar we have only two structures

$$\mathcal{M}_e^{011} = g_{nm,e}(z_2.k_3)(z_3.k_1), \quad \mathcal{M}_o^{011} = g_{nm,o}[-\epsilon(z_2 z_3 k_2)k_3 + \epsilon(z_2 z_3 k_3)k_2] \tag{A.126}$$

this can be generalised to two spin-s and one scalar.

$$\mathcal{M}_e^{0ss} = g_{nm,e}(z_2.k_3)^s(z_3.k_1)^s, \quad \mathcal{M}_o^{0ss} = g_{nm,o}[-\epsilon(z_2 z_3 k_2)k_3 + \epsilon(z_2 z_3 k_3)k_2](z_2.z_3)^{s-1} \tag{A.127}$$

## A.10.2 Outside the triangle inequality

Let us consider one spin-s two scalar amplitude

$$\mathcal{M}_e^{00s} = g_e(z_3.k_1)^s. \tag{A.128}$$

This only has parity-even contribution. For one scalar one spin $s_2$ and one spin $s_3$ amplitude we have

$$\mathcal{M}_e^{0s_2s_3} = g_e(z_2.k_1)^{s_2}(z_3.k_1)^{s_3}, \quad \mathcal{M}_o^{0s_2s_3} = g_o[-\epsilon(z_2 z_3 k_2)k_3 + \epsilon(z_2 z_3 k_3)k_2](z_2.k_1)^{s_2-1}(z_3.k_1)^{s_3-1} \tag{A.129}$$

with $s_3 > s_2$. As an example let us consider spin-3 spin-1 scalar amplitude which is given by

$$\mathcal{M}_e^{013} = g_e(z_2.k_1)(z_3.k_1)^3, \quad \mathcal{M}_o^{013} = g_o[-\epsilon(z_2 z_3 k_2)k_3 + \epsilon(z_2 z_3 k_3)k_2](z_3.k_1)^2. \tag{A.130}$$

Now let us consider two spin-1 and one spin-4 particles which will be useful for our purposes. We have

$$\mathcal{M}_e^{114} = g_{m,e}(z_1.k_2 z_2.z_3 + z_2.k_3 z_3.z_1 + z_3.k_1 z_1.z_2)(z_3.k_1)^3 + g_{nm,e}(z_1.k_2)(z_2.k_3)(z_3.k_1)^3$$
$$\mathcal{M}_o^{114} = g_{m,o}[-\left(\epsilon\left(z_2 z_3 k_2\right) k_3 - \epsilon\left(z_2 z_3 k_3\right) k_2\right)\left(z_1 \cdot z_3\right) + \epsilon\left(z_1 z_2 z_3\right) k_2\left(z_3 \cdot k_1\right)](z_3.k_1)^2$$
$$+ g_{nm,o}[-\epsilon(z_2 z_3 k_2)k_3 + \epsilon(z_2 z_3 k_3)k_2](z_1.k_2)(z_3.k_1)^3. \tag{A.131}$$

## A.11 Various-Identities

In this section, we derive

$$Y_2 Y_3 [O_\epsilon]_2 G = -G V_1 \tag{A.132}$$
$$Y_3 [O_\epsilon]_2 Y_2 = -V_1 \tag{A.133}$$
$$Y_2 Y_3 [O_\epsilon]_1 G = G V_1 \tag{A.134}$$

Consider first

$$Y_2Y_3[O_\epsilon]_2G = (z_2.k_3)(z_3.k_1)\left[(z_1.k_2)\frac{\epsilon(z_2k_2z_3)}{k_2} + (z_3.z_1)\frac{\epsilon(z_2k_2k_3)}{k_2} + (z_3.k_1)\frac{\epsilon(z_2k_2z_1)}{k_2}\right] \quad \text{(A.135)}$$

$$= (z_3.k_1)\left[(z_1.k_2)(z_2.z_3)\frac{\epsilon(z_2k_2k_3)}{k_2} + (z_3.z_1)(z_2.k_3)\frac{\epsilon(z_2k_2k_3)}{k_2} + (z_3.k_1)(z_1.z_2)\frac{\epsilon(z_2k_2k_3)}{k_2}\right] \quad \text{(A.136)}$$

where in the second equality we take $z_2.k_3$ inside the bracket and make use of the Schouten identities

$$\epsilon(z_2k_2z_3)Y_2 = \epsilon(z_2k_2z_3)(z_2.k_3) = \epsilon(z_2k_2k_3)(z_2.z_3) = \epsilon(z_2k_2k_3)Z_1 \quad \text{(A.137)}$$

$$\epsilon(z_2k_2z_1)Y_2 = \epsilon(z_2k_2z_1)(z_2.k_3) = \epsilon(z_2k_2k_3)(z_1.z_2) = \epsilon(z_2k_2k_3)Z_3 \quad \text{(A.138)}$$

to obtain

$$Y_2Y_3[O_\epsilon]_2G = (z_3.k_1)\frac{\epsilon(z_2k_2k_3)}{k_2}G \quad \text{(A.139)}$$

Now, consider the schouten identity

$$\epsilon(z_2k_2k_3)(z_3.k_2) = -\epsilon(z_2z_3k_2)k_2.k_3 + \epsilon(z_2z_3k_3)k_2^2 \quad \text{(A.140)}$$

since, $p_I.p_J = k_Ik_J - k_I.k_J = 0$, the above schouten identity becomes

$$\epsilon(z_2k_2k_3)(z_3.k_2) = -\epsilon(z_2z_3k_2)k_2k_3 + \epsilon(z_2z_3k_3)k_2^2 = -k_2\epsilon(z_2z_3p_2p_3) = k_2V_1 \quad \text{(A.141)}$$

which we now use in (A.139) to get

$$Y_2Y_3[O_\epsilon]_2G = -GV_1 \quad \text{(A.142)}$$

Similarly one can show

$$Y_2Y_3[O_\epsilon]_3G = -GV_1 \quad \text{(A.143)}$$

Similarly, we derive (A.133). Consider the epsilon transform of $Y_2$ as follows

$$Y_3[O_\epsilon]_2Y_2 = (z_3.k_1)\frac{\epsilon(z_2k_2k_3)}{k_2} \quad \text{(A.144)}$$

Now, by using (A.141) in the above, we immediately see

$$Y_3[O_\epsilon]_2Y_2 = -V_1 \quad \text{(A.145)}$$

Similarly, one can show

$$Y_2[O_\epsilon]_3Y_3 = -V_1 \quad \text{(A.146)}$$

To derive (A.134), consider

$$Y_2Y_3[O_\epsilon]_2G = (z_2.k_3)(z_3.k_1)\left[(z_2.z_3)\frac{\epsilon(z_1k_1k_2)}{k_1} + (z_2.k_3)\frac{\epsilon(z_1k_1z_3)}{k_1} + (z_3.k_1)\frac{\epsilon(z_1k_1z_2)}{k_1}\right] \quad \text{(A.147)}$$

Now, we take $z_3.k_1$ inside the bracket and use the following Schouten identities

$$Y_3\epsilon(z_1k_1z_2) = (z_3.k_1)\epsilon(z_1k_1z_2) = -k_1^2\epsilon(z_1z_2z_3) - (z_2.k_3)\epsilon(z_1k_1z_3) \tag{A.148}$$

$$Y_3\epsilon(z_1k_1k_2) = (z_3.k_1)\epsilon(z_1k_1k_2) = k_1^2\epsilon(z_1z_3k_2) + k_1k_2\epsilon(z_1k_1z_3) = -k_1V_2 \tag{A.149}$$

to obtain

$$Y_2Y_3[O_\epsilon]_1G = -Y_2(Z_1V_2 + Y_3W_1) \tag{A.150}$$

Now we use the Schouten identities [204]

$$W_1Y_2Y_3 + V_1(G + Y_1Z_1) = 0 \quad V_1Y_1 = V_2Y_2 = V_3Y_3 \tag{A.151}$$

in (A.150) and simplify to obtain

$$Y_2Y_3[O_\epsilon]_1G = GV_1. \tag{A.152}$$

# Appendix B

## B.1 Notations and Conventions

$$
\begin{aligned}
\text{Metric} &: \eta_{\mu\nu} = \text{diag}(-1,1,1) \\
\text{Gamma Matrices} &: (\gamma^{\mu})_{\alpha}^{\ \beta} = (\sigma_2, -i\sigma_1, i\sigma_3)_{\alpha}^{\ \beta} \quad \Rightarrow \{\gamma^{\mu},\gamma^{\nu}\} = -2\eta^{\mu\nu} I_2 \\
\text{Charge Conjugation} &: C_{\alpha\beta} = -C_{\beta\alpha} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -C^{\alpha\beta} = C^{\beta\alpha} \\
\text{Raising-Lowering} &: \psi^{\alpha} = C^{\alpha\beta}\psi_{\beta} \quad ; \quad \psi_{\alpha} = -C_{\alpha\beta}\psi^{\beta} = \psi^{\beta}C_{\beta\alpha} \\
&\Rightarrow \psi^{+} = i\psi_{-} \quad ; \quad \psi^{-} = -i\psi_{+} \\
\text{Vector} \leftrightarrow \text{Bi-spinor} &: p_{\alpha\beta} = p_{\mu}(\gamma^{\mu})_{\alpha\beta} = \begin{pmatrix} p_0 + p_1 & p_3 \\ p_3 & p_0 - p_1 \end{pmatrix} = \begin{pmatrix} p_{+} & p_3 \\ p_3 & -p_{-} \end{pmatrix} \\
\text{Squared Grassmann variables} &: \theta^2 = \frac{1}{2}\theta^{\alpha}\theta_{\alpha},\ d^2\theta = \frac{1}{2}d\theta^{\alpha}d\theta_{\alpha} \\
\text{Superspace integrals} &: \int d\theta = 0,\ \int d\theta\ \theta = 1 \\
&\int d^2\theta\ \theta^2 = -1,\ \int d^2\theta\ \theta^{\alpha}\theta^{\beta} = C^{\alpha\beta} \\
\text{Grassmann } \delta\text{-function} &: \delta^2(\theta) = -\theta^2 \\
\text{Superfields} &: \Phi = \phi + \theta\psi - \theta^2 F,\ \bar{\Phi} = \bar{\phi} + \theta\bar{\psi} - \theta^2\bar{F} \\
&\bar{\Phi}\Phi = \bar{\phi}\phi + \theta^{\alpha}\left(\bar{\phi}\psi_{\alpha} + \bar{\psi}_{\alpha}\phi\right) - \theta^2\left(\bar{F}\phi + \bar{\phi}F + \bar{\psi}\psi\right)
\end{aligned}
\tag{B.1}
$$

## B.2 Component 3 point functions

In this appendix, we write down the component 3 functions abstractly in term of the functions $\{A_i\}$ appearing in form of full superspace 3 point function determined by supersymmetric Ward identity.

