Seifert Fiber Spaces with Singular Surfaces

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

A Thesis Submitted in Partial Fulfillment of the Requirements
of the Degree of
Doctor of Philosophy

द्वारा/By

नायर रम्या रवींद्रन जयश्री / Nair Ramya Ravindran Jayashri
पंजीकरण सं./Reg No.: 20163482

शोध प्रबंध पर्यवेक्षक/Thesis Supervisor
Dr. Tejas Kalelkar

IISER PUNE

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान पुणे
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

2023

Dedicated to
My Parents & Teachers

Certificate

Certified that the work incorporated in the thesis entitled "*Seifert fiber spaces with singular surfaces*", submitted by *Ramya Nair* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: 26/10/2023

Dr. Tejas Kalelkar
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Date: 26/10/2023

Ramya Nair
Roll Number: 20163482

Acknowledgements

I begin by thanking my advisor Dr. Tejas Kalelkar for his patience and support. This thesis would not be possible without him. I also thank my RAC members Dr. Vivek Mallick and Dr. Subhojoy Gupta for their constructive comments and suggestions throughout my PhD tenure. I would like to express my heartfelt gratitude to all the faculty members of our department who have helped me in various capacities. Also, a special thanks to our office staff Suvarna and Yogesh for making life so much easier.

I would like to thank my teachers, Prof. Ravi Aithal and Prof. Mangala Manohar from University of Mumbai, for instilling in me the love for mathematics and encouragement to pursue research. As also, to Prof. Kumaresan for organizing MTTS where I met amazing people like Oorna, Selva, Arusha, Deeksha and others who have been an integral part of life ever since.

This note would not be complete if I do not mention my friends without whom my stay at IISER would not be nearly as memorable. This list includes (but is not limited to) my batchmates Neha, Debo, Basu, Souptik, Suraj, Sudipa, Kartik, Riju and honorary batchmate Neeraj. As also the Badminton Club comprising of Sriram, Nazia, Adithyan, Arghya, Anjana, Poornima and others for the daily dose of entertainment. Super cool juniors Dhruv, Narayanan, Hitendra, Pranjal, Pavan, Tumpa, Mithesh, Kunal, Tushar, Bittu, Biswanath, Ramandeep, Prachi and Divyashree will all be dearly missed. Also, special thanks to Vivek, Adwait and Ramakrishnan for their company, wisdom (wink, wink) and encouragement to experiment a bit! Loads of love to Namrata and Sandra for always being there and to Visakh for being the wonderful person he is!

I am also thankful for this city which is so awesome that every other place pales in comparison (at least climate-wise). I would also like to thank IISER Pune and the Mathematics Department in particular, for creating an environment conducive to learning and research.

Last but most importantly, I am grateful to my family, especially my parents, for their unconditional love and support.

Ramya Nair

Contents

1

Abstract

Seifert fiber spaces are compact 3-dimensional manifolds that are foliated by circles. Seifert fiber spaces with isolated singular fibers have been well studied. In this thesis, we focus on Seifert fiber spaces which have singular surfaces and extend known results to such manifolds.

Two-sided incompressible surfaces in Seifert fiber spaces with isolated singular fibers can be isotoped to become either horizontal or vertical. Frohman and Rannard have shown that one-sided incompressible surfaces in such manifolds are either pseudo-horizontal or pseudo-vertical. We extend their result to characterize essential surfaces in Seifert fiber spaces which may contain singular surfaces. We also give a complete criterion for the existence of horizontal surfaces in Seifert fiber spaces which may have singular surfaces.

We introduce prism complexes as an analogue of simplicial complexes. And show that while every compact 3-dimensional manifold admits a prism complex structure, it admits a special prism complex structure if and only if it is a Seifert fiber space which has either non-empty boundary or singular surfaces or it is a closed Seifert fiber space with Euler number zero. In particular, a compact 3-dimensional manifold with boundary is a Seifert fiber space if and only if it admits a special prism complex structure.

We will also briefly discuss our future work towards finding families of manifolds that provide evidence for the L-space Conjecture.

2

Introduction

The classification of 3-dimensional manifolds has become one of the central ideas in low dimensional topology research since Thurston [Thu82] proposed the Geometrization Conjecture in 1982. This conjecture was successfully resolved by Perelman in 2002 . Geometrization is an analogue of the uniformization theorem for surfaces for dimension 3. But unlike the case of surfaces, 3-manifolds cannot be classified by themselves but they do admit a canonical decomposition into pieces that possess nice geometric structures. These are the eight geometries described by Thurston. See [Sco83] for a detailed description. Compact 3-manifolds admitting six out of the eight geometries (all except hyperbolic and Sol) have the structure of a special class of 3-manifolds called the Seifert fiber spaces.

Seifert fiber spaces were first studied by Seifert in his paper [Sei33]. They rose to prominence when Johannson [Joh79], Jaco and Shalen [JS79] found these manifolds to be the only examples for non- uniqueness statement of torus decomposition. Study of Seifert fiber spaces are important in their own right for two reasons. Firstly, they coincide with the class of all compact 3-manifolds foliated by circles [Eps72] and secondly they admit a combinatorial description comprising of easily understood invariants.

The objects of study for the most part of this thesis are Seifert fiber spaces. The orientable manifolds in this class have been extensively studied since their discovery in the 1930s. But non-orientable Seifert fiber spaces, especially the ones containing singular surfaces have been largely ignored. Here, we extend some well-known results about orientable Seifert fiber spaces to the non-orientable class and also provide a new way of looking at Seifert fiber spaces via prism complexes.

In Chapter 3, we recall the preliminary definitions and theorems pertaining to 3-manifold topology, triangulations, knot theory and the basic theory of Seifert fiber spaces required to understand the subsequent chapters. There is no original work in this chapter.

In Chapter 4, we introduce the notion of prism complexes as discrete structures to study Seifert fiber spaces akin to cube complexes for hyperbolic manifolds. We also give a combinatorial criterion on such a complex to ensure that the underlying manifold is a Seifert fiber space. Along the way, we extend a well-known criterion for existence of horizontal surfaces in orientable Seifert fiber spaces to the non-orientable cases as well. This chapter is based on our paper [KN23b].

Essential surfaces play a pivotal role in 3-manifold topology. We cut along them to obtain 'simpler' manifolds. It is well known that two-sided essential surfaces in Seifert fiber spaces with isolated singular fibers may be isotoped to a particularly nice form. They can either be isotoped to become transverse to all the circle fibers or to a union of circle fibers. One-sided incompressible surfaces in Seifert manifolds with isolated singular fibers were studied much later by Frohman [Fro86] and Rannard [Ran96]. Rannard showed that any incompressible surface in a Seifert fiber space with only isolated singular fibers can be isotoped to become pseudo-vertical or pseudo horizontal. In Chapter 5, we prove a similar structure theorem for essential surfaces in Seifert fiber spaces which may have singular surfaces not just isolated singular fibers. Along the way we also compile a complete list of incompressible surfaces in a solid Klein bottle. This chapter is based on our paper [KN23a].

In Chapter 6, we discuss ongoing work with Rachel Roberts and her student Jeffrey Norton. There are no original results here. Our objective is to provide evidence for the L-space Conjecture [6.1.3] by constructing taut foliations in an infinite family of non-L-spaces. Based on the work of Tao Li in [Li02], we attempt to construct laminar branched surfaces in torus knot exteriors which carry essential laminations that may be extended to taut foliations in the knot exterior. These can be further extended to the Dehn filled manifold. Torus knot complements are Seifert fibered. Although the L-space conjecture is known to be true for Seifert fiber spaces by the combined work of a number of mathematicians (see [BC17], [BGW13], [BNR97], [CLW13], [HRRW20], [EHN81], [LS09], [BC15]), no algorithm is known for constructing taut foliations in such spaces. So, we hope that a laminar branched surface in torus knot exterior can be modified to construct a branched surface in the exterior of positive n-braids. In this chapter, we show how a potential candidate fails to satisfy the conditions required to apply Li's results about laminar branched surfaces. Our aim is to find a suitable branched surface where such results can be used to construct taut foliations. This constitutes future work.

3

Preliminaries

3.1 Basic definitions and results

In this section, we discuss the basic notions and concepts that are frequently used in the upcoming chapters. Most of the material covered here is based on the books by Schultens [Sch14] and Hatcher [Hat]. Note that all manifolds mentioned in this thesis are compact, connected and of dimension 3 unless stated otherwise. Two dimensional manifolds will be referred to as surfaces. Many of the definitions and theorems in this chapter could be generalised to all manifolds but we will state them only for dimension 3.

Notation 3.1.1. *1. I denotes the closed unit interval $[0, 1]$.*

2. $B^n = \{x \in \mathbb{R}^n | \, ||x|| < 1\}$ denotes the open unit ball in $\mathbb{R}^n$.

3. D^n denotes the closure of B^n.

4. $S^n = \{x \in \mathbb{R}^{n+1} | \, ||x|| = 1\}$ denotes the unit sphere in $\mathbb{R}^{n+1}$.

Definition 3.1.2. *Let M and N be 3-manifolds and $g_i : M \longrightarrow N$, $i = 0, 1$ be continuous maps. The maps g_0 and g_1 are said to be* ***homotopic*** *if there exists a continuous map $G : M \times [0, 1] \longrightarrow N$ such that $G(x, 0) = g_0(x)$ and $G(x, 1) = g_1(x)$ for all $x \in M$. The map G is said to be a homotopy between g_0 and g_1.*

Definition 3.1.3. *Two embeddings $g_i : M \longrightarrow N$, $i = 0, 1$ are said to be* ***isotopic*** *if there exists a continuous map $G : M \times [0, 1] \longrightarrow N$ such that $G(x, 0) = g_0(x)$ and $G(x, 1) = g_1(x)$ for all $x \in M$ and for each $t \in [0, 1]$, the* *map g_t defined by $G(.,t)$ is an embedding. The map G is called an isotopy between g_0 and g_1.*

Two submanifolds are said to be ***isotopic*** *if their inclusion maps are isotopic.*

Definition 3.1.4. *Two smooth submanifolds M_1 and M_2 of a smooth manifold M are* ***transverse*** *at the point $x \in M_1 \cap M_2$ if $T_x(M_1)$ and $T_x(M_2)$ span $T_x(M)$. The submanifolds are said to be transverse if it is transverse at all $x \in M_1 \cap M_2$.*

Let $f : K \longrightarrow M$ be a smooth map and M_2 be a smooth submanifold of M. Then f is said to be transverse to M_2 if for all $a \in K$, $f_*(T_a(K)) + T_p(M_2) = T_p(M)$ where $f(a) = p$ and $f_* : T_a(K) \longrightarrow T_p(M)$.

The theorem below says that transversality can be achieved via small isotopies.

Theorem 3.1.5 (See [GP10])**.** *Let $f : M_1 \longrightarrow M$ be a smooth map and let M_2 be any smooth submanifold of M. Then there exists a smooth map $f' : M_1 \longrightarrow M$ homotopic to f and transverse to M_2. Also, suppose $F : M_1 \times I \longrightarrow M$ is a homotopy between f and f', then $\forall \epsilon > 0$, the map $f_\epsilon(z) = F(z,\epsilon)$ is transverse to M_2.*

Next we give the definition of bundles. They give an interesting decomposition of a given manifold in terms of manifolds of smaller dimensions.

Definition 3.1.6. *A* ***bundle*** *is a quartet (M, F, B, p) where M, F, B are manifolds and $p : M \longrightarrow B$ is a continuous map such that the following holds:*

1. *for every $b \in B$, $p^{-1}(b)$ is homeomorphic to F*
2. *there is an atlas $\{U_\alpha\}$ for B such that for every α there is a homeomorphism $f_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$*
3. *the following diagram commutes:*

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times F \\ {\scriptstyle p}\big\downarrow & & \big\downarrow \\ U_\alpha & \xrightarrow{id} & U_\alpha \end{array}$$

Here M is called the **total space**, F is called the **fiber** and B is the **base space** and p is the **projection**. The quartet (M, F, B, p) is referred to as an **F-bundle over B**.

Definition 3.1.7. *Two bundles* (M, F, B, p) *and* (M', F', B', p') *are said to be* ***isomorphic*** *if there exist homeomorphisms* $h : M \longrightarrow M'$ *and* $g : B \longrightarrow B'$ *such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "p"'] & M' \arrow[d, "p'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

Definition 3.1.8. *A bundle isomorphic to a product bundle i.e.* $F \times B$ *is called* ***trivial bundle****. Any other bundle is said to be* ***non-trivial*** *or* ***twisted****.*

Definition 3.1.9. *A* ***section*** *of a bundle* (M, F, B, p) *is a continuous map* $\sigma : B \longrightarrow M$ *such that* $p \circ \sigma = id_B$*.*

The following definition give an important class of bundles that frequently appear in the later part of this thesis.

Definition 3.1.10. *Let* M *be a* 3*-manifold and let* $f : M \longrightarrow M$ *be a homeomorphism. The manifold obtained from* $M \times [0, 1]$ *by identifying* $(x, 0)$ *to* $(f(x), 1)$ *for all* $x \in M$ *is called* ***the mapping torus of*** f*. The mapping torus of* f *is a bundle with fiber* M *and base space* S^1*.*

This is a related construction to the previous definition.

Definition 3.1.11. *Let* $f : M \longrightarrow N$ *be a continuous map. The* ***mapping cylinder of*** f *is the manifold obtained from* $M \times I \coprod N$ *by identifying* $(x, 0) \in M \times I$ *to* $f(x) \in N$ *for all* $x \in M$*.*

In the remainder of this subsection we define some frequently used terms and mention some relevant results.

Definition 3.1.12. *Let* M *be a* 3*-manifold. Let* K *be a submanifold with* $dim(K) = m$*. A* ***regular neighborhood*** *of* K *is an open submanifold* $N(K)$ *of dimension* 3 *that is the total space of a bundle over* K *with fiber* B^{3-m}*. A regular neighborhood of a* 1*-manifold in a* 3*-manifold is called a* ***tubular*** *neighborhood.*

Theorem 3.1.13 (See [RS82])**.** *For any submanifold* K *of* M*, there exists a regular neighborhood* $N(K)$ *in* M*. Also, any two regular neighborhoods of* K *in* M *are isotopic.*

Definition 3.1.14. *A surface* F *in a* 3*-manifold* M *is said to be* 2***-sided*** *if a regular neighborhood of* F *in* M *is a trivial* I*-bundle and* F *is* 1***-sided*** *if it is twisted.*

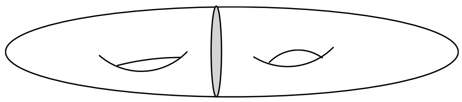


Figure 3.1: The shaded disk is a compressing disk

Definition 3.1.15. *Let M be a manifold and F be a properly embedded surface. To **cut** M **along** F means to consider the manifold $M - N(F)$, where $N(F)$ is a regular neighborhood of F.*

Definition 3.1.16. *Let F be a surface and γ be a simple arc in F. We call γ an **essential** arc if there is no simple arc λ in ∂F such that $\gamma \cup \lambda$ bounds a disk in F.*

Definition 3.1.17. *A surface $F \neq S^2$ in a 3-manifold M is said to have a **compressing disk** if there exists a simple closed curve in F that bounds a disc in M but does not bounds a disc in F (see Figure 3.1). A surface $F \subset M$ is called **incompressible** if it has no compressing disk.*

Definition 3.1.18. *A properly embedded surface F in a 3-manifold M is said to be **boundary incompressible with respect to a surface** G if for every simple arc α in F such that there exists an arc $\beta \in G$ with $\alpha \cup \beta$ bounds a disk in M, α is isotopic to an arc in $F \cap G$. When $G = \partial M$, F is simply said to be **boundary incompressible**.*

Definition 3.1.19. *Let M be a 3-manifold and $F \subset M$ be a surface. We say F is **boundary parallel** if there exists an embedding $\iota : F \times [0, 1] \longrightarrow M$ with $\iota(F \times \{1\}) \subset \partial M$.*

Definition 3.1.20. *A surface F in M is called **essential** if it is incompressible, boundary incompressible, and not boundary parallel.*

The loop theorem is one of the most celebrated results in 3-manifold topology. Loop theorem is in some sense a weaker generalization of Dehn's lemma. Papakyriakopoulos proved these theorems along with Dehn's lemma in his seminal paper [Pap57].

Theorem 3.1.21 (Dehn's Lemma, [Pap57])**.** *If an embedded circle in ∂M is nullhomotopic in M, it bounds a disk in M.*

Theorem 3.1.22 (The Loop Theorem [Sta60],[Pap57])**.** *Let M be a connected 3-manifold. If there is a map $f : (D, \partial D) \longrightarrow (M, \partial M)$ with $f|_{\partial D}$ not nullhomotopic in ∂M, then there is an embedding with the same property.*

Corollary 3.1.23. *Let $F \subset M$ be a 2-sided surface. Then F is incompressible if and only if $\iota_* : \pi_1(F) \longrightarrow \pi_1(M)$ is injective.*

The loop theorem fails for one-sided surfaces, as shown by Stallings in [Sta60]. He constructs an incompressible Klein bottle inside the lens space $L(6, 1)$. Clearly the inclusion map is not π_1-injective. One-sided incompressible surfaces are harder to work as this condition fails.

3.1.1 Manifold Decompositions

The first stage in decomposition of 3-manifolds is prime decomposition where a 3-manifold is repeatedly cut along embedded 2-spheres so that they separate into two 'simpler' 3-manifolds none of which is a 3-ball, and then gluing 3-balls along the resulting boundaries. Kneser [Kne29] showed this process terminates after a finite number of steps and Milnor [Mil62] showed that such a decomposition is unique upto a homeomorphism of the manifold.

The following theorem by Gugenhiem [Gug53] is central to the manifold decomposition described next.

Theorem 3.1.24 ([Gug53])**.** *An orientation preserving homeomorphism of B^3 or S^3 is isotopic to identity.*

Definition 3.1.25. *Let M and M' be two oriented 3-manifolds. Remove open 3-balls B and B' from M and M' respectively. Identify M and M' along the S^2 boundary components via an orientation reversing homeomorphism and the resulting 3-manifold is called the* ***connected sum*** *of M and M' and is denoted by $M \# M'$.*

A connected sum of any two manifolds is well-defined due to Theorem 3.1.24.

Definition 3.1.26. *A manifold M is said to be* ***prime*** *if $M = M_1 \# M_2$ implies that one of the M_i's is homeomorphic to S^3.*

Theorem 3.1.27 (Prime Decomposition, [Kne29], [Mil62])**.** *Let M be a compact, connected, orientable manifold. Then there exits a decomposition of M into a connected sum of prime manifolds and this decomposition is unique upto components that are S^3.*

Definition 3.1.28. *A 3-manifold M is said to* ***irreducible*** *if each sphere in M bounds a ball.*

Remark 3.1.29. *A closed connected prime 3-manifold is either irreducible or $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ where the latter is a mapping torus of the antipodal map of S^2. See Proposition 1.4 in [Hat] for a proof.*

The next step in the decomposition of 3-manifolds involved cutting along incompressible tori and Klein bottles.

Definition 3.1.30. *An irreducible* 3*-manifold is said to be* ***atoroidal*** *if every incompressible torus and Klein bottle is boundary parallel.*

Theorem 3.1.31 (Torus Decomposition, See [Hat])**.** *Let M be a connected, irreducible, compact, orientable* 3*-manifold. Then there exists a finite collection of disjoint incompressible tori $\mathcal{T}$ such that M cut along $\mathcal{T}$ has only atoroidal components.*

But, such a decomposition is not unique. See [Hat] for a counterexample. The astonishing part is that Seifert fiber spaces account for all counterexamples to the uniqueness statement as evident from the next theorem.

Theorem 3.1.32 (JSJ Decomposition, [Joh79], [JS79])**.** *Let M be a compact, orientable, irreducible* 3*-manifold. Then there exists a collection $\mathcal{T} \subset M$ of disjoint incompressible tori such that each component of $M|\mathcal{T}$ is either atoroidal or a Seifert fiber space and a minimal such collection is unique upto isotopy.*

Thurston's geometrization also involves cutting along incompressible tori into atoroidal and Seifert pieces but is not exactly the same as JSJ decomposition, because some pieces in the JSJ decomposition might not admit finite volume geometric structures.

Definition 3.1.33. *A* ***model geometry*** *is a tuple (X, G) where X is a simply connected manifold with a transitive action by a Lie group G such that the stabilizer of each point of X is a compact subgroup of G.*

Definition 3.1.34. *A* ***geometric structure*** *on a* 3*-manifold M is a diffeomorphism $f : M \longrightarrow X/\Gamma$ where X is a model geometry and Γ is a discrete subgroup of G acting freely on X.*

Theorem 3.1.35 (Thurston [Thu82], Perelman [Per02],[Per03b],[Per03a])**.** *Let M be a compact, connected, irreducible manifold. Then M admits a canonical decomposition along tori into pieces that possess geometric structures with finite volume.*

Thurston describes eight model geometries that are needed for classifying 3-manifolds. They are S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Universal cover of $SL(2, \mathbb{R})$ denoted by $\widetilde{SL(2, \mathbb{R})}$, Nil and Sol. Nil is a 3-dimensional Lie group consisting of upper triangular matrices with real entries of the form

$\begin{pmatrix} 1 & x & y \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ under multiplication and is also called the Heisenberg group. These manifolds fiber over $\mathbb{E}^2$. Compact manifolds admitting *Sol* geometry are either the mapping torus of an Anosov map of a torus or quotients of these by groups of order atmost 8.

Compact 3-manifolds admitting all but $\mathbb{H}^3$ and *Sol* geometries admit Seifert fiber structure. We define Seifert fiber spaces in the next section.

3.1.2 Triangulation

Sometimes introducing combinatorial structures to study manifolds makes life much easier. Triangulation is one such commonly used tool to study 3-manifolds. We define triangulation using simplicial complexes.

Definition 3.1.36. *The **standard closed** k**-simplex** is the set*

$$\{c_0v_0 + ... + c_kv_k \colon c_i \geq 0, \sum_{i=0}^{k} c_i = 1\}$$

where v_i is a $k+1$-tuple in $\mathbb{R}^{k+1}$ and is denoted by $[v_0, ..., v_k]$.

Definition 3.1.37. *A k**-simplex** in a topological space Y is a continuous map $f : [v_0, ..., v_k] \longrightarrow Y$ such that $f|_{int([v_0,...,v_k])}$ is a homeomorphism onto its image. Note that $f([v_0, ..., v_k])$ will be referred to as a k-simplex in Y.*

Remark 3.1.38.

1. *The standard 1-simplex is homeomorphic to a compact interval in $\mathbb{R}$.*
2. *The standard 2-simplex is homeomorphic to a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.*

Definition 3.1.39. *A $(k - j)$ dimensional **face** of a standard k-simplex is a subset of $[v_0, ..., v_k]$ given by*

$$\{c_0v_0 + ... + c_kv_k : c_{i_1} = ... = c_{i_j} = 0\}$$

*The faces of a k-simplex $f : [v_0, ..., v_k] \longrightarrow Y$ are images of restriction maps $f|_{[t]}$ where $[t]$ is a face of the standard k-simplex. A 0-dimensional face is called a **vertex** and a 1-dimensional face is called an **edge**.*

Definition 3.1.40. *A **simplicial complex** on a topological space Y is a set of simplices $K = \{f : [v_0, ..., v_k] \longrightarrow Y\}$ satisfying*

1. *For every simplex $f \in K$, all faces of f are also in K*

2. *For any two simplices f_1, f_2 in K, if the images of $f_1|_{int[s_1]}$ and $f_2|_{int[s_2]}$ have non-empty intersection, then they are the same simplex.*

Dimension of K is the maximum of dimensions of the simplices in K. The union of images of simplices in K is called the underlying space of K and is denoted by $|K|$.

Definition 3.1.41 ([Sch14]). *A **triangulated** n**-manifold** is a pair (M, K) where M is a topological n-manifold and K is a simplicial complex on M such that*

1. $|K| = M$
2. *K is locally finite*
3. *for $f, g \in K$ restricted to open simplices, the map $g^{-1} \circ f$ is an affine map on its domain.*

*K is called a **triangulation** of M.*

It is worthwhile to study triangulations as any manifold with dimension less than or equal to 3 can infact be triangulated.

Theorem 3.1.42 (Radó, Kerekjarto, Bing, Moise). *Any manifold with dimension less than or equal to 3 admits a triangulation.*

A proof for dimension 2 can be found in [AS60] and for dimension 3 can be found in [Moi52].

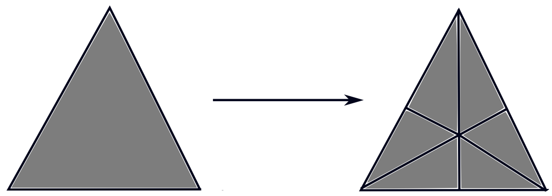
As we study triangulated manifolds, it makes sense to study interesting maps between the manifolds that respect the triangulations on them. The following definition makes this idea precise.

Definition 3.1.43. *Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes. A continuous map $F : |\mathcal{K}| \longrightarrow |\mathcal{L}|$ is said to **simplicial** if for any simplex $f \in \mathcal{K}$, $F \circ f = g$ is a simplex in $\mathcal{L}$. F is a simplicial isomorphism if it is a simplicial map and a homeomorphism, in which case $\mathcal{K}$ and $\mathcal{L}$ are said to be **isomorphic**.*

Definition 3.1.44. *Let $(M_i, \mathcal{K}_i)$, $i = 0, 1$ be two triangulated n-manifolds. They are considered to be **equivalent** if $\mathcal{K}_0$ and $\mathcal{K}_1$ are isomorphic.*

Definition 3.1.45. *A simplicial complex $\mathcal{L}$ is said to be a **subcomplex** of a simplicial complex $\mathcal{K}$ if every simplex in $\mathcal{L}$ is also a simplex in $\mathcal{K}$.*

Definition 3.1.46. *For any k-simplex $\mathcal{K}$, an r**-skeleton** where $r \leq k$, denoted by $\mathcal{K}^r$ is collection of all the simplices in $\mathcal{K}$ whose dimension is less than or equal to r.*



First barycentric subdivision of a triangle

Note that $\mathcal{K}^r$ is a subcomplex of $\mathcal{K}$.

Definition 3.1.47. *Let $[t] = [v_0, ..., v_k]$ be the standard k-simplex and let $v \in [t]$ be any point. Then $v = c_0v_0 + ... + c_kv_k$ for some $(k+1)$-tuple $(c_0, ..., c_k)$ where $c_i \in [0,1]$ for $i = 1, .., k$. This $k+1$-tuple is called the* ***barycentric co-ordinates*** *of v.*

Definition 3.1.48. *The point $b([v_0, ..., v_k]) = \frac{1}{k+1}v_0 + ... + \frac{1}{k+1}v_k$ is called the* ***barycenter*** *of the standard k-simplex.*

Define a partial order on a simplicial complex $\mathcal{K}$ as follows: for simplices $[t_1], [t_2] \in \mathcal{K}$, $[t_1] \leq [t_2]$ if and only if $[t_1]$ is a face of $[t_2]$ and write $[t_1] < [t_2]$ when $[t_1] \leq [t_2]$ and $[t_1] \neq [t_2]$.

Definition 3.1.49. *For a standard k-simplex $[t]$, let $[t_0], ..., [t_m]$ be a collection of its faces with $[t_0] < [t_1] < [t_2] < ... < [t_m]$. The* ***first barycentric subdivision*** *of $[t]$ is the union of all simplices of the form*

$$\{[b([t_0]), ..., b([t_m])] \colon [t_0], ..., [t_m] \text{ faces of } [t] \text{ such that } \\ [t_0] < [t_1] < [t_2] < ... < [t_m]\ \}$$

For a simplicial complex $\mathcal{K}$, the ***first barycentric subdivision*** *denoted by $\mathcal{K}^{(1)}$ is the simplicial complex obtained by taking the union of first barycentric subdivisions of the standard simplices in $\mathcal{K}$. The* ***n-th barycentric subdivision*** *of $\mathcal{K}$ is given by taking the first barycentric division n times denoted by $\mathcal{K}^{(n)} = (((\mathcal{K}^{(1)})^{(1)})...)^{(1)}$.*

See Figure 3.1.2 for the first barycentric subdivision of a standard 2-simplex.

3.1.3 Knot Theory

This subsection on elementary knot theory required to follow the succeeding chapters is primarily based on the books by Schultens [Sch14], Adams [Ada04] and Rolfsen [Rol90].

Definition 3.1.50. *A* ***knot*** $\mathcal{K}$ *in* S^3 *is a smooth isotopy class of smooth embeddings* $k : S^1 \longrightarrow S^3$*. More generally,* $\mathcal{K}$ *is called a link if it is a smooth isotopy class of embeddings of a disjoint union of circles in* S^3*.*

Two knots or links $\mathcal{K}_0$ *and* $\mathcal{K}_1$ *are equivalent if there is an isotopy* $F : S^3 \times I \longrightarrow S^3$ *such that* $F(\mathcal{K}_0, 1) = \mathcal{K}_1$*.*

Example 3.1.51.
1. *The isotopy class of embeddings containing the unknotted circle* $\{(x, y, z)|x^2 + y^2 = 1, z = 0\}$ *is called the unknot or the trivial knot.*

2. *Consider the line* $y = \frac{p}{q}x$ *in* $\mathbb{R}^2$*. The image of this line under the covering map* $\pi : \mathbb{R}^2 \longrightarrow S^1 \times S^1$ *given by* $\pi(x, y) = (e^{2\pi ix}, e^{2\pi iy})$ *is a simple closed curve on a torus. This curve is called a* (p, q)*-torus knot denoted by* $T_{p,q}$*. The knot* $T_{2,3}$ *is called a trefoil knot.*

Let the exterior of a knot $\mathcal{K}$ in S^3 denoted by $X_{\mathcal{K}}$ be given by $\overline{S^3 - N(\mathcal{K})}$ where $N(\mathcal{K})$ is a regular neighborhood of $\mathcal{K}$. Knot complements are an important class of compact 3-manifolds. One of the most fundamental theorems in knot theory is the following:

Theorem 3.1.52 (Gordon-Luecke, [GL89])**.** *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two knots in* S^3*. If* $X_{\mathcal{K}_1}$ *and* $X_{\mathcal{K}_2}$ *are homeomorphic, then* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *are equivalent.*

Such a result is not true for links.

