Harmonic analysis on locally symmetric spaces associated to cocompact discrete subgroups of $SL_2(\mathbb{R})$

A Thesis

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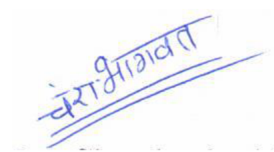
Certificate

This is to certify that this dissertation entitled Harmonic analysis on locally symmetric spaces associated to cocompact discrete subgroups of $SL_2(\mathbb{R})$towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Ajith Nair at Indian Institute of Science Education and Research under the supervision of Dr. Chandrasheel Bhagwat, Assistant Professor, Department of Mathematics , during the academic year 2016-2017.



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This thesis is dedicated to the *IISER brotherhood*.

Declaration

I hereby declare that the matter embodied in the report entitled Harmonic analysis on locally symmetric spaces associated to cocompact discrete subgroups of $SL_2(\mathbb{R})$ are the results of the work carried out by me at the Department of Mathematics, IISER Pune, under the supervision of Dr. Chandrasheel Bhagwat and the same has not been submitted elsewhere for any other degree.

Ajith Nair

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Abstract

In his 1956 paper, Selberg proved the famous Trace Formula for a semisimple Lie group G and its discrete subgroup Γ. The case when $G = SL_2(\mathbb{R})$ is quite well-known. In this thesis, we look at the decomposition of $L^2(\Gamma\backslash G)$ into irreducible unitary representations of G. The multiplicities of the spherical representations correspond to the eigenvalues of the Laplacian on the locally symmetric space $\Gamma\backslash G/K$. Our aim will be to find a finite threshold on the multiplicity spectrum, or equivalently for the eigenvalue spectrum, which determines the entire spectrum.

Contents

Introduction

The main goal of this project is to study a problem which lies in the intersection of representation theory, harmonic analysis and the theory of automorphic forms.

The theory of automorphic forms, pioneered by Klein, Poincare etc., was brought into the limelight by the works of Maass, Selberg, Roelcke etc. In particular, Selberg introduced new techniques from representation theory and spectral theory of self-adjoint operators on Hilbert spaces. The Selberg Trace Formula, which will be discussed here, is a remarkable result which culminated out of these techniques and has found applications in number theory, harmonic analysis and mathematical physics.

Here is a concrete description of the problem we're trying to solve:
Let G be $SL(2, \mathbb{R})$, K be the maximal compact subgroup of G, which is $SO(2)$ and Γ be a cocompact discrete subgroup of G. Let X be the Hilbert space $L^2(\Gamma\backslash G)$ and consider the right regular representation $\mathcal{R}$ of G on X. $\mathcal{R}$ breaks up as a Hilbert direct sum of irreducible unitary representations of G as follows:

$$\mathcal{R} = \hat{\bigoplus} m_\pi \pi, \text{ where } m_\pi \text{ is the multiplicity of } \pi \text{ and } 0 \leq m_\pi < \infty.$$

We are interested in the relation between the multiplicities of the spherical representaions occurring in this decomposition and the spectrum of the non-Euclidean Laplacian acting on the space of smooth functions on the locally symmetric space $\Gamma\backslash G/K$. We would like to investigate whether there is a threshold of finitely many eigenvalues which determine the entire spectrum of the Laplacian. More precisely, our goal is to prove a result of the following kind:

Theorem 0.1. *Let Γ_1 and Γ_2 be two discrete cocompact subgroups of G. Let $m(\pi_s, \Gamma_i)$ be the multiplicity with which the spherical representation π_s occurs in the $L^2(\Gamma_i\backslash G)$ decomposition ($i = 1$ or 2).*
Then, there exists an $M > 0$ such that if

$$m(\pi_s, \Gamma_1) = m(\pi_s, \Gamma_2)$$

for all $s = it$ *such that* $t \leq M$*, then,*

$$m(\pi_s, \Gamma_1) = m(\pi_s, \Gamma_2)$$

for all π_s*.*

Two techniques that we intend to use in attacking this problem are: Selberg Trace Formula for compact quotient and Paley-Wiener theory.
Here's how the thesis is structured. Chapters 1 and 2 serve as an introduction to the subject of harmonic analysis and representation theory. Chapter 3 is about Bargamnn's classification of irreducible unitary representations of $SL_2(\mathbb{R})$ and the principal series representations, which are our spherical representations. In Chapter 4, we discuss the Selberg Trace formula for compact quotient. Chapter 5 gives us the relation between the multiplicities of spherical represenations and eigenvalues of Laplacian on $\Gamma\backslash G/K$. In Chapter 6, we discuss the classical theorems of Paley-Wiener in Fourier analysis. Chapter 7 talks about Harish-Chandra transform which relates bi-K-invariant functions on G and even compactly supported smooth functions on the real line. In Chpater 8, we discuss our strategy in solving the problem. In Chapter 9, we conclude the thesis.

Chapter 1

Preliminaries

In this chapter, we will discuss some general definitions and aspects regarding $SL_2(\mathbb{R})$.
$G = SL_2(\mathbb{R})$, the group of 2×2 real matrices with determinant 1, is a locally compact topological group. Every locally compact topological group has a Haar measure. We will use the Iwasawa decomposition of G to define a Haar measure on G. Let A, N and K be the following subgroups of G:

$$A = \left\{ \begin{bmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{bmatrix} \; : \; u \in \mathbb{R} \right\}$$

$$N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \; : \; x \in \mathbb{R} \right\}$$

$$K = \left\{ \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \; : \; \theta \in [0, 2\pi] \right\}$$

A and N are isomorphic to the group $\mathbb{R}$ (under addition) and K is isomorphic to S^1. Both $\mathbb{R}$ and S^1 have Lebesgue measure as a Haar measure. So, we can pull it back to the groups A, N and K. Thus, we get a Haar measure on G using the Iwasawa decomposition $G = ANK$ as

$$dg = \frac{1}{2\pi} du\, dn\, d\theta$$

The measure dg is both a left as well as a right Haar measure on G. Hence, G is unimodular. We note the following lemma. See [SL] for a proof of this.

Lemma 1.1. *Let G be a locally compact unimodular topological group with Haar measure dg. Let K be a closed subgroup of G which is also unimodular with Haar measure dk. We form the quotient space $K\backslash G$. Then, there exists a unique invariant measure dg' on $K\backslash G$*

such that for any $f \in C_c(G)$,

$$\int_G f(g)dg = \int_{K\backslash G} \left(\int_K f(kg')dk \right) dg'$$

We also have the Cartan decomposition $G = KAK$.

Classification of elements in G: Let $g \in G \backslash \{\pm I\}$. Then, we have the following classification of elements in G.

