Regularity theory of integro-differential operators and its application

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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Dedicated to

My Parents, Brother and My better half

Certificate

Certified that the work incorporated in the thesis entitled "*Regularity theory of integro-differential operators and its application*", submitted by *Mitesh Modasiya* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: September 21, 2023

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Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Mitesh Modasiya

Contents

Abstract

Integro-differential operators arise naturally in biological modeling and mathematical finance. We aim to conduct an in-depth study of integro-differential operators and their regularity properties in this thesis. We start by considering linear integro-differential operators of Lévy type and by studying existence-uniqueness results for the associated boundary-value problems, maximum principles, and generalized eigenvalue problems. As an application of these results, we discuss Faber-Krahn inequality and a one-dimensional symmetry result related to the Gibbons' conjecture.

Next we bring our attention to the boundary regularity of the solutions of linear integro-differential operators over bounded domains and we prove that these solutions are globally $C^{1,\alpha}$ regular. This is also used to study an overdetermined problem. To extend the linear case, we consider fully nonlinear, non-translation invariant integro-differential operators and discuss boundary regularity of solutions which requires a careful construction of a sub and supersolutions and appropriate Harnack type inequality.

At last, we consider fully nonlinear nonlocal operators. We establish Hölder regularity, Harnack inequality and boundary Harnack estimates. As an application of maximum principles, regularity theory and generalized eigenvalue problems, we then discuss one of the most celebrated reaction-diffusion model, known in literature as Fisher-KPP model, in the nonlocal setting. We further establish the existence, uniqueness and multiplicity results of the solutions to the steady state Fisher-KPP equation and long time asymptotic of the solutions of the parabolic counterpart.

1

Introduction

Let us start by defining the Hamilton-Jacobi-Bellman-Isaacs (HJBI) integro-differential operator

$$Iu(x) := \sup_{\theta\in\Theta} \inf_{\nu\in\Gamma} \{L_{\theta\nu}u(x) + f_{\theta\nu}(x)\} = 0, \tag{1.0.1}$$

where

$$L_{\theta\nu}u(x) = \operatorname{Tr} a_{\theta\nu}(x)D^2u(x) + \mathcal{I}_{\theta\nu}u(x) + b_{\theta\nu}(x)\cdot Du(x) + c_{\theta\nu}(x)u(x). \tag{1.0.2}$$

Here Θ, Γ are set of indexes and $\mathcal{I}_{\theta\nu}$ is an integral operator defined as

$$\mathcal{I}_{\theta\nu}u(x) = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_1}(y)Du(x)\cdot y)N_{\theta\nu}(x,y)\,\mathrm{d}y,$$

with Lévy kernel $N_{\theta,\nu}$. We assume that $a_{\theta\nu}$ are non negative definite matrices and $\sup_x \int_{\mathbb{R}^d}(1\wedge|y|^2)N_{\theta\nu}(x,y)\mathrm{d}y < \infty$. Also the coefficient $a_{\theta\nu}(\cdot), b_{\theta\nu}(\cdot), c_{\theta\nu}(\cdot)$ and $f_{\theta\nu}(\cdot)$ are continuous bounded functions on $\mathbb{R}^d$.

Consider equations (1.0.1) and (1.0.2), wherein the operators involved are defined using both the integral operators $\mathcal{I}_{\theta\nu}$ and the differential operators D^2 and D. Thus, it is appropriate to categorize these operators as integro-differential operators due to their combined nature of encompassing both integral and differential components. Also note that $\mathcal{I}_{\theta\nu}$ is a mapping from function to function such that to compute the value of the output function at a given point, information about the input function is required not only in the neighbourhood of a point but in the whole space $\mathbb{R}^d$. This is in contrast with the local operators such as differential operators like D^2, D for which only information about the input function in the neighbourhood of a given point is required to determine the output function's value at that point. Hence this type of operator is sometimes also called a mixed local nonlocal operator.

One can see that I defined in (1.0.1) is a continuous map from $C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$. In fact, I is a Lipschitz map. It also satisfies the global comparison property (GCP).

Definition 1.0.1 (Global comparison property)**.** We say that a map $I : C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ satisfies the global comparison property (GCP), whenever $u \leq v$ in $\mathbb{R}^d$ and $u(x) = v(x)$ implies $Iu(x) \leq Iv(x)$. It says that if a function touches another function from above at some point, then the operator preserves the ordering at that point.

A very natural question would be how general is the max-min form of the operator I in (1.0.1)? For a better understanding, let us start with the linear integro-differential operator $L_{\theta\nu}$ which is clearly a Lipschitz map and satisfies GCP as discussed above. How about the converse? Is it always true that a linear Lipschitz operator L from $C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is of the form given by (1.0.2) if it satisfies GCP? The answer is affirmative; P. Courrege showed this in 1960 [67]. Furthermore, Guillen and Schwab [100] recently prove that any translation invariant Lipschitz map from $C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ satisfying GCP will have the form (1.0.1) with linear operators $L_{\theta\nu}$ having translation invariant coefficients (see [100, Theorem 1.10]). For a more general class of operators, Guillen and Schwab also showed similar results in the same paper [100] under some spatial regularity assumption. This motivates us to study operators of the form (1.0.1).

Another motivation stems from the applications to mathematical finance, more generally, from stochastic control. To cite some of the earlier works, we refer to Soner [157] which consider nonlinear first order Hamiltonian with nonlocal term and Merton [127] where he extended the work of Black and Scholes [39] that revolutionized the theory of corporate liability pricing. For more details, one may read the books [65, 137]. Integro-differential equations involving dispersal type nonlocal kernels arise naturally in the study of population dynamics in biological modelling

[14]. Also, recently Dipierro and Valdinoci [78] showed that sometimes mixed operators such as $\Delta - (-\Delta)^s$ are more suited for predatory modelling. We refer to [77] and reference therein for more information.

This thesis is centred around a detailed study of the regularity properties of integro-differential operators and their subsequent applications. In this regard, the class of integro-differential operators for which certain regularity properties are applicable are enlarged in this thesis. We also obtained global regularity results for a large class of integro-differential operators. We used both analytical and probabilistic tools to study these operators.

1.1 Notations

We start by setting up some conventional notation. We use $B_r(x)$ to denote an open ball of radius $r > 0$ centred at a point $x \in \mathbb{R}^d$ and B_r to denote $B_r(0)$. For any subset $U \subseteq \mathbb{R}^d$, we use $USC(U), LSC(U), C(U)$ and $C_b(U)$ to denote the space of upper semicontinuous, lower semicontinuous, continuous functions and bounded continuous functions on U, respectively.

For any subset $U \subseteq \mathbb{R}^d$ and $\alpha \in (0, 1)$, we define $C^\alpha(U)$ as the space of all bounded, α-Hölder continuous functions equipped with the norm

$$\|f\|_{C^\alpha(U)} := \sup_{x\in U} |f(x)| + \sup_{x,y\in U} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Note that for $\alpha = 1$, $C^{0,1}(U)$ denotes the space of all Lipschitz continuous functions on U. The space of all bounded functions with bounded α-Hölder continuous derivatives is denoted by $C^{1,\alpha}(U)$ with the norm

$$\|f\|_{C^{1,\alpha}(U)} := \sup_{x\in U} |f(x)| + \|Df\|_{C^\alpha(U)}.$$

For any $x \in \mathbb{R}^d$, we say that $u \in C^2(x)$ if u is twice continuously differentiable in some neighborhood of x. $u \in C^2_b(\mathbb{R}^d)$ means that $u \in C^2(\mathbb{R}^d)$ and bounded in $\mathbb{R}^d$. $\mathbf{S}^d$ represents a space of all $d \times d$ real symmetric matrices. $C(\Omega, \mathbf{S}^d) := \{a : \Omega \to \mathbf{S}^d;\ \text{a is continuous}\}$ and we will usually denote $a(x) = (a_{i,j}(x))$. If $\sum_{i,j} \zeta_i a_{i,j} \zeta_j \geq 0$

for any $\zeta = (\zeta_i) \in \mathbb{R}^d$, then we say that matrix $(a_{i,j})$ is non-negative definite. For any $p, q \in \mathbb{R}^d$, we will use $p \wedge q = \min\{p, q\}$ and $p \vee q = \max\{p, q\}$.

A Bernstein function is a non-negative completely monotone function, that is, an element of the set

$$\mathcal{B} = \left\{ f \in C^\infty((0,\infty)) : \ f \geq 0 \ \text{ and } \ (-1)^n \frac{\mathrm{d}^n f}{\mathrm{d}x^n} \leq 0, \text{ for all } n \in \mathbb{N} \right\}.$$

In particular, Bernstein functions are increasing and concave.

1.2 Viscosity solution for integro-differential operator

In this section, we introduce our integro-differential operators in a very general form. We first define the following two collections of functions.

$$\mathfrak{A}_0(\Omega) \coloneqq \{(a_{i,j}(x)) \in C(\Omega, \mathbf{S}^d); (a_{i,j}(x)) \text{ is bounded and non-negative definite } \}$$

and

$$\mathfrak{B}_0(\Omega) \coloneqq \{N(x,y) : \Omega \times \mathbb{R}^d \to \mathbb{R};\ N(x,y) \geq 0 \ \textit{and} \ \int_{\mathbb{R}^d} (1 \wedge |y|^2) N(x,y) \mathrm{d}y < \infty\}.$$

For simplicity, we fix the notation $\mathfrak{A}_0$ and $\mathfrak{B}_0$ for the collection defined on Ω. For $a_{\theta\nu} \in \mathfrak{A}_0$ and $N_{\theta\nu} \in \mathfrak{B}_0$, we define a linear integro-differential operator, denoted $L_{\theta\nu}$, as follows

$$L_{\theta\nu}[x,u] = \mathrm{Tr}(a_{\theta\nu}(x) D^2 u(x)) + \mathcal{I}_{\theta\nu}[x,u], \tag{1.2.1}$$

where $\mathcal{I}_{\theta\nu}$ is a nonlocal operator defined as

$$\mathcal{I}_{\theta\nu} u(x) \coloneqq \mathcal{I}_{\theta\nu}[x,u] = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_1}(y) Du(x) \cdot y) N_{\theta\nu}(x,y) \, \mathrm{d}y. \tag{1.2.2}$$

Let $\mathscr{L}$ be the collection of all linear integro-differential operators $L_{\theta,\nu}$ of the form (1.2.1). We explicitly notice that all $L_{\theta\nu} \in \mathscr{L}$ are defined using functions from the set $\mathfrak{A}_0$ and $\mathfrak{B}_0$. If we take any nonempty subsets $\mathfrak{A} \subset \mathfrak{A}_0$ and $\mathfrak{B} \subset \mathfrak{B}_0$, then $\mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$

represents all linear operators defined by using functions from $\mathfrak{A}$ and $\mathfrak{B}$. Now we use this class of linear operators to define a fully nonlinear integro-differential operator $\mathcal{L}$ as follows

$$\mathcal{L}u(x) := \mathcal{L}[x,u] = \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma} L_{\theta\nu}[x,u] = \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\left\{\mathrm{Tr}(a_{\theta\nu}(x)D^2u(x)) + \mathcal{I}_{\theta\nu}[x,u]\right\}, \tag{1.2.3}$$

for some index sets Θ, Γ. We say that $\mathcal{L}$ is a fully nonlinear integro-differential operator with respect to $\mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$, if $L_{\theta\nu} \in \mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$ for all $\theta \in \Theta$ and $\nu \in \Gamma$. We will always be working with some subclass $\mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$ of $\mathscr{L}_{(\mathfrak{A}_0,\mathfrak{B}_0)}$ where $\mathfrak{A} \subset \mathfrak{A}_0$ and $\mathfrak{B} \subset \mathfrak{B}_0$. We will also assume that the subset $\mathfrak{A}$ and $\mathfrak{B}$ are always non-empty.

Coupling of second order derivatives and nonlocal terms gives rise to some natural difficulties, for instance, mixed order of the derivatives, behavior of solution at infinity and singular nature of the measure appearing in the nonlocal operator. We will work with viscosity solutions throughout this thesis. Next we define the viscosity solution and discuss some of its basic properties.

Definition 1.2.1. A function $u \in USC(\bar{\Omega}) \cap L^\infty(\mathbb{R}^d)$ (resp. $u \in LSC(\bar{\Omega}) \cap L^\infty(\mathbb{R}^d)$) is said to be a viscosity subsolution (resp. supersolution) to $\mathcal{L}u(x) = f(x)$ in Ω and written as $\mathcal{L}u(x) \geq f(x)$ (resp. $\mathcal{L}u(x) \leq f(x)$) in Ω, if the following holds: if a C^2 function ψ touches u at $x \in \Omega$ from above (below) in a small neighbourhood $B_r(x) \Subset \Omega$, i.e., $\psi \geq u$ in $B_r(x)$ and $\psi(x) = u(x)$, then the function v defined by

$$v(y) = \begin{cases} \psi(y) & \text{for } y \in B_r(x), \\ u(y) & \text{otherwise}, \end{cases}$$

satisfies $Iv(x) \geq f(x)$ ($Iv(x) \leq f(x)$, resp.). A function u is said to be a viscosity solution if u is both a viscosity subsolution and a viscosity supersolution.

Let us now recall some of the well-known properties of viscosity solutions.

Lemma 1.2.1. *Assume that u, v are viscosity subsolutions in Ω. Then* $\max\{u, v\}$ *is also a viscosity subsolution in Ω.*

The following lemma is a generalization of the above result when we have different domains.

Lemma 1.2.2. *Let Ω and Ω_1 be bounded domains such that $\bar{\Omega}_1 \subset \Omega$. Suppose that $u \in C(\Omega)$ is a viscosity supersolution in Ω to $\mathcal{L}u(x) = f(x)$ and $v \in C(\bar{\Omega}_1)$ is a viscosity supersolution in $\bar{\Omega}_1$ of $\mathcal{L}v(x) = g(x)$. Assume that $v \geq u$ on Ω_1^c and let*

$$w = \begin{cases} u & \text{in } \Omega \setminus \Omega_1, \\ \inf\{u, v\} & \text{in } \bar{\Omega}_1, \end{cases} \quad \text{and} \quad h = \begin{cases} f & \text{in } \Omega \setminus \Omega_1, \\ \sup\{f, g\} & \text{in } \bar{\Omega}_1, \end{cases}$$

then w is a viscosity supersolution in Ω of $\mathcal{L}w(x) = h(x)$.

Now we define extremal Pucci operators defined on the collection $\mathfrak{A}$ and $\mathfrak{B}$. Let

$$\mathcal{P}^+_{\mathfrak{A}}u(x) := \sup\left\{\mathrm{Tr}(aD^2u(x)) \ : \ a \in \mathfrak{A}\right\},$$
$$\mathcal{P}^-_{\mathfrak{A}}u(x) := \inf\left\{\mathrm{Tr}(aD^2u(x)) \ : \ a \in \mathfrak{A}\right\},$$

and

$$\mathcal{P}^+_{\mathfrak{B}}u(x) := \sup\left\{\int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_1}(y)Du(x)\cdot y)N(x,y)\,\mathrm{d}y \ : \ N \in \mathfrak{B}\right\},$$
$$\mathcal{P}^-_{\mathfrak{B}}u(x) := \inf\left\{\int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_1}(y)Du(x)\cdot y)N(x,y)\,\mathrm{d}y \ : \ N \in \mathfrak{B}\right\}.$$

The following properties are easy to check.

Lemma 1.2.3. *For any $x \in \mathbb{R}^d$ and $r_1, r_2 > 0$, let $\mathfrak{A} \subset \mathfrak{A}_0(B_{r_1}(x))$ and $\mathfrak{B} \subset \mathfrak{B}_0(B_{r_2}(x))$ be any non-empty subcollection. Then for any $u \in L^\infty(\mathbb{R}^d) \cap C^2(x)$, the following assertions hold true,*

- $\mathcal{P}^-_{\mathfrak{A}}u(x) \leq \mathcal{P}^+_{\mathfrak{A}}u(x)$ *and* $\mathcal{P}^-_{\mathfrak{B}}u(x) \leq \mathcal{P}^+_{\mathfrak{B}}u(x)$.
- *For all $\mathcal{L} \in \mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$, we have $\mathcal{P}^-_{\mathfrak{A}}u(x) + \mathcal{P}^-_{\mathfrak{B}}u(x) \leq \mathcal{L}u(x) \leq \mathcal{P}^+_{\mathfrak{A}}u(x) + \mathcal{P}^+_{\mathfrak{B}}u(x)$.*
- *If $\mathfrak{A} \subset \mathfrak{A}'$ and $\mathfrak{B} \subset \mathfrak{B}'$ then $\mathcal{P}^-_{\mathfrak{A}'}u(x) \leq \mathcal{P}^-_{\mathfrak{A}}u(x) \leq \mathcal{P}^+_{\mathfrak{A}}u(x) \leq \mathcal{P}^+_{\mathfrak{A}'}u(x)$ and $\mathcal{P}^-_{\mathfrak{B}'}u(x) \leq \mathcal{P}^-_{\mathfrak{B}}u(x) \leq \mathcal{P}^+_{\mathfrak{B}}u(x) \leq \mathcal{P}^+_{\mathfrak{B}'}u(x)$.*
- $\mathcal{P}^{\pm}_{\mathfrak{A}}(\alpha u)(x) = \alpha\mathcal{P}^{\pm}_{\mathfrak{B}}u(x)$ *for all* $\alpha \geq 0$.

Following lemma in literature refers to as the ellipticity criteria of the operators $\mathcal{L}$.

Lemma 1.2.4. *Let $u, v \in C^2(x) \cap L^\infty(\mathbb{R}^d)$ for some $x \in \mathbb{R}^d$. Then, for any $\mathcal{L} \in \mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$ we have*

$$\mathcal{P}^-_{\mathfrak{A}}(u-v)(x) + \mathcal{P}^-_{\mathfrak{B}}(u-v)(x) \le \mathcal{L}u(x) - \mathcal{L}v(x) \le \mathcal{P}^+_{\mathfrak{A}}(u-v)(x) + \mathcal{P}^+_{\mathfrak{B}}(u-v)(x)\,.$$

The next lemma generalizes the above result by requiring only one of the functions being locally C^2.

Lemma 1.2.5. *If $v \in C^2(x) \cap L^\infty(\mathbb{R}^d)$ for any $x \in \mathbb{R}^d$ and $u \in L^\infty(\mathbb{R}^d)$ then for any $\mathcal{L} \in \mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$ we have*

$$\mathcal{P}^-_{\mathfrak{A}}(u-v)(x) + \mathcal{P}^-_{\mathfrak{B}}(u-v)(x) \le \mathcal{L}u(x) - \mathcal{L}v(x)$$

when u is upper-semicontinuous in the neighbourhood of x, and

$$\mathcal{L}u(x) - \mathcal{L}v(x) \le \mathcal{P}^+_{\mathfrak{A}}(u-v)(x) + \mathcal{P}^+_{\mathfrak{B}}(u-v)(x)$$

when u is lower semicontinuous in the neighbourhood of x.

Proof. Let u be upper semicontinuous in a neighbourhood of x and ϕ be any C^2 test function touching u from above at x. Define

$$w = \begin{cases} \phi & \text{in } B_r(x), \\ u & \text{in } B_r^c(x). \end{cases}$$

Applying the previous lemma on $w - v$ gives us the desired result. □

For the purpose of our analysis we need to investigate scaled operators which we introduce here. It can be easily seen that a function and its scaled version may not always satisfy the same integro-differential equations. For example, let u be some function on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$ and $r > 0$. We can define a scaled function $v(y) = u(r(y-x)+x)$. We wish to study the type of integro-differential equations satisfied by v. Let us first define the scaled domain. Let Ω be any bounded domain in $\mathbb{R}^d$ then for any $0 < r \leq 1$ and $x_0 \in \mathbb{R}^d$ we define scaled domain as

$$\Omega^r(x_0) \coloneqq \{r(x - x_0) + x_0 \ : \ x \in \Omega\}.$$

Now $\mathfrak{A}_0(\Omega^r(x_0))$ and $\mathfrak{B}_0(\Omega^r(x_0))$ can be defined in similar fashion as

$$\mathfrak{A}_0(\Omega^r(x_0)) \coloneqq \{(a_{i,j}(x)) \in C(\Omega^r(x_0), M^d) : \ (a_{i,j}(x)) \text{ is bounded and nonegative definite}\}$$

and

$$\mathfrak{B}_0(\Omega^r(x_0)) \coloneqq \{N(x,y) : \Omega^r(x_0) \times \mathbb{R}^d \to \mathbb{R} : \ N(x,y) \geq 0 \ \textit{and} \ \int_{\mathbb{R}^d} (1 \wedge |y|^2) N(x,y) \mathrm{d}y < \infty\}.$$

Let $\mathfrak{A}(\Omega^r(x_0)) \subset \mathfrak{A}_0(\Omega^r(x_0))$ and $\mathfrak{B}(\Omega^r(x_0)) \subset \mathfrak{B}_0(\Omega^r(x_0))$ be some nonempty subcollection. Then using $\mathfrak{A}(\Omega^r(x_0)), \mathfrak{B}(\Omega^r(x_0))$ one can introduce the following scaled subclasses of $\mathfrak{A}_0$ and $\mathfrak{B}_0$

$$\begin{aligned}
\mathfrak{A}' &= \{a'(x) = a(r(x - x_0) + x_0); a \in \mathfrak{A}(\Omega^r(x_0))\}, \\
\mathfrak{B}' &= \{N'(x,y) = r^{d+2} N(r(x - x_0) + x_0, ry); N \in \mathfrak{B}(\Omega^r(x_0))\}.
\end{aligned}$$

We claim that $\mathfrak{A}'$ and $\mathfrak{B}'$ are actually subclasses of $\mathfrak{A}_0$ and $\mathfrak{B}_0$, respectively. For $\mathfrak{A}'$ it follows directly from the fact that for any $x \in \Omega$, $r(x - x_0) + x_0 \in \Omega^r(x_0)$ and $a(r(x - x_0) + x_0)$ is bounded and nonnegative definite.

Now for the class of nonlocal kernel, let $N' \in \mathfrak{B}'$. First, for any $x' \in \Omega^r(x_0)$, we have

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2) N(x', y) \mathrm{d}y < \infty \ .$$

Hence taking $x' = r(x - x_0) + x_0$ for any $x \in \Omega$ and using the estimate from above and the change of variable, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} (1 \wedge |y|^2) N'(x,y) \mathrm{d}y &= \int_{\mathbb{R}^d} (1 \wedge |y|^2) r^{d+2} N(x', ry) \mathrm{d}y \\
&= \int_{\mathbb{R}^d} (1 \wedge \frac{|y|^2}{r^2}) r^2 N(x', y) \mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} (r^2 \wedge |y|^2) N(x', y) \mathrm{d}y \\
&\leq \int_{\mathbb{R}^d} (1 \wedge |y|^2) N(x', y) \mathrm{d}y \quad .
\end{aligned}
$$

This gives us the correspondence between the sub-collection defined on $\Omega^r(x_0)$ and the sub-collection defined on Ω. This is useful in defining the following scaled operators on Ω using $\mathcal{L}$ which is a fully nonlinear integro-differential operator with respect to class $\mathscr{L}(\mathfrak{A}(\Omega^r(x_0)), \mathfrak{B}(\Omega^r(x_0)))$. Let $0 < r \leq 1$ and $\mathcal{L}$ be a fully nonlinear integro-differential operator with respect to class $\mathscr{L}(\mathfrak{A}(\Omega^r(x_0)), \mathfrak{B}(\Omega^r(x_0)))$ then the scaled operator $\mathcal{L}^r(x_0)$ is defined as

$$
\begin{aligned}
\mathcal{L}^r(x_0)u(x) &= \mathcal{L}^r(x_0)[x, u] \\
&= \sup_{\theta \in \Theta} \inf_{\nu \in \Gamma} \left\{ \mathrm{Tr}(a_{\theta\nu}(r(x - x_0) + x_0) D^2 u(x)) + \mathcal{I}^r(x_0)_{\theta\nu}[x, u] \right\}, \qquad (1.2.4)
\end{aligned}
$$

where $a_{\theta\nu} \in \mathfrak{A}(\Omega^r(x_0))$ and $\mathcal{I}^r(x_0)$ is a scaled nonlocal operator defined as

$$
\begin{aligned}
\mathcal{I}^r_{\theta\nu}(x_0)u(x) &= \mathcal{I}^r_{\theta\nu}(x_0)[x, u] \\
&= \int_{\mathbb{R}^d} (u(x + y) - u(x) - \mathbb{1}_{B_{\frac{1}{r}}}(y) Du(x) \cdot y) N'_{\theta\nu}(x, y) \, \mathrm{d}y,
\end{aligned}
$$

where $N'_{\theta\nu}(x, y) = r^{d+2} N_{\theta\nu}(r(x - x_0) + x_0, ry)$ with $N_{\theta\nu} \in \mathfrak{B}(\Omega^r(x_0))$.