Definition 3.1.53. *Let* α *and* β *be two transverse simple closed curves on a surface. Then*

$$i(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} i_p(\alpha, \beta)$$

denotes their ***algebraic intersection number****, given by the convention in Figure 3.2.*

Definition 3.1.54. *A knot* $\mathcal{K} \in S^3$ *is said to be* ***fibered*** *if* $X_{\mathcal{K}}$ *is a mapping torus* $F \times [0, 1]/\phi$*, where* F *is a compact surface,* $\phi : F \longrightarrow F$ *is a homeomorphism and* $(x, 0) \sim (\phi(x), 1)$*.*

Remark 3.1.55.
1. $\partial X_{\mathcal{K}}$ *is a torus. A simple closed curve on* $\partial X_{\mathcal{K}}$ *that bounds a disk in* $N(\mathcal{K})$ *is called a meridian. All meridians are isotopic. The isotopy class of all meridians is represented by* $[m]$ *or simply* m *when there is no confusion.*

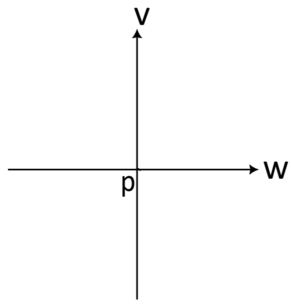


Figure 3.2: $i_p(v, w) = 1$

2. *Any curve l such that $i(l, m) = 1$ is called a longitude. All longitudes are not isotopic. For the unknot, the preferred longitude is the curve l that bounds a disk in $X_{\mathcal{K}}$.*

3. *$H_1(\partial X_{\mathcal{K}})$ is generated by the longitude l and the meridian m.*

The following construction is used to obtain new 3-manifolds by modifying the given 3-manifold. It is often thought of as drilling and filling. This is one of the most important constructions in the theory of 3-manifolds.

Definition 3.1.56. *Let M be a compact 3-manifold and $L = L_1 \cup L_2 \cup ... \cup L_n$ be a link in its interior.*

1. *Let N_i be closed tubular neighborhoods of L_i in the interior of M*

2. *Let c_i be a specified curve on ∂N_i.*

*Let $M' = (M - (\bigcup int(N_i)) \bigcup_h (\bigcup N_i)$ where h is a union of homeomorphisms $h_i : \partial N_i \longrightarrow \partial N_i$ such that it takes the meridian curve m_i to the specified curve c_i. Then M' is said to be obtained by **Dehn surgery** on M along the link L with surgery description 1 and 2.*

*When $M = S^3$, $h_*(m_i) = [c_i] = p_i l_i + q_i m_i$ where l_i is the longitude and m_i is the meridian. We call $r_i = \frac{q_i}{p_i}$ the **surgery co-efficient** associated with the component L_i.*

The importance of Dehn surgery is evident from the following theorem which was first proved by Wallace in 1960 and then a stronger version was proved independently by Lickorish in 1962.

Theorem 3.1.57 (Lickorish-Wallace, [Wal60],[Lic62])**.** *Every closed, orientable, connected 3-manifold is obtained by Dehn surgery on a link in S^3.*

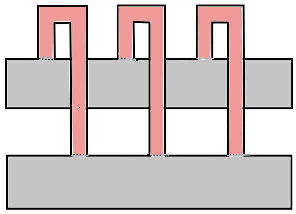


Figure 3.3: Seifert surface of the trefoil knot.

Definition 3.1.58. *Let λ be an essential simple closed curve on a torus. Then the* ***slope*** *of λ is $\frac{i(\lambda,l)}{i(m,\lambda)}$ where $i(\lambda, l)$ and $i(m, \lambda)$ is the algebraic intersection number of λ with l and m respectively.*

The following definition gives a way to construct new knots and links in a manner similar to connected sum.

Definition 3.1.59. *Let (M_1, N_1) and (M_2, N_2) be two pairs of manifolds such that N_i is a locally flat submanifold of M_1. Remove standard ball pair (B_i, B_i') from (M_i, N_i) and sew them back by a homeomorphism $h : (\partial B_2, \partial B_2') \longrightarrow (\partial B_1, \partial B_1')$ to form a pair connected sum.*

In a special case, Let $\mathcal{K}_i$, $i = 0, 1$ be oriented knots in S^3. Then their ***connected sum*** *denoted by $\mathcal{K}_0 \# \mathcal{K}_1$ is defined to be the pair connected sum $(S^3, \mathcal{K}_0)\#(S^3, \mathcal{K}_1)$. This operation is well-defined for oriented knots.*

Definition 3.1.60. *A knot $\mathcal{K}$ is said to be a* ***composite*** *knot if it can be expressed as a connected sum of two non-trivial knots. A knot that cannot be expressed a connected sum of non-trivial knots is called* ***prime****.*

Definition 3.1.61. *A* ***Seifert surface*** *of a knot $\mathcal{K}$in S^3 is a compact, orientable, connected surface F such that $\partial F = \mathcal{K}$.*

Every knot admits a Seifert surface. Seifert's algorithm (See Pg. 110 [Sch14] for instance) gives a construction for a Seifert surface for any knot. See Figure 6.11 for a picture of a Seifert surface of the trefoil knot.

Definition 3.1.62. *The* ***genus*** *of a knot $\mathcal{K}$ is the minimum possible genus of a Seifert surface of $\mathcal{K}$. It is denoted by $g(\mathcal{K})$.*

Definition 3.1.63. *A positive knot in S^3 is an oriented knot whose every crossing is as shown in Figure 3.4.*

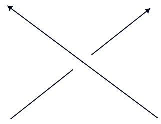


Figure 3.4: Positive crossing

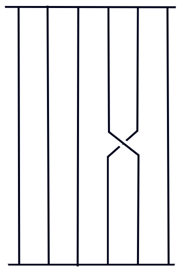


Figure 3.5: σ_4 as an example of 5-braid

Every knot can be represented as the closure of a closed braid. We define braids next.

Definition 3.1.64. *An **n-braid** is a collection of n disjoint strings or arcs that are attached to a horizontal bar at the top and bottom. These strings monotonically connect the top to the bottom. Two n-braids are said to be equivalent if they are isotopic relative to their end-points.*

Definition 3.1.65. *Let σ_i be the braid consisting of only a single positive crossing such that the i-th strand crosses over the $(i+1)$-th one as shown in Figure 3.5.*

Two n-braids can be multiplied by placing one on top of the other. This operation turns n-braids into a group called the braid group with presentation $< \sigma_1, ..., \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \; if \; |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} >$.

Definition 3.1.66. *The **closure of a braid** is the knot diagram obtained from a braid by connecting the top end-points of the strings to the corresponding ones at the bottom by disjoint arcs. Any knot obtained in this fashion is called a **closed braid**.*

Figure 3.6 represents the trefoil knot as a closed 2-braid.

Murasugi sum or generalized plumbing is an operation that relates to two oriented surfaces another oriented surface under certain conditions.

Definition 3.1.67 (See [Gab87b]). *Let F, F_1 and F_2 be compact oriented surfaces in S^3. Then F is said to be a **Murasugi Sum** of F_1 and F_2 if*

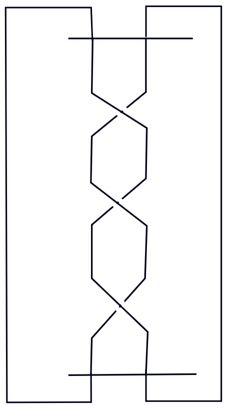


Figure 3.6: Trefoil knot as a closed braid

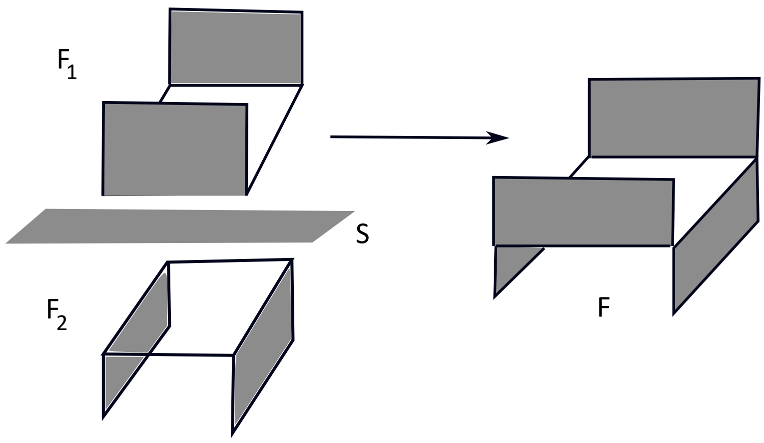


Figure 3.7: Local picture of plumbing of two surfaces

1. *$F = F_1 \cup_D F_2$ where D is a $2n$-gon*

2. *$F_1 \subset B_1$, $F_2 \subset B_2$, where B_1, B_2 are 3-balls and $B_1 \cap B_2 = S$, S is a 2-sphere, $B_1 \cup B_2 = S^3$ and $F_1 \cap S = F_2 \cap S = D$*

When D is a 4-gon, Murasugi sum is known as ***plumbing*** *(See Figure 3.7).*

In [Gab87b], Gabai shows that Murasugi sum is a natural geometric operation which means if F_1 and F_2 satisfy certain geometric properties, then so does F. Let L_i be the oriented link ∂F_i and $L = \partial F$.

Theorem 3.1.68 (Theorem 1,[Gab87b])**.** *The Murasugi sum of incompressible surfaces is incompressible i.e. if for $i = 1, 2$, F_i is incompressible in $S^3 - intN(L_i)$ then F is incompressible in $S^3 - int(L)$.*

Theorem 3.1.69 (Theorem 2, [Gab87b]). *Murasugi sum of minimal genus surfaces is minimal genus i.e. if for $i = 1, 2$ F_i is a minimal genus Seifert surface for L_i then F is the minimal genus Seifert surface for L.*

This generalizes Seifert's result about connected sum of minimal genus surfaces giving minimal genus Seifert surface for the connected sum of their boundaries.

Theorem 3.1.70 (Theorem 3, [Gab87b]). *The Murasugi sum of fibered links is fibered i.e. if for $1 = 1, 2$ L_i are fibered with fiber F_i, then L is fibered with fiber F. Conversely if L is fibered with fiber F, then so are L_i, with fiber F_i.*

3.2 Seifert Fiber Space

3.2.1 Introduction

This section deals with basic definitions pertaining to Seifert fiber spaces and the fiber-preserving classification of these manifolds. The material here is based mostly on [CMMN20],[Sco83] and [Sch14].

Definition 3.2.1. *Let $D^2 = \{re^{2\pi ix} | 0 \leq r \leq 1, x \in [0, 1]\}$. A **fibered solid torus**, denoted by $T(p, q)$ where $p, q \in \mathbb{N} \cup \{0\}$, $0 \leq q < p$ and $gcd(p, q) = 1$, can be constructed by taking $D^2 \times [0, 1]$ and identifying $(re^{2\pi ix}, 0)$ to $(re^{2\pi i(x+\frac{q}{p})}, 1)$. When $p = 1$, $T(p, q)$ is called the **trivial fibered solid torus**.*

Definition 3.2.2. *Let α and β be two simple closed curves on a surface F. The **geometric intersection number**, denoted by $i(\alpha, \beta)$, is defined as the minimum cardinality of $a \cap b$ over all simple closed curves a isotopic to α and b isotopic to β.*

Definition 3.2.3. *Let T be a solid torus. A meridian disk in T is a properly embedded boundary incompressible disk. The boundary of a meridian disk is referred to as a **meridian**, denoted by m. Any curve l on ∂T with $i(m, l) = 1$ is called a **longitude** for ∂T. Upto isotopy there is a unique meridian for ∂T.*

Remark 3.2.4. *Given any curve c on ∂T, we can express $[c] = p[l] + q[m]$ therefore referring to c as a '(p, q)-curve'. Here we are assuming that $gcd(p, q) = 1$ and we mean that c is a representative of the unique isotopy class of simple closed curves representing $p[l] + q[m]$.*

Definition 3.2.5. *A **fibered solid Klein bottle** can be constructed by taking $D^2 \times [0, 1]$ and identifying $(re^{2\pi ix}, 0)$ to $(re^{2\pi ir(x)}, 1)$ where $r : D^2 \longrightarrow D^2$ is a reflection along some diameter of D^2.*

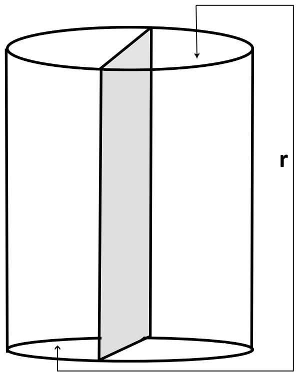


Figure 3.8: The shaded annulus is a singular surface in the fibered solid Klein bottle.

Remark 3.2.6. *As opposed to the infinite number of embedded curves on a torus, there are only four embedded curves on a Klein bottle upto isotopy.*

Definition 3.2.7. *A* ***fiber-preserving homeomorphism*** *between T_1 and T_2 is a homeomorphism $f : T_1 \longrightarrow T_2$ that takes fibers to fibers.*

In a fibered solid torus $T(p, q)$ with $gcd(p, q) = 1$, all the fibers except the central fiber wind along the generator of $\pi_1(T(p, q))$ p times and q times around the central fiber. Hence if there exists a fiber-preserving homeomorphism between $T(p, q)$ and $T(p', q')$ then $p = p'$ and $q = q'$ (mod p).

Since all reflections of a disk are isotopic, there is only one fibered solid Klein bottle upto fiber-preserving homeomorphism.

Definition 3.2.8. *A* ***Seifert fiber space*** *is a 3-manifold M with a decomposition into disjoint circles, called* ***fibers****, such that each circle has a regular neighborhood isomorphic to a fibered solid torus or fibered solid Klein bottle. The description of M in terms of these fibers is called a* ***Seifert fibration*** *of M.*

Definition 3.2.9. *A fiber whose regular neighborhood is a trivial fibered solid torus is called* ***regular****, otherwise the fiber is said to be* ***singular****.*

A fibered solid torus has at most one singular fiber, namely the central fiber. A fibered solid Klein bottle on the other hand, has a continuous family of singular fibers whose union is an annulus as shown in Figure 3.8. Hence in general, for a Seifert fiber space, singular fibers are either isolated or may form subsurfaces. These singular surfaces are either embedded annuli or are closed surfaces obtained by gluing two annuli along their boundaries, i.e. either a torus or a Klein bottle. Note that when the manifold is orientable, it contains no singular surfaces.

Definition 3.2.10. *A codimension-p* ***foliation*** *$\mathcal{F}$ of an n-manifold is a smooth atlas with the following properties:*

1. *for each chart (U_α, h_α), $h_\alpha : U_\alpha \longrightarrow \mathbb{R}^p \times \mathbb{R}^{n-p}$;*
2. *for every $x \in \mathbb{R}^p$ there exists $y \in \mathbb{R}^p$ such that the transition maps $h_{\alpha\beta} : h_\alpha(U_\alpha \cap U_\beta) \longrightarrow h_\beta(U_\alpha \cap U_\beta)$ satisfy $h_{\alpha\beta}(\{x\} \times \mathbb{R}^{n-p}) = \{y\} \times \mathbb{R}^{n-p}$.*

$\mathcal{F}$ is ***co-oriented*** *if its leaves admit locally compatible co-orientations.*

A maximal codimension p injectively immersed submanifold N such that each component of $N \cap U_\alpha$ is $\{c\} \times \mathbb{R}^{n-p}$ whenever the intersection is non-empty is called a **leaf** of the foliation.

Example 3.2.11. *Seifert fiber spaces are a special class of codimension-2 foliations in 3-manifolds with circles as leaves.*

In the original definition by Seifert in [Sei33], he does not include the fibers with solid Klein bottle neighborhoods. This is a more general definition as seen in [Sco83]. Epstein [Eps72] showed that a compact 3-manifold is foliated by circles if and only if it is a Seifert fiber space. This elegant statement would not be true with Seifert's original definition. Also, Epstien's result tells that Seifert fiber spaces account for all compact 3-manifolds foliated by circles.

3.2.2 Orbifolds

Let M be a Seifert fiber space and let B denote the quotient space obtained by collapsing each fiber to a point and $p : M \longrightarrow B$ denote this map. In this subsection we shall explore the structure of B, which shall henceforth be called the **base space** of M.

Let $\mathbb{Z}_p$ denote the cyclic group of order p. $\mathbb{Z}_p$ acts on D^2 by a rotation of angle $\frac{2\pi}{p}$. The orbit space of this action is topologically a disk, but is not a smooth quotient manifold. The quotient map $q : D^2 \longrightarrow D^2/\mathbb{Z}_p$ is a p-fold covering map except at $0 \in D^2$.

Definition 3.2.12. *An n-dimensional* ***orbifold*** *is defined as a Hausdorff, paracompact topological space which is locally homeomorphic to the quotient space of $\mathbb{R}^n$ by a finite group action.*

In general an orbifold may not even be homeomorphic to a manifold. But in dimension 2, any orbifold is homeomorphic to a surface. The only singularities possible in dimension two are as follows:

1. If G is a cyclic group, the quotient space $\mathbb{R}^2/G$ is a cone C with cone angle $\frac{2\pi}{p}$ where p is the order of G. This space inherits a Riemannian metric on $C - \{0\}$ where $\{0\}$ is the vertex of C. A singularity of this type is called a **cone point**.

2. If $G = \mathbb{Z}_2$ generated by reflection in a line $l \subset \mathbb{R}^2$, then $\mathbb{R}^2/G$ is isometric to a half-plane whose boundary is the image of l. Here the quotient space inherits a Riemannian metric away from the image of l. In this case, all of l is singular and is called a **reflector line**.

3. If G is the dihedral group of order $2n$ then $\mathbb{R}^2/G$ is isometric to an infinite wedge with angle $\frac{\pi}{n}$. Here there are two singular semi-infinite boundary lines and singular intersection point of these lines. A singular locus of this kind is called a **corner reflector**.

The base space B of a Seifert fiber space M admits the first two kinds of singularities namely cone points corresponding to the projection of singular fibers in $T(p,q)$ where $p > 1$ and reflector arcs corresponding to the projection of singular annuli in fibered solid Klein bottles. It also has reflector circles corresponding to projection of singular tori and Klein bottles. B does not have corner reflectors. See Section 2 of [Sco83] for more details.

Let $\mathcal{S}$ be the singular locus of B and let $N(\mathcal{S})$ be an open regular neighborhood of $\mathcal{S}$. Then the pre-image of the complement of $N(\mathcal{S})$ is a circle bundle over the compact surface $B - N(\mathcal{S})$.

3.2.3 A Combinatorial Description

A combinatorial description as well as the classification of closed Seifert fiber spaces is given in [Fin76] and was extended to include the manifolds with boundary in [CMMN20]. We briefly discuss the material from Section 2 of [CMMN20] here.

Let $p^* : M^* \longrightarrow B^*$ be a compact Seifert fiber space without any singular fibers, that is, an S^1-bundle over B^*. If $\partial B^* \neq \emptyset$, then B^* is homotopy equivalent to a wedge of circles. Over each generator circle there are only 2 circle bundles possible, namely the torus and Klein bottle. Therefore any circle bundle over B^* determines a homomorphism $\theta : H_1(B^*) \longrightarrow \{-1, 1\}$ defined by $\theta(\alpha) = 1$ if and only if the orientation of a fiber in M^* is preserved as it traverses through α in B^*. This gives a bijection between circle bundles over B^* and the set of all homomorphisms $\{\theta : H_1(B^*) \longrightarrow \{-1,1\}\}$. A precise classification was done by Fintushel in [Fin76].

If B^* has genus g, n boundary components and is orientable, then $H_1(B^*) = \{a_1..., a_g, b_1, ..., b_g, c_1, .., c_n | c_1 + ... + c_n = 0\}$. If B^* is non-orientable, then $H_1(B^*) = \{v_1..., v_g, c_1, .., c_n | c_1 + ... + c_n + 2v_1 + ... + 2v_g = 0\}$.

The S^1-bundle $p^* : M^* \longrightarrow B^*$ is said to be of type:

1. o_1 if $\theta(a_i) = \theta(b_i) = 1$ for all $i = 1, ..., g$
2. o_2 if $\theta(a_i) = \theta(b_i) = -1$ for all $i = 1, ..., g$
3. n_1 if $\theta(v_i) = 1$ for all $i = 1, ..., g$, $g \geq 1$
4. n_2 if $\theta(v_i) = -1$ for all $i = 1, ..., g$, $g \geq 1$
5. n_3 if $\theta(v_1) = 1$ and $\theta(v_i) = -1$ for all $i = 2, ..., g$, $g \geq 2$
6. n_4 if $\theta(v_1) = \theta(v_2) = 1$ and $\theta(v_i) = -1$ for all $i = 3, ..., g$, $g \geq 3$

Theorem 3.2.13 ([Fin76]). *Let B^* be a compact surface with $\partial B^* \neq \emptyset$. The fiber-preserving homeomorphism classes of circle bundles over B^* are in one-to-one correspondence with the pairs (k, ϵ), where $k \in 2\mathbb{Z}^+$ and counts the number of c_j such that $\theta(c_j) = -1$. If $k = 0$, then $\epsilon = o_1$ or o_2 when B^* is orientable and $\epsilon = n_1, n_2, n_3$ or n_4 when B^* is non-orientable. If $k > 0$, then $\epsilon = o$ with $o = o_1 = o_2$ when B^* is orientable and $\epsilon = n$ with $n = n_1 = n_2 = n_3 = n_4$ when B^* is non-orientable.*

If B^* is a closed surface, then in order to determine a bundle over B^*, along with $H_1(B^*) \longrightarrow \{-1, 1\}$ we need an additional invariant b, which is the obstruction to existence of a section of the bundle. When M^* is orientable, $b \in \mathbb{Z}$ otherwise $b = 0, 1$. For a detailed discussion, see [Sco83].

The symbols needed for the combinatorial description of compact Seifert fiber spaces are introduced below.

1. $g, t, k, m_+, m_-, r \in \mathbb{Z}^+$ with $k + m_- \in 2\mathbb{Z}^+$ and $k \leq t$;
2. $\epsilon \in \mathcal{E}$ where $\mathcal{E} = \{o, o_1, o_2, n, n_1, n_2, n_3, n_4\}$ such that $\epsilon = o, n$ if and only if $k + m_- > 0$, if $\epsilon = n_4$ then $g \geq 3$, if $\epsilon = n_3$ then $g \geq 2$ and if $\epsilon = o_2, n, n_1, n_2$ then $g \geq 1$;

3. $h_1, ..., h_{m_+}, k_1, ..., k_{m_-} \in \mathbb{Z}^+$ such that $h_1 \leq ... \leq h_{m_+}$ and $k_1 \leq ... \leq k_{m_-}$;

4. (p_i, q_i) be lexicographically ordered pairs of co-prime integers such that $0 < q_i < p_i$ if $\epsilon = o_1, n_2$ and $0 < q_i < \frac{p_i}{2}$ otherwise, for $i = 1, ..., r$;

5. $b \in \mathbb{Z}$ be arbitrary if $t = m_+ = m_- = 0$ and $\epsilon = o_1, n_2$; $b = 0, 1$ if $t = m_+ = m_- = 0$ and $\epsilon = o_2, n_3, n_4$ and no $p_i = 2$; $b = 0$ otherwise

Let $M = \{b; (\epsilon, g, (t, k)); (h_1, ..., h_{m_+} \mid k_1, ..., k_{m_-}); ((p_1, q_1), ..., (p_r, q_r))\}$ denote the Seifert fiber space with the above mentioned parameters. A construction of such a manifold is given in [CMMN20] and is described briefly in Section 2 of the next chapter.

Remark 3.2.14. *The manifold M is closed if and only if $m_+ + m_- = 0$ and oriented if and only if $\epsilon = o_1$ or n_2, $m_- = t = 0$ and $h_i = 0$ for all $i = 1, ..., m_+$.*

The classification theorem for Seifert fiber spaces is as follows:

Theorem 3.2.15 (Theorem A of [CMMN20])**.** *Every Seifert fiber space is uniquely determined, up to fiber-preserving homeomorphism, by the normalised set of parameters*

$$\{b; (\epsilon, g, (t, k)); (h_1, ..., h_{m_+} \mid k_1, ..., k_{m_-}); ((p_1, q_1), ..., (p_r, q_r))\}$$

Definition 3.2.16. *For a closed Seifert fiber space M, the **Euler number** of the fibering is given by $e(M) = \sum_{i=1}^{r} \frac{q_i}{p_i} + b$.*

Note that if M has no singular fibers, then $e(M)$ is an integer and is the obstruction to the existence of a section of $p : M \longrightarrow B$.

3.2.4 Incompressible surfaces

Essential surfaces in Seifert fiber spaces can be isotoped to a particularly nice form as given below.

Lemma 3.2.17 (Lemma 1.10,[Hat])**.** *Let F be a connected 2-sided incompressible surface in an irreducible 3-manifold such that ∂F is contained in a torus boundary component T of M. Then F is either essential or a boundary parallel annulus.*

Proof. Let D be a boundary compressing disk for F with $\partial D = \alpha \cup \beta$ where $\alpha \subset F$ and $\beta \subset T \subset \partial M$. The circles of $F \cap T$ do not bound disks, for otherwise F would be a disk and disks are boundary incompressible. Let the annulus component of $T|_{\partial F}$ containing β be A. If β cuts off a disk D' from A then $D \cup D'$ is a compressing disk for F. Since, F is incompressible α cuts off a disk from F which contradicts our assumption. Hence, end points of β lie on distinct components of ∂A. If ∂F is a single curve then F is one-sided. So, end-points of β lie in distinct components of ∂F. Let N be a neighborhood of $\partial A \cup \alpha$ in F. Note that $\partial N - \partial F$ bounds a disk outside of F namely $D_1 \cup D_2 \cup D'$ where $\partial N(D) = D_1 \cup D_2 \cup N(\alpha)$ and $D' = A \backslash N(\beta)$. Since F is incompressible, there is another disk in F with the same boundary turning F into an annulus. Surgering $F \cup A$ along D gives a sphere which bounds a ball. So, $F \cup A$ bounds a solid torus and hence F is boundary parallel. $\square$

The following lemma shows that most Seifert fiber spaces are irreducible.

Lemma 3.2.18. *(Proposition 1.12, [Hat]) A compact Seifert fiber space with isolated singular fibers is irreducible except when it is $S^1 \times S^2$, $S^1 \tilde{\times} S^2$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.*

We extend the result to all Seifert fiber spaces in our paper [KN23a] detailed in Lemma 5.2.2 of Chapter 5.

The following result is by Waldhausen [Wal67]. The proof mentioned here is based on the proof of Proposition 1.11 in [Hat].

Theorem 3.2.19 ([Wal67])**.** *In an irreducible connected compact Seifert fiber space M with no singular surfaces, any 2-sided essential surface F is isotopic to a surface which is either vertical i.e. union of regular fibers or horizontal i.e. transverse to each fiber.*

Proof. Remove the neighborhoods of singular fibers (or a single regular fiber if there are no singular ones) from M. We obtain a circle bundle $M_0 \longrightarrow B_0$ where B_0 is the compact surface obtained by deleting disk neighborhoods of points corresponding to the singular fibers. Choose disjoint arcs on B_0 such that deleting those turns it into a disk. Let the collection of annuli that are pre-images of these arcs be called $\mathcal{A}$. Note that M_0 cut along $\mathcal{A}$ is a solid torus M_1. Since, F is essential and M is irreducible, ∂F is non-trivial in ∂M. Hence ∂F may be isotoped to be horizontal or vertical in the torus and Klein bottle boundary components of M such that vertical circles are disjoint from $\mathcal{A}$. Since F may be assumed to be transverse to all the singular fibers, it may be assumed to be transverse to all their regular neighborhoods. So, we have $S_0 = M_0 \cap S$ also has boundary circles either vertical or horizontal in ∂M_0.

Now the trivial circles of $F \cap \mathcal{A}$ can be eliminated by using incompressibilty of F and the irreducibility of M. If $F \cap \mathcal{A}$ has arcs with both end-points on the same component of an annulus in $\mathcal{A}$, those can be eliminated too via an isotopy of F. So, $F \cap \mathcal{A}$ either has vertical circles or horizontal arcs. Now, in $F_1 = F_0|\mathcal{A}$, ∂F_1 consists of horizontal or vertical circles in M_1. F_1 is incompressible in M_1, by using Lemma 3.2.17, it is either essential or a boundary parallel annuli. So F_1 may be isotoped fixing boundary to a collection of meridian disks or boundary parallel annuli. If the boundary parallel annulus has a horizontal boundary, then we can find a ∂-compressing disk D, cutting along which we obtain arcs with both end-points on the same boundary components of annuli in $\mathcal{A}$. So, either we have F_1 to be a collection of meridian disks or boundary parallel annuli with vertical boundaries (since vertical and horizontal circles intersect on ∂M_1). Now, in the former case we may isotope F_1 to be horizontal fixing ∂F_1 thus obtaining an isotopy of F into a horizontal surface. Similarly, in the other case F may be isotoped to be vertical.

$\square$

Vertical surfaces are easy to understand as they are just the union of circle fibers. But existence of a horizontal surface has interesting implications for the manifold. The following existence criterion for horizontal surfaces in orientable Seifert fiber spaces is well-known. We extend this criterion to all Seifert fiber spaces in Theorem 4.1.5 of [KN23b].

Theorem 3.2.20 (Proposition 2.2, [Hat]). *Let M be an orientable Seifert fiber space.*

1. *When $\partial M \neq \emptyset$ then horizontal surfaces exist in M.*

2. *When $\partial M = \emptyset$ then horizontal surfaces exist if and only if $e(M) = 0$.*

The existence of horizontal surfaces in an orientable Seifert fiber space implies that the manifold M is either a surface bundle over circle with fiber the horizontal surface or it is a union of two twisted I-bundles. We extend this result to all Seifert fiber spaces in Corollary 4.2.6 of [KN23b].

Corollary 3.2.21. *Let M be a compact, orientable 3-manifold and let F be a compact 2-sided surface properly embedded in M. The following are equivalent:*

1. *M is a Seifert fiber space and F is a horizontal surface in M that intersects each regular fiber of M n times.*

2. *At least one of the following is true:*

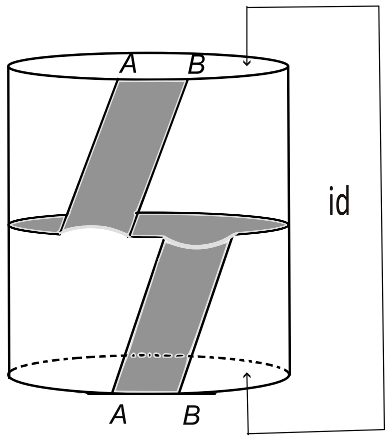


Figure 3.9: A one-sided incompressible surface inside a solid torus

(a) *There exists a homeomorphism ϕ of F such that $M = F \times I/\sim$, where $(x, 1) \sim (\phi(x), 0)$ for all $x \in F$. Furthermore $\phi^n = id$.*

(b) *There exist homeomorphisms ψ_0 and ψ_1 of F such that $M = F \times I/\sim$, where $(x, 0) \sim (\psi_0(x), 0)$ and $(x, 1) \sim (\psi_1(x), 1)$. Furthermore, n is even, $(\psi_0\psi_1)^{n/2} = id$ and both ψ_0 and ψ_1 are fixed-point free involutions.*

The proof of this corollary can be exactly replicated for the general case as well. We do this verification in Corollary 3.2.6 of [KN23b].