1. g is called parabolic if $|\text{Tr } g| = 2$ or g is conjugate to some element in $\pm N$.
2. g is called hyperbolic if $|\text{Tr } g| > 2$ or g is conjugate to some element in $\pm A$.
3. g is called elliptic if $|\text{Tr } g| < 2$ or g is conjugate to some element in K.

We will now discuss some general representation theory definitions.

Definition 1.1. *Let G be a locally compact group. Let H be a Hilbert space over $\mathbb{C}$ and $GL(H)$ be the group of all invertible linear operators on H. A representation (π, H) of the group G is a continuous homomorphism $\pi : G \to GL(H)$ such that the map $v \mapsto \pi(g)v$ is continuous for all $x \in G$ and $v \in H$. A representation is called unitary if $\pi(g)$ is unitary for all $g \in G$.*

Definition 1.2. *Let (π_1, H_1) and (π_2, H_2) be two representations of G. Then, π_1 and π_2 are said to be isomorphic if there exists a continuous linear isomorphism T of H_1 onto H_2 such that*

$$T\pi_1(g) = \pi_2(g)T \ \forall g \in G$$

Definition 1.3. *Let (π, H) be a representation of G. A closed subspace M of H is said to be an invariant subspace of π if $\pi(g)x \in M \ \forall x \in M, \ \forall g \in G$.*

Definition 1.4. *A representation (π, H) of G is said to be irreducible if there are no non-trivial proper π-invariant closed subspaces of H.*

Definition 1.5. *A unitary representation (π, H) of G is said to be completely reducible if there exists a family $\{H_i\}$ of closed mutually orthogonal π-invariant subspaces such that each (π_i, H_i) is irreducible where $\pi_i = \pi|_{H_i}$ and $H = \hat{\bigoplus} H_i$ where $\hat{\bigoplus}$ is the Hilbert space direct sum.*

Chapter 2

Harmonic analysis on upper half-plane

We consider $\mathcal{H} = \{z = x + iy : y > 0\}$ which is the upper half-plane as a model for the hyperbolic plane with the Riemannian metric given by,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

The hyperbolic measure on $\mathcal{H}$ is given by,

$$dz = \frac{dx\,dy}{y^2}$$

The non-euclidean Laplacian is defined as

$$\Delta = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

Δ acts on the space $C^\infty(\mathcal{H})$ of all smooth functions on $\mathcal{H}$.
The group $G = SL_2(\mathbb{R})$ acts on the upper-half plane $\mathcal{H}$ by Mobius transformations. Precisely, an element $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defines a map T_g as follows:

$$T_g(z) := \frac{az + b}{cz + d}$$

One can easily check that the Mobius transformations are orientation-preserving isometries of the upper-half plane $\mathcal{H}$. In fact, the full group of orientation-preserving isometries of $\mathcal{H}$ is $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$.
If we think of $\mathcal{H}$ as embedded in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ then we can extend this action of G to whole of $\hat{\mathbb{C}}$. Under such an action, the upper half-plane $\mathcal{H}$, the lower

half-plane $\overline{\mathcal{H}}$ and the boundary of the upper half-plane $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ are the three distinct orbits.
We get an equivalent classification of elements of G using above as follows:

1. g is parabolic iff T_g has one fixed point on $\hat{\mathbb{R}}$.

2. g is hyperbolic iff T_g has two fixed points on $\hat{\mathbb{R}}$.

3. g is elliptic iff T_g has one fixed point in $\mathcal{H}$ and one in $\overline{\mathcal{H}}$.

Lemma 2.1. *T_g preserves the hyperbolic metric and the hyperbolic measure on $\mathcal{H}$. Also, the Laplacian commutes with the action of G i.e.,*

$$\Delta(f \circ T_g)(z) = (\Delta f)(T_g(z))$$

The action of G on $\mathcal{H}$ is transitive and the stabilizer of the point i is the group $K = SO(2)$. So, we can identify $\mathcal{H}$ with the quotient space G/K and this identification is a homeomorphism.
Also, we have the following:

Lemma 2.2. *Let Γ be a subgroup of G. Then, the following are equivalent:*

1. *Γ is a discrete subgroup of G.*

2. *Γ acts discontinuously on $\mathcal{H}$. i.e. for every $z \in \mathcal{H}$, the orbit of z under Γ has no limit point.*

3. *For any compact subsets A and B of $\mathcal{H}$, the set $\{\gamma \in \Gamma : \gamma A \cap B \neq \phi\}$ is finite.*

A discrete subgroup Γ of G is called a Fuchsian group. A Fuchsian group of the first kind is a Fuchsian group such that the quotient $\Gamma\backslash\mathcal{H}$ has finite volume. A fundamental domain for a Fuchsian group is a connected open set $\mathcal{F} \subset \mathcal{H}$ such that:

1. No two points of $\mathcal{F}$ are equivalent mod Γ.

2. Any point of $\mathcal{H}$ lies in the orbit of some $z \in \overline{\mathcal{F}}$.

A cusp for Γ (or $\mathcal{F}$) is a point which lies in $\overline{\mathcal{F}} \cap \hat{\mathbb{R}}$. Our interest is in Fuchsian groups with compact quotient, which is equivalent to saying that the fundamental domain (or rather its closure) is compact. So, the quotient $\Gamma\backslash\mathcal{H}$ is compact iff Γ has no cusps. Any cusp of Γ is fixed by a parabolic element of Γ. Also, if we choose the fundamental domain so that no two cusps of Γ are equivalent mod Γ, then we have a bijection between the set of cusps and the set $\{z \in \hat{\mathbb{R}} : \gamma z = z \text{ for some parabolic } \gamma \in \Gamma\}$. So, we have the following:

Lemma 2.3. *Let Γ be a Fuchsian group of the first kind. Then, $\Gamma\backslash\mathcal{H}$ is compact iff Γ has no parabolic elements.*

Let $C^\infty(\Gamma\backslash\mathcal{H})$ be the space of all smoooth functions on $\mathcal{H}$ which are left invariant under Γ. Then, because of lemma 2.1, $\phi \in C^\infty(\Gamma\backslash\mathcal{H})$ implies $\Delta\phi \in C^\infty(\Gamma\backslash\mathcal{H})$. Hence, Δ acts on $C^\infty(\Gamma\backslash\mathcal{H})$.
One can define an inner product on this space $C^\infty(\Gamma\backslash\mathcal{H})$ by

$$\langle f, g\rangle := \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}dz \tag{2.0.1}$$

Then, we have the following:

Lemma 2.4. *The Laplacian Δ acts on $C^\infty(\Gamma\backslash\mathcal{H})$ and:*

1. $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$ *i.e Laplacian is a symmetric operator.*
2. $\langle -\Delta f, f\rangle \geq 0$

Note that Laplacian is not a bounded operator though.