Lemma 1.2.6. *Let Ω be a bounded domain, let $x \in \mathbb{R}^d$ and $0 < r \leq 1$. Moreover, let $\mathcal{L}$ be a fully nonlinear integro-differential operator with respect to class $\mathscr{L}(\mathfrak{A}(\Omega^r(x_0)), \mathfrak{B}(\Omega^r(x_0)))$, where $\mathfrak{A}(\Omega^r(x_0)) \subset \mathfrak{A}_0(\Omega^r(x_0))$ and $\mathfrak{B}(\Omega^r(x_0)) \subset \mathfrak{B}_0(\Omega^r(x_0))$ are some nonempty subcollections. Let u be a viscosity subsolution (resp. supersolution) of the equation $\mathcal{L}u(x) = f(x)$ in $\Omega^r(x_0)$ Then, $v(x) = u(r(x - x_0) + x_0)$ is a viscosity subsolution (resp., supersolution) of the equation $\mathcal{L}^r(x_0)v(x) = r^2 f(r(x - x_0) + x_0)$ in Ω, where $\mathcal{L}^r(x_0)$ is a scaled operator defined in (1.2.4).*

1.3 Probabilistic representation of the solution

We start this section by giving a brief introduction to some notion of probability theory that will help us to explain the probabilistic representation of a solution.

If X is a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and integrable with respect to the probability measure $\mathbb{P}$ then its integral value is called expectation and is denoted by

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\,\mathbb{P}(\mathrm{d}\omega).$$

If f is any bounded measurable function on $\mathbb{R}^d$ and X is a $\mathbb{R}^d$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ then

$$\mathbb{E}[f(X)] = \int_{\Omega} f(X(\omega))\,\mathbb{P}(\mathrm{d}\omega) = \int_{\mathbb{R}^d} f(x)\,\mathbb{P}_X(\mathrm{d}x),$$

where $\mathbb{P}_X$ denotes the distribution of X on $\mathbb{R}^d$. A stochastic process is a family of random variables, that is, a family $\{X_t, t \geq 0\}$ of random variables on $\mathbb{R}^d$ parameterized by $t \in [0, \infty)$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is called a stochastic process. An important property concerning a family of a random variable is independence. Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$ and X_j be an $\mathbb{R}^{d_j}$ valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for $j = 1, \ldots, n$. The family $\{X_1, \ldots, X_n\}$ is independent if for every $B_j \in \mathcal{B}(\mathbb{R}^{d_j})$, $j = 1, \ldots, n$,

$$\mathbb{P}[\cap_{j=1}^{n} \{\omega : X_j \in B_j\}] = \mathbb{P}_{X_1}(B_1) \ldots \mathbb{P}_{X_n}(B_n).$$

An infinite family of random variable is independent if every finite subfamily of it is independent. Let $X = \{X_t, t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that it has independent increments if for each $n \in \mathbb{N}$ and each $0 \leq t_1 < t_2 < \cdots < t_{n+1} < \infty$, the random variables $(X_{t_{j+1}} - X_{t_j}, 1 \leq j \leq n)$ are independent. The stochastic process is said to have stationary increments if each $X_{t_{j+1}} - X_{t_j} \stackrel{\mathrm{d}}{=} X_{t_{j+1}-t_j} - X_0$ (that is, $X_{t_{j+1}} - X_{t_j}$ and $X_{t_{j+1}-t_j} - X_0$ are identically distributed). Now we are ready to give the definition of a Lévy process.

Definition 1.3.1 (Lévy processes)**.** We say that a stochastic process X =

$\{X_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lévy process if:

1. $X_0 = 0$ almost surely.

2. X has independent and stationary increments.

3. X is stochastically continuous, that is, for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(|X_t - X_s| > a) = 0.$$

Note that in the presence of (1) and (2), (3) is equivalent to the condition

$$\lim_{t \to 0} \mathbb{P}(|X_t| > a) = 0$$

for all $a > 0$.

1.3.1 Characteristic functions and the Lévy-Khintchine formula

In this section we discuss the relationship between the generator of a Lévy process and its characteristic function. Let X be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ which takes value in $\mathbb{R}^d$ with law $\mathbb{P}_X$. Its characteristic function ϕ_X is defined by

$$\phi_X(u) = \mathbb{E}[e^{iu \cdot X}] = \int_{\mathbb{R}^d} e^{iu \cdot y} \, \mathbb{P}_X(\mathrm{d}y),$$

for each $u \in \mathbb{R}^d$. More generally, if $\mathbb{P}$ is a probability measure on $\mathbb{R}^d$ then its characteristic function is the map $u \to \int_{\mathbb{R}^d} e^{iu \cdot y} \, \mathbb{P}(\mathrm{d}y)$.

Let $X = \{X_t, t \geq 0\}$ be a Lévy process. Then its characteristic function will look like

$$\mathbb{E}[e^{iu \cdot X_t}] = e^{-t\psi(u)} \quad t \geq 0, \; u \in \mathbb{R}^d,$$

where ψ is a characteristic exponent with the following form

$$\psi(u) = \frac{1}{2} \, u \cdot Au + ib \cdot u + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{-iu \cdot y} + iu \cdot y \mathbb{1}_{\bar{B}_1}) \nu(\mathrm{d}y), \; u \in \mathbb{R}^d,$$

where $A = (a_{i,j})$ is a symmetric non-negative definite $d \times d$ matrix, $b \in \mathbb{R}^d$, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|y|^2 \wedge 1)\nu(\mathrm{d}y) < \infty. \tag{1.3.1}$$

The representation of ψ by (A, b, ν) is unique. Conversely, if A is a symmetric non-negative definite $d \times d$ matrix, $b \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying (1.3.1), then there exist a Lévy process whose characteristic exponent is ψ with (A, b, ν). We call A the diffusion coefficient of X, b the drift coefficient of X and ν is the Lévy measure of X. (A, b, ν) is referred to as a Lévy triplet of a Lévy process X. For more details on this topic we refer to the book of Sato [148].

The transition semigroup associated to the Lévy process $X = \{X_t\}$ is defined by

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}[f(x + X_t)].$$

The infinitesimal generator $\mathcal{A}$ of X is defined by

$$\mathcal{A}(u)(x) = \lim_{t \to 0} \frac{P_t u(x) - u(x)}{t},$$

provided that the limit exists. It is well known that $\mathcal{A}u$ is well defined when u is a bounded C^2 function and is represented by

$$\mathcal{A}u(x) = \frac{1}{2}\operatorname{Tr}(AD^2u(x)) + b \cdot Du(x) + \int_{\mathbb{R}^d \setminus \{0\}} (u(x+y) - u(x) - \mathbb{1}_{B_1}(y)Du(x) \cdot y)\nu(\mathrm{d}y). \tag{1.3.2}$$

As we see here, integro-differential operators can be realized as the infinitesimal generator of some Lévy process. There is a great amount of information that can be obtained about an operator from its generating process and used in the analytic study of the integro-differential operator. To cite some specific examples, see [11, 12, 112] where the authors use a probabilistic approach to study different regularity properties. In [30, 32, 34], some properties of the generating Lévy process are exploited to obtain different analytic properties like Hopf's lemma, maximum principles, Liouville-type theorems and Serrin-type rigidity results. overdetermined problem etc. This motivates us to consider the probabilistic representation of a solution u to (1.3.2) over a bounded domain. The probabilistic representation of a solution can be given using the Green function and this representation is going to play a key role in this thesis. We need a few notation to introduce this representation. Let τ be the first exit time of X from Ω, that is,

$$\tau = \inf\{t > 0 \ : \ X_t \notin \Omega\}. \tag{1.3.3}$$

We define the killed process $\{X_t^\Omega\}$ by

$$X_t^\Omega = X_t \quad \text{if } t < \tau, \quad \text{and} \quad X_t^\Omega = \partial \quad \text{if } t \geq \tau,$$

where ∂ denotes a cemetery point. X_t^Ω has transition density $p_\Omega(t, x, y)$ and its transition semigroup $\{P_t^\Omega\}_{t\geq 0}$ is given by

$$P_t^\Omega f(x) = \mathbb{E}_x[f(X_t^\Omega)\mathbb{1}_{\{t<\tau\}}] = \int_\Omega f(y) p_\Omega(t, x, y)\, \mathrm{d}y.$$

The Green function of X^Ω is defined by

$$G^\Omega(x, y) = \int_0^\infty p_\Omega(t, x, y)\, \mathrm{d}t\,.$$

Now we are ready to give the probabilistic representation of a solution.

Definition 1.3.2. A function $u \in C_b(\mathbb{R}^d)$ is said to have a probabilistic representation of a sub-solution to

$$\mathcal{A}u \leq f \quad \text{in} \quad \Omega,$$

whenever for every $x \in \Omega$,

$$u(x) \leq \int_\Omega G^\Omega(x, y) f(y)\, \mathrm{d}y + \mathbb{E}_x[u(X_\tau)] = \mathbb{E}_x\left[\int_0^\tau f(X_t)\, \mathrm{d}t\right] + \mathbb{E}_x[u(X_\tau)] \tag{1.3.4}$$

The main advantage of probabilistic representation (1.3.4) is that we can make use of semi-group property, heat kernel estimate and other probabilistic tools to study analytic properties like Alexandrov-Bakelman-Pucci (ABP) estimate or boundary behavior and transfer those estimates once an appropriate connection between viscosity solution and its probabilistic representation is established.

1.4 Results

In this thesis, the interior and boundary regularity properties of linear and nonlinear integro-differential operators and fully nonlinear nonlocal elliptic operators are developed using analytic and probabilistic methods. The next two chapters, that is, Chapters 2 and 3 deal with linear integro-differential operators whereas Chapter 4 discusses the global regularity of nonlinear nontranslation invariant integro-differential operators. Chapter 5 considers regularity theory of fully nonlinear nonlocal operators and the last chapter, that is, Chapter 6 deals with the nonlocal Fisher-KPP model.

We begin Chapter 2 with linear integro-differential operators obtained by superpositioning Laplacian with a translation invariant nonlocal operator of the form (1.2.2). In Section 2.1, we considered the Dirichlet boundary value problem and studied the existence and uniqueness of the solution. While Mou [131] showed existence of a solution for a large class of integro-differential operators using Perron's method, we do not rely on Perron's method to establish the existence of solution. Instead, we used the probabilistic structure inherited by the linear integro-differential operator and comparison principle to establish the existence of a unique viscosity solution which also has a probabilistic representation. In Section 2.2, we establish preliminary results required to study the eigenvalue problem and symmetry results. The probabilistic representation of viscosity solution plays a very crucial role in establishing ABP maximum principle. We use this ABP maximum principle to obtain a narrow domain maximum principle. We end this section by providing the proof of Hopf's lemma which requires a careful construction of the subsolution. We start Section 2.3 with a discussion on eigenvalue problem. We also prove the existence of a principal eigenvalue and principal eigenfunction by combining Kreĭn-Rutman theorem with boundary estimate and the existence of a unique solution to the Dirichlet boundary value problem of operators $L + c$ where c is a bounded continuous function. We end Section 2.3 by proving a well known Faber-Krahn inequality for linear integro-differential operator. We utilize a probabilistic representation of the principal eigenfunction to get a probabilistic representation of the principal eigenvalue then use the Brascamp-Lieb-Luttinger inequality to obtain Faber-Krahn inequality. In Section 2.4, we aim to study the symmetry properties of the positive solutions of semilinear equations and the one-dimensional symmetry result related to the Gibbons' conjecture by standard method of moving plane without imposing any additional regularity assumption on a solution. The comparison principle and a narrow domain maximum principle are used to establish radial symmetry of the positive solution. The one-dimensional symmetry result related to the Gibbons' conjecture was proved using maximum principle type result Lemma 2.4.1.

We continue with the linear integro-differential operator in Chapter 3 and study the boundary behavior of the solution establishing $C^{1,\alpha}$ regularity up to the boundary. We first show the Lipschitz regularity of a solution up to the boundary. This requires three standard ingredients, interior $C^{1,\alpha}$ regularity estimate, comparison principle and a barrier function to control the behavior of a solution u near the boundary. As it turns out a distance function from the complement of the domain Ω, that is $\delta(x) = \mathrm{dist}(x, \Omega^c)$ gives a barrier function at the boundary (see Lemma 3.1.1). Next we focus on showing global Hölder regularity of Du. As it turns out, the distance function δ as a barrier to the solution is not enough and one needs to show that u/δ is actually Hölder regular up to the boundary. A key step in this analysis is the oscillation lemma (see Proposition 3.2.1) establishing that the oscillation of the function u/δ is controlled near the boundary. Harnack estimates from [90], which was obtained using probabilistic structure inherited by the linear integro-differential operator, play a crucial role in establishing this result. Another difficulty we encounter here is that linear integro-differential operators applied on a distance function δ becomes singular near boundary and thus requires several careful estimates to manage this behavior. At last global Hölder regularity of u/δ combined with interior $C^{1,\alpha}$ regularity estimate gives us the global $C^{1,\alpha}$ regularity of the solution. This is used in combination with results from Chapter 2 to study the overdetermined problem for such operators, ultimately proving that if there exists a positive solution to the class of linear integro-differential operators satisfying both Dirichlet and a constant Neumann boundary data, then the domain must be a ball.

Inspired by the global regularity results for linear integro-differential operators, in Chapter 4 we consider fully nonlinear nontranslation invariant integro-differential operators and established $C^{1,\alpha}$-regularity up to the boundary. Unlike the linear case, most of the analytic and probabilistic tools are unavailable here. For example higher interior regularity results and Harnack type estimates that are uniform with respect to the scaling are developed in this chapter. Furthermore, the standard comparison principle is also unavailable due to the nontranslation invariant nature of the operators considered here. The idea is to try to prove that either a viscosity subsolution or a supersolution is more regular (C^2 locally) and use this to obtain an appropriate comparison between subsolution and supersolution. We also need to construct sub and super-solutions with appropriate singular nature near boundary when integro-differential operator is applied to them. This is due to the fact that operators do not have an order. All these are developed to obtain global regularity properties.

In Chapter 5, the study of regularity theory of fully nonlinear nonlocal operators with kernels obtained by superpositioning α-stable kernel with a lower order (possibly degenerate) kernel are discussed. Such operators do not have a global scaling properties. A nonlocal version of Alexandrov-Bakelman-Pucci (ABP) estimate, weak Harnack inequality and interior Holder estimate is established. A key insight is to use non-degeneracy of α-stable kernel. The Harnack inequality, half Harnack inequality for subsolutions and boundary Harnack estimate are also studied for such operators. The main features of these regularity results are their robustness as compared to the case of stable-like operators, since the operators considered here can have kernels of variable order.

At last in Chapter 6, nonlocal Fisher-KPP model is considered, in particular a class of nonlocal reaction-diffusion equations with a harvesting term are considered. Here the nonlocal operators are generators of a large class of subordinate Brownian motion and can have variable order kernel. First steady state equations are taken into consideration and the existence/non-existence, uniqueness and multiplicity results are established for such equations. One of the critical steps to obtain these results is a comparison principle which was obtained using Hopf's lemma and boundary regularity properties. The existence/non-existence of solution was established using the monotone iteration method, bifurcation results and the eigenvalue theory of the nonlocal operator. At last the long-time asymptotic of the solution is studied for the parabolic counterpart using both analytic and probabilistic arguments.

2

Linear integro-differential equation

In this chapter, we consider a linear operator in $\mathbb{R}^d$ which is a combination of a local and a nonlocal operator. In particular, we consider operators of the form

$$Lu = \Delta u + \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{\{|y|\leq 1\}} y \cdot Du(x)) j(y) \mathrm{d}y, \tag{2.0.1}$$

where $j : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ is a jump kernel satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2) j(y) \mathrm{d}y < \infty. \tag{2.0.2}$$

Operators of the form (2.0.1) appear naturally in the study of Lévy process. More precisely, as shown in equation (1.3.2), the generator of a d-dimensional Lévy process is given by the following general structure

$$\mathcal{A}u = \mathrm{Tr}(aD^2u) + b \cdot Du + \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{\{|y|\leq 1\}} y \cdot Du(x)) \nu(\mathrm{d}y),$$

where a is a non-negative definite matrix, $b \in \mathbb{R}^d$ and ν is a Lévy measure satisfying

$$\int_{\mathbb{R}^d} 1 \wedge |y|^2 \nu(\mathrm{d}y) < \infty.$$

The choice $\nu = 0$ corresponds to the local elliptic operator. For $\nu(\mathrm{d}y) = |y|^{-d-2s}\mathrm{d}y$, the nonlocal part corresponds to the well-studied fractional Laplacian. Here we set

$a = I, b = 0$ and $\nu(\mathrm{d}y) = j(y)\mathrm{d}y$ where j satisfies (2.0.2). There is a large number of works dealing with elliptic operators with both local and nonlocal parts, but most of them restrict the nonlocal term to be the factional Laplacian [1, 5, 7, 8, 19–24, 54, 75]. However, there are many practical situations; for instance, in biology [60, 135], mathematical finance [29, 137], where the Lévy measure need not be of fractional Laplacian type. This gave us enough motivation to consider an integro-differential equation with a general Lévy measure in [37].

Let us now explicitly construct the processes whose generator is given by the linear operator L in (2.0.1). We do this mainly by superpositioning a Brownian motion and a pure jump Lévy process. We will assume that all the processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us denote by

$$\psi(z) = \int_{\mathbb{R}^d} (1 - e^{iz\cdot\xi} + \mathbb{1}_{\{|z|\leq 1\}} iz \cdot \xi)\, j(\xi)\mathrm{d}\xi. \tag{2.0.3}$$

Let Y be a pure-jump Lévy process with Lévy-Khinchine exponent given by ψ and B be a Brownian motion, independent of Y, running twice as fast as the standard d-dimensional Brownian motion. Let $X = B+Y$. It is well-known that X is a strong Markov process and the semigroup generated by X is determined by the generator (2.0.1). Furthermore, the Lévy-Khinchine representation of X is given by

$$\mathbb{E}[e^{iz\cdot X_t}] = e^{-t(|z|^2+\psi(z))} \quad \text{for all } z \in \mathbb{R}^d \quad \text{and } t > 0\,.$$

We impose the following assumption on ψ.

(A1) For some constant $C > 0$, we have $|\mathrm{Im}(\psi(p))| \leq C(|p|^2 + |\mathrm{Re}(\psi(p))|)$ for all p and for all $r > 0$ we have $\sup_{|p|\leq r}(|p|^2 + \mathrm{Re}(\psi(p))) > 0$.

It is easy to see that any symmetric kernel j (that is, $j(y) = j(-y)$) satisfies (A1), since

$$\psi(z) = \int_{\mathbb{R}^d} (1 - \cos(z\cdot\xi))j(\xi)\mathrm{d}\xi \geq 0,$$

and thus, (A1) holds.

In this chapter, our focus will be to study the existence-uniqueness results for the Dirichlet boundary value problems. We will show that the solution of the Dirichlet problem has a unique probabilistic representation. We will use this representation to study maximum principles which will be further helpful in the study of generalized eigenvalue problems. As applications to these results, we obtain Faber-Krahn inequality and a one-dimensional symmetry result related to the Gibbons' conjecture.

2.1 Existence and uniqueness of viscosity solution

In this section, we will study the existence and uniqueness of a Dirichlet boundary value problem. That is, we concern ourselves with the solution of the following boundary value problem

$$Lu = -f \quad \text{in } \Omega, \quad \text{and} \quad u = g \quad \text{in } \Omega^c, \tag{2.1.1}$$

where $f \in C(\bar{\Omega})$ and $g \in C(\Omega^c)$.

As in [68, 69] M.G. Crandall, H. Ishii and P.I. Lions developed a theory of viscosity solution for a nonlinear partial differential equation to prove the existence of a solution. Our goal in this section is to prove the existence of a unique viscosity solution to (2.1.1). The classical approach is to first show the existence of the super solution and subsolution for the Dirichlet boundary value problem and then use Perron's method together with the comparison principle to find the solution in between. For mixed local-nonlocal operators similar things have been done by Mou in [131] to establish the existence of a solution.

Here we present a different approach for the operator L. We will do this by establishing the connection between the viscosity solution and the probabilistic representation of a solution. We will do this in the following way. First we show the uniqueness properties of the viscosity solution for equation (2.1.1). Then we will introduce the probabilistic representation of a solution corresponding to the linear operator L and show that it is indeed a viscosity solution.

We start with following maximum principle type result similar to [50, Lemma 5.10].

Lemma 2.1.1. *Let u be a bounded function on $\mathbb{R}^d$ which is in $USC(\bar{\Omega})$ and satisfies $Lu \geq 0$ in Ω. Then we have $\sup_\Omega u \leq \sup_{\Omega^c} u$.*

Proof. From [131, Lemma 5.5] we can find a non-negative function $\psi \in C^2(\bar{\Omega}) \cap$

$C_b(\mathbb{R}^d)$ satisfying

$$L\psi \leq -1 \quad \text{in } \Omega.$$

Note that, since $\psi \in C^2(\bar{\Omega})$, the above inequality holds in the classical sense. For $\varepsilon > 0$, we let ϕ_M to be

$$\phi_M(x) = M + \varepsilon\psi.$$

Then $L\phi_M \leq -\varepsilon$ in Ω.

Let M_0 be the smallest value of M for which $\phi_M \geq u$ in $\mathbb{R}^d$. We show that $M_0 \leq \sup_{\Omega^c} u$. Suppose, to the contrary, that $M_0 > \sup_{\Omega^c} u$. Then there must be a point $x_0 \in \Omega$ for which $u(x_0) = \phi_{M_0}(x_0)$. Otherwise using the upper semicontinuity of u, we get a $M_1 < M_0$ such that $\phi_{M_1} \geq u$ in $\mathbb{R}^d$, which contradicts the minimality of M_0. Now ϕ_{M_0} would touch u from above at x_0 and thus, by the definition of the viscosity subsolution, we would have that $L\phi_{M_0}(x_0) \geq 0$. This leads to a contradiction. Therefore, $M_0 \leq \sup_{\Omega^c} u$ which implies that for every $x \in \mathbb{R}^d$

$$u \leq \phi_{M_0} \leq M_0 + \varepsilon \sup_{\mathbb{R}^d} \psi \leq \sup_{\Omega^c} u + \varepsilon \sup_{\mathbb{R}^d} \psi.$$

The result follows by taking $\varepsilon \to 0$. □

We remark here that a similar result holds for the more general fully nonlinear integro-differential operator that will be discussed in the coming chapters. Linear integro-differential operator L of the form (2.0.1) enjoys not only a nice probabilistic structure but also has the following useful coupling property between its subsolution and supersolution.

Theorem 2.1.1. *Let Ω be an open bounded set, u and v be two bounded functions such that u is upper-semicontinuous and v is lower-semicontinuous in $\bar{\Omega}$. Also, assume that $Lu \geq f$ and $Lv \leq g$ in the viscosity sense in Ω, for two continuous functions f and g. Then $L(u - v) \geq f - g$ in Ω in the viscosity sense.*

This result combined with the maximum principle in Lemma 2.1.1 gives the uniqueness for the Dirichlet problem. Indeed if we have two different solutions u and v for (2.1.1), then from the theorem above we will have $L(u-v) \geq 0$ and $L(v-u) \geq 0$ in Ω and $u - v = 0$ in Ω^c. Hence using the maximum principle Lemma 2.1.1 we obtain $u = v$. Throughout this chapter Theorem 2.1.1 will play a key role in proving many other results. The readers may compare it with [50, Theorem 5.9].

To prove Theorem 2.1.1, we need the notion of inf and sup convolution. Given a bounded, upper-semicontinuous function u, the sup-convolution approximation u^ε is given by

$$u^\varepsilon(x) = \sup_{y\in\mathbb{R}^d} u(x+y) - \frac{|y|^2}{\varepsilon} = \sup_{y\in\mathbb{R}^d} u(y) - \frac{|x-y|^2}{\varepsilon} = u(x^*) - \frac{|x-x^*|^2}{\varepsilon}.$$

Likewise, for a bounded and lower-semicontinuous function u, the inf-convolution u_ε is given by

$$u_\varepsilon = \inf_{y\in\mathbb{R}^d} u(x+y) + \frac{|y|^2}{\varepsilon} = \inf_{y\in\mathbb{R}^d} u(y) + \frac{|x-y|^2}{\varepsilon}.$$

Lemma 2.1.2. *Let Ω be an open bounded set and f be a continuous function in Ω. If u is a bounded, upper-semicontinuous function such that $Lu \geq f$ in Ω, then $Lu^\varepsilon \geq f - d_\varepsilon$ in $\Omega_1 \Subset \Omega$ where $d_\varepsilon \to 0$ in Ω_1, as $\varepsilon \to 0$, and depends on the modulus of continuity of f.*

An analogous statement also holds for supersolutions.

