Discrepancy results for modular forms

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by

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Certificate

Certified that the work incorporated in the thesis entitled " *Discrepancy results for modular forms*", submitted by *Jishu Das* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

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Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Abstract

Let F be a totally real number field, $r = [F : \mathbb{Q}]$, and $\mathfrak{N}$ be an integral ideal. Let $A_k(\mathfrak{N}, \omega)$ be the space of holomorphic Hilbert cusp forms with respect to $K_1(\mathfrak{N})$, weight $k = (k_1, \ldots, k_r)$ with $k_j > 2$, k_j even for all j and central Hecke character ω. For a fixed level $\mathfrak{N}$, we study the behavior of the Petersson trace formula for the Hecke operators acting on $A_k(\mathfrak{N}, \omega)$ as $k_0 \to \infty$ where $k_0 = \min(k_1, \ldots, k_r)$ subjected to a given condition. We give an asymptotic formula for the Petersson formula under certain conditions. As an application, we generalize a discrepancy result (proved in 2020) for classical cusp forms with squarefree levels by Jung and Sardari to Hilbert cusp forms for F with the ring of integers $\mathcal{O}$ having odd narrow class number 1, and the ideals being generated by numbers belonging to $\mathbb{Z}$.

In the second part, we restrict ourselves to classical cusp forms i.e. the case $F = \mathbb{Q}$. We obtain a generalization for the discrepancy result in the context of levels (of the form $2^a \times b$ with b odd, $a = 0, 1, 2$) and the space of old forms. Then we get a similar kind of lower bound for $\lambda_{p^2}(f)$ for an eigenform f. This is achieved as an application to an asymptotic version for the Petersson formula.

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Statement of Originality

The main results of this thesis that constitute original research are Theorem 3.1.2, 4.1.1, 4.3.5, 4.3.9, 6.0.1, 6.0.3 and 6.1.3.

Lemma 3.0.1, 3.1.1, 3.2.1, 3.2.4, 3.2.5 of Chapter 3, Lemma 4.3.2, 4.3.3, 4.3.4 of Chapter 4 and Lemma 6.1.1, 6.1.2 of Chapter 6 are original subsidiary results towards proving the main results.

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List of Symbols

- The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the set of natural numbers, integers, rational numbers and real numbers respectively.

- Let f, g be two real valued functions. We say

$$f = O_{a,b,c}(g)$$

or

$$f \ll_{a,b,c} g$$

if there exists $C(a,b,c) > 0$ depending on a, b and c such that $|f(x)| \leq C(a,b,c)|g(x)|$ for all x. If $C(a,b,c)$ is an absolute constant then we write

$$f = O(g)$$

or

$$f \ll g.$$

- Let f, g be two real valued functions with $g \neq 0$. We write $f = o(g)$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

- For a set A with finite elements, $|A|$ denotes the cardinality of A.
- Let $\|x\|$ denote the usual Euclidean norm of $x \in \mathbb{R}^r$ for $r \in \mathbb{N}$. This means if $x = (x_1, \dots, x_r)$, then $\|x\| = \sqrt{x_1^2 + \cdots + x_r^2}$.
- The dimension of the vector space F over $\mathbb{Q}$ is denoted by $[F : \mathbb{Q}]$.
- For a natural number n, $\phi(n)$ denotes the number of natural numbers less that n that are coprime to n.
- Given $n \in \mathbb{N}$, $\mu(n)$ denotes the mobius function evaluated at n.
- For a square matrix g of order n, $\det(g)$ denotes the determinant of the matrix g.
- Given a set S and a function f defined on S, $f(S) = \{f(s) : s \in S\}$.
- Given $n \in \mathbb{N}$, $d(n)$ or $\tau(n)$ denotes the number of positive divisors of n.
- Given $a, b, c \in \mathbb{N}$, (a, b) denotes the gcd of a and b. Similarly (a, b, c) denotes the gcd of a, b and c.
- For $m, n \in \mathbb{N}$, $\delta(m, n) = 1$ if $m = n$ otherwise, $\delta(m, n) = 0$.
- $S_k(N)$ denotes the space of cusp forms of even integer weight k and level N.
- $S_k(N)^*$ denotes the space of newforms with weight k and level N.
- $e(x) = e^{2\pi i x}$ for a given $x \in \mathbb{R}$.

1

Introduction

Let $S_k(N)$ denote the space of cusp forms of even integer weight k and level N. Let $\dim(S_k(N))$ denote the dimension of the vector space $S_k(N)$. The n^{th} normalised Hecke operator acting on $S_k(N)$ is given by

$$T_n(f)(z) := n^{\frac{k-1}{2}} \sum_{ad=n, d>0} \frac{1}{d^k} \sum_{b \pmod d} f\left(\frac{az+b}{d}\right)$$

for $(n, N) = 1$. Let $\mathcal{F}_k(N)$ be an orthonormal basis of $S_k(N)$ consisting only of joint eigenfunctions of the Hecke operators T_n with $(n, N) = 1$. For $f \in S_k(N)$, we have Fourier expansion of f at the cusp ∞ which is given by

$$f(z) = \sum_{n=1}^{\infty} a_n(f) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

Let $\lambda_n(f)$ be the n^{th} normalised Hecke eigenvalue of f i.e. $T_n(f) = \lambda_n(f)f$. Let $S_k(N)^*$ be the space of newforms with weight k and level N. Let $\widetilde{T}_n$ be the restriction of Hecke operator T_n from $S_k(N)$ to its subspace $S_k(N)^*$. Let $\mathcal{F}_k(N)^*$ be an orthonormal basis of $S_k(N)^*$ consisting only of joint eigenfunctions of the Hecke operators $\widetilde{T}_n$.

1.1 Discrepancy results for $\lambda_p(f)$

Definition 1.1.1. Let μ be a probability measure on $[a, b]$. A sequence of real numbers $x_n \in [a, b]$ is equidistributed with respect to the measure μ if for any $[a', b'] \subset [a, b]$,

$$\lim_{n\to\infty} \frac{|\{m \leq n : x_m \in [a', b']\}|}{n} = \int_{a'}^{b'} d\mu.$$

Definition 1.1.2. Let μ be a probability measure on $[a, b]$. A sequence of finite multisets A_n with $|A_n| \to \infty$ as $n \to \infty$ are equidistributed with respect to the measure μ if for any $[a', b'] \subset [a, b]$,

$$\lim_{n\to\infty} \frac{|t \in A_n : t \in [a', b']|}{|A_n|} = \int_{a'}^{b'} d\mu.$$

Let us consider the Sato-Tate measure given by

$$\mu_\infty(x) := \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}$$

if $x \in [-2, 2]$. In 2011, Barnet-Lamb, Geraghty, Harris, and Taylor proved the following equidistribution result ([BLGHT11]) for a non-CM newform in $S_k(N)^*$.

Theorem 1.1.3 (Barnet-Lamb, Geraghty, Harris, and Taylor)**.** *Let $f \in \mathcal{F}_k(N)^*$ be a fixed non-CM newform. The sequence $\{\lambda_p(f) : p$ prime, $(p, N) = 1\}$ is equidistributed (see 5.1.1) in $[-2, 2]$ with respect to the measure μ_∞.*

In 1997, Serre considered a vertical Sato Tate conjecture by fixing a prime p and varying N and k. Consider the measure given by

$$\mu_p(x) := \frac{p+1}{\pi} \frac{\left(1 - \frac{x^2}{4}\right)^{\frac{1}{2}}}{(\sqrt{p} + \sqrt{p^{-1}})^2 - x^2} = \frac{p+1}{(\sqrt{p} + \sqrt{p^{-1}})^2 - x^2}\mu_\infty(x)$$

if $x \in [-2, 2]$.

Theorem 1.1.4 (Serre)**.** *Let p be a fixed prime number. If $N_l + k_l \to \infty$ with $(p, N_l) = 1$ and k_l even, then the sequence of multisets $\{\lambda_p(f) : f \in \mathcal{F}_{k_l}(N_l)\}$ are equidistributed (see 5.1.2) in $[-2, 2]$ with respect to the measure μ_p.*

The study of the vertical distribution of Hecke eigenvalues $\lambda_p(f)$ where p is a fixed prime and f varies over suitable cusp forms goes back to the work of Sarnak [Sar87], who derived the above law in the context of Maass cusp forms. Theorem 1.1.4 for the case $N = 1$ was also proved by Conrey, Duke, and Farmer [CDF97]. The generalization of Theorem 1.1.4 for the space of newforms $S_k(N)^*$ was also proved in [Ser97]. Let

$$\mu_{k,N} := \frac{1}{\dim(S_k(N))} \sum_{f \in \mathcal{F}_k(N)} \delta_{\lambda_p(f)}$$

and

$$\mu^*_{k,N} := \frac{1}{\dim(S^*_k(N))} \sum_{f \in \mathcal{F}^*_k(N)} \delta_{\lambda_p(f)}$$

where δ_x is the Dirac measure at x.

Definition 1.1.5. Let μ_1 and μ_2 be two finite measures on a compact set $\Omega \subset \mathbb{R}$. The discrepancy between μ_1 and μ_2 is given by

$$D(\mu_1, \mu_2) := \sup\{|\mu_1(I) - \mu_2(I)| \,:\, I = [a, b] \subset \Omega\}.$$

The discrepancies $D(\mu_{k,N}, \mu_p)$ and $D(\mu^*_{k,N}, \mu_p)$ have been well investigated ([Gol04], [MS09], [MS10]). In [Gol04], the bound

$$D(\mu_{k,1}, \mu_p) = O_\epsilon\left(\frac{1}{(\log k)^{1-\epsilon}}\right)$$

was obtained for any $\epsilon > 0$ in the special case when $N = 1$ and k is a positive, even integer. The above bound was sharpened and generalized in [MS09] for any positive N with $(p, N) = 1$ as follows:

$$D(\mu_{k,N}, \mu_p) = O\left(\frac{1}{\log kN}\right). \tag{1.1}$$

A similar upper bound can also be obtained for $D(\mu^*_{k,N}, \mu_p)$. In [MS10], the following bound was obtained:

$$D(\mu^*_{k,N}, \mu_p) = O\Big(\frac{1}{\log kN}\Big). \tag{1.2}$$

The discrepancy bound in (1.1) was extended to Hilbert modular forms by Lau, Li, and

Wang in [LLW14].

A natural question to ask in this context is whether one can get a lower bound for the discrepancies $D(\mu_{k,N},\mu_p)$ and $D(\mu^*_{k,N},\mu_p)$. For a fixed squarefree level N Jung and Sardari [JS20], obtained a sequence of weights k_n with $k_n \to \infty$ such that

$$D(\mu^*_{k_n,N},\mu_p) \gg \frac{1}{k_n^{\frac{1}{3}} \log^2 k_n}. \tag{1.3}$$

Firstly, they consider the following weighted variants of $\mu_{k,N}$ and $\mu^*_{k,N}$.

For $f \in \mathcal{F}_k(N)$ (respectively $\mathcal{F}_k(N)^*$), define

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} |a_1(f)|^2,$$

$$H_k(N) := \sum_{f \in \mathcal{F}_k(N)} \omega_f,$$

and

$$H_k(N)^* := \sum_{f \in \mathcal{F}_k(N)^*} \omega_f.$$

Further, for any interval $I = [a,b] \subset [-2,2]$, define

$$\nu_{k,N}(I) := \frac{1}{H_k(N)} \sum_{f \in \mathcal{F}_k(N)} \omega_f \, \delta_{\lambda_p(f)}(I),$$

and

$$\nu^*_{k,N}(I) := \frac{1}{H_k(N)^*} \sum_{f \in \mathcal{F}^*_k(N)} \omega_f \, \delta_{\lambda_p(f)}(I). \tag{1.4}$$

Before proceeding we would like to introduce the concept of weak convergence of measures.

Definition 1.1.6. Let Ω be a compact subset of $\mathbb{R}$. A sequence of probability measures $\widetilde{\mu}_n$ converges weakly to a measure $\widetilde{\mu}$ if for every continuous function f on Ω,

$$\int_\Omega f \, d\widetilde{\mu}_n \to \int_\Omega f \, d\widetilde{\mu}.$$

Theorem 1.1.7. *Let $x_n \in [a,b]$ be a sequence of real numbers and*

$$\widetilde{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_n}.$$

The sequence x_n is equidistributed with respect to the measure μ if and only if $\widetilde{\mu}_n$ converges weakly to μ.

Remark 1.1.8. *By virtue of the above theorem, we see that the notion of equidistribution of a bounded real sequence x_n and weak convergence of an associated counting measure*

$$\widetilde{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_n}$$

are equivalent.

In [JS20], Petersson's trace formula is used to show that both $\nu_{k,N}$ and $\nu^*_{k,N}$ converges weakly to μ_∞ as $k+N \to \infty$ with $(p,N)=1$. Equivalently, for any continuous function $g : [-2,2] \to \mathbb{C}$,

$$\lim_{\substack{k+N\to\infty \\ (p,N)=1 \\ k \text{ even}}} \frac{1}{H_k(N)} \sum_{f\in\mathcal{F}_k(N)} \omega_f \, g(\lambda_p(f)) = \int_{-2}^{2} g(x)\, d\mu_\infty(x),$$

and

$$\lim_{\substack{k+N\to\infty \\ (p,N)=1 \\ k \text{ even}}} \frac{1}{H_k(N)^*} \sum_{f\in\mathcal{F}^*_k(N)} \omega_f \, g(\lambda_p(f)) = \int_{-2}^{2} g(x)\, d\mu_\infty(x).$$

Then for a fixed squarefree level N, Jung and Sardari obtain a sequence of weights k_n with $k_n \to \infty$ such that

$$D(\nu^*_{k_n,N}, \mu_\infty) \gg \frac{1}{k_n^{\frac{1}{3}} \log^2 k_n}. \tag{1.5}$$

We now look at the following questions that naturally arise from the above theorems.

Problem 1. Can one also extend the above discrepancy results of [JS20] to Hilbert modular forms?

This problem is addressed in Section 3 of Chapter 4. A result like equation (1.5) is obtained with the help of an explicit asymptotic version of the Petersson trace formula (Theorem 5.1.7), which is an important ingredient in [JS20]. This motivates us to get an asymptotic version of the Petersson trace formula analogous to Theorem 5.1.7 in the context of Hilbert cusp forms.

Problem 2. Can we generalize Theorem 5.1.7 for the setting of Hilbert cusp forms?

We address this problem in Section 2 and Section 3 of Chapter 4.

1.2 Discrepancy results for higher prime powers

The vertical Sato Tate Theorem (Theorem 1.1.4) talks about the distribution of $\lambda_p(f)$ for a fixed prime p. The following theorem talks about the distribution of eigenvalues $\lambda_{p^2}(f)$ for a fixed prime p. Let

$$\mu_{p^2}(x) := \frac{p+1}{2\pi}\frac{1}{(\sqrt{p}+\sqrt{p^{-1}})^2-(x+1)}\sqrt{\frac{3-x}{x+1}}$$

if $x \in [-1,3]$. In 2009, Omar and Mazhouda using Lemma 5.2.2 showed the following equidistribution result.

Theorem 1.2.1 (Theorem 1, [OM09])**.** *Let p be a fixed prime and $k>0$ a fixed even integer. Then the sequence of multisets*

$$\{(\lambda_{p^2}(f))_{f\in\mathcal{F}_k(N)^*} \,:\, N\in\mathbb{N}\}$$

are equidistributed in $[-1,3]$ with respect to the measure μ_{p^2}.

The distribution for $(\lambda_{p^3}(f))$, $(\lambda_{p^4}(f))$ and $(\lambda_{p^r}(f)-\lambda_{p^{r-2}}(f))$ for $r\geq 2$ has been addressed by Tang and Wang in [TW16].

Let

$$\mu_{k,N,2} := \frac{1}{\dim(S_k(N))}\sum_{f\in\mathcal{F}_k(N)}\delta_{\lambda_{p^2}(f)}$$

and

$$\mu^*_{k,N,2} := \frac{1}{\dim(S^*_k(N))}\sum_{f\in\mathcal{F}^*_k(N)}\delta_{\lambda_{p^2}(f)}.$$

We now consider the discrepancy between the following measures. The discrepancies $D(\mu^*_{k,N,2},\mu_{p^2})$ have been well investigated in [TW16].

Theorem 1.2.2 (Theorem 1, [TW16]). *Let $N = 1$. Then we have $D(\mu^*_{k,N,2}, \mu_{p^2}) = O\Big((\log kN)^{-1}\Big)$.*

Consider the following weight variants of $\mu_{k,N,2}$ and $\mu^*_{k,N,2}$ analogue to variants in (1.4). Let

$$\nu_{k,N,2} := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{F}_k(N)} |a_1(f)|^2 \delta_{\lambda_{p^2}(f)},$$

$$\nu^*_{k,N,2} := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{F}^*_k(N)} |a_1(f)|^2 \delta_{\lambda_{p^2}(f)}.$$

Let us consider the measure

$$\mu_{\infty,2}(x) = \frac{1}{2\pi}\sqrt{\frac{3-x}{1+x}}$$

for $x \in [-1, 3]$. One can use Petersson's trace formula to show that both $\nu_{k,N,2}$ and $\nu^*_{k,N,2}$ converge weakly to $\mu_{\infty,2}$ for any fixed even weight k and $N \to \infty$ with $(p, N) = 1$ (see [OM09, Theorem 3]).

Problem 3. Can one get a discrepancy result like equation (1.5) for $\lambda_{p^2}(f)$ as well?

Theorem 6.1.3 gives us the required discrepancy result for $D(\nu_{k,N,2}, \mu_{\infty,2})$. This is basically a generalization of Jung and Sardari's lower bound for the discrepancy $D(\nu_{k,N,2}, \mu_{\infty,2})$. This again necessitates the use of an asymptotic version of the Petersson trace formula that is analogous to Theorem 5.1.7.

Problem 4. Can we generalize Theorem 5.1.7 for classical cusp forms with a better error term? Can we include more levels along with N squarefree?

We answer it partially. More precisely, for level 1 we get a better error term. Our result includes levels for which $8 \nmid N$. Theorem 6.0.1 and Lemma 4.3.7 addresses Problem 4. This has been discussed in Chapter 6.

1.3 Kloosterman sums

We dedicate Chapter 3 to discuss Klooosterman sums specifically as the main term in the asymptotic we obtain contains a Kloosterman sum. The nonvanishing of Kloosterman sums plays an important role in obtaining an asymptotic discussed in Problems 2 and 4. Along with the nonvanishing of the Kloosterman sum, an upper bound of the Kloosterman sum is also necessary for the estimation of the Petersson trace formula. The following lemma for the Kloosterman sum for number fields due to Knightly and Li is helpful.

Lemma 1.3.1 (Knightly, Li)**.** *Let $m_1, m_2 \in \mathfrak{d}^{-1}$, $n \in \hat{\mathcal{O}}$ nonzero and $c \in \hat{\mathfrak{N}} \cap \mathbb{A}^*_{\mathrm{fin}}$. Then*

$$|S_{\omega_{\mathfrak{N}}}(m_1, m_2; n; c)| \leq \mathrm{Nm}(n)\mathrm{Nm}(c).$$

Problem 5. Can we get a sharp bound for local Kloosterman sums as given by Lemma 1.3.1? Can we get an upper bound like Weil bound for classical Kloosterman sum?

Theorem 3.1.2 gives a partial answer to the above problem. We also discuss the nonvanishing aspect of Kloosterman sums in Chapter 3.

Organisation of chapters

The first chapter contains an overview of thesis problems with background and organization of chapters.

In the Second Chapter, we recall the setting of adelic Hilbert modular forms. In the third Section, we discuss Hecke operators. The last Section is dedicated to Petersson's trace formula.

The Third Chapter is entirely dedicated to Kloosteman sums. We provide a sharp upper bound for local Kloosterman sum. The non-vanishing of Kloosterman sum plays an important role in subsequent chapters. Therefore results concerning non-vanishing have also been included in the chapter. The upper bound has been discussed in the first Section and nonvanishing has been discussed in the second Section of the third Chapter.

The primary content of Chapter four is to derive an asymptotic version of Petersson's trace formula for the space $A_k(\mathfrak{N}, \omega)$ (see Section 2.2). This has been obtained in Section one of Chapter four. As an application, we prove an analog of a discrepancy result proved by Jung and Sardari in 2020. This has been achieved in Section 3.

Chapter 5 discusses equidistribution results concerning the eigenvalues of Hecke operators. In Section 2 of this chapter, we discuss equidistribution and discrepancy results for $\lambda_{p^2}(f)$ with $f \in S_k(N)$.

In Chapter 6, we stick to the base case of $F = \mathbb{Q}$. We start with an asymptotic formula for Petersson's trace formula with a better error term. Then in the First Section, we find a sequence of weights for which the lower bound for the discrepancy $D(\nu_{k,N,2}, \mu_{\infty,2})$ holds. Finally, in the last section, we discuss future problems to work on.

2

Adelic Hilbert modular forms

In this chapter, we recall some basic facts about Hilbert modular forms and the Petersson trace formula in this setting. Section 2.1 discusses the basics required for Hilbert modular forms. Section 2.2 and Section 2.3 defines Adelic Hilbert modular forms and Hecke operators respectively. Section 2.4 contains a brief introduction to the Petersson trace formula and its applications that will be useful in a later chapter.

2.1 Review of basics

Let F be a totally real number field and $r = [F : \mathbb{Q}]$. Let the distinct real embeddings be given by $\sigma_1, \ldots, \sigma_r$. This means for each j, σ_j is an injective field homorphism such that $\sigma_j(F) \subset \mathbb{R}$. Let $\mathcal{O}$ be the ring of integers of F and $F^\times = F \setminus \{0\}$. We denote the group of units of $\mathcal{O}$ as $\mathcal{O}^\times$. Let $F^+ = \{x \in F : \sigma_i(x) > 0 \text{ for all } i\}$. Now we state Dirichlet's unit theorem for a totally real field.

Theorem 2.1.1. *Let F be a totally real number field and $r = [F : \mathbb{Q}]$. The group $\mathcal{O}^\times$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{r-1}$.*

Proof. For a proof, refer to [Neu99, Chapter I, Theorem 7.4]. Since F is a totally real field, the roots of unity that lie inside F are 1 and -1. Furthermore, F has r real embeddings and no complex embedding which proves the claim. □

Given an abelian group A which is finite with $\overline{A} := A/A^2$ and $|\overline{A}| = 2^a$, we define $\dim_2(A) := a$. We let $U = \mathcal{O}^\times/(\mathcal{O}^\times)^2 = \overline{\mathcal{O}^\times}$. Note that U is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$. This can be seen by the group homomorphism $g : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{r-1} \to (\mathbb{Z}/2\mathbb{Z})^r$ given by $g((a_1, \ldots, a_r)) = ((-1)^{a_1}, \ldots, (-1)^{a_r})$ with application of first isomorphism theorem. Let $\{u_1, \ldots u_{2^r}\}$ be a fixed set of representatives of U. A fractional ideal I of $\mathcal{O}$ is an $\mathcal{O}$-submodule I of K such that there exists nonzero $t \in \mathcal{O}$ with $tI \subset \mathcal{O}$. Let P_F denote the set of all principal fractional ideals which consists of $\mathcal{O}$- submodule I of K generated by a single nonzero element from K. The ideal class group C of F is defined to be the group of fractional ideals(I_F) modulo the principal fractional ideal P_F i.e. $C = I_F/P_F$. Let P_F^+ be the subgroup of P_F consisting of ideals of the form $a\mathcal{O}$ with $a \in F^+$. The narrow class group C^+ of F is given by $C = I_F/P_F^+$. The class number and narrow class number are defined to be $|C|$ and $|C^+|$ respectively. Let $[\mathfrak{a}]$ denotes the image of fractional ideal $\mathfrak{a}$ in C.

Let $\nu = \nu_{\mathfrak{p}}$ be the discrete valuation associated with the prime ideal $\mathfrak{p}$ in $\mathcal{O}$. More precisely, $\nu : F^\times \to \mathbb{Z}$ given by $\nu(x) = n$ where $x\mathcal{O} = \mathfrak{p}^n\mathfrak{a}$ with $\mathfrak{p} \nmid \mathfrak{a}$. The local field F_ν is the completion of F with respect to the metric induced by the norm given by $\|x\| = \mathfrak{p}^{-\nu(x)}$ for $x \in F^\times$. Let $\mathcal{O}_\nu$ be the ring of integers in the local field F_ν. The Adele ring of F is a subring $\mathbb{A}$ consisting of tuples (a_ν) such that $a_\nu \in \mathcal{O}_\nu$ for all but finitely many ν. We denote the finite adeles as $\mathbb{A}_{\text{fin}}$ which consist of tuples $(a_\nu)_\nu \in \prod_{\nu<\infty} F_\nu$ such that $a_\nu \in \mathcal{O}_\nu$ for all but finitely many ν. We have $\mathbb{A} = F_\infty \times \mathbb{A}_{\text{fin}}$ where F_∞ is isomorphic to $\mathbb{R}^r$. The ideles $\mathbb{A}^\times$ are the units in the ring $\mathbb{A}$. The finite ideles $\mathbb{A}_{\text{fin}}^\times$ denotes the group of units of $\mathbb{A}_{\text{fin}}$ so that $\mathbb{A}^\times = F_\infty^\times \times \mathbb{A}_{\text{fin}}^\times$. For the case $F = \mathbb{Q}$, we refer the reader to section 5 of [KL06] for a concise introduction to adeles and ideles.

We will use the hat notation for considering non-Archimedean valuations. For instance, we let $\hat{\mathcal{O}} = \prod_{\nu<\infty} \mathcal{O}_\nu$. Similarly for a fractional ideal $\mathfrak{a}$, $\hat{\mathfrak{a}} = \prod_{\nu<\infty} \mathfrak{a}_\nu = \mathfrak{a}\hat{\mathcal{O}}$. We use the notation $\text{ord}_\nu(\mathfrak{a})$ and $\mathfrak{a}_\nu$ for its order and localization at the valuation ν. We have $\mathfrak{a}_\nu = \varpi_\nu^{\text{ord}_\nu(\mathfrak{a})}\mathcal{O}_\nu$ where ϖ_ν is a generator of the maximal ideal $\mathfrak{p}_\nu = \mathfrak{p}\mathcal{O}_\nu$ i.e. $\mathfrak{p}_\nu = \varpi_\nu\mathcal{O}_\nu$.

Let $F_\infty^+ \subset F_\infty$ be the collection of vectors whose entries are all positive. The inverse different ideal is given by $\mathfrak{d}^{-1} = \{x \in F : Tr_{\mathbb{Q}}^F(x\mathcal{O}) \subset \mathbb{Z}\}$ Let $\mathfrak{d}_+^{-1} = \mathfrak{d}^{-1} \cap F^+$ and $U^+ = U \cap F^+$.