Chapter 3

Representation Theory of $SL_2(\mathbb{R})$

In this chapter, we will classify the irreducible unitary representations of $G = SL_2(\mathbb{R})$, due to Bargmann, and then discuss an important class of representations known as *Principal Series representations*. Refer [KE] or [BD] for more details.
Classification of irreducible unitary representations of $SL(2, \mathbb{R})$: Any irreducible unitary representation of $SL(2, \mathbb{R})$ is equivalent to exactly one of the following:

1. The *principal series* representations π_t^ϵ , where ϵ is either 0 or 1 and $t \in \mathbb{R}$, and $t \geq 0$ if $\epsilon = 0$ and $t > 0$ if $\epsilon = 1$.

2. The two *mock discrete series* representations.

3. The *discrete series* representations π_n or $\widetilde{\pi}_n$ for $n \in \mathbb{Z}$ and $n \geq 2$.

4. The *complementary series* representations ρ_s for $0 < s < 1$.

5. The trivial representation.

We will now give a construction of the principal series representations.
Consider the Borel subgroup B of all upper-triangular matrices in G.

$$B = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} : \alpha \in \mathbb{R}^\times, \beta \in \mathbb{R} \right\}$$

The only-finite dimensional unitary representations of B are the one-dimensional characters given by

$$\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \mapsto \chi(\alpha)$$

where χ is a character of $\mathbb{R}^\times$. Any character of $\mathbb{R}^\times$ is of the form

$$\chi_{\epsilon,t}(\alpha) = \text{sgn}(\alpha)^\epsilon |\alpha|^{it}, \qquad \epsilon = 0 \text{ or } 1 \text{ and } t \in \mathbb{R}$$

We define principal series representations to be the representations unitarily induced from these characters to the whole group G. More precisely, we consider the following space of functions V_χ on which the group G acts by right translation.

$$V_\chi = \{f : G \to \mathbb{C} : f(bg) = \chi(b)\delta(b)f(g) \quad \forall b \in B,\ \forall g \in G \text{ and } ||f||^2 < \infty\}$$

Here, δ is the following homomorphism from B to $\mathbb{C}$

$$\delta : \begin{bmatrix} \alpha & \beta \\ 0 & \beta^{-1} \end{bmatrix} \mapsto |\alpha|$$

and,

$$||f||^2 := \int_K |f(x)|^2 dx$$

This condition is to make the induced representations unitary. We let $\pi_{\epsilon,t}$ denote $Ind_B^G(\chi_{\epsilon,t})$. Then,

Proposition 3.1.

1. *$\pi_{\epsilon,t}$ is irreducible iff χ is not the character given by $\epsilon = 1$ and $t = 0$.*
2. *$\pi_{\epsilon_1,t_1} \cong \pi_{\epsilon_2,t_2}$ iff $(\epsilon_1, t_1) = (\epsilon_2, t_2)$ or $(\epsilon_1, t_1) = (\epsilon_2, -t_2)$*

Note that, if we have $PSL_2(\mathbb{R})$ instead of $SL_2(\mathbb{R})$, then there will be no dependence on ϵ and hence, there will be just a single class of representations indexed by $s = it,\ t \geq 0$.

Chapter 4

Selberg Trace Formula for Compact Quotient

The Selberg Trace Formula is a beautiful result expressing the equality of certain geometric data (conjugacy classes) and certain spectral data. In this chapter, we will discuss the Selberg Trace Formula when $\Gamma\backslash G$ is compact. Later, in Chapter 8, we will see how Trace Formula can be used to solve our problem.
Our main references for this chapter are [WD] and [BD].
Let G be a unimodular locally compact topological group. e.g $SL_2(\mathbb{R})$. Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. We take the Haar measure on G to be dg and counting measure on Γ. By lemma 1.1, we have a unique measure $d\bar{g}$ on the quotient space $\Gamma\backslash G$. We consider the right regular representation $\mathcal{R}$ on the space of all square-integrable functions on $\Gamma\backslash G$, $L^2(\Gamma\backslash G)$. The action is by right translation as follows:

$$(\mathcal{R}(g)\phi)(x) := \phi(xg)$$

Let $C_c(G)$ be the space of all continuous functions on G. Given a representation (π, H) and

a function $f \in C_c(G)$ we can define a linear operator $\pi(f)$ on H by:

$$(\pi(f))(v) := \int_G f(g)(\pi(g)v)dg$$

$C_c(G)$ is an algebra under convolution operation given by:

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h)dh$$

The map $f \mapsto \pi(f)$ is an algebra homomorphism from $C_c(G)$ to the space of bounded operators on H. That is,

$$\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$$

Also, for the regular representation $\mathcal{R}$ we get

$$(\mathcal{R}(f)\phi)(x) = \int_G f(z)\phi(xz)dz$$

Like matrices, we can calculate the trace of a class of operators called trace-class operators. To understand them, we need a few definitions first.

Definition 4.1. *Let $A : H \to H$ be a bounded linear operator. The quantity*

$$\sum_{v \in \mathcal{B}} ||Ab||^2$$

is independent of the choice of basis $\mathcal{B}$. If it is finite, we say A is Hilbert-Schmidt operator and set the Hilbert-Schmidt norm $||.||_2$ as

$$||A||_2 := \sqrt{\sum_{v \in \mathcal{B}} ||Ab||^2}$$

Definition 4.2. *Let $A : H \to H$ be a bounded linear operator. A is called trace-class if*

$$\sum_{v \in \mathcal{B}} |\langle Av, v \rangle|$$

converges for every orthonormal basis $\mathcal{B}$ of H.

Lemma 4.1. *Let $A : H \to H$ be a trace-class operator. Then, the quantity*

$$\sum_{v \in \mathcal{B}} \langle Av, v \rangle$$

is absolutely convergent independent of the choice of orthonormal basis $\mathcal{B}$.

Thus, we can define the trace of a trace-class operator in the following manner:

Definition 4.3. *Let $A : H \to H$ be a trace class operator. We define the trace of A as:*

$$\operatorname{Tr} A := \sum_{v \in \mathcal{B}} \langle Av, v \rangle$$

where $\mathcal{B}$ is any orthornormal basis.

We note the following:

Proposition 4.2. *1. A trace-class operator is Hilbert-Schmidt.*

2. A Hilbert-Schmidt operator is compact.

3. If A and B are two Hilbert-Schmidt operators, then AB is of trace class and $\operatorname{Tr} AB = \operatorname{Tr} BA$.