Proof. Fix $x_0 \in \Omega_1$, and let φ be a test function that touches u^ε from above at x_0 in some neighbourhood $B_r(x_0) \subset \Omega_1$ and $\varphi = u^\varepsilon$ in $B_r^c(x_0)$. We define

$$Q(x) = \varphi(x + x_0 - x_0^*) + \frac{1}{\varepsilon}|x_0 - x_0^*|^2.$$

We observe from the definition of u^ε that $|x_0 - x_0^*| \leq M$ where $M = (2\|u\|_{L^\infty})^{1/2}$. Hence we can pick ε_1 such that for all $\varepsilon \leq \varepsilon_1$ and $x_0 \in \Omega_1$ we have $x_0^* \in \Omega$. It then follows from the definition that $u(x) \leq u^\varepsilon(x + x_0 - x_0^*) + \frac{1}{\varepsilon}|x_0 - x_0^*|^2$. Thus, for $|x - x_0^*| < r$ we then get

$$u(x) \leq \varphi(x + x_0 - x_0^*) + \frac{1}{\varepsilon}|x_0 - x_0^*|^2 = Q(x)$$

and $u(x_0^*) = Q(x_0^*)$. Hence Q touches u by above at x_0^* in $B_r(x_0^*)$. Define

$$w(x) = \begin{cases} Q(x) & x \in B_r(x_0^*), \\ u(x) & x \in \mathbb{R}^d \setminus B_r(x_0^*). \end{cases}$$

Then by the definition of viscosity subsolution we have $Lw(x_0^*) \geq f(x_0^*)$, that is,

$$\Delta Q(x_0^*) + \int_{\mathbb{R}^d} (w(x_0^* + y) - Q(x_0^*) - \mathbb{1}_{\{|y|\leq 1\}} y \cdot DQ(x_0^*)) j(y) \mathrm{d}y \geq f(x_0^*).$$

Now we observe that $\Delta Q(x_0^*) = \Delta\varphi(x_0)$, $DQ(x_0^*) = D\varphi(x_0)$ and $Q(x_0^*+y) - Q(x_0^*) = \varphi(x_0 + y) - \varphi(x_0)$. Since $\varphi \geq u^\varepsilon \geq u$, we obtain

$$L\varphi(x_0) \geq f(x_0) - |f(x_0^*) - f(x_0)|.$$

Since $|x_0 - x_0^*| \leq M\varepsilon^{\frac{1}{2}}$, choosing $d_\varepsilon(x) = \sup_{y \in B_{M\sqrt{\varepsilon}}(x_0)} |f(x_0^*) - f(x_0)|$ gives us the desired result. $\square$

Now we will show that the difference between sub and supersolution gives us a subsolution. Let us begin by defining the convex envelope and contact set. Let u be a function that is non-negative outside $B_r(x_0)$. The convex envelope Γ_u of u in $B_{2r}(x_0)$ is defined as follows

$$\Gamma_u(x) = \begin{cases} \sup\{p(x) \ : \ p \text{ is a plane satisfying } p \leq u^- \text{ in } B_{2r}(x_0)\} & \text{in } B_{2r}(x_0), \\ 0 & \text{in } B_{2r}^c(x_0). \end{cases}$$

The contact set is defined to be $\Sigma = \{\Gamma_u = u\} \cap B_r(x_0)$.

Lemma 2.1.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $f, g \in C(\Omega)$. Moreover, let u, v be bounded functions such that u is upper-semicontinuous and v is lower-semicontinuous in $\mathbb{R}^d$. Also, let $Lu \geq f$ and $Lv \leq g$ in Ω. Then $L(u - v) \geq f - g$ in Ω.*

Proof. Fix $\Omega_1 \Subset \Omega$ and $\varepsilon > 0$. Let $P \in C_b^2(x_0)$ be such that $u^\varepsilon - v_\varepsilon \leq P$ in $B_r(x_0) \subset \Omega_1$ and $u^\varepsilon(x_0) - v_\varepsilon(x_0) = P(x_0)$. Without loss of generality, we may also assume that P is a paraboloid and $B_{2r}(x_0) \subset \Omega$. Take $\delta > 0$ and define

$$w(x) = v_\varepsilon(x) - u^\varepsilon(x) + \phi(x) + \delta(|x - x_0| \wedge r)^2 - \delta r_1^2,$$

where $0 < r_1 < \delta \wedge \frac{r}{2}$ and

$$\phi(x) = \begin{cases} P(x) & x \in B_r(x_0) \\ u^\epsilon(x) - v_\epsilon(x) & x \in \mathbb{R}^d \setminus B_r(x_0). \end{cases}$$

We see that $w \geq 0$ on $\partial B_{r_1}(x_0)$, $w(x_0) < 0$ and $w > \frac{3\delta}{4} r^2$ on $\mathbb{R}^d \setminus B_r(x_0)$. For any $x \in \overline{B}_{r_1}(x_0)$ there exists a convex paraboloid P^x on opening K such that it touches w from above at x in $B_{r_1}(x)$, where $K(= 4/\varepsilon)$ is a constant independent of x. Using [48, Lemma 3.5] and $w(x_0) < 0$ we obtain

$$0 < \int_{A \cap \{w = \Gamma_w\}} \det D^2 \Gamma_w, \tag{2.1.2}$$

where Γ_w is the convex envelope of w in $B_{2r}(x_0)$ and $A \subset B_{r_1}(x_0)$ satisfies $|B_{r_1}(x_0) \setminus A| = 0$, Γ_w is second order differentiable in A and $\Gamma_w \in C^{1,1}(B_{\frac{r}{2}})$. Furthermore, u^ε, v^ε (and hence w) are punctually second order differentiable in A [48, Theorem 5.1] . Thus $Lu^\varepsilon(x)$, $Lv^\varepsilon(x)$ are defined in the classical sense for $x \in A$ and from Lemma 2.1.2 we have

$$Lu^\varepsilon(x) \geq f(x) - d_\varepsilon \quad \text{and} \quad L(v^\varepsilon)(x) \leq g(x) + d_\varepsilon.$$

We note that since the contact set $\{w = \Gamma_w\}$ are the points of minimum for $w - \Gamma_w$ and w is differentiable at the points of A (as it is punctually twice differentiable), we have $Dw = D\Gamma_w$ on $A \cap \{w = \Gamma_w\}$. Therefore, since Γ_w is convex and $\Gamma_w \leq w$, for all $x \in A \cap \{w = \Gamma_w\}$ we have

$$\Delta w(x) \geq 0 \quad \text{and} \quad \int_{x+y \in B_r(x)} (w(x+y) - w(x) - \mathbb{1}_{\{|y| \leq 1\}} y \cdot Dw(x)) j(y) \mathrm{d}y \geq 0,$$

using the fact that for $x \in A \cap \{w = \Gamma_w\}$ we have

$$w(x+y) - w(x) - \mathbb{1}_{\{|y|\le 1\}} y \cdot Dw(x) \ge \Gamma_w(x+y) - \Gamma_w(x) - \mathbb{1}_{\{|y|\le 1\}} y \cdot Dw(x) \ge 0,$$

for all $x + y \in B_r(x)$.

Now from (2.1.2) it follows that

$$|\{w = \Gamma_w\} \cap A| > 0,$$

and therefore, there is one point $x_1^\delta \in \{w = \Gamma_w\} \cap A$ where $Lu^\varepsilon(x)$, $Lv^\varepsilon(x)$ can be computed classically. At this point we thus have

$$\begin{aligned} f(x_1^\delta) - d_\varepsilon \le Lu^\varepsilon(x_1^\delta) &= Lv_\varepsilon(x_1^\delta) - Lw(x_1^\delta) + L\phi(x_1^\delta) + \delta L(|\bullet - x_0| \wedge r)^2(x_1^\delta) \\ &\le g(x_1^\delta) + d_\varepsilon + L\phi(x_1^\delta) - \int_{|y|\ge r} (w(x_1^\delta + y) - w(x_1^\delta)) j(y) \mathrm{d}y \\ &+ \int_{r\le |y|\le 1} y \cdot D\Gamma_w(x_\delta^1) j(y) \mathrm{d}y + \delta L(|\bullet - x_0| \wedge r)^2(x_1^\delta). \end{aligned}$$

Letting $r_1 \to 0$, we see that $x_1^\delta \to x_0$ and $D\Gamma_w(x_1^\delta) \to D\Gamma(x_0)$. Since Γ_w attains its minimum at x_0 we have $D\Gamma_w(x_0) = 0$. Also, $w(x_1^\delta + y) - w(x_1^\delta) \to w(x_0 + y) - w(x_0)$. Hence, by the dominated convergence theorem, we have

$$\begin{aligned} f(x_0) - d_\varepsilon \le g(x_0) + d_\varepsilon + L\phi(x_0) &- \int_{|y|\ge r} (w(x_0 + y) - w(x_0)) j(y) \mathrm{d}y \\ &+ \delta L(|\bullet - x_0| \wedge r)^2(x_0). \end{aligned}$$

Since $w(x_0 + y) - w(x_0) \ge 0$ for all $|y| \ge r$, we have

$$f(x_0) - d_\varepsilon \le g(x_0) + d_\varepsilon + L\phi(x_0) + \delta L(|\bullet - x_0| \wedge r)^2(x_0).$$

Now, we let $\delta \to 0$ to find that

$$f(x_0) - g(x_0) - 2d_\varepsilon \le L\phi(x_0).$$

This gives us $L(u^\varepsilon - v_\varepsilon) \ge f - g - 2d_\varepsilon$ in Ω_1, in the viscosity sense. At the end, we let $\varepsilon \to 0$ and use the stability property of viscosity solution to obtain our desired result.

□

Now applying a standard approximation argument together with Lemma 2.1.3 we obtain Theorem 2.1.1 (cf. [50, Theorem 5.9])

Now we prove the existence of viscosity solution to (2.1.1).

Lemma 2.1.4. *Assume (A1) and let Ω be a bounded Lipschitz domain. Define*

$$u(x) = \mathbb{E}_x\left[\int_0^\tau f(X_t)\,\mathrm{d}t\right] + \mathbb{E}_x[g(X_\tau)], \quad x \in \Omega,$$

where τ denotes the first exit time of X from Ω as defined in (1.3.3), that is $\tau = \inf\{t > 0 \,:\, X_t \notin \Omega\}$. Then $u \in C_b(\mathbb{R}^d)$ and solves (2.1.1) in the viscosity sense.

Proof. Since $u(x) = g(x)$ in Ω^c, we only need to show that $u \in C(\bar{\Omega})$. Let $x_n \in \Omega$ and $x_n \to z \in \bar{\Omega}$. Define

$$\tau_n = \inf\left\{t > 0 \,:\, X^n_t = x_n + X_t \notin \Omega\right\}.$$

Here τ_n is the first exit time of a process starting from x_n. In a similar manner, one can define the first exit time τ_z of a process starting from z as

$$\tau_z = \inf\left\{t > 0 \,:\, X^z_t = z + X_t \notin \Omega\right\}.$$

First suppose that $z \in \partial\Omega$. Since Ω is Lipschitz, it satisfies the exterior cone condition and hence regular with respect to X [128, 159]. This means $P_z(\tau_{\bar{\Omega}} = 0) = 1$. Therefore, for every $\delta > 0$, X^z intersects $(\bar{\Omega})^c$ before time δ, almost surely. Since

$$\sup_{t\in[0,M]} |x_n + X_t - (z + X_t)| \leq |x_n - z| \to 0, \quad \text{as} \quad n \to \infty, \tag{2.1.3}$$

for every fixed M, it implies that $\tau_n \to 0$ and $X^n_{\tau_n} \to z$, almost surely. Therefore, using Lemma 2.2.1 and dominated convergence theorem, it follows that $u(x_n) \to g(z)$ as $n \to \infty$.

For the remaining part, we assume that $z \in \Omega$. For a fixed $M > 0$, next we show that $\tau_n \wedge M \to \tau_z \wedge M$ almost surely. Denote by Ω_M the event in (2.1.3). Then

$\mathbb{P}(\Omega_M) = 1$. Since Ω is regular we have $P(\tau_z = \overline{\tau}_z) = 1$, where $\overline{\tau}_z$ is the first exit time of a process from $\bar{\Omega}$ starting at z . Denote by $\tilde{\Omega} = \{\tau_z = \overline{\tau}_z\}$. Let $\varepsilon > 0$. We claim that, on $\Omega_M \cap \tilde{\Omega}$, $\tau_n \wedge M \leq \tau_z \wedge M + \varepsilon$ for all large n. Also, we only need to show it on $\{\tau_z < M\}$. For $\omega \in \Omega_M \cap \tilde{\Omega} \cap \{\tau_z < M\}$, there exists $s \in [\tau_z(\omega), \tau_z(\omega) + \frac{\varepsilon}{2}]$ such that $X^z_s(\omega) \in \bar{\Omega}^c$ which implies

$$\mathrm{dist}(X^z_s(\omega), \bar{\Omega}) > 0.$$

By (2.1.3), it then implies that

$$\tau_n(\omega) \leq s \leq \tau_z(\omega) + \frac{\epsilon}{2},$$

for all large n. This proves the claim.

Next we show that, on $\tilde{\Omega} \cap \Omega_M$, we have $\tau_z \wedge M - \varepsilon \leq \tau_n \wedge M$ for all large n. Now since $\tau_z(\omega) \wedge M - \varepsilon < \tau_z(\omega)$, for all $s \in [0, \tau_z(\omega) \wedge M - \varepsilon]$ we have $X^z_s(\omega) \in \Omega^o$. Applying (2.1.3) we get $X^n_s \in \Omega^o$ for all $s \in [0, \tau_z(\omega) \wedge M - \varepsilon]$ and for all large n. Thus $\tau_z \wedge M - \varepsilon \leq \tau_n \wedge M$ for all large n. Thus for every $\varepsilon > 0$ and $\omega \in \Omega_M \cap \tilde{\Omega}$, we have $N(\omega)$ satisfying

$$\tau_z(\omega) \wedge M - \varepsilon \leq \tau_n(\omega) \wedge M \leq \tau_z(\omega) \wedge M + \varepsilon$$

for all $n \geq N(\omega)$. Hence we proved $M \wedge \tau_n \to M \wedge \tau_z$ pointwise in $\omega \in \Omega_M \cap \tilde{\Omega}$, as $n \to \infty$. Since M is arbitrary, this also implies $\tau_n \to \tau_z$ almost surely.

Next we want to show that exit location converges, that is, $X^n_{\tau_n} \to X^z_{\tau_z}$ as $n \to \infty$, almost surely. We recall the Lévy system formula (cf. [43, p. 65]),

$$\mathbb{E}_x \left[\sum_{0 < s \leq t} f(X_{s-}, X_s) \mathbb{1}_{\{X_{s-} \neq X_s\}} \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} f(X_s, y) j(y - X_s) \mathrm{d}y \, \mathrm{d}s \right] \tag{2.1.4}$$

which holds for all non-negative $f \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{R}^d)$. Now we put $f_\varepsilon(x, y) = \mathbb{1}_{\{x \in \Omega\}} \mathbb{1}_{\{|x-y| > \varepsilon\}} \mathbb{1}_{\{y \in \partial\Omega\}}$, in the above formula and using the fact $|\partial\Omega| = 0$ we obtain

$$\mathbb{E}_z \left[\sum_{0 < s \leq t} f_\varepsilon(X_{s_-}, X_s) \mathbb{1}_{\{X_{s-} \neq X_s\}} \right] = 0, \quad \text{for all } \varepsilon > 0.$$

Since ε and t are arbitrary, this gives us

$$\mathbb{P}_z(\{X_{\tau_z} \in \partial\Omega, X_{\tau_z -} \neq X_{\tau_z}\}) = 0. \tag{2.1.5}$$

Again, choosing

$$\hat{f}_\varepsilon(x,y) = \mathbb{1}_{\{x\in\partial\Omega\}}\mathbb{1}_{\{y\in\Omega^+_\varepsilon\}}, \quad \text{for} \quad \Omega^+_\varepsilon = \{y : \mathrm{dist}(y,\Omega) > \varepsilon\}$$

in (2.1.4) and since X_s has transition density (see Lemma 2.2.2), it follows that

$$\mathbb{E}_z\left[\sum_{0<s\leq t} \hat{f}_\varepsilon(X_{s-}, X_s)\mathbb{1}_{\{x_{s-}\neq X_s\}}\right] = 0.$$

Since ε, t are arbitrary, we get

$$\mathbb{P}_z(\{X_{\tau_z^-} \in \partial\Omega, X_{\tau_z} \in (\bar{\Omega})^c\}) = 0. \tag{2.1.6}$$

We claim that for any $M > 0$ we have

$$X^n_{\tau_n\wedge M} \to X^z_{\tau_z\wedge M} \quad \text{as} \quad n\to\infty, \text{ almost surely.}$$

We will be interested in the case where $\tau_z < M$ since, given $t = M$, function $t \mapsto X_t$ is almost surely continuous at $t = M$. Now fix ω so that it is in the complement of the events in (2.1.5) and (2.1.6). Since process, X_t is Càdlàg (right continuous with a finite left limit) and $\tau_n \wedge M \to \tau_z \wedge M$ almost surely, we only need to consider a situation where $\tau_n \wedge M \nearrow \tau_z \wedge M$. On set $\{\tau_z < M\}$, we have $\tau_n \nearrow \tau_z$.

If $t \to X^z_t(\omega)$ is continuous at $\tau_z(\omega) = t$, then we have the claim. So we let $X^z_{\tau_z-}(\omega) \neq X^z_{\tau_z}(\omega)$. Since ω is in the complement of the events in (2.1.5) and (2.1.6), we have $X^z_{\tau_z-}(\omega) \in \Omega^o$ and $X^z_{\tau_z}(\omega) \in \bar{\Omega}^c$. But $x_n \to z$ and $X^z \mid_{[0,\tau_z^-]} (\omega)$ is in Ω°, then $X^n \mid_{[0,\tau_n]} (\omega)$ is in Ω° for large n, contradicting the fact that $X^n_{\tau_n}(\omega) \in \Omega^c$. So this case can not happen and we get that $X^n_{\tau_n\wedge M} \to X^z_{\tau_z\wedge M}$ as $n\to\infty$, almost surely. Since M is arbitrary and $\tau_n \to \tau_z$, it gives us

$$X^n_{\tau_n} \to X^z_{\tau_z} \quad \text{as} \quad n\to\infty, \text{ almost surely.} \tag{2.1.7}$$

Now we are ready to show that $u(x_n) \to u(z)$, Applying dominated convergence theorem and using (2.1.7) we get

$$\mathbb{E}_{x_n}[g(X_\tau)] \to \mathbb{E}_z[g(X_\tau)].$$

Since $\tau_n \wedge M \to \tau_z \wedge M$ almost surely and for all ω, $\sup_{t\in[0,M]} |X_t^n - X_t^z| \to 0$ (see (2.1.3)), we have

$$\int_0^{\tau_n \wedge M} f(X_s^n)\mathrm{d}s \to \int_0^{\tau_z \wedge M} f(X_s^z)\mathrm{d}s,$$

almost surely. Hence by dominated convergence, we have

$$\mathbb{E}\left[\int_0^{\tau_n \wedge M} f(X_s^n)\mathrm{d}s\right] \to \mathbb{E}\left[\int_0^{\tau_z \wedge M} f(X_s^z)\mathrm{d}s\right].$$

Now since M is arbitrary, using Lemma 2.2.1, it follows that

$$\mathbb{E}\left[\int_0^{\tau_n} f(X_s^n)\mathrm{d}s\right] \to \mathbb{E}\left[\int_0^{\tau_z} f(X_s^z)\mathrm{d}s\right].$$

This completes the proof.

It is standard to show that u is a viscosity solution (cf. [32, Remark 3.2] [140, Theorem 2.2]). □

Now we can establish the existence and uniqueness of the solution. Lemma 2.1.4 combined with Theorem 2.1.1 and Lemma 2.1.2 gives us the following result.

Theorem 2.1.2. *Assume (A1). Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^d$. Also, assume that $f \in C(\bar{\Omega})$ and $g \in C_b(\Omega^c)$. Then there exists a unique viscosity solution $u \in C_b(\mathbb{R}^d)$ to*

$$Lu = -f \quad \text{in } \Omega, \quad \text{and} \quad u = g \quad \text{in } \Omega^c. \tag{2.1.8}$$

Furthermore, the unique solution can be written as

$$u(x) = \mathbb{E}_x\left[\int_0^\tau f(X_t)\,\mathrm{d}t\right] + \mathbb{E}_x[g(X_\tau)], \qquad x \in \Omega, \tag{2.1.9}$$

where $\tau = \tau_\Omega$ denotes the first exit time of X from Ω, that is, $\tau_\Omega = \inf\{t > 0$:

$X_t \notin \Omega\}$.

When $f \in C^{\alpha}(\bar{\Omega})$ and $g \in C^{2+\alpha}(\Omega^c)$ for some $\alpha > 0$, the existence of a unique classical solution to (2.1.8) is known from the work of Garroni and Menaldi [96]. In [8] Barles, Chasseigne and Imbert establish the existence of viscosity solutions for a large class of nonlinear integro-differential operators. Unlike ours, [8] (see also [9]) requires the operators to be strictly monotone in the zeroth order term. Recently Mou [131] proved the existence of viscosity solution for fully nonlinear integro-differential operators using Perron's method which also includes the linear operator considered here. What we are able to achieve using the probabilistic structure induced by the linear operator is that we have a unique probabilistic representation of the viscosity solution. Also, this representation of the solution by (2.1.9) will play a crucial role in the subsequent section.

2.2 Maximum principles

In this section we prove a Alexandrov-Bakelman-Pucci (ABP) maximum principle, a narrow domain maximum principle, a Hopf's lemma and a strong maximum principle. We will use the representation (2.1.9) of a solution to establish an Alexandrov-Bakelman-Pucci (ABP) maximum principle. For this purpose, we will begin with the following estimate on the exit time.

Lemma 2.2.1. *Assume (A1). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. For every $k \in \mathbb{N}$ we have*

$$\sup_{x\in D} \mathbb{E}_x[\tau^k] \leq k! \left(\sup_{x\in D} \mathbb{E}_x[\tau]\right)^k .$$

Moreover, there exists a constant $\theta = \theta(d, \mathrm{diam}(\Omega))$, monotonically increasing with respect to $\mathrm{diam}(\Omega)$, such that

$$\sup_{x\in D} \mathbb{E}_x[\tau^k] \leq k!\theta^k.$$

Proof. Proof follows from [32, Lemma 3.1], (A1) and [149, Remark 4.8]. □

Using Lemma 2.2.1 we find an ABP type estimate for the semigroup subsolutions. This is the content of our next lemma.