In his paper [Rub78], Rubinstein proved that any two incompressible surfaces embedded in a lens space $L(2k, q)$ are isotopic. Along the way, he proved that the only 1-sided incompressible surfaces in a solid torus ($D^2 \times S^1$) are once-punctured non-orientable surfaces with boundary a $(2k, q)$-curve. Figure 3.9 shows a one-sided incompressible surface with boundary a $(2, 1)$ curve inside a solid torus.

We have the following list of incompressible surfaces inside a solid torus.

Lemma 3.2.22 (Lemma 3.5,[Ran96])**.** *Let M be $D^2 \times S^1$ and F be a properly embedded connected incompressible surface. Then F is one of the following:*

1. *a boundary parallel disk*

2. *a boundary parallel annulus*

3. *a meridian disk*

4. *a once-punctured non-orientable surface*

We make a similar list of possible incompressible surfaces in a solid Klein bottle in Theorem 4.3.10 of our paper [KN23a].

A structure theorem for 1-sided incompressible surfaces in closed Seifert fiber spaces without singular surfaces was provided by Frohman in [Fro86].

Definition 3.2.23. *Let F be an embedded surface in M. Then F is said to pseudo-vertical if $F \cap M^*$ is vertical where M^* is M with neighborhoods of singular fibers removed , and $F \cap T_i$ (where T_i are the neighborhood of the singular fibers) is either empty or is a once punctured non-orientable surface.*

It is said to be pseudo-horizontal if $F \cap M^*$ is horizontal and $F \cap T_i$ is either a collection of meridian disks or is a once punctured non-orientable surface.

Theorem 3.2.24 (Theorem 2.5, [Fro86])**.** *Every closed* 1*-sided incompressible surface in a compact orientable Seifert fiber space with orientable base is isotopic to a pseudo-horizontal or pseudo-vertical surface.*

He proves this by showing that any 1-sided incompressible surface inside an orientable circle bundle is boundary compressible. If $p : M \longrightarrow B$ is a closed orientable Seifert fiber space, then $p^* : M^* \longrightarrow B^*$ obtained by deleting vertical neighborhoods of singular fibers $T_1, ..., T_k$ is a circle bundle. Any 1-sided incompressible surface F can be isotoped so that it intersects the ∂T_i transversally and minimally upto isotopy (normal intersection). Let $F^* = M^* \cap F$. If F^* is boundary compressible, the compression disk D can be used to isotope F^* so that the arc $\alpha = D \cap F^*$ can be absorbed into T_i. Since ends of the arc complementary to α in ∂D approaches $F \cap \partial T_i$ from opposite sides, this isotopy does not increase the number of components of $F \cap \partial T_i$. Hence, the intersection is still transverse and minimal. Such an isotopy increases $\chi(F^*)$ by 1. After a finite number of steps, F^* becomes boundary incompressible and thereby orientable. By Theorem 2.7 of [Wal67], F^* can be isotoped to horizontal or vertical. Since $F \cap \partial T_i, i = 1, ..., k$ is transverse and minimal, no component of $F \cap T_i$ is a boundary parallel annulus. Therefore F is pseudo-vertical or pseudo-horizontal.

This result was extended by Rannard in [Ran96] by removing orientablity conditions on M and B.

Theorem 3.2.25 (Theorem 4.1, [Ran96])**.** *Any closed incompressible surface in a closed Seifert fiber space with isolated singular fibers is isotopic to a pseudo-horizontal or pseudo-vertical surface.*

Rannard decomposes the manifold into solid tori and shows that the intersection of any incompressible surface with these solid tori are also incompressible. Since, we know all the incompressible surfaces in a solid torus, he proves the result by considering what happens when each case occurs. He defines 'well-embeddedness' for a surface which is used to provide contradiction in many cases. We extend this definition to incorporate the presence of singular surfaces and extend the result to all compact Seifert fiber spaces using similar techniques in our paper [KN23a] which is detailed in Chapter 5.

4

Prism Complexes

In this chapter, we introduce **prism complexes** as an analogue of simplicial complexes and characterize Seifert fiber spaces using a special type of prism complex. Along the way, we also provide a complete criterion for the existence of horizontal surfaces in Seifert fiber spaces which may contain singular surfaces. This chapter is based on our paper [KN23b].

4.1 Introduction

Definition 4.1.1. *A* prism *is the product space $\Delta \times I$ where Δ is a 2-simplex and I a closed interval. We call the edges of $\Delta \times \partial I$* horizontal *and the rest of the edges of the prism we call* vertical. *Similarly, we call the faces in $\Delta \times \partial I$* horizontal *and faces in $\partial\Delta \times I$* vertical.

Definition 4.1.2. *A* prism complex *is a 3-dimensional cell complex in which each cell is a prism, the attaching maps are combinatorial isomorphisms and furthermore, horizontal edges are identified only with horizontal edges.*

Definition 4.1.3. *We call a prism complex* special *if each horizontal edge in the interior of the complex lies in four prisms, each boundary horizontal edge lies in two prisms and no horizontal face lies on the boundary of the complex.*

We prove in this chapter that a special prism complex can be thought of as a discrete version of the local fibration of a Seifert fiber space:

Theorem 4.1.4. *Every compact 3-manifold M admits a prism complex structure. Moreover, it admits a special prism complex structure if and only if it is a Seifert fiber space with $\partial M \neq \emptyset$ or $SE(M) \neq \emptyset$ or $e(M) = 0$.*

So in particular, if M is a compact 3-manifold with boundary, then it admits a special prism complex structure if and only if it is a Seifert fiber space.

Most of the literature on Seifert fiber spaces deals only with oriented Seifert fiber spaces with the corresponding results for non-oriented spaces being folklore. To prove our result for all Seifert fiber spaces, we explicitly give a general criteria for existence of horizontal surfaces. A *horizontal* surface in a Seifert fiber space M is an embedded surface that is transverse to all the circle fibers of M.

Theorem 4.1.5. *Let M be a Seifert fiber space.*

1. *When $\partial M \neq \emptyset$ or $SE(M) \neq \emptyset$ then horizontal surfaces exist in M.*
2. *When $\partial M = \emptyset$ and $SE(M) = \emptyset$ then horizontal surfaces exist if and only if $e(M) = 0$.*

4.2 Seifert fiber spaces

This section deals with the construction of Seifert fiber spaces and a proof of Theorem 4.1.5. A complete combinatorial description for Seifert fiber spaces, which includes the non-orientable spaces, is explained in detail by Cattabriga et al [CMMN20]:

Theorem 4.2.1 (Theorem A of [CMMN20])**.** *Every Seifert fiber space is uniquely determined, up to fiber-preserving homeomorphism, by the normalised set of parameters* $\{b; (\epsilon, g, (t, k)); (h_1, ..., h_{m_+} \mid k_1, ..., k_{m_-}); ((p_1, q_1), ..., (p_r, q_r))\}$.

See Section 2 of [CMMN20] for a description of the parameters in the above theorem and for an explicit construction of a Seifert fiber space with the above parameters. We give below an outline of the construction:

Construction of Seifert fiber space M with given parameters: Let B^* be a compact connected surface of genus g with $m_+ + m_- + (r + 1) + t$ boundary components. B^* is orientable if $\epsilon = o$, o_1 or o_2 and non-orientable otherwise (i.e. if $\epsilon = n, n_1, n_2, n_3, n_4$). Consider B^* as a disk D^* with disjoint arcs σ_i, σ'_i on the boundary identified. As D^* is contractible, a circle bundle over D^* is necessarily the trivial bundle $D^* \times S^1$. Any circle bundle $p : M^* \to B^*$ is then obtained from $D^* \times S^1$ by identifying the disjoint annuli $p^{-1}(\sigma_i)$ and

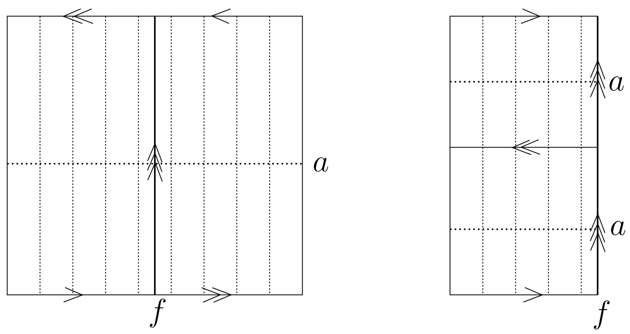


Figure 4.1: Different representations of the same fibered Mobius strip N with exceptional fiber f and a horizontal arc a

$p^{-1}(\sigma_i')$ fiber preservingly. As any homeomorphism of a circle is isotopic to the identity or the reflection map so we may assume that these annuli are identified by the identity or reflection map along the fibers. This pairwise identification is determined by the symbols for ϵ and such that M^* ends up with $(r+1)+(t-k)+m_+$ torus boundary components and $k+m_-$ Klein bottle boundary components. As both the identity and the reflection map on S^1 have a fixed point, so we can identify B^* with a fixed section of this circle bundle. We now obtain M from M^* via the following steps:

Step 1: Let T_i denote the torus boundary components of M^*. On each such boundary component define the meridian μ_i as the curve $T_i \cap \partial B^*$ and choose a regular boundary fiber of T_i as the longitude λ_i. Let V_i be solid tori. Define the meridian on ∂V_i as the unique curve (up to isotopy) that bound a disk in V_i. By a Dehn filling of T_i by V_i along the slope q_i/p_i we mean the attachment of V_i to M^* via a homeomorphism from ∂V_i to T_i that sends the meridian of V_i to the curve $p_i\mu_i + q_i\lambda_i$. Put $(p_{r+1}, q_{r+1}) = (1, b)$. As the first step in our construction, we Dehn fill the first $r+1$ torus boundary components T_i with solid tori V_i along the given slopes q_i/p_i. Let M' be the manifold thus obtained.

Step 2: Let $N = I \times I/(x,0) \sim (1-x,1)$ be a mobius strip foliated by the circles $(x \times I) \cup ((1-x) \times I)/\sim$ as in Figure 4.1(i). Let $\phi_i : S^1 \times \partial N \to T_i$ be the homeomorphism sending $t \times \partial N$ to $\mu_i(t) \times S^1$ in T_i. It is helpful to consider the model of N with the boundary on one side as in Figure 4.1(ii) (which can be obtained from the model in Figure 4.1(i) by cutting along the fiber f, flipping one of the pieces and reattaching along the segment with the double arrows). As a second step in our construction we attach $S^1 \times N$ to the next $(t-k)$ torus boundary components T_i via the attaching map ϕ_i. We call this process capping off T_i via $S^1 \times N$.

Step 3: In each torus T_i of the remaining m_+ torus boundary components let $\{\gamma_j\}_{j=1}^{h_i}$ be h_i many disjoint arcs in μ_i. Let $\psi_{(i,j)} : I \times \partial N \to p^{-1}(\gamma_j) \subset T_i$ be the homeomorphism sending $t \times \partial N$ to $\gamma_j(t) \times S^1$ in T_i. In this step we attach a copy of $(I \times N)$ to each $p^{-1}(\gamma_j) \subset T_i$ via the attaching homeomorphism $\psi_{(i,j)}$. So if $h_i > 0$, then the torus boundary T_i of M' is replaced by h_i many Klein bottle boundary components.

Step 4 Let K_i denote the Klein bottle boundary components of M'. We express K_i as a twisted product $S^1 \widetilde{\times} S^1 = I \times S^1/\sim$ where $(0, z) \sim (1, -z)$ with $B^* \cap K_i$ the curve $\mu_i(t) = (t, e^{\pi i t})$. Let $S^1 \widetilde{\times} N$ denote the twisted product $I \times N/\sim$ where $(0,(x,y)) \sim (1,(1-x, 1-y))$ with $N = I \times I/(x,0) \sim (1-x, 1)$. Let $\phi'_i : S^1 \widetilde{\times} \partial N \to K_i$ be the homeomorphism sending $t \times \partial N$ to the fiber above $\mu_i(t)$ in K_i. In this step, we cap off the first k Klein bottle boundary components of M' by $S^1 \widetilde{\times} N$ via the attaching map ϕ'_i.

Step 5 In each Klein bottle K_i of the remaining m_- Klein bottle boundary components let $\{\gamma'_j\}_{j=1}^{k_i}$ be k_i many disjoint arcs in μ_i. Let $\psi'_{(i,j)} : I \times \partial N \to p^{-1}(\gamma'_j) \subset K_i$ be the homeomorphism sending $t \times \partial N$ to the fiber above $\gamma'_j(t)$ in K_i. In this final step, we attach a copy of $(I \times N)$ to each $p^{-1}(\gamma'_j) \subset K_i$ via the attaching homeomorphism $\psi'_{(i,j)}$ to obtain M. If $k_i > 0$, then the Klein bottle boundary K_i of M' is replaced by k_i many Klein bottle boundary components.

The manifold M is closed if and only if $m_+ + m_- = 0$ and oriented if and only if $\epsilon = o_1$ or n_2, $m_- = t = 0$ and $h_i = 0$ for all $i = 1, ..., m_+$. For a closed Seifert fiber space, the Euler number of the fibering is given by $e(M) = \sum_{i=1}^{r} \frac{q_i}{p_i} + b$.

Let N be a Mobius strip foliated by circles with one exceptional fiber f as in Figure 4.1. We give a sufficient condition below on when a set of curves on the boundary of $S^1 \times N$ or $S^1 \widetilde{\times} N$ bound a horizontal surface. Recall that a horizontal surface is an embedded surface that is transverse to all the fibers of M. Keep Figure 4.2 as reference for Lemma 4.2.2.

Lemma 4.2.2. *Let $N = [0, \frac{1}{2}] \times S^1/\sim$ with $(\frac{1}{2}, z) \sim (\frac{1}{2}, -z)$ be a Mobius strip foliated by the circles $t \times S^1$ with one exceptional fiber $\frac{1}{2} \times S^1$ as in Figure 4.1(ii). Let $M = I \times N$ be foliated by the fibers $s \times t \times S^1$ for $s \in I$, $t \in [0, \frac{1}{2}]$, with an exceptional annulus $I \times \frac{1}{2} \times S^1$. Let Γ be a properly embedded arc in $A = I \times \partial N = I \times 0 \times S^1$ which intersects each fiber $s \times \partial N = s \times 0 \times S^1$ transversely. Γ has a parametrisation $\Gamma(s) = (s, 0, \gamma(s))$, for some arc $\gamma : I \to S^1$. Let $\omega : A \to A$ be the map $(s, 0, z) \to (s, 0, -z)$. Then there exists a horizontal rectangle R_Γ in M such that $R_\Gamma \cap A = \Gamma \cup \omega(\Gamma)$ and for $j = 0, 1$, $R_\Gamma \cap (j \times N) = j \times [0, \frac{1}{2}] \times (\gamma(j) \cup -\gamma(j))/\sim$. Furthermore if Γ' is another properly embedded arc in A disjoint from $(\Gamma \cup \omega(\Gamma))$ then R_Γ is disjoint from $R_{\Gamma'}$.*

Proof. As the arc Γ is transverse to the foliation $s \times 0 \times S^1$ of $A = I \times 0 \times S^1$

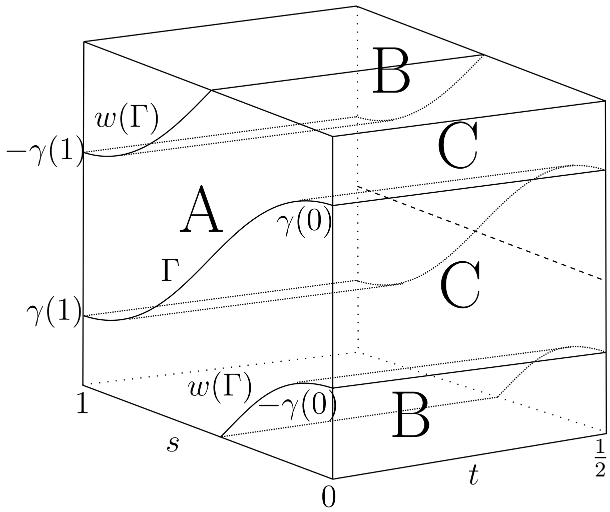


Figure 4.2: $M = I \times N$ depicted as a cube $[0,1] \times [0, \frac{1}{2}] \times [0,1]$ with the top and bottom faces B identified and the two halves C of a vertical faces identified. On the annulus A is shown the curves Γ and $w(\Gamma)$, and in the interior of the cube is the rectangle R_Γ.

so it intersects each fiber exactly once. We can therefore parametrise the arc as $\Gamma(s) = (s, 0, \gamma(s))$ for some arc $\gamma : I \to S^1$. Define the embedding $r_\Gamma : I \times I \to M$ as follows:

$$r_\Gamma(s,t) = \begin{cases} (s, t, \gamma(s)) & \text{if } t \in [0, \frac{1}{2}] \\ (s, 1-t, -\gamma(s)) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Let R_Γ be the image of r_Γ. Then $R_\Gamma \cap A$ is $r_\Gamma(I \times 0) \cup r_\Gamma(I \times 1) = \Gamma \cup \omega(\Gamma)$. And for $j = 0, 1$, $R_\Gamma \cap (j \times N) = r_\Gamma(j \times I) = (j \times I \times \gamma(j)) \cup (j \times I \times -\gamma(j))$. If $\Gamma'(s) = (s, 0, \gamma'(s))$ is disjoint from $\Gamma \cup \omega(\Gamma)$ then for any $s \in I$, $\gamma'(s) \neq \pm\gamma(s)$. Hence the images of r_Γ and $r_{\Gamma'}$ are disjoint as required.

$\square$

Lemma 4.2.3. *Let $M_1 = S^1 \times N$ and let $M_2 = S^1 \widetilde{\times} N = I \times [0, \frac{1}{2}] \times S^1 / \sim$ with $(0, t, z) \sim (1, t, \bar{z})$ and $(s, \frac{1}{2}, z) \sim (s, \frac{1}{2}, -z)$, for $s \in I$, $t \in [0, \frac{1}{2}]$, $z \in S^1$. Take their foliation by the circles $s \times t \times S^1$ which has an exceptional torus $S^1 \times \frac{1}{2} \times S^1$ in M_1 and exceptional Klein bottle $I \times \frac{1}{2} \times S^1/\sim$ in M_2. For $i = 1, 2$, let Λ_i be a simple closed curve on ∂M_i which intersects each fiber $s \times \partial N$ transversely. Let $\omega : \partial M_i \to \partial M_i$ be the map $(s, z) \to (s, -z)$ (as $-\bar{z} = \overline{(-z)}$ so ω is well-defined on ∂M_2). Then there exists a horizontal annulus or horizontal Mobius strip R_{Λ_i} in M_i such that $R_{\Lambda_i} \cap \partial M_i = \Lambda_i \cup \omega(\Lambda_i)$. Furthermore if Λ'_i is another simple closed curve in ∂M_i disjoint from $\Lambda_i \cup \omega(\Lambda_i)$ then R_{Λ_i} is disjoint from $R_{\Lambda'_i}$.*

Proof. Let $M = I \times N = I \times [0, \frac{1}{2}] \times S^1/(s, \frac{1}{2}, z) \sim (s, \frac{1}{2}, -z)$ as in Figure 4.2. Let $M_1 = S^1 \times N = I \times [0, \frac{1}{2}] \times S^1/\sim$ where $(s, \frac{1}{2}, z) \sim (s, \frac{1}{2}, -z)$ and $(0, t, z) \sim (1, t, z)$. There exists a quotient map $q : M \to M_1$ such that $q(1, t, z) = q(0, t, z)$. In particular, $q(1, 0, z) = q(0, 0, w)$ if and only if $z = w$.

The preimage $q^{-1}(\Lambda_1)$ of Λ_1 is a disjoint collection of properly embedded arcs $\{\Gamma_i\}_{i=0}^k$ in $A = I \times 0 \times S^1$ transverse to the fibers $s \times 0 \times S^1$. There exists a parametrisation $\Gamma_i(s) = (s, 0, \gamma_i(s))$ with $\gamma_i : I \to S^1$. Furthermore, $q(\Gamma_i(1)) = q(1, 0, \gamma_i(1)) = q(0, 0, \gamma_{i+1}(0)) = \Gamma_{i+1}(0)$ (taking i modulo $k + 1$). So in particular, $\gamma_i(1) = \gamma_{i+1}(0)$ in S^1. And therefore $q(1, t, \gamma_i(1)) = q(0, t, \gamma_{i+1}(0))$ (i mod $k + 1$).

By Lemma 4.2.2 there exist horizontal rectangles R_i in M such that $R_i \cap A = \Gamma_i \cup \omega(\Gamma_i)$ and for $j = 0, 1$, $R_i \cap (j \times N) = j \times I \times (\gamma_i(j) \cup -\gamma_i(j))$. Let $R_{\Lambda_1} = \cup_{i=0}^k q(R_i)$ in M_1. As $q(A) = \partial M_1$ and $q(\cup_{i=0}^k (\Gamma_i \cup \omega(\Gamma_i))) = \Lambda_1 \cup \omega(\Lambda_1)$ so $R_{\Lambda_1} \cap \partial M_1 = \Lambda_1 \cup \omega(\Lambda_1)$. As $q(1, t, \gamma_i(1)) = q(0, t, \gamma_{i+1}(0))$, so $q(R_i \cap 1 \times N) = q(R_{i+1} \cap 0 \times N)$ (i mod $k + 1$). Hence the edges of $q(R_i)$ that lie on the boundary of M_1 match up to give the horizontal annulus R_{Λ_1}.

Similarly, $M_2 = S^1 \widetilde{\times} N = I \times [0, \frac{1}{2}] \times S^1/\sim$ where $(s, \frac{1}{2}, z) \sim (s, \frac{1}{2}, -z)$ and $(0, t, z) \sim (1, t, \bar{z})$. Let $q : M \to M_2$ be the quotient map so that $q(0, t, z) = q(1, t, \bar{z})$. In particular, $q(0, 0, z) = q(1, 0, w)$ if and only if $z = \bar{w}$. Let $q^{-1}(\Lambda_2)$ be a disjoint collection of properly embedded arcs $\{\Gamma_i\}_{i=0}^k$ in A with a parametrisation $\Gamma_i(s) = (s, 0, \gamma_i(s))$. Furthermore, $q(\Gamma_i(1)) = q(1, 0, \gamma_i(1)) = q(\overline{0, 0, \gamma_{i+1}(0)}) = q(\Gamma_{i+1}(0))$ (taking i modulo $k + 1$). So in particular, $\gamma_i(1) = \overline{\gamma_{i+1}(0)}$ in S^1. And therefore $q(1, t, \gamma_i(1)) = q(0, t, \gamma_{i+1}(0))$ (i mod $k + 1$). Finally using Lemma 4.2.2 as above, there exists a horizontal annulus or horizontal Mobius strip R_{Λ_2} in M_2 such that $\partial R_{\Lambda_2} = \Lambda_2 \cup \omega(\Lambda_2)$ (note that R_{Λ_2} is a Mobius strip when $\omega(\Lambda_2) = \Lambda_2$). $\square$

Lemma 4.2.4. *Let n be an even positive number and let $\theta_0 = 2\pi/n$. Let $w = e^{i\theta_0/2}$ and let $P(n) = \{we^{mi\theta_0} : m \in \mathbb{Z}\} = \{w, we^{i\theta_0}, we^{2i\theta_0}, ..., we^{(n-1)i\theta_0}\}$. Let $\rho : S^1 \to S^1$ be the reflection map $z \to \bar{z}$ and let $\omega : S^1 \to S^1$ be the antipodal map $z \to -z$. Then $\rho(P(n)) = P(n)$ and $\omega(P(n)) = P(n)$.*

Proof. As $e^{ni\theta_0} = 1$, so points in $P(n)$ are of the form $we^{mi\theta_0} = e^{(m+1/2)i\theta_0}$ for $m \in \mathbb{Z}$. In particular $\rho(e^{(m+1/2)i\theta_0}) = e^{-(m+1/2)i\theta_0} = e^{((-m-1)+1/2)i\theta_0}$, so $\rho(P(n)) = P(n)$. And $m\theta_0 + \pi = m\theta_0 + n\theta_0/2 = (m + n/2)\theta_0$. So $-we^{mi\theta_0} = we^{mi\theta_0 + i\pi} = we^{(m+n/2)i\theta_0} \in P(n)$ as n is even. Therefore $\omega(P(n)) = P(n)$. $\square$

Lemma 4.2.5. *Let $0 < q < p$ be coprime integers, and let $n = kp$ be an even number. Let T be a torus $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ with meridian $\mu = S^1 \times 1$ and longitude $\lambda = 1 \times S^1$. Fix a point $z_0 \in S^1$ different from 1. Let $\alpha : I \to S^1$*

and $\beta : I \to S^1$ be the arcs in anti-clockwise direction from 1 to z_0 and from z_0 to 1 respectively. There exists a set of k pairwise disjoint curves Λ of slope q/p such that $\Lambda \cap (\beta(s) \times S^1) = \beta(s) \times P(n)$ and $\Lambda \cap (\alpha(s) \times S^1) = \alpha(s) \times e^{2\pi qsi/p}P(n)$.

Proof. Let $A(s) = \alpha(s) \times e^{2\pi qsi/p}P(n)$ and let $B(s) = \beta(s) \times P(n)$. Let $\theta_0 = 2\pi/n$ as in Lemma 4.2.4. Then $A(0) = 1 \times P(n) = B(1)$ and $A(1) = z_0 \times e^{2\pi iq/p}P(n) = z_0 \times e^{kqi\theta_0}P(n) = z_0 \times P(n) = B(0)$. So $\Lambda = A \cup B$ is a union of pairwise disjoint curves in T.

Both λ and μ intersect Λ transversely with the same sign at every intersection, so the slope of a curve in Λ is given by taking the ratio $|\Lambda \cap \mu|/|\Lambda \cap \lambda|$. To see that the slope of these curves is q/p we shall collapse $\beta \times S^1$ to $1 \times S^1$ so that the annulus $\alpha \times S^1$ becomes a torus T' and Λ goes to curves of slope q/p in T'.

Let $z_0 = e^{i\varphi}$ and let $f : T \to T'$ be the quotient map defined as follows:

$$f(z, w) = \begin{cases} (z^{2\pi/\varphi}, w) & \text{if } z \in \alpha \\ (1, w) & \text{if } z \in \beta \end{cases}$$

As $f(A(0)) = f(A(1)) = 1 \times P(n)$ so $f(A)$ is a pairwise disjoint set of curves in T' parametrised by $s \to (e^{i(2\pi s)}, e^{i(2\pi s)q/p}P(n))$ for $s \in I$. Their lifts in $\mathbb{R}^2$ are parallel straight lines with slope q/p, so the curves in $f(A)$ have slope q/p as required. Let $\lambda' = f(\lambda)$ and let $\mu' = f(\mu)$. Then the slope of $f(\Lambda) = f(A)$ is given by $|f(\Lambda) \cap \mu'|/|f(\Lambda) \cap \lambda'| = q/p$.

Note that f takes both $\lambda = 1 \times S^1$ and $z_0 \times S^1$ to $1 \times S^1$ homeomorphically. Furthermore, $\Lambda \cap \lambda = \Lambda \cap (z_0 \times S^1) = P(n)$. So $|f(\Lambda) \cap f(\lambda)| = |f(\Lambda \cap \lambda)| = |\Lambda \cap \lambda|$. As μ is disjoint from B, so $\Lambda \cap \mu = A \cap \mu \subset \alpha(0,1) \times S^1$. As f restricted to $\alpha(0,1) \times S^1$ is a homeomorphism onto its image and $f(\Lambda) = f(A)$ so $|f(\Lambda) \cap f(\mu)| = |f(A) \cap f(\alpha(0,1) \times 0)| = |A \cap (\alpha(0,1) \times 0)| = |A \cap \mu| = |\Lambda \cap \mu|$. Therefore the slope of curves in Λ, is $|\Lambda \cap \mu|/|\Lambda \cap \lambda| = q/p$ as required. Also as $|\Lambda \cap \lambda| = |P(n)| = n$ so there are $n/p = k$ curves in Λ as required.

□

The criteria for existence of horizontal surfaces in orientable Seifert fiber spaces is well known. We extend this criteria to all Seifert fiber spaces.

Proof of Theorem 4.1.5. To construct the manifold M we proceed as explained in the construction of Seifert fiber spaces with the given parameters at the beginning of this section. Let D be a 2-disk and let $\{\sigma_i, \sigma_i'\}$ be a collection of pairwise disjoint embedded arcs in ∂D. Let $\phi_i : \sigma_i(I) \to \sigma_i'(I)$ be a homeomorphism that is either $\sigma_i(s) \to \sigma_i'(s)$ for all $s \in I$ or $\sigma_i(s) \to \sigma_i'(1-s)$ for all $s \in I$. Let $\psi_i : S^1 \to S^1$ be either the identity map $z \to z$ or the conjugation map $z \to \bar{z}$ for all $z \in S^1 \subset \mathbb{C}$. The number of such arcs σ_i, σ_i' and the choice of ϕ_i and ψ_i is determined by the parameters. See Section 2 of [CMMN20] for details.

Let $D \times S^1$ be a solid torus foliated by the circle leaves $x \times S^1$. Let $M^* = D \times S^1/\sim$ where $(\sigma_i(s), z) \sim (\phi_i(\sigma_i(s)), \psi_i(z))$, i.e., M^* is obtained from the solid torus $D \times S^1$ by identifying the annuli $\sigma_i(I) \times S^1$ with $\sigma_i'(I) \times S^1$ via the maps $\phi_i \times \psi_i$. As ϕ_i and ψ_i send leaves of $D \times S^1$ to leaves of $D \times S^1$ so they induce a foliation of M^* by circle leaves. As $\psi_i(1) = 1$ for all i, so let $B^* = D \times 1/\sim$ be the surface obtained by identifying the arcs $\sigma_i \times 1$ with $\sigma_i' \times 1$ via the map ϕ_i. By construction, B^* intersects each leaf exactly once. Let $f : M^* \to B^*$ be the projection map which collapses each circle leaf of M^* to a point. This gives a circle bundle structure on M^*.