$$\begin{aligned}
\langle J_0^b(p)J_0^b(-p-s)J_0^b(s)\rangle &= 2A_1 \\
\langle J_0^f(p)J_0^f(-p-s)J_0^f(s)\rangle &= 2(A_3p_3^2 + s_3(-A_4p_3 - A_5p_3 + A_2s_3)) \\
\langle J_0^b(p)J_0^f(-p-s)J_0^b(s)\rangle &= 2(A_2 + A_3 + A_4 + A_5) \\
\langle J_0^f(p)J_0^b(-p-s)J_0^f(s)\rangle &= \frac{2}{9}(9A_6 + (p_3 - s_3)(3A_4 - 3A_5 + A_1p_3 - A_1s_3)) \\
\langle \Psi_+(p)J_0^b(-p-s)\Psi_-(s)\rangle &= -\frac{2}{3}(3A_5 + A_1(-p_3 + s_3)) \\
\langle \Psi_+(p)J_0^b(-p-s)\Psi_-(s)\rangle &= -\frac{2}{9}(-9A_6 + p_3(-3(3A_3 + A_4 + 2A_5) + 2A_1p_3) \\
&\quad + (9A_2 + 3A_4 + 6A_5 + 5A_1p_3)s_3 + 2A_1s_3^2)
\end{aligned} \tag{B.2}$$

## B.3 $\langle J_0 J_0 J_s\rangle_{\mathcal{T}_{\kappa,N}}$ via double trace factorization

In this section, we will derive the expression for normalized 3-point coefficient used in subsection (2.7.2) in the main text of the paper. The main idea is to use the fact the supersymmetric theory differs from the regular boson (fermion) theory only via double trace interaction term involving the scalar and spin half operators. This allows one to use large $N$ factorisation to relate the 2 and 3-point function between the supersymmetric and regular boson (fermion) theory[1].

Let us start by writing the action for our $\mathcal{N} = 2$ theory in a way which which makes it easier to compare it with the regular boson (fermion) theory.

$$\begin{aligned}
S_{\mathcal{T}_{\kappa,N}} &= \frac{i\kappa}{4\pi}S_{CS}(A) + S_b(\phi, A) + S_f(\psi, A) + S_{bf}(\phi,\psi) \\
\text{where} \quad S_{CS}(A) &= \int d^3x\ \epsilon_{\mu\nu\rho}\mathrm{Tr}(A^\mu\partial^\nu A^\rho - \frac{2i}{3}A^\mu A^\nu A^\rho) \\
S_b(\phi, A) &= \mathcal{D}_\mu\bar{\phi}\mathcal{D}^\mu\phi \quad , \quad S_f(\psi, A) = -i\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi, \\
S_{bf}(\phi,\psi) &= \int d^3x\left(\frac{4\pi^2}{\kappa^2}(\bar{\phi}\phi)^3 - \frac{4\pi}{\kappa}(\bar{\phi}\phi)(\bar{\psi}\psi) - \frac{2\pi}{\kappa}(\bar{\psi}\phi)(\bar{\phi}\psi)\right).
\end{aligned} \tag{B.3}$$

Similarly, the action for regular boson (fermion) theory in term of these building blocks can be written as follows

$$\begin{aligned}
S_{\mathcal{B}_{\kappa,N}} &= \frac{i\kappa}{4\pi}S_{CS}(A) + S_b(\phi, A) + \frac{\lambda_6}{3!N^2}(\bar{\phi}\phi)^3 \\
S_{\mathcal{F}_{\kappa,N}} &= \frac{i\kappa}{4\pi}S_{CS}(A) + S_f(\phi, A).
\end{aligned} \tag{B.4}$$

Note that the regular boson theory above has an extra parameter, $\lambda_6$. To leading order in the 't Hooft large $N$ limit, of interest in this paper, $\lambda_6$ is exactly marginal while it develops a non-trivial beta function at subleading orders. The question of beta function and fixed points structure for this deformations have been studied in details in [137, 140, 142]. The particular value of $\lambda_6$ for the regular bosonic theory that will be relevant for us in this paper is the one in supersymmetric theory, namely

$$\lambda_6 = 24\pi^2\lambda^2. \tag{B.5}$$

Henceforth, in this paper 'regular boson theory' should be understood as with this values of $\lambda_6$

[1]From the diagrammatic point of view one might wonder as to how is possible to derive any such relation since the supersymmetric theory contain more fields which can run in the internal loops of Feynman diagrams in supersymmetric theory. It is easy to see that in these Chern-Simons vector models any diagrams which has gauge boson converting into matter in the loops is suppressed in the large $N$ 't Hooft limit of interest in this paper.

coupling.

For notational convenience, we will use the subscripts $\mathcal{T}_{\kappa,N}, \mathcal{B}_{\kappa,N}$ and $\mathcal{F}_{\kappa,N}$ to refer to quantities computed in the supersymmetric, regular boson (with (B.5)) and regular fermion theory respectively.

For later use, let us further define

$$S_{(\mathcal{BF})_{\kappa,N}} = \frac{i\kappa}{4\pi} S_{CS}(A) + S_b(\phi, A) + S_f(\psi, A) + \frac{4\pi^2}{\kappa^2}\int (\bar{\phi}\phi)^3 \, . \tag{B.6}$$

As discussed in section 2.5, our supersymmetric theory consists of a pair of approximately conserved single trace higher spin operators at each value of half integer spin. At any integers values '$s$' of the spin, the two currents can be taken to be the ones existing in theories $\mathcal{B}_{\kappa,N}$ and $\mathcal{F}_{\kappa,N}$. We will refer to these current operators as $J_s^b$ and $J_s^f$ respectively. The explicit expressions for these currents for low value of spins can be found in [71, 77, 79].

Let us first consider $\langle J_s^b J_0^b J_0^b\rangle_{\mathcal{T}_{\kappa,N}}$. Taylor expanding the double trace interaction terms in the action, the path integral expression for the correlator can be written as follows

$$\begin{aligned}
&\langle J_s^b J_0^b J_0^b\rangle_{\mathcal{T}_{\kappa,N}} \\
&= \int [D\Phi]\, e^{-S_{\mathcal{BF}}} \left( J_s^b(p_1) J_0^b(p_2) J_0^b(p_3)\; e^{\int d^3q \left(\frac{4\pi}{\kappa} J_0^b(q) J_0^f(-q) + \frac{2\pi}{\kappa}(\bar{\psi}\phi)(q)(\bar{\phi}\psi)(-q)\right)}\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle J_s^b(p_1) J_0^b(p_2) J_0^b(p_3)\left[\int d^3q \left(\frac{4\pi}{\kappa} J_0^b(q) J_0^f(-q) + \frac{2\pi}{\kappa}(\bar{\psi}\phi)(q)(\bar{\phi}\psi)(-q)\right)\right]^n\right\rangle_{(\mathcal{BF})_{\kappa,N}} \\
&\equiv \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{4\pi}{\kappa}\right)^n \left\langle J_s^b(p_1) J_0^b(p_2) J_0^b(p_3) \prod_{i=1}^{n}\left(\int d^3q_i\; J_0^b(q_i) J_0^f(-q_i)\right)\right\rangle_{(\mathcal{BF})_{\kappa,N}}
\end{aligned} \tag{B.7}$$

In the third line above we dropped the fermion double trace terms $((\bar{\psi}\phi)(\phi\bar{\psi}))$ since they do not contribute to the leading order result. The leading $\mathcal{O}(N)$ contribution from the last line of (B.7) can be computed using large $N$ factorization as we outline now. Let's look at the general n-th term in the sum

$$\left\langle J_s^b(p_1) J_0^b(p_2) J_0^b(p_3) \prod_{i=1}^{n}\left(\int d^3q_i J_0^b(q_i) J_0^f(-q_i)\right)\right\rangle_{(\mathcal{BF})_{\kappa,N}} \tag{B.8}$$

The leading $\mathcal{O}(N)$ contribution from this term comes from its factorization into a product of (n+1) correlators, namely n 2-point functions and one 3-point function. Since $S_{\mathcal{BF}}$ doesn't have any explicit interaction term between fermions and bosons, this can only happen for even values of $n$ (say $n = 2m$) in the 't Hooft limit, in which case the factorized contribution (schematically, suppressing the argument momenta) looks like

$$\langle J_s^b J_0^b J_0^b\rangle_{(\mathcal{BF})_{\kappa,N}} \langle J_0^b J_0^b\rangle^n_{(\mathcal{BF})_{\kappa,N}} \langle J_0^f J_0^f\rangle^n_{(\mathcal{BF})_{\kappa,N}}$$

More precisely, there are three different type of such factorized contribution which are represented in figure B.1. The contribution from each of these type of factorization channels is exactly the same[2]. Carefully counting the numerical factor for each and summing up gives the

[2]This is because of the fact that the product $\langle J_0^b(q) J_0^b(-q)\rangle\langle J_0^f(q) J_0^f(-q)\rangle$ is independent of the momenta $q$.

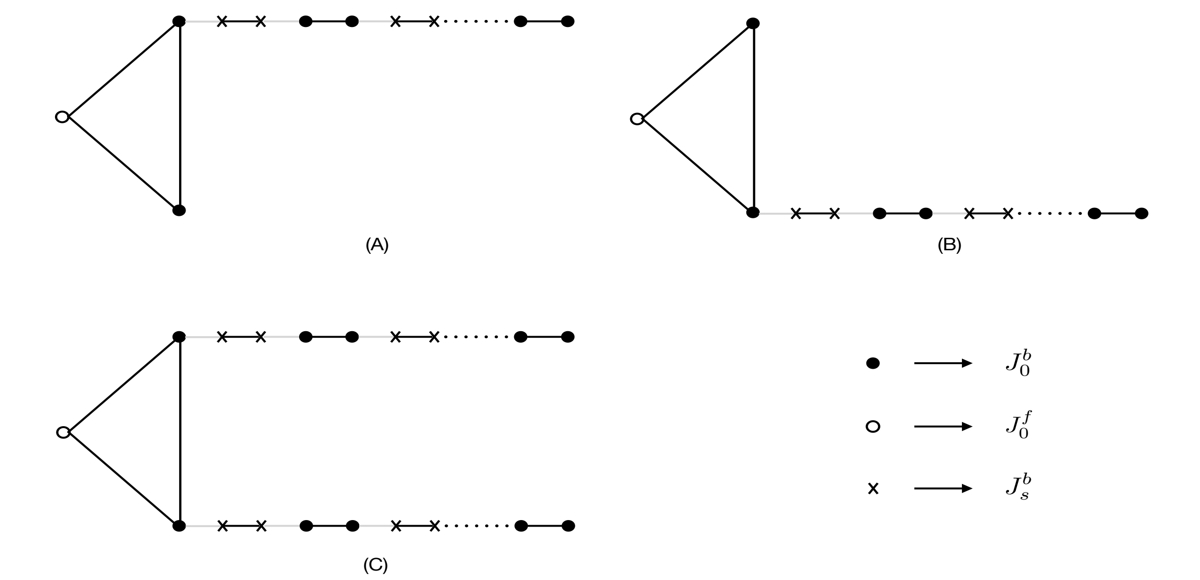



**Figure B.1:** Schematic representation of 3 type of diagrams contributing to the factorization via the double trace term $J_0^b J_0^f$ in the action. The dots (crosses) connected with solid lines are factorized correlation functions while the grey line connecting a dot with a cross means the corresponding operators have same momenta.