Lemma 2.2.2. *Let Ω be any bounded domain and $u : \mathbb{R}^d \to \mathbb{R}$ be a bounded function satisfying*

$$u(x) \leq \mathbb{E}_x[u(X_\tau)] + \mathbb{E}_x\left[\int_0^\tau f(X_t)\,\mathrm{d}t\right] \quad \text{for all } x \in \Omega,$$

with $f \in L^p(\Omega)$, for some $p > \frac{d}{2}$, where τ denotes the first exit time from Ω. Then there exists a constant $C_1 = C_1(p, d, \mathrm{diam}(\Omega))$ such that

$$\sup_{\Omega} u^+ \leq \sup_{\Omega^c} u^+ + C_1\,\|f\|_{L^p(\Omega)}.$$

Proof. For simplicity of notation we extend f by zero outside of Ω. From the given condition it is easily seen that

$$u(x) \leq \sup_{\Omega^c} u^+ + \mathbb{E}_x\left[\int_0^\tau |f(X_s)|\mathrm{d}s\right], \quad x \in \Omega.$$

Thus, we only need to estimate the rightmost term in the above expression. Recall that $X = B + Y$ where B is a d-dimensional Brownian motion, running twice as fast as standard Brownian motion, and is independent of Y. Let ν_t be the transition probability of Y_t, starting from 0, that is,

$$\nu_t(A) = \mathbb{P}\left(Y_t \in A\right),$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$. Let

$$p_t^B(y) = (4\pi t)^{-d/2} e^{-\frac{|y|^2}{4t}},$$

be the transition density of B_t starting from 0. Then the transition density of X_t, starting from x, is given by

$$p_t(x, y) = \int_{\mathbb{R}^d} p_t^B(z - x - y)\nu_t(\mathrm{d}z), \quad t > 0.$$

In particular, for any $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned} p_t(x, y) &= \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y-z|^2}{4t}} \nu_t(\mathrm{d}z) \\ &\leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \nu_t(\mathbb{R}^d) = \frac{1}{(4\pi t)^{\frac{d}{2}}}. \end{aligned} \tag{2.2.1}$$

Next we write

$$\begin{aligned}\mathbb{E}_x\left[\int_0^\tau |f(X_s)|\mathrm{d}s\right] &= \mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{\{\tau>s\}}|f(X_s)|\mathrm{d}s\right]\\ &\le \mathbb{E}_x\left[\int_0^1 |f(X_s)|\mathrm{d}s\right] + \mathbb{E}_x\left[\int_1^\infty \mathbb{1}_{\{\tau>s\}}|f(X_s)|\mathrm{d}s\right]. \qquad (2.2.2)\end{aligned}$$

We estimate the first term on the RHS as

$$\begin{aligned}\mathbb{E}_x\left[\int_0^1 |f(X_s)|\mathrm{d}s\right] &= \int_0^1\int_{\mathbb{R}^d} |f(y)|p_s(x,y)\mathrm{d}y\,\mathrm{d}s \le \|f\|_{L^p(\Omega)}\int_0^1 \|p_s(x,\cdot)\|_{p'}\mathrm{d}s\\ &\le \|f\|_{L^p(\Omega)}\int_0^1 \left[\int_{\mathbb{R}^d} (p_s(x,y))^{p'}\mathrm{d}y\right]^{\frac{1}{p'}}\mathrm{d}s\\ &\le \frac{1}{(4\pi)^{d/2p}}\|f\|_{L^p(\Omega)}\int_0^1 \left[(s^{-\frac{d}{2}})^{p'-1}\int_{\mathbb{R}^d}(p_s(x,y))\mathrm{d}y\right]^{\frac{1}{p'}}\mathrm{d}s\\ &\le \frac{1}{(4\pi)^{d/2p}}\|f\|_{L^p(\Omega)}\int_0^1 s^{-\frac{d}{2p}}\mathrm{d}s = \frac{1}{(4\pi)^{d/2p}}\frac{2p}{2p-d}\|f\|_{L^p(\Omega)},\end{aligned}$$

where in the third line we use (2.2.1) and p, p' are Hölder conjugates. To deal with the rightmost term in (2.2.2) we choose $k \in \mathbb{N}$ with $k > p'$. Using Lemma 2.2.1 we then calculate

$$\begin{aligned}\mathbb{E}_x\left[\int_1^\infty \mathbb{1}_{\{\tau>s\}}|f(X_s)|\mathrm{d}s\right] &\le \int_1^\infty (\mathbb{P}_x(\tau>s))^{\frac{1}{p'}}\,\mathbb{E}_x\left[|f(X_s)^p|\right]^{\frac{1}{p}}\mathrm{d}s\\ &\le \frac{1}{(4\pi)^{d/2p}}\|f\|_{L^p(\Omega)}\int_1^\infty (\mathbb{P}_x(\tau>s))^{\frac{1}{p'}}\mathrm{d}s\\ &\le \frac{1}{(4\pi)^{d/2p}}\|f\|_{L^p(\Omega)}\int_1^\infty s^{-\frac{k}{p'}}\,\mathbb{E}_x\left[\tau^k\right]^{\frac{1}{p'}}\mathrm{d}s\\ &\le \frac{1}{(4\pi)^{\frac{d}{2p}}}\|f\|_{L^p(\Omega)}\frac{p'}{k-p'}(k!\theta^k)^{\frac{1}{p'}}.\end{aligned}$$

Combining these estimates in (2.2.2) completes the proof. □

Using Lemma 2.2.2 and the arguments of [34, Theorem 3.1] then gives us the ABP maximum principle.

Theorem 2.2.1 (ABP-maximum principle)**.** *Assume (A1). Let* $f : \Omega \to \mathbb{R}$ *be* *continuous and* $u \in C_b(\mathbb{R}^d)$ *be a viscosity subsolution to*

$$Lu = -f \quad in\,\{u > 0\} \cap \Omega, \quad and \quad u \le 0 \quad in\, \Omega^c.$$

Then for every $p > \frac{d}{2}$*, there exists a constant* $C = C(d, p, \mathrm{diam}(\Omega))$ *satisfying*

$$\sup_{\Omega} u \le C\, \|f^+\|_{L^p(\Omega)}.$$

In [132, Theorem 3.2] Mou and Święch consider the Pucci extremal operators and establish the ABP estimate for strong solutions. Similar estimate for viscosity solutions can be found in Mou [131]. It should be observed that the ABP estimates in [131, 132] holds for $p > p_0$ where p_0 is some number in $[d/2, d)$. Theorem 2.2.1 shows that we can choose $p_0 = d/2$ for L. Also, compare this result with [46, Theorem 1.9]. Recently, Sobolev regularity and maximum principles for the operator $-\Delta + (-\Delta)^s$, $s \in (0, 1)$, are studied by Biagi et. al. in [20]. We also mention the work of Alibaud et. al. [2] where the authors provide a complete characterization of the translation-invariant integro-differential operators that satisfy the Liouville property in the whole space.

As an application of Theorem 2.2.1 we obtain a narrow domain maximum principle.

Corollary 2.2.1 (Maximum principle for narrow domains)**.** *Assume (A1). Let* $u \in C_b(\mathbb{R}^d)$ *be a viscosity subsolution to*

$$Lu + c\,u = 0 \quad in\, \Omega, \quad and \quad u \le 0 \quad in\, \Omega^c,$$

for some continuous bounded function $c : \Omega \to \mathbb{R}$*. Then there exists a constant* $\varepsilon = \varepsilon(d, \|c\|_{L^\infty(\Omega)}, \mathrm{diam}(\Omega))$ *such that* $u \le 0$ *in* $\mathbb{R}^d$*, whenever* $|\Omega| < \varepsilon$*.*

Proof. Since $u \le 0$ in Ω^c, it is enough to show that $\sup_\Omega u \le 0$. Suppose, to the contrary, that $\sup_\Omega u > 0$. Note that, moving cu on the RHS, we can take $f = \|c\|_{L^\infty(\Omega)} u^+$ on $\{u > 0\}$ in Theorem 2.2.1. Then applying Theorem 2.2.1, we obtain

$$\sup_{\Omega} u \le C\, \|f^+\|_{L^p(\Omega)} \le C \|c\|_{L^\infty(\Omega)} \sup_{\Omega} u |\Omega|^{\frac{1}{p}},$$

for some $p > \frac{d}{2}$. This gives us that $1 \leq C\|c\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{p}}$. Now choosing $\varepsilon^p = \frac{1}{2C\|c\|_{L^\infty(\Omega)}}$ gives us the contradiction. Therefore, we have $\sup_\Omega u \leq 0$ which gives us the desired results. $\square$

Corollary 2.2.2. *Assume (A1) and let Ω be a bounded open set. Let $u, v \in C_b(\mathbb{R}^d)$ satisfy*

$$Lu + cu \geq f, \quad Lv + cv \leq g \quad \textit{in } \Omega,$$

for some $c, f, g \in C(\Omega)$. Also, assume that $c \leq 0$ and $f \geq g$ in Ω. Then, if $u \leq v$ in Ω^c, we have $u \leq v$ in $\mathbb{R}^d$.

Proof. Using Theorem 2.1.1 we get that $(L+c)w \geq 0$ in Ω with $w \leq 0$ in Ω^c where $w = u - v$. Note that, moving cw on the RHS, we can take $f = 0$ on $\{w > 0\}$ in Theorem 2.2.1. Then applying Theorem 2.2.1, we obtain $w \leq 0$ in $\mathbb{R}^d$. Hence the proof. $\square$

Next we prove the Hopf's lemma. At this point, we mention a recent interesting work of Klimsiak and Komorowski [116] where an abstract Hopf's type lemma is obtained for the semigroup solutions of a general integro-differential operator.

Theorem 2.2.2 (Hopf's lemma)**.** *Let $Lu + cu \leq 0$ in Ω where $u \in C_b(\mathbb{R}^d)$ and $c \in C_b(\bar{\Omega})$. Suppose that $u > 0$ in Ω and non-negative in $\mathbb{R}^d$. Then there exists $\eta > 0$ such that for any $x_0 \in \partial\Omega$ with $u(x_0) = 0$ we have*

$$\frac{u(x)}{(r - |x - z|)} \geq \eta,$$

for all $x \in B_r(z) \cap B_{\frac{r}{2}}(x_0)$, where $B_r(z) \subset \Omega$ is a ball that touches $\partial\Omega$ at x_0.

Proof. Since $u > 0$ in Ω, without any loss of generality, we may assume that $c \leq 0$. Let $K = B_r(z) \cap B_{\frac{r}{2}}(x_0)$ and define

$$v(x) = \mathrm{e}^{-\alpha q(x)} - \mathrm{e}^{-\alpha r^2},$$

where $q(x) = |z - x|^2 \wedge 9r^2$. Clearly $v > 0$ in $B_r(z)$, $v(x) = 0$ on $\partial B_r(z)$, and $v \leq 0$ in $\mathbb{R}^d \setminus B_r(z)$. For $x \in B_{2r}(z)$ we have

$$\Delta v = \alpha \mathrm{e}^{-\alpha|x-z|^2}\left(4\alpha|x - z|^2 - 2d\right).$$

Fix any $x \in B_{2r}(z)$. Using the convexity of $x \mapsto \mathrm{e}^x$ we first note that, for $|y| \leq 1$,

$$
\begin{aligned}
& v(x+y) - v(x) - \mathbb{1}_{\{|y|\leq 1\}} y \cdot Dv(x) \\
&\quad = \mathrm{e}^{-\alpha|x+y-z|^2} - \mathrm{e}^{-\alpha|x-z|^2} + 2\alpha \mathbb{1}_{\{|y|\leq 1\}} y \cdot (x-z)\mathrm{e}^{-\alpha|x-z|^2} \\
&\quad \geq -\alpha \mathrm{e}^{-\alpha|x-z|^2}\Big(|x+y-z|^2 - |x-z|^2 - 2\mathbb{1}_{\{|y|\leq 1\}} y \cdot (x-z)\Big) \\
&\quad \geq -\alpha \mathrm{e}^{-\alpha|x-z|^2}|y|^2.
\end{aligned}
$$

Therefore, for $x \in B_{2r}(z)$, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^d} (v(x+y) - v(x) - \mathbb{1}_{\{|y|\leq 1\}} y \cdot Dv(x)) j(y) \mathrm{d}y \\
&\quad = \int_{|y|\leq 1} (v(x+y) - v(x) - \mathbb{1}_{\{|y|\leq 1\}} y \cdot Dv(x)) j(y) \mathrm{d}y \\
&\qquad + \int_{|y|>1} (v(x+y) - v(x)) j(y) \mathrm{d}y \\
&\quad \geq -\alpha \mathrm{e}^{-\alpha|x-z|^2} \int_{|y|\leq 1} |y|^2 j(y)\,\mathrm{d}y + \int_{|y|>1} (v(x+y) - v(x)) j(y) \mathrm{d}y \\
&\quad \geq -\alpha \mathrm{e}^{-\alpha|x-z|^2} \int_{|y|\leq 1} |y|^2 j(y)\,\mathrm{d}y + \mathrm{e}^{-\alpha|x-z|^2} \int_{|y|>1} \Big(\mathrm{e}^{-9\alpha r^2} - 1\Big) j(y)\,\mathrm{d}y,
\end{aligned}
$$

where in the last line we used $|x-z+y|^2 \wedge 9r^2 \leq |x-z|^2 + 9r^2$. Thus, using (2.0.2) and $x \in B_{2r}(z)$, we obtain

$$
\begin{aligned}
& Lv(x) + c(x)v(x) \\
&\quad \geq \alpha \mathrm{e}^{-\alpha|x-z|^2}\Big[4\alpha|x-z|^2 - 2d - \int_{|y|\leq 1} |y|^2 j(y) \mathrm{d}y + \alpha^{-1} \int_{|y|>1} \Big(\mathrm{e}^{-9\alpha r^2} - 1\Big) j(y)\,\mathrm{d}y \\
&\qquad - \|c\|_{L^\infty(\Omega)} \alpha^{-1}(1 - \mathrm{e}^{-\alpha(r^2 - |x-z|^2)})\Big].
\end{aligned}
$$

For $|x-z| \geq r/2$, we can choose α large enough so that

$$
Lv + cv > 0 \quad \text{for } \frac{r}{2} \leq |x-z| < 2r. \tag{2.2.3}
$$

Let $m = \min_{\Omega^-_{\frac{r}{2}}} u$ where $\Omega^-_{\frac{r}{2}} = \{y \in \Omega \ : \ \mathrm{dist}(y, \partial\Omega) \geq \frac{r}{2}\}$. Defining $w = mv$, we have $L(w-u) + c(w-u) \geq 0$ in $B_r(z) \setminus B_{r/2}(z)$, by Theorem 2.1.1, and $w - u \leq 0$

in $(B_r(z) \setminus B_{r/2}(z))^c$. Thus as a consequence of Corollary 2.2.2, we obtain

$$\begin{aligned} u &\geq mv \\ &= m\mathrm{e}^{-\alpha r^2}(\mathrm{e}^{\alpha(r^2-|x-z|^2)} - 1) \\ &\geq m\mathrm{e}^{-\alpha r^2}\alpha(r^2 - |x-z|^2). \end{aligned}$$

This completes the proof. □

As a by-product of the proof of Hopf's lemma above we obtain a strong maximum principle (compare with Ciomaga [63]).

Theorem 2.2.3 (Strong maximum principle). *Assume (A1) and let $c \in C_b(\Omega)$. Let $Lu + cu \leq 0$ in Ω and $u \in C_b(\mathbb{R}^d)$ be non-negative in $\mathbb{R}^d$. Then either $u > 0$ in Ω or it is identically 0 in Ω.*

Proof. Since u is non-negative, without any loss of generality, we may assume that $c \leq 0$. If we take $K = \overline{\{u = 0\} \cap \Omega}$, then we want to show that $K \cap \Omega$ is either empty or Ω. Suppose, to the contrary, $K \cap \Omega$ is non-empty and is not equal to the set Ω. This means $\Omega \setminus K$ is also non-empty. Hence we can find a point $z \in \Omega \setminus K$ and r small, such that $x_0 \in \partial B_r(z)$ for some $x_0 \in K \cap \Omega$, $B_{2r}(z) \subset \Omega$ and $u > 0$ in $B_r(z)$.
Now we consider the function v that we constructed in Theorem 2.2.2, that is

$$v(x) = \mathrm{e}^{-\alpha q(x)} - \mathrm{e}^{-\alpha r^2},$$

where $q(x) = |z - x|^2 \wedge 9r^2$. Again we have $v > 0$ in $B_r(z)$, $v(x) = 0$ on $\partial B_r(z)$, and $v \leq 0$ in $\mathbb{R}^d \setminus B_r(z)$. Also by (2.2.3) we have

$$Lv + cv > 0 \quad \text{in} \quad B_{2r}(z) \setminus B_{r/2}(z). \tag{2.2.4}$$

Let $m = \min_{B_{\frac{r}{2}}(z)} u$ and define $w = mv$. Then following similar argument as in Theorem 2.2.2 we have $u \geq w$ in $\mathbb{R}^d$. Since $w \in C^2(B_{2r}(z))$ and $w(x_0) = u(x_0) = 0$, we use w as a test function and define

$$\phi(y) := \begin{cases} w(y) & \text{for } y \in B_r(x_0), \\ u(y) & \text{otherwise.} \end{cases}$$

Then by the definition of viscosity supersolution we have

$$L\phi(x_0) + c(x_0)\phi(x_0) \leq 0.$$

Clearly, $w \leq \phi$ in $\mathbb{R}^d$, $D\phi(x_0) = Dw(x_0)$ and $\Delta\phi(x_0) = \Delta w(x_0)$. Thus $Lw(x_0) + c(x_0)w(x_0) \leq 0$ which contradicts the fact that $Lw(x_0)+c(x_0)w(x_0) > 0$ (see (2.2.4)). Hence $K \cap \Omega$ is either empty or Ω. This completes the proof. □

2.3 Principal eigenvalue and Faber-Krahn inequality

In this section, we study the generalized eigenvalue problem of L and then we establish a Faber-Krahn inequality. In view of Theorems 2.1.2 and 2.2.1 we can define a generalized principal eigenvalue for L in the spirit of Berestycki, Nirenberg and Varadhan [16]. By $C_+(\Omega)$ ($C_{b,+}(\mathbb{R}^d)$) we denote the set of all positive (bounded and non-negative) continuous functions in Ω (in $\mathbb{R}^d$, respectively). Given any bounded domain Ω, the (Dirichlet) generalized principal eigenvalue of L in Ω is defined to be

$$\lambda_\Omega = \sup\{\lambda \,:\, \exists\, v \in C_+(\Omega) \cap C_{b,+}(\mathbb{R}^d) \text{ satisfying } Lv + cv + \lambda v \leq 0 \text{ in } \Omega\},$$

where $c \in C_b(\Omega)$.

We begin with the following boundary estimate which will be useful. We will show that a solution to the Dirichlet problem behaves like a distance function to $\partial\Omega$ near the boundary of Ω.

Lemma 2.3.1. *Assume (A1). Let Ω be a bounded domain satisfying a uniform exterior sphere condition with radius $r > 0$. Let $u \in C_b(\mathbb{R}^d)$ be a viscosity solution to*

$$Lu = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{in} \quad \Omega^c,$$

for some $f \in L^\infty(\Omega)$. Then there exists a constant C, dependent on $r, d, \mathrm{diam}(\Omega)$, satisfying

$$|u(x)| \leq C\|f\|_{L^\infty(\Omega)}\, \mathrm{dist}(x, \partial\Omega) \quad \text{for } x \in \Omega.$$

Proof. Let B be ball containing Ω. Then $v(x) = \mathbb{E}_x[\uptau_B]$ solves (see Theorem 2.1.2)

$$Lv = -1 \quad \text{in } B, \quad v = 0 \quad \text{in } B^c.$$

Applying coupling property, Theorem 2.1.1 and Lemma 2.1.1, it then follows that

$$|u(x)| \le \|f\|_{L^\infty(\Omega)} v(x), \quad x \in \mathbb{R}^d. \tag{2.3.1}$$

Without any loss of generality we may assume $r \in (0, 1)$. By [131, Lemma 5.4] there exists a bounded, Lipschitz continuous function φ, with Lipschitz constant r^{-1}, satisfying

$$\begin{cases} \varphi = 0, & \text{in } \bar{B}_r, \\ \varphi > 0, & \text{in } \bar{B}_r^c, \\ \varphi \ge \varepsilon, & \text{in } B^c_{(1+\delta)r}, \\ L\varphi \le -1, & \text{in } B_{(1+\delta)r}, \end{cases}$$

for some constant ε, δ, where B_r denotes the ball of radius r around 0. Now for any point $y \in \partial\Omega$ we can find another point $z \in \Omega^c$ such that $\overline{B}_r(z)$ touches $\partial\Omega$ at y. Defining $w(x) = \varepsilon^{-1}\|f\|_{L^\infty(\Omega)}\|v\|_{L^\infty(\Omega)}\varphi(x - z)$ and using (2.3.1) it follows that $|u(x)| \le w(x)$ in $B_{(1+\delta)r}(z) \cap \Omega$, by Theorem 2.1.1 and Lemma 2.1.1. This relation holds for any $y \in \partial\Omega$. Now for any point $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \delta r$ we can find $y \in \partial\Omega$ satisfying $\operatorname{dist}(x, \partial\Omega) = |x - y| < \delta r$. By the previous estimate we obtain

$$\begin{aligned} |u(x)| \le \varepsilon^{-1}\|f\|_{L^\infty(\Omega)}\|v\|_{L^\infty(\Omega)}\varphi(x - z) &\le \varepsilon^{-1}\|f\|_{L^\infty(\Omega)}\|v\|_{L^\infty(\Omega)}(\varphi(x - z) - \varphi(y - z)) \\ &\le \varepsilon^{-1}\|f\|_{L^\infty(\Omega)}\|v\|_{L^\infty(\Omega)}\, r^{-1} \operatorname{dist}(x, \partial\Omega). \end{aligned}$$

Now the proof follows from (2.3.1). $\square$

Now fix a Lipschitz domain Ω satisfying a uniform exterior sphere condition. In view of Theorem 2.1.2 we can define a map $\mathcal{T} : C(\bar{\Omega}) \to C_0(\Omega)$ where $C_0(\Omega)$ denotes the space of continuous function in $\bar{\Omega}$ vanishing on the boundary, as follows: $\mathcal{T}[f] = u$ where u is the unique viscosity solution to

$$Lu = -f \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{in} \quad \Omega^c.$$

Lemma 2.3.2. *Under (A1), the map $\mathcal{T}$ is a bounded linear, compact operator.*

Proof. It is evident that $\mathcal{T}$ is a bounded linear operator. So we only need to show that $\mathcal{T}$ is compact. Let $\mathcal{K}$ be a bounded subset of $C(\bar{\Omega})$, that is, there exists a constant κ such that $\|f\|_\infty \leq \kappa$ for all $f \in \mathcal{K}$. Define $\mathcal{G} = \{u \,:\, u = \mathcal{T}[f] \text{ for some } f \in \mathcal{K}\}$. By Lemma 2.3.1, $\mathcal{G}$ is bounded in $C_0(\Omega)$. Let us now show that $\mathcal{G}$ is also equicontinuous. Consider $\varepsilon > 0$. For $\delta_1 > 0$ let us define

$$\Omega^-_{\delta_1} = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > \delta_1\}.$$

Using Lemma 2.3.1, we can then choose $\delta_1 > 0$ small enough to satisfy

$$\sup_{\Omega\setminus\Omega^-_{2\delta_1}} |u(x)| < \varepsilon/2 \quad \text{for all } u \in \mathcal{G}. \tag{2.3.2}$$

Again, by [131, Theorem 4.2], there exists $\alpha > 0$ satisfying

$$\sup_{x\neq y, x,y\in\Omega^-_{\delta_1}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq \kappa_2 \quad \text{for all } u \in \mathcal{G}, \tag{2.3.3}$$

for some constant κ_2. Choose $\delta \in (0, \delta_1)$ satisfying $\kappa_2\delta^\alpha < \varepsilon$. Then, from (2.3.2) and (2.3.3), we obtain

$$|u(x) - u(y)| < \varepsilon \quad \text{for all } u \in \mathcal{G}\ x, y \in \Omega, \ \ |x-y| \leq \delta.$$

Therefore $\mathcal{G}$ is equicountinuous. Hence by Arzelà-Ascoli theorem we have $\mathcal{G}$ compact, completing the proof. □

From Lemma 2.3.2 and Corollary 2.2.2 we get the following existence result.

Lemma 2.3.3. *Grant (A1) and let Ω be a bounded Lipschitz domain satisfying a uniform exterior sphere condition. Suppose $c \in C(\bar{\Omega})$ with $c \leq 0$. Then for any $f \in C(\bar{\Omega})$ there exists a unique solution u to*

$$Lu + cu = -f \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{in } \Omega^c. \tag{2.3.4}$$

Proof. Fixing $f \in C(\bar{\Omega})$, we define a map $F : C_0(\Omega) \to C_0(\Omega)$ as

$$u = F(v) = \mathcal{T}[f + cv],$$

where $\mathcal{T}$ is same as in Lemma 2.3.2. From Lemma 2.3.2 it follows that F is continuous and compact. Consider a set

$$E = \{v \in C_0(\Omega) : v = \lambda F(v) \quad \text{for some } \lambda \in [0,1]\}.$$

Claim: E is bounded in $C_0(\Omega)$.

Suppose, to the contrary, that E is not bounded. Then there exist $v_n \in E$, for $n \in \mathbb{N}$, such that $\|v_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$. So we have tuples (v_n, λ_n) satisfying $v_n = \lambda_n F(v_n)$ which gives us

$$Lv_n = \lambda_n(-f - cv_n) \quad \text{in } \Omega.$$

Letting $w_n = \|v_n\|^{-1}_{L^\infty(\Omega)} v_n$ we get from above that

$$Lw_n = \lambda_n \frac{-f}{\|v_n\|_{L^\infty(\Omega)}} - c\lambda_n w_n.$$

Since $\|w_n\|_{L^\infty(\Omega)} = 1$ for all n we see that

$$\|\lambda_n \frac{f}{\|v_n\|_{L^\infty(\Omega)}} + c\lambda_n w_n\|_{L^\infty(\Omega)} \leq \kappa_2,$$

for some $\kappa_2 > 0$. Therefore, using Lemma 2.3.2, $\{w_n : n \geq 1\}$ is equicontinuous and hence, up to a subsequence, $w_n \to w$ in $C_0(\Omega)$. By the stability property of the viscosity solutions we obtain

$$Lw = 0 - c\lambda w \quad \text{in } \Omega \quad \text{and} \quad w = 0 \quad \text{in } \Omega^c,$$

for some $\lambda \in [0,1]$. Hence w solves the equation

$$\begin{aligned} Lw + c\lambda w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{in } \Omega^c. \end{aligned}$$

From Corollary 2.2.2 we see that $w = 0$ in $\mathbb{R}^d$. But this contradicts the fact that $\|w\|_{L^\infty(\Omega)} = 1$, and this proves our claim.