The manifold M is now obtained from the circle bundle M^* as follows: First Dehn fill $r+1$ torus boundary components T_i to obtain the manifold M', as described in Step 1 of the construction. Then cap off some torus boundary components of M' by $S^1 \times N$ and some Klein bottle boundary components of M' by $S^1 \widetilde{\times} N$ as explained in Step 2 and Step 4 of the construction. And lastly attach copies of $I \times N$ along its boundary to disjoint fibered annuli $I \times S^1$ in some of the boundary components of M', as detailed in Steps 3 and 5 of the construction.

Case I: $\partial M \neq \emptyset$. Let $(p_{r+1}, q_{r+1}) = (1, b)$ and let $n = 2p_1 \ldots p_{r+1}$. Our aim is to construct a horizontal surface S which intersects each regular fiber of M n times.

Constructing horizontal surface S^* in M^*: As $\partial M \neq \emptyset$, there exists an arc d in $B^* \cap \partial M$. Let $P(n)$ be the set of points $e^{i(m+1/2)2\pi/n}$ in S^1 for $m \in \mathbb{Z}$, as in Lemma 4.2.4. Let $\mathcal{D} = D \times P(n) \subset D \times S^1$. By Lemma 4.2.4, $\psi_i(P(n)) = P(n)$ for all i, so the arcs $\sigma_i \times P(n) \subset \partial\mathcal{D}$ are identified with the arcs $\sigma_i' \times P(n) \subset \partial\mathcal{D}$ to give a horizontal surface in M^* that we denote by S^*. The map $f : S^* \to B^*$ is an n-to-1 covering projection.

Constructing horizontal surface S' in M': Let $c_1, \ldots, c_{r+1}$ be the $r+1$ boundary components of B^* whose preimages $T_j = f^{-1}(c_j)$ are tori which are Dehn filled by solid tori V_j. Let $\beta_1, \ldots, \beta_{r+1}$ be disjoint arcs in B^* with both end points on d which cut out from d disjoint arcs α_j. Furthermore, they cut out from B^* annuli with disjoint interiors and with boundary curves $\alpha_j \cup \beta_j$ and c_j. Let B_0 be the component of B^* outside of all these annuli (i.e, B_0 is disjoint from all the c_j). Let $M_0 = p^{-1}(B_0)$ be its pre-image in M^*. Let $T_j' = f^{-1}(\alpha_j \cup \beta_j)$ be a torus parallel to $T_j = f^{-1}(c_j)$. As in Lemma 4.2.5 let Λ_j be a set of n/p_j pairwise disjoint curves of slope q_j/p_j in T_i' such that $\Lambda_j \cap f^{-1}(\beta_j) = \beta_j \times P(n)$ and $\Lambda_j \cap f^{-1}(\alpha_j(s)) = \alpha_j(s) \times e^{2\pi i s(q_j/p_j)} P(n)$. In Step 1 of our construction, we obtained M' by Dehn filling M^* along the

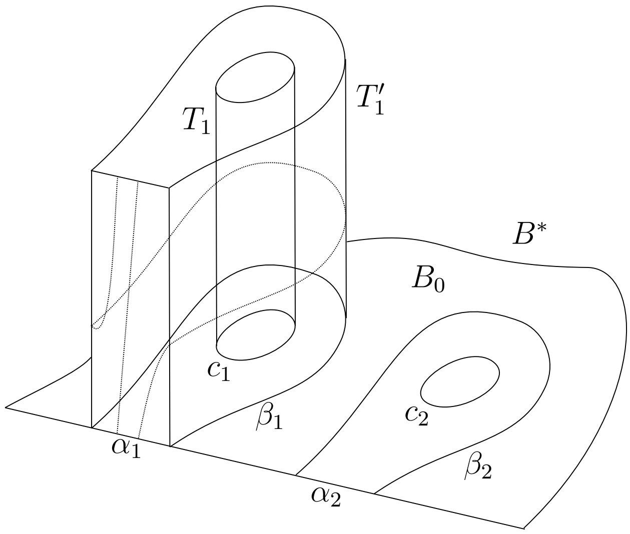


Figure 4.3: A piece of the section B^* in M^* where T_1 is the torus boundary above c_1, T_1' is the parallel torus above $\alpha_1 \cup \beta_1$ and the dotted curve on T' represents the surgery slope $\frac{1}{2}$.

slopes q_j/p_j of T_j. We can instead construct M' by attaching the meridians of the solid tori V_j to T_j' along the curves in Λ_j. See Figure 4.3.

Let $S_0 = M_0 \cap S^*$ be a horizontal surface in M_0 which intersects each fiber $f^{-1}(b)$ of the circle bundle $f : M_0 \to B_0$ in $b \times P(n)$. In particular, S_0 intersects each annulus $f^{-1}(\beta_i)$ in $\beta_i \times P(n)$.

Let $\mathcal{D}_i$ be a union of n/p_j disjoint meridian disks in the solid torus V_j. To construct M' from M_0 we attach V_j to M_0 along $f^{-1}(\beta_j)$ by homeomorphisms h_j from ∂V_j to T_j' that send $\partial \mathcal{D}_j$ to Λ_j. As $S_0 \cap f^{-1}(\beta_j) = \beta_j \times P(n) = h_j(\mathcal{D}_j) \cap f^{-1}(\beta_j)$ so $S' = S_0 \cup h_j(\mathcal{D}_j)$ is a horizontal surface in M'. Let F be a boundary component of M' and let η be a fiber of F. If η is a fiber over a point in some α_j then $S' \cap \eta = e^{i\theta}P(n)$ for some angle θ. By Lemma 4.2.4, $\omega(e^{i\theta}P(n)) = e^{i\theta}\omega(P(n)) = e^{i\theta}P(n)$. If η is not a fiber over a point in some α_j, then $\eta \cap S' = P(n)$. So in either case $\omega(S' \cap \eta) = S' \cap \eta$.

Extending S' to a horizontal surface S in M: Let F be a boundary component of M' that is capped off by $S^1 \times N$ (or $S^1 \widetilde{\times} N$) as in Step 2 (or Step 4) of the construction of Seifert fiber spaces with given parameters. Let Γ be the disjoint union of curves $\Gamma = S' \cap F$. For each fiber η of F, $\omega(\Gamma \cap \eta) = \Gamma \cap \eta$ and so by Lemma 4.2.3, there exists a horizontal surface $\mathcal{A}_F$ in $S^1 \times N$ (or in $S^1 \widetilde{\times} N$) such that $\mathcal{A}_F \cap F = \Gamma$.

Let γ be an arc in ∂B^* disjoint from all the α_i and c_i. Assume that we need to attach a copy of $I \times \partial N$ along $f^{-1}(\gamma)$ as in Step 3 or Step 5 of the construction of Seifert fiber spaces with given parameters. Then

$S' \cap f^{-1}(\gamma) = \gamma \times P(n)$. By Lemma 4.2.2, there exists a horizontal surface $\mathcal{A}_\gamma$ in $I \times N$ such that $\mathcal{A}_\gamma \cap f^{-1}(\gamma) = \gamma \times P(n)$.

We can therefore extend the surface S' to $S = S' \cup_F \mathcal{A}_F \cup_\gamma \mathcal{A}_\gamma$ where F varies over all boundary surfaces of M' which are capped off by an $S^1 \times N$ or $S^1 \tilde{\times} N$ and γ varies over all arcs in ∂B^* such that an $I \times N$ is attached to $f^{-1}(\gamma)$. By construction if η is a fiber of ∂M then $\omega(S \cap \eta) = S \cap \eta$.

Case II: $\partial M = \emptyset$ and $SE(M) \neq \emptyset$. If $SE(M)$ has an annulus then $\partial M \neq \emptyset$, as such annuli can only be obtained as the exceptional set of an $I \times N$ attached to $\partial M'$. So we may assume that $SE(M)$ only has torus and Klein bottle components. These are obtained as the exceptional sets of $S^1 \times N$ or $S^1 \tilde{\times} N$ attached to M' along boundary components.

Assume that $SE(M)$ has a torus exceptional set obtained by attaching $P = S^1 \times N$ along a torus boundary component T of M'. Let $W = M \backslash int(P)$. As W is a Seifert fiber space with boundary T so by Case I, W contains a horizontal surface S_W. Furthermore for each fiber η of T, $\omega(S_W \cap \eta) = S_W \cap \eta$. So by Lemma 4.2.3, there exists a horizontal surface $\mathcal{A}_P$ in P such that $\mathcal{A}_P \cap T = S_W \cap T$. Suppose $SE(M)$ has a Klein bottle exceptional set which lies in $Q = S^1 \tilde{\times} N$ attached to a Klein bottle boundary component K of M'. Proceeding similarly, we get a horizontal surface $\mathcal{A}_Q$ in Q such that $\mathcal{A}_Q \cap K = S_W \cap K$. Therefore either $S = S_W \cup_T \mathcal{A}_P$ or $S = S_W \cup_K \mathcal{A}_Q$ is the required horizontal surface in M.

Case III: Suppose $\partial M = \emptyset$ and $SE(M) = \emptyset$. The proof here is identical to the closed orientable case (see Pg 26-27 of [Hat]). We reproduce here the details for completion. Remove a solid torus neighbourhood V of a regular fiber of M to get a manifold W with a torus boundary component T. Proceed as in Case I, to obtain a horizontal surface S_W. It is now enough to show that S_W intersects T in curves of slope $e(M) = \sum_{i=1}^{r+1} q_i/p_i$: If $e(M) = 0$, we can extend the horizontal surface S_W to a horizontal surface on all of M by attaching meridian disks of the solid torus V that we Dehn fill in at T with slope zero. Conversely, given a horizontal surface S in M, the intersection of S with T bounds disks in V and hence must have slope zero, so $e(M) = 0$.

Claim: Slope of $S \cap T$ is $e(M)$. As S is horizontal it meets each fiber of M_0 the same number of times, say n times. Intersections of S with B_0 on the boundary we count with sign according to whether the slope of ∂S at such an intersection point is positive or negative. The signed total number of intersections we get is zero as points at the end of an arc of $S \cap B_0$ have opposite sign. The slope of S on the torus boundary containing c_i is by definition the ratio of the signed intersection with ∂B_0 and the signed intersection with a regular fiber. As this slope is q_i/p_i, it gives the signed intersection of S with B_0 on c_i as $n(q_i/p_i)$ for $i = 1...(r+1)$. So the slope of $S \cap T$ is $(\sum_{i=1}^{r+1} nq_i/p_i)/n = \sum_{i=1}^{r} q_i/p_i + b = e(M)$.

□

The below Corollary 4.2.6 now follows from standard arguments for horizontal surfaces (see the discussion on Pg 17-18 of [Hat]):

Corollary 4.2.6. *Let M be a compact 3-manifold and let F be a compact 2-sided surface properly embedded in M. The following are equivalent:*

1. *M is a Seifert fiber space and F is a horizontal surface in M that intersects each regular fiber of M n times.*

2. *At least one of the following is true:*

   (a) *There exists a homeomorphism ϕ of F such that $M = F \times I/\sim$, where $(x, 1) \sim (\phi(x), 0)$ for all $x \in F$. Furthermore $\phi^n = id$.*

   (b) *There exist homeomorphisms ψ_0 and ψ_1 of F such that $M = F \times I/\sim$, where $(x, 0) \sim (\psi_0(x), 0)$ and $(x, 1) \sim (\psi_1(x), 1)$. Furthermore, n is even, $(\psi_0\psi_1)^{n/2} = id$ and both ψ_0 and ψ_1 are fixed-point free involutions.*

Proof. Let M be a Seifert fiber space and let F be an embedded 2-sided horizontal surface in M that intersects each regular fiber n times. Each fiber of M has a fibered neighbourhood fiber-preserving homeomorphic to the fibered solid torus $D \times I/\sim$ where $(x, 1) \sim (h(x), 0)$ for some homeomorphism h of D. So $M \setminus F$ is an I-bundle. Let $p : M \setminus F \to G$ be the I-bundle projection, with G the base surface. As F is 2-sided, the associated ∂I-subbundle is two copies of F. Let $N(F)$ denote the tubular neighbourhood of F. Then $p : \partial(M \setminus N(F)) = F \sqcup F \to G$ is a 2-sheeted covering projection.

If $M \setminus F$ is connected then G is connected and the covering map $p : F \cup F \to G$ is the identity on each copy of F. So $M \setminus F = F \times I$ and hence $M = F \times I/\phi$ for some homeomorphism $\phi : F \to F$. Let $x \in F$ and let η_x be the fiber above x. As $F \times \frac{1}{2}$ intersects η_x n times, so η_x is divided by F into n segments with end points $(\phi^i(x), \frac{1}{2})$ and $(\phi^{i+1}(x), \frac{1}{2})$. So in particular, $\phi^n(x) = x$ as required.

If $M \setminus F$ is disconnected then each of the two components of $M \setminus F$ are I-bundles with a copy of F as the associated ∂I-subbundle. The base surface G has two components G_0 and G_1 and the projection map restricted to each copy of F is a 2-sheeted cover $p_i : F \to G_i$ for $i = 0, 1$. Let $\psi_i : F \times i \to F \times i$ be the non-trivial deck transformation corresponding to p_i. As the group of deck transformations is $\mathbb{Z}_2$ so $\psi_i^2 = id$. As F is 2-sided, by thickening F to

$F \times I$ and collapsing the two I bundles along the fibers, we get $M = F \times I/\psi_i$ as required. Let $x \in F$ and let η_x be the fiber above x. As before, $F \times \frac{1}{2}$ intersects η_x n times so η_x is divided by $F \times \frac{1}{2}$ into n segments along the points $(x, \frac{1}{2})$, $(\psi_1(x), \frac{1}{2})$, $(\psi_0\psi_1(x), \frac{1}{2})$, $(\psi_1\psi_0\psi_1(x), \frac{1}{2})$, ..., $(\psi_1(\psi_0\psi_1)^{(n/2)-1}(x), \frac{1}{2})$. In particular, n is even and $(\psi_0\psi_1)^{n/2}(x) = x$ as required. As ψ_i is a non-trivial deck transformation so it is fixed-point free.

Conversely, let $M = F \times I/\phi$ be a surface bundle with periodic monodromy ϕ of period n. $F \times I$ is foliated by the leaves $x \times I$ for $x \in F$. For each $x \in F$, let U_x be a neighbourhood of x in F homeomorphic to $\mathbb{R}^2$. Then $U_x \times [0, \frac{1}{2}) \cup (\phi(U_x) \times (\frac{1}{2}, 1])$ is a fibered neighbourhood fiber-wise homeomorphic to $\mathbb{R}^2 \times \mathbb{R}$ with the leaves $x \times \mathbb{R}$. Therefore, the leaves $x \times I/\phi$ give a foliation of M with 1-dimensional leaves. As $\phi^n(x) = x$, so each leaf $\cup_{i=0}^{\infty}(\phi^i(x) \times I)$ is a circle.

Similarly, assume $M = F \times I/\psi_i$ with ψ_i fixed-point free, $\psi_i^2 = id$ and $(\psi_0\psi_1)^{n/2} = id$. As ψ_i is fixed-point free, for each point $(x, i) \in F \times i$, there exists a neighbourhood $U_{(x,i)} \subset F$ homeomorphic to $\mathbb{R}^2$ such that $\psi_i(U_{(x,i)}) \cap U_{(x,i)} = \emptyset$. If this were not true we would obtain a sequence of points $x_n \to x$ in F such that $\psi_i(x_n) \to x$. As ψ_i is an involution so it follows that $x_n \to \psi_i(x)$ and hence $x = \psi_i(x)$. $U_{(x,1)} \times (0, 1] \cup \psi_1(U_{(x,1)}) \times (0, 1]$ is then a fibered neighbourhood fiber-wise homeomorphic to $\mathbb{R}^2 \times \mathbb{R}$ (with the leaves $x \times \mathbb{R}$). And similarly $U_{(x,0)} \times [0, 1) \cup \psi_0(U_{(x,0)}) \times [0, 1)$ is a fibered neighbourhood fiber-wise homeomorphic to $\mathbb{R}^2 \times \mathbb{R}$ with the leaves $x \times \mathbb{R}$. M is therefore foliated by the circle leaves $(x \times I) \cup (\psi_1(x) \times I) \cup (\psi_0\psi_1(x) \times I) \cup ... \cup (\psi_1(\psi_0\psi_1)^{(n/2)-1} \times I)$.

We now use a result of Epstein [Eps72] which says that that any compact 3-manifold foliated by circles is a Seifert fiber space to conclude that M is Seifert fibered. Furthermore, it contains the 2-sided horizontal surface $F \times \frac{1}{2}$ which intersects each regular fiber n times. By Theorem 4.1.5, it must have $\partial M \neq \emptyset$ or $SE(M) \neq \emptyset$ or be closed with $e(M) = 0$. $\square$

4.3 Prism complexes

Let F be a surface with a Riemannian metric g. We begin this section with an overview of the existence and uniqueness of a Riemannian center of mass for small enough convex geodesic polyhedra in (F, g). The Euclidean center of mass of points $p_1, ..., p_k \in \mathbb{R}^n$ is the point $\frac{1}{k}\sum p_i$. The Riemannian center of mass, also known as the Karcher mean, is a generalisation of this affine notion and was extensively studied by Karcher [Kar77]. We present here the treatment as in [DVW15].

Definition 4.3.1. *For $x \in F$, let $B(x, r)$ denote the set of points of F at a distance less than r from x, and denote by $\overline{B(x, r)}$ its closure. The injectivity* *radius of M at a point $x \in M$ is the supremum of the radii r of Euclidean balls $B(0,r) \subset T_x(M)$ that project down diffeomorphically to balls $B(x,r)$ in M via the exponential map exp_x. The injectivity radius of M, denoted by $i(M)$, is the infimum of the injectivity radius at all points of M.*

We call a set $B \subset M$ convex if any two points $p, q \in B$ are connected by a minimising geodesic that is unique in M and which lies entirely in B.

Lemma 4.3.2 (Theorem IX.6.1 of [Cha06])**.** *Let M be a Riemannian manifold with sectional curvatures bounded above by K_+ and let $i(M)$ be its injectivity radius. If*

$$r < \min\left\{\frac{i(M)}{2}, \frac{\pi}{2\sqrt{K_+}}\right\}$$

then $\overline{B(x,r)}$ is a convex set. (If $K_+ \leq 0$ then we take $1/\sqrt{K_+}$ to be infinite.)

Let B be an open set in M such that $\overline{B}$ is convex. Let $P \subset B$ be a geodesic convex polyhedron with vertices $\{p_1, ...p_k\}$. Let d denote the Riemannian distance function in M. Let $\epsilon : \overline{B} \to \mathbb{R}$ be the smooth function

$$\epsilon(x) = \frac{1}{2k}\sum_{i=1}^{k} d(x, p_i)^2$$

The gradient of ϵ is given by

$$\operatorname{grad}(\epsilon)(x) = -\frac{1}{k}\sum_{i} exp_x^{-1}(p_i)$$

At any point $x \in \partial P$, this gradient is therefore a vector pointing outward from P. And hence a minimum of ϵ lies in the interior of P. Karcher proved that when B is small enough, ϵ is convex and hence this minimum is unique. He in fact proved this in more generality for sets of measure 1 (as opposed to a set of k points with point measure $1/k$) and with an explicit bound on the convexity of ϵ. The following lemma follows from Theorem 1.2 of [Kar77] (see also Lemma 3 of [DVW15]):

Lemma 4.3.3. *If $\{p_1, ..., p_k\} \subset B(x,r) \subset M$ with*

$$r < \rho = \min\left\{\frac{i(M)}{2}, \frac{\pi}{4\sqrt{K_+}}\right\}$$

Then the function ϵ has a unique minimum in $B(x,r)$.

Definition 4.3.4. *Given a convex geodesic polyhedron $P \subset B(x,\rho) \subset M$ with vertices $\{p_1, ..., p_k\}$, we call this unique minimum $b(P)$ of ϵ in P the barycenter of P.*

Lemma 4.3.5. *Let ϕ be an isometry of (F, g), let $P \subset B(x_0, \rho)$ be a convex geodesic polyhedron and let $Q = \phi(P)$. Then $\phi(b(P)) = b(Q)$.*

Proof. Let $V(P) = \{p_1, ..., p_k\}$ and $V(Q) = \{q_1, ..., q_k\}$ be the set of vertices of P and Q respectively. Then $\phi(V(P)) = V(Q)$ and so for all $x \in P$,

$$\epsilon_Q(\phi(x)) = \frac{1}{2k}\sum_{i=1}^{k} d(\phi(x), q_i)^2 = \frac{1}{2k}\sum_{i=1}^{k} d(\phi(x), \phi(p_i))^2$$

As ϕ is an isometry, so it follows that $\epsilon_Q \circ \phi = \epsilon_P$. Let $\epsilon_P(b(P)) = m$ be the minimum value of ϵ_P. So $\epsilon_Q(\phi(b(P))) = \epsilon_P(b(P)) = m$ and as $\epsilon_Q = \epsilon_P \circ \phi^{-1}$ so the minimum value of ϵ_Q is also m. As $b(Q)$ is the unique minima of ϵ_Q so $\phi(b(P)) = b(Q)$. □

We show below that a periodic surface automorphism is simplicial with respect to some triangulation.

Lemma 4.3.6. *Let H be a finite subgroup of the group of automorphisms of a compact surface F. There exists a triangulation τ of F such that each $\phi \in H$ is a simplicial map with respect to τ.*

Proof. Let n be the order of H. Let g_0 be a Riemannian metric on F and let $g = \sum_{h \in H} h^*(g_0)$. Any $\varphi \in H$ acts on H as $\varphi(h) = h \circ \varphi$ for all $h \in H$ to give a bijection of H. So $\varphi^* g = \sum_{h \in H}(h \circ \varphi)^*(g_0) = g$, i.e., φ is an isometry of (F, g).

Let τ_0 be a geodesic triangulation of F such that each simplex lies in a convex ball of radius less than ρ as defined in Lemma 4.3.3. Let Π denote the polyhedral complex obtained by intersecting the simplexes of $h(\tau_0)$ for $h \in H$. In other words, if $H = \{h_1, ..., h_n\}$ then each cell P of Π is obtained by taking n triangles δ_1, ..., δ_n in τ_0 (possibly with repetition) and taking the intersection $P = \cap_{i=1}^{n} h_i(\delta_i)$. As each $h_i(\delta_i)$ is convex so P is a convex polyhedron. As φ induces a permutation of H so φ is a map sending polyhedra of Π to polyhedra.

For each polyhedron P of Π let $V(P)$ be its set of vertices and let $b(P) \in int(P)$ denote its Riemannian center of mass. P can be subdivided into the triangulation $\tau_P = b(P) \star \partial P$ by dividing along edges joining $b(P)$ to the vertices of P. Let τ be the triangulation obtained by replacing each polyhedron $P \in \Pi$ with the triangulated polyhedron τ_P.

For any $\varphi \in H$ as φ is a polyhedral map on Π, so if P is a polyhedron in Π so is $Q = \varphi(P)$. As $\varphi(V(P)) = V(Q)$ and by Lemma 4.3.5 $\varphi(b(P)) = b(Q)$, so φ is in fact a simplicial map from τ_P to τ_Q. Hence φ is simplicial over τ as required. □

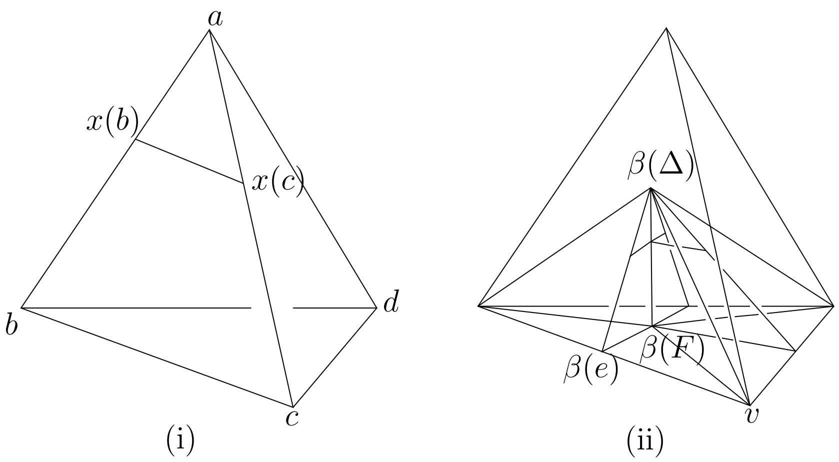


Figure 4.4: (i) Converting a 3-simplex into a prism (ii) Consistently doing so in the barycentric subdivision of a 3-simplex

We now prove the main theorem of this chapter:

Proof of 4.1.4. Given a 3-simplex δ with vertices a, b, c, d we can convert it to a prism that we denote as $[a; b, c; d]$ by introducing a vertex $x(b)$ in the edge $[a, b]$ a vertex $x(c)$ in the edge $[a, c]$ and an edge $[x(b), x(c)]$ that divides the face $[a, b, c]$ into a triangle which contains a and a quadrilateral which contains b and c, as shown in Figure 4.4(i). This gives a prism with triangular faces $[a, x(b), x(c)]$ and $[b, c, d]$ and quadrilateral faces $[b, c, x(c), x(b)]$, $[c, d, a, x(c)]$ and $[d, b, x(b), a]$.

To change tetrahedra to prisms consistently, we work instead with the barycentric subdivision $\beta(\tau)$ of a simplicial triangulation τ of M. Let $\beta(\sigma)$ denote the barycenter of a simplex σ. Any 3-simplex in $\beta(\tau)$ is of the form $[\beta(\delta), \beta(F), \beta(e), v]$ where δ is a 3-simplex of τ, F a 2-simplex of δ, e an edge of F and v a vertex of e. To obtain a prism complex structure we change each such simplex to the prism $[\beta(\delta); \beta(F), \beta(e); v]$, by introducing a vertex $x(F)$ on the edge $[\beta(\delta), \beta(F)]$, a vertex $x(e)$ on the edge $[\beta(\delta), \beta(e)]$ and by splitting the face $[\beta(\delta), \beta(F), \beta(e)]$ along an edge $[x(F), x(e)]$. See Figure 4.4(ii) for such a construction on the 3-simplexes of $\beta(\tau)$ in δ which contain the 2-simplexes of $\beta(F)$, for a fixed face F of δ. Such a change on each 3-simplex of $\beta(\tau)$ is consistent. Also any horizontal face is either of the form $[\beta(\delta), x(F), x(e)]$ and lies in the interior of a 3-simplex of τ or is of the form $[\beta(F), \beta(e), v]$. Varying δ, $F \in \delta$ and $e \in F$, the union of the faces $[\beta(F), \beta(e), v]$ gives the barycentric subdivision of the 2-skeleton of τ. In either case, horizontal edges only meet other horizontal edges, so this construction transforms a simplicial complex to a prism complex. As every compact 3-manifold has a simplicial complex structure it therefore has a prism complex structure. This construction does not however give a special prism complex structure, in particular, the interior horizontal edge $[x(F), x(e)]$ lies in only two prisms.

Suppose M admits a special prism complex structure. Foliate each prism $\delta \times I$ by intervals $x \times I$. Each face in the interior of the complex is shared by exactly two prisms while each boundary face lies in one prism. Furthermore as horizontal edges are identified only with horizontal edges, so horizontal faces are identified only with horizontal faces. Therefore points in the interior of faces have fibered neighborhoods that are fiber-wise homeomorphic to the fibered product $D \times I$ if the point is in the interior of M and $D^+ \times I$ if the point is on the boundary of M. The star of an edge is the union of all prisms which contain the edge. The dual graph of the star of an interior edge of the complex is regular of degree 2 and is therefore a circuit. So points in the interior of vertical edges have neighborhoods fiber-wise homeomorphic to $D \times I$. Similarly points on a vertical edge on the boundary has neighbourhoods fiber-wise homeomorphic to $D^+ \times I$. Exactly 4 prisms meet at a horizontal edge so exactly 2 horizontal faces meet along a horizontal edge. Consequently, the union of all horizontal faces gives a triangulated surface S. The interior of the union of all triangles containing a vertex v in S is a disk. All the prisms with a horizontal face on this disk which lie on the same side of the disk, share the vertical edge containing v. And so the union of all prisms containing such a vertex in M contains a fibered neighbourhood of v fiber-wise homeomorphic to $D \times I$. Therefore M is foliated by 1-dimensional leaves.

As no horizontal face lies on the boundary so the dual graph of the prism complex with edges corresponding to horizontal faces and vertices corresponding to the prisms is also a circuit. The union of the corresponding prisms is then either a solid torus or a solid Klein bottle foliated by circles. This shows that the 1-dimensional foliation constructed above has only circle leaves. Epstein [Eps72] has shown that any compact 3-manifold foliated by circles is a Seifert fiber space. As the surface S consisting of horizontal faces is transverse to this foliation, so by Theorem 4.1.5 either $\partial M \neq \emptyset$, $SE(M) \neq \emptyset$ or M is closed with $e(M) = 0$ as required.

Conversely, if M is a Seifert fiber space with $\partial M \neq \emptyset$, $SE(M) \neq \emptyset$ or $e(M) = 0$ then by Corollary 4.2.6, $M = F \times I/\phi$ where $\phi : F \times \{1\} \to F \times \{0\}$ is a periodic monodromy or $M = F \times I/\psi_i$ i=0,1 where $\psi_i : F \times \{i\} \to F \times \{i\}$ is an involution.

If $M = F \times I/\phi$ then let H be the finite subgroup of $Aut(F)$ generated by ϕ. By Lemma 4.3.6, there exists a triangulation τ of F with respect to which ϕ is simplicial. Let $\tau \times I$ be a prism complex structure on $F \times I$ where each prism is of the form $\delta \times I$ for δ a triangle of τ. Then $\Pi = \tau \times I/\phi$ is the required prism triangulation M. Similarly, if $M = F \times I/\psi_i$ then let H be the finite subgroup of $Aut(F)$ generated by ψ_1 and ψ_2. By Lemma 4.3.6, there exists a triangulation τ of F with respect to which both ψ_1 and ψ_2 are simplicial. Then $\Pi = \tau \times I/\psi_i$ is the required prism triangulation of M. $\square$

5

Essential Surfaces in Seifert Fiber Spaces

This chapter is based on our paper [KN23a] where we prove a structure theorem for essential surfaces in Seifert fiber spaces. Along the way we provide a complete list of incompressible surfaces in a solid Klein bottle i.e. $N \times I$.