total contribution to be

$$\begin{aligned} &\left\langle J_s^b(p_1)J_0^b(p_2)J_0^b(p_3)\prod_{i=1}^{n}\left(\int d^3q_i J_0^b(q_i)J_0^f(-q_i)\right)\right\rangle_{(\mathcal{BF})_{\kappa,N}} \\ &= (n+1)\langle J_s^b(p_1)J_0^b(p_2)J_0^b(p_3)\rangle_{(\mathcal{BF})_{\kappa,N}}\left(\langle J_0^b J_0^b\rangle_{(\mathcal{BF})_{\kappa,N}}\langle J_0^f J_0^f\rangle_{(\mathcal{BF})_{\kappa,N}}\right)^n \end{aligned} \tag{B.9}$$

Now we further notice that the absence of explicit interaction terms between bosons and fermions[3] in the action $S_{\mathcal{BF}}$ implies the following relations in the large $N$ limit

$$\begin{aligned} \langle J_0^b(q)J_0^b(-q)\rangle_{(\mathcal{BF})_{\kappa,N}} &= \langle J_0^b(q)J_0^b(-q)\rangle_{\mathcal{B}_{\kappa,N}} \\ \langle J_0^f(q)J_0^f(-q)\rangle_{(\mathcal{BF})_{\kappa,N}} &= \langle J_0^f(q)J_0^f(-q)\rangle_{\mathcal{F}_{\kappa,N}} \\ \langle J_s^b(p_1)J_0^b(p_2)J_0^b(p_3)\rangle_{(\mathcal{BF})_{\kappa,N}} &= \langle J_s^b(p_1)J_0^b(p_2)J_0^b(p_3)\rangle_{\mathcal{B}_{\kappa,N}} \end{aligned} \tag{B.10}$$

Combining (B.7), (B.9) and (B.10) and summing the series over n, we arrive at the following expression for the supersymmetric correlator

$$\langle J_s^b J_0^b J_0^b\rangle_{\mathcal{T}_{\kappa,N}} = \langle J_s^b J_0^b J_0^b\rangle_{\mathcal{B}_{\kappa,N}}\left[\sum_{n=0}^{\infty}\left(\left(\frac{4\pi}{\kappa}\right)^2\langle J_0^b J_0^b\rangle_{\mathcal{B}}\langle J_0^f J_0^f\rangle_{\mathcal{F}_{\kappa,N}}\right)^n\right]^2 \tag{B.11}$$

Further using the relation [96][4]

$$\langle J_0^b J_0^f\rangle_{\mathcal{T}_{\kappa,N}} = \frac{\kappa}{4\pi}\sum_{n=1}^{\infty}\left(\left(\frac{4\pi}{\kappa}\right)^2\langle J_0^b J_0^b\rangle_{\mathcal{B}_{\kappa,N}}\langle J_0^f J_0^f\rangle_{\mathcal{F}_{\kappa,N}}\right)^n, \tag{B.12}$$

[3] i.e. $\bar{\phi}\phi\bar{\psi}\psi$ and $\bar{\phi}\psi\phi\bar{\psi}$ terms

[4] This can also be derived in a very similar fashion using the large $N$ factorization via double trace ($J_0^b J_0^f$) interaction term in the SUSY lagrangian.

we can write (B.11) as

$$\langle J_s^b(p_1)J_0^b(p_2)J_0^b(p_3)\rangle_{\mathcal{T}_{\kappa,N}} = \langle J_s^b(p_1)J_0^b(p_2)J_0^b(p_3)\rangle_{\mathcal{B}_{\kappa,N}}\left[1+\frac{4\pi}{\kappa}\langle J_0^b J_0^f\rangle_{\mathcal{T}_{\kappa,N}}\right]^2. \tag{B.13}$$

Following exactly the same procedure, one can also derive the following relation[5]

$$\langle J_s^f(p_1)J_0^f(p_2)J_0^f(p_3)\rangle_{\mathcal{T}_{\kappa,N}} = \langle J_s^f(p_1)J_0^f(p_2)J_0^f(p_3)\rangle_{\mathcal{F}_{\kappa,N}}\left[1+\frac{4\pi}{\kappa}\langle J_0^b J_0^f\rangle_{\mathcal{T}_{\kappa,N}}\right]^2. \tag{B.14}$$

The correlators $\langle J_s^f(p_1)J_0^f(p_2)J_0^f(p_3)\rangle_{\mathcal{F}_{\kappa,N}}$ and $\langle J_s^b(p_1)J_0^b(p_2)J_0^b(p_3)\rangle_{\mathcal{B}_{\kappa,N}}$ are known from [74] where the authors determined the all 3-point correlators of single trace operators in *quasi bosonic* and *quasi fermionic* theories in term of two abstract parameters $\tilde{\lambda}$ and $\tilde{N}$ using the constraints of weakly broken higher spin symmetry in these theories. The result for the 2-point and 3-point functions relevant to our analysis are as follows

$$\begin{aligned}
\langle \tilde{J}_0(x_1)\tilde{J}_0(x_2)\rangle_{\mathcal{B}_{\kappa,N}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\langle \tilde{J}_0(x_1)\tilde{J}_0(x_2)\rangle_{bos}\\
\langle \tilde{J}_0(x_1)\tilde{J}_0(x_2)\rangle_{\mathcal{F}_{\kappa,N}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\langle \tilde{J}_0(x_1)\tilde{J}_0(x_2)\rangle_{fer}\\
\langle \tilde{J}_s(x_1)\tilde{J}_s(x_2)\rangle_{\mathcal{B}_{\kappa,N}} &= \langle \tilde{J}_s(x_1)\tilde{J}_s(x_2)\rangle_{\mathcal{F}_{\kappa,N}} = \tilde{N}\langle \tilde{J}_s(x_1)\tilde{J}_s(x_2)\rangle_{bos}\\
\langle \tilde{J}_s(x_1)\tilde{J}_0(x_2)\tilde{J}_0(x_3)\rangle_{\mathcal{B}_{\kappa,N}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\langle \tilde{J}_s(x_1)\tilde{J}_0(x_2)\tilde{J}_0(x_3)\rangle_{bos}\\
\langle \tilde{J}_s(x_1)\tilde{J}_0(x_2)\tilde{J}_0(x_3)\rangle_{\mathcal{F}_{\kappa,N}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\langle \tilde{J}_s(x_1)\tilde{J}_0(x_2)\tilde{J}_0(x_3)\rangle_{fer}
\end{aligned} \tag{B.15}$$

Here the subscript *bos* (*fer*) refers to the quantity computed in theory of a free single real boson (Majorana fermion) respectively. Further, in above relation we denote the operators with a *tilde* on top to emphasize that the normalization used in [74] is in general different from the usual normalization used for these operators in Chern Simons vector models.

The exact relation between these operators normalizations and the abstract parameters $(\tilde{\lambda}, \tilde{N})$ to the parameter $(\lambda, N)$ of the regular boson theory $(\mathcal{B}_{\kappa,N})$ were obtained in [77] while the equivalent relations for the regular fermion theory $(\mathcal{F}_{\kappa,N})$ were obtained in [79] via explicit computation of 3 point function for some of the low spin operators. These relations are as follows

$$\begin{aligned}
\mathcal{B}_{\kappa,N} \quad &: \quad (\tilde{J}_0,\tilde{J}_s) = \left(\frac{J_0^b}{1+\tilde{\lambda}^2}, J_s^b\right)\\
\mathcal{F}_{\kappa,N} \quad &: \quad (\tilde{J}_0,\tilde{J}_s) = \left(\frac{J_0^f}{1+\tilde{\lambda}^2}, J_s^f\right)\\
\text{where} \quad & (\tilde{\lambda},\tilde{N}) = \left(\tan\left(\frac{\pi\lambda}{2}\right), 2N\frac{\sin(\pi\lambda)}{\pi\lambda}\right)
\end{aligned} \tag{B.16}$$

Combining (B.13),(B.14),(B.15) and (B.16), we get the following expression for our desired 3-point function in supersymmetric theory $\mathcal{T}_{\kappa,N}$

$$\begin{aligned}
\langle J_s^b(x_1)J_0^b(x_2)J_0^b(x_3)\rangle_{\mathcal{T}_{\kappa,N}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\langle \tilde{J}_s(x_1)\tilde{J}_0(x_2)\tilde{J}_0(x_3)\rangle_{bos}\\
\langle J_s^f(x_1)J_0^f(x_2)J_0^f(x_3)\rangle_{\mathcal{T}_{\kappa,N}} &= \frac{\tilde{N}}{1+\tilde{\lambda}^2}\langle \tilde{J}_s(x_1)\tilde{J}_0(x_2)\tilde{J}_0(x_3)\rangle_{fer}
\end{aligned} \tag{B.17}$$

[5]We have difference in signs compared to [96] due to spinor convention difference.

with $\tilde{\lambda}$ and $\tilde{N}$ as in (B.16).

Now that we have all the requisite 2 and 3 point functions, we can compute the normalization independent squared 3-point function coefficients to be

$$\begin{aligned} C^{2(BBB)}_{0,susy} &= \frac{(1-\tilde{\lambda}^2)^2}{\tilde{N}(1+\tilde{\lambda}^2)^2} C^2_{0,fb} \\ C^{2(BBB)}_{s,susy} &= \frac{1}{\tilde{N}(1+\tilde{\lambda}^2)^2} C^2_{s,fb} \qquad s = 2,4,6\ldots \end{aligned} \tag{B.18}$$

where $C^2_{s,fb}$ ($C^2_{s,ff}$) denote the corresponding coefficients in a free real scalar (majorana fermion) theory. The normalized coefficients above and in the rest of the paper can formally be defined as follows. Conformal invariance uniquely fixes the position dependence of all the 2 point functions and the relevant 3 point functions we are interested in, namely of the type $\langle J_0(x_1)J_0(x_2)J^{(s)}(x_3)\rangle$. Lets define the normalization $N_*$ and 3 point function coefficient $C_{***}$ as our operators to be

$$\begin{aligned} &\langle J^{(s)}(x_1,\lambda_1)J^{(s)}(x_2,\lambda_2)\rangle = N_s^2 \frac{P_3^{2s}}{|x_{12}|^2}, \\ &\langle J^{(s)}(x_1,\lambda_1)J_0(x_2)J_0(x_3)\rangle = \tilde{C}_{s00}\frac{Q_1^s}{|x_{12}||x_23|^{2\Delta_0-1}|x_{31}|}, \\ &\text{where} \quad P_3 = \frac{\lambda_1 X_{12}\lambda_2^{2s}}{|x_{12}|^2} \quad , \quad Q_1 = \frac{\lambda_1 X_{12}X_{23}X_{31}\lambda_1}{x_{12}^2 x_{31}^2} \text{ with } X = x_i\sigma^i. \end{aligned} \tag{B.19}$$

We refer the reader to [202] for further details of the conformally invariant structures involved in 2 and 3 point functions. The relevant normalized 3-point function coefficient squares we are interested in are then defined as

$$C^2_{ijk} = \frac{\tilde{C}^2_{ijk}}{N_i^2 N_j^2 N_k^2} \tag{B.20}$$

where the $i,j,k$ are just labels for the operators involved.