Now we consider an equation in the ideal class group which will be helpful in subsequent chapters. An integral ideal I is a fractional ideal $I \subset \mathcal{O}$. For a fixed integral ideal

$\mathfrak{n}$, consider the following equation

$$[\mathfrak{b}]^2[\mathfrak{n}] = 1. \tag{2.1}$$

The solutions $\mathfrak{b}_i$ can be taken as integral ideals. Since the ideal class group is finite, the number of solutions for the above equation is also finite. Now we will see an example of solutions to the equation.

Example 2.1.1. Let F have class number 2. Then the equation (2.1) has a solution if and only if $\mathfrak{n}$ is a principal integral ideal. In the case of a solution existing, the solutions can be taken as $\mathfrak{b}_1 = \mathcal{O}$ and $\mathfrak{b}_2$ to be any non-principal integral ideal.

Example 2.1.2. Let F have an odd class number. The equation (2.1) has a unique solution.

Proof. For proof, we refer to [KL08, Example 5.16]. □

2.2 Adelic Hilbert modular forms

Let $\mathfrak{N}$ be an integral ideal and $k = (k_1, \ldots, k_r)$ where k_j are positive integers with $k_j > 2$. Let $\omega : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be an unitary Hecke character (see section 12.1 of [KL06]). We have $\omega = \prod_\nu \omega_\nu$ where ω_ν can be defined as follows. Let $x \in \mathbb{A}^\times$ be given by $x = (1, \ldots, x_\nu, \ldots, 1, \ldots)$ such that $x_\nu \in F_\nu$. Then $\omega_\nu(x_\nu) = \omega(x)$. For infinite valuation and $j = 1, \ldots, r$, define $\omega_{\infty_j}(x) = \mathrm{sgn}(x)^{k_j}$. We further let the conductor of ω divide $\mathfrak{N}$. This means that ω_ν is trivial on $1 + \mathfrak{N}_\nu$ for all $\mathfrak{p}|\mathfrak{N}$, and unramified for all $\mathfrak{p} \nmid \mathfrak{N}$.

Let $\bar{G} = GL_2/Z(GL_2)$ where $Z(GL_2)$ is the centre of GL_2. Let $L^2(\omega) = L^2(\bar{G}(F) \backslash \bar{G}(\mathbb{A}), \omega)$ denote the space of left $\mathrm{GL}_2(\mathrm{F})$ invariant functions on $\mathrm{GL}_2(\mathbb{A})$ which transform by ω under the center and are square integrable on $\bar{G}(F)\backslash\bar{G}(\mathbb{A})$. Let $L_0^2(\omega)$ be the subspace of cuspidal functions.

Let K_{fin} denote the maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$. We have

$$K_{\mathrm{fin}} = \prod_{\nu<\infty} K_\nu = \prod_{\nu<\infty} GL_2(\mathcal{O}_\nu) = GL_2(\hat{\mathcal{O}}).$$

Given an integral ideal $\mathfrak{N}$, we now define the groups $K_0(\mathfrak{N})$ and $K_1(\mathfrak{N})$.

$$K_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\text{fin}} \,\middle|\, c \in \mathfrak{N}\hat{\mathcal{O}} \right\}$$

and

$$K_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\text{fin}} \,\middle|\, c \in \mathfrak{N}\hat{\mathcal{O}}, d \in 1 + \mathfrak{N}\hat{\mathcal{O}} \right\}.$$

Note that $K_0(\mathfrak{N})$ and $K_1(\mathfrak{N})$ are analogues of $\Gamma_0(N)$ and $\Gamma_1(N)$ with level N for the classical setting of $F = \mathbb{Q}$. Let $A_k(\mathfrak{N}, \omega)$ be the set of $\phi \in L_0^2(\omega)$ satisfying the given three conditions.

1. $\phi(gk_{\text{fin}}) = \phi(g)$ for every $k_{\text{fin}} \in K_1(\mathfrak{N})$,
2. $\phi(gk_\infty) = \prod_{j=1}^r e^{ik_j\theta_j}\phi(g)$ for all $k_\infty = \prod_j k_{\theta_j} \in K_\infty = SO_2(\mathbb{R})^r$,
3. For any fixed $x \in GL_2(\mathbb{A})$ and for all j, the function $g_{\infty_j} \mapsto \phi(xg_{\infty_j})$ is annihilated by $R(E^-)$, where $E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in GL_2(\mathbb{C})$ and R denotes the right regular action of $GL_2(F_{\infty_j})$.

In condition (2) above, θ_j are such that

$$SO_2(\mathbb{R})^r = \prod_{j=1}^r \left\{ \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} : \theta_j \in \mathbb{R} \right\}.$$

For details, see Proposition 12.5 and Theorem 12.6 of [KL06].

Theorem 2.2.1. *Let $\phi \in A_k(\mathfrak{N}, \omega)$. Then ϕ is a continuous function on $GL_2(\mathbb{A})$.*

Proof. For a proof, we refer to Lemma 3.3 of [Li09]. □

Theorem 2.2.2. *The space $A_k(\mathfrak{N}, \omega)$ is a finite-dimensional vector space.*

Proof. The space $A_k(\mathfrak{N}, \omega)$ is finite- dimensional by a general theorem of Harish-Chandra. We refer the reader to the first chapter of [HC68] (see also [BJ79]) for a proof. □

Definition 2.2.3. Let $\theta : \mathbb{A} \to \mathbb{C}^\times$ denote the standard character of $\mathbb{A}$. Concretely, $\theta(x) = \theta_\infty(x_\infty) \cdot \prod_{\nu<\infty} \theta_\nu(x_\nu)$, where

1. $\theta_\infty : F_\infty \to \mathbb{C}^\times$ is defined by $\theta_\infty(x_\infty) = e^{-2\pi i(x_1+\cdots+x_r)}$ with $x_\infty = (x_1, ..., x_r)$, and

2. for $\nu < \infty$, $\theta_\nu : F_\nu \to \mathbb{C}^\times$ is given by $\theta_\nu(x_\nu) = e^{2\pi i(\mathrm{Tr}_\nu(x_\nu))}$. Here $\mathrm{Tr}_\nu(x_\nu)$ is obtained by composing the following maps: $\mathrm{Tr}^{F_\nu}_{\mathbb{Q}_p} : F_\nu \to \mathbb{Q}_p$, going modulo p-adic integers: $\mathbb{Q}_p \to \mathbb{Z}_p$, and identifying $\mathbb{Q}_p/\mathbb{Z}_p$ with $\mathbb{Q}/\mathbb{Z}$. More precisely, let

$$\mathrm{Tr}^{F_\nu}_{\mathbb{Q}_p}(x_\nu) = \sum_{n=n_0}^{\infty} a_n p^n$$

where $n_0 \in \mathbb{Z} \cup \{\infty\}$, $a_{n_0} \neq 0$, $\langle p \rangle = \mathfrak{p} \cap \mathbb{Z}$ and $0 \leq a_n < p$. Then

$$\mathrm{Tr}_\nu(x_\nu) = \sum_{n=n_0}^{-1} a_n p^n.$$

The map θ_ν is well-defined since $e^{2\pi i \mathbb{Z}} = 1$.

Note that the kernel of θ_ν is the local inverse different $\mathfrak{d}_\nu^{-1} = \{x \in F_\nu \mid \mathrm{Tr}^{F_\nu}_{\mathbb{Q}_p}(x) \in \mathbb{Z}_p\}$.

Let $m \in \mathfrak{d}_+^{-1}$ and $\phi \in A_k(\mathfrak{N}, \omega)$. The mth Fourier coefficient of ϕ is given by W_m^ϕ as in [KL08, §3.4]. Consider the unipotent subgroup $\widetilde{N} = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$ of GL_2. For any $g \in GL_2(\mathbb{A})$, the map $n \mapsto \phi(ng)$ is a continuous function on $\widetilde{N}(F)\backslash\widetilde{N}(\mathbb{A})$. We have a Fourier expansion that is

$$\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \frac{1}{\sqrt{d_F}} \sum_{m \in F} W_m^\phi(g)\theta_m(x).$$

The coefficients are Whittaker functions given by

$$W_m^\phi(g) = \int_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\theta(mx)dx. \tag{2.2}$$

For $y \in \mathbb{A}^\times$, we have

$$W_m^\phi(y) = W_m^\phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right).$$

When $y \in \mathbb{A}_{\text{fin}}^{\times}$, we identify $\mathbb{A}_{\text{fin}}^{\times}$ with $\{1_\infty\} \times \mathbb{A}_{\text{fin}}^{\times} \subset \mathbb{A}^{\times}$. Let $\mathfrak{n}$ be an integral ideal coprime to the integral ideal $\mathfrak{N}$. Let ϕ be an eigenvector of the Hecke operator $T_{\mathfrak{n}}$ (see section 2.3), then let $T_{\mathfrak{n}}\phi = \lambda_{\mathfrak{n}}(\phi)\phi$. Let $\{a_1, \dots, a_r\}$ be a basis of $\mathcal{O}$ as a $\mathbb{Z}$ module. The discriminant of the number field F is equal to the determinant of the matrix $M = (m_{ij})$ where $m_{ij} = \sigma_i(a_j)$. We denote the discriminant of F as d_F. We now recall the following result which we use later.

Lemma 2.2.4 ([KL08, Cor. 4.8])**.** *Let $\widetilde{d} \in \mathbb{A}_{\text{fin}}^{\times}$ such that $\widetilde{d}\hat{\mathcal{O}} = \hat{\mathfrak{d}}$ and $(m\mathfrak{d}, \mathfrak{N}) = 1$. Then for any $T_{m\mathfrak{d}}$-eigenvector $\phi \in A_k(\mathfrak{N}, \omega)$ with $W_1^{\phi}(1/\widetilde{d}) = 1$ and $T_{m\mathfrak{d}}\phi = \lambda_{m\mathfrak{d}}\phi$, we have*

$$W_m^{\phi}(1) = \frac{e^{2\pi r} \prod_{j=1}^{r} \sigma_j(m)^{(k_j/2)-1}}{d_F e^{2\pi \mathrm{Tr}_{\mathbb{Q}}^{F}(m)}} \lambda_{m\mathfrak{d}}.$$

Lemma 2.2.4 gives a relation between a Fourier coefficient and an eigenvalue of the Hecke operator.

2.3 Hecke operators

Let $g \in \bar{G}(\mathbb{R})$. For $\det(g) > 0$ let

$$f_{\infty_j}(g) = \frac{k_j - 1}{4\pi} \frac{(\det(g))^{\frac{k_j}{2}} (2i)^{k_j}}{((c-b) + (a+d)i)^{k_j}}$$

and $f_{\infty_j}(g) = 0$ otherwise (see [KL06, Theorem 14.5]). We have f_{∞_j} is integrable over $\bar{G}(\mathbb{R})$ for $k_j > 2$ (see [KL06, Proposition 14.3]). Let

$$f_\infty = \prod_{j=1}^{r} f_{\infty_j}.$$

For non-archimedean valuation, we proceed as follows. Let $\mathfrak{n}_\nu, \mathfrak{N}_\nu$ be two coprime integral ideals in $\mathcal{O}_\nu$ and

$$M(\mathfrak{n}_\nu, \mathfrak{N}_\nu) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_\nu) : c \in \mathfrak{N}_\nu, (ad - bc)\mathcal{O}_\nu = \mathfrak{n}_\nu \right\}.$$

Suppose $\nu \mid \mathfrak{N}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{N})_\nu$ where

$$K_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_\nu) \,\middle|\, c \in \mathfrak{N}_\nu \right\}.$$

Then we let $\omega_\nu(g) = \omega_\nu(d)$. We observe that ω_ν is a character of $K_0(\mathfrak{N})_\nu$. Let $m \in M(\mathfrak{n}_\nu, \mathfrak{N}_\nu)$. We now define $f_{\mathfrak{n}_\nu}$ as per condition $\nu \mid \mathfrak{N}$ or $\nu \nmid \mathfrak{N}$. Let $z \in F_\nu^\times$. When $\nu \nmid \mathfrak{N}$, take $f_{\mathfrak{n}_\nu}(zm) = (\omega_\nu(z))^{-1}$. When $\nu \mid \mathfrak{N}$, take $f_{\mathfrak{n}_\nu}(zm) = \psi(\mathfrak{N}_\nu)(\omega_\nu(z))^{-1}(\omega_\nu(m))^{-1}$ where

$$\psi(\mathfrak{N}_\nu) = (\mathrm{Nm}(\mathfrak{p}))^{\mathrm{ord}_\nu(\mathfrak{N})}\left(1 + \frac{1}{\mathrm{Nm}(\mathfrak{p})}\right).$$

Now we define the global Hecke operator for an integral ideal $\mathfrak{n}$ that is coprime to the integral ideal $\mathfrak{N}$. Let $f_\mathfrak{n}$ be defined on $\mathbb{A}_{\text{fin}}$ such that $f_\mathfrak{n} = \prod_\nu f_{n_\nu}$. Note that $f_\mathfrak{n}$ is bi-$K_1(\mathfrak{N})$ invariant and supported on $\mathbb{A}_{\text{fin}}^\times M(\mathfrak{n}, \mathfrak{N})$ with $M(\mathfrak{n}, \mathfrak{N}) = \prod_{\nu<\infty} M(\mathfrak{n}_\nu, \mathfrak{N}_\nu)$.

Proposition 2.3.1. Let $z \in \mathbb{A}_{\text{fin}}^\times$ and $g \in \mathrm{GL}_2(\mathbb{A}_{\text{fin}})$. Then

$$f_\mathfrak{n}(zg) = (\omega_{\text{fin}}(z))^{-1} f_\mathfrak{n}(g).$$

Let $f \in L^1(\mathrm{GL}_2(\mathbb{A}), \omega^{-1})$ and $\phi \in L^2(\omega)$. The right regular action R of f on $L^2(\omega)$ is given by

$$R(f)(\phi(x)) = \int_{\bar{G}(\mathbb{A})} f(g)\phi(xg)\, dg.$$

We define the operator $T_\mathfrak{n} := R(f_\infty \times f_\mathfrak{n})$ on $L_0^2(\omega)$.

Theorem 2.3.1 ([KL08],Proposition 4.1)**.** *Let f_∞ be defined as earlier. Let f_{fin} be a bi-$K_1(\mathfrak{N})$ invariant function on $\mathrm{GL}_2(\mathbb{A}_{\text{fin}})$ satisfying $f_{\text{fin}}(zg) = (\omega_{\text{fin}}(z))^{-1} f_{\text{fin}}(g)$. Further let the support of f_{fin} be compact modulo $\mathbb{A}_{\text{fin}}^\times$. Then $R(f)$ vanishes on the orthogonal complement of $A_k(\mathfrak{N}, \omega)$ in $L^2(\omega)$ and its image is a subspace of $A_k(\mathfrak{N}, \omega)$.*

Proof. For a proof, we refer the reader to [KL08, Proposition 4.1]. □

By virtue of Theorem 2.3.1 and Proposition 2.3.1, $T_\mathfrak{n}$ can be viewed as an operator on the space $A_k(\mathfrak{N}, \omega)$.

Proposition 2.3.2. There exists an orthogonal basis for $A_k(\mathfrak{N}, \omega)$ consisting of eigenfunctions for the Hecke operator $T_{\mathfrak{n}}$.

Proof. The vector space $A_k(\mathfrak{N}, \omega)$ is finite-dimensional. The result follows from applying Theorem 7.4.1 of [MDG15] to the commutative ring of Hecke operators acting on $A_k(\mathfrak{N}, \omega)$. $\square$

2.4 Petersson's trace formula

Recall that $\mathfrak{n}$ is an integral ideal coprime to the integral ideal $\mathfrak{N}$. Petersson's trace formula gives us a weighted orthogonality relation between Fourier coefficients of an eigenfunction for the Hecke operator $T_{\mathfrak{n}}$. The formula under special cases yields a weighted formula for the trace of the Hecke operator $T_{\mathfrak{n}}$ (see Corollary 2.4.4, Proposition 5.1.1 and Theorem 5.1.5). Note that Corollary 2.4.3 of the formula also gives us a weighted orthogonality relation between eigenvalues of an eigenfunction for the Hecke operator $T_{\mathfrak{n}}$.

Before we get to Petersson's trace formula, we define Kloosterman sums, first locally and then globally. For any finite valuation ν of F, let $\mathfrak{n}_\nu \in \mathcal{O}_\nu \setminus \{0\}$ and $m_{1\nu}, m_{2\nu} \in \mathfrak{d}_\nu^{-1}$. For $c_\nu \in \mathfrak{N}_\nu \setminus \{0\}$, we define a local Kloosterman sum by

$$S_{\omega_\nu}(m_{1\nu}, m_{2\nu}; \mathfrak{n}_\nu; c_\nu) = \sum_{\substack{s_1, s_2 \in \mathcal{O}_\nu / c_\nu \mathcal{O}_\nu \\ s_1 s_2 \equiv \mathfrak{n}_\nu \bmod c_\nu \mathcal{O}_\nu}} \theta_\nu\left(\frac{m_{1\nu} s_1 + m_{2\nu} s_2}{c_\nu}\right) \omega_\nu(s_2)^{-1}. \tag{2.3}$$

The sum is equal to 1 if $c_\nu \in \mathcal{O}_\nu^\times$. Let $\mathfrak{n} \in \hat{\mathcal{O}} \cap \mathbb{A}_{\text{fin}}^\times$, $c \in \hat{\mathfrak{N}} \cap \mathbb{A}_{\text{fin}}^\times$, and $m_1, m_2 \in \hat{\mathfrak{d}}^{-1}$,

$$S_{\omega_{\mathfrak{N}}}(m_1, m_2; \mathfrak{n}; c) = \sum_{\substack{s_1, s_2 \in \hat{\mathcal{O}} / c\hat{\mathcal{O}} \\ s_1 s_2 \equiv \mathfrak{n} \bmod c\hat{\mathcal{O}}}} \theta_{\text{fin}}\left(\frac{m_1 s_1 + m_2 s_2}{c}\right) \omega_{\mathfrak{N}}(s_2)^{-1}$$

where

$$\omega_{\mathfrak{N},\nu} = \begin{cases} \omega_\nu, & \text{if } \nu | \mathfrak{N} \\ 1, & \text{if } \nu \nmid \mathfrak{N}. \end{cases}$$

Also, let

$$\omega_{\mathfrak{N}} = \prod \omega_{\mathfrak{N},\nu} = \prod_{\nu | \mathfrak{N}} \omega_\nu.$$

We have the following relation between the global and local Kloosterman sums,

$$S_{\omega_{\mathfrak{N}}}(m_1, m_2; \mathfrak{n}; c) = \prod_{\nu<\infty} S_{\omega_{\mathfrak{N},\nu}}(m_{1\nu}, m_{2\nu}; \mathfrak{n}_\nu; c_\nu). \tag{2.4}$$

Note that the product over all finite valuations is well-defined because $c_\nu \in \mathcal{O}_\nu^*$ except for finitely many ν.

We encounter the J-Bessel function of the first kind in the Petersson trace formula. The Bessel functions of the first kind $J_a(x)$ for $a \geq 0$, can be defined by the following power series representation

$$J_a(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(a+j+1)} \left(\frac{x}{2}\right)^{a+2j},$$

where $\Gamma(x)$ denotes the Gamma function evaluated at x. For our purpose, we will need various estimates of the J-Bessel function which we recall in the next lemma.

Lemma 2.4.1. *We have the following estimates of the J-Bessel function.*
(i) If $a \geq 0$ and $0 < x \leq 1$, we have

$$1 \leq \frac{J_a(ax)}{x^a J_a(a)} \leq e^{a(1-x)}.$$

(ii) $0 < J_a(a) \ll \frac{1}{a^{\frac{1}{3}}}$ as $a \to \infty$.
(iii) If $|d| < 1$, then

$$\frac{1}{a^{\frac{1}{3}}} \ll J_a(a + da^{\frac{1}{3}}) \ll \frac{1}{a^{\frac{1}{3}}}.$$

(iv) For $x \in \mathbb{R}$ and $a > 0$, $|J_a(x)| \leq \min(ba^{\frac{-1}{3}}, c|x|^{\frac{-1}{3}})$ where $b = 0.674885...$ and $c = 0.7857468704...$.
(v) For $\frac{1}{2} \leq x < 1$, we have the following uniform bound

$$J_a(ax) \ll \frac{1}{(1-x^2)^{1/4} a^{1/2}}.$$

We refer to [JS20, Section 2.1.1] for the Lemma except for Lemma 2.4.1 (iv). For Lemma 2.4.1 (iv), we refer to [Lan00, Section 1]. It is worth noting that Lemma 2.4.1 (iii) is important for finding a lower bound for Theorem 4.1.1. The geometric origin of

Lemma 2.4.1 (iii) has been discussed in [JS20, Section 5].

In 1932, Petersson expressed a weighted sum of $a_m(f)\overline{a_n(f)}$ over f with $f \in S_k(N)$ in terms of the Bessel function and a Kloosterman sum (see [Pet32]). This is the first relative trace formula, about 22 years earlier than the unweighted trace formula which was proved by Selberg [Sel56] in 1956. In 2009, Knightly and Li obtained the following generalization of Petersson's trace formula in the Hilbert modular forms setting for the space $A_k(\mathfrak{N}, \omega)$.

Theorem 2.4.2 ([KL08, Thm. 5.11]). *Let $\mathfrak{n}$ and $\mathfrak{N}$ be two coprime integral ideals. Let $k = (k_1, ..., k_r)$ with all $k_j > 2$. Let $\mathcal{F}$ be an orthogonal basis for $A_k(\mathfrak{N}, \omega)$ consisting of eigenfunctions for the Hecke operator $T_\mathfrak{n}$. Then given $m_1, m_2 \in \mathfrak{d}_+^{-1}$, we have*

$$\frac{e^{2\pi \mathrm{Tr}_{\mathbb{Q}}^F(m_1+m_2)}}{\psi(\mathfrak{N})}\left[\prod_{j=1}^r \frac{(k_j-2)!}{(4\pi\sqrt{\sigma_j(m_1m_2)})^{k_j-1}}\right]\sum_{\phi\in\mathcal{F}}\frac{\lambda_\mathfrak{n}(\phi)W^\phi_{m_1}(1)\overline{W^\phi_{m_2}(1)}}{\|\phi\|^2}$$
$$= \hat{T}(m_1, m_2, \mathfrak{n})\frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_\mathfrak{N}(m_1/s)\omega_f(s)}$$
$$+\sum_{i=1}^t \sum_{\substack{u\in U\\ \eta_i u\in F^+}} \sum_{\substack{s\in\mathfrak{b}_i\mathfrak{N}/\pm\\ s\neq 0}} \Bigg\{\omega_f(sb_i^{-1})S_{\omega_\mathfrak{N}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1})$$
$$\times\frac{\sqrt{\mathrm{Nm}(\eta_i u)}}{\mathrm{Nm}(s)} \times \prod_{j=1}^r \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\left(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\right)\Bigg\}.$$

- *where $\hat{T}(m_1, m_2, \mathfrak{n})$ is 1 if there exists $s \in \hat{\mathfrak{d}}^{-1}$ such that $m_1m_2 \in s\hat{\mathcal{O}}$ with $m_1m_2\hat{\mathcal{O}} = s^2\hat{\mathfrak{n}}$, and 0 otherwise,*

- *U is a set of representatives for $\mathcal{O}^\times/\mathcal{O}^{\times 2}$,*

- *$b_i\hat{\mathcal{O}} = \hat{\mathfrak{b}}_i$ for $\mathfrak{b}_i$ for $i = 1, ..., t$, where the b_i are distinct solution(s) of the equation $[b]^2[\mathfrak{n}] = 1$ in the ideal class group of F,*

- *$\eta_i \in F$ generates the principal ideal $\mathfrak{b}_i^2\mathfrak{n}$,*

- *$\omega_\mathfrak{N} = \prod_{v|\mathfrak{N}} \omega_v \prod_{v\nmid\mathfrak{N}} 1$,*

- *and, $\psi(\mathfrak{N}) = \mathrm{Nm}(\mathfrak{N})\prod_{\mathfrak{p}|\mathfrak{N}}\left(1 + \frac{1}{\mathrm{Nm}(\mathfrak{p})}\right)$.*

Corollary 2.4.3 ([BDS23], Corollary 6)**.** *Let $\widetilde{d} \in \mathbb{A}_f^\times$ such that $\widetilde{d}\hat{\mathcal{O}} = \hat{\mathfrak{d}}$. Let $m_1, m_2 \in \mathfrak{d}_+^{-1}$ such that $(m_1\mathfrak{d}, \mathfrak{N}) = (m_2\mathfrak{d}, \mathfrak{N}) = 1$ with $W_1^\phi(1/\widetilde{d}) = 1$ for $\phi \in \mathcal{F}$, where $\mathcal{F}$ is an orthogonal basis for $A_k(\mathfrak{N}, \omega)$. Then*

$$\frac{e^{4\pi r}}{\psi(\mathfrak{N}) d_F^2 \sqrt{\mathrm{Nm}(m_1 m_2)}} \left[\prod_{j=1}^{r} \frac{(k_j-2)!}{(4\pi)^{k_j-1}}\right] \sum_{\phi\in\mathcal{F}} \frac{\lambda_{m_1\mathfrak{d}}(\phi)\overline{\lambda_{m_2\mathfrak{d}}(\phi)}}{\|\phi\|^2} = \hat{T}(m_1, m_2, \mathcal{O}) \frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_f(s)}$$
$$+ \sum_{i=1}^{t} \sum_{\substack{u\in U \\ \eta_i u\in F^+}} \sum_{\substack{s\in\mathfrak{b}_i\mathfrak{N}/\pm \\ s\neq 0}} \left\{ \omega_f(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\mathrm{Nm}(\eta_i u)}}{\mathrm{Nm}(s)} \right.$$
$$\left. \times \prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\left(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|} \right) \right\}.$$

Proof. This corollary is proved by substituting the expressions for $W_{m_1}^\phi(1)$ and $W_{m_2}^\phi(1)$ obtained from Lemma 2.2.4 into Theorem 2.4.2. To see this

$$W_{m_1}^\phi(1) = \frac{e^{2\pi r} \prod_{j=1}^{r} \sigma_j(m_1)^{(k_j/2)-1}}{d_F e^{2\pi \mathrm{Tr}_{\mathbb{Q}}^F(m_2)}} \lambda_{m_1\mathfrak{d}}(\phi)$$

and

$$W_{m_2}^\phi(1) = \frac{e^{2\pi r} \prod_{j=1}^{r} \sigma_j(m_2)^{(k_j/2)-1}}{d_F e^{2\pi \mathrm{Tr}_{\mathbb{Q}}^F(m_2)}} \lambda_{m_2\mathfrak{d}}(\phi).$$