5.1 Introduction

Recall that we call a closed 3-manifold M *irreducible* if every embedded 2-sphere in M bounds a 3-ball. The prime decomposition phenomenon of 3-manifolds allows us to uniquely express every closed 3-manifold as a connected sum of manifolds which are either irreducible or $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$. The geometrisation of 3-manifolds further allows us to cut these irreducible summands along a canonical collection of incompressible tori and Klein bottle into pieces which are one of three possible types: either they are Seifert fiber spaces or they are finitely covered by torus bundles or they have interiors which admit a complete hyperbolic metric. Seifert fiber spaces are precisely those compact 3-manifolds which admit a foliation by circles. They are an important class of 3-manifolds that are fairly well-understood as they are completely determined by a finite collection of invariants.

Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and let $D^+ = \{(x, y) \in D : x \geq 0\}$. A *model fibered solid torus* is the monodromy fibering of a solid torus given by $D \times I/\sim_\rho$ or $D^+ \times I/\sim_\rho$. Where $\rho : D \times 1 \to D \times 0$ is a rational rotation map (possibly identity) and $\rho : D^+ \times 1 \to D^+ \times 0$ is the identity map. We call the model fibered solid torus *regular* if ρ is the identity and *non-regular* otherwise. Similarly, *a model fibered Klein bottle* is the monodromy fibering given by $D \times I/ \sim_r$ or $D^+ \times I/ \sim_r$ where $r : D \times 1 \to D \times 0$ or $r : D^+ \times 1 \to D^+ \times 0$ is the reflection along the x-axis.

Epstein [Eps72] has shown that every circle fiber f in a Seifert fiber space has a fibered neighbourhood isomorphic to a model fibered solid torus or a model fibered Klein bottle with f identified with the fiber $0 \times S^1$ in the model. We call f *regular* if it has a fibered neighbourhood that is isomorphic to a regular model fibered solid torus and *singular* otherwise. An isolated singular fiber has a fibered neighbourhood isomorphic to a non-regular model solid torus while the non-isolated singular fibers have fibered neighbourhoods isomorphic to a model solid Klein bottle. The union of the non-isolated singular fibers give a collection of annuli, tori and Klein bottles that we call *singular surfaces*. Let N denote a Mobius strip. The model fibered neighbourhood of a singular surface C is either a solid Klein bottle $N \times I$ (when C is an annulus), $N \times S^1$ (when C is a torus) or $N \tilde{\times} S^1$ (when C is a Klein bottle). A good exposition for Seifert fiber spaces is the survey paper by Scott [Sco83] and the preprint of a book by Hatcher [Hat]. A good account of the non-orientable Seifert fibered spaces with singular surfaces is given by Cattabriga et al [CMMN20].

Let M denote a Seifert fiber space possibly with singular surfaces. Let S denote a properly embedded surface in M. We call an embedded disk D in M a *compressing disk* if $D \cap S = \partial D$ which is an essential curve in S. We call an embedded disk D in M a *boundary compressing disk* if ∂D is the union of arcs α and β where $\alpha = D \cap S$ is an arc in S that is not boundary-parallel and $\beta = D \cap \partial M$. For $E \subset \partial M$, we say that S is *boundary-compressible with respect to* E if β lies in E. We say that S is *incompressible* or *boundary-incompressible* if it does not have any compressing disks or boundary-compressing disks respectively. We say that S is *essential* if it is neither S^2 nor a boundary-parallel disk and it is both incompressible and boundary-incompressible in M. We now define the notions of horizontal, vertical, pseudo-horizontal and pseudo-vertical below.

Definition 5.1.1. *Let C be a fibered annulus, Mobius strip, torus or Klein bottle. We say that a properly embedded arc or curve in C is* horizontal *if it is transverse to the fibration. We say that a curve in C is* vertical *if it is a fiber of the fibration.*

We say that a properly embedded surface S in M is horizontal *if it is transverse to the fibration of M and we call it* vertical *if it is composed of a union of the fibers of M.*

Definition 5.1.2. *Let S be a properly embedded surface in a Seifert fiber space M. Let C be an isolated singular fiber or a singular surface of M and* *let W be a model neighbourhood of C. We say that the intersection of S with W is* exceptional *if $S \cap W$ is one of the following:*

1. *A once-punctured non-orientable surface with horizontal boundary in ∂W when C is an isolated fiber.*

2. *A pair of pants with vertical boundary in ∂W when C is an annulus.*

3. *A once-punctured torus with vertical boundary in ∂W when C is a torus.*

4. *A once-punctured Klein bottle with vertical boundary in ∂W when C is a Klein bottle.*

Definition 5.1.3. *Let S be a properly embedded surface in a Seifert fiber space M. We say that S is* pseudo-horizontal *if it is a horizontal surface outside model neighbourhoods of the isolated singular fibers and intersects the model neighbourhoods of the isolated singular fibers horizontally or exceptionally.*

We say that S is pseudo-vertical *if it is a vertical surface outside model neighbourhoods of the singular surfaces and isolated singular fibers and it intersects the model neighbourhoods of the singular surfaces and isolated singular fibers vertically or exceptionally.*

It is well-known that in Seifert fiber spaces without singular surfaces, any two-sided essential surface can be isotoped to become vertical or horizontal (see Hatcher [Hat]). Frohman [Fro86] showed that a closed one-sided incompressible surface in an orientable Seifert fiber space with orientable base can be isotoped to become pseudo-horizontal or pseudo-vertical. Rannard [Ran96] extended this result to closed incompressible surfaces in non-orientable Seifert fiber spaces without singular surfaces. In this chapter, we extend Rannard's proof to Seifert fiber spaces which may have boundary and singular surfaces by listing out the incompressible surfaces in a solid Klein bottle. The main theorem we prove in this chapter is the following:

Theorem 5.1.4. *Let M be a Seifert fiber space (possibly with singular surfaces) which has at least one singular fiber and let S be a connected properly embedded essential surface in M. Then S can be isotoped to a surface that is pseudo-horizontal or pseudo-vertical.*

If M has no singular fibers, i.e., M is an S^1-bundle over a surface then Rannard [Ran96] has shown that S can be isotoped either to a vertical surface or to a surface that is horizontal outside the model neighbourhood of one regular fiber and intersects the model neighbourhood horizontally or exceptionally. Note that if M has singular surfaces then it always has a horizontal surface (Theorem 1.2 of [KN23b]).

5.2 Properties of Seifert fiber spaces

In this section we prove that most Seifert fibered spaces are irreducible and have incompressible boundary and we fix a partition of the manifold into solid tori and solid Klein bottle.

We will reserve the letter N to denote a fibered Mobius strip with monodromy fibering given by $[-1,1]\times[-1,1]/(x,1)\sim(-x,-1)$. The solid Klein bottle $N\times I$ has an induced fibering which is the same as that of the model fibered solid Klein bottle. The boundary Klein bottle has a fibering given by the cores of the two Mobius strips $N\times 0$ and $N\times 1$ that we denote by l_1 and l_2 and fibers parallel to $\partial N\times t$ that we denote by d. We denote by m the boundary of a meridian disk of the solid Klein bottle (see Figure 5.2).

Let M be a Seifert fiber space with base space B and projection map $p: M\to B$. Let M_1 be the union of disjoint model neighbourhoods of its singular surfaces, i.e., M_1 is a (possibly empty) union of $N\times I$, $N\times S^1$ and $N\tilde{\times}S^1$ components. Let B_1 denote the projection of M_1 on B. Let $B_0=\overline{B\setminus B_1}$ denote the base space of the Seifert fiber space $M_0=\overline{M\setminus M_1}$ which has only isolated singular fibers.

Lemma 5.2.1. *Let M be an irreducible Seifert fiber space with compressible boundary. Then M is a solid torus or a solid Klein bottle.*

Proof. Assume that M has a compressible torus or Klein bottle boundary component T. Let D be a compressing disk for ∂M. Let $N(D)\simeq D\times I$ be a regular neighbourhood of D in M. Then $(T\setminus(\partial D\times I))\cup(D\times\partial I)$ is an embedded sphere in M. As M is given to be irreducible so this sphere bounds a ball that does not contain $D\times(int(I)))$. Hence M is obtained from two balls by identifying a pair of disks on their boundaries so M is either a solid torus or a solid Klein bottle. □

The following Theorem 5.2.2 is known for Seifert fibered spaces with isolated singular fibers (see Proposition 1.12 of Hatcher [Hat]). We extend this result to Seifert fibered spaces which may have singular surfaces.

Theorem 5.2.2. *Let M be a Seifert fibered space (possibly with singular surfaces). Then M is irreducible unless it is $S^2\times S^1$, $S^2\tilde{\times}S^1$ or $\mathbb{RP}^3\#\mathbb{RP}^3$.*

Proof. Let M_0 and M_1 be as described above. If M_0 is reducible then it must have a horizontal reducing sphere. So M_0 is either an S^2-bundle over S^1 or an S^2-semibundle over I. Hence M_0 is closed and so $M = M_0$ must be either $S^2 \times S^1$, $S^2 \tilde{\times} S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$. See Proposition 1.12 of Hatcher [Hat] for details. We shall henceforth assume that M_0 is irreducible.

Let $P = M_0 \cap M_1$ be a union of tori, Klein bottle and annuli. Let S be a reducing sphere of M that intersects P minimally in its isotopy class. If S does not intersect P then it lies either in M_0 or in an $N \times I$, $N \times S^1$ or $N \tilde{\times} S^1$ component of M_1 all of which are irreducible. Therefore S bounds a ball in M contradicting the fact that S is reducing.

Let D be an innermost disk of $S \cap P$ in S. If D lies in a solid Klein bottle component $N \times I$ of M_1 then it is either a meridian disk or it is parallel to a disk in $\partial N \times I$. As $D \subset S$ does not intersect $N \times \partial I \subset \partial M$ so D can not be a meridian disk. Isotoping D across $\partial N \times I$ reduces the number of components of $S \cap P$ which is a contradiction. So we may assume that D does not lie in an $N \times I$ component of M_1.

As M_0 has only isolated singular fibers so it is not a solid Klein bottle. Assume that M_0 is not a solid torus, so by Lemma 5.2.1 M_0 is an irreducible manifold with incompressible boundary. As fibers are essential in M_i so the annuli components of P which are all vertical are incompressible in M_i. Hence all the components of P are incompressible in M_i, so ∂D bounds a disk D' in P. And as M_i is irreducible so $D \cup D'$ bounds a ball in M_i. There exists an isotopy which sweeps D across this ball and off D' to reduce the number of components of $S \cap P$, which again contradicts the minimality of this intersection.

Finally if M_0 is a solid torus and D is a compressing disk for ∂M_0 in M_0, then it is a meridian disk of M_0 with horizontal boundary. So ∂M_0 lies in the interior of M and hence M is the union of a solid torus M_0 and $M_1 = N \times S^1$ along their boundary tori. In this case, we claim that M is $S^2 \tilde{\times} S^1$. Let D_1 be a horizontal meridian disks of M_0. Let γ_1 be the boundary of D_1 in ∂M_1. Let α be a horizontal straight arc in $N_0 = N \times t_0$ connecting a point of $\gamma_1 \cap \partial N_0$ to a point p of ∂N_0 which is not on γ_1. Let D_2 be another meridian disk of M_0 with a boundary γ_2 which passes through p and is parallel to γ_1 on ∂M_1. Let A be the horizontal annulus in M_1 with boundary $\gamma_1 \cup \gamma_2$ obtained by sweeping α across γ_i. Then $S = D_1 \cup A \cup D_2$ is a horizontal sphere in M. The complement of S in M is an I-bundle so M is an S^2-bundle over S^1 or an S^2-semibundle over I (see Corollary 2.6 of [KN23b]). As M is non-orientable so it must be $S^2 \tilde{\times} S^1$.

□

There are no essential surfaces in $S^2 \times S^1$, $S^2 \tilde{\times} S^1$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$ so we

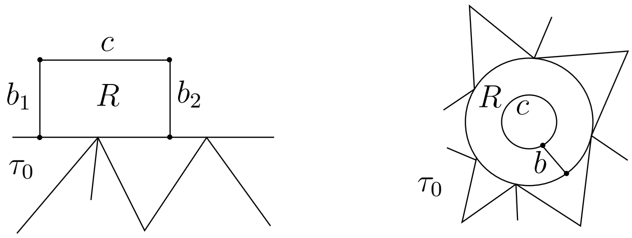


Figure 5.1: The CW-complex of the base space B obtained by attaching to τ_0 the rectangles and annuli corresponding to the projection of the model neighbourhoods of the singular surfaces.

shall henceforth assume that M is a Seifert fibered space which is either a solid torus, a solid Klein bottle or an irreducible manifold with incompressible boundary.

Partition of M into solid tori and solid Klein bottle. We first give a CW-complex structure on B. Let τ_0 be a simplicial triangulation of B_0. Assume that τ_0 is fine enough so that projections of the singular fibers lie in the interiors of distinct disjoint triangles which do not meet the boundary of B_0. Each $N \times I$ component of M_1 projects down to a rectangle R with one edge c given by the projection of the singular annulus of $N \times I$, two edges b_1, b_2 given by the projection of $N \times \partial I$ and an edge a given by the projection of $\partial N \times I$ to ∂B_0. Attach one such rectangle to τ_0 for each $N \times I$ component of M_1 by introducing the 2-cell R, the edges b_1, b_2 and c and the four corner vertices (see Figure 5.1). Similarly, each $N \times S^1$ or $N \tilde{\times} S^1$ component of M_1 projects down to an annulus A. Give a cell-structure to A by introducing an edges b corresponding to the projection of $N \times t_0$ for some fixed $t_0 \in S^1$, a boundary edge-loop c corresponding to the projection of the singular torus (in $N \times S^1$) or singular Klein-bottle (in $N \tilde{\times} S^1$) and a boundary edge-loop a that lies in ∂B_0. Let R denote the 2-cell $A \setminus \{a, b, c\}$. For each $N \times S^1$ and $N \tilde{\times} S^1$ component of M_1, insert such a 2-cell R, the edge b and c and the two end-vertices of b (see Figure 5.1). Call all the edges corresponding to b, b_1 and b_2 as Mobius edges because their pre-images in M are Mobius strips and call all the edges corresponding to c as singular surface edges as their pre-images in M are singular surfaces. Let τ be the cell structure of B obtained by attaching these cells to τ_0.

We will henceforth ignore the singular-surface edges of τ. Let $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$ denote the collection of preimages of the vertices, the edges which are not singular surface edges and the faces of τ respectively. Each $V \in \mathcal{V}$ is a fiber of M. Each $E \in \mathcal{E}$ is a vertical Mobius strip when E is a Mobius edge and is a vertical annulus otherwise. And each $F \in \mathcal{F}$ is model regular solid torus when it does not contain any singular fibers, is a model non-regular solid torus when it contains an isolated singular fiber and is a model solid Klein bottle $N \times I$ otherwise.

5.3 Essential surfaces

In this section we study the essential surfaces in Seifert fiber spaces and prove Theorem 5.1.4. The main original contribution here is a list of the incompressible surfaces in a solid Klein bottle given in Theorem 5.3.10. The rest of this section extends the proof of Rannard [Ran96] in order to apply it to the case when M may have singular surfaces and non-empty boundary.

Definition 5.3.1. *Let S be a properly embedded surface in M that intersects all the annuli and mobius strips $E \in \mathcal{E}$ transversely. Define the complexity $\xi(S)$ of S to be $\sum_{E\in\mathcal{E}} |\pi_0(S \cap E)|$. We say that S is minimal if it has minimal complexity in its isotopy class. We say that S is well-embedded if it is minimal and it intersects each $E \in \mathcal{E}$ horizontally or vertically.*

Applying the below Lemma 5.3.2 to each $E \in \mathcal{E}$ shows that every essential surface in M can be isotoped to a well-embedded surface. This Lemma 5.3.2 combines the arguments of Lemma 3.1 and Lemma 3.2 of Rannard [Ran96] and extends it to manifolds with boundary.

Lemma 5.3.2. *Let S be a connected essential minimal surface in M. For each $E_0 \in \mathcal{E}$ there exists an isotopy of S in an arbitrarily small neighbourhood of E_0 that fixes ∂E_0 and takes S to a minimal surface which intersects E_0 horizontally or vertically.*

Proof. We study the various possibilities for the intersection of the surface S with the annulus or Mobius strip E_0.
Case I: A component of $S \cap E_0$ is a closed curve that bounds a disk in E_0. By taking the innermost such disk D in E_0 we may assume that S does not intersect the interior of D. As S is incompressible there exists a disk $D_0 \subset S$ with $\partial D = \partial D_0$. As M is irreducible, $D \cup D_0$ bounds a ball in M. When E_0 does not lie in ∂M we may reduce $\xi(S)$ by isotoping D_0 across this ball and off D. This contradicts the minimality of S. If $E \subset \partial M$ then as S is connected $D_0 = S$ is a boundary parallel disk, which contradicts the fact that S is essential. So we may assume that no component of $S \cap E_0$ is a trivial closed curve in E_0.

Case II: A component of $S \cap E_0$ is a boundary-parallel arc in E_0. By taking the outermost disk D cut off by such arcs we may assume that the interior of D is disjoint from S. Let β be the arc $D \cap \partial E_0$ which lies in a component V of ∂E_0.

If V lies in the interior of M or if $E_0 \subset \partial M$ then we claim that isotoping S across D and off β reduces $\xi(S)$. Let $E \in \mathcal{E}$ be such that $E \neq E_0$ and $V \subset \partial E$. Let α_1 and α_2 be components of $S \cap E$ that meet the end-points of α (α_1 may be equal to α_2). After this isotopy, $\{\alpha_1, \alpha_2\}$ is replaced in the list of connected components of $S \cap E$ by the component $\alpha_1 \cup \beta \cup \alpha_2$. So the number of components of $S \cap E$ does not increase under such an isotopy when $V \subset \partial E$. If $E \in \mathcal{E}$ is such that V does not lie in ∂E then this local isotopy does not change $S \cap E$. And lastly, this isotopy reduces the number of components of $S \cap E_0$. Therefore the complexity $\xi(S)$ of S reduces after this isotopy, contradicting the minimality of S.

Assume that V lies in ∂M and E_0 does not lie in ∂M. The disk D can not be a boundary-compressing disk for S which is given to be essential. So $D \cap S$ is an arc α that is boundary-parallel in S. Hence there exists a disk $D_0 \subset S$ with $\partial D_0 = \alpha \cup \gamma$ where γ is an arc in ∂S. As ∂M is incompressible in M so there exists a disk $D_1 \subset \partial M$ such that $\partial D_1 = \partial(D \cup D_0) = \beta \cup \gamma$. As β is vertical, so there exists an $E_1 \in \mathcal{E}$ that lies in ∂M such that γ cuts off a disk $D_1' \subset D_1$ from E_1. By the above argument applied to $S \cap E_1$ isotoping S across D_1' reduces the complexity of S, contradicting the minimality of S. So we may assume that no component of $S \cap E_0$ is a boundary-parallel arc in E_0.

Case III: A component of $S \cap E_0$ is an essential closed curve in E_0. By the above cases, there are no trivial closed curves or boundary-parallel arcs in $S \cap E_0$. If some component of $S \cap E_0$ is an essential closed curve in the annulus or Mobius strip E_0 then $S \cap E_0$ is a union of essential closed curves. There is an isotopy of E_0 fixing ∂E_0 which takes these closed curves to fibers of E_0. We can extend this isotopy of E_0 to a local isotopy of an arbitrarily small neighbourhood of E_0, which fixes ∂E_0 and does not increase $\xi(S)$.

Case IV: No component of $S \cap E_0$ is a closed curve or a boundary-parallel arc in E_0. If there are no closed curves and no boundary-parallel arcs in $S \cap E_0$ then there exists a local isotopy of E_0 fixing ∂E_0 which takes $S \cap E_0$ to a union of horizontal arcs. Again, we can extend this isotopy of E_0 to a local isotopy in an arbitrarily small neighbourhood of E_0 which fixes ∂E_0 and does not increase $\xi(S)$.

□

We now focus on the intersection of an essential surface with the solid tori and solid Klein-bottles in $\mathcal{F}$. The following lemmas lead up to Theorem 5.3.9 and Theorem 5.3.10. The below Lemma 5.3.3 is an extension of Lemma 3.6 of Rannard [Ran96] to include boundary-parallel Mobius strips in solid

Klein bottle.

Lemma 5.3.3. *Let S be a well-embedded essential surface in M. Fix $F \in \mathcal{F}$ and let S_0 be a component of $S \cap F$. Assume that S_0 is a boundary-parallel annulus or Mobius strip in F. Then S_0 can be isotoped fixing ∂F to a vertical surface.*

Proof. As S_0 is boundary-parallel, there exists a surface $S_0' \subset \partial F$ isotopic to S_0 in F with $\partial S_0' = \partial S_0$. As S is well-embedded, ∂S_0 is either horizontal or vertical in ∂F. If ∂S_0 is vertical then S_0' is a vertical annulus or Mobius strip in ∂F. Pushing the interior of S_0' into the interior of F we get a properly-embedded vertical surface as required. Assume henceforth that $\partial S_0 = \partial S_0'$ is horizontal in ∂F.

If $S_0' \subset \partial F$ is a Mobius strip then F is a solid Klein bottle and as $\partial S_0'$ is an embedded closed curve transverse to the fibration of the Klein bottle so it must be isotopic to m. But as m is non-zero in $H_1(\partial F, \mathbb{Z}_2)$ so there can not exist a Mobius strip in ∂F with boundary isotopic to m.

If $S_0' \subset \partial F$ is an annulus then there exists a solid torus Q in F with $\partial Q = S_0 \cup S_0'$. As $\partial S_0'$ is a horizontal curve so it intersects each $E \in \mathcal{E}$ that lies in ∂F in horizontal arcs. As there is more than one cell in the cell-structure τ of the base space B, so ∂F is not a subset of ∂M. Take some $E_0 \in \mathcal{E}$ that lies in ∂F and does not lie in ∂M. As $\partial S_0' \cap E_0$ is a union of horizontal arcs so there exists a vertical arc $\beta \subset S_0' \cap int(E_0)$ with $\beta \cap \partial S_0 = \partial \beta$. Let α be the arc in S_0 isotopic relative boundary to β in Q. Let D be the meridian disk in the solid torus Q with $\partial D = \alpha \cup \beta$. Let δ_1 and δ_2 be the horizontal arcs of $S_0 \cap E_0$ which contain $\partial \beta$. Let $\beta \times I$ denote a thickening of β in E_0 with $\partial \beta \times I \subset \delta_1 \cup \delta_2$. Isotoping S across D and off β replaces the two components δ_1 and δ_2 of $S \cap E_0$ with the two components given by $(\delta_1 \cup \delta_2 \cup \beta \times \partial I) \setminus (\partial \beta \times int(I))$. These arcs cut off disks from E_0, however the complexity of S does not change under such a local isotopy. As in Lemma 5.3.2 Case II, by isotoping off such a disk cut off from E_0 the complexity $\xi(S)$ of S can be reduced, contradicting the minimality of S.

□

Any pair of disjoint essential curves on a torus are isotopic curves. This is however not true on a Klein bottle. The below Lemma 5.3.4 says that if the boundary components of a boundary-compressible incompressible surface are isotopic then the surface must be a boundary-parallel annulus.

Lemma 5.3.4. *Let T be a torus or Klein bottle boundary component of M. Let S be a connected incompressible surface in M and let γ_1 and γ_2 be distinct components of $S \cap T$. Let D be a boundary compressing disk of S with $D \cap S$*

an arc with endpoints on both γ_1 and γ_2. If γ_1 is isotopic to γ_2 in T then S is a boundary-parallel annulus.

Proof. If a component of ∂S is trivial in T then let D_0 be the innermost disk bounded by ∂S in T. As S is incompressible, there exists a disk $D_1 \subset S$ with $\partial D_1 = \partial D_0$. As S is connected $S = D_1$. And as M is irreducible so the sphere $D_0 \cup D_1$ bounds a ball in M. Hence S is parallel to the disk $D_0 \subset \partial M$. This contradicts the fact that ∂S has at least two components. So we may assume that γ_1 and γ_2 are essential curves in T.

Let $\partial D = \alpha \cup \beta$ where α and β are the arcs $D \cap S$ and $D \cap T$ respectively. If T is a torus then the closure of any component of the complement of any two disjoint essential closed curves in T is an annulus. Assume that T is a Klein bottle and refer to Figure 5.2 for the labels of the closed embedded curves on T. Any two curves on a Klein bottle which are isotopic to l_1 (or to l_2) must intersect. If the curves γ_i are isotopic to m then the closure of any component of their complement in T is an annulus. If the γ_i are isotopic to d then the closures of their complementary components gives two Mobius strips and an annulus. In either case as the endpoints of β are in distinct components of $S \cap T$, so the closure of the component of $T \setminus \partial S$ which contains $int(\beta)$ must be an annulus A with $\partial A = \gamma_1 \cup \gamma_2$. Note that if γ_1 were not given to be isotopic to γ_2, for example $\gamma_1 = l_1$ and $\gamma_2 = d$, then the $int(\beta)$ may lie in an open annulus but its closure will not remain an annulus.

Let $N(D) = D \times [-1, 1]$ be a regular neighbourhood of D in $M \setminus S$. The boundary of $N(D)$ decomposes as $N(\alpha) \cup N(\beta) \cup (D \times \{-1, 1\})$. Let D' be the disk obtained by taking the closure of $A \setminus N(\beta)$. Then $D_0 = D' \cup (D \times \{-1, 1\})$ is an embedded disk with boundary in S. As S is incompressible so there exists a disk $D_1 \subset S$ with $\partial D_1 = \partial D_0$. As M is irreducible so $D_0 \cup D_1$ bounds a ball B in M with $B \cap N(D) = D \times \{-1, 1\}$. And $S = D_1 \cup N(\alpha)$ is the union of two disks along a pair of arcs $\alpha \times \{-1, 1\}$ on their boundary, so it is either an annulus or a Mobius strip. But as S has at least two boundary components so it can not be a Mobius strip. Furthermore S cuts off the solid torus $B \cup N(D)$ from M so it is boundary-parallel. □

Lemma 5.3.5. *Let F be a fibered solid torus or fibered solid Klein bottle. Let S be a connected incompressible surface in F with horizontal or vertical boundary. There exists a boundary relative isotopy which takes S to a horizontal disk or to a surface that intersects a horizontal disk transversely in a non-empty collection of arcs which are not boundary-parallel in S.*

Proof. Let D be a meridian disk of F. We claim that there exists a boundary-relative isotopy, which takes S to a surface that does not intersect the interior of D in any circles. To see this, let D_0 be an innermost disk in D bounded by the circles in $S \cap D$. As S is incompressible, ∂D_0 bounds a disk D_1 in S. By the irreducibility of F, $D_0 \cup D_1$ bounds a ball in F. Therefore D_1 can be isotoped off D_0 through this ball to reduce the number of circles in $S \cap D$. The claim then follows from induction on the number of circle components of $S \cap D$ in the interior of D.

If S does not intersect the interior of D then it is an incompressible surface in the ball $B = F \setminus N(D)$ and so it must be a boundary-parallel disk in B. If ∂S is a trivial closed curve in the fibered annulus $\partial D \times I \subset \partial B$ then it can not be horizontal or vertical, which contradicts our assumption. So $\gamma = \partial S$ separates the two copies of D in ∂B. Hence there exists a fiber-preserving isotopy of F which takes ∂D to the horizontal curve γ. Such an isotopy takes D to a horizontal disk D' with $\partial D' = \gamma$. Applying the above arguments to $S \cap D'$ we can conclude that S does not intersect the interior of D' in any circles and so $S \cap D' = \partial S$. Therefore by incompressibility of S and irreducibility of M, S is isotopic relative boundary to the horizontal disk D' in M.

Assume that S intersects D in a non-empty collection of arcs. If any of these arcs is boundary-parallel in S then choose an outermost such arc α that cuts off a disk D_0 from D. Let $\partial D_0 = \alpha \cup \beta$ with β as the arc $D_0 \cap \partial F$. As α is boundary-parallel in S so there exists a disk $D_1 \subset S$ such that $\partial D_1 = \alpha \cup \gamma$ with γ the arc $D_1 \cap \partial F$. The embedded disk $D' = D_0 \cup D_1$ is either a meridian disk or a boundary parallel disk of F. If D' is a meridian disk then after a slight perturbation we may assume that it is disjoint from S. And so by the arguments above, S lies in the ball $F \setminus N(D')$ and hence is isotopic relative boundary to a horizontal disk. If D' is a boundary parallel disk then it bounds a ball B' in F. There exists an isotopy of F that is identity oustide a neighbourhood of B' which sweeps D_0 across B' and off D_1. Restricting this isotopy to D takes D to a meridian disk with fewer number of components of intersection with S.

$\square$

We state below the kind of non-orientable incompressible surfaces that exist in a solid torus, following the characterisation of such surfaces given by Rubinstein [Rub78].

Lemma 5.3.6 (Corollary 2.2 [Fro86])**.** *A one-sided incompressible surface in a solid torus has as boundary a single $(2k, q)$-curve with $k \neq 0$. Conversely every $(2k, q)$-curve with $k \neq q$ on the boundary of a solid torus is the boundary of a one-sided incompressible surface in the solid torus.*

For the sake of exposition we expand on a proof of the below result that has been proved in Rannard [Ran96].

Lemma 5.3.7 (Lemma 3.7, [Ran96]). *Let S be a connected incompressible non-orientable surface in a regular solid torus $F \in \mathcal{F}$. Assume that the boundary of S is horizontal or vertical in ∂F. Then S can be reduced to a meridian disk by a sequence of boundary compressions with respect to any $E \in \mathcal{E}$ with $E \subset \partial F$.*

Proof. By Lemma 5.3.6, S is a connected incompressible surface with connected boundary. If S is not a disk then we will show that there exists a boundary compression of S along E that increases its Euler characteristic by one while keeping S a connected incompressible surface with connected boundary.

By Lemma 5.3.5, there exists a boundary-relative isotopy that takes S to a surface which intersects a horizontal disk D in a non-empty collection of arcs which are not boundary-parallel in S. Let α be an outermost arc which cuts off a disk D_0 from D. Let $\partial D_0 = \alpha \cup \beta$ where β is the arc $\partial F \cap D_0$. Assume that β lies in E. Let $N(\partial S)$ be an annulus neighbourhood of ∂S in S and let $N(\alpha)$ be a rectangular neighbourhood of α in S. Translating a normal vector to S pointing into D_0 along α from $\alpha(0)$ to $\alpha(1)$ followed by a translation along ∂S from $\alpha(1)$ to $\alpha(0)$ reverses the original vector. So the 1-handle $N(\alpha)$ is attached to the annulus $N(\partial S)$ with a twist, giving a once-punctured Mobius strip Q. The complement of α in Q is connected and hence its complement in S is also connected. So compressing S along the disk D_0 gives the required connected incompressible surface with connected boundary and with Euler characteristic one more than that of S.