The mixed correlators $\langle J_s^f J_0^b J_0^b\rangle$ and $\langle J_s^b J_0^f J_0^f\rangle$ of our theory cannot directly be related to correlators of $\mathcal{B}_{\kappa,N}$ or $\mathcal{F}_{\kappa,N}$ theories via double trace type factorization used above. We will instead use the self duality of our theory to determine these correlators. Under the self duality transformation (2.54) the operators in our theory map in the following way [96]

$$J_0^b \leftrightarrow J_0^b \quad , \quad J_0^f \leftrightarrow J_0^f \quad , \quad J_s^b \leftrightarrow (-1)^s J_s^f. \tag{B.21}$$

Thus, we have following relations for the mixed 3-point functions

$$\begin{aligned} \langle J_0^b J_0^b J_s^f\rangle_{\mathcal{T}_{\kappa,N}} &= (-1)^s \langle J_s^b J_0^b J_0^b\rangle_{\mathcal{T}_{-\kappa,|\kappa|-N}} \\ \langle J_0^f J_0^f J_s^b\rangle_{\mathcal{T}_{\kappa,N}} &= (-1)^s \langle J_s^f J_0^f J_0^f\rangle_{\mathcal{T}_{-\kappa,|\kappa|-N}} \end{aligned} \tag{B.22}$$

The 2-point functions are, of course, invariant under the duality while the parameters $\tilde{\lambda}$ and $\tilde{N}$ transform as follows

$$\tilde{N} \to \tilde{N} \quad , \quad \tilde{\lambda} \to \tilde{\lambda}^{-1}. \tag{B.23}$$

Using (B.22), the result of our explicit computation (2.74) for the mixed 3-point function $\langle J_0^b J_0^b J_0^f\rangle$ and the duality transformation (B.23), we can determine the other 3 point function coefficients, $C^{2(BBF)}_{0,susy}$ and $C^{2(BBF)}_{s,susy}$ to be

$$\begin{aligned} C^{2(BBF)}_{0,susy} &= \frac{2}{\pi^2}\frac{(2\tilde{\lambda})^2}{\tilde{N}(1+\tilde{\lambda}^2)^2}C^2_{fb} \\ C^{2(BBF)}_{s,susy} &= \frac{\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2}C^2_{s,fb} \end{aligned} \tag{B.24}$$

Note that since our result for the 2 and 3-point function (2.73) and (2.74) are obtained in the momentum space, in order to compare $C^{2(BBF)}_{0,susy}$ with $C^2_{fb}$ (as we have done in the first line of (B.24)) we need to read out the 3-point function coefficient in position space by taking the appropriate Fourier transform of our result to go to the position space expression. This can be implemented in a straightforward manner, e.g. using the Fourier transform result in [41]. This leads to the extra factor of $(2/\pi^2)$ in the first line of (B.24).

Using the method described above the relevant normalized 3-point function coefficients required for the $J^f_0$ 4-point function can also be computed. We simply quote the results below

$$\begin{aligned} C^{2(FFF)}_{s,susy} &= \frac{1}{\tilde{N}(1+\tilde{\lambda}^2)^2}C^2_{s,ff} \\ C^{2(FFB)}_{s,susy} &= \frac{\tilde{\lambda}^4}{\tilde{N}(1+\tilde{\lambda}^2)^2}C^2_{s,ff} \end{aligned} \tag{B.25}$$

We do not write down the coefficients $C^{2(FFF)}_{0,susy}$ and $C^{2(FFB)}_{0,susy}$ since the corresponding 3-point functions are contact terms.

## B.4 Comments on direct computation of $J^{(0)}$ 4 point function

In this appendix, we describe the relevant diagrams, and corresponding integrals, constructed using the exact 4 point vertex which contribute to the full $J^{(0)}$ four point function. Figure B.2 shows the exact 4-point vertex used to construct all the relevant diagrams in Figure B.3.

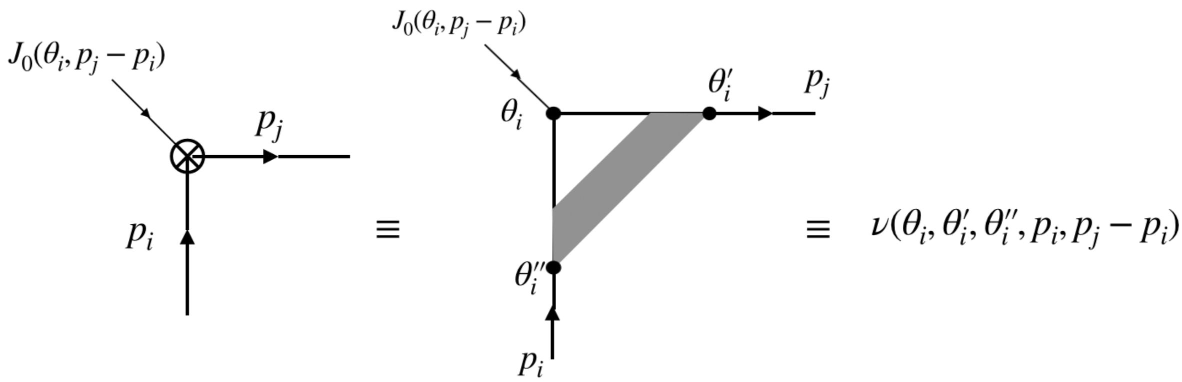



**Figure B.2:** Convention for the definition of each vertex in the 4-point function. The 'internal' Grassmann variables, $\theta'_i, \theta''_i$ that are explicitly shown here are suppressed in Figure B.3 to avoid clutter. These internal variables are integrated over in the computation of the correlation functions. The convention of various momenta entering or leaving the vertex is also demonstrated here.

For diagrams in Figure B.3, note that the exact vertex (2.71) is a function of two internal grassmann variables ($\theta'_i, \theta''_i$ as depicted in Figure B.2). The internal propagators in figure (B.3) that emanate from/to the exact vertices connect these internal Grassmann variables, which

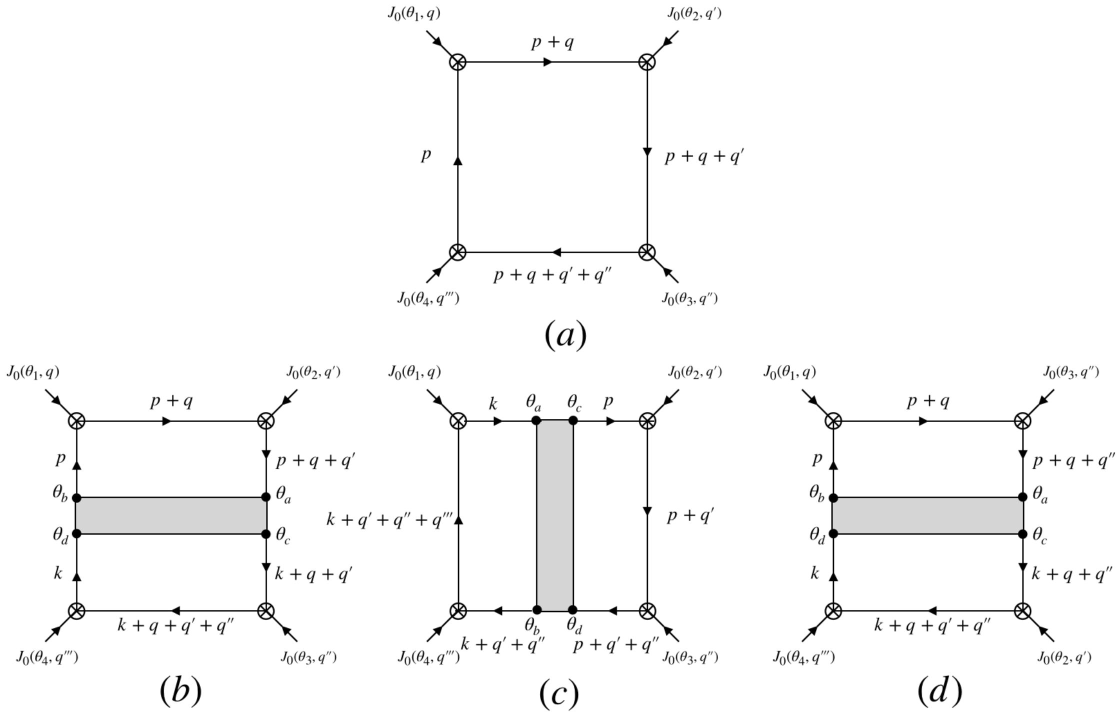



**Figure B.3:** The contributing diagrams for the four point function of currents. The first diagram is diagram type (a). The grey blob in (b), (c) , (d) represents the all loop four point correlator. The remaining diagrams are obtained by permutations of the external operators.

are integrated over in the computation of the relevant diagrams. In figure (B.3) the value of diagram (a) is given by

$$V^{(A)}(q, q', q'', \theta_1, \theta_2, \theta_3, \theta_4)$$
$$= N \int \frac{d^3p}{(2\pi)^3} d^2\theta_1' d^2\theta_1'' d^2\theta_2' d^2\theta_1'' d^2\theta_3' d^2\theta_3'' d^2\theta_4' d^2\theta_4''$$
$$\Big( P(\theta_1', \theta_4'', p+q) P(\theta_4', \theta_3'', p-q'-q'') P(\theta_3', \theta_2'', p-q') P(\theta_2', \theta_1'', p)$$
$$\mathcal{V}_3(\theta_1, \theta_1', \theta_1'', q, p) \mathcal{V}_3(\theta_2, \theta_2', \theta_2'', q', p-q') \mathcal{V}_3(\theta_3, \theta_3', \theta_3'', q'', p-q'-q'') \mathcal{V}_3(\theta_4, \theta_4', \theta_4'', -q-q'-q'', p+q) \Big) \tag{B.26}$$