Multiplying the first with the conjugate of the second one, we get

$$\begin{aligned} W_{m_1}^\phi(1)\overline{W_{m_2}^\phi(1)} &= \frac{e^{4\pi r} \prod_{j=1}^{r} \sigma_j(m_1 m_2)^{(k_j/2)-1}}{d_F^2 e^{2\pi \mathrm{Tr}_{\mathbb{Q}}^F(m_1+m_2)}} \lambda_{m_1\mathfrak{d}}(\phi)\overline{\lambda_{m_2\mathfrak{d}}(\phi)} \\ &= \frac{e^{4\pi r} \prod_{j=1}^{r} \sigma_j(m_1 m_2)^{((k_j-1)/2)}}{d_F^2 \sqrt{\prod_{j=1}^{r} \sigma_j(m_1 m_2)} e^{2\pi \mathrm{Tr}_{\mathbb{Q}}^F(m_1+m_2)}} \lambda_{m_1\mathfrak{d}}(\phi)\overline{\lambda_{m_2\mathfrak{d}}(\phi)}. \end{aligned}$$

Thus we have

$$\frac{e^{2\pi \mathrm{Tr}_{\mathbb{Q}}^F(m_1+m_2)}}{\prod_{j=1}^{r} \sigma_j(m_1 m_2)^{((k_j-1)/2)}} W_{m_1}^\phi(1)\overline{W_{m_2}^\phi(1)} = \frac{e^{4\pi r}}{d_F^2 \sqrt{\mathrm{Nm}(m_1 m_2)}} \lambda_{m_1\mathfrak{d}}(\phi)\overline{\lambda_{m_2\mathfrak{d}}(\phi)}.$$

□

Corollary 2.4.4. *Let $\mathfrak{p}$ be a prime ideal coprime to the level $\mathfrak{N}$. Let $k = (k_1, ..., k_r)$ with all $k_j > 2$ and $\ell \geq 0$. Let $\mathcal{F}$ be an orthogonal basis for $A_k(\mathfrak{N}, \omega)$ consisting of eigenfunctions for the Hecke operator $T_{\mathfrak{p}^\ell}$. Then, given $m \in \mathfrak{d}_+^{-1}$, we have*

$$\frac{e^{4\pi \mathrm{Tr}_{\mathbb{Q}}^F(m)}}{\psi(\mathfrak{N})} \left[\prod_{j=1}^r \frac{(k_j-2)!}{(4\pi\sigma_j(m))^{k_j-1}} \right] \sum_{\phi \in \mathcal{F}} \frac{\lambda_{\mathfrak{p}^\ell}(\phi) |W_m^\phi(1)|^2}{\|\phi\|^2} = \hat{T}(m, m, \mathfrak{p}^\ell) \frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{p}^\ell)}}{\omega_{\mathfrak{N}}(m/s)\omega_f(s)}$$
$$+ \sum_{i=1}^t \sum_{\substack{u \in U \\ \eta_i u \in F^+}} \sum_{\substack{s \in \mathfrak{b}_i \mathfrak{N}/\pm \\ s \neq 0}} \Bigg\{ \omega_f(s b_i^{-1}) S_{\omega_{\mathfrak{N}}}(m, m; \eta_i u b_i^{-2}; s b_i^{-1}) \frac{\sqrt{\mathrm{Nm}(\eta_i u)}}{\mathrm{Nm}(s)}$$
$$\times \prod_{j=1}^r \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\left(\frac{4\pi |\sigma_j(m)| \sqrt{\sigma_j(\eta_i u)}}{|\sigma_j(s)|} \right) \Bigg\}.$$

Proof. We obtain this corollary by taking $\mathfrak{n} = \mathfrak{p}^\ell$ and $m_1 = m_2 = m$ in Theorem 2.4.2.
□

3

Non-vanishing and bounds on Kloosterman sums

In this chapter, we discuss the nonvanishing of Kloosterman sums and a conditional upper bound for a local Kloosterman sum. The non-vanishing of a Kloosterman sum is an integral part of obtaining an asymptotic formula like Theorem 4.3.5. More precisely, for finding a lower bound for the main term in Theorem 4.3.5. The conditional upper bound is not used in deriving the asymptotic formula. However, the result gives a better bound as compared to the bound given by Lemma 6.1 of [KL08] whose proof essentially uses triangle inequality. Section 3.1 contains results regarding an upper bound for a local Kloosterman sum and Section 3.2 contains results about the nonvanishing of the classical Kloosterman sum.

In 1926, Kloosterman sums were introduced as an application to solve the following problem. Given natural numbers a, b, c, d and n, Kloosterman obtained an asymptotic formula in [Klo27] for the number of representations of n in the form $ax^2+by^2+cz^2+dt^2$. Given integer m, n and a natural number c, the Kloosterman sum is defined to be

$$S(m,n;c) = \sum_{x (mod\ c), \gcd(x,c)=1} e\left(\frac{mx+n\overline{x}}{c}\right)$$

where $\bar{x}$ denotes the multiplicative inverse of x modulo c and $e(x) = e^{2\pi i x}$. The sums for the special case of $m = 0$ or $n = 0$ are called Ramanujan sums. Kloosterman sums also appear in the Kuznetsov trace formula along with Petersson's trace formula. Therefore estimates for Kloosterman sums are important to consider. In 1948, Andre Weil [Wei48] gave a sharp bound for $S(a,b,p)$ where p is a prime number. By the multiplicativity property of Kloosterman sums, the bound is enough to show

$$|S(m,n,c)| \leq \tau(n)\sqrt{\gcd(m,n,c)}\sqrt{c}. \tag{3.1}$$

A slightly stronger bound is given by equation (2.13) of [ILS00]. For squarefree N, the main term in Theorem 5.1.7 is

$$2\pi i^{-k}\frac{\mu(N)}{N}\prod_{p|N}(1-p^{-2})J_{k-1}(4\pi\sqrt{mn}).$$

Note that $\mu(N) = S(1,0,N)$ (see equation (3.4) of [IK04]) which is nonzero if and only if N is squarefree. Thus in order to have a Kloosterman sum as the main term, we need to ensure the nonvanishing of that particular Kloosterman sum.

The definition of local and global Kloosterman sums for the totally real field F is given by equation (2.3) and (2.4). We start with a lemma for the nonvanishing of a Kloosterman sum satisfying certain conditions.

Lemma 3.0.1 ([BDS23], Lemma 13)**.** *Let $\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s} \in \mathbb{N}$, $\tilde{s}$ be squarefree and $\omega_{\mathfrak{N}}$ be trivial. Then*

$$S_{\omega_{\mathfrak{N}}}(m_1, m_2; 1; \tilde{s}) \neq 0.$$

Proof. We prove the lemma in two steps. Consider the prime factorisation of the ideal $\tilde{s}\mathcal{O} = \prod_{l=1}^{t'}\mathfrak{p}_l$ for distinct prime ideals $\mathfrak{p}_l$. Let ν_l be the valuation for the prime ideal $\mathfrak{p}_l$ and $p_l = \mathfrak{p}_l \cap \mathbb{Z}$. The first step shows,

$$e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right)\right) \in \mathbb{Z}\left[e^{\frac{2\pi i}{p_l}}\right].$$

In the second step we use this fact and method of contradiction to prove the claim.

Step (1)**.** Note that

$$S_{\omega_{\mathfrak{N}}}(m_1, m_2; 1; \tilde{s}) = \prod_{\nu<\infty} S_{\omega_{\mathfrak{N},\nu}}(m_{1\nu}, m_{2\nu}; 1; (\tilde{s}\mathcal{O})_\nu).$$

If ϖ_{ν_l} is a generator of the maximal ideal $(\mathfrak{p}_l)_{\nu_l} = \mathfrak{p}_l \mathcal{O}_{\nu_l}$, then we get

$$\prod_{\nu<\infty} S_{\omega_{\mathfrak{N},\nu}}(m_{1\nu}, m_{2\nu}; 1; (\tilde{s}\mathcal{O})_\nu) = \prod_{l=1}^{t'} S_{\omega_{\mathfrak{N},\nu_l}}(m_{1\nu_l}, m_{2\nu_l}; 1; \varpi_{\nu_l}).$$

We have $\langle p_l \rangle = \mathfrak{p}_l \cap \mathbb{Z}$ and $p_l\mathcal{O} = \mathfrak{p}_l \prod_{i=1}^{s'} \mathfrak{q}_i$ for distinct q_i. We have

$$p_l\mathcal{O}_{\nu_l} = \mathfrak{p}_l\mathcal{O}_{\nu_l} = \varpi_{\nu_l}\mathcal{O}_{\nu_l}.$$

Hence,

$$\begin{aligned} S_{\omega_{\mathfrak{N},\nu_l}}(m_{1\nu_l}, m_{2\nu_l}; 1; \varpi_{\nu_l}) &= \sum_{\substack{s_1,s_2\in\mathcal{O}_{\nu_l}/\varpi_{\nu_l}\mathcal{O}_{\nu_l} \\ s_1s_2\equiv 1 (\mathrm{mod}) \varpi_{\nu_l}\mathcal{O}_{\nu_l}}} \theta_{\nu_l}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right) \\ &= \sum_{\substack{s_1,s_2\in\mathcal{O}_{\nu_l}/\varpi_{\nu_l}\mathcal{O}_{\nu_l} \\ s_1s_2\equiv 1 (\mathrm{mod}) \varpi_{\nu_l}\mathcal{O}_{\nu_l}}} e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right)\right) \end{aligned}$$

where $e(x) = e^{2\pi i x}$. Consider the following

$$\left[e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right)\right)\right]^{p_l} = e\left(p_l \cdot \mathrm{Tr}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right)\right)$$

$$= e\left(\mathrm{Tr}\left(p_l \cdot \frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right)\right) = e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}p_ls_1 + m_{2\nu_l}p_ls_2}{\varpi_{\nu_l}}\right)\right)$$

$$= e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}p_ls_1}{\varpi_{\nu_l}}\right)\right) \cdot e\left(\mathrm{Tr}\left(\frac{m_{2\nu_l}p_ls_1}{\varpi_{\nu_l}}\right)\right)$$

However $p_l\mathcal{O}_{\nu_l} = \varpi_{\nu_l}\mathcal{O}_{\nu_l}$ imply that both $\left(\frac{m_{1\nu_l}p_ls_1}{\varpi_{\nu_l}}\right)$, $\left(\frac{m_{2\nu_l}p_ls_2}{\varpi_{\nu_l}}\right)$ belong to local inverse different. Since θ_ν is trivial on the local inverse different, we conclude

$$\left[e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right)\right)\right]^{p_l} = 1.$$

Step (2)**.** Suppose $S_{\omega_{\mathfrak{N},\nu_l}}(m_{1\nu_l}, m_{2\nu_l}; 1; \varpi_{\nu_l}) = 0$. Note that $\mathbb{Z}\left[e^{\frac{2\pi i}{p_l}}\right]$ is isomorphic to $\mathbb{Z}[x]/(\Phi_{p_l}(x))$ where $e^{\frac{2\pi i}{p_l}}$ gets mapped to $x + (\Phi_{p_l}(x))$. Further more $\mathbb{Z}[x]/(\Phi_{p_l}(x), p_l) =$

$\mathbb{Z}[x]/((x-1)^{p_l}, p_l)$ by Exercise 8 of section 13.6 of [DF04]. Consider the ring homomorphism given by

$$\mathbb{Z}[x]/(\Phi_{p_l}(x)) \to \mathbb{Z}[x]/((x-1)^{p_l}, p_l) \to \mathbb{Z}[x]/((x-1), p_l) \to \mathbb{F}_{p_l}$$

where

$$x + (\Phi_{p_l}(x)) \mapsto x + ((x-1)^{p_l}, p_l) \mapsto x + ((x-1), p_l)$$
$$= 1 + [x-1] + ((x-1), p_l) = 1 + ((x-1), p_l) \mapsto 1.$$

Thus $e^{\frac{2\pi i}{p_l}}$ gets mapped to 1 via the ring homomorphism. As shown earlier in step 1, $e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\varpi_{\nu_l}}\right)\right)$ lies in $\mathbb{Z}\left[e^{\frac{2\pi i}{p_l}}\right]$. Hence $e\left(\mathrm{Tr}\left(\frac{m_{1\nu_l}s_1 + m_{2\nu_l}s_2}{\pi_{\nu_l}}\right)\right)$ gets mapped to 1 via the ring homomorphism. This implies that $S_{\omega_{\mathfrak{N},\nu_l}}(m_{1\nu_l}, m_{2\nu_l}; 1; \varpi_{\nu_l})$ will get mapped to $p_l^f - 1 = -1 \in \mathbb{F}_{p_l}$ with $\left|\mathcal{O}_{\nu_l}/\varpi_{\nu_l}\mathcal{O}_{\nu_l}\right| = p_l^f$ for some natural number f. This is a contradiction as the image of $S_{\omega_{\mathfrak{N},\nu_l}}(m_{1\nu_l}, m_{2\nu_l}; 1; \varpi_{\nu_l})$ should be 0.

□

3.1 An upper bound for a local Kloosterman sum

We prove the following sharp bound for a special case of the local Kloosterman sum (see Lemma 1.3.1 or Lemma 6.1 of [KL08]). We assume the Hecke character to be trivial for the next two lemmas. Let $c_\nu = u_\nu \varpi_\nu^k$ for a fixed uniformizer ϖ_ν and a unit u_ν and $\mathfrak{d}_\nu^{-1} = \langle \varpi_\nu^{-\delta} \rangle$ for some $\delta \geq 0$. Further let $\varpi_\nu^\delta m_{1\nu} = m'_{1\nu}$ and $\varpi_\nu^\delta m_{2\nu} = m'_{2\nu}$ where $m'_{1\nu}, m'_{2\nu} \in \mathcal{O}_\nu$. Let $\overline{u_\nu}$ denote the multiplicative inverse of u_ν in $\mathcal{O}_\nu/c_\nu\mathcal{O}_\nu$. Note that $S_\nu(m_{1\nu}, m_{2\nu}; c_\nu) = S_\nu(\overline{u_\nu}m_{1\nu}, \overline{u_\nu}m_{2\nu}; 1; \varpi_\nu^k)$. Thus it is sufficient to prove bounds for $S_\nu(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k)$ since $\mathrm{Nm}(\overline{u_\nu}) = 1$. First, we consider an analogue of Ramanujan sum which is $S(m_{1\nu}, 0; 1; \varpi_\nu^k)$. We consider three cases to show the following bound.

Lemma 3.1.1. *We have*

$$|S(m_{1\nu}, 0; 1; c_\nu)| \leq \tau(\mathrm{Nm}(c_\nu))\sqrt{\gcd(\mathrm{Nm}(m'_{1\nu}), \mathrm{Nm}(c_\nu))}\sqrt{\mathrm{Nm}(c_\nu)}$$

if $\mathrm{Nm}(c_\nu)$ *is an odd natural number.*

Proof. If $c_\nu\mathcal{O}_\nu$ is a prime ideal for some ν, then $\mathcal{O}_\nu/c_\nu\mathcal{O}_\nu$ is a finite field with $\mathrm{Nm}(c_\nu)$ elements. Now using 11.11 (Weil Theorem) of [IK04], we have $|S(m_{1\nu},0;1;c_\nu)| \leq 2\sqrt{\mathrm{Nm}(c_\nu)}$. Thus we focus when $c_\nu\mathcal{O}_\nu$ is not necessarily a prime ideal. We now consider three cases as per the value of $\nu(m_{1\nu})$ to prove the lemma.

Case (1)**.** When $\nu(m_{1\nu}) \geq k-\delta$.
We have $\nu(m_{1\nu}s_1\varpi_\nu^{-k}) \geq -\delta$ which implies θ_ν becomes trivial for each term in the sum, so $|S(m_{1\nu},0;1;\varpi_\nu^k)| = |(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times|$. In such case, $\nu(m'_{1\nu}) \geq k$, which implies $\gcd(\mathrm{Nm}(m'_{1\nu}),\mathrm{Nm}(\varpi_\nu^k)) = \mathrm{Nm}(\varpi_\nu^k)$. Hence

$$\tau(\mathrm{Nm}(\varpi_\nu^k))\sqrt{\gcd(\mathrm{Nm}(m'_{1\nu}),\mathrm{Nm}(\varpi_\nu^k))}\sqrt{\mathrm{Nm}(\varpi_\nu^k)}$$

$$= \tau(\mathrm{Nm}(\varpi_\nu^k))\mathrm{Nm}(\varpi_\nu^k) \geq \mathrm{Nm}(\varpi_\nu^k).$$

Thus

$$|S(m_{1\nu},0;1;\varpi_\nu^k)| = |(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times| \leq \mathrm{Nm}(\varpi_\nu^k)$$
$$\leq \tau(\mathrm{Nm}(\varpi_\nu^k))\sqrt{\gcd(\mathrm{Nm}(m'_{1\nu}),\mathrm{Nm}(\varpi_\nu^k))}\sqrt{\mathrm{Nm}(\varpi_\nu^k)}.$$

Case (2)**.** When $\nu(m_{1\nu}) = k-\delta-1$.
we have $s_1 \in (\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times$ if and only if $\gcd(s_1,\varpi_\nu)=1$. Hence

$$(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times =$$

$$(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu) \setminus \{s'_1 \in (\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu) \mid s'_1 = \varpi_\nu s''_1, s''_1 \in (\mathcal{O}_\nu/\varpi_\nu^{k-1}\mathcal{O}_\nu)\}.$$

Therefore

$$S(m_{1\nu},0;1;\varpi_\nu^k) = \sum_{s_1\in(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times} \theta_\nu\Big(\frac{m_{1\nu}s_1}{\varpi_\nu^k}\Big)$$
$$= \sum_{s_1\in(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)} \theta_\nu\Big(\frac{m_{1\nu}s_1}{\varpi_\nu^k}\Big) - \sum_{s_1\in(\mathcal{O}_\nu/\varpi_\nu^{k-1}\mathcal{O}_\nu)} \theta_\nu\Big(\frac{m_{1\nu}s_1}{\varpi_\nu^{k-1}}\Big).$$

Now we try to see that

$$\sum_{s_1\in(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)} \theta_\nu\Big(\frac{m_{1\nu}s_1}{\varpi_\nu^k}\Big) = 0$$

On taking $\nu(s_1)=0$ in particular, we get $\nu(\frac{m_{1\nu}s_1}{\varpi_\nu^k}) = -\delta-1$. This makes the required sum to be a nontrivial character sum over an additive group which is equal to 0. The second sum is given by

$$\sum_{s_1 \in (\mathcal{O}_\nu/\varpi_\nu^{k-1}\mathcal{O}_\nu)} \theta_\nu\left(\frac{m_{1\nu}s_1}{\varpi_\nu^{k-1}}\right) = \mathrm{Nm}(\varpi_\nu^{k-1})$$

since $\nu(\frac{m_{1\nu}s_1}{\varpi_\nu^{k-1}}) \geq -\delta$ and θ_ν becomes trivial for each term in the sum. Finally $S(m_{1\nu}, 0; 1; \varpi_\nu^k) = -\mathrm{Nm}(\varpi_\nu^{k-1})$. In this case $\nu(m'_{1\nu}) = k-1$, $\gcd(\mathrm{Nm}(m'_{1\nu}), \mathrm{Nm}(\varpi_\nu^k)) = \mathrm{Nm}(\varpi_\nu^{k-1})$ so that

$$\tau(\mathrm{Nm}(\varpi_\nu^k))\sqrt{\gcd(\mathrm{Nm}(m'_{1\nu}), \mathrm{Nm}(\varpi_\nu^k))}\sqrt{\mathrm{Nm}(\varpi_\nu^k)}$$

$$= \tau(\mathrm{Nm}(\varpi_\nu^k))\mathrm{Nm}(\varpi_\nu^k)\mathrm{Nm}(\varpi_\nu)^{-\frac{1}{2}} \geq \mathrm{Nm}(\varpi_\nu^{k-1}).$$

Case (3)**.** When $-\delta \leq \nu(m_{1\nu}) \leq k - \delta - 2$.

$$S(m_{1\nu}, 0; 1; \varpi_\nu^k) = \sum_{s_1 \in (\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times} \theta_\nu\left(\frac{m_{1\nu}s_1}{\varpi_\nu^k}\right)$$

$$= \sum_{s_1 \in (\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)} \theta_\nu\left(\frac{m_{1\nu}s_1}{\varpi_\nu^k}\right) - \sum_{s_1 \in (\mathcal{O}_\nu/\varpi_\nu^{k-1}\mathcal{O}_\nu)} \theta_\nu\left(\frac{m_{1\nu}s_1}{\varpi_\nu^{k-1}}\right).$$

Similar to the previous case

$$\sum_{s_1 \in (\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)} \theta_\nu\left(\frac{m_{1\nu}s_1}{\varpi_\nu^k}\right) = 0$$

as we can take s_1 such that $\nu(s_1) = 0$. Choosing s'_1 such that $\nu(s'_1) = 0$, we get $\nu(\frac{m_{1\nu}s'_1}{\varpi_\nu^{k-1}}) = -\delta - 1$. Hence

$$\sum_{s'_1 \in (\mathcal{O}_\nu/\varpi_\nu^{k-1}\mathcal{O}_\nu)} \theta_\nu\left(\frac{m_{1\nu}s'_1}{\varpi_\nu^{k-1}}\right) = 0$$

and $S(m_{1\nu}, 0; 1; \varpi_\nu^k) = 0$.

□

Theorem 3.1.2. *We have*

$$|S(m_{1\nu}, m_{2\nu}; 1; c_\nu)| \leq \tau(\mathrm{Nm}(c_\nu))\sqrt{\gcd(\mathrm{Nm}(m'_{1\nu}), \mathrm{Nm}(m'_{2\nu}), \mathrm{Nm}(c_\nu))}\sqrt{\mathrm{Nm}(c_\nu)}$$

if $\mathrm{Nm}(c_\nu)$ *is an odd natural number.*

Proof. If $c_\nu\mathcal{O}_\nu$ is a prime ideal, then using Theorem 11.11 (Weil Theorem) of [IK04] again we get

$$|S(m_{1\nu}, m_{2\nu}; 1; c_\nu)| \le 2\sqrt{\mathrm{Nm}(c_\nu)}.$$

Note that $S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) = S(m'_{1\nu}, m'_{2\nu}; 1; \varpi_\nu^{k+\delta})$. Now we consider three cases to prove the theorem.

Case (1)**.** When ϖ_ν neither divides $m'_{1\nu}$ nor $m'_{2\nu}$.
For a given $s \in (\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times$, let $\overline{s}$ be such that $s\overline{s} \equiv 1 \pmod{\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times}$. Let us consider the following sum for a given function f on $(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times$.

$$\sum_{s_1\in(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times} f(s_1) = \sum_{r\in(\mathcal{O}_\nu/\varpi_\nu^n\mathcal{O}_\nu)^\times} \sum_{t\in\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu} f(r+\varpi_\nu^n t),$$

with $\frac{k}{2} \le n < k$. If $\frac{k}{2} \le n < k$, then $\overline{(r+\varpi_\nu^n t)} = \overline{r} - \overline{r}^2\varpi_\nu^n t$. This is because

$$(r+\varpi_\nu^n t)(\overline{r} - \overline{r}^2\varpi_\nu^n t) = r\overline{r} + \overline{r}\varpi_\nu^n t(1-r\overline{r}) + \overline{r}^2\varpi_\nu^{2n}t = 1 + \overline{r}\varpi_\nu^n t(0) + 0 = 1.$$

Taking $f(s_1) = \theta_\nu\left(\frac{m_{1\nu}s_1 + m_{2\nu}\overline{s_1}}{\varpi_\nu^k}\right)$ in the identity mentioned we get

$$\begin{aligned} S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) &= \sum_{r\in(\mathcal{O}_\nu/\varpi_\nu^n\mathcal{O}_\nu)^\times} \sum_{t\in\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu} \theta_\nu\left(\frac{(r+\varpi_\nu^n t)m_{1\nu} + \overline{(r+\varpi_\nu^n t)}m_{2\nu}}{\varpi_\nu^k}\right) \\ &= \sum_{r\in(\mathcal{O}_\nu/\varpi_\nu^n\mathcal{O}_\nu)^\times} \sum_{t\in\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu} \theta_\nu\left(\frac{(r+\varpi_\nu^n t)m_{1\nu} + (\overline{r} - \overline{r}^2\varpi_\nu^n t)m_{2\nu}}{\varpi_\nu^k}\right) \\ &= \sum_{r\in(\mathcal{O}_\nu/\varpi_\nu^n\mathcal{O}_\nu)^\times} \theta_\nu\left(\frac{rm_{1\nu} + \overline{r}m_{2\nu}}{\varpi_\nu^k}\right) \sum_{t\in\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu} \theta_\nu\left(\frac{(m'_{1\nu} - \overline{r}^2 m'_{2\nu})t}{\varpi_\nu^{k-n+\delta}}\right) \end{aligned}$$

Now let us focus on the inner sum. The inner sum is $\mathrm{Nm}(\varpi_\nu^{k-n})$ or 0 according to $\nu(m'_{1\nu} - \overline{r}^2 m'_{2\nu}) \ge k-n$ or not respectively. Hence

$$|S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k)| \le \mathrm{Nm}(\varpi_\nu^{k-n}) \sum_{\substack{r\in(\mathcal{O}_\nu/\varpi_\nu^n\mathcal{O}_\nu)^\times \\ m'_{1\nu} - \overline{r}^2 m'_{2\nu} \in \mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu}} 1$$

$$= \mathrm{Nm}(\varpi_\nu^n) \sum_{\substack{r \in (\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu)^\times \\ m'_{1\nu} - \bar{r}^2 m'_{2\nu} \in \mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu}} 1$$

In the last summation contribution is 0 if ϖ_ν divides either $m'_{1\nu}$ or $m'_{2\nu}$. Therefore we need to consider when $r^2 \equiv m'_{2\nu}\overline{m'_{1\nu}} \pmod{\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu}$. The number of solutions of this equation in r is bounded by 2 whenever $\mathrm{Nm}(\varpi_\nu)$ is odd. To see this we consider the following. Let $r^2 = r'^2 \pmod{\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu}$ with $r = a + b\varpi_\nu, r' = a' + b'\varpi_\nu$ where $a, a' \in \mathcal{O}_\nu/\varpi_\nu\mathcal{O}_\nu$ and $b, b' \in \mathcal{O}_\nu/\varpi_\nu^{k-n-2}\mathcal{O}_\nu$. We have

$$(a^2 - a'^2) + 2(b - b')\varpi_\nu + (b^2 - b'^2)\varpi_\nu^2 = 0$$

modulo $\mathcal{O}_\nu/\varpi_\nu^{k-n}\mathcal{O}_\nu$. This implies $b = b'$ as $\mathrm{Nm}(\varpi_\nu)$ odd and $a^2 = a'^2$. Let us write $a = \alpha + \beta p$ and $a' = \alpha' + \beta' p$, where $\mathrm{Nm}(\varpi_\nu) = p^f$ with $\alpha, \alpha' \in \mathbb{Z}_p/p\mathbb{Z}_p$. Proceeding similarly as above we end up getting $\beta = \beta'$ and $, \alpha^2 = \alpha'^2$. Hence we can have two possibilities given by $\alpha' = \alpha$, and $\alpha' = -\alpha$. Finally on taking $n = \frac{k}{2}$, we get

$$|S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k)| \leq 2\sqrt{\mathrm{Nm}(\varpi_\nu^k)}.$$

Case (2)**.** When ϖ_ν divides $m'_{2\nu}$ and ϖ_ν does not divide $m'_{1\nu}$.
We have $\gcd(m'_{1\nu}, \varpi_\nu) = 1$ which implies $\gcd(m'_{1\nu}, \varpi_\nu^k) = 1$. We claim that $\{s'_1 + m'_{2\nu}m'_{1\nu}\overline{s'_1} \mid \gcd(s'_1, \varpi_\nu^k) = 1\}$ sums over all elements of $(\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times$. Note that $\gcd(s'_1 + m'_{2\nu}m'_{1\nu}\overline{s'_1}, \varpi_\nu) = 1$. To see this suppose $\gcd(s'_1 + m'_{2\nu}m'_{1\nu}\overline{s'_1}, \varpi_\nu) \neq 1$, then ϖ_ν divides $m'_{2\nu}m'_{1\nu}\overline{s'_1}$. This implies ϖ_ν divides s'_1 since ϖ_ν divides $m'_{2\nu}$. This shows $\gcd(s'_1, \varpi_\nu) \neq 1$ which contradicts our assumption.