4. $|\operatorname{Tr} AB| \leq ||A||_2 \, ||B||_2$

5. If A is a trace-class operator, then A^* is also a trace-class operator and $\operatorname{Tr} A^* = \overline{\operatorname{Tr} A}$

Next, we will define integral operators.

Lemma 4.3. *Let (X, μ) be a locally compact measure space. Let $H = L^2(X, \mu)$. Assume H is separable. Let $K(x, y) \in L^2(X \times X, \mu \otimes \mu)$. Then we say $A_K : H \to H$ is an integral operator with kernel K where A_K is defined as:*

$$(A_K f)(x) := \int_X K(x, y) f(y) dy$$

for $f \in L^2(X, \mu)$. We note that A_K is a Hilbert-Schmidt operator and $||A_K||_2 = ||K||_{L^2}$ where $||.||_{L^2}$ denotes the L^2-norm.

Now, we can realize the operator $\mathcal{R}(f)$ defined earlier as an integral operator in the following way:

$$\begin{aligned}
(\mathcal{R}(f)\phi)(x) &= \int_G f(z)\phi(xz)dz \\
&= \int_G f(x^{-1}y)\phi(y)dy \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\phi(\gamma y)dy
\end{aligned}$$

Hence, $\mathcal{R}(f)$ is an integral operator with kernel $K_f(x, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$. f vanishes outside a compact set and only finitely many γ would lie in a compact set. So, there are only finitely many terms in the sum. So, $K(x, y)$ is continuous and hence square-integrable. Therefore, $\mathcal{R}(f)$ is a Hilbert-Schmidt operator on $L^2(\Gamma\backslash G)$ because of Lemma 4.3.

Lemma 4.4. *Let $f = f_1 * f_2$ where $f_1, f_2 \in C_c(G)$. Then,*

$$\mathcal{R}(f) = \mathcal{R}(f_1) \circ \mathcal{R}(f_2)$$

Since, $\mathcal{R}(f_1)$ and $\mathcal{R}(f_2)$ are Hilbert-Schmidt operators, $\mathcal{R}(f)$ is a trace-class operator with trace as

$$\operatorname{Tr}\mathcal{R}(f) = \int_{\Gamma\backslash G} K_f(x,x)dx$$

Also $\mathcal{R}(f)$ is given by the kernel,

$$K_f(x,y) = \int_{\Gamma\backslash G} K_{f_1}(x,z)K_{f_2}(z,y)dz$$

We now calculate the geometric side of the trace formula. Let $\{\Gamma\}$ denote a set of representatives of conjugacy classes in Γ. Let Γ_γ be the centralizer of an element γ in Γ and G_γ be the centralizer of an element γ in G. Then, we have:

Proposition 4.5. *Let $f = f_1 * f_2$ where $f_1, f_2 \in C_c(G)$. Assume, G_γ is unimodular for every $\gamma \in \Gamma$. Then,*

$$\operatorname{Tr}\mathcal{R}(f) = \sum_{\gamma\in\{\Gamma\}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx$$

Any C^∞ function with compact support on a Lie group can be written as a finite sum of convolutions of continuous functions with compact support. Thus, the geometric side of the trace formula holds for any C_c^∞ function on $G = SL_2(\mathbb{R})$.

We will now discuss the spectral side of the trace formula.

Theorem 4.6. *The right regular representation of G, $L^2(\Gamma\backslash G)$ decomposes into a discrete Hilbert direct sum of irreducible unitary representations of G with each of them occurring with finite multiplicities i.e.*

$$L^2(\Gamma\backslash G) \;\cong\; \hat{\bigoplus_{\pi\in\hat{G}}} m_\pi\pi, \qquad 0 \leq m_\pi < \infty$$

We present the proof of the above theorem as given in [BD]. The proof uses the following lemma:

Lemma 4.7. *Let (π, H) be a unitary representation of G. Then, there exists an $f \in C_c(G)$ such that $\pi(f)$ is non-zero on H and $\pi(f)$ is self-adjoint.*

Here's a proof of Theorem 4.6.

Proof. Let H be a closed non-zero G-invariant subspace of $L^2(\Gamma\backslash G)$. We will show that H contains a closed irreducible subspace.

Then, by lemma 4.7, there exists an $f \in C_c(G)$ such that the operator $\mathcal{R}(f)$ is non-zero when restricted to H and is self-adjoint. Now, $\mathcal{R}(f)$ is a Hilbert-Schmidt operator on H and hence compact. So, by the spectral theorem for compact self-adjoint operators, $\mathcal{R}(f)$ has a nonzero eigenvalue, say λ, on H and let the correspnding eigenspace be $H(\lambda)$ which is finite dimensional.
Consider the set of all invariant subspaces M of H. We choose a subspace from this set such that $\dim(M \cap H(\lambda))$ is positive but minimal. Let us call it M_0. The existence of M_0 is assured by the fact that $H(\lambda)$ is finite dimensional.
Let V be the intersection of all closed invariant subspaces M of H such that $M_0 = M \cap H(\lambda)$. We will show that V is irreducible.
Let us assume to the contrary. Then, $V = V_1 \oplus V_2$. Let $v \in H(\lambda)$ be a non-zero vector. Then, $v \in V$. Suppose, $v = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$. Since, V_1 and V_2 are G-invariant subspaces, they are also invariant under $\mathcal{R}(f)$. So, $\mathcal{R}(f)v_1 - \lambda v_1 \in V_1$ and $\mathcal{R}(f)v_2 - \lambda v_2 \in V_2$. Then,

$$(\mathcal{R}(f)v_1 - \lambda v_1) + (\mathcal{R}(f)v_2 - \lambda v_2) = \mathcal{R}(f)v - \lambda v = 0$$

Hence, v_1 and v_2 are eigenvectors of $\mathcal{R}(f)$. Let us assume $v_1 \neq 0$. Then, $v_1 \in H(\lambda) \cap V_1 \subset M_0$. Since, M_0 is minimal with respect to this property, $H(\lambda) \cap V_1 = M_0$. But, V was defined to be the intersection of all closed invariant subspaces M of H such that $M_0 = M \cap H(\lambda)$ and V_1 is a proper subspace of V. Hence, we arrived at a contradiction.
Now, by Zorn's lemma, choose a maximal element S_0 in the set of all sets S of closed irreducible invariant subspaces of $L^2(\Gamma\backslash G)$ such that elements in S are orthogonal to each other. Then, $L^2(\Gamma\backslash G) = \hat{\bigoplus}_{\mathbb{V} \in S_0} V$ because otherwise, if it is proper, we can consider the orthogonal complement and by the previous argument, it contains an irreducible closed subspace contradicting the maximality of S_0.
The finiteness of the multiplicities m_π follows from the fact that every eigenvalue of $\mathcal{R}(f)$ has finite multiplicity.