There are a total of 6 additional diagrams due to permutations of the operators. and the interaction part is given by

$$
\begin{aligned}
&V_4^{(B)}(q,q',q'',\theta_1,\theta_2,\theta_3,\theta_4)\\
&=N^2\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}d^2\theta_a d^2\theta_b d^2\theta_c d^2\theta_d d^2\theta'_1 d^2\theta''_1 d^2\theta'_2 d^2\theta''_1 d^2\theta'_3 d^2\theta''_3 d^2\theta'_4 d^2\theta''_4\\
&\Big(P(\theta'_1,\theta''_4,p+q)P(\theta'_4,\theta_a,p-q'-q'')P(\theta_c,\theta''_3,k-q'-q'')P(\theta'_3,\theta''_2,k-q')P(\theta'_2,\theta_d,k)P(\theta_b,\theta''_1,p)\\
&\mathcal{V}_3(\theta_1,\theta'_1,\theta''_1,q,p)\mathcal{V}_3(\theta_2,\theta'_2,\theta''_2,q',k-q')\mathcal{V}_3(\theta_3,\theta'_3,\theta''_3,q'',k-q'-q'')\mathcal{V}_3(\theta_4,\theta'_4,\theta''_4,-q-q'-q'',p+q)\\
&\mathcal{V}_4(\theta_a,\theta_b,\theta_c,\theta_d,p,-q'-q'',k)\Big)
\end{aligned}
\tag{B.27}
$$

The bosonic and fermionic correlators for the diagram figure (B.3) are given by

$$
\begin{aligned}
\langle J_0^b(q)J_0^b(q')J_0^b(q'')J_0^b(-q-q'-q'')\rangle &= V_4^{(1)}(q,q',q'',\theta_1,\theta_2,\theta_3,\theta_4)\Big|_{\theta_1\to0,\theta_2\to0,\theta_3\to0,\theta_4\to0}\\
\langle J_0^f(q)J_0^f(q')J_0^f(q'')J_0^f(-q-q'-q'')\rangle &= \prod_{i=1}^{4}\frac{\partial}{\partial\theta_{\alpha i}}\frac{\partial}{\partial\theta_i^{\alpha}}V_4^{(1)}(q,q',q'',\theta_1,\theta_2,\theta_3,\theta_4)
\end{aligned}
\tag{B.28}
$$

Although we were able to successfully perform the integrals for the components $p_3,\theta_p$ and $k_3,\theta_k$ in the expression for $V_4^{(B)}$ given by (B.27) $k_s$ and $p_s$ integrals out be intractable analytically. Due to this difficulty we were not able to obtain a closed form expression for the four point function of the scalar operators$J_0^b$ and $J_0^f$ in (B.28).

## B.5 AdS Contact diagrams

### B.5.1 Closed-form

$$
\begin{aligned}
&\bar{D}_{1111}(z,\bar{z}) = \frac{1}{z-\bar{z}}[\ln(z\bar{z})\ln\Big(\frac{1-z}{1-\bar{z}}\Big)+2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar{z})]\\
&\bar{D}_{2222}(z,\bar{z}) = \frac{12uv}{(z-\bar{z})^5}+\frac{1+u+v}{(z-\bar{z})^3}[\ln(z\bar{z})\ln\Big(\frac{1-z}{1-\bar{z}}\Big)+2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar{z})]\\
&+\frac{6}{(z-\bar{z})^4}\Big((1+u-v)v\ln v+(1+v-u)u\ln u\Big)+\frac{2}{(z-\bar{z})^2}(\ln uv+1)\\
&\bar{D}_{3333}(u,v) =\\
&\Big(\frac{1680u^2v^2}{(z-\bar{z})^9}+\Big(\frac{240uv}{(z-\bar{z})^7}+\frac{24}{(z-\bar{z})^5}\Big)(1+u+v)+\frac{4}{(z-\bar{z})^3}\Big)[\ln(z\bar{z})\ln\Big(\frac{1-z}{1-\bar{z}}\Big)+2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar{z})]\\
&+\Big(\Big(\frac{840u}{(z-\bar{z})^8}+\frac{100}{(z-\bar{z})^6}\Big)v^2(1+u-v)+\frac{480uv}{(z-\bar{z})^6}+\frac{12(1+u)+76v}{(z-\bar{z})^4})\Big)\ln v+u\leftrightarrow v\\
&+\frac{260uv}{(z-\bar{z})^6}+\frac{26}{(z-\bar{z})^4}(1+u+v)
\end{aligned}
\tag{B.29}
$$

$$
\begin{aligned}
\bar{D}(u,v)_{3322} &= -\partial_u\bar{D}_{2222}(u,v)\\
\bar{D}(u,v)_{4433} &= -\partial_u\bar{D}_{3333}(u,v)
\end{aligned}
\tag{B.30}
$$

### B.5.2 Decomposition in terms of conformal blocks

The contact diagrams may be written as an expansion in conformal blocks [236]

$$D_{\Delta\Delta\Delta'\Delta'}(x_i) = \sum_m a_m^{\Delta\Delta}\alpha_m^{\Delta'\Delta'}\mathcal{W}_{\Delta_m,0}(x_i) + \sum_n a_n^{\Delta\Delta}\alpha_n^{\Delta'\Delta'}\mathcal{W}_{\Delta_n,0}(x_i) \tag{B.31}$$

$$D_{\Delta\Delta\Delta\Delta}(x_i) = \sum_n 2a_n^{\Delta\Delta}(\sum_{m\neq n}\frac{a_m^{\Delta\Delta}}{m_n^2 - m_m^2})\mathcal{W}_{\Delta_n,0}(x_i) + \sum_n (a_n^{\Delta\Delta})^2\frac{\partial}{\partial m_n^2}\mathcal{W}_{\Delta_n,0}(x_i) \tag{B.32}$$

where $\mathcal{W}_{\Delta,0} = \beta_{\Delta 34}\beta_{\Delta 12}W_{\Delta,0}$. For $\Delta_i = \Delta$

$$D_{\Delta\Delta\Delta\Delta}(x_i) = \sum_n 2a_n^{\Delta\Delta}\eta_n^{\Delta\Delta}\mathcal{W}_{\Delta_n,0}(x_i) + \sum_n (a_n^{\Delta\Delta})^2\frac{\partial}{\partial m_n^2}\mathcal{W}_{\Delta_n,0}(x_i) \tag{B.33}$$

$$= \sum_n [(2a_n^{\Delta\Delta}\eta_n^{\Delta\Delta} + (a_n^{\Delta\Delta})^2)\beta^2_{\Delta_n\Delta\Delta} + \frac{\partial}{\partial m_n^2}\beta^2_{\Delta_n\Delta\Delta}]W_{\Delta_n,0}(x_i)$$
$$+ \sum_n (a_n^{\Delta\Delta})^2\beta^2_{\Delta_n\Delta\Delta}\frac{\partial}{\partial m_n^2}W_{\Delta_n,0}(x_i) \tag{B.34}$$

with

$$\eta_n^{\Delta\Delta} = \sum_{m\neq n}\frac{a_m^{\Delta\Delta}}{m_n^2 - m_m^2} \tag{B.35}$$

$$\beta_{\Delta 34} \equiv \frac{\Gamma\left(\frac{\Delta+\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta-\Delta_{34}}{2}\right)}{2\Gamma(\Delta)} \tag{B.36}$$

$$m^2_{\Delta_k} = \Delta_k(\Delta_k - d)$$

$$a_m^{12} = \frac{(-1)^m}{\beta_{\Delta_m 12}m!}\frac{(\Delta_1)_m(\Delta_2)_m}{(\Delta_1+\Delta_2+m-d/2)_m} \tag{B.37}$$

with the anomalous dimension being proportional to the coefficient of the third term which involves derivative of the conformal block. Writing the above in terms of the $\bar{D}$ functions

$$\bar{D}_{\Delta\Delta\Delta\Delta}(u,v) = \frac{1}{u^\Delta}[\sum_n [(2a_n^{\Delta\Delta}\eta_n^{\Delta\Delta})\beta^2_{\Delta_n\Delta\Delta} + (a_n^{\Delta\Delta})^2\frac{\partial}{\partial m_n^2}\beta^2_{\Delta_n\Delta\Delta}]G_{\Delta_n,0}(u,v)$$
$$+ \sum_n (a_n^{\Delta\Delta})^2\beta^2_{\Delta_n\Delta\Delta}\frac{\partial}{\partial m_n^2}G_{\Delta_n,0}(u,v)] \tag{B.38}$$

We will re-label

$$P_1^{(\Delta)}(n,0) = (2a_n^{\Delta\Delta}\eta_n^{\Delta\Delta} + (a_n^{\Delta\Delta})^2)\beta^2_{\Delta_n\Delta\Delta} + \frac{\partial}{\partial m_n^2}\beta^2_{\Delta_n\Delta\Delta}$$
$$P_0^{(\Delta)}(n,0)\gamma_1^{(\Delta)}(n,0) = 2(a_n^{\Delta\Delta})^2\beta^2_{\Delta_n\Delta\Delta} \tag{B.39}$$

so that

$$\bar{D}_{\Delta\Delta\Delta\Delta}(u,v) = \frac{2\Gamma(\Delta)^4}{\Gamma(2\Delta-d/2)}\frac{1}{u^\Delta}\sum_n [P_1^{(\Delta)}(n,0)G_{\Delta_n,0}(u,v)$$
$$+ \frac{1}{2}P_0^{(\Delta)}(n,0)\gamma_1^{(\Delta)}(n,0)\frac{\partial}{\partial m_n^2}G_{\Delta_n,0}(u,v)] \tag{B.40}$$

satisfying [237]

$$P_1^{(\Delta)}(n,0) = \frac{1}{2}\partial_n (P_0^{(\Delta)}(n,0)\gamma_1^{(\Delta)}(n,0)) \tag{B.41}$$