Let $s'_1 \neq t'_1 \in \mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu$ and $\gcd(s'_1, \varpi_\nu) = \gcd(t'_1, \varpi_\nu) = 1$. Our claim will be proved once we show that $s'_1 + m'_{2\nu}m'_{1\nu}\overline{s'_1} \neq t'_1 + m'_{2\nu}m'_{1\nu}\overline{t'_1} \in (\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu)^\times$. On the contrary, suppose $s'_1 + m'_{2\nu}m'_{1\nu}\overline{s'_1} = t'_1 + m'_{2\nu}m'_{1\nu}\overline{t'_1}$. Then ϖ_ν^k divides

$$(s'_1 - t'_1) + m'_{2\nu}m'_{1\nu}(\overline{s'_1} - \overline{t'_1}) = (s'_1 - t'_1) + m'_{2\nu}m'_{1\nu}(t'_1\overline{t'_1 s'_1} - s'_1\overline{s'_1 t'_1})$$

$$= (s'_1 - t'_1)(1 - m'_{2\nu}m'_{1\nu}\overline{s'_1 t'_1}).$$

Since ϖ_ν divides $m'_{2\nu}$, ϖ_ν does not divide $(1 - m'_{2\nu}m'_{1\nu}\overline{s'_1 t'_1})$. Hence ϖ_ν divides $s'_1 - t'_1$, which is a contradiction to the assumption $s'_1 \neq t'_1$ in $\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu$.

Therefore by taking $s_1' = m_{1\nu} s_1$ we have

$$S_\nu(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) = \sum_{\substack{s_1', s_2' \in \mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu \\ s_1' s_2' \equiv 1 \,(\mathrm{mod})\, \varpi_\nu^k \mathcal{O}_\nu}} \theta_\nu\left(\frac{s_1' + m_{2\nu}' m_{1\nu}' \overline{s_1'}}{\varpi_\nu^{k+\delta}}\right)$$

$$\sum_{\substack{s_1'', s_2'' \in \mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu \\ s_1'' s_2'' \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{s_1''}{\varpi_\nu^{k+\delta}}\right) = \sum_{\substack{s_1'', s_2'' \in \mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu \\ s_1'' s_2'' \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{m_{1\nu}' s_1''}{\varpi_\nu^{k+\delta}}\right) = S_\nu(m_{1\nu}, 0; 1; \varpi_\nu^k).$$

This is an analogue of the Ramanujan sum for which the bound is given by Lemma 3.1.1.

Case (3)**.** When ϖ_ν divides both $m_{1\nu}'$ and $m_{2\nu}'$.
We have four sub-cases for this case. Let $m_{1\nu}' = t_1 \varpi_\nu^{k_1}$ and $m_{2\nu}' = t_2 \varpi_\nu^{k_2}$ with $\gcd(t_1, \varpi_\nu) = \gcd(t_2, \varpi_\nu) = 1$. Without loss of generality we can assume $k_2 \geq k_1$ since $S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) = S(m_{2\nu}, m_{1\nu}; 1; \varpi_\nu^k)$.

Subcase (1)**.** When $k_2 \geq k_1 \geq k$.
We have

$$S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) = \sum_{\substack{s_1 \in \mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{t_1 \varpi_\nu^{k_1} s_1 + t_2 \varpi_\nu^{k_2} \overline{s_1}}{\varpi_\nu^{k+\delta}}\right)$$

$$= \sum_{\substack{s_1 \in \mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(t_1 s_1 \varpi_\nu^{k_1 - k - \delta} + t_2 \varpi_\nu^{k_2 - k - \delta} \overline{s_1}\right)$$

Note that $\nu(t_1 s_1 \varpi_\nu^{k_1 - k - \delta}) \geq -\delta$ and $\nu(t_2 s_2 \varpi_\nu^{k_1 - k - \delta}) \geq -\delta$ and θ_ν is trivial on $\mathfrak{d}_\nu^{-1}$. The above sum hence reduces to

$$\sum_{\substack{s_1 \in \mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} 1 = |(\mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu)^\times|.$$

Therefore $\gcd(m_{1\nu}, m_{2\nu}, \varpi_\nu^k) = \varpi_\nu^k$ implies

$$|S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k)| = |(\mathcal{O}_\nu/\varpi_\nu^k \mathcal{O}_\nu)^\times| \leq \mathrm{Nm}(\varpi_\nu^k)$$

$$\leq \tau(\mathrm{Nm}(\varpi_\nu^k)) \gcd(\mathrm{Nm}(m_{1\nu}), \mathrm{Nm}(m_{2\nu}), \mathrm{Nm}(\varpi_\nu^k)) \sqrt{\mathrm{Nm}(\varpi_\nu^k)}.$$

Subcase (2)**.** When $k > k_1 = k_2$.

We have

$$\begin{aligned} S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) &= \sum_{\substack{s_1 \in \mathcal{O}_\nu / \varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{t_1 \varpi_\nu^{k_1} s_1 + t_2 \varpi_\nu^{k_1} \overline{s_1}}{\varpi_\nu^{k+\delta}} \right) \\ &= \sum_{\substack{s_1 \in \mathcal{O}_\nu / \varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{t_1 s_1 + t_2 \overline{s_1}}{\varpi_\nu^{k-k_1+\delta}} \right) = S(t_1, t_2, \varpi_\nu^{k-k_1+\delta}). \end{aligned}$$

Since $\gcd(t_1, \varpi_\nu) = \gcd(t_2, \varpi_\nu) = 1$ reduces to the case 1.

Subcase (3)**.** When $k \geq k_2 > k_1$.
We have

$$\begin{aligned} S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) &= \sum_{\substack{s_1 \in \mathcal{O}_\nu / \varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{t_1 \varpi_\nu^{k_1} x + t_2 \varpi_\nu^{k_2} \overline{s_1}}{\varpi_\nu^{k+\delta}} \right) \\ &= \sum_{\substack{s_1 \in \mathcal{O}_\nu / \varpi_\nu^{k+\delta} \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{t_1 s_1 + t_2 \varpi_\nu^{k_2-k_1} \overline{s_1}}{\varpi^{k-k_1+\delta}} \right) = S(t_1, t_2 \varpi_\nu^{k_2-k_1}; 1; \varpi_\nu^{k-k_1+\delta}) \end{aligned}$$

This reduces to Case 2. This is because $\gcd(t_1, \varpi_\nu) = 1$ is equivalent to ϖ_ν does not divide t_1 and ϖ_ν divides $t_2 \varpi_\nu^{k_2-k_1}$.

Subcase (4)**.** When $k_2 > k > k_1$.
We have

$$\begin{aligned} S(m_{1\nu}, m_{2\nu}; 1; \varpi_\nu^k) &= \sum_{\substack{s_1 \in \mathcal{O}_\nu / \varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(\frac{t_1 \varpi_\nu^{k_1} s_1 + t_2 \varpi_\nu^{k_2} \overline{s_1}}{\varpi_\nu^{k+\delta}} \right) \\ &= \sum_{\substack{s_1 \in \mathcal{O}_\nu / \varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(t_2 \varpi_\nu^{k_2-k-\delta} \overline{s_1} + \frac{t_1 s_1}{\varpi_\nu^{k-k_1+\delta}} \right) \\ &= \sum_{\substack{s_1 \in \mathcal{O}_\nu / \varpi_\nu^k \mathcal{O}_\nu \\ s_1 s_2 \equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k \mathcal{O}_\nu)}} \theta_\nu\left(t_2 \varpi_\nu^{k_2-k-\delta} \overline{s_1} \right) \theta_\nu\left(\frac{t_1 s_1}{\varpi_\nu^{k-k_1+\delta}} \right) \end{aligned}$$

since $\nu(t_2\varpi_\nu^{k_2-k-\delta}\overline{s_1}) \geq -\delta$ the sum simplifies to

$$\sum_{\substack{s_1\in\mathcal{O}_\nu/\varpi_\nu^k\mathcal{O}_\nu \\ s_1s_2\equiv 1 \,(\mathrm{mod}\, \varpi_\nu^k\mathcal{O}_\nu)}} \theta_\nu\left(\frac{t_1s_1}{\varpi_\nu^{k-k_1+\delta}}\right).$$

This subcase reduces to an analogue of the Ramanujan for which the bound is given by Lemma 3.1.1.

□

Remark 3.1.3. *We follow [Est61] for the argument given in Case(1) for the proof of Theorem 3.1.2.*

3.2 Non-vanishing of classical Kloosterman sums

Let p be a prime number and $N = q_1^{\beta_1} \ldots q_t^{\beta_t}$ where $q_1, \ldots, q_t$ are distinct prime numbers with $(p, q_i) = 1$ for all $i = 1, \ldots t$. Let $q_i^{\beta_i} = l_i$ and

$$c_i = \frac{N}{\prod_{j=1}^{i} l_j}$$

for $i \in \{1, \ldots, t\}$. Further let $m_0 = 1$, $m_i = \prod_{j=1}^{i} \overline{l_j}^{c_j}$ for $i \in \{1, \ldots, t-1\}$ where $l_j\overline{l_j}^{c_j} \equiv 1 \pmod{c_j}$ and $c_j\overline{c_j}^{l_j} \equiv 1 \pmod{l_j}$. Now we demonstrate a lemma regarding the multiplicative property of the Kloosterman sum. The lemma helps us to show the non-vanishing of a Kloosterman sum by reducing the task to show the nonvanishing for prime powers.

Lemma 3.2.1 ([Das23], Lemma 3.1)**.** *Let n be a positive integer. Then*

$$S(1, p^{2n}, N) = \prod_{i=1}^{t} S(m_{i-1}\overline{c_i}^{l_i}, m_{i-1}\overline{c_i}^{l_i}p^{2n}, l_i).$$

Proof. We use the multiplicative property of the Kloosterman sum (see equation (1.59) of [IK04]) repetitively which is given by

$$S(a,b,cd) = S(a\overline{c}^d, b\overline{c}^d, d)S(a\overline{d}^c, b\overline{d}^c, c) \tag{3.2}$$

where $(c,d)=1$, $c\overline{c}^d \equiv 1 \pmod d$ and $d\overline{d}^c \equiv 1 \pmod c$. Consider

$$S(m_{t-1}, m_{t-1}p^{2n}, l_t)S(m_{t-2}\overline{c_{t-1}}^{l_{t-1}}, m_{t-2}\overline{c_{t-1}}^{l_{t-1}}p^{2n}, l_{t-1})$$

$$= S(m_{t-2}\overline{l_{t-1}}^{c_{t-1}}, m_{t-2}\overline{l_{t-1}}^{c_{t-1}}p^{2n}, l_t)S(m_{t-2}\overline{c_{t-1}}^{l_{t-1}}, m_{t-2}\overline{c_{t-1}}^{l_{t-1}}p^{2n}, l_{t-1}).$$

As $\overline{c_{t-1}}^{l_{t-1}} l_t = \overline{l_t}^{l_{t-1}} l_t = 1 \pmod{l_{t-1}}$ and $\overline{l_{t-1}}^{c_{t-1}} l_{t-1} = 1 \pmod{c_{t-1}}$, on using equation (3.2) the above product is equal to

$$S(m_{t-2}, m_{t-2}p^{2n}, l_{t-1}l_t).$$

Again using equation (3.2),

$$S(m_{t-i}, m_{t-i}p^{2n}, l_{t-i+1}\dots l_t)S(m_{t-i-1}\overline{c_{t-i}}^{l_{t-i}}, m_{t-i-1}\overline{c_{t-i}}^{l_{t-i}}p^{2n}, l_{t-i})$$

$$= S(m_{t-i-1}, m_{t-i-1}p^{2n}, l_{t-i}l_{t-i+1}\dots l_t)$$

for any $i \in \{2, \dots, t\}$. Now we justify why equation (3.2) is applicable. This is because

$$\overline{c_{t-i}}^{l_{t-i}}(l_{t-i+1}\dots l_t) = \overline{\left(\frac{N}{l_1\dots l_{t-i}}\right)}^{l_{t-i}}(l_{t-i+1}\dots l_t)$$

$$= \overline{(l_{t-i+1}\dots l_t)}^{l_{t-i}}(l_{t-i+1}\dots l_t) = 1$$

modulo l_{t-i}) and $m_{t-i} = m_{t-i-1}\overline{l_{t-i}}^{c_{t-i}}$. Therefore

$$S(1,p^{2n},N) = \prod_{i=1}^{t} S(m_{i-1}\overline{c_i}^{l_i}, m_{i-1}\overline{c_i}^{l_i}p^{2n}, l_i).$$

□

Remark 3.2.2. *A similar result like Lemma 3.2.1 for $S(a,b,N)$ with $a,b \in \mathbb{N}$ can be derived. However, keeping equation (6.6) in mind, we consider only $S(1,p^{2n},N)$.*

The next lemma considers the non-vanishing of the Kloosterman sum when N is an odd powerful number.

Lemma 3.2.3 ([Das23], Lemma 3.2)**.** *Let $n \in \mathbb{N}$, $\beta_1, \dots, \beta_t$ be such that $\beta_i > 1$ and q_i is odd for all i. Then $S(1, p^{2n}, N) \neq 0$.*

Proof. Let i be fixed with $l_i = q_i^{\beta_i}$. Using Lemma 3.2.1, it is sufficient to show that

$$S(m_{i-1}\overline{c_i}^{l_i}, m_{i-1}\overline{c_i}^{l_i}p^{2n}, l_i) \neq 0$$

for all i. Firstly, we show that q_i does not divide $2(m_{i-1}\overline{c_i}^{l_i})^2 p^{2n}$. This is achieved in Step 1 and Step 2.

Step (1)**.** If possible, suppose $q_i | 2(m_{i-1}\overline{c_i}^{l_i})^2 p^{2n}$. Note q_i divides $(m_{i-1}\overline{c_i}^{l_i})^2$ as $q_i \nmid 2$ and $q_i \nmid p^{2n}$. Now suppose $q_i | (\overline{c_i}^{l_i})^2$. $c_i\overline{c_i}^{l_i} \equiv 1 \pmod{l_i}$ implies

$$c_i^{2\beta_i}(\overline{c_i}^{l_i})^{2\beta_i} \equiv 1 \pmod{l_i}. \tag{3.3}$$

However by our assumption $q_i | (\overline{c_i}^{l_i})^2$. Hence $q_i^{\beta_i} | (\overline{c_i}^{l_i})^{2\beta_i}$ which implies

$$q_i^{\beta_i} | c_i^{2\beta_i}(\overline{c_i}^{l_i})^{2\beta_i}.$$

By equation (3.3), $q_i^{\beta_i} | 1$ a contradiction. Therefore $q_i | m_{i-1}^2$. q_i being a prime number implies $q_i | m_{i-1}$ where $m_{i-1} = \prod_{j=1}^{i-1} \overline{l_j}^{c_j}$ which has been defined earlier in the section.

Step (2)**.** In this step we show that $q_i \nmid m_{i-1}$ using the method of contradiction. The case $i = 1$ is impossible as this would mean q_1 divides $m_0 = 1$. Let $l_j\overline{l_j}^{c_j} \equiv 1 \pmod{c_j}$ denote the equation (j) for $j = 1, 2, \dots, i-1$. Multiplying $(l_2 \dots l_j)$ for $j = 2, \dots n-1$ gives

$$l_j\overline{l_j}^{c_j}(l_2 \dots l_j) \equiv (l_2 \dots l_j) \pmod{(l_2 \dots l_j c_j)}$$
$$\Rightarrow l_j\overline{l_j}^{c_j}(l_2 \dots l_j) \equiv (l_2 \dots l_j) \pmod{c_1}$$

let the equation be $(j)'$ for $j = 2, \dots, i-1$. On considering equations $(1), (2)', \dots, (i-1)'$ which are modulo c_1 and multiplying them yields

$$(l_1 \dots l_{i-1})(\overline{l_1}^{c_1} \dots \overline{l_{i-1}}^{c_{i-1}})(l_2^{i-2}l_3^{i-3} \dots l_{i-1}) \equiv (l_2^{i-2}l_3^{i-3} \dots l_{i-1}) \pmod{c_1}. \tag{3.4}$$

Since $q_i | c_1$ for $i \geq 2$, q_i divides

$$(l_1 \ldots l_{i-1})(\overline{l_1}^{c_1} \ldots \overline{l_{i-1}}^{c_{i-1}})(l_2^{i-2} l_3^{i-3} \ldots l_{i-1}) - (l_2^{i-2} l_3^{i-3} \ldots l_{i-1}).$$

But the assumption $q_i | m_{i-1}$ implies $q_i | l_2^{i-2} l_3^{i-3} \ldots l_{i-1}$. In particular, $\gcd(q_i, l_2^{i-2} l_3^{i-3} \ldots l_{i-1}) = q_i$ which is absurd since $\gcd(q_i, l_j) = 1$ for $j = 2, \ldots, i-1$. Hence q_i does not divide m_{i-1}.

Step (3)**.** By Step 1 and Step 2, the Kloosterman sum $S(m_{i-1}\overline{c_i}^{l_i}, m_{i-1}\overline{c_i}^{l_i} p^{2n}, l_i)$ satisfy the hypothesis for Exercise 1 of [IK04, Chapter 12]. Let $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. Using the exercise,

$$S(m_{i-1}\overline{c_i}^{l_i}, m_{i-1}\overline{c_i}^{l_i} p^{2n}, l_i) = 2\left(\frac{l'}{l_i}\right)\sqrt{l_i}\,\Re\Big(\epsilon_{l_i} e^{\frac{4\pi i l'}{l_i}}\Big)$$

where $l'^2 \equiv (m_{i-1}\overline{c_i}^{l_i} p^n)^2 \pmod{l_i}$, $\epsilon_{l_i} = 1$ or i for $l_i \equiv 1$ or $l_i \equiv 3 \pmod 4$ respectively. By the congruence relation, $\left(\frac{l'}{l_i}\right) = 1$.
We show $l' \notin \{\frac{\alpha l_i}{8} \,|\, \alpha \in \mathbb{Z}\}$. Suppose not, then $l' = \frac{\alpha_0 l_i}{8}$ for some integer α_0. The congruence relation implies

$$\left(\frac{\alpha_0 l_i}{8}\right)^2 = (m_{i-1}\overline{c_i}^{l_i} p^n)^2 + l_i t$$

with $t \in \mathbb{Z}$. Equivalently,

$$\alpha_0^2 l_i^2 - 64 l_i t = 64(m_{i-1}\overline{c_i}^{l_i} p^n)^2.$$

Now q_i divides $64(m_{i-1}\overline{c_i}^{l_i} p^n)^2$ as q_i divides l_i. However $q_i \nmid 64p^{2n}$ and $q_i \nmid \overline{c_i}^{l_i}$ imply $q_i | m_{i-1}^2$. This is a contradiction by following arguments for $q_i \nmid m_{i-1}^2$ in step 2. Since $\frac{l'}{l_i} \notin \{\frac{\alpha}{8} \,|\, \alpha \in \mathbb{Z}\}$, $\cos\left(\frac{4\pi l'}{l_i}\right)$ and $\sin\left(\frac{4\pi l'}{l_i}\right)$ are not equal to zero implying $e^{\frac{4\pi i l'}{l_i}} \notin \mathbb{R} \cup \{iy : y \in \mathbb{R}\}$. Hence

$$\Re\Big(\epsilon_{l_i} e^{\frac{4\pi i l'}{l_i}}\Big) \neq 0$$

and the proof is complete.

□

Lemma 3.2.4. *Let p be an odd prime number. Then $S(1,a,p) \neq 0$ for $a \in \mathbb{Z}$.*

Proof. The proof is quite similar to Step 2 of proof for Lemma 3.0.1. □

Lemma 3.2.5. *Let N be a natural number given by $2^a b$ with $a = 0,1,2$ and b an odd number. If $n \in \mathbb{N}$, then $S(1,p^{2n},N) \neq 0$.*

Proof. Lemma 3.2.1 implies

$$S(1,p^{2n},N) = \prod_{i=1}^{t} S(m_{i-1}\overline{c_i}^{l_i}, m_{i-1}\overline{c_i}^{l_i}p^{2n}, l_i) = S(1,(m_{i-1}\overline{c_i}^{l_i}p^n)^2, l_i).$$

When l_i is odd, $S(1,(m_{i-1}\overline{c_i}^{l_i}p^n)^2, l_i) \neq 0$ by Lemma 3.2.3 and Lemma 3.2.4. We now focus when l_i is an even number. Note that $m_{i-1}\overline{c_i}^{l_i}p^n$ is odd follows from the proof of Lemma 3.2.3. We have $S(1,a^2,2^b) = S(1,1,2^b)$ for a odd and $b = 1,2,3$. On computing the values of $S(1,1,2), S(1,1,4)$ and $S(1,1,8)$ are equal to $1,-2$ and 0 respectively. Hence for $l_i = 2,4$, we also have $S(1,(m_{i-1}\overline{c_i}^{l_i}p^n)^2, l_i) \neq 0$. □

One can compute $S(1,p^{2n},2^a)$ for $a > 3$ and check for the nonvanishing and vanishing of it independently. We do it till a=3 as $S(1,1,8) = 0$.

4

An asymptotic formula for Petersson trace formula

In this chapter, we derive an asymptotic formula for the Petersson formula under certain conditions. This is achieved in Section 4.2. Section 4.1 contains preliminary lemmas and the statement of the asymptotic formula. Section 4.3 considers the asymptotic formula under special cases. As an application, we generalize equation (1.5) (Jung and Sardari's bound for $D(\nu^*_{k_n,N}, \mu_\infty)$) for $A_k(\mathfrak{N}, \omega)$ with F having odd narrow class number 1 which addresses Problem 1. Problem 2 has been discussed in Section 4.2 and Section 4.3.

4.1 Preliminary lemmas and statement of the Main Theorem

As before, F is a totally real field and $\sigma_1, \ldots, \sigma_r$ denote the distinct real embeddings of F. Let $\|x\|$ denote the usual Euclidean norm of $x \in \mathbb{R}^r$ and $\sigma : F \to \mathbb{R}^r$ be given by $\sigma(s) = (\sigma_1(s), \ldots, \sigma_r(s))$. Consider the infimum of the given set for an integral ideal $\mathfrak{M}$.

$$\inf\{\|\sigma(s)\| \mid s \in \mathfrak{M}/\pm \setminus \{0\}\} = \delta_0. \tag{4.1}$$

δ_0 basically represents the smallest distance of $\sigma(\mathfrak{M}) \setminus \{0\}$ from the origin. Keeping the notation used in Theorem 2.4.2 in mind let $\delta = \frac{\delta_0}{2\sqrt{r}}$ and

$$A_i = \cap_{j=1}^{r} \{s \in \mathfrak{b}_i \mathfrak{N}/\pm \,:\, |\sigma_j(s)| \leq 2\delta, s \neq 0\}. \tag{4.2}$$

For a fixed i, A_i represents a cube along with its interior in $\mathbb{R}^r$. Since $\sigma(b_i\mathfrak{N})$ is a discrete set in $\mathbb{R}^r$, A_i is in fact finite. Note that δ depends upon i.

The notation A_i is kept reserved throughout the thesis. Let $\gamma_j = \max(\{\sqrt{\sigma_j(\eta_i u)} : i = 1, \dots, t, u \in U, \eta_i u \in F^+\})$ and $\beta_j = \min(\{\sqrt{\sigma_j(\eta_i u)} : i = 1, \dots, t, u \in U, \eta_i u \in F^+\})$. We take $\epsilon_j = \frac{\gamma_j}{\beta_j}$ for all $j = 1, \dots, r$. Theorem 4.1.1 is one of the main theorems in the submitted article [BDS23, Theorem 1].