We will now obtain such a boundary-compressing disk when β is not in E. We first claim that β can be isotoped to lie arbitrarily close to a vertical arc via an isotopy through arcs with endpoints on $\gamma = \partial S$. Let A be an annulus such that ∂F is obtained by identifying the interior of A with $\partial F \setminus \gamma$ and ∂A with γ. A has an I-fibering induced by the fibering of ∂F. Let γ_0 and γ_1 denote the boundary components of A, containing $\beta(0)$ and $\beta(1)$ respectively. As ∂D and hence β is horizontal so $\beta(0)$ and $\beta(1)$ do not lie on the same I-fiber of A. Let $\overline{\beta}$ be the projection of β on γ_1. As ∂D and hence β intersects each fiber of ∂F at most once so the vertical I-fiber of A at $\beta(0)$ followed by $\overline{\beta}$ followed by the reverse of β is an embedded closed curve that bounds a disk in A. This gives an isotopy in A taking β to the I-fiber at $\beta(0)$ via arcs with one endpoint fixed at $\beta(0)$ and the other on $\overline{\beta}$. No point of $\overline{\beta}$ other than possibly $\overline{\beta}(0)$ is identified with $\beta(0)$ in γ as β intersects each fiber of ∂F at most once. If $\overline{\beta}(0)$ is identified with $\beta(0)$ in γ then γ intersects each fiber of ∂F exactly once and so the two boundary components of A are identified by full twists, otherwise they are identified via some rational twist. In either case we have an isotopy in ∂F that takes β close to a vertical arc via arcs with endpoints on ∂S. Following this up with a rotation in A takes β to an arc δ in $E \cap A$ via some isotopy $H : I \times I \to \partial F$.

Let $P = \partial F \times I$ be a neighbourhood of ∂F in F, such that $S \cap P = \partial S \times I$, $D \cap P = \partial D \times I$, $D_0 \cap P = \beta \times I$. Let F' be the closure of $F \setminus P$ i.e. the inner solid torus. Let $D_0' = F' \cap D_0$, $\beta' = \partial F' \cap D_0'$, $\alpha' = F' \cap \alpha$ and $S' = F' \cap S$. Define $G : I \times I \to \partial F \times I = P$ by $G(s,t) = (H(s,t),t)$. G is then an embedding of a rectangle with boundary consisting of a pair of opposite edges $\eta_1 \cup \eta_2$ on $\partial S \times I = S \cap P$ and the remaining pair of edges as $\beta' \cup \delta$. Let $\alpha'' = \alpha' \cup \eta_1 \cup \eta_2$ and let $D_1 = D_0' \cup G(I \times I)$. Then the disk D_1 has boundary $\alpha'' \cup \delta$ with α'' the arc $\partial D_1 \cap S$ and δ the arc $\partial D_1 \cap \partial F$. As α'' is isotopic on S to α which is not boundary-parallel in S so D_1 is a boundary-compression of S with respect to E.

□

Lemma 5.3.8. *Let P be a properly embedded 2-sided surface in M that is a union of some $E \in \mathcal{E}$. Let S be an incompressible minimal surface in M. Let M' be the closure of a component of $M \setminus P$. Then $S \cap M'$ is an incompressible surface in M'.*

Proof. Suppose that $S' = S \cap M'$ is compressible. Let D be a compressing disk for S' in M'. As S is incompressible in M, so there exists a disk $E \subset S$ such that $\partial D = \partial E$. As ∂D is essential in S' so E does not lie in S' and must therefore intersect P. As M is irreducible, the sphere $D \cup E$ bounds a ball in M. Isotoping E across this ball to D reduces the number of components of $S \cap P$. As P is a union of some $E \in \mathcal{E}$ so this isotopy reduces $\xi(S)$, contradicting the fact that S is minimal. Therefore S' is incompressible in M'.

□

We list below the possible incompressible surfaces in a solid torus $F \in \mathcal{F}$. Note that the solid torus may have non-regular fibering. The following Theorem 5.3.9 follows from Lemma 3.5 and Lemma 3.6 of Rannard [Ran96]. We give a proof here for the sake of exposition and to ensure that the isotopy is boundary-relative.

Theorem 5.3.9. *Let S be a connected well-embedded essential surface in M let $F \in \mathcal{F}$ be a solid torus. Then there is an isotopy of S which pointwise fixes all $E \in \mathcal{E}$, and takes $S \cap F$ to a union of the following components:*

1. *A vertical boundary-parallel annulus*

2. *A horizontal disk*

3. *A once-punctured non-orientable surface whose boundary is neither a meridian nor a longitude of F*

Proof. Let P be the union of all $E \in \mathcal{E}$ that lie in ∂F and do not lie on ∂M. By Lemma 5.3.8, $S \cap F$ is an incompressible surface. And as S is well-embedded so the boundary of $S \cap F$ is either horizontal or vertical. Let S_0 be a component of $S \cap F$. By Lemma 5.3.5, there exists an isotopy which fixes ∂F and takes S_0 to either a horizontal disk or to a surface that intersects a horizontal disk D in arcs that are not boundary-parallel in S_0. Assume that S_0 is not isotopic to a horizontal disk.

Let D_0 be the outermost disk cut off by arcs of $S_0 \cap D$ in D. Let $\partial D_0 = \alpha \cup \beta$ where $\alpha = S_0 \cap D$ and $\beta = \partial F \cap D$. Let γ_1 and γ_2 be the components of ∂S_0 that contain $\partial\alpha$. If γ_1 and γ_2 are distinct curves then by Lemma 5.3.4 S_0 can be isotoped fixing boundary to a boundary-parallel annulus. By Lemma 5.3.3, S_0 can be isotoped fixing boundary to a vertical annulus and in particular, ∂S_0 is a vertical curve. If $\gamma_1 = \gamma_2$, then we claim that S_0 is a one-sided surface in a solid torus and is therefore non-orientable. To see this observe that the normal vector field along α pointing into D_0, followed by the normal vector field along γ_i from one end-point of α to the next gives a normal vector field along a closed curve on S_0 that reverses the direction of the initial vector. By Lemma 5.3.6, the boundary of S_0 is connected and is neither a meridian nor a longitude of F. S_0 is therefore isotopic relative boundary to either a horizontal disk, a vertical boundary-parallel annulus or to a once-punctured non-orientable surface whose boundary is neither a meridian nor a longitude. We shall now see that there exists a boundary-relative isotopy of F which takes all components of $S \cap F$ simultaneously to one of these three types.

As the Dehn surgery slope at an isolated singular fiber of M is finite, so the slope of a fiber of ∂F is non-zero even when F is a non-regular solid. And by Lemma 5.3.6, the slope of the boundary of a non-orientable surface is also non-zero. Therefore if some component of $S \cap F$ is isotopic relative boundary to a horizontal disk then every component of $S \cap F$ is isotopic relative boundary to a horizontal disk. So a single isotopy fixing ∂F exists taking $S \cap F$ to a union of horizontal disks.

Assume that $S \cap F$ is a union of boundary-parallel annuli and non-orientable surfaces. Any two one-sided surfaces in F with isotopic boundary slopes must intersect, so there is at most one such surface. As any boundary parallel annuli separates F into two solid tori, so only one of these pieces can contain the non-orientable surface component. So as in Lemma 5.3.3 there is an isotopy of F which isotopes all the boundary-parallel annuli with vertical boundary into a neighbourhood of ∂F where they are vertical. $\square$

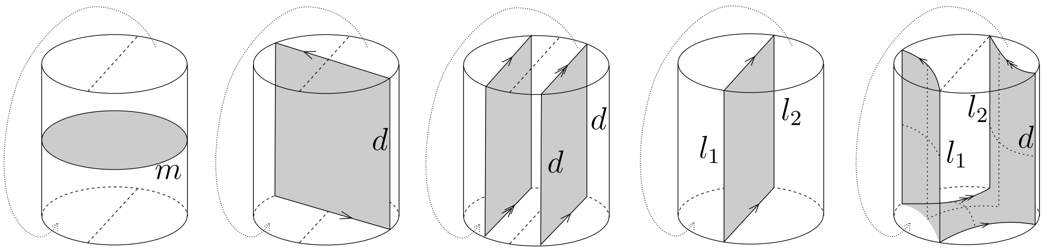


Figure 5.2: Incompressible surfaces in a solid Klein Bottle represented as $\mathbb{D}^2 \times I$ with the monodromy reflection along a diameter: (i) Horizontal disk with boundary m, (ii) Vertical boundary-parallel Mobius strip with boundary d, (iii) Vertical boundary-parallel annulus with boundary $d \cup d$, (iv) Vertical one-sided annulus with boundary $l_1 \cup l_2$ and (v) One-sided pair of pants with boundary $l_1 \cup l_2 \cup d$

We now list the possible incompressible surfaces in a solid Klein bottle.

Theorem 5.3.10. *Let $F = N \times I$ be a fibered solid Klein bottle and let S be an incompressible surface in F. Assume that the boundary of S is horizontal or vertical in ∂F. Then S can be isotoped relative boundary to a union of the following components (see Figure 5.2):*

1. *A horizontal disk*
2. *A vertical boundary-parallel Mobius strip with boundary d*
3. *A vertical boundary-parallel annulus with boundary two copies of d*
4. *A vertical one-sided annulus with boundary $l_1 \cup l_2$*
5. *A one-sided pair of pants with boundary $l_1 \cup l_2 \cup d$*

Proof. Assume that S is connected. If ∂S is horizontal then there exists a horizontal disk D whose boundary is disjoint from ∂S. And so by Lemma 5.3.5, S is isotopic relative boundary to a horizontal disk. (See Figure 5.2 (i))

Assume that ∂S is vertical in ∂F. The Klein bottle ∂F is the union of two Mobius strips so the boundary of S consists of components that are either the cores l_1 or l_2 of these Mobius strips or copies of the common boundary d of the Mobius strips. By Lemma 5.3.5 S intersects a horizontal disk D in arcs that are not boundary-parallel in S.

Let $B = D \times I$ and let F be obtained from B by identifying $D_1 = D \times 1$ with $D_0 = D \times 0$ via a reflection along the diameter of D joining l_1 and l_2. We claim that $S' = S \cap B$ is an incompressible surface in B. To see this let E_0 be a compressing disk of S' in B. After an isotopy we may assume that E_0 does not intersect D_0, i.e., E_0 lies in the interior of B. As S is incompressible so there exists a disk E_1 in S with $\partial E_1 = \partial E_0$. But as S does not intersect D_0 in any circles, so interior of E_1 is disjoint from ∂B, i.e., $E_1 \subset S \cap B = S'$ which contradicts the fact that ∂E_0 is essential in S'. Therefore S' is incompressible in a ball and hence is a disjoint collection of disks. Note that S can be reconstructed from S' by identifying arcs of $S' \cap D_1$ with their reflections in $S' \cap D_0$. We now analyse the system of arcs in $S' \cap D_1$.

Case I: $S' \cap D_1$ contains an outermost arc with endpoints on distinct copies of d. Such an outermost arc cuts off a compressing disk E from D_1 which satisfies the conditions of Lemma 5.3.4, so S is a boundary-parallel annulus. As ∂S is vertical so there exists a boundary-relative isotopy that takes S to a vertical surface. See Figure 5.2 (iii).

Case II: $S' \cap D_1$ contains an outermost arc with both end points on the same curve d. Let d' and d'' be the two vertical arcs given by the intersection of S' with $\partial D \times I \subset \partial B$. Let $\alpha_1 = S' \cap D_1$ be an arc in D_1 connecting d' and d'' and let $\alpha_0 = S' \cap D_0$ be its reflection in D_0 which also connects d' and d''. Then $\gamma = \alpha_0 \cup \alpha_1 \cup d' \cup d''$ is a closed curve in $\partial S'$. So S' is a single disk with boundary γ. But as α_0 is identified with α_1 via a reflection in S and S is connected so S is a boundary-parallel Mobius strip with boundary d as in Figure 5.2 (ii). Again as ∂S is vertical so there exists a boundary-relative isotopy that takes S to a vertical Mobius strip.

Case III: $S' \cap D_1$ does not contain any outermost arc with both end points on copies of d. Any outermost arc of $S' \cap D_1$ must have an end point on either l_1 or l_2 so there are at most two such arcs. If there is only one outermost arc α_1, with end points on both l_1 and l_2 then its reflection on D_0 is an arc α_0 that also joins l_1 and l_2. Arguing as in Case II then S' is a disk with boundary $\alpha_0 \cup \alpha_1 \cup l_1 \cup l_2$. As α_0 is identified with α_1 via an orientation-preserving map in S and S is connected so S is an annulus with boundary components $l_1 \cup l_2$, as in Figure 5.2 (iv). The complement of $l_1 \cup l_2$ in ∂F is connected so such an annulus is one-sided.

Assume that there are two outermost arcs in $S' \cap D_1$, one of which has an end point on l_1 and the other has an end point on l_2. Let $\partial S = l_1 \cup l_2 \cup (\cup_{i=1}^{k} d_i)$ for some $k \geq 1$. Then the pattern of arcs U on D_1 is as in Figure 5.3 (i) with outermost arcs from a point in l_1 to d_1 and l_2 to d_k, and parallel arcs from points in d_i to d_{i+1}. The pattern of arcs L on D_0 is a reflection across the diameter $l_1 l_2$ of the pattern U (see Figure 5.3 (ii)). Each boundary

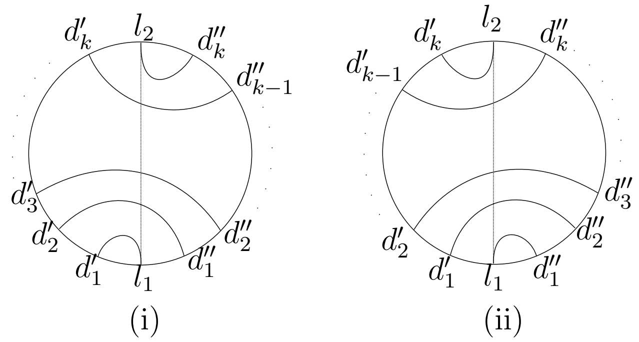


Figure 5.3: The pattern of arcs in (i) $U = S \cap D_1$ and its reflection (ii) $L = S \cap D_0$

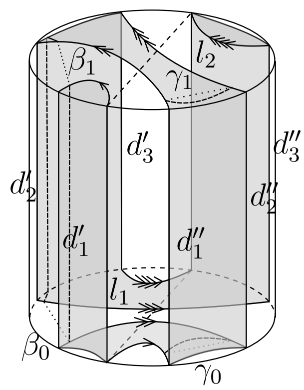


Figure 5.4: Shaded disk S' in B containing the vertical arcs of l_1, l_2, d_1, d_2 and d_3 (i.e., $k = 3$). The dashed curve represents the boundary of a compressing disk of S.

component d_i of S splits into two vertical arcs in S', with the vertical arc on the left of the diameter $l_1 l_2$ in D_1 denoted by d'_i and the vertical arc on the right by d''_i and ordering of the d'_i and d''_i given via a path on ∂D_1 from l_1 to l_2. The boundary of S' alternates between the horizontal arcs in U, the vertical arcs l_1, l_2, d'_i or d''_i and the horizontal arcs in L.

To see that $\partial S'$ is connected we trace one such curve starting with l_1. If k is odd then starting from l_1, $\partial S'$ traces the following vertical arcs in the given order l_1, d'_1, d''_2, d'_3, d''_4, ..., d'_k, l_2, d''_k, d'_{k-1}, d''_{k-2}, ..., d''_1 and back to l_1. If k is even then starting from l_1, it traces out the vertical arcs d'_1, d''_2, d'_3, d''_4, ..., d''_k, l_2, d'_k, d''_{k-1}, d'_{k-2}, ..., d''_1 and back to l_1. In either case it runs through all the vertical arcs of $\partial S'$ and it is therefore connected. Hence, S' is a single disk.

Let E be the disk in the sphere ∂B bounded by $\partial S'$ which contains the disk cut off from D_1 by the outermost arc $l_1 d'_1$ (see Figure 5.4). Note that

S' is a disk in B parallel to E. To reconstruct S from E we need to push the interior of E into the interior of B and identify U with L, i.e., identify the horizontal arc $l_1 d_1'$ in U with its reflection $l_1 d_1''$ in L, the arcs $d_i'' d_{i+1}'$ in U with $d_i' d_{i+1}''$ in L and the arc $d_k'' l_2$ with $d_k' l_2$ if k is odd ($d_k' l_2$ with $d_k'' l_2$ if k is even).

When $k = 1$ then S is a pair of pants with boundary $l_1 \cup l_2 \cup d$. To see this observe that E is an octagon with ∂E composed of the arcs l_1, $l_1 d'$ in U, d', $d' l_2$ in L, l_2, $l_2 d''$ in U, d'' and $d'' l_1$ in L (see Figure 5.2 (v)). Identifying the horizontal arcs $l_1 d'$ with $l_1 d''$ and $d' l_2$ with $l_2 d''$ gives a pair of pants. Any simple closed curve in a pair of pants is isotopic to one of its boundary components. But as l_1, l_2 and d (the boundary components of S) are all fibers and therefore non trivial in F so S does not have any compressing disk in F.

We shall show that if $k > 1$ then S does in fact have a compressing disk which contradicts the incompressibility of S. As before, let E be the disk in ∂B parallel to S' (see Figure 5.4). Let G be the disk in $\partial D \times I$ cut off by $d_1' \cup d_2'$ and $\partial D \times \{0, 1\}$. The boundary of G consists of the vertical arcs d_1', d_2' and the horizontal arcs β_i in $\partial D \times \{i\}$ connecting d_1' and d_2' (which do not pass through l_i). Observe that the arcs γ_i joining d_1'' and d_2'' in ∂D_i (which do not pass through l_1 and l_2) lie in E. As S' is parallel to E we may assume that γ_1 and γ_2 lie in S'. After identifying D_1 with D_0 in F, γ_i is identified with β_i and so $G \cap S = \partial G$. To see that ∂G is essential in S observe that it intersects the horizontal arc $l_1 d_1'$ in U exactly once. As it intersects a properly embedded arc of S exactly once so it can not bound a disk in S.

Suppose that S is not connected. We have shown that each component of S is isotopic to one of the five possibilities listed in the statement of this Lemma. We now argue that a single isotopy can be used to simultaneously make horizontal or vertical all the components of S which are not a pair of pants.

If any component of S has horizontal boundary, then it intersects any vertical curve. So if one component of S is isotopic to a horizontal disk then S is a union of disks that can be isotoped to be horizontal. Hence there exists a single isotopy of F that makes all components of S horizontal.

Assume that no component of S has horizontal boundary. If a component C of S is a boundary-parallel annulus with boundary two copies of d, then its complement in F is a solid Klein bottle containing l_1 and l_2 and a solid torus T. The only components of S that can lie in T are other annuli parallel to C. Assume that C is the innermost such annuli. There is an isotopy defined in a neighbourhood of T which takes C into ∂F. Push the interiors of these annuli back into the interior of F to get an isotopy that fixes all other components of S and takes all the annuli in T to vertical surfaces. Repeating this process for all boundary-parallel innermost annuli we get an isotopy fixing ∂F and taking all boundary-parallel annuli to vertical surfaces.

We now take the solid Klein bottle F' in the complement of these boundary-parallel annuli. And let C be a component of S that is a boundary-parallel Mobius strip in M'. Such a component intersects any surface whose boundary contains l_1 and l_2, so the only remaining components of S in F' are similar boundary parallel Mobius strips. A single isotopy of F' exists that takes all such components to vertical surfaces while fixing the boundary of F'.

As the one-sided annulus and one-sided pair of pants have intersecting boundaries so there can be at most one such component. If a component C of S in F' is a one-sided annulus with vertical boundary, then there is an isotopy of F' fixing the boundary which takes C to a vertical surface. Combining all these isotopies gives an isotopy which takes S to a vertical surface. □

Corollary 5.3.11. *Let M be the fibered manifold $N \times S^1$ or $N \tilde{\times} S^1$. Let S be a connected incompressible surface in M. Assume that for some $t_0 \in S^1$, $S \cap (N \times t_0)$ is horizontal or vertical and that ∂S is either horizontal or vertical in ∂M. Then S can be isotoped fixing ∂S to be either horizontal or vertical or a one-sided once-punctured torus with vertical boundary when $M = N \times S^1$ or a one-sided once-punctured Klein bottle with vertical boundary when $M = N \tilde{\times} S^1$.*

Proof. Let N be a fibered Mobius strip $I \times I/(x, 0) \sim (1 - x, 1)$ and let r denote the reflection of N along the arc $I \times \frac{1}{2}$. The manifold M can be obtained from $K = N \times I$ by identifying $N_1 = N \times 1$ with $N_0 = N \times 0$ via the identity if $M = N \times S^1$ or via the reflection r if $M = N \tilde{\times} S^1$. In either case, the fibration on ∂M is induced by the circles $\partial N \times t$. We may assume that S is either disjoint from or transversely intersects the mobius strip $N \times t_0$ which we identify with $N = N_0 = N_1$.

Let $S' = S \cap K$. Assume that S' has a compressing disk D in K. As S is incompressible so there exists a disk $D_0 \subset S$ with $\partial D_0 = \partial D \subset int(K)$. As ∂D is non-trivial in S' so D_0 intersects N in some closed curves. This contradicts the fact that $S \cap N$ is horizontal or vertical. Therefore S' is incompressible in K.

The fibration of the Klein bottle ∂K (given by the fibration of ∂M and N) is a union of curves parallel to d, the curve l_1 and the curve l_2. By assumption, $\partial S'$ is either horizontal or vertical in ∂K. Applying Theorem 5.3.10 to S', we know that each component of S' is either a horizontal disk (meridian disk of the solid Klein bottle K), vertical annulus, vertical Mobius strip or a pair of pants with boundary curves $l_1 \cup l_2 \cup d$.

A horizontal disk of K intersects any surface in K which has vertical boundary curves, so if a component of S' is horizontal then S' is a collection of horizontal disks attached in pairs along arcs on their boundaries. As it is a covering space of an annulus which is the base space of M so S' must be a horizontal annulus.

If S' is a collection of vertical annuli or Mobius strips then S is obtained from S' by attaching them along their boundaries. So S is a vertical annulus, Mobius strip, torus or Klein bottle.

At most one component of S' is a pair of pants. Assume that some component C of S' is a pair of pants. The boundary components l_1 and l_2 of C are identified in S when N_1 is stuck to N_0 via the identity or via the reflection map r. If the third boundary component d of C lies on ∂M, then S is obtained from a pair of pants by identifying two of the boundary curves l_1 and l_2 via identity if $M = N \times S^1$ and reflection if $M = N \tilde{\times} S^1$. S is therefore a one-sided punctured torus with vertical boundary when $M = N \times S^1$ and a one-sided punctured Klein bottle with vertical boundary when $M = N \tilde{\times} S^1$.

Assume that the third boundary component d of C lies on N. A boundary parallel Mobius strip in K separates l_1 and l_2 and therefore must intersect C. So by Theorem 5.3.10 the component of S' that meets the pair of pants along the curve d on N must be a vertical annulus with boundary two copies of d. If both these boundary components lie on N then by the same reasoning it is adjacent to another vertical annulus. Repeating this argument finitely many times, we get a vertical annulus with one boundary curve d attached to the boundary of the pair of pants and the other boundary curve a fiber of ∂M. S is therefore obtained from a pair of pants by identifying two of the boundary curves l_1 and l_2 via identity if $M = N \times S^1$ and reflection if $M = N \tilde{\times} S^1$ and by identifying the third boundary curve to a boundary component of a vertical annulus. And so again S is a one-sided punctured torus with vertical boundary when $M = N \times S^1$ and a one-sided punctured Klein bottle with vertical boundary when $M = N \tilde{\times} S^1$.

□

We now prove Theorem 5.1.4 by replicating the proof of Theorem 4.1 of Rannard [Ran96], with modifications to take into account singular surfaces and boundary components.

Proof. As M contains an essential surface so it is not $S^2 \times S^1$, $S^2 \tilde{\times} S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$. So M is a solid torus, solid Klein bottle or an irreducible manifold with incompressible boundary. Isotope S to have minimal complexity in its isotopy class. After a further isotopy of S near each $E \in \mathcal{E}$ using Lemma 5.3.2, we may assume that it is well-embedded. Let M_1 denote the disjoint union of model neighbourhoods of the singular surfaces and let $M_0 = \overline{M \setminus M_1}$. If M_0 is empty, then M is a either $N \times I$, $N \times S^1$ or $N \tilde{\times} S^1$ and so by Corollary 5.3.11 M, S can be isotoped to be pseudo-horizontal or psuedo-vertical. Assume that M_0 is non-empty.

When M_1 is non-empty, let $P = \partial M_0 \cap \partial M_1$ be a properly embedded surface that is a union of some $E \in \mathcal{E}$. So by Lemma 5.3.8, $S \cap M_0$ is an incompressible surface in M_0.

By Theorem 5.3.9 for each solid torus $F \in \mathcal{F}$, there exists an isotopy pointwise fixing ∂F and taking $S \cap F$ to either a union of vertical annuli, a union of horizontal disks or a once-punctured non-orientable surface. Combining these isotopies we get an isotopy of S that pointwise fixes $\mathcal{E}$ and takes S to a well-embedded surface that intersects each solid torus $F \in \mathcal{F}$ in vertical annuli, horizontal disks or a once-punctured non-orientable surface.

Case I: For some solid torus $F \in \mathcal{F}$, a component of $S \cap F$ is a boundary-parallel vertical annulus.

As S is well-embedded, it intersects ∂F in a union of fibers. Suppose there exists a solid torus F' that is adjacent to F along some $E \in \mathcal{E}$ that intersects S. If F' is a regular solid torus then by Theorem 5.3.9 and the fact that fibers are longitudes of regular solid tori, $S \cap F'$ must also be a vertical annulus. If F' is a non-regular solid torus then as slopes at singular fibers can not be infinite so fibers can not be the boundary of a meridian disk. Therefore $S \cap F'$ is either a vertical annulus or a once-punctured non-orientable surface. So $S \cap (F \cup F')$ is a pseudo-vertical surface. Repeating this argument for adjacent solid tori we can conclude by induction that $S \cap M_0$ is a pseudo-vertical surface with vertical boundary in ∂M_0.

Case II: For all solid tori $F \in \mathcal{F}$, $S \cap F$ consists of horizontal disks and once-punctured non-orientable surfaces.

Fix a solid torus $F_0 \in \mathcal{F}$. Suppose that for some regular solid torus $F \in \mathcal{F}$, $S \cap F$ is a punctured non-orientable surface. There is a path of regular solid tori from F to F_0. We will use Lemma 5.3.7 repeatedly to reduce the intersection of S with each solid tori in this path (except possibly F_0) to meridian disks.

Let $F' \in \mathcal{F}$ be a regular solid torus adjacent to F along some $E \in \mathcal{E}$. By Lemma 5.3.7, we may compress S along E finitely many times to reduce $S \cap F$ to a meridian disk. These compressions give local isotopies that change S only in the interior of $F \cup F'$. As these isotopies fix $S \cap \mathcal{V}$ so the complexity $\xi(S)$ does not change and so S is still of minimal complexity. By Lemma 5.3.2, after an isotopy near E in $F \cup F'$, S is a well-embedded surface. Repeat this process along a path of regular solid tori from from F to F_0. Eventually the surface S is isotoped to a well-embedded surface that intersects each regular solid torus in meridian disks. As S intersects ∂F_0 horizontally so by Theorem 5.3.9, $S \cap F_0$ must be a non-orientable surface or a union of meridian disk.

If M has an isolated singular fiber, then take F_0 to be a non-regular solid torus containing such a fiber so that $S \cap M_0$ becomes a pseudo-horizontal surface in M_0 with horizontal boundary in ∂M_0. If M has no isolated singular fibers then by assumption it must have singular surfaces, i.e, M_1 is non-empty. Take F_0 to be a regular solid torus that intersects ∂M_1 along some $E_0 \in \mathcal{E}$. Assume that $S \cap F_0$ is a once-punctured non-orientable surface. By Lemma 5.3.7, repeated boundary compressions of $S \cap F_0$ along E_0 reduces $S \cap F_0$ to a meridian disk and such an isotopy does not change $\xi(S)$. Again the surface $S \cap M_0$ has been isotoped to a pseudo-horizontal surface in M_0 with horizontal boundary in ∂M_0.

If M_1 is empty, then we have shown that S is a pseudo-horizontal or pseudo-vertical surface. When M_1 is non-empty, take $P = \partial M_0 \cap \partial M_1$ and by Lemma 5.3.8, $S \cap M_1$ is an incompressible surface. By Lemma 5.3.2, there is an isotopy in a neighbourhood of $E \in \mathcal{E}$ which makes S a well-embedded surface. By Theorem 5.3.10 and Corollary 5.3.11, when $S \cap P$ is vertical as in Case I, for each component W of M_1, $S \cap W$ is either vertical, a pair of pants (when $W = N \times I$), a once-punctured torus (when $W = N \times S^1$) or a once punctured Klein bottle (when $W = N \tilde{\times} S^1$). Therefore S is a pseudo-vertical surface in M. And when $S \cap P$ is horizontal as in Case II, then $S \cap M_1$ must be a horizontal surface. And therefore S is a pseudo-horizontal surface in M.

$\square$

6

Taut Foliations

This chapter is based on ongoing work with Rachel Roberts and her student Jeffrey Norton of Washington University in St. Louis. There are no original results here.

6.1 L-space Conjecture

Definition 6.1.1. *A transversely orientable codimension-*1 *foliation $\mathcal{F}$ of a* 3*-manifold M is said to be* ***taut*** *if there exists a transverse circle in M which intersects each leaf of $\mathcal{F}$.*

The existence of taut foliations has interesting topological implications. For instance, if M contains a taut foliation and has no sphere leaf, then M is irreducible [Nov65]. Or that the existence of taut foliations implies that $\pi_1(M)$ is infinite [[Nov65], [GO89],[Hae62]].

In [Gab83], Gabai shows that any compact irreducible manifold which is not a a rational homology 3-sphere ($\mathbb{Q}HS$) admits a taut foliation. A rational homology sphere is a 3-manifold which has the same homology groups as S^3 when computed with rational co-efficients. So, when does a $\mathbb{Q}HS$ admit a taut foliation? The L-space Conjecture 6.1.3 attempts to answer this very question.