Applying Leray-Schauder theorem we must have a fixed point of F. This gives the existence of solution for (2.3.4). Uniqueness again follows from the above arguments. □

Now we are ready to prove the existence of principal eigenfunction.

Theorem 2.3.1. *Assume (A1) and let Ω be a bounded domain satisfying a uniform exterior sphere condition. Let $c \in C(\bar{\Omega})$. There exists a unique $\psi_\Omega \in C_b(\mathbb{R}^d)$, satisfying*

$$\begin{aligned} L\psi_\Omega + c\psi_\Omega &= -\lambda_\Omega \psi_\Omega \quad \textit{in } \Omega, \\ \psi_\Omega &= 0 \quad \textit{in } \Omega^c, \\ \psi_\Omega &> 0 \quad \textit{in } \Omega, \quad \psi_\Omega(0) = 1. \end{aligned}$$

Moreover, if $u \in C_{b,+}(\mathbb{R}^d)$ is positive in Ω and satisfies

$$Lu + cu \leq -\lambda u \quad \textit{in} \quad \Omega,$$

for some $\lambda \in \mathbb{R}$ then $\lambda \leq \lambda_\Omega$. Furthermore, if $\lambda = \lambda_\Omega$ and $u = 0$ in Ω^c, then we have $u = k\psi_\Omega$ for some $k > 0$. Furthermore, λ_Ω is the only Dirichlet eigenvalue with a positive eigenfunction.

Proof. The proof technique is quite standard and follows by combining Kreĭn-Rutman theorem with Lemmas 2.3.2 and 2.3.3. Replacing c by $c - \|c\|_{L^\infty(\Omega)}$ we can assume that $c \leq 0$. Using Lemma 2.3.3 we define a map $\mathcal{T}_1 : C_0(\Omega) \to C_0(\Omega)$ as follows: $\mathcal{T}_1[u] = v$ if and only if

$$Lv + cv = -u \quad \text{in } \Omega, \quad \text{and} \quad v = 0 \quad \text{in } \Omega^c.$$

Since, by coupling property Theorem 2.1.1 and Corollary 2.2.2, we have $\|v\| \leq \|u\| \max\{\|w_+\|, \|w_-\|\}$ where

$$Lw_\pm + cw_\pm = \pm 1 \quad \text{in } \Omega, \quad \text{and} \quad w_\pm = 0 \quad \text{in } \Omega^c,$$

it follows from Lemma 2.3.2 that $\mathcal{T}_1$ is a compact, bounded linear map. Again, if $u_1 \leq u_2$ in $C_0(\Omega)$, by coupling property Theorem 2.1.1 and Corollary 2.2.2, we get $\mathcal{T}_1[u_1] \leq \mathcal{T}_1[u_2]$. Furthermore, if $u_1 \lneq u_2$, then $\mathcal{T}_1[u_1] < \mathcal{T}_1[u_2]$ in Ω by Theorem 2.2.3. Let $f \gneq 0$ be a nonzero compactly supported function in Ω. Then, for $\mathcal{T}_1[f] = v$, we have $v > 0$ in Ω and therefore, we can find $M > 0$ satisfying $M\mathcal{T}_1[f] > f$ in Ω. Denote by $\mathcal{P}$ the cone of non-negative functions in $C_0(\Omega)$. From Theorem 2.2.3, it is easily seen that $\mathcal{T}_1(\mathcal{P}) \subset \mathcal{P}$. Therefore, Kreĭn-Rutman applies to $\mathcal{T}_1$ and we find $\lambda_\Omega > 0$ and $\psi_D \in C_0(\Omega)$ with $\psi_\Omega > 0$ in Ω satisfying

$$\begin{aligned} L\psi_\Omega + c\psi_\Omega + \lambda_\Omega\psi_\Omega &= 0 \quad \text{in } \Omega, \\ \psi_\Omega &= 0 \quad \text{in } \Omega^c. \end{aligned} \tag{2.3.5}$$

Now we focus on the second part of the theorem. Consider a non-negative function $u \in C_b(\mathbb{R}^d) \cap C_+(\Omega)$ satisfies

$$Lu + cu + \lambda u \leq 0 \quad \text{in } \Omega,$$

for some $\lambda \in \mathbb{R}$. Suppose, to the contrary, that $\lambda > \lambda_\Omega$. Then using Corollary 2.2.1, Theorem 2.2.3 and the proof of [34, Theorem 3.2] we find $\mathfrak{z} > 0$ satisfying $u = \mathfrak{z}\psi_\Omega$ in Ω. Since minimum of two viscosity supersolutions is also a supersolution, we have $\mathfrak{z}\psi_\Omega = \min\{u, \mathfrak{z}\psi_\Omega\}$ and

$$L(\mathfrak{z}\psi_\Omega) + (c + \lambda)(\mathfrak{z}\psi_\Omega) \leq 0 \quad \text{in } \Omega.$$

Applying Theorem 2.1.1, we see that $(\lambda - \lambda_\Omega)\psi_\Omega \leq 0$ in Ω which is a contradiction. Hence $\lambda \leq \lambda_\Omega$. Rest of the proof follows from [34, Theorem 3.2]. □

For some recent works dealing with generalized eigenvalue problems of integro-differential operator we refer [27, 34, 142]. Our next aim is to prove Faber-Krahn inequality and to do so we need certain continuity property of the principal eigenvalue with respect to the domains. To do so we need the following condition.

(A2) The domain Ω is Lipschitz and bounded with uniform exterior sphere condition of radius r. Furthermore, there exists a collection of bounded, Lipschitz decreasing domains $\{\Omega_n\}$ such that $\cap_n\Omega_n = \bar{\Omega}$ and each Ω_n satisfies uniform exterior sphere condition of radius r.

It can be easily seen that convex domains, $C^{1,1}$ domains satisfy the above condition. In the next lemma we prove the result on continuity of λ_Ω.

Lemma 2.3.4. *Assume (A1) and (A2). Denote by $\lambda_n = \lambda_{\Omega_n}$. Then $\lambda_n \to \lambda_\Omega$ as $n \to \infty$.*

Proof. From Theorem 2.3.1 we notice that $\lambda_n \leq \lambda_{n+1}$. Let $\lim_{n\to\infty} \lambda_n = \lambda$. Evidently, $\lambda \leq \lambda_\Omega$. Using condition (A2), Lemma 2.3.1 and the proof of Lemma 2.3.2 it follows that $\{\psi_n\}$ is equicontinuous in $\mathbb{R}^d$ where ψ_n is the principal eigenfunction corresponding to λ_n. We also normalize ψ_n to satisfy $\|\psi_n\|_{L^\infty(\mathbb{R}^d)} = 1$. Applying Arzelá-Ascoli we can extract a subsequence of ψ_n converging to ψ and by the stability property of the viscosity solution we obtain

$$L\psi + (c+\lambda)\psi = 0 \quad \text{in } \Omega, \quad \text{and} \quad \psi = 0 \quad \text{in } \Omega^c.$$

Since $\psi \gneq 0$, by strong maximum principle Theorem 2.2.3, we must have $\psi > 0$ in Ω. Then, by Theorem 2.3.1, we must have $\lambda = \lambda_\Omega$. Hence the proof. $\square$

Next we find a representation of the principal eigenvalue which is crucial for the proof of Faber-Krahn inequality.

Lemma 2.3.5. *Consider the setting of Lemma 2.3.4 and let $c = 0$. Let λ_Ω be the corresponding principal eigenvalue. Then*

$$\lambda_\Omega = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x(\tau > t) \quad \textit{for all } x \in \Omega. \tag{2.3.6}$$

Proof. The principal eigenpair $(\psi_\Omega, \lambda_\Omega)$ satisfies

$$L\psi_\Omega + \lambda_\Omega\psi_\Omega = 0 \quad \text{in } \Omega, \quad \text{and} \quad \psi_\Omega = 0 \quad \text{in } \Omega^c.$$

From the arguments of [28, Lemma 3.1] we then have

$$\psi_\Omega(x) = \mathbb{E}_x\left[e^{\lambda_\Omega t}\psi_\Omega(X_t)\mathbb{1}_{\{\tau>t\}}\right], \quad x \in \Omega. \tag{2.3.7}$$

Using (2.3.7), Lemma 2.3.4 and the proof of [32, Corollary 4.1] we get (2.3.6). This completes the proof. □

Now we are ready to prove the Faber-Krahn inequality.

Theorem 2.3.2 (Faber-Krahn inequality)**.** *Let $z \mapsto j(z)$ be isotropic and radially decreasing. Let Ω be any bounded domain satisfying $|\partial\Omega| = 0$. Then*

$$\lambda_\Omega \geq \lambda_B, \tag{2.3.8}$$

where B is ball around 0 satisfying $|B| = |\Omega|$.

Proof. By the assumption on j, (A1) holds. We note that $p_t(x, y) = p_t(y-x)$ where

$$\int_{\mathbb{R}^d} e^{i\xi\cdot x} p_t(x)\mathrm{d}x = e^{-t(|\xi|^2+\psi(\xi))}.$$

From [162] we know that p_t is isotropic unimodal, that is, p_t is radially decreasing. We first assume that Ω is a smooth domain. Without any loss of generality we may also assume $0 \in \Omega$. Then by Markov property

$$\begin{aligned}
\mathbb{P}_0(\tau > t) &= \lim_{m\to\infty} \mathbb{P}_0(X_{\frac{t}{m}} \in \Omega, X_{\frac{2t}{m}} \in \Omega, \ldots, X_{\frac{mt}{m}} \in \Omega) \\
&= \lim_{m\to\infty} \int_\Omega \int_\Omega \cdots \int_\Omega p_{\frac{t}{m}}(x_1) p_{\frac{t}{m}}(x_2 - x_1) \cdots p_{\frac{t}{m}}(x_m - x_{m-1}) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_m \\
&\leq \lim_{m\to\infty} \int_B \int_B \cdots \int_B p_{\frac{t}{m}}(x_1) p_{\frac{t}{m}}(x_2 - x_1) \cdots p_{\frac{t}{m}}(x_m - x_{m-1}) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_m \\
&= \mathbb{P}_0(\tau_B > t),
\end{aligned}$$

where in the third line we used Brascamp-Lieb-Luttinger inequality [44, Theorem 3.4]. Therefore,

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_0(\tau > t) \geq -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_0(\tau_B > t).$$

Applying Lemma 2.3.5 we then have

$$\lambda_\Omega \geq \lambda_B\,. \tag{2.3.9}$$

Now given a bounded domain Ω with $|\partial\Omega| = 0$ we consider a decreasing sequence of smooth domains Ω_n such that $\cap_{n\geq 1}\Omega_n = \bar{\Omega}$ and $|\Omega_n| \to |\Omega|$ as $n \to \infty$. Let B_n be a ball centered at 0 and $|B_n| = |\Omega_n|$. It is also easily seen that B and $\{B_n\}$ satisfies condition (A2). Using (2.3.9) and monotonicity of λ_Ω with respect to domains, we get that

$$\lambda_\Omega \geq \lambda_{\Omega_n} \geq \lambda_{B_n}.$$

Now let $n \to \infty$ and apply Lemma 2.3.4 to conclude

$$\lambda_\Omega \geq \lambda_B.$$

Hence the proof. $\square$

As well-known Faber-Krahn inequality was first proved independently by Faber [79] and Krahn [119] for the Laplacian. See also [102, Chapter 2]. Very recently, Biagi et. al. [19] establish Faber-Krahn inequality for the operator $-\Delta + (-\Delta)^s$ for $s \in (0, 1)$. Their method uses Schwarz symmetrization combined with the Polya-Szegö inequality and [91, Theorem A.1]. Since the inequality in [91, Theorem A.1] holds for a more general class of kernel j, it might be possible to mimic the proof of [19] in an appropriate variational set-up and Sobolev space to establish (2.3.8). However, our viscosity solution approach does not impose any additional regularity on the solution. We found a probabilistic representation of the principal eigenvalue using Theorem 2.3.1 which together with the Brascamp-Lieb-Luttinger inequality gave us Theorem 2.3.2. Also, note that our condition on j is very general and our proof works in dimension one.

2.4 Symmetry of positive solutions and Gibbons' problem

One useful application of ABP maximum principle and Theorem 2.1.1 is to study symmetry properties of the positive solutions of semilinear equations. Thanks to Theorem 2.1.1, symmetry of the positive solutions can be established using the standard method of moving plane [34, 88, 98]. Another interesting application of

Theorem 2.1.1 is the one-dimensional symmetry result related to the Gibbons' conjecture.

Theorem 2.4.1. *Assume (A1). Also, assume that j is radially decreasing in $\mathbb{R}^d \setminus \{0\}$ and strictly decreasing in a neighbourhood of 0. Suppose that Ω is convex in the direction of the x_1 axis, and symmetric about the plane $\{x_1 = 0\}$. Also, let $f : [0, \infty) \to \mathbb{R}$ be locally Lipschitz continuous. Consider any solution $u \in C_b(\mathbb{R}^d)$ of*

$$\begin{aligned} Lu = f(u) &\quad \text{in } \Omega, \\ u = 0 &\quad \text{in } \Omega^c, \\ u > 0 &\quad \text{in } \Omega. \end{aligned}$$

Then u is symmetric with respect to $x_1 = 0$ and strictly decreasing in the x_1 direction.

Proof. We use the method of moving plane appeared in the seminal work Gidas, Ni and Nirenberg [98] which was motivated by a work of Serrin [152]. The proof can be easily completed following the arguments of [88]. We provide proof for the sake of completeness. Define

$$\Sigma_\lambda = \left\{ x = (x_1, x^{'}) \in \Omega : x_1 > \lambda \right\} \quad \text{and} \quad T_\lambda = \left\{ x = (x_1, x^{'}) \in \mathbb{R}^d : x_1 = \lambda \right\},$$
$$u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x),$$

where $x_\lambda = (2\lambda - x_1, x^{'})$. For a set A we denote by $\mathscr{R}_\lambda A$ the reflection of A with respect to the plane T_λ. Also, define

$$\lambda_{\max} = \sup \{ \lambda > 0 : \Sigma_\lambda \neq \emptyset \}.$$

We note that for any $\lambda \in (0, \lambda_{\max})$, u_λ is a viscosity solution of

$$Lu_\lambda = f(u_\lambda) \quad \text{in } \Sigma_\lambda,$$

and therefore, from Theorem 2.1.1 we obtain

$$Lw_\lambda = f(u_\lambda) - f(u) \quad \text{in } \Sigma_\lambda.$$

Define $\Sigma_\lambda^- = \{x \in \Sigma_\lambda : w_\lambda < 0\}$. Since $w_\lambda \geq 0$ on $\partial\Sigma_\lambda$, it follows that $w_\lambda = 0$ on $\partial\Sigma_\lambda^-$. Hence the function

$$v_\lambda = \begin{cases} w_\lambda & \text{in } \Sigma_\lambda^-, \\ 0 & \text{elsewhere,} \end{cases}$$

is in $C_b(\mathbb{R}^d)$. We claim that for every $\lambda \in (0, \lambda_{\max})$

$$Lv_\lambda \leq f(u_\lambda) - f(u) \quad \textit{in} \quad \Sigma_\lambda^-, \tag{2.4.1}$$

in the viscosity sense. To see this, let φ be a test function that touches v_λ from below at a point $x \in \Sigma_\lambda^-$. Then $\varphi + (w_\lambda - v_\lambda) \in C_b(x)$ and touches w_λ at x from below. Denote $\zeta_\lambda(x) = w_\lambda - v_\lambda$. It then follows that

$$L(\phi + \zeta_\lambda)(x) \leq f(u_\lambda(x)) - f(u(x)).$$

Since $\zeta_\lambda = 0$ in a small neighbourhood of x in Σ_λ^- we have $\Delta\zeta_\lambda(x) = 0$. Again, since j is radial, to show (2.4.1) it is enough to show that

$$\int_{\mathbb{R}^d} (\zeta_\lambda(x+z) - \zeta_\lambda(x)) j(z) \mathrm{d}z \geq 0.$$

This can be done by following the argument of [88, p. 8] and the fact j is radially decreasing.

If $\lambda < \lambda_{\max}$ is sufficiently close to $\lambda_{\max}$, then $w_\lambda > 0$ in Σ_λ. Indeed, note that if $\Sigma_\lambda^- \neq \emptyset$, then v_λ satisfies (2.4.1). Denoting

$$c(x) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)},$$

it then follows that

$$Lv_\lambda - c(x)v_\lambda \leq 0 \quad \text{in} \quad \Sigma_\lambda^-.$$

Thus choosing λ sufficiently close to $\lambda_{\max}$ it follows from Corollary 2.2.1 that $v_\lambda \geq 0$ in $\mathbb{R}^d$. Hence $\Sigma_\lambda^- = \emptyset$ and we have a contradiction. To show that $w_\lambda > 0$ in Σ_λ, we assume to the contrary that $w_\lambda(x_0) = 0$ for some $x_0 \in \Sigma_\lambda$. Consider a non-negative test function $\varphi \in C_b(x_0)$ crossing w_λ from below with the property that $\varphi = 0$ in

$B_r(x_0) \Subset \Sigma_\lambda$ and $\varphi = w_\lambda$ in $B^c_{2r}(x_0)$. Furthermore, choose r small enough such that $B_{2r}(x_0) \Subset \Sigma_\lambda$ and $\varphi \geq 0$ in Σ_λ. Then we obtain

$$L\varphi(x_0) \leq 0. \tag{2.4.2}$$

Since $\Delta\varphi(x_0) = 0$ we get

$$I[\varphi](x_0) := \frac{1}{2}\int_{\mathbb{R}^d} (\varphi(x_0 + z) + \varphi(x_0 - z) - 2\varphi(x_0))j(z)\mathrm{d}z \leq 0.$$

Next we compute $I[\varphi](x_0)$. Note that $\varphi \geq 0$ in $R_\lambda := \left\{x \in \mathbb{R}^d \,:\, x_1 \geq \lambda\right\}$. We have

$$\begin{aligned}
I[\varphi](x_0) &= \int_{\mathbb{R}^d} \varphi(z)j(|x_0 - z|)\mathrm{d}z \\
&= \int_{R_\lambda} \varphi(z)j(|x_0 - z|)\mathrm{d}z + \int_{\mathscr{R}_\lambda R_\lambda} w_\lambda(z)j(|x_0 - z|)\mathrm{d}z \\
&= \int_{R_\lambda} \varphi(z)j(|x_0 - z|)\mathrm{d}z + \int_{R_\lambda} w_\lambda(z_\lambda)j(|x_0 - z_\lambda|)\mathrm{d}z \\
&= \int_{R_\lambda} \varphi(z)j(|x_0 - z|)\mathrm{d}z - \int_{R_\lambda} w_\lambda(z)j(|x_0 - z_\lambda|)\mathrm{d}z \\
&\geq \int_{R_\lambda \setminus B_{2r}(x_0)} w_\lambda(z)\left(j(|x_0 - z|) - j(|x_0 - z_\lambda|)\right)\mathrm{d}z \\
&\qquad - \int_{B_{2r}(x_0)} w_\lambda(z)j(|x_0 - z_\lambda|)\mathrm{d}z.
\end{aligned}$$

Since $|z_\lambda - x_0| > |z - x_0|$ and $j(|z_\lambda - x_0|) \geq j(|z - x_0|)$ thus the first term in the above expression is non-negative. In fact, since

$$\lim_{r\to 0} \int_{B_{2r}(x_0)} w_\lambda(z)j(|x_0 - z_\lambda|)\mathrm{d}z = 0,$$

using (2.4.2), we obtain

$$\lim_{r\to 0} \int_{R_\lambda \setminus B_{2r}(x_0)} w_\lambda(z)\left(j(|x_0 - z|) - j(|x_0 - z_\lambda|)\right)\mathrm{d}z \leq 0. \tag{2.4.3}$$

Since w_λ is continuous and j is strictly decreasing in a neighbourhood of 0, from (2.4.3) we get $w_\lambda = 0$ in $B_\delta(x_0)$ for some $\delta > 0$. Thus $\{w_\lambda = 0\} \cap \Sigma_\lambda$ is an open set.

Hence $\{w_\lambda = 0\}$ forms a connected component of Σ_λ which in turn, implies that $\{w_\lambda = 0\} \cap \partial\Sigma_\lambda \cap \partial\Omega \neq \emptyset$. This is a contradiction. Hence we must have $w_\lambda > 0$ in Σ_λ.

Now from the above argument and Step 2 in [88, p. 10] we can show that $\inf\{\lambda > 0 \ :\ w_\lambda > 0 \quad \text{in } \Sigma_\lambda\} = 0$. Also, strict monotonicity of u in the x_1 direction can be obtained by following the calculations in Step 3 of [88]. □

As a consequence of Theorem 2.4.1 we obtain.

Corollary 2.4.1. *Grant the setting of Theorem 2.4.1. Then every solution to*

$$Lu = f(u) \quad \textit{in } B_1(0), \quad u = 0 \quad \textit{in } B_1^c(0), \quad \textit{and} \quad u > 0 \quad \textit{in } B_1(0),$$

is radial and strictly decreasing in $|x|$.

The remaining part of this section is devoted to the Gibbons' problem. Let $u : \mathbb{R}^d \to \mathbb{R}$ be a solution to the problem

$$\begin{cases} Lu(x) = f(u(x)), & \text{for } x \in \mathbb{R}^d, \\ \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, & \text{uniformly for } \ x' \in \mathbb{R}^{d-1}. \end{cases} \tag{2.4.4}$$

We also suppose that $f \in C^1(\mathbb{R})$ satisfying

$$\inf_{|r| \geq 1} f'(r) > 0\,. \tag{2.4.5}$$

We show that u is one-dimensional.

Theorem 2.4.2. *Assume (A1). Let $u \in C_b(\mathbb{R}^d)$ solve (2.4.4) where f satisfies (2.4.5). Then there exists a strictly increasing function $u_0 : \mathbb{R} \to \mathbb{R}$ satisfying*

$$u(y,t) = u_0(t) \quad \textit{for all } y \in \mathbb{R}^d,\ t \in \mathbb{R}.$$

We need the following lemma to prove Theorem 2.4.2.

Lemma 2.4.1. *Let $w \in C_b(\mathbb{R}^d)$ satisfy*

$$Lw - c(x)w = 0 \quad \textit{in } \mathbb{R}^d,$$

with

$$w(x) \geq 0 \quad \text{in } \mathbb{R}^d \setminus U \text{ and } c(x) \geq \kappa \text{ in } U$$

for some open set $U \subseteq \mathbb{R}^d$ and some constant $\kappa > 0$. Also, assume that $c \in C_b(\mathbb{R}^d)$. Then

$$w(x) \geq 0 \;\; \text{for all } x \in \mathbb{R}^d.$$

Proof. Suppose, to the contrary, that $m = \inf_{\mathbb{R}^d} w < 0$. If m is attained then the proof can be completed from the maximum principle. In general, without loss of generality, we may choose a sequence $x_k \in \mathbb{R}^d$ satisfying

$$\lim_{k\to\infty} w(x_k) = m \quad \text{and} \quad w(x_k) \leq \frac{m}{2} < 0 \quad \forall\, k \in \mathbb{N}. \tag{2.4.6}$$

By given condition, for every $k \in \mathbb{N}$, we have

$$x_k \in U \quad \text{and} \quad c(x_k) \geq \kappa > 0.$$

By [131, Theorem 4.1] there exist $\hat{\kappa}, \alpha > 0$, dependent on $j, d, \|c\|_{L^\infty(\mathbb{R}^d)}$, such that

$$\sup_{x \neq y: x,y \in B_{1/2}(0)} \frac{|w(x) - w(y)|}{|x-y|^\alpha} \leq \hat{\kappa}\, \|w\|_{L^\infty(\mathbb{R}^d)}. \tag{2.4.7}$$

Translating the center of the ball it is evident from (2.4.7) that $w \in C^\alpha_b(\mathbb{R}^d)$. We note that for some $\delta > 0$ we have $\mathrm{dist}(x_k, U^c) > \delta$. Otherwise, along some subsequence, we must have $|x_k - z_k| \to 0$ as $k \to \infty$, for some $z_k \in U^c$. Since $w \in C^\alpha_b(\mathbb{R}^d)$ and $w \geq 0$ in U^c, we get

$$w(x_k) \geq w(z_k) - \hat{\kappa}|x_k - z_k|^\alpha \to 0, \quad \text{as } k \to \infty\,.$$

This is a contradiction to (2.4.6). Thus, we must find $\delta > 0$ so that $B_\delta(x_k) \Subset U$.