□

As a corollary of Theorem 4.6 , we get the spectral side of the trace formula:

Corollary 4.8. *We have*

$$L^2(\Gamma\backslash G) \cong \hat{\bigoplus_{\pi \in \hat{G}}} m_\pi \pi, \qquad 0 \leq m_\pi < \infty$$

Thus, for $\mathcal{R}(f)$ in trace-class, we get

$$\text{Tr } \mathcal{R}(f) = \sum_{\pi \in \hat{G}} m_\pi \text{Tr } \pi(f)$$

From Proposition 4.5 and Corollary 4.8, we get the trace formula.

Theorem 4.9. *Let $f \in C_c^\infty(G)$. Then $\mathcal{R}(f)$ is of trace-class and,*

$$\sum_{\pi \in \hat{G}} m_\pi \mathrm{Tr}\ \pi(f) = \mathrm{Tr}\ \mathcal{R}(f) = \sum_{\gamma \in \{\Gamma\}} \mathrm{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx \tag{4.0.1}$$

Chapter 5

Spherical representations and duality theorem

In this chapter we will demonstrate the duality between the representation spectrum of $L^2(\Gamma\backslash G)$ and the eigenvalue spectrum of the Laplacian on $\Gamma\backslash G/K$. Our references are [WD] and [BD].

Definition 5.1. *Let π be an irreducible unitary representation of G. Then, π is said to be spherical if there exists a non-zero K-fixed vector v i.e. $\pi(k)v = v\ \forall k \in K$.*

The only spherical representations of $G = SL_2(\mathbb{R})$ are the trivial representation and the principal series representations. We discuss the following result:

Theorem 5.1. *Let H be a closed, irreducible G-invariant subspace of $L^2(\Gamma\backslash G)$. Let, H^K be the subspace of H which is K-fixed. Then, H^K is at most one-dimensional. Also, if $0 \neq \phi \in H^K$, then $\phi \in C^\infty(\Gamma\backslash G)$. As a function on $\Gamma\backslash\mathcal{H}$, ϕ satisfies*

$$\Delta\phi = -\lambda\phi$$

where $\lambda \in \mathbb{R}$ depends only on the isomorphism class of H.

A function f on G is said to *bi-K-invariant* if $f(k_1xk_2) = f(x)\ \forall k_1, k_2 \in K$ and $x \in G$. We denote by $C_c^\infty(G//K)$ the space of all smooth, compactly supported bi-K-invariant functions on G. $C_c^\infty(G//K)$ forms an algebra under convolution. To prove the theorem, we will need the following lemmas:

Lemma 5.2. *Let $f \in C_c^\infty(G//K)$. Then,*

$$f(g) = f(g^T)$$

for all $g \in G$. (g^T means the transpose of g).

Lemma 5.3. *The algebra $C_c^\infty(G//K)$ is commutative.*

Lemma 5.4. *Let (π, H) be a unitary representation of G. Suppose there exists a K-fixed vector v in H. Then, there exists an $f \in C_c^\infty(G//K)$ such that $\pi(f)$ is self-adjoint and $\pi(f)v \neq 0$.*

Definition 5.2. *Let (π, H) be a representation of G. Then, π is called admissible if,*

$$\pi|_K = \bigoplus_{\rho \in \hat{K}} m_\rho \rho$$

with $0 \leq m_\rho < \infty$

Proposition 5.5. *Let (π, H) be an irreducible representation of G appearing in $L^2(\Gamma \backslash G)$. Then, π is admissible and H^K is at most one-dimensional.*

We will now give a representation theoretic definition of the Laplacian. The Lie algebra $\mathfrak{g}$ of the group $G = SL_2(\mathbb{R})$ consists of all 2×2 matrices with trace zero. The Lie bracket operation is given by,

$$[X, Y] = XY - YX$$

G acts on $C^\infty(G)$ by right translations. Similarly, $\mathfrak{g}$ acts on $C^\infty(G)$ as follows:

$$(dXf)(g) := \frac{d}{dt} f(ge^{tX}) \Big|_{t=0}$$

Thus, the elements of the Lie algebra can be thought of as differential operators on the space of smooth functions on G. The action of $\mathfrak{g}$ on $C^\infty(G)$ satisfies the following:

$$dX \circ dY - dY \circ dX = d[X, Y]$$

We will now extend this definition to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The construction of $U(\mathfrak{g})$ is as follows: Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. i.e.

$$T(\mathfrak{g}) := \bigoplus_{k=0}^{\infty} \otimes^k \mathfrak{g}$$

Here the multiplication is given by,

$$(v_1 \otimes v_2 \otimes ... \otimes v_k) \times (w_1 \otimes w_2 ... \otimes w_l) = v_1 \otimes v_2 \otimes ... \otimes v_k \otimes w_1 \otimes w_2 ... \otimes w_l$$

Let I be the ideal of $T(\mathfrak{g})$ generated by elements of the form

$$X \otimes Y - Y \otimes X - [X, Y]$$

for $X, Y \in \mathfrak{g}$.
Then we define,

$$U(\mathfrak{g}) := T(\mathfrak{g})/I$$

Thus, we can define the action of $U(\mathfrak{g})$ on $C^\infty(G)$ by

$$d(X_1 \otimes X_2 \otimes ... \otimes X_n)f := d(X_1) \circ d(X_2) \circ ... \circ d(X_n)f$$

This enables us to realize the elements in the universal enveloping algebra to be left-invariant differential operators on G. Also, an element in the center of $U(\mathfrak{g}$ can be realized as a right-invariant differential operator.
Now, consider the following elements of $\mathfrak{g}$:

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These form a basis of $\mathfrak{g}$. Define an element C called as the Casismir element to be

$$C = -\frac{1}{4}(H \otimes H + 2R \otimes L + 2L \otimes R)$$

Then, we note the following:

Lemma 5.6. *The element C lies in the center of $U(\mathfrak{g})$.*

Let (π, H) be a representation of G. A vector $v \in H$ is said to be C^1 if

$$d\pi(X)v = \frac{d}{dt}\pi(e^{tX})v\Big|_{t=0}$$

exists for all $X \in \mathfrak{g}$. We say v is C^k if v is C^1 and $d\pi(X)v$ is C^{k-1} for all $X \in \mathfrak{g}$. v is called a smooth vector if v is C^k for all k. Let H^∞ denote the space of all smooth vectors in H. Then, the following lemma tells us that H^∞ is stable under the action of G.:

Lemma 5.7. *H^∞ is invariant under the action of G.*

In particular, for $H = L^2(\Gamma\backslash G)$, the space of smooth vectors H^∞ coincides with $C^\infty(\Gamma\backslash G)$, the space of smooth funcions on $\Gamma\backslash G$.