Similarly, for (141)

$$\begin{aligned}\bar{D}_{\Delta+1\Delta+1\Delta\Delta}(u,v) = &\frac{2\Gamma(\Delta)^2\Gamma(\Delta+1)^2}{\Gamma(2\Delta+1-d/2)}\frac{1}{u^\Delta}\\ &[\sum_m \bar{P}_1^{(\Delta)}(n,0)G_{\Delta_m,0}(u,v) + \frac{1}{2}\bar{P}_0^{(\Delta)}(n,0)\bar{\gamma}_1^{(\Delta)}(n,0)\frac{\partial}{\partial m_n^2}G_{\Delta_n,0}(u,v)]\\ &+\beta_{2\Delta}^2 a_0^{\Delta\Delta}\eta_0^{\Delta+1\Delta+1}G_{2\Delta,0}(u,v)\end{aligned} \tag{B.42}$$

$$\begin{aligned}\bar{P}_1^{(\Delta)}(n,0) &= (a_n^{\Delta+1\Delta+1}\eta_n^{\Delta\Delta} + a_n^{\Delta\Delta}\eta_n^{\Delta+1\Delta+1})\beta_{\Delta_n\Delta\Delta}^2 + a_n^{\Delta\Delta}a_n^{\Delta+1\Delta+1}\frac{\partial}{\partial m_n^2}\beta_{\Delta_n\Delta\Delta}^2\\ \bar{P}_0^{(\Delta)}(n,0)\bar{\gamma}_1^{(\Delta)}(n,0) &= 2a_n^{\Delta\Delta}a_n^{\Delta+1\Delta+1}\beta_{\Delta_n\Delta\Delta}^2\end{aligned} \tag{B.43}$$

#### B.5.2.1 Examples

$$\begin{aligned}\bar{D}_{1111}(u,v) &= \frac{2}{\pi^{1/2}u}\sum_n [P_1^{(1)}(n,0)G_{2+2n,0}(u,v) + \frac{1}{2}P_0^{(1)}(n,0)\gamma_1^{(1)}(n,0)\frac{\partial_n G_{2+2n,0}(u,v)}{8n+2}]\\ \bar{D}_{2222}(u,v) &= \frac{8}{3\pi^{1/2}u^2}\sum_n [P_1^{(2)}(n,0)G_{4+2n,0}(u,v) + \frac{1}{2}P_0^{(2)}(n,0)\gamma_1^{(2)}(n,0)\frac{\partial_n G_{4+2n,0}(u,v)}{8n+10}]\\ \bar{D}_{3333}(u,v) &= \frac{256}{105\pi^{1/2}u^3}\sum_n [P_1^{(3)}(n,0)G_{6+2n,0}(u,v) + \frac{1}{2}P_0^{(3)}(n,0)\gamma_1^{(3)}(n,0)\frac{\partial_n G_{6+2n,0}(u,v)}{8n+18}]\end{aligned} \tag{B.44}$$

$$\begin{aligned}&\bar{D}_{3322}(u,v) = \frac{64}{15\pi^{1/2}u^3}[\sum_m \bar{P}_1^{(3)}(m,0)G_{6+2m,0}(u,v) + \frac{1}{2}\bar{P}_0^{(3)}(m,0)\bar{\gamma}_1^{(3)}(m,0)\frac{\partial}{\partial m_n^2}G_{6+2m,0}(u,v)\\ &+\beta_4^2 a_0^{22}\eta_0^{33}G_{4,0}(u,v)]\\ &\bar{D}_{4433}(u,v) = \frac{1024}{105\sqrt{\pi}u^4}[\sum_m \bar{P}_1^{(4)}(m,0)G_{8+2m,0}(u,v) + \frac{1}{2}\bar{P}_0^{(4)}(m,0)\bar{\gamma}_1^{(4)}(m,0)\frac{\partial}{\partial m_n^2}G_{8+2m,0}(u,v)\\ &+\beta_6^2 a_0^{33}\eta_0^{44}G_{6,0}(u,v)]\end{aligned} \tag{B.45}$$

**Contact terms for bosonic correlator**

$$\begin{aligned}G_{\phi^4}^{AdS} &= \bar{D}_{1111}(u,v)\\ G_{(\partial\phi)^4}^{AdS} &= (1+u+v)\bar{D}_{2222}(u,v)\\ G_{\phi^2(\partial^3\phi)^2}^{AdS} &= 2(u^2\bar{D}_{3322}(u,v) + v^2\bar{D}_{3322}(v,u) + \frac{1}{v^3}\bar{D}_{3322}(1/v,u/v))\end{aligned} \tag{B.46}$$

**Contact terms for fermionic correlator**

$$\begin{aligned}G_{\phi^4}^{AdS} &= \bar{D}_{2222}(u,v)\\ G_{(\partial\phi)^4}^{AdS} &= (1+u+v)\bar{D}_{3333}(u,v)\\ G_{\phi^2(\partial^3\phi)^2}^{AdS} &= 2(u^2\bar{D}_{4433}(u,v) + v^2\bar{D}_{4433}(v,u) + \frac{1}{v^3}\bar{D}_{4433}(1/v,u/v))\end{aligned} \tag{B.47}$$

## B.6 Perturbative Computations in the Bosonic Theory

Here, we compute the anomalous dimension of $j_0 = \bar{\phi}\phi$ in the regular bosonic theory, i.e., $SU(N)_k$ Chern Simons theory coupled to a single complex scalar field, to two loops. (The leading $1/N$ correction to the anomalous dimension is the same whether one considers the $U(N)$ or $SU(N)$ theories, although subleading corrections may differ.) This provides an additional check of our conjecture. Our computation closely follows the calculation of the anomalous dimension of $j_0$ in the $O(N)$ theory carried out in [72]. All our calculations in this appendix are in the bosonic theory, so we drop the subscript $b$ in what follows. We also refer to related perturbative computations in Chern-Simons theory is which appear in [159, 238–240]. The Lagrangian is given by

$$S = S_{CS} + S_{RB} \tag{B.48}$$

$$S_{RB} = \int d^3x\ |D_\mu \phi|^2 + \frac{\lambda_6}{3!N^2}(\phi^\dagger\phi)^3 \tag{B.49}$$

$$S_{CS} = \frac{ik}{4\pi}\int d^3x \quad \mathrm{Tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big) \tag{B.50}$$

$$= \frac{ik}{8\pi}\int d^3x\ \epsilon_{\mu\nu\lambda}[A^a_\mu \partial\nu A^a_\lambda - \frac{i}{3}f^{abc}A^a_\mu A^b_\nu A^c_\lambda] \tag{B.51}$$

In expanding the Chern-Simons action, we used $\mathrm{Tr}\Big(T^aT^b\Big) = \frac{1}{2}\delta_{ab}$ as our convention for group generators. We will express all the divergent diagrams that contribute to the anomalous dimension in terms of $C_1$, $C_2$ and $C_3$ which are defined by following relations

$$\mathrm{Tr}\Big(T^aT^b\Big) = \delta_{ab}C_1 \tag{B.52}$$

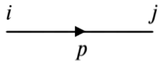
$$f^{acd}f^{bcd} = \delta^{ab}C_2 \tag{B.53}$$

$$T^aT^a = IC_3. \tag{B.54}$$

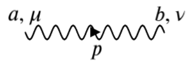
In the normalization that we have chosen for $SU(N)$ generators,

$$C_1 = \frac{1}{2}, \quad C_2 = -N, \quad C_3 = \frac{1}{2}(N - \frac{1}{N}). \tag{B.55}$$

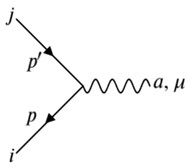
If we work in Landau gauge, as in [72], we obtain the following Feynman rules:



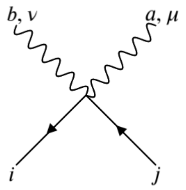

$$= \frac{\delta_{ij}}{p^2}$$



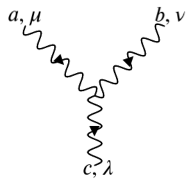

$$= G_{\nu\mu}(p)\delta_{ab}$$




$$= (p' + p)^\mu T^a_{ij}$$




$$= \frac{1}{2}\{T^a, T^b\}_{ij}$$




$$= \frac{k}{24\pi}\epsilon^{\mu\nu\lambda} f^{abc}$$

The gluon propagator is given by

$$G_{\mu\nu} = -\frac{4\pi}{k}\frac{\epsilon^{\mu\nu\delta} p_\delta}{p^2}. \tag{B.56}$$

Ghosts do not contribute at this order.

# Appendix C

## C.1 Consistency with OPE limit

In this section we follow discussion in section 2.1 of [184]. In momentum space, there are four solutions to conformal ward identity for scalar three point function. One can show that all these four solutions can be combined to give most general correlation function to be given by

$$\langle OOO\rangle = a_1 f\left(k_1+k_2+k_3\right) + a_2 f\left(-k_1+k_2+k_3\right) + a_3 f\left(k_1-k_2+k_3\right) + a_4 f\left(k_1+k_2-k_3\right). \tag{C.1}$$

Permutation symmetry implies that $a_2 = a_3 = a_4$. Let us consider simple case of scalar operator $O$ with $\Delta = 2$. For this case we have $f(k_1+k_2+k_3) = \ln(k_1+k_2+k_3)$. The same correlator in position space in the OPE limit, $x_{23} \to 0$, goes like $\langle OOO\rangle \sim \frac{1}{x_{23}^2 x_{12}^4}$ which in momentum space leads to $\langle OOO\rangle \sim \frac{k_1}{k_3}$ with $k_2 \approx k_3 \gg k_1$ where $k_1 \to 0$. It is easy to check that this can be only be reproduced by $\ln(k_1+k_2+k_3)$. Hence, singularity of the form $f(k_i - k_j + k_k)$ is not consistent with the OPE limit and only singularity structure $E = k_1+k_2+k_3 \to 0$ is consistent. So we conclude that consistency with OPE limit restricts the correlator to only have a total energy pole that is a pole in $E = k_1+k_2+k_3$.

## C.2 Shadow Transform in dS correlator

In dS, correlators for boundary operators are related to the correlators of the bulk field through shadow transform. Though this is a standard and well-known technique [26, 28], we will elaborate with a simple example for completeness.

Consider a bulk field $\phi(x)$ in dS background and $O(x)$ is the corresponding dual operator in the 'boundary CFT'. The wave-function of the universe (using in-in formalism) is written in terms of this bulk field as

$$\Psi[\phi(x)] = \exp\left(-\frac{1}{2!}\int d^3x d^3y \phi(x)\phi(y)\langle O(x)O(y)\rangle + \frac{1}{3!}\int d^3x d^3y d^3z \phi(x)\phi(y)\phi(z)\langle O(x)O(y)O(z)\rangle + \cdots\right) \tag{C.2}$$

The interpretation of this wave function is that one can access all information about the dynamics of this bulk field which can be related to boundary operator $O(x)$ with dimension $\Delta$. This similar fact is also seen in more conventional shadow transform in AdS correlators. One can calculate various moments of the bulk field using,

$$\langle \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n}\rangle = \frac{\int \mathcal{D}\phi\ \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n} |\Psi[\phi]|^2}{\int \mathcal{D}\phi |\Psi[\phi]|^2} \tag{C.3}$$

which reduces to the in-in formalism correlator (eq (3.18)). Using perturbation of the bulk field we can obtain series of relations between moments of $\phi(x)$ and the set of correlators of the boundary dual insertion $O(x)$. Below we will state the relations

$$\begin{aligned}\langle\phi_{\vec{k}_1}\phi_{\vec{k}_2}\rangle &= \frac{1}{2\mathrm{Re}\langle O_{\vec{k}_1}O_{\vec{k}_2}\rangle}\\ \langle\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\rangle &= \frac{\mathrm{Re}\langle O_{\vec{k}_1}O_{\vec{k}_2}O_{\vec{k}_3}\rangle}{2\prod_{i=1}^{3}\mathrm{Re}\langle O_{\vec{k}_i}O_{-\vec{k}_i}\rangle}\\ &\cdots\end{aligned} \tag{C.4}$$

The two function above is obtained by doing a standard Gaussian path integral over the bulk field. On the other hand, the other side is obtained using the expansion of the wave function up to cubic order and using the standard rules of wick contraction.