Theorem 4.1.1 ([BDS23], Theorem 1)**.** *Let $\mathfrak{N}$ and $\mathfrak{n}$ be fixed integral ideals. Let δ and A_is be as defined by equation (4.1),(4.2) and $k_0 = \min\{k_j \,|\, 1 \le j \le r\}$. Further, assume that*

$$\frac{2\pi\gamma_j\sqrt{\sigma_j(m_1m_2)}}{\delta} \in \Big((k_j-1)-(k_j-1)^{\frac{1}{3}}, (k_j-1)\Big)$$

for all j. Then as $k_0 \to \infty$ we have

$$\frac{e^{2\pi tr_{\mathbb{Q}}^F(m_1+m_2)}}{\psi(\mathfrak{N})}\left[\prod_{j=1}^{r}\frac{(k_j-2)!}{(4\pi\sqrt{\sigma_j(m_1m_2)})^{k_j-1}}\right]\sum_{\phi\in\mathcal{F}}\frac{\lambda_{\mathfrak{n}}^{\phi}W_{m_1}^{\phi}(1)\overline{W_{m_2}^{\phi}(1)}}{\|\phi\|^2}$$

$$= \hat{T}(m_1,m_2,\mathfrak{n})\frac{\sqrt{d_F\mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)}$$

$$+\sum_{i=1}^{t}\sum_{u\in U,\eta_i u\in F^+}\sum_{s\in A_i}\Bigg\{\omega_{\mathrm{fin}}(sb_i^{-1})S_{\omega_{\mathfrak{N}}}(m_1,m_2;\eta_i u b_i^{-2};sb_i^{-1})$$

$$\frac{\sqrt{\mathrm{Nm}(\eta_i u)}}{\mathrm{Nm}(s)}\times\prod_{j=1}^{r}\frac{2\pi}{(\sqrt{-1})^{k_j}}J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\Bigg\} + o\Big(\prod_{j=1}^{r}\big(k_j-1\big)^{-\frac{1}{3}}\Big).$$

Remark 4.1.2. *Theorem 4.1.1 is valid for any given number field F, integral ideal $\mathfrak{N}$, $\mathfrak{n}$, and character ω. By Theorem 2.1.1, the set U is finite. This implies that the triple sum involved is indeed a finite sum. In the later part, we discuss whether the triple sum can also be taken to be a single term on certain restrictions on $F, \mathfrak{N}$, and ω.*

Remark 4.1.3. *We note that Theorem 4.1.1 is an analogue of Theorem 5.1.7 for the space $A_k(\mathfrak{N},\omega)$ that can be seen in the following manner. The term $\hat{T}(m_1,m_2,\mathfrak{n})\frac{\sqrt{d_F\mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)}$*

in the above theorem takes the role of $\delta(m,n)$ in Theorem 5.1.7. The term

$$\sum_{i=1}^{t}\sum_{u\in U,\eta_i u\in F^+}\sum_{s\in A_i}\Bigg\{\omega_{\text{fin}}(sb_i^{-1})S_{\omega_{\mathfrak{N}}}(m_1,m_2;\eta_i u b_i^{-2};sb_i^{-1})\frac{\sqrt{\text{Nm}(\eta_i u)}}{\text{Nm}(s)}$$

$$\times\prod_{j=1}^{r}\frac{2\pi}{(\sqrt{-1})^{k_j}}J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\Bigg\}$$

is an analogue of $2\pi i^{-k}\frac{\mu(N)}{N}\prod_{p|N}(1-p^{-2})J_{k-1}(4\pi\sqrt{mn})$ in Theorem 5.1.7.

We now consider some lemmas which will help us to prove Theorem 4.1.1.

Lemma 4.1.4. *Let M,ϵ be two given real numbers $M>2e$ and $\epsilon>0$. We have,*

$$\lim_{k\to\infty}\left[\int_M^\infty e^{(k-1)\left(1-\frac{16}{9y\epsilon}+\log\left(\frac{2}{y}\right)\right)}dy\right]=0.$$

Proof. Let us consider the integral

$$\int_M^\infty e^{(k-1)\left(1-\frac{16}{9y\epsilon}+\log\left(\frac{2}{y}\right)\right)}dy$$

first. By the substitution $\frac{16}{9y\epsilon}=x$, we have $-\frac{16}{9y^2\epsilon}dy=dx$ so that the integral becomes

$$\left(\frac{9\epsilon}{4}\right)\int_0^{\frac{16}{9M\epsilon}}e^{-(k-1)x}\left(\frac{9x\epsilon}{8}\right)^{(k-3)}dx$$

$$\leq\left(\frac{9\epsilon}{4}\right)\int_0^{\frac{16}{9M\epsilon}}e^{-(k-1)x}\left(\frac{2}{M}\right)^{(k-3)}dx$$

$$=\left(\frac{9\epsilon}{4}\right)\left(\frac{2}{M}\right)^{(k-3)}\int_0^{\frac{16}{9M\epsilon}}e^{-(k-1)x}dx=\left(\frac{9\epsilon}{4}\right)\left(\frac{2}{M}\right)^{(k-3)}\left(\frac{e^{-(k-1)\frac{16}{9M\epsilon}}}{-(k-1)}+\frac{1}{k-1}\right)$$

Hence

$$\lim_{k\to\infty}\int_M^\infty e^{(k-1)\left(1-\frac{16}{9y\epsilon}+\log\left(\frac{2}{y}\right)\right)}dy$$

$$\leq\left(\frac{9\epsilon}{4}\right)\lim_{k\to\infty}\left(\frac{2e}{M}\right)^{(k-3)}\left(\frac{e^{-(k-1)\frac{16}{9M\epsilon}}}{-(k-1)}+\frac{1}{k-1}\right)=0$$

for $M > 2e$. □

Lemma 4.1.5. *Let* $A_i' = \{s \in \mathfrak{b}_i\mathfrak{N}/\pm \setminus \{0\} \,:\, s \notin A_i\}$*. Then*

$$\Bigg| \sum_{u\in U,\eta_i u\in F^+} \sum_{s\in A_i'} \omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\mathrm{Nm}(\eta_i u)}}{\mathrm{Nm}(s)}$$

$$\prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\Bigg|$$

$$\leq \sum_{u\in U,\eta_i u\in F^+} \sum_{s\in A_i'} \mathrm{Nm}(\eta_i u)^{\frac{3}{2}} \mathrm{Nm}(b_i^{-3}) \prod_{j=1}^{r} 2\pi \Bigg| J_{k_j-1}\Big(\frac{\sqrt{4\pi\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\Bigg|.$$

Proof. Using Lemma 1.3.1 we get

$$\Big| S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1})\Big| \leq \mathrm{Nm}(\eta_i u b_i^{-2})\mathrm{Nm}(sb_i^{-1}).$$

Thus

$$\Bigg|\omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\mathrm{Nm}(\eta_i u)}}{\mathrm{Nm}(s)}\Bigg| \leq \mathrm{Nm}(\eta_i u)^{\frac{3}{2}} \mathrm{Nm}(b_i^{-3}).$$

Using triangle inequality, the proof of the lemma is complete. □

The next three lemmas consist of bounds for the Bessel function necessary to prove Theorem 4.1.1.

Lemma 4.1.6. *Let i be fixed. Let g be an index such that $|\sigma_g(s)| > 2\delta, k_g > 28$ and*

$$\frac{2\pi\gamma_g\sqrt{\sigma_g(m_1 m_2)}}{\delta} \in \Big((k_g-1) - (k_g-1)^{\frac{1}{3}}, (k_g-1)\Big).$$

Then,

$$\Bigg|J_{k_g-1}\Big(\frac{4\pi\sqrt{\sigma_g(\eta_i u m_1 m_2)}}{|\sigma_g(s)|}\Big)\Bigg| \leq (k_g-1)^{\frac{-1}{3}} e^{(k_g-1)\left(1 - \frac{16\delta}{(9\epsilon_g)|\sigma_g(s)|} + \log\left(\frac{2\delta}{|\sigma_g(s)|}\right)\right)}. \tag{4.3}$$

Proof. Since $k_g > 28$ and $\frac{2\pi\gamma_g\sqrt{\sigma_g(m_1m_2)}}{\delta} \in \Big((k_g-1)-(k_g-1)^{\frac{1}{3}}, (k_g-1)\Big)$, we have

$$\frac{8}{9} < \left(1 - \frac{1}{(k_g-1)^{\frac{2}{3}}}\right) < \left|\frac{2\pi\gamma_g\sqrt{\sigma_g(m_1m_2)}}{(k_g-1)\delta}\right| < 1. \tag{4.4}$$

It follows from Lemma 2.4.1(i) that

$$\begin{aligned}\left|J_{k_g-1}\Big(\frac{4\pi\sqrt{\sigma_g(\eta_i u m_1 m_2)}}{|\sigma_g(s)|}\Big)\right| &= \left|J_{k_g-1}\left((k_g-1)\frac{4\pi\sqrt{\sigma_g(\eta_i u m_1 m_2)}}{(k_g-1)|\sigma_g(s)|}\right)\right| \\ &\leq e^{a(1-x)}x^a J_a(a)\end{aligned}$$

where $a = k_g - 1$ and $x = \frac{4\pi\sqrt{\sigma_g(\eta_i u m_1 m_2)}}{(k_g-1)|\sigma_g(s)|}$ since

$$x < \frac{2\pi\gamma_g\sqrt{\sigma_g(m_1m_2)}}{(k_g-1)\delta} < 1$$

by equation (4.4). But by 2.4.1(ii),

$$J_a(a) \ll \frac{1}{a^{\frac{1}{3}}} = \frac{1}{(k_g-1)^{\frac{1}{3}}}. \tag{4.5}$$

Note that the equation (4.4) also implies $x < \frac{2\delta}{|\sigma_g(s)|}$ and $x > \frac{16\delta}{(9\epsilon_g)|\sigma_g(s)|}$. Hence

$$e^{a(1-x)}x^a < e^{a\left(1-\frac{16\delta}{(9\epsilon_g)|\sigma_g(s)|}+\log\left(\frac{2\delta}{|\sigma_g(s)|}\right)\right)} \tag{4.6}$$

since $e^{a(1-x)}x^a = e^{a(1-x+\log x)}$. The proof is complete by (4.4) and (4.5). □

Lemma 4.1.7. *Let i be fixed. Let l be an index such that $|\sigma_l(s)| \leq 2\delta, k_l > 28$ and*

$$\frac{2\pi\gamma_l\sqrt{\sigma_g(m_1m_2)}}{\delta} \in \Big((k_l-1)-(k_l-1)^{\frac{1}{3}}, (k_l-1)\Big).$$

Then,

$$\left|J_{k_l-1}\Big(\frac{4\pi\sqrt{\sigma_l(\eta_i u m_1 m_2)}}{|\sigma_l(s)|}\Big)\right| \ll_{F,\mathfrak{N},\mathfrak{n}} (k_l-1)^{\frac{-1}{3}}. \tag{4.7}$$

Proof. For index l, we use the uniform bound given by the Lemma 2.4.1(iv) as follows.

$$\left|J_{k_l-1}\Big(\frac{4\pi\sqrt{\sigma_l(\eta_i u m_1 m_2)}}{|\sigma_l(s)|}\Big)\right| \leq \min\left((k_l-1)^{\frac{-1}{3}}, \Big(\frac{4\pi\sqrt{\sigma_l(\eta_i u m_1 m_2)}}{|\sigma_l(s)|}\Big)^{\frac{-1}{3}}\right)$$

$$\ll_{F,\mathfrak{N}} \min\left((k_l-1)^{\frac{-1}{3}}, \left(\sqrt{\sigma_l(\eta_i u m_1 m_2)}\right)^{\frac{-1}{3}}\right)$$

as $|\sigma_l(s)| \leq 2\delta$. By equation (4.4) we have $\sqrt{\sigma_l(\eta_i u m_1 m_2)} \geq \frac{4\sqrt{\sigma_l(\eta_i u)}(k_l-1)\delta}{9\pi\gamma_l}$ which implies $\sqrt{\sigma_l(\eta_i u m_1 m_2)} \gg_{F,\mathfrak{N}} (k_l-1)$. Hence we have

$$\left|J_{k_l-1}\Big(\frac{4\pi\sqrt{\sigma_l(\eta_i u m_1 m_2)}}{|\sigma_l(s)|}\Big)\right| \ll_{F,\mathfrak{N}} \min((k_l-1)^{\frac{-1}{3}}, (k_l-1)^{\frac{-1}{3}}) = (k_l-1)^{\frac{-1}{3}}. \tag{4.8}$$

□

Lemma 4.1.8. *Let i be fixed and $M > 2e$. Let g' be an index such that $2\delta < |\sigma_{g'}(s)| \leq (M+1)\delta, k_{g'} > 27$ and*

$$\frac{2\pi\gamma_{g'}\sqrt{\sigma_{g'}(m_1 m_2)}}{\delta} \in \Big((k_{g'}-1) - (k_{g'}-1)^{\frac{1}{3}}, (k_{g'}-1)\Big).$$

Then,

$$J_{k_{g'}-1}\Big(\frac{4\pi\sqrt{\sigma_{g'}(\eta_i u m_1 m_2)}}{|\sigma_{g'}(s)|}\Big) = o((k_{g'}-1)^{\frac{-1}{3}}).$$

Proof. We apply the bounds in the following manner. Let

$$x' = \frac{4\pi\sqrt{\sigma_{g'}(\eta_i u m_1 m_2)}}{(k_{g'}-1)|\sigma_{g'}(s)|} < 1.$$

We consider three cases according to when $0 < x' < \frac{1}{3}$, $\frac{1}{3} \leq x' < \frac{1}{2}$ or $\frac{1}{2} \leq x' < 1$. For each part we use bounds as follows. When $\frac{1}{2} \leq x' < 1$ we use the uniform bound given by Lemma 2.4.1(v). This gives us

$$J_{k_{g'}-1}((k_{g'}-1)x') \ll \frac{1}{(1-x'^2)^{\frac{1}{4}}(k_{g'}-1)^{\frac{1}{2}}} = o((k_{g'}-1)^{\frac{-1}{3}}).$$

For $\frac{1}{3} \leq x' < \frac{1}{2}$, using Lemma 2.4.1(i),(ii) we have

$$J_{k_{g'}-1}((k_{g'}-1)x') \ll e^{(k_{g'}-1)(1-x'+\log x')} \cdot \frac{1}{(k_{g'}-1)^{\frac{-1}{3}}}$$

$$\leq e^{(k_{g'}-1)(1-\frac{1}{3}+\log\frac{1}{2})} \cdot \frac{1}{(k_{g'}-1)^{\frac{1}{3}}} = o((k_{g'}-1)^{\frac{1}{3}}).$$

Similarly for $0 < x' < \frac{1}{3}$,

$$J_{k_{g'}-1}((k_{g'}-1)x') \ll e^{(k_{g'}-1)(1+\log\frac{1}{3})} \cdot \frac{1}{(k_{g'}-1)^{\frac{1}{3}}} = o((k_{g'}-1)^{\frac{-1}{3}}).$$

□

4.2 Main Theorem

Proposition 4.2.1. Let $k \in \mathbb{N}$ and $\epsilon > 1$ be fixed. Then the function

$$f(y) = e^{(k-1)\left(-\frac{16}{(9\epsilon)y}+\log\left(\frac{2}{y}\right)\right)}$$

is monotonically decreasing for $y > \frac{16}{9}$.

Proof. Note that

$$f(y) = e^{(k-1)\left(-\frac{16}{(9\epsilon)y}+\log\left(\frac{2}{y}\right)\right)} = \left(\frac{2}{y}\right)^{k-1} e^{\frac{-16(k_g-1)}{(9\epsilon)y}}.$$

Hence

$$f'(y) = 2^{k-1} \cdot \frac{y^{k-3} \times \frac{16(k-1)}{9\epsilon} \times e^{\frac{-16(k-1)}{(9\epsilon)y}} - (k-1) \times y^{k-2}}{y^{2k-2}}$$

$$= 2^{k-1} \cdot (k-1) \cdot y^{-k-1}\Big(\frac{16}{9\epsilon} - y\Big).$$

The proof is complete since $f'(y) < 0$ for $y > \frac{16}{9\epsilon} > \frac{16}{9}$. $\square$

Proof.[Theorem 4.1.1]

First, we try to estimate the triple sum appearing in the Petersson trace formula (Theorem 2.4.2). Let i be fixed in the triple sum. Let us take $A_i' = \{s \in \mathfrak{b}_i\mathfrak{N}/\pm : s \neq 0, s \notin A_i\}$. We have,

$$\sum_{u\in U,\eta_i u\in F^+} \sum_{s\in\mathfrak{b}_i\mathfrak{N}/\pm,s\neq 0} \omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\text{Nm}(\eta_i u)}}{\text{Nm}(s)}$$

$$\prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)$$

$$= \sum_{u\in U,\eta_i u\in F^+} \sum_{s\in A_i} \omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\text{Nm}(\eta_i u)}}{\text{Nm}(s)}$$

$$\prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)$$

$$+ \sum_{u\in U,\eta_i u\in F^+} \sum_{s\in A_i'} \omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\text{Nm}(\eta_i u)}}{\text{Nm}(s)}$$

$$\prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big).$$

We estimate the sum when $s \in A_i'$ and show that as $k_0 \to \infty$, the sum is in fact $o\Big(\prod_{j=1}^{r}\Big(k_j - 1\Big)^{\frac{-1}{3}}\Big)$. Using Lemma 4.1.5 we get,

$$\Bigg| \sum_{u\in U,\eta_i u\in F^+} \sum_{s\in A_i'} \omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\text{Nm}(\eta_i u)}}{\text{Nm}(s)}$$

$$\prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\Bigg|$$

$$\leq \sum_{u\in U,\eta_i u\in F^+} \sum_{s\in A_i'} \mathrm{Nm}(\eta_i u)^{\frac{3}{2}}\mathrm{Nm}(b_i^{-3}) \prod_{j=1}^{r} 2\pi\left|J_{k_j-1}\Big(\frac{\sqrt{4\pi\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\right|.$$

Let us take $u \in U$ and $\eta_i u \in F^+$ to be fixed. We now estimate

$$\sum_{s\in A_i'} \prod_{j=1}^{r}\left|J_{k_j-1}\Big(\frac{\sqrt{4\pi\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\right|$$

with using the estimates from previous section. Let $h = \sum_{j=1}^{r} a_j 2^{j-1}$ where $a_j \in \{0,1\}$. There is a one to one correspondence between h and r-tuple $(a_1,\dots,a_r)$ with $a_j \in \{0,1\}$. To see this consider the map $(a_1,\dots,a_r) \mapsto \sum_{j=1}^{r} a_j 2^{j-1}$. We partition the set A_i' as per the given correspondence, i.e.

$$A_i' = \cup_{h=1}^{2^r-1} A_{i,h}'$$

where $A_{i,h}' = \{s \in A_i' : h = \sum_{j=1}^{r} a_j 2^{j-1},\ a_g = 1, a_l = 0$ for $g,l \in \{1,\dots,r\}$ with $|\sigma_g(s)| > 2\delta$ and $|\sigma_l(s)| \leq 2\delta\}$. Hence

$$\sum_{s\in A_i'} \prod_{j=1}^{r}\left|J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\right|$$

$$\sum_{h=1}^{2^r-1}\sum_{s\in A_{i,h}'} \prod_{j=1}^{r}\left|J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)\right|.$$

Now let us fix the value of h. Corresponding to $h = \sum_{j=1}^{r} a_j 2^{j-1}$, let $g_1,\dots,g_v,\ g_1',\dots,g_{v'}'$, $l_1,\dots,l_w$ be a permutation of $1,\dots,r$ with the property that $a_{g_1},\dots,a_{g_v} = 1,\ a_{g_1'},\dots,a_{g_{v'}'} = 1$ and $a_{l_1},\dots,a_{l_w} = 0$. Let M be a fixed real number with $M > 2e$. We distinguish between the index g_α and g_α' with $|\sigma_{g_\alpha}(s)| > (M+1)\delta$ and $|\sigma_{g_\alpha'}(s)| \leq (M+1)\delta$ respectively. Let $\delta_1,\dots,\delta_r$ denote the length of sides of the fundamental parallelopiped of the lattice $\sigma(\mathfrak{b}_\mathfrak{i}\mathfrak{N})$ for a fixed i.

Let us take $\widetilde{\delta} = \min(\delta_1,\dots,\delta_r)$ and $\epsilon = \left(\frac{\widetilde{\delta}}{2}\right)^r$. For the choice of ϵ, it is clear that a cube of volume ϵ can contain at most one lattice point of $\sigma(\mathfrak{b}_\mathfrak{i}\mathfrak{N})$. Using Lemma 4.1.8 we have,

$$\sum_{s\in A_{i,h}'} \prod_{\alpha=1}^{v}\left|J_{k_{g_\alpha}-1}\left(\frac{4\pi\sqrt{\sigma_{g_\alpha}(\eta_i u m_1 m_2)}}{|\sigma_{g_\alpha}(s)|}\right)\right| \prod_{g_{\alpha'}',\alpha'=1}^{v'}\left|J_{k_{g_{\alpha'}'}-1}\left(\frac{4\pi\sqrt{\sigma_{g_{\alpha'}'}(\eta_i u m_1 m_2)}}{|\sigma_{g_{\alpha'}'}(s)|}\right)\right|$$

$$\prod_{\beta=1}^{w}\left|J_{k_{l_\beta}-1}\left(\frac{4\pi\sqrt{\sigma_{l_\beta}(\eta_i u m_1 m_2)}}{|\sigma_{l_\beta}(s)|}\right)\right|$$

$$\ll_{F,\mathfrak{N},\mathfrak{n}} \prod_{g'_{\alpha'},\alpha'=1}^{v'} f(k_{g'_{\alpha'}}-1) \sum_{(M+1)\delta<|\sigma_{g_1}(s)|} \dots \sum_{(M+1)\delta<|\sigma_{g_v}(s)|}$$

$$\sum_{\substack{(m_1-1)\epsilon^{\frac{1}{r}}<|\sigma_{l_1}(s)|\le m_1\epsilon^{\frac{1}{r}} \\ m_1\le\left[\frac{2\delta}{\epsilon^{\frac{1}{r}}}\right]+1}} \dots \sum_{\substack{(m_w-1)\epsilon^{\frac{1}{r}}<|\sigma_{l_w}(s)|\le m_w\epsilon^{\frac{1}{r}} \\ m_w\le\left[\frac{2\delta}{\epsilon^{\frac{1}{r}}}\right]+1}} \left(\prod_{\alpha=1}^{v}\left|J_{k_{g_\alpha}-1}\left(\frac{4\pi\sqrt{\sigma_{g_\alpha}(\eta_i u m_1 m_2)}}{|\sigma_{g_\alpha}(s)|}\right)\right|\times\right.$$

$$\left.\prod_{\beta=1}^{w}\left|J_{k_{l_\beta}-1}\left(\frac{4\pi\sqrt{\sigma_{l_\beta}(\eta_i u m_1 m_2)}}{|\sigma_{l_\beta}(s)|}\right)\right|\right)$$

where $f(k_{g'_{\alpha'}}-1)=o((k_{g'_{\alpha'}}-1)^{\frac{-1}{3}})$ for all $\alpha'=1,2,...,v'$. Using Lemma 4.7 and Lemma 4.8 the above expression is

$$\ll_{F,\mathfrak{N},\mathfrak{n}} \prod_{g'_{\alpha'},\alpha'=1}^{v'} f(k_{g'_{\alpha'}}-1)\prod_{\beta=1}^{w}(k_{l_\beta}-1)^{\frac{-1}{3}}$$

$$\sum_{\substack{(M+1)\delta+(n_1-1)\epsilon^{\frac{1}{r}}\le|\sigma_{g_1}(s)|<(M+1)\delta+n_1\epsilon^{\frac{1}{r}} \\ n_1\in\mathbb{N}}} \dots \sum_{\substack{(M+1)\delta+(n_v-1)\epsilon^{\frac{1}{r}}\le|\sigma_{g_v}(s)|<(M+1)\delta+n_v\epsilon^{\frac{1}{r}} \\ n_v\in\mathbb{N}}}$$

$$\sum_{\substack{(m_1-1)\epsilon^{\frac{1}{r}}<|\sigma_{l_1}(s)|\le m_1\epsilon^{\frac{1}{r}} \\ m_1\le\left[\frac{2\delta}{\epsilon^{\frac{1}{r}}}\right]+1}} \dots \sum_{\substack{(m_w-1)\epsilon^{\frac{1}{r}}<|\sigma_{l_w}(s)|\le m_w\epsilon^{\frac{1}{r}} \\ m_w\le\left[\frac{2\delta}{\epsilon^{\frac{1}{r}}}\right]+1}}$$

$$\left[\prod_{\alpha=1}^{v}(k_{g_\alpha}-1)^{\frac{-1}{3}}\cdot e^{(k_{g_\alpha}-1)\left(1-\frac{16\delta}{(9\epsilon_{g_\alpha})|\sigma_{g_\alpha}(s)|}+\log\left(\frac{2\delta}{|\sigma_{g_\alpha}(s)|}\right)\right)}\right]$$

$$\le \prod_{g'_{\alpha'},\alpha'=1}^{v'} f(k_{g'_{\alpha'}}-1)\prod_{\alpha=1}^{v}(k_{g_\alpha}-1)^{\frac{-1}{3}}\prod_{\beta=1}^{w}(k_{l_\beta}-1)^{\frac{-1}{3}}\left[\sum_{\alpha=1}^{v}\sum_{\substack{(M+1)\delta+(n_\alpha-1)\epsilon^{\frac{1}{r}}\le|\sigma_{g_\alpha}(s)|<(M+1)\delta+n_\alpha\epsilon^{\frac{1}{r}} \\ n_\alpha\in\mathbb{N}}}\right.$$

$$\sum_{\beta=1}^{w} \sum_{\substack{(m_\beta-1)\epsilon^{\frac{1}{r}}<|\sigma_{l_\beta}(s)|\leq m_\beta\epsilon^{\frac{1}{r}} \\ m_\beta\leq\left[\frac{2\delta}{\epsilon^{\frac{1}{r}}}\right]+1}} \prod_{\alpha=1}^{v} e^{(k_{g_\alpha}-1)\left(1-\frac{16\delta}{(9\epsilon_{g_\alpha})|\sigma_{g_\alpha}(s)|}+\log\left(\frac{2\delta}{|\sigma_{g_\alpha}(s)|}\right)\right)}\Bigg].$$

For simplicity let us take $y_\alpha = \frac{|\sigma_{g_\alpha}(s)|}{\delta}$ and $z_\beta = \frac{|\sigma_{l_\beta}(s)|}{\delta}$. Thus we have

$$\leq \prod_{g'_{\alpha'},\alpha'=1}^{v'} f(k_{g'_{\alpha'}}-1)\prod_{\alpha=1}^{v}(k_{g_\alpha}-1)^{\frac{-1}{3}}\prod_{\beta=1}^{w}(k_{l_\beta}-1)^{\frac{-1}{3}}\Bigg[\sum_{\alpha=1}^{v}\sum_{\substack{(M+1)+(n_\alpha-1)\frac{e^{\frac{1}{r}}}{\delta}\leq y_\alpha<(M+1)+n_\alpha\frac{e^{\frac{1}{r}}}{\delta} \\ n_\alpha\in\mathbb{N}}}$$

$$\sum_{\beta=1}^{w}\sum_{\substack{(m_\beta-1)\frac{e^{\frac{1}{r}}}{\delta}<z_\beta\leq m_\beta\frac{e^{\frac{1}{r}}}{\delta} \\ m_\beta\leq\left[\frac{2}{\epsilon^{\frac{1}{r}}}\right]+1}}\prod_{\alpha=1}^{v} e^{(k_{g_\alpha}-1)\left(1-\frac{16}{(9\epsilon_{g_\alpha})y_\alpha}+\log\left(\frac{2}{y_\alpha}\right)\right)}\Bigg]$$

$$=\frac{\delta^r}{\epsilon}\times\prod_{\alpha=1}^{v}(k_{g_\alpha}-1)^{\frac{-1}{3}}\prod_{\beta=1}^{w}(k_{l_\beta}-1)^{\frac{-1}{3}}\Bigg[\sum_{\alpha=1}^{v}\sum_{\substack{(M+1)+(n_\alpha-1)\frac{e^{\frac{1}{r}}}{\delta}\leq y_\alpha<(M+1)+n_\alpha\frac{e^{\frac{1}{r}}}{\delta} \\ n_\alpha\in\mathbb{N}}}$$

$$\sum_{\beta=1}^{w}\sum_{\substack{(m_\beta-1)\frac{e^{\frac{1}{r}}}{\delta}<z_\beta\leq m_\beta\frac{e^{\frac{1}{r}}}{\delta} \\ m_\beta\leq\left[\frac{2}{\epsilon^{\frac{1}{r}}}\right]+1}}\prod_{\alpha=1}^{v} e^{(k_{g_\alpha}-1)\left(1-\frac{16}{(9\epsilon_{g_\alpha})y_\alpha}+\log\left(\frac{2}{y_\alpha}\right)\right)}\frac{\epsilon}{\delta^r}\Bigg].$$

By Proposition 4.2.1 and the integral test the above sum is bounded by the multiple integral

$$\leq\frac{\delta^r}{\epsilon}\prod_{g'_{\alpha'},\alpha'=1}^{v'} f(k_{g'_{\alpha'}}-1)\prod_{\alpha=1}^{v}(k_{g_\alpha}-1)^{\frac{-1}{3}}\prod_{\beta=1}^{w}(k_{l_\beta}-1)^{\frac{-1}{3}}\int_M^\infty\ldots\int_M^\infty$$

$$\int_0^{2+\frac{e^{\frac{1}{r}}}{\delta}}\ldots\int_0^{2+\frac{e^{\frac{1}{r}}}{\delta}}\left[\prod_{\alpha=1}^{v} e^{(k_{g_\alpha}-1)\left(1-\frac{16}{(9\epsilon_{g_\alpha})y_\alpha}+\log\left(\frac{2}{y_\alpha}\right)\right)}\right]dy_1\ldots dy_v dz_1\ldots dz_w$$

$$\ll_{F,\mathfrak{N},\mathfrak{n}} \prod_{g'_{\alpha'},\alpha'=1}^{v'} f(k_{g'_{\alpha'}}-1) \prod_{\alpha=1}^{v}(k_{g_\alpha}-1)^{\frac{-1}{3}} \prod_{\beta=1}^{w}(k_{l_\beta}-1)^{\frac{-1}{3}}$$

$$\int_M^\infty \dots \int_M^\infty \left[\prod_{\alpha=1}^{v} e^{(k_{g_\alpha}-1)\left(1+\log\left(\frac{2}{y_\alpha}\right)\right)}\right] dy_1 \dots dy_v$$

$$= \prod_{g'_{\alpha'},\alpha'=1}^{v'} f(k_{g'_{\alpha'}}-1) \prod_{\alpha=1}^{v}(k_{g_\alpha}-1)^{\frac{-1}{3}} \prod_{\beta=1}^{w}(k_{l_\beta}-1)^{\frac{-1}{3}} \prod_{\alpha=1}^{v} \int_M^\infty e^{(k_{g_\alpha}-1)\left(1-\frac{16}{(9\epsilon_{g_\alpha})y_\alpha}+\log\left(\frac{2}{y_\alpha}\right)\right)} dy_\alpha.$$

However by Lemma 4.1.4,

$$\lim_{k_{g_\alpha}\to\infty} \int_M^\infty e^{(k_{g_\alpha}-1)\left(1-\frac{16}{(9\epsilon_{g_\alpha})y_\alpha}+\log\left(\frac{2}{y_\alpha}\right)\right)} dy_\alpha = 0$$

for all possible α.