Lemma 5.8. *Let $\mathcal{R}$ be the representation of G on $L^2(\Gamma\backslash G)$. Then, $v \in L^2(\Gamma\backslash G)^\infty$ iff $v \in C^\infty(\Gamma\backslash G)$.*

We also have the following:

Lemma 5.9. *Let (π, H) be a representation of G. Then, we have a Lie algebra representation of $\mathfrak{g}$ on the space H^∞. More precisely, we have $d\pi : \mathfrak{g} \to \mathrm{End}(H^\infty)$ such that*

$$d\pi(X) \circ d\pi(Y)v - d\pi(Y) \circ d\pi(X)v = d\pi([X, Y])v$$

for all $X, Y \in \mathfrak{g}$ and all $v \in H^\infty$.
Also, for $g \in G$, $X \in \mathfrak{g}$ and $v \in H^\infty$, we have

$$\pi(g)d\pi(X)\pi(g)^{-1}v = d\pi(\mathrm{Ad}(g)X)v$$

where $\mathrm{Ad}(g)X = gXg^{-1}$.
So, for $g \in G$ and D in the center of $U(\mathfrak{g})$, we have

$$\pi(g) \circ d\pi(D)v = d\pi(D) \circ \pi(g)v$$

for all $v \in H^\infty$.

Lemma 5.10. *Let G act on $C^\infty(G)$ by right translations. Suppose $\phi \in C^\infty(G)$ is right invariant under K. Then, we can consider ϕ to be a function on $\mathcal{H}$ and we have,*

$$\Delta\phi = -C\phi$$

where Δ is the non-Euclidean Laplacian on $\mathcal{H}$ and C is the Casimir element.

We now present the proof of theorem 5.1. We follow [WD].

Proof. Let H be a closed, irreducible G-invariant subspace of $L^2(\Gamma\backslash G)$. Then, by Proposition 5.5, H^K is atmost one dimensional. Let $0 \neq \phi \in H^K$. By lemma 5.4, there exists $f \in C_c^\infty(G//K)$ such that $\mathcal{R}(f)\phi \neq 0$. Since, f is smooth, $\mathcal{R}(f)\phi$ smooth. Also, for $f \in C_c^\infty(G//K)$, $\mathcal{R}(f)$ preserves H^K. Therefore, $\mathcal{R}(f)\phi = \lambda\phi$ for some non-zero λ. Hence, ϕ is also smooth.
Now, by lemma 5.9, we can show that $C\phi \in H^K$. Hence, we have, $C\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$. $\phi \in H^K$ and hence, we can think of ϕ as a smooth function on $\Gamma\backslash\mathcal{H}$. Then, by lemma 5.10, we get $\Delta\phi + \lambda\phi = 0$. □

We will now give an explicit relation between the multiplicities of the spherical representations occurring in $L^2(\Gamma\backslash G)$ and the multiplicities of the eigenvalues of Laplacian on $\Gamma\backslash G/K$. We know that the only non-trivial spherical representations of G are the principal series representations.
The principal series representations of $G = PSL_2(\mathbb{R})$ are indexed by $s = it$ where $t \geq 0$. So,

let π_s be a spherical representation. We will compute the action of the Casimir element on the K-fixed vector of π_s, say ϕ. Assume that ϕ is normalized so that $\phi(k) = 1$ for all $k \in K$. Then,

$$\phi\left(\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} k\right) = |\alpha|^{s+1}$$

Next, if we consider ϕ as a function on the upper-half plane, we have,

$$\phi(x+iy) = y^{\frac{s+1}{2}}$$

By lemma 5.10, Casimir acts as $-\Delta$ on the functions on uppper-half plane. Therefore,

$$\Delta\phi = y^2 \frac{\partial^2 \phi}{\partial y^2} = \frac{s^2-1}{4} y^{\frac{s+1}{2}} = \frac{s^2-1}{4}\phi$$

Thus, we have the following:

Theorem 5.11. *Let Γ be a discrete cocompact subgroup of G. Then,*

$$L^2(\Gamma\backslash G) \;\cong\; \hat{\bigoplus_{\pi \in \hat{G}}} m_\pi \pi, \qquad 0 \le m_\pi < \infty$$

For $s \in \mathbb{C}$, we take $\lambda_s = \frac{1-s^2}{4}$. Then,

$$\text{Dim}\{\phi \in C^\infty(\Gamma\backslash\mathcal{H}) : \Delta\phi + \lambda_s\phi = 0\} = m_{\pi_s}$$

Now, the following lemma gives an expression for the trace formula for any $f \in C_c^\infty(G//K)$.

Lemma 5.12. *Let $s \in \mathbb{C}$ and $f \in C_c^\infty(G//K)$. Then,*

$$\text{Tr } \pi_s(f) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} f\left(\begin{bmatrix} e^{u/2} & x \\ 0 & e^{-u/2} \end{bmatrix}\right) e^{us/2} \, du \, dx$$

Chapter 6

Paley-Wiener theorems

In this chapter we will discuss the classical theorems of Paley-Wiener in Fourier analysis. We hope to use some version of Paley-Wiener theorem in solving the problem. Our main reference is [SR].
The Fourier transform of an integrable (L^1) function f on $\mathbb{R}$ is defined as follows:

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}dt$$

The Schwartz space $\mathcal{S}(\mathbb{R})$ is the space of all smooth functions on $\mathbb{R}$ such that the function and all its derivatives tend to zero at infinity more rapidly than any inverse power of x. More rigorously,

$$\mathcal{S}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \ : \ \sup_{x\in\mathbb{R}} |x^m f^{(n)}(x)| < \infty \ \ \forall m, n \geq 0\}$$

Fourier transform is an isometry on the Schwartz space $\mathcal{S}(\mathbb{R})$ with respect to the L^2 norm. In fact, by Plancherel's theorem, the Fourier transform extends to an isometry on the space of all square-integrable functions on $\mathbb{R}$.