Similar kinds of relations can be obtained for higher spin field correlators. Also, the above relations can be inverted systematically to write boundary correlators in terms of bulk field correlators.

## C.3 Cosmological correlation function

In this Appendix, we compute several cosmological correlation functions which play important role in the main text.

### C.3.1 $\langle TOO\rangle$

To compute two scalars and one graviton amplitude we need to consider

$$H_{int} = \int d^4x\sqrt{-g}\; g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{C.5}$$

Using the above general vacua mode expansion (3.8),(3.14), along with (3.18) we obtain the following time integral

$$\begin{aligned}(z_1.k_2)^2\,\mathrm{Im}\Big[\int_{-\infty}^{0}\frac{d\eta}{\eta^2}\Big(&-2f_{k_3}(A,B)u_{k_3}(\eta)[\bar{f}_{k_1}(C,D)f_{k_2}(A,B)\bar{\gamma}_{k_1}(\eta)\bar{u}_{k_2}(\eta)\\ &-2f_{k_1}(C,D)f_{k_2}(A,B)\gamma_{k_1}(\eta)u_{k_2}(\eta)]\\ &+2\bar{f}_{k_1}(C,D)\bar{f}_{k_2}(A,B)\bar{\gamma}_{k_1}(\eta)\bar{u}_{k_2}(\eta)[-\bar{f}_{k_3}(A,B)\bar{u}_{k_3}(\eta)+f_{k_3}(A,B)u_{k_3}(\eta)]\Big)\Big]\end{aligned} \tag{C.6}$$

where

$$f_k(A,B) = \frac{1}{\sqrt{k^3}}(A+B) \tag{C.7}$$

$$\gamma_k(\eta) = \frac{1}{\sqrt{2k^3}}\left[e^{ik\eta}(1-ik\eta)C + e^{-ik\eta}(1+ik\eta)D\right] \tag{C.8}$$

The $\langle TOO\rangle$ correlator can be found after computing the time integral and performing the shadow transform can be shown to be given by

$$\langle TO_3O_3\rangle = c_1\langle TO_3O_3\rangle_1 + c_2\langle TO_3O_3\rangle_2 + c_3\left(\langle TO_3O_3\rangle_3 + \langle TO_3O_3\rangle_4\right) \tag{C.9}$$

where $c_i$ are given by

$$
\begin{aligned}
c_1 &= \mathrm{Re}\left\{\frac{1}{2(C+D)(\bar{A}+\bar{B})^2}\left[\frac{\bar{A}^2+\bar{B}^2}{\bar{C}+\bar{D}}+\frac{((2BD+A(C+D))\bar{A}+(2AC+B(C+D))\bar{B})}{(A+B)^2}\right]\right\} \\
c_2 &= \mathrm{Re}\left\{\frac{1}{2(C+D)(\bar{A}+\bar{B})^2}\left[-\frac{\bar{A}^2+\bar{B}^2}{\bar{C}+\bar{D}}+\frac{(2BC+A(C+D))\bar{A}+(2AD+B(C+D))\bar{B}}{(A+B)^2}\right]\right\} \\
c_3 &= \mathrm{Re}\left\{\frac{1}{2(C+D)(\bar{A}+\bar{B})^2}\left[\frac{2\bar{A}\bar{B}}{\bar{C}+\bar{D}}-\frac{(-B\bar{A}+A\bar{B})(C-D)}{(A+B)^2}\right]\right\}
\end{aligned} \tag{C.10}
$$

For the special case when we consider $A=C, B=D$ we have

$$
\langle TO_3O_3\rangle = a\langle TO_3O_3\rangle_1 + b\left(\langle TO_3O_3\rangle_2+\langle TO_3O_3\rangle_3+\langle TO_3O_3\rangle_4\right) \tag{C.11}
$$

where the form of $a, b$ in (C.11) are given by

$$
\begin{aligned}
a &= \frac{1}{2\mathcal{N}^3(A,B)}[(2A^2+3AB+3B^2)(\bar{A}^2+\bar{B}^2+\bar{A}\bar{B})+(A-B)\bar{B}(|A|^2+|B|^2+A\bar{B})] \\
b &= \frac{1}{2\mathcal{N}^3(A,B)}[A(A-B)B(\bar{A}^3+\bar{B}^3)+(A^3+6A^2B+B^3)(\bar{A}^2\bar{B}-\bar{A}\bar{B}^2)]
\end{aligned} \tag{C.12}
$$

and

$$
\begin{aligned}
\langle TO_3O_3\rangle_{R_1} &= \left[k_1+k_2+k_3-\frac{k_1k_2+k_2k_3+k_3k_1}{k_1+k_2+k_3}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right](z_1.k_2)^2 \\
\langle TO_3O_3\rangle_{R_2} &= \left[-k_1+k_2+k_3-\frac{-k_1k_2+k_2k_3-k_3k_1}{-k_1+k_2+k_3}+\frac{k_1k_2k_3}{(-k_1+k_2+k_3)^2}\right](z_1.k_2)^2 \\
\langle TO_3O_3\rangle_{R_3} &= \left[k_1-k_2+k_3-\frac{-k_1k_2-k_2k_3+k_3k_1}{k_1-k_2+k_3}+\frac{k_1k_2k_3}{(k_1-k_2+k_3)^2}\right](z_1.k_2)^2 \\
\langle TO_3O_3\rangle_{R_4} &= \left[k_1+k_2-k_3-\frac{k_1k_2-k_2k_3-k_3k_1}{k_1+k_2-k_3}+\frac{k_1k_2k_3}{(k_1+k_2-k_3)^2}\right](z_1.k_2)^2
\end{aligned} \tag{C.13}
$$

The Ward identity for a correlator in the above form is by brute computation can be shown to be

$$
\begin{aligned}
k_1^\mu z^\nu\langle T_{\mu\nu}O_3O_3\rangle &= (a+b)(z_1.k_2)(\langle O_3(k_2)O_3(-k_2)\rangle_{BD}-\langle O_3(k_3)O_3(-k_3)\rangle_{BD}) \\
&= \frac{1}{(A+B)(\bar{A}+\bar{B})}(z_1.k_2)(\langle O_3(k_2)O_3(-k_2)\rangle_{BD}-\langle O_3(k_3)O_3(-k_3)\rangle_{BD})
\end{aligned} \tag{C.14}
$$

where we have used (C.12) to obtain

$$
a+b = \frac{1}{(A+B)(\bar{A}+\bar{B})} \tag{C.15}
$$

Let us note that WT identity should be consistent with

$$
\begin{aligned}
k_1^\mu z^\nu\langle T_{\mu\nu}O_3O_3\rangle &= (z_1.k_2)(\langle O_3(k_2)O_3(-k_2)\rangle_\alpha-\langle O_3(k_3)O_3(-k_3)\rangle_\alpha) \\
&= \frac{1}{(A+B)(\bar{A}+\bar{B})}(z_1.k_2)(\langle O_3(k_2)O_3(-k_2)\rangle_{BD}-\langle O_3(k_3)O_3(-k_3)\rangle_{BD})
\end{aligned} \tag{C.16}
$$

where in the last line we have used (A.1) i.e.

$$\langle O(k_2)O(-k_2)\rangle_\alpha = \frac{1}{(A+B)(\bar{A}+\bar{B})}\langle O(k_2)O(-k_2)\rangle_{BD} \tag{C.17}$$

We observe that (C.16) with (C.14) are identical as should be the case. Alternative one can use this consistency condition to fix the normalization of $A, B$ to be as given in (3.9) [1].

### C.3.2 $\langle TTO_3\rangle$

To calculate two graviton and one scalar amplitude we consider the following interaction term

$$H_{int} = \int d^4x\,\sqrt{-g}\varphi\mathcal{W}_{\rho\sigma\alpha\beta}\mathcal{W}_{\rho\sigma\alpha\beta} \tag{C.21}$$

Using the mode expansion (3.14) along with (3.8) in (C.21) the following time integral can be obtained

$$\begin{aligned}\mathrm{Im}\Bigg[\int_{-\infty}^{0} d\eta[2\bar{f}_{k_1}(C,D)v_{k_1}^{\alpha\beta\mu\nu}(\eta)(\bar{f}_{k_2}(C,D)v_{k_2}^{\alpha\beta\mu\nu}(\eta) - f_{k_2}(C,D)\bar{v}_{k_2}^{\alpha\beta\mu\nu}(\eta))\\ + (\bar{f}_{k_2}(C,D)v_{k_2}^{\alpha\beta\mu\nu}(\eta) + f_{k_2}(C,D)\bar{v}_{k_2}^{\alpha\beta\mu\nu}(\eta))(\bar{f}_{k_1}(C,D)v_{k_1}^{\alpha\beta\mu\nu}(\eta) - f_{k_1}(C,D)\bar{v}_{k_1}^{\alpha\beta\mu\nu}(\eta))]\\ (\bar{f}_{k_3}(A,B)u_{k_3}(\eta) - f_{k_3}(A,B)\bar{u}_{k_3}(\eta))\Bigg]\end{aligned} \tag{C.22}$$

where $f_k$ and $u_k(\eta)$ has been defined in (C.7)and $v_k^{\alpha\beta\mu\nu}(\eta)$ has been defined in (C.29). See appendix **??** for details on Weyl tensor in terms of mode expansion. The above gives

$$\langle TTO_3\rangle_\alpha = d_1\langle TTO_3\rangle_{R_1} + d_2\langle TTO_3\rangle_{R_4} + d_3(\langle TTO_3\rangle_{R_2} + \langle TTO_3\rangle_{R_3}) \tag{C.23}$$

where

$$\begin{aligned}d_1 &= \mathrm{Re}\left[\frac{-|B|^2C^2(\bar{C}+\bar{D})^2 + \bar{A} - B|C|^4 - 2BC\bar{D}|C|^2 + (A(C+D)^2 + BD(2C+D)\bar{D}^2)]}{6(A+B)(C+D)^2(\bar{A}+\bar{B})(\bar{C}+\bar{D})^2}\right]\\ d_2 &= a(A \leftrightarrow B)\\ d_3 &= \mathrm{Re}\left[\frac{\Big((A-B)CD(\bar{A}+\bar{B})(\bar{C}^2+\bar{D}^2) + \textbf{c.c.}\Big) + 4|C|^2|D|^2(|A|^2-|B|^2)}{6(A+B)(C+D)^2(\bar{A}+\bar{B})(\bar{C}+\bar{D})^2}\right]\end{aligned} \tag{C.24}$$