Therefore for each fixed i and fixed $u \in U$ with $\eta_i u \in F^+$ we obtain,

$$\sum_{s\in\mathfrak{b}_i\mathfrak{N}/\pm, s\neq 0} \omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\text{Nm}(\eta_i u)}}{\text{Nm}(s)}$$

$$\prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big)$$

$$= \sum_{s\in A_i} \omega_{\text{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1}) \frac{\sqrt{\text{Nm}(\eta_i u)}}{\text{Nm}(s)}$$

$$\prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|}\Big) + o\Big(\prod_{j=1}^{r}\Big(k_j-1\Big)^{\frac{-1}{3}}\Big)$$

Finally, varying i from 1 to t and $u \in U$ with $\eta_i u \in F^+$, we get

$$\frac{e^{2\pi tr_{\mathbb{Q}}^F(m_1+m_2)}}{\psi(\mathfrak{N})}\left[\prod_{j=1}^{r} \frac{(k_j-2)!}{(4\pi\sqrt{\sigma_j(m_1 m_2)})^{k_j-1}}\right] \sum_{\phi\in\mathcal{F}} \frac{\lambda_{\mathfrak{n}}^{\phi} W_{m_1}^{\phi}(1)\overline{W_{m_2}^{\phi}(1)}}{\|\phi\|^2}$$

$$= \hat{T}(m_1, m_2, \mathfrak{n}) \frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)}$$

$$+ \sum_{i=1}^{t} \sum_{u \in U, \eta_i u \in F^+} \sum_{s \in A_i} \Bigg\{ \omega_{\mathrm{fin}}(sb_i^{-1}) S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_i u b_i^{-2}; sb_i^{-1})$$

$$\frac{\sqrt{\mathrm{Nm}(\eta_i u)}}{\mathrm{Nm}(s)} \times \prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j - 1}\Big(\frac{4\pi \sqrt{\sigma_j(\eta_i u m_1 m_2)}}{|\sigma_j(s)|} \Big) \Bigg\} + o\Big(\prod_{j=1}^{r} \big(k_j - 1\big)^{\frac{-1}{3}} \Big).$$

□

4.3 Reformulation of Main Theorem in some special cases

We remind the reader that the notation $F, \mathfrak{N}, \mathfrak{n}$ remains the same throughout this section and has been taken from Petersson trace formula for $A_k(\mathfrak{N}, \omega)$ (Theorem 2.4.2). In this section, we focus on $\mathcal{O}$ having an odd narrow class number.

Lemma 4.3.1 ([EMP86, Prop. 2.4])**.** *Let $\mathcal{O}$ have an odd narrow class number. Then we have $t = 1$ and $|U^+| = 1$.*

Proof. Since the class number divides the narrow class number, the class number is odd. By the Example 2.1.2, the equation $[\mathfrak{b}]^2[\mathfrak{n}] = 1$ has a unique solution implying $t = 1$. By Proposition 2.4 of [EMP86], we have $\dim_2(U^+) \leq \dim_2(C^+) = 0$ (see section 2.1 for notation). Hence $|U^+| = 1$. □

As $t = 1$, we can take $\eta_1 = 1$. Lemma 4.3.1 also helps us to take $|\{u \in U : \eta_i u \in F^+$ for some $i = 1, \ldots, t\}| = 1$. Recall that $\inf\{\|\sigma(s)\| : s \in \mathfrak{b}_1\mathfrak{N}/\pm \setminus \{0\}\} = \delta_0$. For a general number field F and integral ideal $\mathfrak{N}$, $|\{s \in \mathfrak{b}_1\mathfrak{N}/\pm \setminus \{0\} : \|\sigma(s)\| = \delta_0\}|$ is not necessarily equal to 1. For instance if we take $b_1 = \mathcal{O}$, $\mathfrak{N} = \mathbb{Z}\sqrt{d}$ with $d \equiv 2, 3 \pmod 4$ and square free, we have a unique s_0. Whereas by taking the ideal $(3 + \sqrt{3})\mathcal{O} \subset \mathbb{Z}[\sqrt{3}]$ we get $|A_1| = 3$. Hence we ask the question whether there is a necessary and sufficient condition for $|\{s \in \mathfrak{b}_1\mathfrak{N}/\pm \setminus \{0\} : \|\sigma(s)\| = \delta_0\}| = 1$. This will make the triple sum in Theorem 4.1.1 a single term.

Along this line, we provide sufficient conditions in the proceeding three lemmas.

Lemma 4.3.2. *Suppose the infimum* $\inf\{\|\sigma(s)\| : s \in \mathfrak{b}_1\mathfrak{N}/\pm\setminus\{0\}\} = \delta_0$ *be attained for some* s_0 *with* $\sigma(s_0) = (a, a, \dots, a) = \left(\frac{\delta_0}{\sqrt{r}}, \frac{\delta_0}{\sqrt{r}}, \dots, \frac{\delta_0}{\sqrt{r}}\right)$. *We have* $A_1 = \left\{\frac{\delta_0}{\sqrt{r}}\right\}$.

Proof. Note that $A_1 \subset S_{\delta_0}$ where $S_{\delta_0} := \{s \in \mathfrak{b}_1\mathfrak{N}/\pm : \sigma_1^2(s) + \dots + \sigma_r^2(s) = \delta_0^2\}$. The set A_1 basically represents a cube with centre $(0, \dots, 0)$ in r-dimensional euclidean space along with its interior where as S_{δ_0} denotes a sphere with radius δ_0 and centre $(0, \dots, 0)$. Hence if $s' \in A_1 \cap S_{\delta_0}$, then $s' \in \partial A_1 \cap S_{\delta_0}$. This implies $|\sigma_j(s')| = \frac{\delta_0}{\sqrt{r}}$ for all $j = 1, \dots, r$. Therefore s' is equal to $\left((-1)^{m_1}\frac{\delta_0}{\sqrt{r}}, (-1)^{m_2}\frac{\delta_0}{\sqrt{r}}, \dots, (-1)^{m_r}\frac{\delta_0}{\sqrt{r}}\right)$ where $m_j = 0, 1$ for $j = 1, \dots, r$. If $m_j = 0$ or $m_j = 1$ for all j, then $s' = s$. In any different case, we have $m_{j_0} = 0$ for some j_0 which yields $\sigma_{j_0}(s) = \sigma_{j_0}(s')$. Thus $s' = s_0 = \frac{\delta_0}{\sqrt{r}}$ as σ_{j_0} is injective. $\square$

Lemma 4.3.3. *Let* $\mathfrak{b}_1\mathfrak{N} = \mathcal{O}$ *for the set* A_1. *We have* $\delta_0 = \sqrt{r}$ *and* $A_1 = \{1\}$.

Proof. Let us consider minimizing $\inf\{\|\sigma(s)\| : s \in \mathcal{O}/\pm\setminus\{0\}\}$. This is equivalent to minimize the quantity $\|\sigma(s)\|^2 = \sigma_1^2(s) + \dots + \sigma_r^2(s)$. For a given s, if

$$\sigma(s') = \left((-1)^{m_1}\sigma_1(s), (-1)^{m_2}\sigma_2(s), \dots, (-1)^{m_r}\sigma_r(s)\right)$$

for $m_j \in \mathbb{N}$, then $\|\sigma(s')\| = \|\sigma(s)\|$. Without loss of generality we consider minimizing $\|\sigma(s)\|^2$ on the set $\{s : s \in \mathcal{O}/\pm\setminus\{0\}, \sigma_j(s) \geq 0 \text{ for } j = 1, \dots, r\}$. The Cauchy-Schwartz inequality implies

$$\sqrt{(1^2 + \dots + 1^2)(\sigma_1^2(s) + \dots + \sigma_r^2(s))} \geq \sigma_1(s) + \dots + \sigma_r(s).$$

Further using AM-GM inequality, we get

$$\frac{\sigma_1(s) + \dots + \sigma_r(s)}{r} \geq (\sigma_1(s) \times \dots \times \sigma_r(s))^{\frac{1}{r}} = \mathrm{Nm}(s)^{\frac{1}{r}}.$$

Note that the equality holds when $\sigma_i(s) = \sigma_j(s)$ for all possible i, j. In such scenario, $\sigma_1^2(s) + \dots + \sigma_r^2(s) = r\sigma_1^2(s) = r(\mathrm{Nm}(s))^{\frac{2}{r}}$. However $\mathrm{Nm}(s) \in \mathbb{N}$, so that the minimum value of $r(\mathrm{Nm}(s))^{\frac{2}{r}} = r$. This can happen when $\mathrm{Nm}(s) = 1$ and $\sigma_1^2(s) = 1$. Therefore $\delta_0 = \sqrt{r}$ and on applying Lemma 4.3.2 with taking $s_0 = \frac{\delta_0}{\sqrt{r}}$, we get $A_1 = \{1\}$. $\square$

Lemma 4.3.4. *Let $\mathfrak{b}_1\mathfrak{N}$ be an ideal in $\mathcal{O}$ such that $\mathfrak{b}_1\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s} \in \mathbb{Z}$, then $\delta_0 = |\tilde{s}|\sqrt{r}$ and $A_1 = \{|\tilde{s}|\}$.*

Proof. The proof is quite similar to the proof of Lemma 4.3.3. We again have to minimize $\inf\{\|\sigma(s)\| : s \in \mathfrak{b}_1\mathfrak{N}/\pm\setminus\{0\}\}$. Without loss of generality this is equivalent minimizing $\|\sigma(s)\|^2 = \sigma_1^2(s) + \ldots + \sigma_r^2(s)$ on the set $B = \{s : s \in \mathfrak{b}_1\mathfrak{N}/\pm\setminus\{0\}, \sigma_j(s) \geq 0$ for $j = 1, .., r$ $\}$. Proceeding similar to the arguments given in Lemma 4.3.3, the possible minimum value of $\|\sigma(s)\|^2$ is $r(\mathrm{Nm}(s))^{\frac{2}{r}}$. If $s \in B$, we have $s = \tilde{s}s'$ for some $s' \in \mathcal{O}$. This implies $\mathrm{Nm}(s) = \mathrm{Nm}(\tilde{s})\mathrm{Nm}(s')$ consequently, $\mathrm{Nm}(\tilde{s})$ divides $\mathrm{Nm}(s)$. Thus the minimum value of $\mathrm{Nm}(s)$ is $\mathrm{Nm}(\tilde{s})$. Hence $r(\mathrm{Nm}(s))^{\frac{2}{r}}$ has minimum value $r(\mathrm{Nm}(\tilde{s}))^{\frac{2}{r}}$. This proves $\|\sigma(s)\|^2$ has minimum value $r(\mathrm{Nm}(\tilde{s}))^{\frac{2}{r}}$ which can happen if $\sigma(s) = (|\tilde{s}|, \ldots, |\tilde{s}|)$. However by Lemma 4.3.2, we have $A_1 = \{|\tilde{s}|\}$. □

We observe that for an ideal $\mathfrak{b}_1\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s} \notin \mathbb{Z}$ may or may not satisfy the hypothesis of Lemma 4.3.2. For instance the ideal $(1+\sqrt{3})\mathcal{O} \subset \mathbb{Z}[\sqrt{3}]$ satisfy the hypothesis for Lemma 4.3.2 with $s_0 = 2$. However if we take $(3+\sqrt{3})\mathcal{O} \subset \mathbb{Z}[\sqrt{3}]$, we can show that the ideal does not satisfy the hypothesis of Lemma 4.3.2. This illustrates hypothesis of Lemma 4.3.2 is not necessary to have $|\{s \in \mathfrak{b}_1\mathfrak{N}/\pm\setminus\{0\} : \|\sigma(s)\| = \delta_0\}| = 1$. Nonetheless the hypothesis of $\mathfrak{b}_1\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s} \in \mathbb{N}$ in Theorem 4.3.5 can be replaced by any ideal $\mathfrak{b}_1\mathfrak{N}$ with the property $|\{s \in \mathfrak{b}_1\mathfrak{N}/\pm\setminus\{0\} : \|\sigma(s)\| = \delta_0\}| = 1$ in order to have a Theorem like Theorem 4.3.5.

Theorem 4.3.5 ([BDS23], Theorem 2)**.** *Let F have an odd narrow class number and assumptions of Theorem 4.1.1 hold true. Further let $\mathfrak{b}_1\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s} \in \mathbb{N}$.*

- *Then the main term in Theorem 4.1.1 reduces to*

$$\hat{T}(m_1, m_2, \mathfrak{n})\frac{\sqrt{d_F\mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)} + \Bigg\{\omega_{\mathrm{fin}}(sb_1^{-1})S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_1 b_1^{-2}; sb_1^{-1})\frac{\sqrt{\mathrm{Nm}(\eta_1)}}{\mathrm{Nm}(s)} \times \prod_{j=1}^{r}\frac{2\pi}{(\sqrt{-1})^{k_j}}J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_1 m_1 m_2)}}{|\sigma_j(s)|}\Big)\Bigg\}$$

  where $s = \tilde{s}$.

- *Further assume that $S_{\omega_{\mathfrak{N}}}(m_1, m_2; \eta_1 b_1^{-2}; sb_1^{-1}) \neq 0$ for some m_1 and m_2. Then as*

$k_0 \to \infty$, we have

$$\frac{e^{2\pi tr_{\mathbb{Q}}^F(m_1+m_2)}}{\psi(\mathfrak{N})} \prod_{j=1}^{r} \frac{(k_j-2)!}{(4\pi\sqrt{|\sigma_j(m_1m_2)|})^{k_j-1}} \sum_{\phi\in\mathcal{F}} \frac{\lambda_{\mathfrak{n}}^{\phi} W_{m_1}^{\phi}(1)\overline{W_{m_2}^{\phi}(1)}}{\|\phi\|^2}$$
$$-\hat{T}(m_1,m_2,\mathfrak{n})\frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)} \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_j-1)^{-\frac{1}{3}}.$$

Proof. Using Theorem 4.1.1, Lemma 4.3.1 and Lemma 4.3.4, we get

$$\Bigg|\frac{e^{2\pi tr_{\mathbb{Q}}^F(m_1+m_2)}}{\psi(\mathfrak{N})}\left[\prod_{j=1}^{r} \frac{(k_j-2)!}{(4\pi\sqrt{\sigma_j(m_1m_2)})^{k_j-1}}\right] \sum_{\phi\in\mathcal{F}} \frac{\lambda_{\mathfrak{n}}^{\phi} W_{m_1}^{\phi}(1)\overline{W_{m_2}^{\phi}(1)}}{\|\phi\|^2}$$
$$-\hat{T}(m_1,m_2,\mathfrak{n})\frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)}\Bigg|$$
$$=\Bigg|\omega_{\mathrm{fin}}(sb_1^{-1})S_{\omega_{\mathfrak{N}}}(m_1,m_2;\eta_1 b_1^{-2};sb_1^{-1})$$
$$\frac{\sqrt{\mathrm{Nm}(\eta_1)}}{\mathrm{Nm}(s)} \times \prod_{j=1}^{r}\frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_1m_1m_2)}}{|\sigma_j(s)|}\Big)\Bigg| + o\Big(\prod_{j=1}^{r}\big(k_j-1\big)^{\frac{-1}{3}}\Big)$$
$$\gg_{F,\mathfrak{N},\mathfrak{n}} \prod_{j=1}^{r} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_1m_1m_2)}}{|\sigma_j(s)|}\Big) + o\Big(\prod_{j=1}^{r}\big(k_j-1\big)^{\frac{-1}{3}}\Big)$$

The last step of $\gg_{F,\mathfrak{N},\mathfrak{n}}$ is justified in the following part. Note that

$$\frac{4\pi\sqrt{\sigma_j(\eta_1m_1m_2)}}{|\sigma_j(s)|} = \frac{4\pi\sqrt{\sigma_j(\eta_1m_1m_2)}}{(\delta_0/\sqrt{r})} = \frac{2\pi\sqrt{\sigma_j(\eta_1m_1m_2)}}{\delta}.$$

By the given condition

$$\frac{2\pi\gamma_j\sqrt{\sigma_j(m_1m_2)}}{\delta} \in \Big((k_j-1)-(k_j-1)^{\frac{1}{3}}, \big(k_j-1\big)\Big),$$

which implies

$$\frac{2\pi\gamma_j\sqrt{\sigma_j(m_1m_2)}}{\delta} = (k_j-1)+d(k_j-1)^{\frac{1}{3}}$$

with $d \in (-1,0)$. Using Lemma 2.4.1(iv)

$$\prod_{j=1}^{r} J_{k_j-1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_1 m_1 m_2)}}{|\sigma_j(s)|}\Big) \gg \prod_{j=1}^{r}\Big(k_j-1\Big)^{\frac{-1}{3}}.$$

Therefore

$$\Bigg|\frac{e^{2\pi tr_{\mathbb{Q}}^{F}(m_1+m_2)}}{\psi(\mathfrak{N})}\Bigg[\prod_{j=1}^{r}\frac{(k_j-2)!}{(4\pi\sqrt{\sigma_j(m_1m_2)})^{k_j-1}}\Bigg]\sum_{\phi\in\mathcal{F}}\frac{\lambda_{\mathfrak{n}}^{\phi}W_{m_1}^{\phi}(1)\overline{W_{m_2}^{\phi}(1)}}{\|\phi\|^2}$$

$$-\hat{T}(m_1,m_2,\mathfrak{n})\frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)}\Bigg| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_j-1)^{\frac{-1}{3}}.$$

□

Remark 4.3.6. *For the case of $S_{\omega_{\mathfrak{N}}}(m_1,m_2;\eta_1 b_1^{-2};sb_1^{-1}) = 0$ for some m_1 and m_2, the main term in Theorem 4.1.1 reduces to $\hat{T}(m_1,m_2,\mathfrak{n})\frac{\sqrt{d_F\mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)}$. The assumption $S_{\omega_{\mathfrak{N}}}(m_1,m_2;\eta_1 b_1^{-2};sb_1^{-1}) \neq 0$ is very important to have a lower bound like Theorem 4.3.5. For instance, the main term of Theorem 5.1.7 is $J_{k-1}(4\pi\sqrt{mn})\frac{\mu(N)}{N}\prod_{p|N}\left(1-p^{-2}\right)$. When N is squarefree, $\mu(N)\neq 0$ so that $J_{k-1}(4\pi\sqrt{mn})\frac{\mu(N)}{N}\prod_{p|N}\left(1-p^{-2}\right)\neq 0$ in Theorem 5.1.7. Therefore the non-vanishing of the Kloosterman sum in Theorem 4.3.5 is a crucial assumption in getting a lower bound (see Theorem 3.0.1 also).*

Note that we can similarly obtain a result like Theorem 4.3.5 under the assumption that $\frac{2\pi\gamma_j\sqrt{\sigma_j(m_1m_2)}}{\delta} \in \Big((k_j-1)-(k_j-1)^{\frac{1}{3}},(k_j-1)\Big)$ for all j, with fixed $d \in F^*$ and $\gamma_j = \frac{\sqrt{\sigma_j(\eta_1)}}{|\sigma_j(d)|}$. This is shown in the next lemma and the lemma will help us to get our desired discrepancy result.

Lemma 4.3.7. *Let F has odd narrow class number and $\mathfrak{b}_1\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s}\in\mathbb{N}$. Let $d\mathcal{O}=\mathfrak{d}$ and $\frac{2\pi\gamma_j\sqrt{\sigma_j(m_1m_2)}}{\delta} \in \Big((k_j-1)-(k_j-1)^{\frac{1}{3}},(k_j-1)\Big)$ for all j, where $\gamma_j = \frac{\sqrt{\sigma_j(\eta_1)}}{|\sigma_j(d)|}$. Further assume that $S_{\omega_{\mathfrak{N}}}(m_1,m_2;\eta_1 b_1^{-2};sb_1^{-1}) \neq 0$ for some m_1 and m_2. Then*

$$\Bigg|\frac{e^{2\pi tr_{\mathbb{Q}}^{F}(m_1+m_2)}}{\psi(\mathfrak{N})}\Bigg[\prod_{j=1}^{r}\frac{(k_j-2)!}{(4\pi\sqrt{\sigma_j(m_1m_2)})^{k_j-1}}\Bigg]\sum_{\phi\in\mathcal{F}}\frac{\lambda_{\mathfrak{n}}^{\phi}W_{m_1}^{\phi}(1)\overline{W_{m_2}^{\phi}(1)}}{\|\phi\|^2}$$

$$- \hat{T}(m_1, m_2, \mathfrak{n}) \frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}(m_1/s)\omega_{\mathrm{fin}}(s)} \Bigg| \gg_{F,\mathfrak{N},\mathfrak{n}} \prod_{j=1}^{r} (k_j - 1)^{\frac{-1}{3}}.$$

Proof. To see this replace the new $\gamma_j = \frac{\sqrt{\sigma_j(\eta_1)}}{|\sigma_j(d)|}$ in place of the old $\gamma_j = \sqrt{\sigma_j(\eta_1)}$ in the proof of Theorem 4.1.1. The rest of the argument follows through proof of Theorem 4.3.5 similarly. $\square$

Corollary 4.3.8. *Let F have an odd narrow class number equal to 1 and l be an odd natural number. Let $\mathfrak{b}_1\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s} \in \mathbb{N}$ and squarefree. Further let $\omega_{\mathfrak{N}}$ be trivial with $\tilde{p}\mathcal{O} = \mathfrak{p}$ and $d\mathcal{O} = \mathfrak{d}$. Let $\frac{2\pi\gamma_j\sqrt{\sigma_j(\tilde{p}^l)}}{\delta|\sigma_j(d)|} \in \Big((k_j - 1) - (k_j - 1)^{\frac{1}{3}}, (k_j - 1)\Big)$ for all j, with $\gamma_j = \frac{\sqrt{\sigma_j(\eta_1)}}{|\sigma_j(d)|}$. Then we have*

$$\left| \frac{1}{\sqrt{\mathrm{Nm}(\tilde{p}^l)}} \left[\prod_{j=1}^{r} \frac{(k_j - 2)!}{(4\pi)^{k_j - 1}} \sum_{\phi \in \mathcal{F}} \frac{\lambda_{\mathfrak{p}^l}^{\phi}}{\|\phi\|^2} \right] \right| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r} (k_j - 1)^{\frac{-1}{3}}.$$

Proof. We use Corollary 2.4.3 by taking $m_1 = \frac{\tilde{p}^l}{d}$ and $m_2 = \frac{1}{d}$. Hence

$$\frac{e^{4\pi r}\mathrm{Nm}(d)}{\psi(\mathfrak{N}) d_F^2 \sqrt{\mathrm{Nm}(\tilde{p}^l)}} \left[\prod_{j=1}^{r} \frac{(k_j - 2)!}{(4\pi)^{k_j - 1}} \right] \sum_{\phi \in \mathcal{F}} \frac{\lambda_{\mathfrak{p}^l}^{\phi}}{\|\phi\|^2}$$

$$= \hat{T}\Big(\frac{\tilde{p}^l}{d}, \frac{1}{d}, \mathcal{O}\Big) \frac{\sqrt{d_F \mathrm{Nm}(\mathfrak{n})}}{\omega_{\mathfrak{N}}\Big(\frac{\tilde{p}^l}{sd}\Big)\omega_{\mathrm{fin}}(s)} + \sum_{s \in \mathfrak{b}_1\mathfrak{N}/\pm, s \neq 0} \Bigg\{ \omega_{\mathrm{fin}}(s b_1^{-1}) S_{\omega_{\mathfrak{N}}}\Big(\frac{\tilde{p}^l}{d}, \frac{1}{d}; 1; \tilde{s}\Big) \times$$

$$\frac{\sqrt{\mathrm{Nm}(\eta_1)}}{\mathrm{Nm}(s)} \times \prod_{j=1}^{r} \frac{2\pi}{(\sqrt{-1})^{k_j}} J_{k_j - 1}\Big(\frac{4\pi\sqrt{\sigma_j(\eta_1 \tilde{p}^l)}}{|\sigma_j(sd)|}\Big) \Bigg\}.$$

l being an odd natural number , $\hat{T}\Big(\frac{\tilde{p}^l}{d}, \frac{1}{d}, \mathcal{O}\Big) = 0$. However by Lemma 3.0.1, we get $S_{\omega_{\mathfrak{N}}}\Big(\frac{\tilde{p}^l}{d}, \frac{1}{d}; 1; \tilde{s}\Big) \neq 0$. Now apply Lemma 4.3.7 to have

$$\left| \frac{e^{4\pi r}\mathrm{Nm}(d)}{\psi(\mathfrak{N}) d_F^2 \sqrt{\mathrm{Nm}(\tilde{p}^l)}} \left[\prod_{j=1}^{r} \frac{(k_j - 2)!}{(4\pi)^{k_j - 1}} \right] \sum_{\phi \in \mathcal{F}} \frac{\lambda_{\mathfrak{p}^l}^{\phi}}{\|\phi\|^2} \right|$$

$$\gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_j-1)^{\frac{-1}{3}}.$$

Therefore

$$\left|\frac{1}{\sqrt{\mathrm{Nm}(\widetilde{p}^l)}}\left[\prod_{j=1}^{r}\frac{(k_j-2)!}{(4\pi)^{k_j-1}}\sum_{\phi\in\mathcal{F}}\frac{\lambda^{\phi}_{\mathfrak{p}^l}}{\|\phi\|^2}\right]\right| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_j-1)^{-\frac{1}{3}}.$$

□

Theorem 4.3.9 ([BDS23], Theorem 3)**.** *Let F have a narrow class number equal to 1. Let $\mathfrak{b}_1\mathfrak{N} = \widetilde{s}\mathcal{O}$ with $\widetilde{s} \in \mathbb{Z}$ and $|\widetilde{s}|$ being squarefree. Further let $\omega_{\mathfrak{N}}$ be trivial. Then there exists an infinite sequence of weights $k_l = (k_{l1}, ..., k_{lr})$ with $(k_l)_0 \to \infty$ such that*

$$D(\widetilde{\nu}_{k_l,\mathfrak{N}}, \mu_\infty) \gg \frac{1}{\left(\log k_{lj}\right)^2 \times \prod_{i=1}^{r}(k_{li}-1)^{\frac{1}{3}}}.$$

for all $j \in \{1, \dots, r\}$.