As we know, the Fourier transform reverses differentiation and multiplication (by polynomials, for example). Thus, smoother the function, more rapidly decreasing it's Fourier transform will be and vice versa.
Compact support is like the ultimate in rapidly decreasing nature while analyticity is the ultimate in smoothness. The classical theorems of Paley-Wiener deal with support conditions on f to ensure analyticity of $\hat{f}$.
Let us first try to define a complex Fourier transform of $f \in L^1(\mathbb{R})$. If we define it by just

replacing the real variable x by a complex variable $z = x + iy$, then

$$\hat{f}(z) = \hat{f}(x+iy) = \int_{\infty}^{\infty} f(t)e^{-it(x+iy)}dt$$
$$= \int_{\infty}^{\infty} f(t)e^{-itx}e^{ty}dt$$
$$= \hat{g}(x)$$

where, $g(t) = e^{ty}f(t)$. Since, e^{ty} grows rapidly at infinity, the integral won't converge. So, we will take f to be compactly supported in which case the complex Fourier transform is defined as above.
It can be easily verified that $\hat{f}(z)$ satisfies the Cauchy-Riemann equations and hence, $\hat{f}(z)$ is an entire function. But, not every entire function can be written as the complex Fourier transform of a compactly supported function because $\hat{f}$ satisfies certain estimates.
Take f to be an integrable function supported in the interval $[-A, A]$ where $A \geq 0$. Then,

$$|\hat{f}(z)| = \left| \int_{\infty}^{\infty} f(t)e^{-it(x+iy)dt} \right| = \left| \int_{-A}^{A} f(t)e^{-it(x+iy)dt} \right|$$
$$\leq \int_{-A}^{A} |f(t)e^{-it(x+iy)}|dt$$
$$= \int_{-A}^{A} |f(t)e^{ty}|dt$$
$$\leq e^{A|y|} \int_{-A}^{A} |f(t)|dt = Ce^{A|y|}$$

Thus, $|\hat{f}(z)| \leq Ce^{A|y|}$. We say $\hat{f}$ is an entire function of exponential type A.
The important content in the Paley-Wiener theorems is that the converse of the above is also true. Here, we state two versions of Paley-Wiener theorem without proof. See [SR] for more details.

Theorem 6.1. *Let f be a complex-valued square-integrable function with support as $[-A, A]$. Define $\hat{f}(z) = \hat{f}(x + iy) := \int_{-\infty}^{\infty} f(t)e^{-it(x+iy)}dt$.*
Then, $\hat{f}(z)$ is an entire function of exponential type A and $\hat{f}(x)$ is a square-integrable function.
Conversely, if $F(z)$ is an entire function of exponential type A and if $F(x)$ is a square-integrable function, then $F = \hat{f}$ for some such function f.

Theorem 6.2. *Let f be a smooth function with support as $[-A, A]$. Then, $\hat{f}(z)$, as defined above, is an entire function of exponential type A and $\hat{f}(x)$ lies in the Schwartz space i.e. $\hat{f}(x)$ is also rapidly decreasing.*
Conversely, if $F(z)$ is an entire function of exponential type A and $F(x)$ is rapidly decreasing, then $F = \hat{f}$ for some such function f.

Chapter 7

Plancherel Formula and Weyl's Law

The aim of this chapter will be to give a concrete realization of $C_c^\infty(G//K)$ using the Harish-Chandra transform and give a version of Plancherel formula. Please refer [WD] for more details.
We have set,

$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{R}_{>0} \right\}$$

Let $f \in C_c^\infty(G//K)$. We define the Harish-Chandra transform $Hf \in C_c^\infty(A)$ as follows:

$$Hf \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \int_{\mathbb{R}} f\left(\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} \right) dx$$

We let $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then,

$$w^{-1} \begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} w = \begin{bmatrix} a^{-1} & 0 \\ -x & a \end{bmatrix}$$

Since, $w \in K$,

$$f\left(\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} \right) = f\left(\begin{bmatrix} a^{-1} & 0 \\ -x & a \end{bmatrix} \right)$$

Also, by lemma 5.2, $f(g) = f(g^T)$. Hence,

$$f\left(\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} \right) = f\left(\begin{bmatrix} a^{-1} & -x \\ 0 & a \end{bmatrix} \right)$$

Thus,

$$Hf\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) = Hf\left(\begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix}\right)$$

We will denote this as $Hf \in C_c^\infty(A)^w$.

Theorem 7.1. *The map $H : C_c^\infty(G//K) \to C_c^\infty(A)^w$ is an algebra isomorphism.*

The proof of this theorem uses the following lemmas:

Lemma 7.2. *Let $f \in C_c^\infty(\mathbb{R}_{>0})$ such that $f(x) = f(x^{-1})$. Define F on $C_c(\mathbb{R}_{\geq 1})$ by,*

$$F\left(\frac{x^2 + x^{-2}}{2}\right) = f(x)$$

Then, $F \in C_c^\infty(\mathbb{R}_{\geq 1})$.

Lemma 7.3. *Let $f \in C_c^\infty(G//K)$. Then, there exists $F_f \in C_c^\infty(\mathbb{R}_{\geq 1})$ such that,*

$$f(g) = F_f(\frac{1}{2}\text{Tr } g^T g)$$

Conversely, given $F \in C_c^\infty(\mathbb{R}_{\geq 1})$, if we define f by $f(g) = F(\frac{1}{2}\text{Tr } g^T g)$, then $f \in C_c^\infty(G//K)$.

So, by the previous lemma, we have,

$$\begin{aligned} Hf\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) &= \int_{\mathbb{R}} f\left(\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix}\right) dx \\ &= \int_{\mathbb{R}} F_f\left(\frac{a^2 + a^{-2} + x^2}{2}\right) dx \end{aligned}$$

Lemma 7.4. *Let $F \in C_c^\infty(\mathbb{R}_{\geq 1})$. Define, for $a \geq 1$,*

$$H(a) = \int_{\mathbb{R}} F(a + \frac{x^2}{2}) dx$$

Then, $H \in C_c^\infty(\mathbb{R}_{\geq 1})$ and

$$F(a) = -\frac{1}{2\pi} \int_{\mathbb{R}} H'(a + \frac{x^2}{2}) dx$$

The converse is also true.

Theorem 7.1 now follows from the above lemmas.

So, given $f \in C_c^\infty(G//K)$, Harish-Chandra transform gives us $Hf \in C_c^\infty(A)^w$ and using the above lemmas, we get,

$$Hf\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) = h_f\left(\frac{a^2 + a^{-2}}{2}\right)$$

Let us now define $g_f \in C_c^\infty(\mathbb{R})^{even}$ by,

$$g_f(u) = Hf\left(\begin{bmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{bmatrix}\right) = h_f\left(\frac{e^u + e^{-u}}{2}\right) = h_f(cosh(u))$$

Thus, we get the following version of Plancherel formula:

Theorem 7.5. *Let $f \in C_c^\infty(G//K)$. Then,*

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2\pi} \int_0^\infty \hat{g_f}(u) u \, tanh(\pi u) du$$

where, $\hat{g_f}(u) = \int_{\mathbb{R}} g_f(t) e^{-iut} dt$ is the Fourier transform of g_f.

We also have the following:

Lemma 7.6. *Let $f \in C_c^\infty(G//K)$ and $s = it \in \mathbb{C}$. Then,*

$$\text{Tr } \pi_s(f) = \hat{g_f}(-s/2i)$$

We now briefly discuss Weyl's law.