Definition 6.1.2. *A closed* 3*-manifold M is called an L-**space** if $H_1(M, \mathbb{Q}) = 0$ and $\widehat{H(M)}$ (Heegard Floer homology) is the free abelian group of rank $\left|H_1(M, \mathbb{Z})\right|$.*

L-spaces are a generalisation of lens spaces. They have the simplest possible Heegard-Floer homology. The L-space conjecture predicts a curious relationship between Heegard-Floer homology, orderability of the fundamental group and the existence of taut foliations in a $\mathbb{Q}HS$. The precise statement of the conjecture is as follows:

Conjecture 6.1.3 (L-space Conjecture, [BGW13], [Juh15])**.** *Let M be an irreducible $\mathbb{Q}HS$. Then the following are equivalent*

1. *M is not an L-space.*
2. *$\pi_1(M)$ is left-orderable i.e. there is a total order on $\pi_1(M)$ that is invariant under left multiplication*
3. *M admits a taut foliation*

The conjecture is fully resolved for graph manifolds (i.e either a Solv manifold or a manifold with only Seifert pieces in their JSJ decomposition or a connected sum of these two categories) by the combined work of a lot of mathematicians ([BC17],[BGW13],[BNR97],[CLW13],[HRRW20], [EHN81],[LS09],[BC15]).

We are interested in (1) if and only if (3) part of the conjecture. The (3) implies (1) part of the conjecture was resolved by the combined work of Ozsváth, Szabo, Bowden, Kazez and Roberts.

Theorem 6.1.4 ([OS04], [Bow16], [KR17])**.** *If M is an L-space, then it does not admit taut foliations*

Our objective here is to provide evidence for the converse of the above statement by explicitly constructing taut foliations in $\mathbb{Q}HS$ that are not L-spaces. One way to produce L-space knots is via Dehn surgery.

Definition 6.1.5. *A non-trivial knot $\mathcal{K}$ is said to be an **L-space knot** if some non-trivial surgery along this knot produces an L-space.*

Example 6.1.6. *Torus knots are L-space knots. More generally all closed positive n-braids are L-space knots. See [KMOS07],[RR17],[OS05] for details.*

Definition 6.1.7 ([DR21])**.** *A knot $\mathcal{K}$ is said to be **persistently foliar** if for every rational boundary slope (except the meridional slope), there is a co-oriented taut foliation meeting the boundary of the knot exterior in a foliation by curves of that slope.*

Definition 6.1.8. *Let $\mathcal{K} \subset S^3$ be a non-trivial knot. Let $S^3_r(\mathcal{K})$ denote the manifold obtained by Dehn surgery on S^3 along $\mathcal{K}$ with surgery co-efficient r. Then $S^3_r(\mathcal{K})$ is said to be reducible if it contains a sphere that does not bound a ball and the slope r is called a **reducible surgery slope**.*

Conjecture 6.1.9 (L-space knot conjecture)**.** *A knot $\mathcal{K} \subset S^3$ is persistently foliar if and only if $\mathcal{K}$ is not an L-space knot and has no reducible surgeries, in other words, it has taut foliations realizing all possible slopes if and only if every non-trivial Dehn surgery gives a manifold that is not an L-space but is irreducible.*

Krcatovich [Krc15] showed that composite knots can never be L-space knots. Roberts and Delman in their paper [DR21] confirm Conjecture 6.1.9 for a sub-class of composite knots.

Theorem 6.1.10 ([DR21])**.** *A connected sum of knots in S^3 is persistently foliar if at least one of the summands is persistently foliar. Also any connected sum of fibered knots is persistently foliar.*

Theorem 6.1.11 ([DR20])**.** *Any knot $\mathcal{K} \subset S^3$ with a minimal Seifert surface F which is the plumbing of surfaces F_1 and F_2 where F_2 is an unknotted band with even number (greater than 4) of twists, is persistently foliar.*

Along similar lines, we hope to provide evidence for L-space conjecture by constructing taut foliations in an infinite family of non-L-spaces. One way of obtaining non-L-spaces is as follows:

Given an L-space knot $\mathcal{K}$, infinitely many surgeries along $\mathcal{K}$ gives rise to L-spaces. It is known that for any non-trivial positive knot $\mathcal{K} \subset S^3$, the set of L-space slopes is either empty or is $[2g-1, \infty) \cap \mathbb{Q}$ where g is the Seifert genus of $\mathcal{K}$ [KMOS07],[RR17],[OS05]. Therefore, Dehn surgery along any positive knot with slope $r \in (-\infty, 2g-1) \cap \mathbb{Q}$ yields a non-L-space. Conjecture 6.1.3 predicts existence of taut foliations in them. Krishna confirms this prediction by constructing taut foliations in $\mathbb{Q}HS$ obtained by Dehn surgery along knots realized as closures of 3-braids in [Kri20].

Theorem 6.1.12 (Theorem 1.2 [Kri20])**.** *Let $\mathcal{K} \subset S^3$ be a non-trivial positive knot realized as the closure of a 3-braid. Then the manifold obtained as a result of Dehn surgery along $\mathcal{K}$ with slope any $r \in (-\infty, 2g-1) \cap \mathbb{Q}$ admits taut foliation.*

We hope to construct taut foliations in manifolds obtained by suitable Dehn surgery along closed positive n-braids. The precise statement in mentioned is the conjecture below. This constitutes future work.

Conjecture 6.1.13. *Let $\mathcal{K} \subset S^3$ be a closed positive n-braid, $n \geq 4$. Then any manifold obtained by Dehn surgery along $\mathcal{K}$ with slope any $r \in (-\infty, 2g-1) \cap \mathbb{Q}$ admits a taut foliation, where g is the genus of $\mathcal{K}$.*

In this chapter, we show why the construction in [Kri20] cannot be generalised to all closed positive n-braids.

Modus Operandi: Motivated by the work of Rachel Roberts and Tao Li in [Rob95], [Rob01a], [Rob01b], [LR14],[Li02] and [Li03], we hope to build branched surfaces without sink disks in the exterior of non-trivial knots that are realized as closures of positive n-braids. By Li's work in [Li02]and [Li03], such a branched surface will carry essential laminations. These laminations can be extended to taut foliations in the knot exterior, and then to taut foliations in the Dehn filled manifold by capping off the foliation by disks whenever the slopes are rational.

The preliminaries required to make sense of the above paragraph are covered in the following section.

6.2 Preliminaries

6.2.1 Train Tracks and Branched Surfaces

Train tracks were popularized as a means to study simple closed curves on a surface by Thurston (see [Thu22]). Branched surfaces are a generalization of train tracks and are convenient tools in the study of incompressible surfaces and their generalizations. We define train tracks and branched surfaces below.

Definition 6.2.1. *A **train track** τ on a surface S is a subspace locally modeled on the object in the left side of Fig 6.1. The non-manifold points on τ are called **switches** and the closure of components of the complement of switches are called **branches** of τ.*

Figure 6.1 shows a local train track model and its I-fibered neighborhood.

Definition 6.2.2. *A curve λ is said to be **carried** by τ if it can be isotoped into a I-fibered regular neighborhood of τ transverse to the I-fibers and is said to be **fully carried** by τ if it intersects every I-fiber of $N(\tau)$.*

Definition 6.2.3. *A **measure** μ on a train track τ is an assignment of a non-negative real number (called weights) to each branch of τ satisfying the*

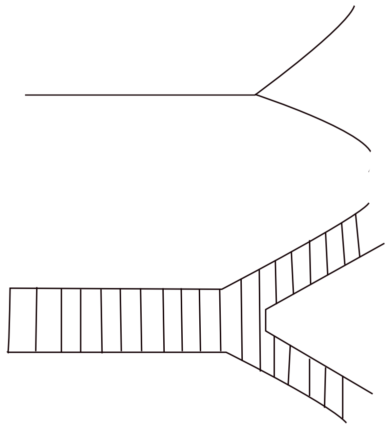


Figure 6.1: Train track model and its I-fibered neighborhood

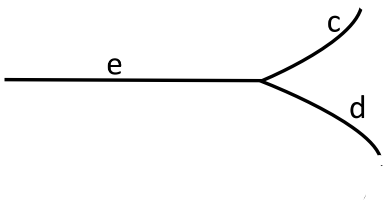


Figure 6.2: Model measured train track

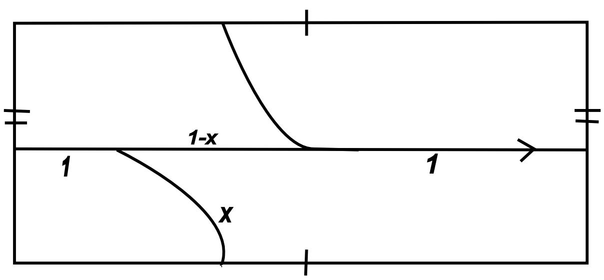


Figure 6.3: Measured train track on a torus

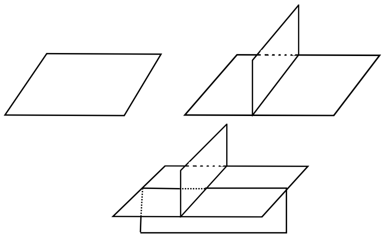


Figure 6.4: Standard spine models

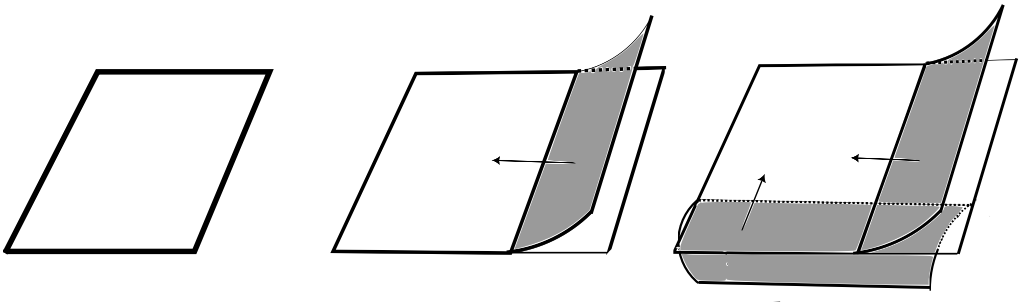


Figure 6.5: Local model of a branched surface

branching equation $e = c + d$ for each branch point of τ as shown in Figure 6.2.

Figure 6.3 shows a measured train track on a torus. Let $\tau(\omega)$ be this measured train track. Then the slopes realised by this train track are given by

$$\frac{i(\tau(\omega),l)}{i(\tau(\omega),m)} = \frac{x}{1}$$

Since all weights should be non-negative, both x and $1-x$ are non-negative. So, varying x in $[0,1]$, we get $\tau(\omega)$ carries all rational slopes in $[0,1]$ and fully carries $[0,1)$.

Definition 6.2.4. *A **standard spine** is a space locally modeled on one of the spaces in Figure 6.4.*

*A **branched surface** $\mathcal{B}$ is a union of a finite number of compact surfaces locally modeled on Figure 6.5 which is obtained by smoothening a standard spine. $\mathcal{B}$ intersects ∂M in a train track.*

Let $N(\mathcal{B})$ denote a regular neighborhood of $\mathcal{B}$ as in Figure 6.6.The boundary $\partial N(\mathcal{B})$ has two parts, $\partial_h N(\mathcal{B})$ which is the horizontal boundary transverse to all the $I-$fibers and the vertical boundary $\partial_v N(\mathcal{B})$ which is made up of subarcs of the I-fibers.

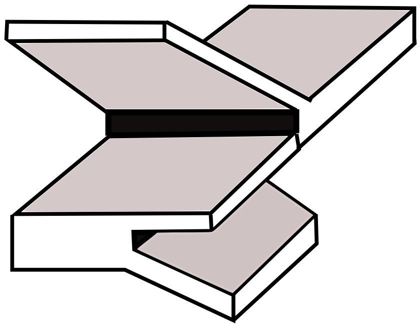


Figure 6.6: The region shaded in black is $\partial_v(N(\mathcal{B}))$ and the region shaded lightly is $\partial_h(N(\mathcal{B}))$

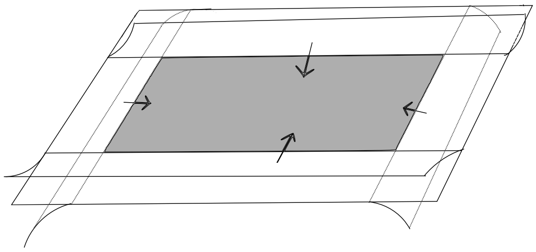


Figure 6.7: The shaded region is a sink disk

Definition 6.2.5. *The* ***branch locus*** *of $\mathcal{B}$ denoted by $\mathcal{L}$ is the set of all non-manifold points of $\mathcal{B}$ i.e points that do not have a neighborhood homeomorphic to $\mathbb{R}^2$. A* ***branch sector*** *of $\mathcal{B}$ is any connected component of $\overline{\mathcal{B} - \mathcal{L}}$ under the path metric.*

In Figure 6.5 the points on the branched locus in the middle branched surface are called double points. The intersection point of the two loci in the rightmost branched surface is called a triple point.

Let $\mathcal{Z}$ be the set of double points in $\mathcal{L}$. Associate a normal vector to each component of $\mathcal{L} - \mathcal{Z}$ (in $\mathcal{B}$) pointing towards the cusp as in Figure 6.5. This is called the branch direction of this arc.

Definition 6.2.6. *A disk D which is a branch sector of $\mathcal{B}$ is called a* ***sink disk*** *if every arc in its boundary points into the disk and $D \cap \partial M = \emptyset$. We call D a* ***half sink disk*** *if all arcs in $\partial D - \partial M$ points into D. See Figure 6.7.*

Definition 6.2.7 ([Oer88])**.** *An invariant measure on a branched surface $\mathcal{B}$ is a function that assigns non-negative real numbers to all the sectors of $\mathcal{B}$*

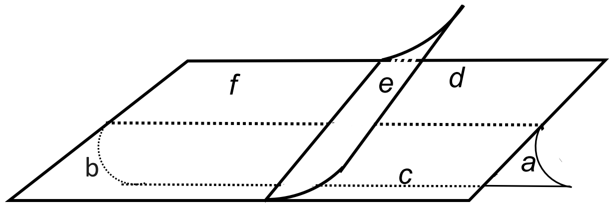


Figure 6.8: Measured branched surface

satisfying the branching equations $f = a + b = d + e$, $b = c + e$ *and* $d = c + a$ *as shown in Figure 6.8. A branched surface with an invariant measure is called a* ***measured*** *branched surface.*

Definition 6.2.8. *Let* M *be a* 3*-manifold. A codimension-*1 ***lamination*** Λ *is a closed, foliated subset of* M *where* Λ *is covered by open sets* $\mathcal{U}$ *of* M *of the form* $\mathbb{R}^2 \times \mathbb{R}$ *such that* $\Lambda \cap \mathcal{U} = \mathbb{R}^2 \times \mathcal{C}$, *where* $\mathcal{C}$ *is closed in* $\mathbb{R}$.

Definition 6.2.9. *A lamination (or foliation)* Λ *is said to be* ***carried by*** $\mathcal{B}$ *if it can be isotoped into* $N(\mathcal{B})$ *transverse to the* I*-fibers of* $N(\mathcal{B})$*. It is* ***fully carried by*** $\mathcal{B}$ *if it intersects every* I*-fiber of* $N(\mathcal{B})$.

Essential laminations were introduced by Gabai and Oertel in their seminal paper [GO89] as a generalisation of incompressible surfaces. In [Hat92], Hatcher constructs essential laminations in a number of irreducible 3-manifolds which have no incompressible surfaces.

Definition 6.2.10 (See [GO89])**.** *A lamination* Λ *of* M *is said to be an essential lamination if*

1. *The inclusion of the leaves of the* Λ *into* M *is* π_1*-injective.*

2. M_Λ *is irreducible, where* M_Λ *is the metric completion of* $M \backslash \Lambda$ *with the path metric inherited from a Riemannian metric on* M *.*

3. Λ *has no* S^2 *leaves.*

4. Λ *is end-incompressible*

Laminar branched surfaces were introduced in [Li02] by changing one condition from that of essential branched surfaces defined in [GO89]. The existence of a laminar branched surface ensures the existence of essential laminations.

Definition 6.2.11. *Let $\mathcal{B}$ be branched surface in a closed 3-manifold and $N(\mathcal{B})$ be its I-fibered neighborhood. A disk $D \subset M$ is called a* ***monogon*** *if $D \subset M \backslash int(N(\mathcal{B}))$ with $\partial D = D \cap N(\mathcal{B}) = \alpha \cup \beta$, where $\alpha \subset \partial_v N(\mathcal{B})$ is in an interval fiber of $\partial_v N(\mathcal{B})$ and $\beta \subset \partial_h N(\mathcal{B})$.*

Definition 6.2.12. *Let D_1, D_2 be the two disk components of $\partial_h(N(\mathcal{B}))$ of a $D^2 \times I$ region of $M \backslash int(N(\mathcal{B}))$. Let $p : N(\mathcal{B}) \longrightarrow \mathcal{B}$ be the projection identifying each I-fiber to a point. If the intersection of any I-fiber of $N(\mathcal{B})$ with $int(D_1) \cup int(D_2)$ is either empty or a single point, we call $p(D_1 \cup D_2)$ a* ***trivial bubble*** *in $\mathcal{B}$.*

Definition 6.2.13. *A branched surface $\mathcal{B}$ in a closed 3-manifold M is called* ***laminar*** *if it satisfies the following conditions:*

1. *$\partial_h N(\mathcal{B})$ is incompressible in $M \backslash int(N(\mathcal{B}))$, no component of $\partial_h N(\mathcal{B})$ is a sphere and $M \backslash \mathcal{B}$ is irreducible.*
2. *There is no monogon in $M \backslash int(N(\mathcal{B}))$*
3. *There is no Reeb component i.e., $\mathcal{B}$ does not carry a torus that bounds a solid torus in M.*
4. *$\mathcal{B}$ has no trivial bubbles.*
5. *$\mathcal{B}$ has no sink disk or half sink disks.*

The following theorem of Tao Li gives a nice and easy way to detect essential laminations in a compact 3-manifold.

Theorem 6.2.14 (Theorem 2, [Li02], [Li03])**.** *Let M be a compact oriented 3-manifold. Then*

1. *Any laminar branched surface fully carries an essential lamination.*
2. *Any essential lamination that is not a lamination by planes is fully carried by a laminar branched surface.*

6.2.2 Sutured Manifolds

Gabai introduced the concept of sutured manifolds in [Gab83]. For a nice exposition see [Sch90]. In [Gab87b], he constructs a branched surface in knot complements using sutured manifold decomposition. This branched surface intersects the boundary of the manifold in curves of slope zero. If such a branched surface carries taut foliations, then it will meet the boundary of the manifold in curves of slope zero. Tao Li and Rachel Roberts [LR14] modify Gabai's sutured manifold hierarchy to obtain a branched surface carrying taut foliations realizing an interval of rational slopes containing zero.

Definition 6.2.15. *Let $\mathcal{K}$ be a knot in S^3. A* ***longitude*** *of $\mathcal{K}$ is the unique (up to isotopy) essential simple closed curve l in $\partial N(\mathcal{K})$ such that $[l] = 0$ in $H_1(X_{\mathcal{K}})$, where $X_{\mathcal{K}}$ denotes the exterior of K.*

Definition 6.2.16. *A manifold M is said to be obtained by* ***zero frame surgery on*** *$\mathcal{K}$ if it is obtained by performing Dehn surgery along the longitude.*

Definition 6.2.17. *A* ***sutured manifold*** *(M, Γ) is a compact oriented 3-manifold M together with a set $\Gamma \subset \partial M$ of pairwise disjoint annuli $A(\Gamma)$ and tori $T(\Gamma)$ such that the interior of each component of $A(\Gamma)$ contains a* ***suture****, a homologically non-trivial oriented simple closed curve. Let $\mathcal{R} = \partial M \backslash int(\Gamma)$ and define $\mathcal{R}_+$ (or $\mathcal{R}_-$) to be the component of $\mathcal{R}$ that is oriented so that its normal vectors are pointing outwards (or inwards) and so that it induces the same orientation on the sutures. Let the set of sutures be denoted by $s(\Gamma)$.*

Definition 6.2.18. *Let (M, Γ) be a sutured manifold and let F be a properly embedded surface such that for every component c of $F \cap \Gamma$, one of the following is true:*

1. *c is a properly embedded non-separating arc in Γ.*
2. *c is a simple closed curve in an annulus $A \in \Gamma$ such that $[c] = [A \cap s(\Gamma)]$ homologically.*
3. *c is a non-trivial curve in a torus $T \in \Gamma$ and $[c] = [\lambda]$ if λ is any other component of $T \cap F$.*

Then F defines a ***sutured manifold decomposition*** *$(M, \Gamma) \xrightarrow{F} (M', \Gamma')$ where*

1. *$M' = M \backslash int(N(F))$*
2. *$\Gamma' = (\Gamma \cap M') \cup N(F'_+ \cap \mathcal{R}_-) \cup N(F'_- \cap \mathcal{R}_+)$*
3. *$\mathcal{R}'_+ = ((\mathcal{R}_+ \cap M') \cup F'_+) \backslash int(\Gamma')$*
4. *$\mathcal{R}'_- = ((\mathcal{R}_- \cap M') \cup F'_-) \backslash int(\Gamma')$*

where $F'_+(F'_-)$ is the component of $\partial N(F) \cap M'$ where the normal vectors point out of (into) M'.

A schematic representation of sutured manifold decomposition one dimension lower as given in [Sch90] is shown in Figure 6.9.

Gabai shows that there is a nice sutured manifold decomposition where you end up with product manifolds at the last stage. The following definition leads upto that result.

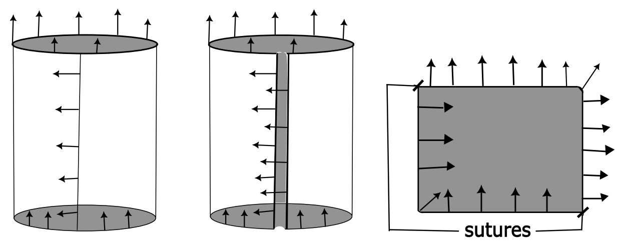


Figure 6.9: Decomposition of an annulus by an oriented arc

Definition 6.2.19 ([Gab87a]). *A sutured manifold decomposition* $(M, \Gamma) \xrightarrow{F} (M', \Gamma')$ *is called* ***well-groomed*** *if for every component* W *of* $\mathcal{R}$, $F \cap W$ *is a union of parallel, coherently oriented, non-separating closed curves and arcs.*

Theorem 6.2.20 (Lemmas 3.6, 5.1 in [Gab87b]). *Let* $\mathcal{K}$ *be a knot in* S^3 *and* $M = X_{\mathcal{K}}$. *There is a well-groomed sutured manifold sequence of* (M, Γ) *where* $\Gamma = \partial M$

$$(M, \Gamma) \xrightarrow{F_1} (M_1, \Gamma_1) \xrightarrow{F_2} ... \xrightarrow{F_n} (M_n, \Gamma_n) = (F \times I, \partial F \times I)$$

such that ∂F_i *intersects the boundary of* M *in a union of circles for each* $1 \leq i \leq n$, F_1 *is a minimal genus Seifert surface of* $\mathcal{K}$ *and* F *is compact and oriented.*

6.2.3 Product Disks

Stallings [Sta78] showed that positive braid closures are fibered links. Gabai [Gab86] proved the same result using the theory of sutured manifolds (via disk decomposition).

Definition 6.2.21. *Let* (M, Γ) *be a sutured manifold. A product disc is a disc* D *properly embedded in* M *such that* $\partial D \cap \Gamma$ *consists of two essential arcs in* Γ. *A sutured manifold decomposition* $(M, \Gamma) \xrightarrow{D} (M', \Gamma')$ *where* D *is a product disk is called a* ***disk decomposition***.

Definition 6.2.22. (M, Γ) *is a product sutured manifold if* $M = S \times I$ *and* $\Gamma = \partial F \times I$, *where* F *is a compact surface.*

Theorem 6.2.23 (Theorem 1.9, [Gab86]). *Let* $\mathcal{L}$ *be a link in* S^3. $\mathcal{L}$ *is fibered with fiber* F *if and only if there exists a sutured manifold sequence*

$$(\overline{S^3 - (F \times I)}, \partial F \times I) \xrightarrow{D_1} ... \xrightarrow{D_n} (M_n, \Gamma_n)$$

such that D_i are product disks and (M_n, Γ_n) is a collection of product sutured balls $(D^2 \times I, S^1 \times I)$.

When $\mathcal{L}$ is a knot, $(M_n, \Gamma_n) = (B^3, S^1 \times I)$. Let $\mathcal{K} \subset S^3$ be a fibered knot with monodromy ϕ and fiber surface F, then such a disk decomposition determines the image of properly embedded arcs on F. For instance, let β be a properly embedded essential arc on F. Look at β as an arc on $F_+ \subset F \times I$ by calling it β_+. As $(F \times I, \partial F \times I)$ is a trivial product sutured manifold, all information regarding ϕ seems to be captured by the complementary sutured manifold. So, pushing β through $(\overline{S^3 - (F \times I)}, \partial F \times I)$ we get a disk D that intersects the suture exactly twice and $\overline{\partial D - (\partial F \times I)} = \beta_+ \cup \beta_-$ where $\beta_+ \subset F_+$ and $\beta_- \subset F_-$. Here D is a product disk and ϕ takes β_+ to β_-.

Definition 6.2.24. *A branched surface $\mathcal{B}$ is said to be* ***foliar*** *if*

1. *It does not carry a torus leaf.*
2. *Complementary regions of the neighborhood of the branched surface are sutured manifold products.*
3. *It has no sink disks.*
4. *It has no trivial bubbles.*

Foliar branched surfaces are laminar. If complement of a laminar branched surface are product regions, then it will carry essential lamination that can be extended to a foliation. If this foliation has no compact leaves, then it is taut. Hence, existence of a foliar branched surface guarantees the existence of a taut foliation by [Li02] and [Li03].

6.3 An Example

In this section, we describe a construction of a foliar branched surface in the exterior of the trefoil knot (See [Rob01a] and [Kri20]) which satisfies the required conditions. Let $\mathcal{K}$ be a positive trefoil knot (see Figure 6.10). $\mathcal{K}$ has genus 1. The interval of slopes that are realized by taut foliations in its exterior is $(-\infty, 2g - 1)$ which in this case is $(-\infty, 1)$.

A Seifert surface F of $\mathcal{K}$ can be drawn as in Figure 6.11 applying Seifert's algorithm. We obtain F by attaching bands to Seifert disks. Since F is orientable, we can call one side F_+ and the other one F_-. We opt for the convention that we can see only the positive side of F as in [Rud93].

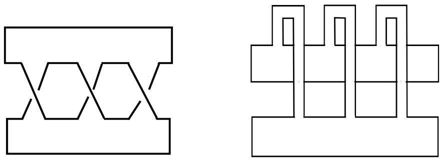


Figure 6.10: Two ways of drawing a positive trefoil knot

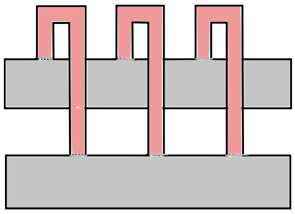


Figure 6.11: Seifert surface of trefoil knot.

It is well known that $\mathcal{K}$ is fibered with fiber a once-punctured torus. Hence, $X_{\mathcal{K}} = F \times I/\phi$ with $\phi : F \longrightarrow F$ the monodromy. Let $\partial N(\mathcal{K}) = \partial X_{\mathcal{K}} = T$. Let m be the meridian of T i.e. the curve on T that bounds an essential disk in $N(\mathcal{K})$ and $l = \partial F$ be the longitude. Note that $i(m, l) = 1$.

Let D be a disk as shown in Figure 6.12. Let $\alpha_+ = \partial D \cap F_+$ and $\alpha_- = \partial D \cap F_-$. Note that $\partial D \subset F_+ \cup F_- \cup T$. Also, $\alpha_+ \cup \alpha_-$ is outlined in red on the fiber surface F in 6.13. Sticking to our conventions, α_+ is the solid arc and α_- is the dotted arc. We now construct a branched surface in $X_{\mathcal{K}}$ which intersects T in a train track which carries all rational slopes in $(-\infty, 1)$.

The spine for the branched surface $\mathcal{B}$ is constructed from $F \times \{\frac{1}{2}\} \cup D$. We have to assign an orientation to D so as to obtain the desired branched surface using the smoothing conventions shown in Figure 6.14. We then show that $\mathcal{B}$ is foliar and intersects T in a train track carrying all rational slopes in $(-\infty, 1)$. Then we extend essential laminations to taut foliations in $X_{\mathcal{K}}$ and then to a taut foliation in the manifold obtained by Dehn filling $X_{\mathcal{K}}$.