The exponent $\frac{1}{3}$ in Theorem 4.3.9 shows that one can not achieve

$$D(\widetilde{\nu}_{k_l,\mathfrak{N}}, \mu_\infty) = O_{\epsilon,N}\left(\prod_{j=1}^{r}(k_j-1)^{-\frac{1}{2}+\epsilon}\right)$$

for every even weight $k = (k_1, \dots, k_r)$ (also refer to [JS20, (1.9)]).

Proof. Let k_{lj} be such that $\frac{2\pi\gamma_j\sqrt{\sigma_j(\widetilde{p}^l)}}{\delta|\sigma_j(d)|} \in \Big((k_{lj}-1)-(k_{lj}-1)^{\frac{1}{3}}, (k_{lj}-1)\Big)$. In particular we can take $k_{lj} = \left[\frac{2\pi\gamma_j\sqrt{\sigma_j(\widetilde{p}^l)}}{\delta|\sigma_j(d)|}\right]-1$. where $[x]$ denotes the greatest integer part of x. Using Corollary 4.3.8 we have

$$\left|\frac{1}{\sqrt{\mathrm{Nm}(\widetilde{p}^l)}}\left[\prod_{j=1}^{r}\frac{(k_{lj}-2)!}{(4\pi)^{k_{lj}-1}}\sum_{\phi\in\mathcal{F}}\frac{\lambda^{\phi}_{\mathfrak{p}^l}}{\|\phi\|^2}\right]\right| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_{lj}-1)^{\frac{-1}{3}} \tag{4.9}$$

Recall that $\kappa^{\phi}_{\mathfrak{p}^l} = \frac{\lambda^{\phi}_{\mathfrak{p}^l}}{\sqrt{\mathrm{Nm}(\mathfrak{p}^l)}}$ and $\kappa^{\phi}_{\mathfrak{p}^l} \in [-2, 2]$. By Proposition 4.5 of [KL08] we have

$$\kappa^{\phi}_{\mathfrak{p}^l} = X_l(\kappa^{\phi}_{\mathfrak{p}})$$

where $X_l(2\cos\theta) = \frac{\sin(l+1)\theta}{\sin\theta}$ is the Chebyshev polynomial of second kind with degree l.

Rewriting equation (4.9)

$$\left|\prod_{j=1}^{r}\frac{(k_{lj}-2)!}{(4\pi)^{k_{lj}-1}}\sum_{\phi\in\mathcal{F}}\frac{\kappa_{\mathfrak{p}^l}^{\phi}}{\|\phi\|^2}\right| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_{lj}-1)^{\frac{-1}{3}}$$

equivalently

$$\left|\prod_{j=1}^{r}\frac{(k_{lj}-2)!}{(4\pi)^{k_{lj}-1}}\sum_{\phi\in\mathcal{F}}\frac{X_l(\kappa_{\mathfrak{p}}^{\phi})}{\|\phi\|^2}\right| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_{lj}-1)^{\frac{-1}{3}}.$$

Hence

$$\left|\int_{-2}^{2} X_l(x)\, d\widetilde{\nu}_{k_l,\mathfrak{N}}(x)\right| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_{lj}-1)^{\frac{-1}{3}}.$$

Using equation (5.1),

$$\int_{-2}^{2} X_l(x)\, d\mu_\infty = 0.$$

This helps us to deduce

$$\left|\int_{-2}^{2} X_l(x)\, d(\widetilde{\nu}_{k_l,\mathfrak{N}}-\mu_\infty)(x)\right| = \left|\int_{-2}^{2} X_l(x)\, d\widetilde{\nu}_{k_l,\mathfrak{N}}(x) - \int_{-2}^{2} X_l(x)\, d\mu_\infty(x)\right|$$

$$= \left|\int_{-2}^{2} X_l(x)\, d\widetilde{\nu}_{k_l,\mathfrak{N}}(x)\right| \gg_{F,\mathfrak{N}} \prod_{j=1}^{r}(k_{lj}-1)^{\frac{-1}{3}}. \tag{4.10}$$

Using integration by parts and $\left|X_l'(x)\right| \ll l^2$ we get

$$\left|\int_{-2}^{2} X_l(x)\, d(\widetilde{\nu}_{k_l,\mathfrak{N}}-\mu_\infty)(x)\right| \ll l^2 \left|\int_{-2}^{2} d(\widetilde{\nu}_{k_l,\mathfrak{N}}-\mu_\infty)(x)\right| \tag{4.11}$$

Consider

$$D(\widetilde{\nu}_{k_l,\mathfrak{N}},\mu_\infty) \geq \left|\widetilde{\nu}_{k_l,\mathfrak{N}}([-2,2]) - \mu_\infty([-2,2])\right|$$

$$= \left|\int_{-2}^{2} d(\widetilde{\nu}_{k_l,\mathfrak{N}}-\mu_\infty)(x)\right| \gg \frac{1}{l^2}\left|\int_{-2}^{2} X_l(x)\, d(\widetilde{\nu}_{k_l,\mathfrak{N}}-\mu_\infty)(x)\right|$$

by equation (4.11). Equation (4.10) then implies

$$D(\widetilde{\nu}_{k_l,\mathfrak{n}}, \mu_\infty) \gg \frac{1}{l^2 \times \prod_{j=1}^{r} (k_{lj} - 1)^{\frac{1}{3}}}.$$

However $k_{lj} = \left[\frac{2\pi\gamma_j \sqrt{\sigma_j(\widetilde{p}^l)}}{|\sigma_j(d)|\delta} \right] - 1$, so that $\frac{2\pi\gamma_j \sqrt{\sigma_j(\widetilde{p}^l)}}{|\sigma_j(d)|\delta} < 2k_{lj}$. This implies $\frac{k_{lj}\delta|\sigma_j(d)|}{\pi\gamma_j} > \left(\sqrt{\sigma_j(\widetilde{p})}\right)^l$ and $\log k_{lj} \gg l$. Therefore

$$D(\widetilde{\nu}_{k_l,\mathfrak{n}}, \mu_\infty) \gg \frac{1}{\left(\log k_{lj_0}\right)^2 \times \prod_{j=1}^{r} (k_{lj} - 1)^{\frac{1}{3}}}.$$

for any $j_0 \in \{1, \ldots, r\}$. $\square$

5

Equidistribution results for eigenvalues

In this chapter, we discuss some results involving equidistribution of eigenvalues. In section 5.1 we consider equidistribution results for $\lambda_p(f)$. Section 5.2 consists of equidistribution results for $\lambda_{p^2}(f)$.

5.1 Equidistribution results for $\lambda_p(f)$

Let $S_k(N)$ denote the space of cusp forms of even integer weight k and level N. Let $\dim(S_k(N))$ denote the dimension of the vector space $S_k(N)$. The n^{th} normalised Hecke operator acting on $S_k(N)$ is given by

$$T_n(f)(z) := n^{\frac{k-1}{2}} \sum_{ad=n, d>0} \frac{1}{d^k} \sum_{b \pmod d} f\left(\frac{az+b}{d}\right).$$

Let $\mathcal{F}_k(N)$ be an orthonormal basis of $S_k(N)$ consisting only of joint eigenfunctions of the Hecke operators T_n with $(n, N) = 1$. For $f \in S_k(N)$, we have Fourier expansion of f at the cusp ∞ which is given by

$$f(z) = \sum_{n=1}^{\infty} a_n(f) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

we denote $\lambda_n(f)$ to be the n^{th} normalised Hecke eigenvalue of f i.e. $T_n(f) = \lambda_n(f)f$. We have $a_n(f) = a_1(f)\lambda_n(f)$. From the Ramanujan-Deligne bound [Del74] we have

$$|\lambda_n(f)| \leq \tau(n)$$

where $\tau(n)$ denotes the divisor function.

Consider the subspace of $S_k(N)$ spanned by the set

$$\left\{f(mz) \,:\, d|N,\, d < N,\, f \in S_k(d), m \text{ divides } \frac{N}{d}\right\}.$$

The subspace obtained is the space of oldforms in $S_k(N)$ denote by $S_k^{\text{old}}(N)$. The space of newforms $S_k(N)^*$ is defined to be the orthogonal complement of $S_k^{\text{old}}(N)$ in $S_k(N)$ with respect to the Petersson inner product. The subspace $S_k^{\text{old}}(N)$ and $S_k(N)^*$ are stable under the Hecke operator T_n for all $n \in \mathbb{N}$ (see Proposition 5.6.2 of [DS05]). Let T_n^* be the restriction of Hecke operator T_n from $S_k(N)$ to $S_k(N)^*$. Let $\mathcal{F}_k(N)^*$ be an orthonormal basis of $S_k(N)^*$ consisting only of joint eigenfunctions of the Hecke operators T_n^* (see Corollary 5.6.3 of [DS05]).

Definition 5.1.1. Let μ be a probability measure on $[a, b]$. A sequence of real numbers $x_n \in [a, b]$ is equidistributed with respect to the measure μ if for any $[a', b'] \subset [a, b]$,

$$\lim_{n\to\infty} \frac{|\{m \leq n \,:\, x_m \in [a', b']\}|}{n} = \int_{a'}^{b'} d\mu.$$

Definition 5.1.2. Let μ be a probability measure on $[a, b]$. A sequence of finite multisets A_n with $|A_n| \to \infty$ as $n \to \infty$ are equidistributed with respect to the measure μ if for any $[a', b'] \subset [a, b]$,

$$\lim_{n\to\infty} \frac{|t \in A_n \,:\, t \in [a', b']|}{|A_n|} = \int_{a'}^{b'} d\mu.$$

Let us recall the Sato$-$Tate measure given by

$$\mu_\infty(x) := \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}$$

if $x \in [-2, 2]$. We recall the following two crucial theorems concerning the distribution of eigenvalues which we already discussed in Chapter 1.

Theorem 5.1.3 (Barnet-Lamb, Geraghty, Harris, and Taylor)**.** *Let $f \in S_k(N)^*$ be a fixed non-CM newform. The sequence $\{\lambda_p(f) : p$ prime, $(p, N) = 1\}$ is equidistributed in $[-2, 2]$ with respect to the measure μ_∞.*

Proof. We refer to [BLGHT11, Theorem B(3)]. □

Theorem 5.1.4 (Serre)**.** *Let p be a fixed prime number. The familiy of multisets $A_l = \{\lambda_p(f) : (p, N_l) = 1, f \in S_{k_l}(N_l)$ and $N_l + k_l \to \infty\}$ are equidistributed in $[-2, 2]$ with respect to the measure μ_p.*

Proof. We refer to [Ser97, Theroem 1]. □

Recall

$$\mu_{k,N} := \frac{1}{\dim(S_k(N))} \sum_{f \in \mathcal{F}_k(N)} \delta_{\lambda_p(f)}$$

and

$$\mu^*_{k,N} := \frac{1}{\dim(S^*_k(N))} \sum_{f \in \mathcal{F}^*_k(N)} \delta_{\lambda_p(f)}$$

where δ_x is the Dirac measure at x. We would also like the reader to recall the notion of Discrepancy given by Definition 1.1.5. The main goal of Problem 1 is to get a discrepancy result like equation (1.5) for Hilbert cusp forms. A result like equation (1.5) is obtained with the help of an explicit asymptotic version of the Petersson trace formula (Theorem 5.1.7), which is an important ingredient in [JS20].

Before considering the asymptotic version of the Petersson trace formula, we consider the Petersson trace formula for $S_k(N)$. Let $\rho_n(f) := \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right)^{\frac{1}{2}} a_n(f)$ and $\Delta_{k,N}(m, n) := \sum_{f \in} \overline{\rho_m(f)}\rho_n(f)$.

Proposition 5.1.1. Let $n \in \mathbb{N}$. Then

$$\Delta_{k,N}(1, n) = \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right) \sum_{f \in \mathcal{F}_k(N)} |a_1(f)|^2 \lambda_n(f).$$

Proof. Take $m = 1$ in definition of $\Delta_{k,N}(m, n)$ to get

$$\Delta_{k,N}(1, n) = \sum_{f \in \mathcal{F}_k(N)} \overline{\rho_1(f)}\rho_n(f) = \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right) \sum_{f \in \mathcal{F}_k(N)} \overline{a_1(f)} a_n(f).$$

The proposition follows from the fact $a_n(f) = a_1(f)\lambda_n(f)$. □

The proposition also justifies the choice of variants $\nu_{k,N}$ and $\nu^*_{k,N}$ (see (1.4)).

Theorem 5.1.5 ([ILS00, Proposition 2.1]). ***Petersson's Trace formula*** *Let m, n be natural numbers. We have*

$$\Delta_{k,N}(m,n) = \sum_{f \in \mathcal{F}_k(N)} \overline{\rho_m(f)}\rho_n(f) = \delta(m,n) + 2\pi i^k \sum_{N|c, c>0} \frac{S(m,n;c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big).$$

We would now like to consider the following asymptotic result of $\Delta_{k,N}(m,n)$ for future reference.

Theorem 5.1.6 ([ILS00], Corollary 2.3). *Let N be a fixed positive integer. Let m, n be such that $\frac{4\pi\sqrt{mn}}{N} \leq \frac{k}{3}$. We have*

$$\Delta_{k,N}(m,n) = \delta(m,n) + O_N\left(\frac{k^{1+\epsilon}}{2^k}\right)$$

Proof. For a proof, we refer the reader to the statement of Corollary 2.3 of [ILS00]. Since N is fixed,

$$\frac{(m,n,N)}{\big((m,N)+(n,N)\big)^{\frac{1}{2}}} = O_N(1).$$

The result follows as

$$d(mn) = O_N(k^\epsilon).$$

□

Let

$$\Delta^*_{k,N}(m,n) := \sum_{f \in \mathcal{F}^*_k(N)} \overline{\rho_m(f)}\rho_n(f).$$

In 2020, Jung and Sardari obtained the following estimate for $\Delta^*_{k,N}(m,n)$ with a fixed squarefree level N.

Theorem 5.1.7 ([JS20], Theorem 1.7). *Let N be a fixed squarefree positive integer. Let*

$|4\pi\sqrt{mn} - k| < 2k^{\frac{1}{3}}$ *with* $\gcd(mn, N) = 1$. *Then*

$$\Delta^*_{k,N}(m,n) = \delta(m,n) + 2\pi i^{-k}\frac{\mu(N)}{N}\prod_{p|N}(1-p^{-2})J_{k-1}(4\pi\sqrt{mn}) + O_N(k^{-\frac{1}{2}}).$$

Proof. For a proof, we refer the reader to proof of Theorem 1.7 of [JS20]. □

Remark 5.1.8. *Since* $|4\pi\sqrt{mn} - k| < 2k^{\frac{1}{3}}$ *in Theorem 5.1.7, using Lemma 2.4.1 (iii) we have*

$$|J_{k-1}(4\pi\sqrt{mn})| \gg_N k^{-\frac{1}{3}}.$$

This makes $2\pi i^{-k}\frac{\mu(N)}{N}\prod_{p|N}(1-p^{-2})J_{k-1}(4\pi\sqrt{mn})$ *to be the main term in the above theorem and* $|\Delta^*_{k,N}(m,n) - \delta(m,n)| \gg_N k^{-\frac{1}{3}}$. *Note that Corollary 2.2 of [ILS00] gives us an asymptotic result for* $\Delta_{k,N}(m,n)$ *which Jung and Sardari in [JS20] use to prove Theorem 5.1.7.*

5.2 Equidistribution results for $\lambda_{p^2}(f)$

For a given non negative integer n, the Chebyshev polynomial of the second kind $X_n(x)$ is given by $X_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$. The generating function for X_n is given by

$$\sum_{n=0}^{\infty} X_n(x)t^n = \frac{1}{1-xt+t^2}.$$

Proposition 5.2.1. Let $n \in \mathbb{N} \cup \{0\}$. Then X_n are orthogonal with respect to the measure μ_∞.

Proof. For a proof, we refer to [CDF97, Lemma 3]. □

In particular, for $n \geq 1$, we have

$$\int_{-2}^{2} X_n(x)\, d\mu_\infty(x) = \int_{-2}^{2} X_0(x)X_n(x)\, d\mu_\infty(x) = 0. \tag{5.1}$$

Let $Y_n(x)$ be defined by the generating function given by

$$\sum_{n=0}^{\infty} Y_n(x)t^n = \frac{1+t}{1-(x-1)t+t^2}.$$

We have the following relation between X_n and Y_n.

Lemma 5.2.1. *Let $n \in \mathbb{N}$, then $X_{2n} = Y_n \circ X_2$.*

Proof. We have $X_2(x) = x^2 - 1$ which implies

$$\sum_{n=0}^{\infty} Y_n(X_2(x))t^n = \frac{1+t}{1-((x^2-1)-1)t+t^2} = \frac{1+t}{(1+t)^2 - x^2 t}.$$

However for generating function of X_{2n}, we calculate

$$\sum_{n=0}^{\infty} X_{2n}(x)t^{2n} = \frac{1}{2}\Big(\sum_{n=0}^{\infty} X_n(x)t^n + \sum_{n=0}^{\infty} X_n(x)(-t)^n\Big) = \frac{1}{2}\Big(\frac{1}{1-xt+t^2} + \frac{1}{1+xt+t^2}\Big)$$

$$= \frac{1+t^2}{(1-xt+t^2)(1+xt+t^2)} = \frac{1+t^2}{(1+t^2)^2 - x^2t^2}.$$

On comparing the generating function of X_{2n} and $Y_n \circ X_2$, we get $X_{2n} = Y_n \circ X_2$. $\square$

Let

$$Q_{2n+1} = Y_1 + Y_3 + \cdots + Y_{2n+1}$$

and

$$Q_{2n} = Y_0 + Y_2 + \cdots + Y_{2n}.$$

Then by Lemma 2 of [OM09],

$$\sum_{n=0}^{\infty} Q_n(x)t^n = \frac{1}{(1-t)(1-(x-1)t+t^2)}.$$

The Ramanujan-Hecke identity for powers of $\lambda_p(f)$ is given by

$$(\lambda_p(f))^n = X_n\Big(\lambda_p(f)\Big). \tag{5.2}$$

For reference, we consider Lemma 3 of [CDF97] (see [Ser97]). The following Lemma due to Omar and Mazouda talks about powers of $\lambda_{p^2}(f)$.

Lemma 5.2.2 ([OM09], Lemma 4)**.** *Let $n \in \mathbb{N}$. Then*

$$(\lambda_{p^2}(f))^n = Q_n(\lambda_{p^2}(f)).$$

Proof. For proof, we refer to [OM09, Lemma 4]. □

Recall that

$$\mu_{p^2}(x) = \frac{p+1}{2\pi} \frac{1}{(\sqrt{p}+\sqrt{p^{-1}})^2 - (x+1)} \sqrt{\frac{3-x}{x+1}}$$

if $x \in [-1,3]$. In 2009, Omar and Mazhouda, using Lemma 5.2.2, showed the following equidistribution results.

Theorem 5.2.3 ([OM09], Theorem 1)**.** *Let p be a fixed prime and $k > 0$ a fixed even integer. The sequence*

$$\{(\lambda_{p^2}(f))_{f \in S_k(N)^*} : N \in \mathbb{N}\}$$

is equidistributed with respect to the measure μ_{p^2}.

Proof. We refer the reader to the proof of [OM09, Theorem 1]. □

The distribution for $(\lambda_{p^3}(f))$, $(\lambda_{p^4}(f))$ and $(\lambda_{p^r}(f) - \lambda_{p^{r-2}}(f))$ for $r \geq 2$ has been discussed in [TW16].

Let

$$\mu_{k,N,2} := \frac{1}{\dim(S_k(N))} \sum_{f \in \mathcal{F}_k(N)} \delta_{\lambda_{p^2}(f)}$$

and

$$\mu^*_{k,N,2} := \frac{1}{\dim(S^*_k(N))} \sum_{f \in \mathcal{F}^*_k(N)} \delta_{\lambda_{p^2}(f)}.$$

We now consider the discrepancy between the following measures. The discrepancies $D(\mu^*_{k,N,2}, \mu_{p^2})$ have been well investigated in [TW16].

Theorem 5.2.4 ([TW16], Theorem 1)**.** *Let $N = 1$. Then we have $D(\mu^*_{k,1,2}, \mu_{p^2}) = O\big((\log k)^{-1}\big)$.*

Proof. On taking $r = 2$ and $N = 1$ in [TW16, Theorem 1], we get $D(\mu^*_{k,1,2}, \mu_{p^2}) = O\big((\log k)^{-1}\big)$. □

Definition 5.2.5. Let Ω be a compact subset of $\mathbb{R}$. A sequence of probability measures $\widetilde{\mu}_n$ converges weakly to a measure $\widetilde{\mu}$ if for every continuous function f on Ω,

$$\int_\Omega f \, d\widetilde{\mu}_n \to \int_\Omega f \, d\widetilde{\mu}.$$

Theorem 5.2.6. *Let $x_n \in [a, b]$ be a sequence of real numbers and*

$$\widetilde{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_n}.$$

The sequence x_n is equidistributed with respect to the measure μ if and only if $\widetilde{\mu}_n$ converges weakly to μ.

Proof. For a proof, we refer [KN74, Chapter 3, Theorem 1.2]. □

Remark 5.2.7. *By virtue of the above theorem, we see that the notion of equidistribution of a bounded real sequence x_n and weak convergence of an associated counting measure*

$$\widetilde{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_n}$$

are equivalent.

Consider the following weight variants of $\mu_{k,N,2}$ and $\mu^*_{k,N,2}$. Let

$$\nu_{k,N,2} := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{F}_k(N)} |a_1(f)|^2 \delta_{\lambda_{p^2}(f)},$$

and

$$\nu^*_{k,N,2} := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{F}^*_k(N)} |a_1(f)|^2 \delta_{\lambda_{p^2}(f)}.$$

One can use Petersson's trace formula to show both $\nu_{k,N,2}$ and $\nu^*_{k,N,2}$ converges weakly to $\mu_{\infty,2}$ any positive N with $(p, N) = 1$ (see [OM09, Theorem 3]).

We conclude the chapter with the following upper bound for Q'_n that will be used in the last chapter and play a role in solving Problem 5.

Lemma 5.2.8. *Let $\epsilon > 0$ be given and $n \in \mathbb{N}$. Then for $x \in (-1, 3]$,*

$$|Q'_n(x)| \ll_\epsilon n^{3+\epsilon}$$

and

$$|Q_n(3)| \ll n^2.$$

Proof. We prove for n even. The case when n is odd can be dealt with similarly. If not, then for some fixed $x_0 \in (-1, 3]$ we have $|Q'_{2n}(x_0)| \gg_\epsilon n^{3+\epsilon}$. There exists some $y_0 \in [-2, 2] \setminus \{0\}$ such that $X_2(y_0) = y_0^2 - 1 = x_0$. From proof of Theorem 1.7 of [JS20] we have $|X'_\alpha(x)| \ll \alpha^2$ for all $x \in [-2, 2]$ and $\alpha \in \mathbb{N}$. This implies

$$\left| \sum_{i=0}^{n} X'_{4i}(y_0) \right| \ll n^3$$

Using Lemma 5.2.1 we have,

$$\left| X'_2(y_0) \sum_{i=0}^{n} Y'_{2i}(x_0) \right| \ll n^3$$
$$\Rightarrow \left| 2y_0 \right| \left| \sum_{i=0}^{n} Y'_{2i}(x_0) \right| \ll n^3.$$

Since y_0 is fixed, we get $\left| Q'_{2n}(x_0) \right| \ll n^3$. This is a contradiction to our assumption. Therefore $|Q'_{2n}(x)| \ll_\epsilon n^{3+\epsilon}$ for $x \in (-1, 3]$.

$$Q_{2n}(3) = \sum_{i=0}^{n} Y_{2i}(3) = \sum_{i=0}^{n} X_{4i}(2).$$

Noting $X_{2n}(2) = 2n + 1$ yields $|Q_{2n}(3)| \ll n^2$. □

6

Discrepancy results for classical case

In this chapter, we restrict ourselves to the classical setting of $F = \mathbb{Q}$. First we derive an asymptotic formula similar to Theorem 5.1.7 with better estimates. This will help us in obtaining a discrepancy result like Theorem 6.0.3 for more levels. In Section 6.1, we discuss a discrepancy result for $\lambda_{p^2}(f)$ analogue to equation (1.5). The last section contains details about future work.