[1]Note that the CFT ward identity constraints the $A, B$. This makes the connection to alpha vacuum clear from CFT side. In position space, $\langle TOO\rangle$ is given by (see [17])

$$\langle T_{\mu\nu}(x_2)O_\Delta(x_2)O_\Delta(x_3)\rangle = \frac{1}{x_{12}^d x_{23}^{2\Delta-d}x_{31}^d}\frac{dN\Delta}{(1-d)S_d}h_{\mu\nu}^1(\hat{X}_{23}) \tag{C.18}$$

where

$$h_{\mu\nu}^1 = \hat{X}_{23\mu}\hat{X}_{23\nu} - \frac{1}{d}\delta_{\mu\nu} \quad \hat{X}_{23\mu} = \frac{X_{23\mu}}{X_{23}^2}, X_{23} = \frac{x_{21}}{x_{21}^2} - \frac{x_{31}}{x_{31}^2} \tag{C.19}$$

and

$$\langle O_\Delta(x_1)O_\Delta(x_2)\rangle = \frac{N}{x_{12}^{2\Delta}} \tag{C.20}$$

So the three-point function OPE coefficient is determined in terms of the two-point function $N$. This is a reflection of the fact that $\langle TOO\rangle$ is non-homogeneous.

and $\langle TTO_3 \rangle_{R_i}$ has been defined in (3.74) and (3.75). Notice, if we put $C = A, D = B$, then, we find

$$d_1 = a \quad d_2 = d_3 = b \tag{C.25}$$

where $a, b$ are same as that appear in (3.31). To summarise for this special case we have

$$\langle TTO_3 \rangle_\alpha = c_1 \langle TTO_3 \rangle_{R_1} + c_2 \left( \langle TTO_3 \rangle_{R_4} + \langle TTO_3 \rangle_{R_2} + \langle TTO_3 \rangle_{R_3} \right). \tag{C.26}$$

The explicit expressions for $\langle TTO_3 \rangle_{R_i}$ are complicated. They are best written in spinor helicity variables. We give their explicit forms in section 3.3.3.

### C.3.3 $\langle TTT \rangle$

The three graviton amplitude can get contribution from two different sources, the Einstein Hilbert term, and the $Weyl^3$ term. Let us start with the Weyl tensor contribution.

#### $Weyl^3$ contribution

To calculate the three graviton amplitude we need to consider the following interaction

$$S^{(3)}_{\gamma, \mathcal{W}^3} = \int d^4x \sqrt{-g}\, \mathcal{W}^3. \tag{C.27}$$

Using the mode expansion (3.14) in (C.21) the following time integral can be obtained

$$\begin{aligned}
&\mathrm{Im} \Big[ \int d\eta\, \eta^2 \, (-f_{k_3}(C,D) v^{\alpha\beta\gamma\delta}_{k_3}(\eta) + \bar{f}_{k_3}(C,D) v^{\alpha\beta\gamma\delta}_{k_3}(\eta) f_{k_2}(C,D) f_{k_1}(C,D) \bar{v}^{\gamma\delta\eta\zeta}_{k_2}(\eta) \bar{v}^{\eta\zeta\alpha\beta}_{k_1}(\eta) \\
&+ (-f_{k_2}(C,D) \bar{v}^{\alpha\beta\gamma\delta}_{k_2}(\eta) + \bar{f}_{k_2}(C,D) v^{\alpha\beta\gamma\delta}_{k_2})(\eta) |f(C,D)|^2 \bar{v}^{\gamma\delta\eta\zeta}_{k_1}(\eta) v^{\eta\zeta\alpha\beta}_{k_3}(\eta) \\
&+ (-f_{k_1}(C,D) v^{\alpha\beta\gamma\delta}_{k_1}(\eta) + \bar{f}_{k_1}(C,D) v^{\alpha\beta\gamma\delta}_{k_1}(\eta)) \bar{f}_{k_2}(C,D) \bar{f}_{k_3}(C,D) v^{\gamma\delta\eta\zeta}_{k_2}(\eta) v^{\eta\zeta\alpha\beta}_{k_3}(\eta) \Big]
\end{aligned} \tag{C.28}$$

where $f_k(\alpha)$ has been defined in (C.7) and $v_k^{\alpha\beta\gamma\delta}(\eta)$ is defined in (C.29). See appendix **??** for details on Weyl tensor in terms of mode expansion. We also have

$$v_k^{\alpha\beta\gamma\delta}(\eta) = \frac{1}{\sqrt{k^3}} [e^{ik\eta}(-ik\eta)[\mathcal{W}^+]_{\alpha\beta\gamma\delta} A - B e^{-ik\eta}(ik\eta)[\mathcal{W}^-]_{\alpha\beta\gamma\delta}] \tag{C.29}$$

$$[\mathcal{W}^+]^\dagger_{\alpha\beta\gamma\delta} = [\mathcal{W}^-]_{\alpha\beta\gamma\delta} \tag{C.30}$$

which when evaluated gives exactly the result in Section 3.3.4. Due to the form of (3.8), the following contractions of $W$ appear in the time integral

$$\mathcal{M}^{\pm\pm\pm}_{\mathcal{W}^3} = \mathcal{W}^{\pm}_{\alpha\beta\gamma\delta}(k_1) \mathcal{W}^{\pm}_{\gamma\delta\eta\zeta}(k_2) \mathcal{W}^{\pm}_{\eta\zeta\alpha\beta}(k_3) \tag{C.31}$$

where

$$\mathcal{W}^{\pm}_{\eta\zeta\alpha\beta}(k^\mu) \equiv \mathcal{W}_{\eta\zeta\alpha\beta}(\pm k, \vec{k}) \tag{C.32}$$

However, only four of the sign combinations are unique

$$\mathcal{M}^{-++}_{\mathcal{W}^3} = \mathcal{M}^{+--}_{\mathcal{W}^3} \equiv \mathcal{M}^2_{\mathcal{W}^3} \tag{C.33}$$

$$\mathcal{M}^{+-+}_{\mathcal{W}^3} = \mathcal{M}^{-+-}_{\mathcal{W}^3} \equiv \mathcal{M}^3_{\mathcal{W}^3} \tag{C.34}$$

$$\mathcal{M}^{++-}_{\mathcal{W}^3} = \mathcal{M}^{--+}_{\mathcal{W}^3} \equiv \mathcal{M}^4_{\mathcal{W}^3} \tag{C.35}$$

$$\mathcal{M}^{---}_{\mathcal{W}^3} = \mathcal{M}^{+++}_{\mathcal{W}^3} \equiv \mathcal{M}^1_{\mathcal{W}^3} \tag{C.36}$$

The direct computation of the time integral (C.28) along with the shadow tranform gives the full homogenous part in momentum space,

$$\langle TTT\rangle_{h,\alpha} = a\langle TTT\rangle_{h,1} + b(\langle TTT\rangle_{h,2} + \langle TTT\rangle_{h,3} + \langle TTT\rangle_{h,4})] \tag{C.37}$$

where $a, b$ are precisely what we given by

$$\begin{aligned} a &= \frac{1}{3\mathcal{N}^3(C,D)}[(2C^2+3CD+3D^2)(\bar{C}^2+\bar{D}^2+\bar{C}\bar{D})+(C-D)\bar{D}(|C|^2+|D|^2+C\bar{D})] \\ b &= \frac{1}{3\mathcal{N}^3(C,D)}[(C^2+6CD+D^2)\bar{C}\bar{D}+C\bar{D}^2(-C+D)+(C-D)D\bar{C}^2] \end{aligned} \tag{C.38}$$

We also have

$$\langle TTT\rangle_{h,i} = \langle TTT\rangle_{h,R_i} \tag{C.39}$$

where $\langle TTT\rangle_{h,R_i}$ appear in (3.119), (3.120).

**Two-derivative interaction the Einstein-Hilbert contribution**

Consider now the interaction

$$S^{(3)}_{\gamma,EG} = \int d^4x\sqrt{-g}\ R \tag{C.40}$$

The time integral due to the above interaction is then given by

$$\begin{aligned} \mathcal{M}_{EG}\,\mathrm{Im}\Bigg[\int_{-\infty}^{0}\frac{d\eta}{\eta^2}\Big(&-2f_{k_3}(C,D)\gamma_{k_3}(\eta)[\bar{f}_{k_1}(C,D)\bar{f}_{k_2}(C,D)\bar{\gamma}_{k_1}(\eta)\bar{\gamma}_{k_2}(\eta)-2f_{k_1}(C,D)f_{k_2}(C,D)\gamma_{k_1}(\eta)\gamma_{k_2}(\eta)] \\ &+2\bar{f}_{k_1}(C,D)\bar{f}_{k_2}(C,D)\bar{\gamma}_{k_1}(\eta)\bar{\gamma}_{k_2}(\eta)[-\bar{f}_{k_3}(C,D)\bar{\gamma}_{k_3}(\eta)+f_{k_3}(C,D)\gamma_{k_3}(\eta)]\Big)\Bigg] \end{aligned} \tag{C.41}$$

where

$$\mathcal{M}_{EG} = (z_1.z_2z_3.k_1 + z_2.z_3z_1.k_2 + z_3.z_1z_2.k_3)^2 \tag{C.42}$$

Therefore, in momentum space, the full non-homogeneous part is then given by

$$\langle TTT\rangle_{nh,\alpha} = a\langle TTT\rangle_{nh,1} + b(\langle TTT\rangle_{nh,2} + \langle TTT\rangle_{nh,3} + \langle TTT\rangle_{nh,4})] \tag{C.43}$$

where $a, b$ is precisely given by

$$\begin{aligned} a &= \frac{1}{3\mathcal{N}^3(C,D)}[(2C^2+3CD+3D^2)(C\bar{C}^3-D\bar{D}^3)+(C^3-D^3)(3\bar{C}^2\bar{D}+\bar{C}\bar{D}^2)] \\ b &= \frac{1}{3\mathcal{N}^3(C,D)}[(C^2+6CD+D^2)\bar{C}\bar{D}+C\bar{D}^2(-C+D)+(C-D)D\bar{C}^2] \end{aligned} \tag{C.44}$$

$$\langle TTT\rangle_{nh,i} = \langle TTT\rangle_{nh,R_i} \quad i=1,2,3,4 \tag{C.45}$$

where $\langle TTT\rangle_{nh,R_i}$'s appear in (3.123).

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