Let us now define $v_k(x) = w(x_k + x)$. Using (2.4.6) and (2.4.7), we restrict δ small enough so that

$$w(y) < \frac{m}{4} \quad \text{in } B_\delta(x_k), \quad \text{for all } k.$$

Thus, by the given condition on c, it follows that

$$Lv_k \leq \kappa \frac{m}{4} \quad \text{in } B_\delta(0). \tag{2.4.8}$$

Since $\{v_k\}$ forms a equicontinuous family, using Arzelà-Ascoli theorem, we can find a $v \in C_b(\mathbb{R}^d)$ satisfying $v_k \to v$ along some subsequence, uniformly over compacts. Using the stability property of viscosity supersolutions, it follows from (2.4.8) that

$$Lv \leq \kappa \frac{m}{4} \quad \text{in } B_\delta(0). \tag{2.4.9}$$

On the other hand

$$v(0) = \lim_{k\to\infty} v_k(0) = \lim_{k\to\infty} w(x_k) = m \leq \lim_{k\to\infty} w(x + x_k) = v(x),$$

for all $x \in \mathbb{R}^d$. Thus $x = 0$ is a minimum point for v in $\mathbb{R}^d$. Then $\varphi \equiv m$ is a bonafide test function at $x = 0$. From (2.4.9) we then obtain

$$0 > \frac{m}{4}\kappa \geq L\varphi(0) \geq 0,$$

which is a contradiction. Therefore, we must have $m \geq 0$ which completes the proof. □

Now we can complete the proof of Theorem 2.4.2.

Proof of Theorem 2.4.2. We broadly follow the idea of [24,87] without imposing any stronger regularity assumption on u. Fix a unit vector $\nu = (\nu_1, \ldots, \nu_d)$ such that $\nu_d > 0$ and we write $\nu = (\nu', \nu_d)$. We also define

$$\Gamma_h[u](x) = u(x + h\nu) - u(x).$$

We first show that $\Gamma_h[u](x) > 0$ for all $x \in \mathbb{R}^d$ and for all $h > 0$. Observe that

$$f(u(x + h\nu)) - f(u(x)) = c_h(x)\Gamma_h[u](x),$$

where

$$c_h(x) = \int_0^1 f'\left((1-t)u(x) + tu(x + h\nu)\right) \mathrm{d}t.$$

Using (2.4.5) we can choose $\delta \in (0, \frac{1}{2})$ such that $f' \geq \kappa_1$ in $(-\infty, -1+\delta] \cup [1-\delta, \infty)$ for some $\kappa_1 > 0$. Again, by (2.4.4), we may take $M > 0$ satisfying

$$u(x) \geq 1 - \delta \quad \text{for} \quad x_d \geq M, \quad \text{and} \quad u(x) \leq -1 + \delta \quad \text{for} \quad x_d \leq -M. \tag{2.4.10}$$

Claim: If $x \in \{\Gamma_h[u] < 0\} \cap \{|x_d| \geq M\}$, then $c_h(x) \geq \kappa_1$.

If $x \in \{\Gamma_h[u] < 0\} \cap \{x_d \leq -M\}$, then $u(x + h\nu) < u(x) \leq -1 + \delta$, by (2.4.10), which implies

$$(1-t)u(x) + tu(x + h\nu) \leq -1 + \delta,$$

for all $t \in [0, 1]$. Similarly, if $x \in \{\Gamma_h(u) < 0\} \cap \{x_d \geq M\}$, then $u(x) > u(x+h\nu) \geq 1 - \delta$, since $x_d + h\nu_d > x_d \geq M$. Thus

$$(1-t)u(x) + tu(x + h\nu) \geq 1 - \delta,$$

for all $t \in [0, 1]$. Hence we have $c_h(x) \geq \kappa_1$ for $x \in \{\Gamma_h[u] < 0\} \cap \{|x_d| \geq M\}$.

Next we claim that if $h \geq \frac{2M}{\nu_d}$, then $\Gamma_h[u](x) > 0$ for any $x \in \mathbb{R}^d$. Fix any $h > \frac{2M}{\nu_d}$ and let $U = \{\Gamma_h(u) < 0\}$. For $x_d = -M$ we have $x_d + h\nu_d \geq M$ and therefore,

$$\Gamma_h[u](x) \geq \inf_{x_d \geq M} u(x) - \sup_{x_d \leq -M} u(x) \geq 1 - \delta - (-1 + \delta) = 2(1-\delta) > 0.$$

Thus $U \subset \{x_d = -M\}^c$. We write $U = U^- \cup U^+$ where $U^- = U \cap \{x_d < -M\}$ and $U^+ = U \cap \{x_d > -M\}$. By above claim, $c_h(x) \geq \kappa_1$ for $x \in U^-$. Again, for $x \in U^+$ we have $x_d + h\nu_d > M$ by our choice of h and hence, we have $u(x) > u(x+h\nu) \geq 1-\delta$. Thus

$$(1-t)u(x) + tu(x + h\nu) \geq 1 - \delta,$$

for all $t \in [0, 1]$. Hence we have $c_h(x) \geq \kappa_1$ for all $x \in U$ and $\Gamma_h[u] \geq 0$ in U^c. Applying Theorem 2.1.1 we also have

$$L\Gamma_h[u] = c_h \Gamma_h[u] \quad \text{in } \mathbb{R}^d.$$

By Lemma 2.4.1 we then obtain $\Gamma_h[u] \geq 0$ in $\mathbb{R}^d$. We can apply strong maximum principle Theorem 2.2.3 to get $\Gamma_h(u)(x) > 0$ in $\mathbb{R}^d$, since $\Gamma_h[u] > 0$ on $\{x_d = -M\}$.

This proves the claim that $\Gamma_h[u] > 0$ in $\mathbb{R}^d$ and for $h \geq \frac{2M}{\nu_d}$. Define

$$h_\circ = \inf\{h > 0 \ : \ \Gamma_s[u](x) > 0 \text{ for all } x \in \mathbb{R}^d \text{ with } |x_d| \leq M, \text{ for all } s \geq h\}. \tag{2.4.11}$$

From the above argument we have $h_\circ \in [0, \frac{2M}{\nu_d}]$. We show that $h_\circ = 0$. Suppose, to the contrary, that $h_\circ > 0$. Then for any $\varepsilon \in (0, h_\circ)$

$$\Gamma_{h_\circ+\varepsilon}[u](x) = u(x + (h_\circ + \varepsilon)\nu) - u(x) > 0 \text{ for all } x \in \{|x_d| \leq M\},$$

and

$$\Gamma_{h_\circ-\varepsilon_k}[u](x_k) = u(x_k + (h_\circ - \varepsilon_k)\nu) - u(x_k) \leq 0 \text{ for some } x_k \in \{|x_d| \leq M\},$$

for any sequence $\varepsilon_k \to 0$ as $k \to \infty$. Since u is continuous, we obtain

$$\Gamma_{h_\circ}[u](x) = \lim_{\varepsilon \to 0} u(x + (h_\circ + \varepsilon)\nu) - u(x) \geq 0, \tag{2.4.12}$$

for any $x \in \{|x_d| \leq M\}$. Repeating an argument similar to above would give $\Gamma_{h_\circ}[u] \geq 0$ in $\mathbb{R}^d$. Now we define $w_k = u(x + x_k)$. Since $u \in C^\alpha_b(\mathbb{R}^d)$ by (2.4.7), $\{w_k\}$ is equicontinuous and therefore, $w_k \to w_\infty$ along some subsequence, uniformly on compacts. As a consequence,

$$\begin{aligned} c_k(x) := c_{h_\circ-\varepsilon_k}(x + x_k) &= \int_0^1 f'\left((1-t)w_k(x) + t w_k(x + (h_\circ - \varepsilon_k)\nu)\right) \mathrm{d}t \\ &\to \int_0^1 f'\left((1-t)w_\infty(x) + t w_\infty(x + h\nu)\right) \mathrm{d}t := c_\infty(x) \end{aligned}$$

uniformly over compacts. From the stability property of the viscosity solution, we then obtain

$$L\,\Gamma_{h_\circ}[w_\infty](x) = f(w_\infty(x + h_\circ\nu)(x)) - f(w_\infty(x)) = c_\infty(x)\Gamma_{h_\circ}[w_\infty](x) \quad \text{in } \mathbb{R}^d.$$

Again, by (2.4.12)

$$\Gamma_{h_\circ}[w_\infty](x) = \lim_{k \to \infty} \Gamma_{h_\circ}[u](x + x_k) \geq 0.$$

Also, from (2.4.7) we have

$$\begin{aligned}\Gamma_{h_\circ}[w_\infty](0) &= \lim_{k\to\infty} u(x_k + h_\circ\nu) - u(x_k) \\ &\le \lim_{k\to\infty} u(x_k + (h_\circ - \varepsilon_k)\nu) - u(x_k) + (\varepsilon_k)^\alpha \|u\|_{C^\alpha} \\ &\le 0.\end{aligned}$$

Hence $\Gamma_{h_\circ} w_\infty(0) = 0$. By strong maximum principle (Theorem 2.2.3) we must have $\Gamma_{h_\circ} w_\infty(x) = 0$ for all $x \in \mathbb{R}^d$. By a simple iteration this also gives

$$w_\infty(x + jh_\circ\nu) = w_\infty(x)$$

for any x and $j \in \mathbb{Z}$. Choosing $j \in \mathbb{N} \cap [\frac{2M}{h_\circ\nu_d}, \infty)$ we see that $jh_\circ\nu_d + (x_d)_k \ge M$ and $-jh_\circ\nu_d + (x_d)_k \le -M$ (since $x_k \in \{|x_d| \le M\}$) and therefore, by (2.4.10) we obtain $u(x_k + jh_\circ\nu) \ge 1 - \delta$ and $u(x_k - jh_\circ\nu) \le -1 + \delta$ for all k. Hence

$$\begin{aligned}2(1-\delta) &\le \lim_{k\to\infty} u(x_k + jh_\circ\nu) - u(x_k - jh_\circ\nu) \\ &= w_\infty(jh_\circ\nu) - w_\infty(-jh_\circ\nu) = 0,\end{aligned}$$

which is a contradiction. Thus $h_\circ$ in (2.4.11) must be 0. In other words, for any $h > 0$, $\Gamma_h[u] > 0$ in $\{|x_d| \le M\}$. Since $c_h(x) > \kappa_1$ for any $x \in \{\Gamma_h[u] < 0\} \cap \{|x_d| \ge M\}$, by the same argument as above we have $\Gamma_h[u] > 0$ in $\mathbb{R}^d$.

Thus we have proved that $\Gamma_h^\nu[u](x) \coloneqq u(x+h\nu) - u(x) \ge 0$ for all $\nu \in S^{d-1}$ with $\nu_d > 0$ and all $h \ge 0$. Taking $\mu = -\nu$ we obtain for all $h \ge 0$ that

$$\Gamma_h^\mu[u] \le 0 \quad \text{for all } x \in \mathbb{R}^d, \text{ and for all } \mu \in \mathbb{S}^{d-1} \text{ with } \mu_d < 0,$$

as

$$\begin{aligned}\Gamma_h^\mu[u](x) &= u(x + h\mu) - u(x) = u(\tilde{x}) - u(\tilde{x} + h(-\mu)) \\ &= -(u(\tilde{x} + h\nu) - u(\tilde{x})) = -\Gamma_h^\nu[u](\tilde{x}) \le 0\,,\end{aligned}$$

where $\tilde{x} = x + h\mu$. Now letting $\mu_d \nearrow 0$ and $\nu_d \searrow 0$ it follows from above that

$$\Gamma_h^\omega[u] = 0 \text{ for all } x \in \mathbb{R}^d, \text{ and for all } \omega \in \mathbb{S}^{d-1} \text{ with } \omega_d = 0.$$

In particular, this gives $\partial_{x_i} u = 0$ for $i = 1, 2, \ldots, d-1$. Again, u_0 is strictly increasing follows from (2.4.11) and the fact that $h_\circ = 0$. This completes the proof of the theorem.

□

The above problem is inspired by a conjecture of G. W. Gibbons [97] which was formulated for the classical Laplacian operator. The classical Gibbons' conjecture was proved by several researchers using different approaches; see for instance, [10, 15, 84]. In [87] Farina and Valdinoci prescribed a unified approach to this problem which also works for several other classes of operators. Using the approach of [87], a similar problem is treated in [24] for the operator $-\Delta + (-\Delta)^s$ with $s \in (0, 1)$. For the proof of Theorem 2.4.2 we also broadly follow the approach of [87] but, thanks to Theorem 2.1.1, we do not impose any additional regularity assumption on u.

3

Regularity theory of linear intergo-differential equation

This chapter will be focused on the study of the regularity theory of linear mixed local-nonlocal operators and its applications. We are interested in the integro-differential operator L of the form

$$Lu = \Delta u + Iu,$$

where I is a nonlocal operator given by

$$\begin{aligned} Iu(x) &= \int_{\mathbb{R}^d} \Big(u(x+y) - u(x) - \mathbb{1}_{\{|y| \le 1\}} y \cdot Du(x) \Big) k(y) \mathrm{d}y \\ &= \frac{1}{2} \int_{\mathbb{R}^d} (u(x+y) + u(x-y) - 2u(x)) k(y) \mathrm{d}y \end{aligned}$$

for some nonnegative, symmetric kernel k, that is, $k(y) = k(-y) \ge 0$ for all y. Here the second representation of the nonlocal operator I as an integration of symmetric difference of u around a point x times a kernel k is due to the fact that k is symmetric. As we have seen in Chapter 2, operator L appears as a generator of a Lévy process which is obtained by superimposing a Brownian motion, running twice as fast as standard n-dimensional Brownian motion, and an independent pure-jump Lévy process corresponding to the nonlocal operator I. Throughout this chapter, we impose the following assumptions on the kernel k.

Assumption 3.0.1.

(a) There exist constants $\alpha \in (0,2)$, $\Lambda > 0$ and a non-negative and measurable function $J \in L^1(B_1^c)$ such that

$$r^{d+\alpha} k(ry) \le \widehat{k}(y) = \frac{\Lambda}{|y|^{d+\alpha}} \mathbb{1}_{B_1}(y) + J(y) \mathbb{1}_{B_1^c}(y) \quad \forall\, y \in \mathbb{R}^d,\ r \in (0,1]$$

(b) There exists $\beta > 0$ such that for any $r \in (0,1]$ and $x_0 \in \mathbb{R}^d$ the following holds: for all $x, y \in B_{\frac{r}{2}}(x_0)$ and $z \in B_r^c(x_0)$ we have

$$k(x-z) \le \varrho\, k(y-z) \quad \text{for } |y-z| < \beta,$$

for some $\varrho > 1$.

Assumption 3.0.1(a) will be used to study certain scaled operators and to find the exact behavior of $I\delta(x)$ near the boundary where $\delta(x)$ denotes the distance function from Ω^c. Here scaled operator will be much more manageable due to the linear and translation-invariant nature of the operator L. Assumption 3.0.1(b) will be used to apply the Harnack estimate from [90]. It should be noted that [90] uses a stronger hypothesis compared to Assumption 3.0.1(b). Assumption 3.0.1 is satisfied by a large class of nonlocal kernels as shown in the examples below.

Example 3.0.1. The following class of nonlocal kernels satisfy Assumption 3.0.1.

(i) $k(y) = \frac{1}{|y|^{d+\alpha}}$ for some $\alpha \in (0,2)$. More generally, we may take $k(y) = \frac{1}{|y|^{d+\alpha}} \mathbb{1}_B(y)$ for some ball B centered at the origin.

(ii) $k(y) \asymp \frac{\Psi(|y|^{-2})}{|y|^d}$ where Ψ is Bernstein function vanishing at 0. In particular, Ψ is strictly increasing and concave. This class of nonlocal kernels correspond to a special class of Lévy processes, known as subordinate Brownian motions (see [150]). Assume that Ψ satisfies a global weak scaling property with parameters $\mu_1, \mu_2 \in (0,1)$, that is,

$$\lambda^{\mu_1} \Psi(s) \lesssim \Psi(\lambda s) \lesssim \lambda^{\mu_2} \Psi(s) \quad \text{for } s > 0, \lambda \ge 1.$$

Then it is easily seen that

$$r^{d+2\mu_2}k(ry) \asymp r^{2\mu_2}\frac{\Psi(|ry|^{-2})}{|y|^d} \lesssim \frac{\Psi(|y|^{-2})}{|y|^d} \lesssim \mathbb{1}_{B_1}(y)\frac{\Psi(1)}{|y|^{d+2\mu_2}} + \mathbb{1}_{B_1^c}(y)\frac{\Psi(1)}{|y|^{d+2\mu_1}}$$

for all $r \in (0,1]$. Thus Assumption 3.0.1(a) holds. Using the weak scaling property we can also check that Assumption 3.0.1(b) holds.

Let Ω be a bounded C^2 domain in $\mathbb{R}^d$. Let $u \in C(\mathbb{R}^d)$ be a viscosity solution to

$$\begin{aligned} Lu + C_0|Du| &\geq -K \quad \text{in } \Omega, \\ Lu - C_0|Du| &\leq K \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \end{aligned} \tag{3.0.1}$$

for some nonnegative constants C_0, K. Though the results in this Chapter are obtained for viscosity solutions, the result can be also applied for weak solutions, see Remark 3.3.2 below for more details. Our equations in (3.0.1) are motivated by the operators of the form

$$Lu + H(Du, x) := Lu + \inf_{\mu}\sup_{\nu}\{b_{\mu,\nu}(x)\cdot Du(x) + f_{\mu,\nu}(x)\} = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c. \tag{3.0.2}$$

Such equations arise in the study of stochastic control problems where the control can influence the dynamics only through the drift $b_{\mu,\nu}$.

On the topic of regularity theory for linear elliptic equations, Hölder estimates play a key role and it can be obtained by using Harnack inequality. The pioneering contributions are by DeGiorgi-Nash-Moser [73,130,136] who proved C^α regularity for solutions to second order elliptic equations in divergence form with measurable coefficients under the assumption of uniform ellipticity. For equations of non-divergence form, the corresponding regularity theory was established by Krylov and Safonov [121]. In [122], Krylov studied the boundary regularity for local second order elliptic equations in non-divergence form with bounded measurable coefficients. He obtained the Hölder regularity of $\frac{u}{\delta}$ up to the boundary where δ denotes the distance function, i.e, $\delta(x) = \mathrm{dist}(x, \Omega^c)$.

For linear equations, one may recover up to the boundary $C^{1,\gamma}$ regularity of u from the $W^{2,p}$ regularity (cf. [96, Theorem 3.1.22]). Recently, inspired by [50, 51], interior regularity of the solutions of (3.0.2) are studied in [131, 133]. Let us also mention the recent works [72, 94] where interior Hölder regularity of the gradient is established for the weak solutions of degenerate elliptic equations of mixed type. Here we are interested in the boundary regularity of the solutions. This study was carried out in [38]. It should also be noted that we are dealing with solutions to inequations (integro-differential inequalities).

Remark 3.0.1. When we say that u is a viscosity solution to the inequations or integro-differential inequalities (3.0.1), what we mean is that u is a viscosity sub-solution to the equation $Lu + C_0|Du| = -K$ and a viscosity supersolution to the equation $Lu - C_0|Du| = K$ in Ω and $u = 0$ in Ω^c.

We will start with the interior regularity theory of the solutions of the equation (3.0.1). Here regularity of the solution is a direct consequence of the ellipticity of the Laplacian. As we can see in [131], the Harnak estimate is an essential tool to get C^α-interior regularity result. $C^{1,\alpha}$-interior regularity result is obtain in [133] using blowup and approximation technique. We will state a version of interior $C^{1,\alpha}$ regularity result for linear integro-differential operator L. See [133] for more details.

Lemma 3.0.1. *Let $u \in C_b(\mathbb{R}^d)$ solves the in-equations*

$$\begin{aligned} Lu + C_0|Du| &\geq -K \quad in \;\; B_2, \\ Lu - C_0|Du| &\leq K \quad in \;\; B_2, \end{aligned} \tag{3.0.3}$$

in the viscosity sense. Then there exist constants $0 < \gamma < 1$ and $C > 0$, such that

$$||u||_{C^{1,\gamma}(B_1)} \leq C\Big(||u||_{L^\infty(\mathbb{R}^d)} + K\Big),$$

where γ and C depend only on d, C_0 and $\int_{\mathbb{R}^d}(1 \wedge |y|^2)\widehat{k}(y)\mathrm{d}y$.

We mention here that the proof in [133, Theorem 4.1] is stated for equations but it is easily seen that the same proof also works for a system of inequalities as in (3.0.3). In fact, by making minor changes to the technique involved in proving [133, Theorem 4.1], we will generalize this result uniformly for inequations involving scaled fully nonlinear nontranslation invariant integro-differential operators. We will discuss this in more detail in the next chapter where we will deal with such operators (see Theorem 4.1.1).

3.1 Lipschitz regularity up to the boundary

In this section, we will prove Lipschitz regularity of the solution of the inequation (3.0.1) over a $C^{1,1}$ domain Ω. To prove such a regularity result we need two standard ingredients, interior estimates Lemma 3.0.1 and barrier function. It can be easily shown that $\mathrm{dist}(\cdot, \Omega^c)$ gives a barrier function to u at the boundary. We also need coupling property and maximum principle throughout this chapter to compare appropriate sub and super solutions. Note that the operator in Lemma 2.1.1 and Theorem 2.1.1 does not have any gradient term, but the same proof (using the supersolution obtained in [131, Lemma 5.5]) also gives a comparison principle for the above operator. For reference, we state the following comparison principle with gradient term.

Theorem 3.1.1. *Let Ω be a bounded open set, and let $C_0 \geq 0$. Then, the following facts hold.*

(i) If u is a viscosity subsolution of $Lu + C_0|Du| = 0$ in Ω, then we have $\sup_{\Omega} u \leq \sup_{\Omega^c} u$.

(ii) If $f, g \in C(\Omega)$ and is u, v are, respectively, a viscosity subsolution of $Lu \pm C_0|Du| = f$ and a viscosity supersolution of $Lv \pm C_0|Dv| = g$ in Ω, then $L(u-v) \pm C_0|D(u-v)| \geq f - g$ in Ω in the viscosity sense.

To show the Lipschitz regularity up to the boundary, we begin by showing that δ is a barrier function to u at the boundary.

Lemma 3.1.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that Assumption 3.0.1(a) holds and u be a viscosity solution to (3.0.1). Then there exists a constant C, dependent only on $d, C_0, \mathrm{diam}(\Omega)$, radius of exterior sphere and $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\widehat{k}(y)\mathrm{d}y$, such that*

$$|u(x)| \leq CK\delta(x) \quad \textit{for all } x \in \Omega, \tag{3.1.1}$$

where $\delta(x) = \mathrm{dist}(x, \Omega^c)$.

Proof. We first show that

$$|u(x)| \leq \kappa K \quad x \in \mathbb{R}^d, \tag{3.1.2}$$

for some constant κ. From [131, Lemma 5.5] we can find a non-negative function $\chi \in C^2(\bar{\Omega}) \cap C_b(\mathbb{R}^d)$, with $\inf_{\mathbb{R}^d} \chi > 0$, satisfying

$$L\chi + C_0|D\chi| \leq -1 \quad \text{in} \quad \Omega.$$

Note that, since $\chi \in C^2(\bar{\Omega})$, the above equation holds in the classical sense. Defining $\psi = 2(K+\varepsilon)\chi$, $\varepsilon > 0$, we have that $\inf_{\mathbb{R}^d} \psi > 0$ and

$$L\psi + C_0|D\psi| \leq -2(K+\varepsilon) \quad \text{in} \quad \Omega. \tag{3.1.3}$$

We claim that $u \leq \psi$ in $\mathbb{R}^d$. Suppose, on the contrary, that $(u-\psi)(z) > 0$ at some point in $z \in \Omega$. Define

$$\theta = \inf\{t : u \leq t + \psi \text{ in } \mathbb{R}^d\}.$$

Since $(u-\psi)(z) > 0$, we must have $\theta \in (0, \infty)$. Again, since $u = 0$ in Ω^c, there must be a point $x_0 \in \Omega$ such that $u(x_0) = \theta + \psi(x_0)$ and $u \leq \theta + \psi$ in $\mathbb{R}^d$. Since ψ is C^2 in Ω, we get from the definition of viscosity subsolution that

$$-K \leq L(\theta+\psi)(x_0) + C_0|D\psi(x_0)| = L\psi(x_0) + C_0|D\psi(x_0)| \leq -2(K+\varepsilon),$$

using (3.1.3). But this is a contradiction. This proves the claim that $u \leq \psi$ in $\mathbb{R}^d$. Similar calculation using $-u$ will also give us $-u \leq \psi$ in $\mathbb{R}^d$. Thus

$$|u| \leq 2 \sup_{\mathbb{R}^d} |\chi| \, (K+\varepsilon) \quad \text{in } \mathbb{R}^d.$$

Since ε is arbitrary, we get (3.1.2).