Theorem 6.0.1 ([Das23], Theorem 4)**.** *Let* $\left|\frac{4\pi\sqrt{mn}}{N} - (k-1)\right| < (k-1)^{\frac{1}{3}}$ *for* $k \geq 28$ *and* $m, n \in \mathbb{N}$. *We have*

$$\Delta_{k,N}(m,n) = \delta(m,n) + 2\pi i^k \frac{S(m,n,N)}{N} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{N}\Big) + O_N\left(e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))}\right).$$

Furthermore, if $S(m,n,N) \neq 0$, *then*

$$\Big|\Delta_{k,N}(m,n) - \delta(m,n)\Big| \gg_N (k-1)^{-\frac{1}{3}}.$$

Proof. Let us consider Petersson's trace formula

$$\Delta_{k,N}(m,n) = \delta(m,n) + 2\pi i^k \sum_{N|c, c>0} \frac{S(m,n;c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big)$$

$$= \delta(m,n) + 2\pi i^k \frac{S(m,n;N)}{N} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{N}\Big) + 2\pi i^k \sum_{b\geq 2} \frac{S(m,n;bN)}{bN} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{bN}\Big).$$

Using the triangle inequality and a trivial bound $|S(m,n;bN)| \leq bN$, we get

$$\left|\sum_{b\geq 2} \frac{S(m,n;bN)}{bN} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{bN}\Big)\right|$$

$$\leq \sum_{b\geq 2}\left|\frac{S(m,n;bN)}{bN} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{bN}\Big)\right| \leq \sum_{b\geq 2}\left|J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{bN}\Big)\right|. \tag{6.1}$$

Using the assumption,

$$1-\frac{1}{(k-1)^{\frac{2}{3}}} < \frac{4\pi\sqrt{mn}}{(k-1)N} < 1+\frac{1}{(k-1)^{\frac{2}{3}}}. \tag{6.2}$$

Note that for $b>1$ we can use the Lemma 2.4.1(i) as $\left|\frac{4\pi\sqrt{mn}}{(k-1)bN}\right| < 1$. Hence

$$\left|J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{bN}\Big)\right| = \left|J_{k-1}\bigg((k-1)\frac{4\pi\sqrt{mn}}{(k-1)bN}\bigg)\right|$$

$$\leq e^{a(1-x)}x^a J_a(a)$$

where $a=k-1$ and $x=\frac{4\pi\sqrt{mn}}{(k-1)bN}$. By virtue of equation (6.2) and $k\geq 28$,

$$\frac{8}{9b} < \frac{4\pi\sqrt{mn}}{(k-1)bN} < \frac{10}{9b}$$

We conclude

$$e^{a(1-x)}x^a = e^{(k-1)(1-x+\log x)}$$

$$\ll_N e^{(k-1)\left(1-\frac{8}{9b}+\log\frac{10}{9b}\right)}.$$

Proceeding from equation (6.1) and using Lemma 2.4.1(ii) we get the following bound.

$$\sum_{b\geq 2}\left|J_{k-1}\left(\frac{4\pi\sqrt{mn}}{bN}\right)\right| \ll_N \frac{e^{(k-1)}}{(k-1)^{\frac{1}{3}}}\sum_{b\geq 2} e^{-(k-1)\left(\frac{8}{9b}+\log\frac{9b}{10}\right)}$$
$$\ll_N \frac{e^{-(k-1)\left(-1+\frac{8}{18}+\log\frac{18}{10}\right)}}{(k-1)^{\frac{1}{3}}} + \frac{e^{-(k-1)\left(-1+\frac{8}{27}+\log\frac{27}{10}\right)}}{(k-1)^{\frac{1}{3}}}$$

$$+\frac{e^{-(k-1)\left(-1+\frac{8}{36}+\log\frac{36}{10}\right)}}{(k-1)^{\frac{1}{3}}} + \frac{e^{(k-1)}}{(k-1)^{\frac{1}{3}}}\left(\sum_{b\geq 5} e^{-(k-1)\left(\frac{8}{9b}+\log\frac{9b}{10}\right)}\right). \tag{6.3}$$

Since $\Big(-1+\frac{8}{18}+\log\frac{18}{10}\Big), \Big(-1+\frac{8}{27}+\log\frac{27}{10}\Big), \Big(-1+\frac{8}{36}+\log\frac{36}{10}\Big) \leq \Big(1-\frac{4}{9}-\log(\frac{9}{5})\Big)$, the first three terms in the equation (6.3) are equal to $O_N(e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))})$. We use the integral test to estimate

$$\left(\sum_{b\geq 5} e^{-(k-1)\left(\frac{8}{9b}+\log\frac{9b}{10}\right)}\right).$$

Similar to the proof of Proposition 4.2.1, we can show $e^{-(k-1)\left(\frac{8}{9x}+\log\frac{9x}{10}\right)}$ is strictly decreasing for $x>2$. Therefore

$$\left(\sum_{b\geq 5} e^{-(k-1)\left(\frac{8}{9b}+\log\frac{9b}{10}\right)}\right) \leq \int_4^\infty e^{-(k-1)\left(\frac{8}{9x}+\log\frac{9x}{10}\right)}\,dx. \tag{6.4}$$

Let $y=\frac{8}{9x}$, then $dy=\frac{-8}{9x^2}dx$ and

$$e^{-(k-1)\left(\frac{8}{9x}+\log\frac{9x}{10}\right)} = \frac{e^{-(k-1)y}}{(\frac{9x}{10})^{k-3}}\cdot\frac{100}{81x^2} = \frac{e^{-(k-1)y}}{(\frac{4}{5y})^{k-3}}\cdot\frac{100}{81x^2}.$$

This yields

$$\int_4^\infty e^{-(k-1)\left(\frac{8}{9x}+\log\frac{9x}{10}\right)}\,dx = \frac{25}{18}\int_0^{\frac{2}{9}} e^{-(k-1)y}\left(\frac{5y}{4}\right)^{k-3}\,dy$$
$$\leq \frac{25}{18}\int_0^{\frac{2}{9}}\left(\frac{5}{18}\right)^{k-3} e^{-(k-1)y}\,dy = \frac{25}{18}\left(\frac{5}{18}\right)^{k-3}\left(\frac{e^{\frac{-2(k-1)}{9}}}{-(k-1)}+\frac{1}{(k-1)}\right)$$

as $\left(\frac{5y}{4}\right)^{k-3} \leq \left(\frac{5}{18}\right)^{k-3}$. Applying the above bound for equation (6.4) and considering equation (6.1), we get

$$\sum_{b\geq 2}\left|J_{k-1}\left(\frac{4\pi\sqrt{mn}}{bN}\right)\right| \ll_N A_1(k) + \frac{e^{(k-1)}}{(k-1)^{\frac{1}{3}}}\left(\frac{5}{18}\right)^{k-3}\left(\frac{e^{\frac{-2(k-1)}{9}}}{-(k-1)} + \frac{1}{(k-1)}\right)$$

where $A_1(k) = O_N\left(e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))}\right)$. However

$$\frac{e^{(k-1)}}{(k-1)^{\frac{1}{3}}}\left(\frac{5}{18}\right)^{k-3}\left(\frac{e^{\frac{-2(k-1)}{9}}}{-(k-1)} + \frac{1}{(k-1)}\right) \leq \frac{2e^{(k-1)}}{(k-1)^{\frac{4}{3}}}\left(\frac{5}{18}\right)^{k-3}$$

$$= \frac{2e^2}{(k-1)^{\frac{4}{3}}}\left(\frac{5e}{18}\right)^{k-3}$$

Therefore

$$\sum_{b\geq 2}\left|J_{k-1}\left(\frac{4\pi\sqrt{mn}}{bN}\right)\right| \ll_N A_1(k) + A_2(k) \tag{6.5}$$

where $A_2(k) = \frac{2e^2}{(k-1)^{\frac{4}{3}}}\left(\frac{5e}{18}\right)^{k-3} = O_N\left(e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))}\right)$. Using equation (6.5) in equation (6.1) we get

$$\Delta_{k,N}(m,n) = \delta(m,n) + 2\pi i^k \frac{S(m,n,N)}{N} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{N}\right) + O_N\left(e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))}\right)$$

which proves the first part.

For second part, let $\frac{4\pi\sqrt{mn}}{N} = (k-1) + d(k-1)^{\frac{1}{3}}$ with $|d| < 1$. On applying Lemma 2.4.1 (iii),

$$J_{k-1}\left(\frac{4\pi\sqrt{mn}}{N}\right) \gg \frac{1}{(k-1)^{\frac{1}{3}}}.$$

This implies

$$\left|\Delta_{k,N}(m,n) - \delta(m,n)\right|$$

$$\gg_N \frac{1}{(k-1)^{\frac{1}{3}}} + o_N\left((k-1)^{-\frac{1}{3}}\right) \gg_N (k-1)^{-\frac{1}{3}}$$

as $e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))} = o_N\left((k-1)^{-\frac{1}{3}}\right)$. $\square$

Remark 6.0.2. *Since* $(1-\frac{4}{9}-\log(\frac{9}{5})) < -0.03$*, we have* $e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))} = o_N((k-1)^{\frac{-1}{3}})$. *If* $S(m,n,N) = 0$*, then*

$$\Delta_{k,N}(m,n) = \delta(m,n) + O_N(e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))}).$$

For level $N = 1$*, Theorem 6.0.1 gives a better error term as compared to* $O_N(k^{-\frac{1}{2}})$ *of Theorem 5.1.7. That is we have*

$$\Delta_{k,1}(m,n) = \delta(m,n) + 2\pi i^k J_{k-1}\Big(4\pi\sqrt{mn}\Big) + O_N\Big(e^{(k-1)(1-\frac{4}{9}-\log(\frac{9}{5}))}\Big).$$

Theorem 6.0.1 also expands on the remark of Theorem 1.7 of [JS20] which states that Theorem 5.1.7 can be generalised for the condition

$$\left|\frac{4\pi\sqrt{mn}}{N} - (k-1)\right| < (k-1)^{\frac{1}{3}}.$$

Note that Theorem 5.1.6 gives an asymptotic of $\Delta_{k,N}(m,n)$ *for* $\frac{4\pi\sqrt{mn}}{N} \leq \frac{k}{3}$*, whereas Theorem 6.0.1 gives an asymptotic of* $\Delta_{k,N}(m,n)$ *for*

$$(k-1) - (k-1)^{\frac{1}{3}} < \frac{4\pi\sqrt{mn}}{N} < (k-1) + (k-1)^{\frac{1}{3}},$$

where N *is a fixed level.*

Recall that

$$\nu_{k,N} = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{F}_k(N)} |a_1(f)|^2 \delta_{\lambda_p(f)}.$$

On taking $k_n = \left[\frac{4\pi p^n}{N}\right]$ and using Theorem 6.0.1, we have

$$\int_{-2}^{2} X_{2n}(x)\, d\nu_{k_n,N} = \Delta_{k_n,N}(1,p^{2n}) = 2\pi i^k \frac{S(1,p^{2n},N)}{N} J_{k_n-1}\Big(\frac{4\pi p^n}{N}\Big) + o((k_n-1)^{-\frac{1}{3}}). \tag{6.6}$$

Theorem 6.0.3 ([Das23], Theorem 1)**.** *Let N be a natural number given by $2^a b$ with $a = 0, 1, 2$ and b be an odd number. There exists an infinite sequence of weights k_n with $k_n \to \infty$ such that*

$$D(\nu_{k_n,N}, \mu_\infty) \gg \frac{1}{(k_n-1)^{\frac{1}{3}}(\log k_n)^2}.$$

Proof. Let $k_n = \left[\frac{4\pi p^n}{N}\right]$. By Theorem 6.0.1(ii), Lemma 3.2.5, Equation (6.6) and (5.1) we have

$$\int_{-2}^{2} X_{2n}(x)\, d(\nu_{k_n,N} - \mu_\infty)(x) = \int_{-2}^{2} X_{2n}(x)\, d\nu_{k_n,N}(x) - \int_{-2}^{2} X_{2n}(x)\, d\mu_\infty(x)$$

$$= \int_{-2}^{2} X_{2n}(x)\, d\nu_{k_n,N}(x) \gg_N (k_n - 1)^{-\frac{1}{3}}.$$

Using integration by parts

$$\left| \int_{-2}^{2} X_{2n}(x)\, d(\nu_{k_n,N} - \mu_\infty)(x) \right|$$

$$= \left| \Big[X_{2n}(x)\, (\nu_{k_n,N} - \mu_\infty)(x) \Big]_{-2}^{2} - \int_{-2}^{2} X_n'(x)\, (\nu_{k_n,N} - \mu_\infty)(x) dx \right|$$

$$\leq \left| \Big(X_{2n}(2) + X_{2n}(-2) \Big) (\nu_{k_n,N} - \mu_\infty)([-2,2]) \right| + \left| \int_{-2}^{2} X_n'(x)\, (\nu_{k_n,N} - \mu_\infty)(x) dx \right|$$

$$\ll n^2 \left| (\nu_{k_n,N} - \mu_\infty)([-2,2]) \right|$$

as $\left| X_{2n}'(x) \right| \ll n^2$ and $|X_{2n}(2)| = |X_{2n}(-2)| = 2n + 1$. Hence we have

$$D(\nu_{k_n,N}, \mu_\infty) \geq \left| \nu_{k_n,N}([-2,2]) - \mu_\infty([-2,2]) \right| = \left| (\nu_{k_n,N} - \mu_\infty)([-2,2]) \right|$$

$$\gg \frac{1}{n^2} \left| \int_{-2}^{2} X_{2n}(x)\, d(\nu_{k_n,N} - \mu_\infty)(x) \right| \gg_N \frac{1}{n^2 (k_n - 1)^{\frac{1}{3}}}.$$

However $k_n = \left[\frac{4\pi\sqrt{p^{2n}}}{N}\right]$ implies $\frac{4\pi\sqrt{p^{2n}}}{N} < 2k_n$. So $k_n \gg_N p^n$ which implies $\log k_n \gg_N n$.

Therefore

$$D(\nu_{k_n,N}, \mu_\infty) \gg \frac{1}{(k_n-1)^{\frac{1}{3}}(\log k_n)^2}.$$

□

Remark 6.0.4. *Theorem 1.6 of [JS20] gives us a sequence of weights for squarefree levels such that lower bound like equation (1.5) holds. Theorem 6.0.3 generalizes a version of Theorem 1.6 of [JS20] for old forms to more levels of the form $2^a b$ with b odd and $a = 0, 1, 2$. Note that the natural density of squarefree integers is $\frac{6}{\pi^2}$ whereas the natural density of the levels for Theorem 6.0.3 is 0.875.*

6.1 Discrepancy result for $\lambda_{p^2}(f)$

We now consider a Lemma that will be crucial for obtaining the discrepancy result for $\lambda_{p^2}(f)$.

Lemma 6.1.1. *Let N be a natural number given by $2^a b$ with $a = 0, 1, 2$ and b be an odd number. Given $n \in \mathbb{N}$ and $\left|\frac{4\pi p^{2n+1}}{N} - (k_n - 1)\right| < (k_n-1)^{\frac{1}{3}}$, we have*

$$(i) \quad \sum_{i=0}^{n-1} \Delta_{k_n,N}(1, p^{4i+2}) = O_N\left(\frac{\log k_n}{(k_n-1)^{\frac{1}{3}}} \cdot \left(\frac{5e}{18}\right)^{k_n-1}\right)$$

$$(ii) \quad \Delta_{k_n,N}(1, p^{4n+2}) \gg_N (k_n-1)^{-\frac{1}{3}}.$$

Proof. Let $i \in \{1, 2, \ldots, n-1\}$ be given. Using Petersson's trace formula, we get

$$\Delta_{k_n,N}(1, p^{4i+2}) = 2\pi i^k \sum_{N|c, c>0} \frac{S(1, p^{4i+2}; c)}{c} J_{k_n-1}\left(\frac{4\pi p^{2i+1}}{c}\right).$$

Note that

$$\frac{4\pi p^{2i+1}}{(k_n-1)N} < \frac{1}{p^2}(1 + (k_n-1)^{-\frac{2}{3}})$$

which implies $\frac{4\pi p^{2i+1}}{(k_n-1)N} < \frac{1}{4} \cdot \frac{10}{9} = \frac{5}{18}$ as $p \geq 2$ and $k_n \geq 28$. Using Lemma 2.4.1 we have,

$$\left|\Delta_{k_n,N}(1, p^{4i+2})\right| \leq \left|J_{k_n-1}\left(\frac{4\pi p^{2i+1}}{N}\right)\right| + \sum_{b=2}^{\infty}\left|J_{k_n-1}\left(\frac{4\pi p^{2i+1}}{bN}\right)\right|$$

$$\leq e^{(k_n-1)}\left(\frac{5}{18}\right)^{k_n-1} J_{k_n-1}(k_n-1) + e^{(k_n-1)} J_{k_n-1}(k_n-1)\sum_{b=2}^{\infty}\left(\frac{5}{18b}\right)^{k_n-1}$$

as $\frac{4\pi p^{2i+1}}{(k_n-1)bN} < \frac{5}{18b}$ for $b \geq 2$.

$$\ll_N \left(\frac{5e}{18}\right)^{k_n-1}(k_n-1)^{-\frac{1}{3}}\left[1 + \int_1^{\infty}\left(\frac{1}{x}\right)^{(k_n-1)} dx\right]$$

$$\ll_N \left(\frac{5e}{18}\right)^{k_n-1}(k_n-1)^{-\frac{1}{3}}\left(1 + \lim_{t\to\infty}\left[\frac{x^{-k_n+2}}{-k_n+2}\right]_1^t\right)$$

$$\ll_N \left(\frac{5e}{18}\right)^{k_n-1}(k_n-1)^{-\frac{1}{3}}\left(1 + \frac{1}{(k_n-2)}\right) \ll_N \left(\frac{5e}{18}\right)^{k_n-1}(k_n-1)^{-\frac{1}{3}}.$$

Now by triangle inequality

$$\left|\sum_{i=0}^{n-1}\Delta_{k_n,N}(1, p^{4i+2})\right| \leq \sum_{i=0}^{n-1}\left|\Delta_{k_n,N}(1, p^{4i+2})\right|$$

$$\ll_N n\left(\frac{5e}{18}\right)^{k_n-1}(k_n-1)^{-\frac{1}{3}} \ll_N \log k_n\left(\frac{5e}{18}\right)^{k_n-1}(k_n-1)^{-\frac{1}{3}}.$$

Using Theorem 6.0.1(ii) and observing $\log k_n\left(\frac{5e}{18}\right)^{k_n-1}(k_n-1)^{-\frac{1}{3}} = o((k_n-1)^{-\frac{1}{3}})$, we derive

$$|\Delta_{k_n,N}(1, p^{4n+2})| \gg_N (k_n-1)^{-\frac{1}{3}}.$$

□

Lemma 6.1.2. *Let N be a natural number given by $2^a b$ with $a = 0, 1, 2$ and b be an odd number. There is a sequence of $k_n \to \infty$ such that*

$$\left|\int_{-1}^{3}(Q_{2n+1})(t)\, d\nu_{k_n,N,2}(t)\right| \gg_N \frac{1}{(k_n-1)^{\frac{1}{3}}}.$$

Proof. Let us consider

$$
\begin{aligned}
&\int_{-1}^{3} Q_{2n+1}(t)\, d\nu_{k_n,N,2}(t) \\
&= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathfrak{F}_{k,N}} |a_1(f)|^2 Q_{2n+1}(\lambda_{p^2}(f)) \\
&= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathfrak{F}_{k,N}} |a_1(f)|^2 Q_{2n+1}(X_2(\lambda_p(f))) \\
&= \int_{-2}^{2} Q_{2n+1}\circ X_2(t)\, d\nu_{k_n,N}(t) \\
&= \sum_{i=0}^{n} \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathfrak{F}_{k,N}} |a_1(f)|^2 X_{4n+2}(\lambda_p(f))
\end{aligned}
$$

as $Q_{2\alpha} = Y_0 + Y_2 + \cdots + Y_{2\alpha}$ with $Y_\alpha \circ X_2 = X_{2\alpha}$ (see section 5.2). Now using Proposition 5.1.1 and equation (5.2), the above expression

$$
= \sum_{i=0}^{n} \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathfrak{F}_{k,N}} |a_1(f)|^2 \lambda_{p^{4n+2}}(f)
$$

$$
= \sum_{i=0}^{n} \Delta_{k,N}(1, p^{4n+2}).
$$

Let $k_n = \left[\frac{4\pi p^{2n+1}}{N}\right]$ be the chosen sequence for $n \in \mathbb{N}$. The above sequence satisfies the hypothesis of Lemma 6.1.1. Therefore we get

$$
\left| \int_{-1}^{3} (Q_{2n+1})(t)\, d\nu_{k_n,N,2}(t) \right| \gg_N \frac{1}{(k_n-1)^{\frac{1}{3}}}.
$$

□

Theorem 6.1.3 ([Das23], Theorem 2)**.** *Let $\epsilon > 0$ be given. Then there exists a sequence of weights k_n with $k_n \to \infty$ such that*

$$
D(\mu_{\infty,2}, \nu_{k_n,N,2}) \gg_{N,\epsilon} \frac{1}{(\log k_n)^{3+\epsilon}(k_n-1)^{\frac{1}{3}}}
$$

Proof. The proof is similar to that of Theorem 6.0.3. From equation (4.2) of [OM09],

$$\int_{-1}^{3} Q_{2n+1}(t)\, d\mu_{\infty,2}(t) = \int_{-2}^{2} Q_{2n+1} \circ X_2\, d\mu_{\infty}(t) = \delta(2n+1, pair) = 0. \tag{6.7}$$

By equation (6.7) and 6.1 we have

$$\int_{-1}^{3} Q_{2n+1}(x)\, d(\nu_{k_n,N,2} - \mu_{\infty,2})(x) = \int_{-1}^{3} Q_{2n+1}(x)\, d\nu_{k_n,N,2}(x) \gg_N (k_n - 1)^{-\frac{1}{3}}.$$

Using integration by parts

$$\left| \int_{-1}^{3} Q_{2n+1}(x)\, d(\nu_{k_n,N,2} - \mu_{\infty,2})(x) \right|$$

$$= \left| \Big[Q_{2n+1}(x)\, (\nu_{k_n,N,2} - \mu_{\infty,2})(x) \Big]_{-1}^{3} - \int_{-1}^{3} Q'_{2n+1}(x)\, (\nu_{k_n,N,2} - \mu_{\infty,2})(x) dx \right|$$

$$\leq \left| \Big(Q_{2n+1}(-1) + Q_{2n+1}(3) \Big) (\nu_{k_n,N,2} - \mu_{\infty,2})([-1,3]) \right| + \left| \int_{-1}^{3} Q'_{2n+1}(x)\, (\nu_{k_n,N,2} - \mu_{\infty,2})(x) dx \right|$$

We now use Lemma 5.2.8 which yields

$$\ll_{\epsilon} n^{3+\epsilon} \Big| (\nu_{k_n,N,2} - \mu_{\infty,2})([-1,3]) \Big|.$$

Hence we have

$$D(\nu_{k_n,N,2}, \mu_{\infty,2}) \geq \Big| \nu_{k_n,N,2}([-1,3]) - \mu_{\infty,2}([-1,3]) \Big| = \Big| (\nu_{k_n,N,2} - \mu_{\infty,2})([-1,3]) \Big|$$

$$\gg \frac{1}{n^{3+\epsilon}} \left| \int_{-2}^{2} Q_{2n+1}(x)\, d(\nu_{k_n,N,2} - \mu_{\infty,2})(x) \right| \gg_N n^{-(3+\epsilon)} (k_n - 1)^{-\frac{1}{3}}.$$

For the chosen sequence $k_n = \left[\frac{4\pi p^{2n+1}}{N} \right]$, $k_n \gg_N p^n$ which implies $\log k_n \gg_N n$. Finally

$$D(\nu_{k_n,N,2}, \mu_{\infty,2}) \gg_{N,\epsilon} \frac{1}{(\log k_n)^{3+\epsilon} (k_n - 1)^{\frac{1}{3}}}.$$

□

Remark 6.1.4. *Theorem 6.1.3 gives us an analogue of Theorem 1.6 of [JS20] (see equation (1.5)) for the discrepancy* $D(\nu_{k_n,N,2}, \mu_{\infty,2})$ *corresponding to* $\lambda_{p^2}(f)$ *with more generalized levels.*

6.2 Future work

In this section, we would like to consider some of the future prospects to work upon. Note that Theorem 4.3.5 and Theorem 4.3.9 holds true if F has an odd narrow class number and $\mathfrak{b}_1\mathfrak{N} = \tilde{s}\mathcal{O}$ with $\tilde{s} \in \mathbb{N}$. First, we consider generalization of Theorem 4.3.5 in the aspects given by the following problem.

Problem 6. Can we generalize the lower bound result of Theorem 4.3.5 to any integral ideal for a field with an odd narrow class number? Can we generalize Theorem 4.3.5 to any arbitrary totally real field?

A similar generalization remark can be made for Theorem 4.3.9.

Problem 7. Can we generalize Theorem 4.3.9 to any integral ideal for a field with an odd narrow class number? Can we generalize Theorem 4.3.9 to any arbitrary totally real field?

Recall that Theorem 4.3.9 is a generalization of equation (1.5) for Hilbert modular form setting. Equation (1.3) gives us a lower bound for the Discrepancy $D(\mu^*_{k,N}, \mu_p)$.

Problem 8. Can we get a lower bound like $D(\mu^*_{k,N}, \mu_p)$ for Hilbert cusp forms?

For $f \in S_k(N)$, the discrepancy $D(\nu^*_{k_n,N}, \mu_\infty)$ corresponds to the distribution of $\{\lambda_p(f)\}$. Theorem 6.1.3 gives us lower bound for $D(\mu_{\infty,2}, \nu_{k_n,N,2})$ which corresponds to the distribution of $\{\lambda_{p^2}(f)\}$. It is natural to consider the following problem.

Problem 9. Can we generalise Theorem 6.1.3 for $\{\lambda_{p^r}(f)\}$ with $r > 2$ and $f \in S_k(N)$?

Upper bounds and nonvanishing of Kloosterman sums play a central role in proving Theorem 4.3.5, Theorem 6.0.3. In this regard, we can have some of the following directions. Theorem 3.1.2 gives a conditional upper bound for the trivial character.

Problem 10. Can we get an unconditional upper bound of Kloosterman sums for non-trivial characters as well?

Another direction is to explore the non-vanishing of Kloosterman sums for a general scenario. As seen in the proof of Lemma 3.2.3, getting explicit formula for a Kloosterman sum might be a good way to show the nonvanishing of a Kloosterman sum. As a starting point, We can consider generalizing some explicit formulas which are available for the classical case.

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