By Lemma 3.4 in [Kri20], D can isotoped such that α_+ and α_- lie entirely in the Seifert disks as shown in Figure 6.15 (ignore the arrows for now). Let us choose positive orientation on D. Then, following the cusping conventions

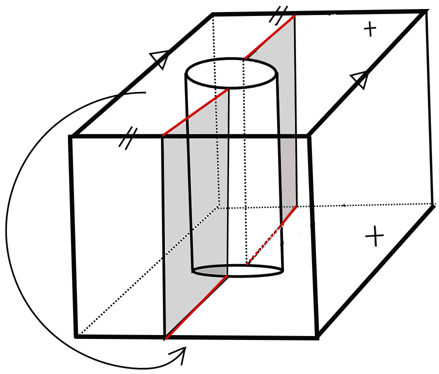


Figure 6.12: Product disk in $X_{\mathcal{K}}$ modulo ϕ

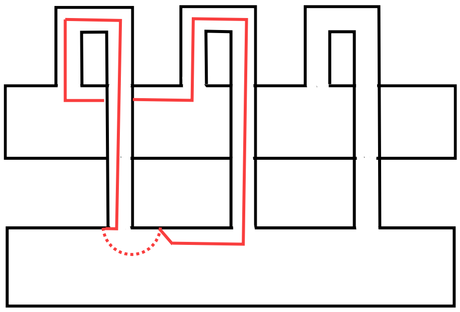


Figure 6.13: $\alpha_+ \cup \alpha_- \subset \partial D$ on Seifert surface of $\mathcal{K}$

Figure 6.14: Smoothing convention

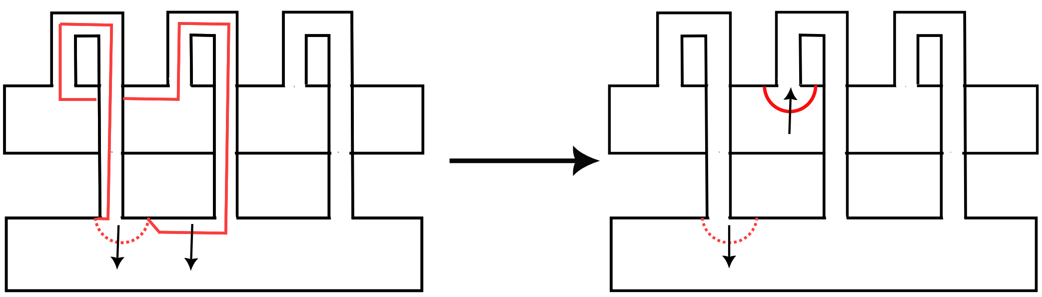


Figure 6.15: Isotoping D

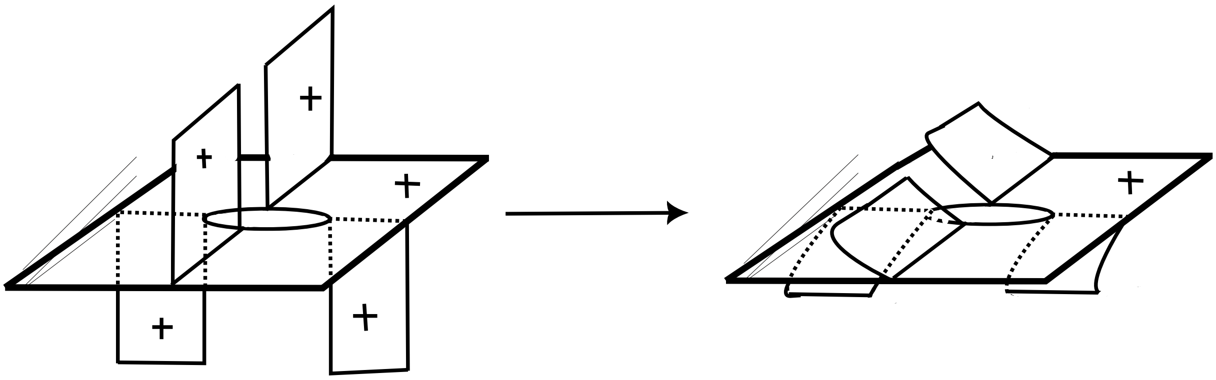


Figure 6.16: Local model of $\mathcal{B}$

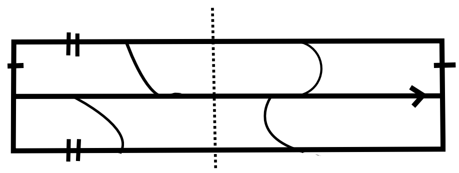


Figure 6.17: Train track $\mathcal{B} \cap \partial X_{\mathcal{K}}$

in Figure 6.14 we get a branched surface $\mathcal{B}$ which looks locally like Figure 6.16 in the knot complement picture. Note that since F is co-oriented, ∂F inherits a co-orientation from F using the right hand rule. The arcs α_+ and α_- are co-oriented as in Figure 6.15. Then $\tau = \mathcal{B} \cap \partial X_{\mathcal{K}}$ is the train track shown in Figure 6.17. The sector of τ on the right along with ∂F carries all rational slopes in $[0, 1)$ and the sector on the left along with ∂F carries all slopes between $(-\infty, 0)$. So, together they carry all slopes in $(-\infty, 1)$. Therefore, we have a branched surface $\mathcal{B}$ that intersects $\partial X_{\mathcal{K}}$ in a train track that carries all curves with rational slopes in $(-\infty, 1)$.

Evidently, we do not have any sink disks or half sink disks in $\mathcal{B}$. By Propsition 3.11 of [Kri20], such a branched surface is laminar. Applying Theorem 2.5 of [Li03], we have a family of essential laminations Λ_r that meet $\partial X_{\mathcal{K}}$ in all rational slopes $r \in (-\infty, 1)$. Such laminations can be extended to taut foliations $\mathcal{F}_r$ in $X_{\mathcal{K}}$ that foliate $\partial X_{\mathcal{K}}$ in curves of slope r (See Proposition 3.18 in [Kri20]). Then we obtain taut foliation in $S^3_r(\mathcal{K})$ for each $r \in (-\infty, 1)$ by doing a Dehn filling that sends the meridian to a curve with slope r. These manifolds are non-L-spaces and we have produced explicit taut foliations in them, just as the L-space Conjecture predicts.

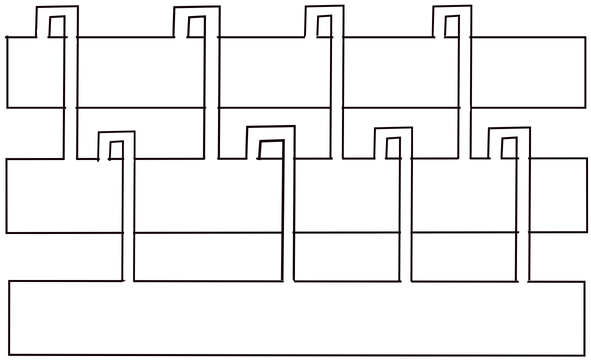


Figure 6.18: Torus knot $T_{3,4}$

6.4 A Potential Candidate

It is tempting to believe that the above construction of branched surface can be generalised to all torus knots $T_{p,q}$ and then to all positive closed n-braids as Krishna did for closed 3-braids in [Kri20]. Torus knot complements are Seifert fiber spaces. The motivation behind attempting to construct taut foliations in torus knot exteriors is twofold. Although the L-space Conjecture 6.1.3 is fully resolved for Seifert fiber spaces, an explicit foliation has not been constructed. Also, given that $T_{p,q}$ for $0 \leq p < q$ are the most complex of the p-braid closures, so finding a suitable branched surface in this case also opens up the possibility of extending this construction to the general case with minor modifications.

If our branched surface is constructed from a copy of the fiber surface and a bunch of product disks and it is sink disk-free, then it will always be laminar (See Proposition 3.11 in [Kri20]). But unfortunately, as the complexity of the knot increases, a naive generalization of the construction in [Kri20] results in combinatorial issues as we shall explain in the remainder of this section.

Applying Seifert's algorithm to $T_{p,q}$, we obtain a Seifert surface F with p disks and $q(p-1)$ bands as shown in Figure 6.18. Call the disks $S_1, ..., S_p$. We number them going from down to up. There are q bands attached between S_i and S_{i+1}. Call them $b_{i,j}$, where $j = 1, 2, .., q$. Now $T_{p,q}$ is fibered with fiber the obvious Seifert surface of the knot.

Define $\mathcal{P}$ to be the collection of disks in $X_{T_{p,q}}$ satisfying the following property: Let $D_{i,j} \in \mathcal{P}$ be a disk such that $\partial D_{i,j} \subset S_i \cup S_{i+1} \cup b_{i,j} \cup b_{i,j+1} \cup \mathcal{A}$ where $\mathcal{A} = \partial F \times I \subset X_{T_{p,q}}$. All disks in $\mathcal{P}$ intersect the $\partial F \times I$ in exactly two components and are therefore product disks. See Figure 6.13 for the outline of such a disk. Note that for a torus knot $T_{p,q}$, $|\mathcal{P}| = q(p-1)$. We start numbering the disks from bottom left. So, the first disk in Figure 6.18 on the bottom left is $D_{1,1}$, the adjacent disk in the same row will be $D_{1,2}$ and so on. The last disk in the first row $D_{1,q}$ has its boundary partially contained in bands $b_{1,q} \cup b_{1,1}$.

In order to generalize Krishna's techniques of using foliar branched surfaces on closed 3-braids to arbitrary $T_{p,q}$ knots we need to answer the following question.

Question 6.4.1. *Can we obtain a branched surface $\mathcal{B}$ in $X_{\mathcal{K}}$ where $\mathcal{K} = T_{p,q}$, $0 \leq p \leq q$, using only a copy of the fiber surface and $2g-1$ product disks obtained from $\mathcal{P}$ (with appropriate co-orientations) such that $\mathcal{B}$ intersects $\partial X_{\mathcal{K}}$ in a train track that carries all rational slopes in $(-\infty, 2g-1)$, where g is the Seifert genus of $\mathcal{K}$?*

We discuss the answer to this question in the remainder of this section. This is work in progress with Prof. Rachel Roberts and Jeffrey Norton of Washington University in St. Louis.

6.4.1 Constraint Analysis

The train track in the left half of Figure 6.17 along with the longitude is said to be of Y-type and the one on the right side in Figure 6.17 along with the longitude is said to be of X-type. An X-type sub-train track fully carries all rational slopes in $(-\infty, 0]$ and the Y-type fully carries all rational slopes in $[0, 1)$. So, if a train track τ contains one sub-train track of X-type and $2g-1$ sub-tracks of Y-type, then τ fully carries all rational slopes in $(-\infty, 2g-1)$. Note that each product disk contributes one sub-train track of Y-type and one sub-train track of X-type.

The diagram convention is that we can see the positive side of F. If a disk $D_{i,j}$ is given positive orientation we get smoothing as shown in Figure 6.15 and it contributes the train track in Figure 6.17. If $D_{i,j}$ is given negative orientation then the resulting train track would have an X-type followed by a Y-type. After attaching $2g-1$ product disks from $\mathcal{P}$ and assigning them orientations and orienting the knot, we will get a train track with $2g-1$

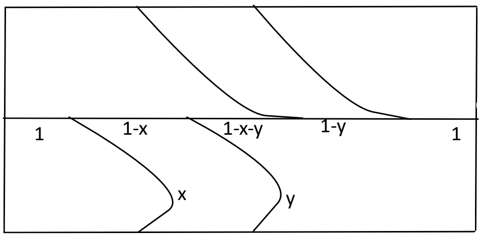


Figure 6.19: Overlapping Y-type sub-tracks on a torus with usual identifications

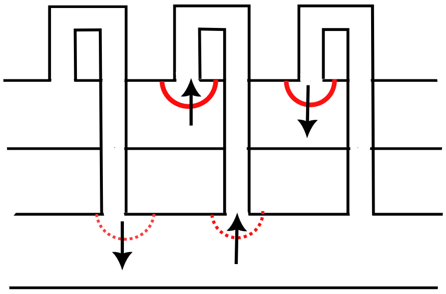


Figure 6.20: Horizontal Constraint

X-type and Y-type sub-tracks.

Now, we enumerate some constraints on the co-orientations we can assign to the product disks in order to ensure that they do not give rise to sink disks and that they carry the required slopes. For convenience, think of a $T_{p,q}$ knot as a $(p \times q)$ grid with the first and last vertical lines identified.

1. **Y-type overlap:** If no two Y-type sub-tracks overlap, we would have a train track that carries the required slopes. Overlapping Y-types do not increase the interval of slope, thus making it redundant. So, avoiding such an overlap is desired. As shown in Figure 6.19, if two Y-type sub- train tracks overlap, the interval of slopes fully carried by the entire train track is still $[0, 1)$ as $1 - x - y \geq 0$ forces $x + y \leq 1$.

2. **Horizontal Constraint:** To ensure that the middle band in Figure 6.20 does not become a half-sink disk, two adjacent disks $D_{i,j}$ and $D_{i,j+1}$ cannot have '$(+, -)$' configuration for $(D_{i,j}, D_{i,j+1})$.

3. **Diagonal Constraint:** If both $D_{i,j}$ and $D_{i+1,j+1}$ are attached then they must have opposite orientations or else one of the polygonal disk sectors shown in Figure 6.21 will be a sink.

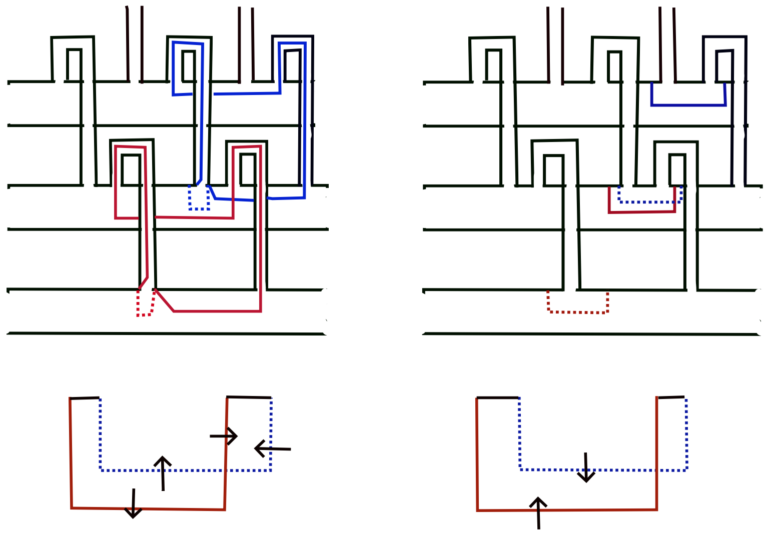


Figure 6.21: Diagonal Constraint

4. **Vertical Constraint:** If $D_{i,j}$ and $D_{i+1,j}$ are both attached as in Figure 6.22, then $(-,+)$ co-orientation for $(D_{i,j}, D_{i+1,j+1})$ is not allowed as it leads to Y-type overlap.

5. 4**-cell Constraint** If $D_{i,j}$, $D_{i,j+1}$, $D_{i+1,j}$, $D_{i+1,j+1}$ are all attached then we call such a configuration a 4-cell. The only possible co-orientations for $(D_{i,j}, D_{i,j+1}, D_{i+1,j}, D_{i+1,j+1})$ are $(+,+,-,-)$ or $(-,+,-,+)$ after applying the above constraints. See Figure 6.23 for why the other choices of orientations do not work. The red ellipses indicate constraints. For instance, a vertical ellipse means we have a vertical constraint there and hence cannot have that configuration.

6. 9**-cell Constraint:** Using the 4-cell constraint along with the horizontal, vertical and diagonal constraints, we can conclude that we cannot have a fully filled 9-cell. For instance, if we choose the $(+,+,-,-)$ for $D_{i,j}$, $D_{i,j+1}$,$D_{i+1,j}$,$D_{i+1,j+1}$ orientation, then $D_{i+2,j+1}$ cannot have '+' orientation owing to vertical constraint and cannot be '−' due to diagonal constraint as seen from the first row of Figure 6.24. So, there is no choice if we start with this orientation. Similarly, if we choose the $(-,+,-,+)$, then $D_{i+2,j+1}$ is forced to be '+' by diagonal constraint and $D_{i+2,j}$ is forced to '−' by vertical constraint. Now $D_{i+2,j+2}$ cannot be '+' by diagonal constraint and cannot be '−' by horizontal constraint as illustrated by Figure 6.24. Hence, we cannot have a fully filled 9-cell.

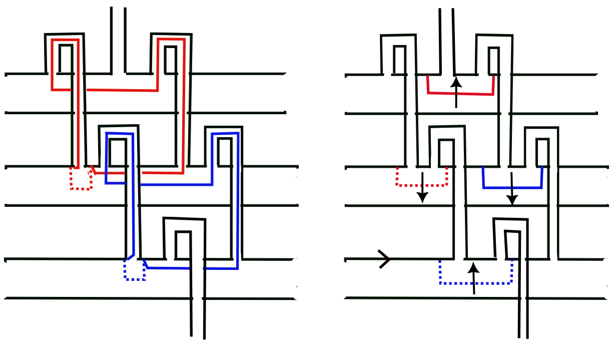


Figure 6.22: Vertical Constraint

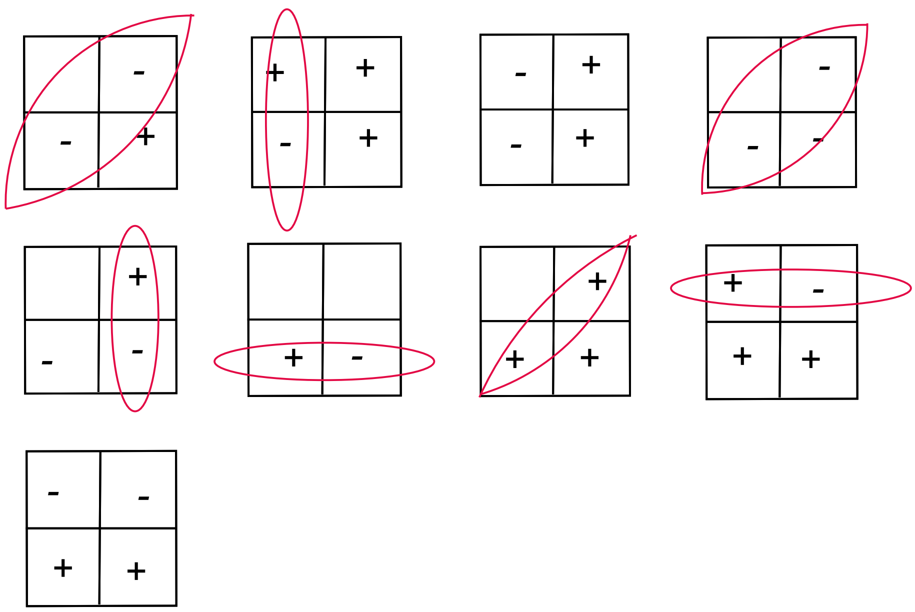


Figure 6.23: 4-cell Constraint

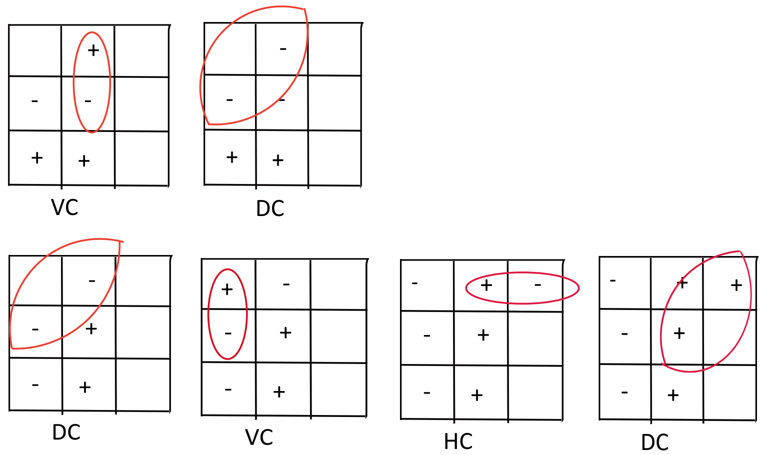


Figure 6.24: 9- cell Constraint

6.4.2 A counterexample

For $\mathcal{K} = T_{p,q}$, let F be a minimal genus Seifert surface of $\mathcal{K}$ and g is the genus of F. Note that $\chi(F) = 1 - 2g = p - (p-1)q$. So, the total number of product disks required for $X_{\mathcal{K}}$ is $pq - p - q$ and the total number of product disks from $\mathcal{P}$ is $(p-1)q$.

Let $\mathcal{K}$ be $T_{13,108}$. Let F be its Seifert surface along which we have to attach $2g - 1$ product disks. Visualize $T_{13,108}$ as a (13×108) grid with appropriate identifications. We have established above that a fully filled 9-cell is not allowed while trying to attach the product disks. When $p = 13$ and $q = 108$, we can tile a (12×108) sub-grid of the (13×108) using 144 (3×3) sub-grids. As 9-cells are not allowed, the maximum number of disks from the 144 (3×3) sub-grids is $144 \times 8 = 1152$ whereas $2g - 1 = 1283$, the difference being 131. But we only have 108 slots left to put in the disks and hence do not have enough slots for this construction to work thereby answering Question 6.4.1 negatively.

6.4.3 Conclusion

The above discussion shows us that the methods used in [Kri20] cannot be generalised to all closed positive n-braids. We have to find a suitable branched surface without sink disks in the complement of these closed braid exteriors so that we can apply Li's results to construct taut foliations in them. We can then extend it to all rational homology spheres obtained by performing a Dehn filling for all $r \in (-\infty, 2g-1) \cap \mathbb{Q}$, where g is the genus of the closed braid, thus providing further evidence for the L-space Conjecture. This constitutes future work!

7

List of Publications

1. *Prism complexes*; Tejas Kalelkar and Ramya Nair, published in *Topology Proceedings*, 62:45–63, 2023
2. *Essential Surfaces in Seifert Fiber Spaces with Singular Surfaces*; Tejas Kalelkar and Ramya Nair, published in *Topology and its Applications*, 337, 2023, Paper No. 108627.

Bibliography

[Ada04] Colin C. Adams. *The Knot Book*. American Mathematical Society, Providence, RI, 2004. An elementary introduction to the mathematical theory of knots, Revised reprint of the 1994 original.

[AS60] Lars V. Ahlfors and Leo Sario. *Riemann surfaces*. Princeton Mathematical Series, No. 26. Princeton University Press, Princeton, NJ, 1960.

[BC15] Steven Boyer and Adam Clay. Slope detection, foliations in graph manifolds, and L-spaces. *arXiv: 1510.02378*, 2015.

[BC17] Steven Boyer and Adam Clay. Foliations, orders, representations, L-spaces and graph manifolds. *Adv. Math.*, 310:159–234, 2017.

[BGW13] Steven Boyer, Cameron McA. Gordon, and Liam Watson. On L-spaces and left-orderable fundamental groups. *Math. Ann.*, 356(4):1213–1245, 2013.

[BNR97] Mark Brittenham, Ramin Naimi, and Rachel Roberts. Graph manifolds and taut foliations. *J. Differential Geom.*, 45(3):446–470, 1997.

[Bow16] Jonathan Bowden. Approximating C^0-foliations by contact structures. *Geom. Funct. Anal.*, 26(5):1255–1296, 2016.

[Cha06] Isaac Chavel. *Riemannian geometry*, volume 98 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2006. A modern introduction.

[CLW13] Adam Clay, Tye Lidman, and Liam Watson. Graph manifolds, left-orderability and amalgamation. *Algebr. Geom. Topol.*, 13(4):2347–2368, 2013.

[CMMN20] Alessia Cattabriga, Sergei Matveev, Michele Mulazzani, and Timur Nasybullov. On the complexity of non-orientable Seifert fibre spaces. *Indiana Univ. Math. J.*, 69(2):421–451, 2020.

[DR20] Charles Delman and Rachel Roberts. Taut foliations from double-diamond replacements. In *Characters in low-dimensional topology*, volume 760 of *Contemp. Math.*, pages 123–142. Amer. Math. Soc., [Providence], RI, [2020] ©2020.

[DR21] Charles Delman and Rachel Roberts. Persistently foliar composite knots. *Algebr. Geom. Topol.*, 21(6):2761–2798, 2021.

[DVW15] Ramsay Dyer, Gert Vegter, and Mathijs Wintraecken. Riemannian simplices and triangulations. *Geom. Dedicata*, 179:91–138, 2015.

[EHN81] David Eisenbud, Ulrich Hirsch, and Walter Neumann. Transverse foliations of Seifert bundles and self-homeomorphism of the circle. *Comment. Math. Helv.*, 56(4):638–660, 1981.

[Eps72] D. B. A. Epstein. Periodic flows on three-manifolds. *Ann. of Math. (2)*, 95:66–82, 1972.

[Fin76] Ronald Fintushel. Local S^1 actions on 3-manifolds. *Pacific J. Math.*, 66(1):111–118, 1976.

[Fro86] Charles Frohman. One-sided incompressible surfaces in Seifert fibered spaces. *Topology Appl.*, 23(2):103–116, 1986.

[Gab83] David Gabai. Foliations and the topology of 3-manifolds. *J. Differential Geom.*, 18(3):445–503, 1983.

[Gab86] David Gabai. Detecting fibred links in S^3. *Comment. Math. Helv.*, 61(4):519–555, 1986.

[Gab87a] David Gabai. Foliations and the topology of 3-manifolds. II. *J. Differential Geom.*, 26(3):461–478, 1987.

[Gab87b] David Gabai. Foliations and the topology of 3-manifolds. III. *J. Differential Geom.*, 26(3):479–536, 1987.

[GL89] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *Bull. Amer. Math. Soc. (N.S.)*, 20(1):83–87, 1989.

[GO89] David Gabai and Ulrich Oertel. Essential laminations in 3-manifolds. *Ann. of Math. (2)*, 130(1):41–73, 1989.

[GP10] Victor Guillemin and Alan Pollack. *Differential topology.* AMS Chelsea Publishing, Providence, RI, 2010. Reprint of the 1974 original.

[Gug53] V. K. A. M. Gugenheim. Piecewise linear isotopy and embedding of elements and spheres. I, II. *Proc. London Math. Soc. (3)*, 3:29–53, 129–152, 1953.

[Hae62] André Haefliger. Variétés feuilletées. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 16:367–397, 1962.

[Hat] A. Hatcher. Notes on basic 3-manifold topology. *https://pi.math.cornell.edu/ hatcher/3M/3M.pdf.*

[Hat92] Allen Hatcher. Some examples of essential laminations in 3-manifolds. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):313–325, 1992.

[HRRW20] Jonathan Hanselman, Jacob Rasmussen, Sarah Dean Rasmussen, and Liam Watson. L-spaces, taut foliations, and graph manifolds. *Compos. Math.*, 156(3):604–612, 2020.

[Joh79] Klaus Johannson. *Homotopy equivalences of* 3*-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics.* Springer, Berlin, 1979.

[JS79] William H. Jaco and Peter B. Shalen. Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.*, 21(220):viii+192, 1979.

[Juh15] András Juhász. A survey of Heegaard Floer homology. In *New ideas in low dimensional topology*, volume 56 of *Ser. Knots Everything*, pages 237–296. World Sci. Publ., Hackensack, NJ, 2015.

[Kar77] H. Karcher. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math.*, 30(5):509–541, 1977.

[KMOS07] P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó. Monopoles and lens space surgeries. *Ann. of Math. (2)*, 165(2):457–546, 2007.

[KN23a] Tejas Kalelkar and Ramya Nair. Essential surfaces in Seifert fiber spaces with singular surfaces. *Topology Appl.*, 337:Paper No. 108627, 2023.

[KN23b] Tejas Kalelkar and Ramya Nair. Prism complexes. *Topology Proc.*, 62:45–63, 2023.

[Kne29] Hellmuth Kneser. Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 38:248–259, 1929.

[KR17] William H. Kazez and Rachel Roberts. C^0 approximations of foliations. *Geom. Topol.*, 21(6):3601–3657, 2017.

[Krc15] David Krcatovich. The reduced knot Floer complex. *Topology Appl.*, 194:171–201, 2015.

[Kri20] Siddhi Krishna. Taut foliations, positive 3-braids, and the L-space conjecture. *J. Topol.*, 13(3):1003–1033, 2020.

[Li02] Tao Li. Laminar branched surfaces in 3-manifolds. *Geom. Topol.*, 6:153–194, 2002.

[Li03] Tao Li. Boundary train tracks of laminar branched surfaces. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 269–285. Amer. Math. Soc., Providence, RI, 2003.

[Lic62] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540, 1962.

[LR14] Tao Li and Rachel Roberts. Taut foliations in knot complements. *Pacific J. Math.*, 269(1):149–168, 2014.

[LS09] Paolo Lisca and András I. Stipsicz. On the existence of tight contact structures on Seifert fibered 3-manifolds. *Duke Math. J.*, 148(2):175–209, 2009.

[Mil62] J. Milnor. A unique decomposition theorem for 3-manifolds. *Amer. J. Math.*, 84:1–7, 1962.

[Moi52] Edwin E. Moise. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. *Ann. of Math. (2)*, 56:96–114, 1952.

[Nov65] S. P. Novikov. The topology of foliations. *Trudy Moskov. Mat. Obšč.*, 14:248–278, 1965.

[Oer88] Ulrich Oertel. Measured laminations in 3-manifolds. *Transactions of the American Mathematical Society*, 305(2):531–573, 1988.

[OS04] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334, 2004.

[OS05] Peter Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44(6):1281–1300, 2005.

[Pap57] C. D. Papakyriakopoulos. On Dehn's lemma and the asphericity of knots. *Ann. of Math. (2)*, 66:1–26, 1957.

[Per02] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications. *arXiv: math/0211159*, 2002.

[Per03a] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. *arXiv: math/0307245*, 2003.

[Per03b] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv: math/0303109*, 2003.

[Ran96] Richard Rannard. Incompressible surfaces in Seifert fibered spaces. *Topology Appl.*, 72(1):19–30, 1996.

[Rob95] Rachel Roberts. Constructing taut foliations. *Comment. Math. Helv.*, 70(4):516–545, 1995.

[Rob01a] Rachel Roberts. Taut foliations in punctured surface bundles. I. *Proc. London Math. Soc. (3)*, 82(3):747–768, 2001.

[Rob01b] Rachel Roberts. Taut foliations in punctured surface bundles. II. *Proc. London Math. Soc. (3)*, 83(2):443–471, 2001.

[Rol90] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[RR17] Jacob Rasmussen and Sarah Dean Rasmussen. Floer simple manifolds and L-space intervals. *Adv. Math.*, 322:738–805, 2017.

[RS82] Colin Patrick Rourke and Brian Joseph Sanderson. *Introduction to piecewise-linear topology*. Springer Study Edition. Springer-Verlag, Berlin-New York, 1982. Reprint.

[Rub78] J. H. Rubinstein. One-sided Heegaard splittings of 3-manifolds. *Pacific J. Math.*, 76(1):185–200, 1978.

[Rud93] Lee Rudolph. Quasipositivity as an obstruction to sliceness. *Bull. Amer. Math. Soc. (N.S.)*, 29(1):51–59, 1993.

[Sch90] Martin G. Scharlemann. Lectures on the theory of sutured 3-manifolds. In *Algebra and topology 1990 (Taejon, 1990)*, pages 25–45. Korea Adv. Inst. Sci. Tech., Taejŏn, 1990.

[Sch14] Jennifer Schultens. *Introduction to 3-manifolds*, volume 151 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2014.

[Sco83] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[Sei33] H. Seifert. Topologie Dreidimensionaler Gefaserter Räume. *Acta Math.*, 60(1):147–238, 1933.

[Sta60] John Stallings. On the loop theorem. *Ann. of Math. (2)*, 72:12–19, 1960.

[Sta78] John R. Stallings. Constructions of fibred knots and links. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 55–60. Amer. Math. Soc., Providence, R.I., 1978.

[Thu82] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[Thu22] William P. Thurston. *The geometry and topology of three-manifolds. Vol. IV.* American Mathematical Society, Providence, RI, [2022] ©2022. Edited and with a preface by Steven P. Kerckhoff and a chapter by J. W. Milnor.

[Wal60] Andrew H. Wallace. Modifications and cobounding manifolds. *Canadian J. Math.*, 12:503–528, 1960.

[Wal67] Friedhelm Waldhausen. Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I. *Invent. Math.*, 3:308–333, 1967.