Now we can prove (3.1.1). In view of (3.1.2), it is enough to consider the case $K > 0$. Since Ω belongs to the class $C^{1,1}$, it satisfies a uniform exterior sphere condition from outside. Let $r_\circ$ be a radius satisfying uniform exterior condition. From [131, Lemma 5.4] there exists a bounded, Lipschitz continuous function φ, Lipschitz constant being $r_\circ^{-1}$, satisfying

$$\varphi = 0 \quad \text{in} \quad \bar{B}_{r_\circ},$$

$$\begin{aligned} \varphi > 0 \quad &\text{in} \quad \bar{B}^c_{r_\circ}, \\ \varphi \geq \varepsilon \quad &\text{in} \quad B^c_{(1+\delta)r_\circ}, \\ L\varphi + C_0|D\varphi| \leq -1 \quad &\text{in} \quad B_{(1+\delta)r_\circ} \setminus \bar{B}_{r_\circ}, \end{aligned}$$

for some constants ε, δ, dependent on C_0 and $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\widehat{k}(y)\mathrm{d}y$. Furthermore, φ is C^2 in $B_{(1+\delta)r_\circ} \setminus \bar{B}_{r_\circ}$. For any point $y \in \partial\Omega$, we can find another point $z \in \Omega^c$ such that $\overline{B}_{r_\circ}(z)$ touches $\partial\Omega$ at y. Let $w(x) = \varepsilon^{-1}\kappa K\varphi(x-z)$. Also $L(w) + C_0|Dw| \leq -K$. Then

$$L(u-w) + C_0|D(u-w)| \geq 0 \quad \text{in } B_{(1+\delta)r_\circ}(z) \cap \Omega.$$

Since, by (3.1.2), $u - w \leq 0$ in $(B_{(1+\delta)r_\circ}(z) \cap \Omega)^c$, from comparison principle Theorem 3.1.1 it follows that $u(x) \leq w(x)$ in $\mathbb{R}^d$. Repeating a similar calculation for $-u$, we can conclude that $-u(x) \leq w(x)$ in $\mathbb{R}^d$. This relation holds for any $y \in \partial\Omega$. For any point $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < r_\circ$ we can find $y \in \partial\Omega$ satisfying $\operatorname{dist}(x, \partial\Omega) = |x-y| < r_\circ$. By previous estimate we then obtain

$$|u(x)| \leq \varepsilon^{-1}\kappa K\varphi(x-z) \leq \varepsilon^{-1}\kappa K(\varphi(x-z) - \varphi(y-z)) \leq \varepsilon^{-1}\kappa K\, r_\circ^{-1} \operatorname{dist}(x, \partial\Omega).$$

This gives us (3.1.1). □

Now we are ready to prove that $u \in C^{0,1}(\mathbb{R}^d)$.

Theorem 3.1.2. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that Assumption 3.0.1(a) holds and $u \in C(\mathbb{R}^d)$ is a viscosity solution to (3.0.1). Then, for some constant C, dependent only on $d, \Omega, C_0, \widehat{k}$, we have*

$$\|u\|_{C^{0,1}(\mathbb{R}^d)} \leq CK. \tag{3.1.4}$$

Proof. Let $x \in \Omega$ and $r \in (0,1)$ be such that $4r = \operatorname{dist}(x, \partial\Omega) \wedge 1$. Without loss of any generality, we assume $x = 0$. Define $v(y) = u(ry)$ in $\mathbb{R}^d$. From Lemma 3.1.1, we get

$$|v(y)| \leq CK\delta(ry) \leq CK(\operatorname{diam}(\Omega))^{1-\alpha/2}\delta^{\alpha/2}(ry).$$

We also have that for any $y \in \mathbb{R}^d$ and $x \in \Omega$

$$\delta(ry) \leq |ry - x| + \delta(x).$$

We also have that $\delta(0) \leq 4r(\mathrm{diam}(\Omega) \vee 1)$. Hence, taking $x = 0$ in above equation gives us that $\delta(ry) \leq Cr(1 + |y|)$. We then have

$$|v(y)| \leq C_1 K \min\{r^{\alpha/2}(1 + |y|^{\alpha/2}), r(1 + |y|)\} \qquad y \in \mathbb{R}^d, \tag{3.1.5}$$

for some constant C_1. We let

$$I^r f(x) = r^2 \frac{1}{2} \int_{\mathbb{R}^d} (f(x + y) + f(x - y) - f(x))k(ry)r^d \mathrm{d}y,$$

and $L^r f = \Delta f + I^r f$. Let us compute $L^r v(x) + C_0 r|Dv|$ in B_2. Clearly, we have $\Delta v(x) = r^2 \Delta u(rx)$ and $Dv(x) = rDu(rx)$. Also

$$\begin{aligned} I^r v(x) &= r^2 \frac{1}{2} \int_{\mathbb{R}^d} (u(rx + ry) + u(rx - ry) - 2u(rx))k(ry)r^d \mathrm{d}y \\ &= r^2 Iu(rx). \end{aligned}$$

Thus, it follows from (3.0.1) that

$$\begin{aligned} L^r v + C_0 r|Dv| &\geq -Kr^2 \quad \text{in} \quad B_2, \\ L^r v - C_0 r|Dv| &\leq Kr^2 \quad \text{in} \quad B_2. \end{aligned} \tag{3.1.6}$$

Now consider a smooth cut-off function $\varphi, 0 \leq \varphi \leq 1$, satisfying

$$\varphi = \begin{cases} 1 & \text{in } B_{3/2}, \\ 0 & \text{in } B_2^c. \end{cases}$$

Let $w = \varphi v$. Clearly, $((\varphi - 1)v)(y) = 0$ for all $y \in B_{3/2}$, which gives $D((\varphi - 1)v) = 0$ and $\Delta((\varphi - 1)v) = 0$ in $x \in B_{3/2}$. Since $w = v + (\varphi - 1)v$, we obtain

$$\begin{aligned} L^r w + C_0 r|Dw| &\geq -Kr^2 - |I^r((\varphi - 1)v)| \quad \text{in} \quad B_1, \\ L^r w - C_0 r|Dw| &\leq Kr^2 + |I^r((\varphi - 1)v)| \quad \text{in} \quad B_1, \end{aligned} \tag{3.1.7}$$

from (3.1.6). Again, since $(\varphi - 1)v = 0$ in $B_{3/2}$, we have in B_1 that

$$|I^r((\varphi - 1)v)(x)| = r^{d+2} \frac{1}{2} \left| \int_{|y| \geq 1/2} ((\varphi - 1)v)(x + y) + ((\varphi - 1)v)(x - y)k(ry)\mathrm{d}y \right|$$

$$
\begin{aligned}
&\le r^{d+2}\int_{|y|\ge 1/2}|v(x+y)|k(ry)\mathrm{d}y\\
&\le r^{d+2}\int_{\frac{1}{2}\le|y|\le\frac{1}{r}}|v(x+y)|k(ry)dy + r^{d+2}\int_{r|y|>1}|v(x+y)|k(ry)\mathrm{d}y,\\
&:= I_{r,1}+I_{r,2}.
\end{aligned}
$$

By Assumption 3.0.1(a)

$$
\begin{aligned}
I_{r,1} &= r^{d+2}\int_{\frac{1}{2}\le|y|\le\frac{1}{r}}|v(x+y)|k(ry)\mathrm{d}y\\
&\le r^{2-\alpha}\int_{\frac{1}{2}\le|y|\le\frac{1}{r}}|v(x+y)|\widehat{k}(y)\mathrm{d}y\\
&= \Lambda r^{2-\alpha}\int_{\frac{1}{2}\le|y|\le\frac{1}{r}}|v(x+y)|\frac{1}{|y|^{d+\alpha}}\mathrm{d}y\\
&\le r^{2-\alpha}\Lambda 3^{d+\alpha}\int_{|y|\ge 1/2}\frac{|v(x+y)|}{1+|x+y|^{\alpha/2}}\frac{1+|x+y|^{\alpha/2}}{1+|y|^{d+\alpha}}\mathrm{d}y\\
&\le \kappa_2 K r^{2-\alpha/2}\int_{|y|\ge 1/2}\frac{1+|x+y|^{\alpha/2}}{1+|x+y|^{d+\alpha}}\mathrm{d}y\\
&\le \kappa_3 K r^{2-\alpha/2},
\end{aligned}
$$

for some constants κ_2, κ_3, and in the fifth line we use (3.1.5). Again, by (3.1.2), we have

$$
\begin{aligned}
I_{r,2} \le \kappa r^2 K\int_{r|y|>1} r^d k(ry)\mathrm{d}y &= \kappa r^2 K\int_{|y|>1}k(y)\mathrm{d}y\\
&\le \kappa r^2 K\int_{|y|>1}\widehat{k}(y)\mathrm{d}y \le \kappa r^2 K\int_{|y|>1}J(y)\mathrm{d}y \le \kappa_4 r^2 K,
\end{aligned}
$$

for some constant κ_4. Therefore, putting the estimates of $I_{1,r}$ and $I_{2,r}$ in (3.1.7) we obtain

$$
\begin{aligned}
L^r v + C_0 r|Dv| &\ge -\kappa_5 K r^{2-\alpha/2} \quad \text{in} \quad B_1,\\
L^r v - C_0 r|Dv| &\ge \kappa_5 K r^{2-\alpha/2} \quad \text{in} \quad B_1,
\end{aligned}
\tag{3.1.8}
$$

for some constant κ_5. Applying Lemma 3.0.1 we obtain from (3.1.8)

$$\|v\|_{C^1(B_{\frac{1}{2}})} \leq \kappa_6\Big(\|v\|_{L^\infty(B_2)} + r^{2-\alpha/2}K\Big), \tag{3.1.9}$$

for some constant κ_6. From (3.1.5) and (3.1.9) we then obtain

$$\sup_{y\in B_{r/2}(x), y\neq x} \frac{|u(x)-u(y)|}{|x-y|} \leq \kappa_7 K, \tag{3.1.10}$$

for some constant κ_7.

Now we can complete the proof. Not that if $|x-y| \geq \frac{1}{8}$, then

$$\frac{|u(x)-u(y)|}{|x-y|} \leq 2\kappa K,$$

by (3.1.2). So we consider $|x-y| < \frac{1}{8}$. If $|x-y| \geq 8^{-1}(\delta(x)\vee\delta(y))$, then using Lemma 3.1.1 we get

$$\frac{|u(x)-u(y)|}{|x-y|} \leq 4CK(\delta(x)+\delta(y))(\delta(x)\vee\delta(y))^{-1} \leq 8CK.$$

Now let $|x-y| < 8^{-1}\min\{\delta(x)\vee\delta(y), 1\}$. Then either $y \in B_{\frac{\delta(x)\wedge 1}{8}}(x)$ or $x \in B_{\frac{\delta(y)\wedge 1}{8}}(y)$. Without loss of generality, we suppose $y \in B_{\frac{\delta(x)\wedge 1}{8}}(x)$. From (3.1.10) we get

$$\frac{|u(x)-u(y)|}{|x-y|} \leq \kappa_7 K.$$

This completes the proof. □

Remark 3.1.1. With the help of Theorem 3.1.2 we may choose $\beta = \infty$ in Assumption 3.0.1(b). Since u is globally Lipschitz, choosing $\mathscr{J}(y) = |y|^{-d-\zeta}$, $\zeta \in (0, 1\wedge\alpha)$, we see from Theorem 3.1.2 that

$$\Big|\int_{\mathbb{R}^d}(u(x+y)+u(x-y)-2u(x))\mathscr{J}(y)\mathrm{d}y\Big| \leq \kappa\|u\|_{C^{0,1}(\mathbb{R}^d)} \leq \kappa CK \tag{3.1.11}$$

for some constant κ. Let $\tilde{k}(y) = k(y)\mathbb{1}_{\{|y|\leq\beta'\}} + \mathscr{J}(y)$, where $\beta' < \frac{2}{3}\beta$. It is easy to see that $\tilde{k}$ satisfies Assumption 3.0.1(a).

We now show that Assumption 3.0.1(b) also holds for the kernel $\tilde{k}$ with $\tilde{\beta} = \infty$.

Fix $r \in (0,1]$ and $x_0 \in \mathbb{R}^d$ and choose $x, y \in B_{\frac{r}{2}}(x_0)$ and $z \in B_r^c(x_0)$. Without any loss of generality we may assume that the kernel k satisfies the Assumption 3.0.1(b) for some $\beta \in (0, 1/2)$. If $\max\{|x-z|, |y-z|\} \leq \beta'$, we have from Assumption 3.0.1(b) that

$$\tilde{k}(x-z) = k(x-z) + \mathscr{J}(x-z) \leq \varrho k(y-z) + 3^{d+\zeta}\mathscr{J}(y-z) \leq (\varrho \vee 3^{d+\zeta})\tilde{k}(y-z),$$

using the fact

$$\frac{1}{3}|x-z| \leq |y-z| \leq 3|x-z|.$$

Also, if $|x-z| > \beta'$, then

$$\tilde{k}(x-z) = \mathscr{J}(x-z) \leq 3^{n+\zeta}\tilde{k}(y-z).$$

Suppose $|x-z| \leq \beta'$ and $|y-z| > \beta'$. Note that $|y-z| \leq \frac{3}{2}\beta' < \beta$. Using Assumption 3.0.1(a) we find that

$$\begin{aligned}
\tilde{k}(x-z) \leq \varrho k(y-z) + 3^{d+\zeta}\mathscr{J}(y-z) &\leq \varrho\Lambda|y-z|^{-n-\alpha} + 3^{d+\zeta}\mathscr{J}(y-z) \\
&\leq \left(\varrho\Lambda(\beta')^{-\alpha+\zeta} + 3^{d+\zeta}\right)\mathscr{J}(y-z) \\
&= \left(\varrho\Lambda(\beta')^{-\alpha+\zeta} + 3^{d+\zeta}\right)\tilde{k}(y-z).
\end{aligned}$$

Thus, the kernel $\tilde{k}$ satisfies Assumption 3.0.1(b) for $\tilde{\beta} = \infty$.

On the other hand, replacing the kernel k by $\tilde{k}$ and using Theorem 3.1.2, (3.1.11), we obtain from (3.0.1) that

$$\begin{aligned}
\Delta u + I_{\tilde{k}}u + C_0|Du| &\geq -C_1 K \quad \text{in } \Omega, \\
\Delta u + I_{\tilde{k}}u - C_0|Du| &\leq C_1 K \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega^c,
\end{aligned} \tag{3.1.12}$$

for some constant C_1, dependent on $\widehat{k}, \zeta$. This modification of the nonlocal kernel would be useful to apply the Harnack inequality from [90].

3.2 Fine boundary regularity of u/δ

In this section, we investigate the finer regularity property of u near $\partial\Omega$. Let Ω be bounded C^2 domain and $\delta(x) = \mathrm{dist}(x, \Omega^c)$ be the distance function from the boundary. Modifying $\delta(x)$ inside Ω, if required, we may assume that $\delta \in C^2(\bar{\Omega})$ (cf. [74, Theorem 5.4.3]). We want to study the regularity of u/δ in Ω. Since u is Lipschitz, using the estimate (3.1.4) we may write (3.0.1) as follows

$$|Lu| = |\Delta u + Iu| \leq CK \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{in } \Omega^c, \tag{3.2.1}$$

where C is a constant depending on $\widehat{k}, C_0$. Also, in view of Remark 3.1.1, we can assume that k satisfies Assumption 3.0.1(b) for $\beta = \infty$. The aim of this section is to establish Hölder regularity of u/δ up to the boundary.

For elliptic operators, Hölder regularity of u/δ up to the boundary is obtained by Krylov [122]. Boundary estimate for fractional Laplacian operators are studied by Ros-Oton and Serra in [144–146]. Result of [144] has been extended for nonlocal operators with kernel of variable orders by Kim et. al. [112] whereas extension to the fractional p-Laplacian operator can be found in [105]. Here we follow the approach of [144] which is inspired by a method of Caffarelli [107, p. 39]. Clearly from Lemma 3.1.1 we know that u/δ is bounded in Ω. A key step in this analysis is the oscillation lemma (see Proposition 3.2.1) for u/δ which involves computation of $L((u - \kappa\delta)^+)$ for some suitable constant κ. Since if we can control the oscillation of u/δ near $\partial\Omega$ appropriately then one can easily get Hölder regularity of u/δ up to the boundary. Note that, by Theorem 3.1.2, Iu is bounded in Ω for $\alpha \in (0, 1)$, and therefore, in this case we can follow the standard approach of local operators to get the regularity estimate on u/δ. But for $\alpha \in [1, 2)$, $I\delta$ becomes singular near $\partial\Omega$. So we have to do several careful estimates to apply the method of [144].

Our first goal is to get the oscillation estimate Proposition 3.2.1. To obtain this result we need auxiliary Lemmas 3.2.1 to 3.2.5. Following lemma gives an existence of appropriate subsolution.

Lemma 3.2.1. *There exists a constant $\tilde{\kappa}$, dependent on $d, \widehat{k}$, such that for any*

$r \in (0,1]$, *we have a bounded radial function* ϕ_r *satisfying*

$$\begin{cases} L\phi_r \geq 0 & \text{in } B_{4r} \setminus \bar{B}_r, \\ 0 \leq \phi_r \leq \tilde{\kappa} r & \text{in } B_r, \\ \phi_r \geq \frac{1}{\tilde{\kappa}}(4r - |x|) & \text{in } B_{4r} \setminus B_r, \\ \phi_r \leq 0 & \text{in } \mathbb{R}^d \setminus B_{4r}. \end{cases}$$

Moreover, $\phi_r \in C^2(B_{4r} \setminus \bar{B}_r)$.

Proof. Fix $r \in (0,1]$ and define $v_r(x) = \mathrm{e}^{-\eta q(x)} - \mathrm{e}^{-\eta(4r)^2}$, where $q(x) = |x|^2 \wedge 2(4r)^2$ and $\eta > 0$. Clearly, $1 \geq v_r(0) \geq v_r(x)$ for all $x \in \mathbb{R}^d$. Thus

$$v_r(x) \leq 1 - \mathrm{e}^{-\eta(4r)^2} \leq \eta(4r)^2, \tag{3.2.2}$$

using the fact that $1 - e^{-s} \leq s$ for all $s \geq 0$. Again, for $x \in B_{4r} \setminus B_r$, we have

$$\begin{aligned} v_r(x) = \mathrm{e}^{-\eta(4r)^2}(\mathrm{e}^{\eta((4r)^2 - q(x))} - 1) &\geq \eta \mathrm{e}^{-\eta(4r)^2}((4r)^2 - |x|^2) \\ &= \eta \mathrm{e}^{-\eta(4r)^2}(4r + |x|)(4r - |x|) \\ &\geq 5\eta r \mathrm{e}^{-\eta(4r)^2}(4r - |x|). \end{aligned} \tag{3.2.3}$$

Now we estimate Lv_r in $B_{4r} \setminus \bar{B}_r$. Fix $x \in B_{4r} \setminus \bar{B}_r$. Then

$$\Delta v_r = \eta \mathrm{e}^{-\eta|x|^2} \left(4\eta|x|^2 - 2n\right),$$

and, since $Iv_r = I(v_r + e^{-\eta(4r)^2})$, using the convexity of exponential map we obtain

$$\begin{aligned} I(e^{-\eta q(\cdot)})(x) &\geq -\eta e^{-\eta|x|^2} \int_{\mathbb{R}^d} \Big(q(x+y) + q(x-y) - 2q(x)\Big) k(y)\mathrm{d}y \\ &\geq -\eta e^{-\eta|x|^2} \Big[\int_{B_1} \Big(q(x+y) + q(x-y) - 2q(x)\Big) k(y)\mathrm{d}y \\ &\qquad + \int_{B_1^c} (8r)^2 k(y)\mathrm{d}y\Big] \\ &\geq -\eta e^{-\eta|x|^2} \left[\int_{|y|<r} \frac{2|y|^2}{|y|^{d+\alpha}}\mathrm{d}y + \int_{r<|y|<1} \frac{(8r)^2}{|y|^{d+\alpha}}\mathrm{d}y + (8r)^2 \int_{|y|>1} J(y)\mathrm{d}y\right] \\ &\geq -\eta e^{-\eta|x|^2} \kappa r^{2-\alpha}, \end{aligned}$$

for some constant κ, independent of η. Combining the above estimates we see that, for $x \in B_{4r} \setminus \bar{B}_r$,

$$Lv_r(x) \geq \eta e^{-\eta|x|^2}\Big[4\eta|x|^2 - 2d - \kappa r^{2-\alpha}\Big] \geq \eta e^{-\eta|x|^2}\Big[4\eta r^2 - 2n - \kappa r^{2-\alpha}\Big].$$

Thus, letting $\eta = \frac{1}{r^2}(n+\kappa)$, we obtain

$$Lv_r > 0 \quad \text{in } B_{4r} \setminus \bar{B}_r.$$

We set $\phi_r = \frac{v_r}{r}$ and the result follows from (3.2.2)-(3.2.3). □

Let us now define the sets that we use for our oscillation estimates. We borrow the notations of [144].

Definition 3.2.1. Let $\kappa \in (0, \frac{1}{16})$ be a fixed small constant and let $\kappa' = 1/2 + 2\kappa$. Given a point $x_0 \in \partial\Omega$ and $R > 0$, we define

$$D_R = D_R(x_0) = B_R(x_0) \cap \Omega,$$

and

$$D^+_{\kappa' R} = D^+_{\kappa' R}(x_0) = B^+_{\kappa' R}(x_0) \cap \{x \in \Omega : (x - x_0) \cdot \mathrm{n}(x_0) \geq 2\kappa R\},$$

where $\mathrm{n}(x_0)$ is the unit inward normal at x_0. Using the C^2 regularity of the domain, there exists $\rho > 0$, depending on Ω, such that the following inclusions hold for each $x_0 \in \partial\Omega$ and $R \leq \rho$:

$$B_{\kappa R}(y) \subset D_R(x_0) \qquad \text{for all } y \in D^+_{\kappa' R}(x_0), \tag{3.2.4}$$

and

$$B_{4\kappa R}(y^* + 4\kappa R\mathrm{n}(y^*)) \subset D_R(x_0), \quad \text{and} \quad B_{\kappa R}(y^* + 4\kappa R\mathrm{n}(y^*)) \subset D^+_{\kappa' R}(x_0) \tag{3.2.5}$$

for all $y \in D_{R/2}$, where $y^* \in \partial\Omega$ is the unique boundary point satisfying $|y - y^*| = \mathrm{dist}(y, \partial\Omega)$. Note that, since $R \leq \rho$, $y \in D_{R/2}$ is close enough to $\partial\Omega$ and hence the point $y^* + 4\kappa R\,\mathrm{n}(y^*)$ belongs to the line joining y and y^*.

Remark 3.2.1. In the remaining part of this section, we fix $\rho > 0$ to be a small constant depending only on Ω, so that (3.2.4)-(3.2.5) hold whenever $R \leq \rho$ and $x_0 \in \partial\Omega$. Also, every point on $\partial\Omega$ can be touched from both inside and outside Ω by balls of radius ρ. We also fix $\gamma > 0$ small enough so that for $0 < r \leq \rho$ and $x_0 \in \partial\Omega$ we have

$$B_{\eta r}(x_0) \cap \Omega \subset B_{(1+\sigma)r}(z) \setminus \bar{B}_r(z) \quad \text{for} \quad \eta = \sigma/8,\, \sigma \in (0, \gamma),$$

for any $x' \in \partial\Omega \cap B_{\eta r}(x_0)$, where $B_r(z)$ is a ball contained in $\mathbb{R}^d \setminus \Omega$ that touches $\partial\Omega$ at point x'.

We first treat the case $\alpha \in (0,1)$. Note that in this situation Iu can be defined in the classical sense and is bounded in Ω, by Theorem 3.1.2.

Lemma 3.2.2. *Let $\alpha \in (0,1)$ and Ω be a bounded C^2 domain. Let u be such that $u \geq 0$ in $\mathbb{R}^d$, and $|Lu| \leq C_2$ in D_R, for some constant C_2. Then, there exists a positive constant C, depending only on $d, \Omega, \widehat{k}$, such that*

$$\inf_{D^+_{\kappa' R}} \left(\frac{u}{\delta}\right) \leq C \left(\inf_{D_{\frac{R}{2}}} \frac{u}{\delta} + C_2 R \right) \tag{3.2.6}$$

for all $R \leq \rho_0$, where the constant ρ_0 depends only on d, Ω and $\int_{\mathbb{R}^d} (|y|^2 \wedge 1)\widehat{k}(y)\mathrm{d}y$.