Let S be a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂S. Consider the Euclidean Laplacian given by,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}$$

We look for functions ϕ on S satisfying

$$\Delta\phi + \lambda\phi = 0$$

and $\phi|_{\partial S} \equiv 0$. Let $N(T)$ be the number of linearly independent solutions with $\lambda \leq T$. Note that $\lambda \geq 0$. Weyl proved that,

$$N(T) \sim \frac{\text{Area}(S)}{4\pi} T$$

as $T \to \infty$.
Using the trace formula, one can extend the Weyl's law to the quotients of upper-half plane. More precisely, we have,

$$N(T) \sim \frac{\text{Area}(\Gamma\backslash\mathcal{H})}{4\pi}T$$

as $T \to \infty$.

Chapter 8

The Problem

We now finally come to the problem we intended to solve.
Let $G = PSL_2(\mathbb{R})$. We consider $PSL_2(\mathbb{R})$ instead of $SL_2(\mathbb{R})$ because in this case, the spherical representations are indexed by $s = it$ and there is no dependence on ϵ. We know,

$$L^2(\Gamma\backslash G) \;\cong\; \hat{\bigoplus_{\pi\in\hat{G}}} m_\pi \pi, \qquad 0 \leq m_\pi < \infty$$

We consider the spherical spectrum i.e the multiplicities of the spherical representations π_s in the above decomposition. In [BR], Chandrasheel Bhagwat and C.S. Rajan prove an analogous result of strong multiplicity one theorem in the case of spherical spectrum. They conclude that if all but finitely many multiplicities of spherical representations agree, then the spectra are same.
Along similar lines, we ask the following question - Does there exist a threshold, say $M > 0$, such that if the spherical spectrum of two different discrete cocompact subgroups Γ_1 and Γ_2 of G agree till M, then the entire spherical spectra is the same? The threshold we are looking for should ideally be independent of the subgroups Γ_1 and Γ_2.
Equivalently, if we consider the setting of $\Gamma\backslash G/K$, then we are looking for a threshold M such that if the multiplicities of the eigenvalues of the non-Euclidean Laplacian Δ for two different $\Gamma_1\backslash G/K$ and $\Gamma_2\backslash G/K$ agree until M, then both the Laplacian spectra should be identical.

In more precise terms, we would like to establish a result of the following kind:

Theorem 8.1. *Let Γ_1 and Γ_2 be two discrete cocompact subgroups of G. Let $m(\pi_s, \Gamma_i)$ be the multiplicity with which the spherical representation π_s occurs in the $L^2(\Gamma_i\backslash G)$ decomposition ($i = 1$ or 2).*

Then, there exists an $M > 0$ such that if

$$m(\pi_s, \Gamma_1) = m(\pi_s, \Gamma_2)$$

for all $s = it$ such that $t \leq M$, then,

$$m(\pi_s, \Gamma_1) = m(\pi_s, \Gamma_2)$$

for all π_s.

Since, $L^2(\Gamma\backslash G)$ decomposes into a discrete sum, only finitely many π_s will be there such that $t \leq M$.
Our approach is to use the trace formula and the Paley-Wiener estimates to find such an M. Let us write down the trace formula.

$$\sum_{\pi \in \hat{G}} m_\pi \text{Tr } \pi(f) = \text{Tr } \mathcal{R}(f) = \sum_{\gamma \in \{\Gamma\}} \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx$$

Let us choose $f \in C_c^\infty(G//K)$. Then, $\text{Tr}\pi(f) = 0$ if π is not spherical. Hence,

$$\sum_{\pi_s \in \hat{G}_s} m_\pi \text{Tr } \pi_s(f) = \sum_{\gamma \in \{\Gamma\}} \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx$$

where $\hat{G}_s$ denotes the set of all equivalence classes of irreducible unitary spherical representations of G. Now, comparing the equations for Γ_1 and Γ_2, we get,

$$\sum_{\pi_s \in \hat{G}_s} (m(\pi_s, \Gamma_1) - m(\pi_s, \Gamma_2)) \text{Tr } \pi_s(f) = \sum_{\gamma \in \{\Gamma_1\} \cup \{\Gamma_2\}} \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx$$

Now, let S be the set of all π_s such that $s = it$ and $t > M$. Then, by our assumption we have,

$$\sum_{\pi_s \in S} (m(\pi_s, \Gamma_1) - m(\pi_s, \Gamma_2)) \text{Tr } \pi_s(f) = \sum_{\gamma \in \{\Gamma_1\} \cup \{\Gamma_2\}} \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx$$

Then, by lemma 9.6, we get,

$$\sum_{\pi_s \in S} (m(\pi_s, \Gamma_1) - m(\pi_s, \Gamma_2)) \hat{g_f}(-s/2i) = \sum_{\gamma \in \{\Gamma_1\} \cup \{\Gamma_2\}} \text{Vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx$$

Our next step is to construct an $f \in C_c^\infty(G//K)$ whose support contains exactly one of the conjugacy classes from $\{\Gamma_1\} \cup \{\Gamma_2\}$. Then, the right hand side would be a constant while on the left hand side, we have a convergent series. Our approach then would be to use Paley-Wiener estimates or using Weyl's law to help us in estimating the left hand side. But, we have not yet succeeded in finding a suitable function f as desired.

Chapter 9

Conclusion

We have studied here the Selberg Trace Formula in the compact quotient case, the duality of spherical representations and the Laplacian spectrum on $\Gamma\backslash G/K$ and briefly discussed Paley-Wiener theorems.

We haven't been successful in solving the problem yet but we hope to use the techniques as outlined in the last chapter to establish Theorem 8.1 or its appropriate modification.

Bibliography

[WD] Whitehouse, David. An Introduction to the Trace Formula. [Notes from Internet], 2010.

[BD] Bump, Daniel. Automorphic forms and representations. Vol. 55. Cambridge University Press, 1998.

[LS] Lang, Serge. $SL(2, \mathbb{R})$. Graduate Texts in Mathematics 105 (1975).

[IH] Iwaniec, Henryk. Spectral methods of automorphic forms. Vol. 53. Providence: American Mathematical Society, 2002.

[KE] Kowalski, Emmanuel. Representation Theory.

[RW] Rudin, Walter. Real and complex analysis. Tata McGraw-Hill Education, 1987.

[SR] Strichartz, Robert S. A guide to distribution theory and Fourier transforms. Singapore: World Scientific, 2003.

[BR] Bhagwat, Chandrasheel, and C. S. Rajan. "On a spectral analog of the strong multiplicity one theorem." International Mathematics Research Notices (2010): rnq243.