Proof. We split the proof in two steps.
Step 1. Suppose $C_2 = 0$ and $R \leq \rho$, where ρ is given by Remark 3.2.1. Define $m = \inf_{D^+_{\kappa' R}} u/\delta \geq 0$. By (3.2.4),

$$u \geq m\delta \geq m\,(\kappa R) \quad \text{in} \;\; D^+_{\kappa' R}. \tag{3.2.7}$$

Again, by (3.2.5), for any $y \in D_{R/2}$, we have either $y \in D^+_{\kappa' R}$ or $\delta(y) < 4\kappa R$. If $y \in D^+_{\kappa' R}$ it follows from the definition of m that $m \leq u(y)/\delta(y)$. Now let $\delta(y) < 4\kappa R$. Let y^* be the nearest point to y on $\partial\Omega$ and $\tilde{y} = y^* + 4\kappa R\,\mathrm{n}(y^*)$. Again by (3.2.5), we have $B_{4\kappa R}(\tilde{y}) \subset D_R$ and $B_{\kappa R}(\tilde{y}) \subset D^+_{\kappa' R}$. Recall that $Lu = 0$ in D_R and $u \geq 0$ in $\mathbb{R}^d$.

Now take $r = \kappa R$ and let ϕ_r be the subsolution in Lemma 3.2.1. Define $\tilde{\phi}_r(x) = \frac{1}{\tilde{\kappa}}\phi_r(x - \tilde{y})$. Using (3.2.7) and the comparison principle Theorem 3.1.1 in $B_{4r}(\tilde{y}) \setminus$

$\bar{B}_r(\tilde{y})$ it follows that $u(x) \geq m\tilde{\phi}_r(x)$ in all of $\mathbb{R}^d$. In particular, we have $u/\delta \geq \frac{1}{(\tilde{\kappa})^2}m$ on the segment joining y^* and $\tilde{y}$, that contains y. Hence

$$\inf_{D^+_{\kappa' R}} \left(\frac{u}{\delta}\right) \leq C \inf_{D_{\frac{R}{2}}} \frac{u}{\delta}\,.$$

Step 2. Suppose $C_2 > 0$. Define $r' = \eta r$ for $r \leq \rho$ and $\eta \leq 1$ to be chosen later. Let $\tilde{u}$ to be the solution of (cf. Theorem 2.1.2)

$$\begin{cases} L\tilde{u} = 0 & \text{in } D_{r'}\,, \\ \tilde{u} = u & \quad \text{in } \mathbb{R}^d \setminus D_{r'}\,. \end{cases}$$

From step 1, we see that $\tilde{u}$ satisfies (3.2.6). Define $w = \tilde{u} - u$. Applying Theorem 3.1.1, we obtain that $|Lw| \leq C_2$ in $D_{r'}$ and $w = 0$ in $\mathbb{R}^d \setminus D_{r'}$. Since $r \leq \rho$, points of $\partial\Omega$ can be touched by an exterior ball of radius r. Thus for any point $y \in \partial\Omega$ we can find another point $z \in \Omega^c$ such that $\bar{B}_r(z)$ touches $\partial\Omega$ at y. From the proof of [131, Lemma 5.4] there exists a bounded, Lipschitz continuous function φ_r, with Lipschitz constant r^{-1}, that satisfies

$$\begin{cases} \varphi_r = 0, & \text{in} \quad \bar{B}_r, \\ \varphi_r > 0, & \text{in} \quad \bar{B}^c_r, \\ L\varphi_r \leq -\frac{1}{r^2}, & \text{in} \quad B_{(1+\sigma)r} \setminus \bar{B}_r, \end{cases}$$

for some constant σ, independent of r. Without any loss of generality we may assume $\sigma \leq \gamma$ (see Remark 3.2.1). We set $\eta = \frac{\sigma}{8}$. Then $D_{r'} \subset B_{(1+\sigma)r}(z) \setminus \overline{B}_r(z)$. Letting $v(x) = C_2 r^2 \varphi_r(x - z)$ will give us a desired supersolution and therefore, by comparison principle Theorem 3.1.1 we get $|w| \leq v$ in $\mathbb{R}^d$. For any point $x \in D_{r'}$ we can find $y \in \partial\Omega$ satisfying $\operatorname{dist}(x, \partial\Omega) = |x - y|$. By above estimate we obtain

$$|w(x)| \leq C_2 r^2 \varphi_r(x-z) \leq C_2 r^2(\varphi_r(x-z) - \varphi_r(y-z)) \leq C_2 r \operatorname{dist}(x, \partial\Omega) = C_2 r\delta(x).$$

Thus we obtain

$$|w(x)| \leq C_2 \frac{r'}{\eta}\delta(x) \qquad \text{in } D_{r'}.$$

Combining with step 1 we have

$$\inf_{D^+_{\kappa' r'}} \left(\frac{u}{\delta}\right) \leq \frac{C}{\eta}\left(\inf_{D_{\frac{r'}{2}}} \frac{u}{\delta} + C_2 r'\right).$$

Setting $\rho_0 = \eta\rho$ and $R = r'$ we have the desired result. □

Next we obtain a similar estimate when $\alpha \in [1, 2)$.

Lemma 3.2.3. *Let Ω be a bounded C^2 domain and u be such that $u \geq 0$ in all of $\mathbb{R}^d$ and $|Lu| \leq C_2 g$ in D_R for some positive constant C_2 and g is given by*

$$g(x) = \begin{cases} (\delta(x))^{1-\alpha} & \text{if } \alpha > 1, \\ -\log(\delta(x)) + C_3 & \text{if } \alpha = 1, \end{cases}$$

for some constant C_3. Set $\hat{\alpha} = 2 - \alpha$ for $\alpha \in (1, 2)$ and for $\alpha = 1$, $\hat{\alpha}$ is any number in $(0, 1)$. Then there exists a positive constant C, depending on Ω, n and $\widehat{k}$, such that

$$\inf_{D^+_{k' R}} \frac{u}{\delta} \leq C\Big(\inf_{D_{R/2}} \frac{u}{\delta} + C_2 R^{\hat{\alpha}}\Big) \tag{3.2.8}$$

for all $R < \rho_0$, where ρ_0 is a positive constant depending only on $\Omega, d, \hat{\alpha}$ and $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\widehat{k}(y)\mathrm{d}y$.

Proof. When $C_2 = 0$, the proof follows from Step 1 of Lemma 3.2.2. So we let $C_2 > 0$. As before, we consider $\tilde{u}$ to be the solution of

$$\begin{aligned} L\tilde{u} &= 0 \quad \text{in } D_R, \\ \tilde{u} &= u \quad \text{in } \mathbb{R}^d \setminus D_R. \end{aligned}$$

Then

$$\inf_{D^+_{k' R}} \frac{\tilde{u}}{\delta} \leq C \inf_{D_{R/2}} \frac{\tilde{u}}{\delta}$$

holds, by step 1 of Lemma 3.2.2. Defining $w = \tilde{u} - u$, we get $|Lw| \leq C_2 g$ in D_R by using Theorem 3.1.1 and $w = 0$ in D^c_R. As before, we would consider an appropriate supersolution and then apply the comparison principle to establish (3.2.8).

For this construction of supersolution we take inspiration from [131, Lemma 5.8]. We set

$$\tilde{\psi}(s) = \int_0^s 2\mathrm{e}^{-ql-q\int_0^l \Theta(\tau)\mathrm{d}\tau}\mathrm{d}l - s,$$

where $q > 0$ is to be chosen later and Θ is given by

$$\Theta(s) = \int_{|z|>s} \min\{1, |z|\}\widehat{k}(z)\mathrm{d}z.$$

Since Θ is integrable in a neighbourhood of 0, there exists a sufficiently small constant $s(q) > 0$ such that, for $0 < s < s(q)$, $\tilde{\psi}'(s) = 2e^{-qs-q\int_0^s \Theta(\tau)d\tau} - 1 \geq \frac{1}{2}$. Set $\sigma_1 = \min\{\frac{s(q)}{8}, 1, \gamma\}$. For any $r \in (0, 1)$, we define

$$\psi_{r,z}(x) = \begin{cases} \tilde{\psi}\left(\frac{d_{B_r(z)}(x)}{r}\right) & \text{if } d_{B_r(z)}(x) < r\sigma_1, \\ \tilde{\psi}(\sigma_1) & \text{if } d_{B_r(z)}(x) \geq r\sigma_1, \end{cases} \tag{3.2.9}$$

where $d_{B_r(z)}(x) = \operatorname{dist}(x, B_r(z))$. Let $\eta = \frac{\sigma_1}{8}$, $0 < r \leq \rho$ and $B_{\eta r}(x_0) \cap \Omega = D_{\eta r}$. We define

$$\Phi_r(x) = \begin{cases} \tilde{\psi}\left(\frac{\delta(x)}{r}\right), & \text{if } \delta(x) < r\sigma_1, \\ \tilde{\psi}(\sigma_1) & \text{if } \delta(x) \geq r\sigma_1. \end{cases}$$

For $x \in D_{\eta r}$ then we have $x^* \in \partial\Omega$ such that $\delta(x) = |x - x^*|$. Let $z_x^* = z$ be a point in Ω^c such that $B_r(z)$ touches $\partial\Omega$ at x^*. From Remark 3.2.1 we have that

$$D_{\eta r} \subset B_{(1+\sigma_1)r}(z) \setminus B_r(z).$$

Since $\tilde{\psi}'' < 0$ and $|D\delta(x)| \geq \kappa > 0$ for $\delta(x) \in (0, \rho_1)$, ρ_1 sufficiently small, it follows that

$$\Delta\Phi_r(x) \leq \frac{C}{r} + \tilde{\psi}''(\frac{\delta(x)}{r})\frac{\kappa^2}{r^2}. \tag{3.2.10}$$

Consider $\psi_{r,z}$ from (3.2.9) and notice that $\psi_{r,z}(x) = \Phi_r(x)$ and $\delta(x+y) \leq d_{B_r(z)}(x+y)$ for all $y \in \mathbb{R}^d$. Hence

$$\psi_{r,z}(x + y) + \psi_{r,z}(x - y) - 2\psi_{r,z}(x) \geq \Phi_r(x + y) + \Phi_r(x - y) - 2\Phi_r(x).$$

This readily gives (see [131, Lemma 5.8])

$$I\Phi_r(x) \leq I\psi_{r,z}(x) \leq \frac{C}{r}\left(1+\Theta\left(\frac{d_{B_r(z)}(x)}{r}\right)\right) = \frac{C}{r}\left(1+\Theta\left(\frac{\delta(x)}{r}\right)\right), \tag{3.2.11}$$

using the fact $\delta(x) = |x-x^*| = d_{B_r(z)}(x)$. Combining (3.2.10) and (3.2.11) we have

$$L\Phi_r \leq \frac{C}{r}\Big(1+\Theta\Big(\frac{\delta(x)}{r}\Big)\Big) - \frac{2\kappa^2 q}{r^2}\Big(1+\Theta\Big(\frac{\delta(x)}{r}\Big)\Big).$$

for all $x \in D_{\eta r}$. Now choose $q = \frac{1}{2\kappa^2}(C+1)$ in the expression of $\tilde{\psi}$ we obtain

$$L\Phi_r \leq -\frac{1}{r^2}\Big(1+\Theta\Big(\frac{\delta(x)}{r}\Big)\Big) \leq -\frac{1}{r^2}\Theta\Big(\frac{\delta(x)}{r}\Big), \tag{3.2.12}$$

for all $x \in D_{\eta r}$.

Next we estimate the function Θ in $D_{\eta r}$. For $\xi \in (0,1]$, we see that

$$\begin{aligned}\Theta(\xi) = \int_{|z|>\xi} \min\{1,|z|\}\widehat{k}(z)\mathrm{d}z &= \Lambda\int_{\xi<|z|\leq 1}\frac{|z|}{|z|^{d+\alpha}}\mathrm{d}z + \int_{|z|\geq 1} J(z)\mathrm{d}z \\ &= \Lambda\omega_d\int_\xi^1 \frac{r^d}{r^{d+\alpha}}\mathrm{d}r + \kappa_1 \\ &= \begin{cases} \frac{\omega_d\Lambda}{\alpha-1}\Big[\xi^{1-\alpha}-1\Big]+\kappa_1 & \text{for } \alpha\in(1,2), \\ \omega_d\Lambda(-\log\xi)+\kappa_1 & \text{for } \alpha = 1, \end{cases}\end{aligned} \tag{3.2.13}$$

for some positive constant κ_1. Here ω_d denotes the surface area of the unit sphere in $\mathbb{R}^d$. Since $\frac{\delta(x)}{r} < \frac{1}{2}$ in $D_{\eta r}$, we get from the above estimate that

$$\begin{aligned}\Theta\left(\frac{\delta(x)}{r}\right) = \frac{\omega_d\Lambda}{\alpha-1}\Big[\left(\frac{\delta(x)}{r}\right)^{1-\alpha}-1\Big]+\kappa_1 &\geq \Lambda\omega_d\left(\frac{\delta(x)}{r}\right)^{1-\alpha}\Big(\frac{2^{\alpha-1}-1}{2^{\alpha-1}(\alpha-1)}\Big) \\ &\geq \kappa_2\left[\frac{\delta(x)}{r}\right]^{1-\alpha}\end{aligned}$$

for $\alpha\in(1,2)$, where the constant κ_2 is independent of α. Again, for $\alpha = 1$, we have

$$\Theta\Big(\frac{\delta(x)}{r}\Big) = \Lambda\omega_d\log\big(\frac{r}{\delta(x)}\big)+\kappa_1 \tag{3.2.14}$$

for $x \in D_{\eta r}$. We claim that for any $0 < \zeta < 1$, there exists a $r_\theta < 1$ such that for all $r < r_\theta$

$$\log(rz) \geq r^\zeta \log(z) \tag{3.2.15}$$

for all $z \geq \frac{1}{\theta r}$, where $0 < \theta < 1$ is a fixed positive constant. To prove the claim, we let

$$h(z) = \frac{\log(rz)}{\log(z)}.$$

By our choice of parameters z, r, θ, we have $h(z) > 0$. Since $\log z \geq \log(rz)$, we have

$$h'(z) = \frac{(\log z - \log(rz))}{z(\log z)^2} > 0$$

for $z > \frac{1}{\theta r}$. Thus h is strictly increasing in $[(\theta r)^{-1}, \infty)$, and therefore,

$$h(z) \geq h((\theta r)^{-1}) = \frac{\log\left(\frac{1}{\theta}\right)}{\log\left(\frac{1}{\theta r}\right)} = \frac{\log\theta}{\log(r\theta)} \geq r^\zeta,$$

for all $r \in (0, r_\theta)$, where r_θ depends only on θ and ζ. This gives us (3.2.15). Putting (3.2.15) in (3.2.14) we have

$$\Theta\Big(\frac{\delta(x)}{r}\Big) \geq \Lambda\omega_d r^\zeta \log(\frac{1}{\delta(x)}) + \kappa_1 \geq C r^\zeta \left(\log\left(\frac{1}{\delta(x)}\right) + \kappa_1\right),$$

in $D_{\eta r}$, for all $r \leq r_\theta$. Using the above estimate and (3.2.12), we define the super-solutions as

$$v(x) = \mu\, r^{1+\hat{\alpha}} \Phi_r(x),$$

where the constant μ is chosen suitably so that $Lv \leq -g$ in $D_{\eta r}$, for all $r \leq r_\theta$. Thus, in $x \in D_{\eta r}$, we have $L(C_2 v)(x) \leq -C_2 g(x)$ and $C_2 v(x) \geq 0$ in $\mathbb{R}^d$. Using comparison principle Theorem 3.1.1 we then obtain $C_2 v(x) \geq w(x)$ in $\mathbb{R}^d$. Repeating the same argument with $-w$, we get $|w(x)| \leq C_2 v(x)$ in $D_{\eta r}$. Now we can complete the proof by repeating the same argument as in Lemma 3.2.2. □

Lemma 3.2.4. *Let Ω be a bounded C^2 domain, and u be a bounded continuous* *function such that $u \geq 0$ in all of $\mathbb{R}^d$, and $|Lu| \leq C_2(1+\mathbb{1}_{[1,2)}(\alpha)g)$ in D_R, for some constant C_2. Let $\hat{\alpha} = 1 \wedge (2-\alpha)$ for $\alpha \neq 1$, and for $\alpha = 1$, $\hat{\alpha}$ be any number in $(0,1)$. Then, there exists a positive constant C, depending only on d, Ω and $\widehat{k}$, such that*

$$\sup_{D^+_{\kappa' R}} \left(\frac{u}{\delta}\right) \leq C\left(\inf_{D^+_{\kappa' R}} \frac{u}{\delta} + C_2 R^{\hat{\alpha}}\right) \tag{3.2.16}$$

for all $R \leq \rho_0$, where constant ρ_0 depends only on $\Omega, d, \hat{\alpha}$ and $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\widehat{k}(y)\mathrm{d}y$.

Proof. Recall from Remark 3.1.1 that we may take $\beta = \infty$ in Assumption 3.0.1(b). This property will be useful to apply the Harnack inequality from [90]. We split the proof into two steps.
Step 1. Let $C_2 = 0$. In this case (3.2.16) follows from the Harnack inequality for L. Let $R \leq \rho$. Then for each $y \in D^+_{\kappa' R}$ we have $B_{\kappa R}(y) \subset D_R$. Hence we have $Lu = 0$ in $B_{\kappa R}(y)$. Without loss of generality, we may assume $y = 0$. Let $r = \kappa R$ and define $v(x) = u(rx)$ for all $x \in \mathbb{R}^d$. Then, it can be easily seen that

$$r^2 Lu(rx) = L^r v(x) := \Delta v(x) + r^2 \frac{1}{2}\int_{\mathbb{R}^d}(v(x+y)+v(x-y)-2v(x))k(ry)r^n \mathrm{d}y,$$

for all $x \in B_1$. This gives $L^r v(x) = 0$ in B_1 and $v \geq 0$ in whole $\mathbb{R}^d$. From the stochastic representation of v [37, Theorem 1.1], it follows that v is also a harmonic function in the probabilistic sense as considered in [90]. Hence by the Harnack inequality [90, Theorem 2.4] we obtain

$$\sup_{B_{\frac{1}{2}}} v \leq C \inf_{B_{\frac{1}{2}}} v,$$

where constant C does not depend on r. This of course, implies

$$\sup_{B_{\frac{\kappa R}{2}}} u \leq C \inf_{B_{\frac{\kappa R}{2}}} u.$$

Now cover $D^+_{\kappa' R}$ by a finite number of balls $B_{\kappa R/2}(y_i)$, independent of R, to obtain

$$\sup_{D^+_{\kappa' R}} u \leq C \inf_{D^+_{\kappa' R}} u.$$

(3.2.16) follows since $\kappa R/2 \le \delta \le 3\kappa R/2$ in $D^+_{\kappa' R}$.

Step 2. Let $C_2 > 0$. The proof follows by combining Step 1 above and Step 2 of Lemmas 3.2.2 and 3.2.3. $\square$

Next we compute $L\delta$ in Ω.

Lemma 3.2.5. *We have $|L\delta(x)| \le Cg(x)$, where g is given by*

$$g(x) = \begin{cases} (\delta(x) \wedge 1)^{1-\alpha} & \textit{for} \quad \alpha > 1, \\ \log(\frac{1}{\delta(x)\wedge 1}) + 1 & \textit{for} \quad \alpha = 1, \\ 1 & \textit{for} \quad \alpha \in (0,1). \end{cases} \tag{3.2.17}$$

Proof. Since $\delta \in C^{0,1}(\mathbb{R}^d) \cap C^2(\bar{\Omega})$ [74, Theorem 5.4.3], (3.2.17) easily follows for the case $\alpha \in (0,1)$. Let $\Omega_{\rho_0} = \Omega \cap \{\delta < \rho_0\}$ where $\rho_0 < 1$. It is enough to show that

$$|L\delta(x)| \le C\Theta(\delta(x)) \quad \text{for } x \in \Omega_{\rho_0}, \tag{3.2.18}$$

where Θ is defined as before

$$\Theta(\xi) = \int_{|z|>\xi} \min\{1, |z|\}\hat{k}(z)\mathrm{d}z.$$

First of all

$$|L\delta(x)| \le |\Delta\delta(x)| + |I\delta(x)| \le \kappa + |I\delta(x)|, \tag{3.2.19}$$

for some constant κ, depending on Ω. Again,

$$\begin{aligned} I\delta(x) &= \int_{\mathbb{R}^d} \Big(\delta(x+z) + \delta(x-z) - 2\delta(x)\Big)k(z)\mathrm{d}z \\ &= \int_{|z|\le\delta(x)/2} + \int_{|z|>\delta(x)/2}. \end{aligned}$$

Since $\delta(x+z) + \delta(x-z) - 2\delta(x) \le \kappa_2|z|^2$ for $|z| \le \delta(x)/2$, we have

$$\int_{|z|\le\delta(x)/2} \Big(\delta(x+z) + \delta(x-z) - 2\delta(x)\Big)k(z)\mathrm{d}z \le \kappa_2 \int_{|z|\le\delta(x)/2} |z|^2\,\hat{k}(z)\mathrm{d}z \le \kappa_3,$$

for some constant κ_3. Since δ is Lipschitz, it follows that

$$\delta(x+z) + \delta(x-z) - 2\delta(x) \le 2(\mathrm{diam}(\Omega) \vee 1) \min\{|z|, 1\}.$$

Thus

$$\begin{aligned}\int_{|z|>\delta(x)/2} \Big(\delta(x+z) + \delta(x-z) - 2\delta(x)\Big) k(z)\mathrm{d}z &\le \kappa_4 \int_{|z|>\delta(x)/2} \min\{|z|, 1\}\widehat{k}(z)\mathrm{d}z \\ &= \kappa_4 \Theta(\delta(x)/2),\end{aligned}$$

for some constant κ_4. Inserting these estimates in (3.2.19) we obtain

$$|L\delta(x)| \le \kappa_5(1 + \Theta(\delta(x)/2)) \quad \text{for all } x \in \Omega_{\rho_0},$$

for some constant κ_5. Choosing ρ_0 sufficiently small, (3.2.18) follows from (3.2.13). □

Now we are ready to prove the oscillation estimate which is a key estimate towards the regularity of u/δ.

Proposition 3.2.1. *Let u be a bounded continuous function such that $|Lu| \le K$ in Ω, for some constant K, and $u = 0$ in Ω^c. Given any $x_0 \in \partial\Omega$, let D_R be as in the Definition 3.2.1. Then for some $\tau \in (0, 1 \wedge (2-\alpha))$ there exists C, dependent on Ω, d, α and $\widehat{k}$ but not on x_0, such that*

$$\sup_{D_R} \frac{u}{\delta} - \inf_{D_R} \frac{u}{\delta} \le CKR^{\tau} \tag{3.2.20}$$

for all $R \le \rho_0$, where $\rho_0 > 0$ is a constant depending only on $\Omega, d, \hat{\alpha}$ and $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\widehat{k}(y)\mathrm{d}y$.

Proof. For the proof we follow a standard method, similar to [144], with the help of Lemmas 3.2.3 to 3.2.5. Fix $x_0 \in \partial\Omega$ and consider $\rho_0 > 0$ to be chosen later. With no loss of generality, we assume $x_0 = 0$. In view of (3.1.2), we only consider the case $K > 0$. By considering u/K instead of u, we may assume that $K = 1$, that is, $|Lu| \le 1$ in Ω. From Lemma 3.1.1 we note that $|u|_{C^{0,1}(\mathbb{R}^d)} \le C_1$. For $\alpha \in (0, 1)$, we can calculate Iu classically and $|Iu| \le \tilde{C}$ in Ω, we can combine the nonlocal term on the RHS and only deal with Δu. In this case the proof is simpler and can be done following the same method as for the local case (the proof below also works with minor modifications). Therefore, we only deal with $\alpha \in [1,2)$.

We show that there exists $G > 0$, $\rho_1 \in (0, \rho_0)$ and $\tau \in (0,1)$, dependent only on Ω, d and $\widehat{k}$, and monotone sequences $\{M_k\}$ and $\{m_k\}$ such that, for all $k \geq 0$,

$$M_k - m_k = \frac{1}{4^{k\tau}}, \quad -1 \leq m_k \leq m_{k+1} < M_{k+1} \leq M_k \leq 1, \tag{3.2.21}$$

and

$$m_k \leq G^{-1}\frac{u}{\delta} \leq M_k \quad \text{in} \quad D_{R_k} = D_{R_k}(x_0), \quad \text{where} \quad R_k = \frac{\rho_1}{4^k}. \tag{3.2.22}$$

Note that (3.2.22) is equivalent to the following

$$m_k\delta \leq G^{-1}u \leq M_k\delta, \quad \text{in} \quad B_{R_k} = B_{R_k}(x_0), \quad \text{where} \quad R_k = \frac{\rho_1}{4^k}. \tag{3.2.23}$$

Next we construct monotone sequences $\{M_k\}$ and $\{m_k\}$ by induction.

The existence of M_0 and m_0 such that (3.2.21) and (3.2.23) hold for $k = 0$ is guaranteed by Lemma 3.1.1. Assume that we have the sequences up to M_k and m_k. We want to show the existence of M_{k+1} and m_{k+1} such that (3.2.21)-(3.2.23) hold. We set

$$u_k = \frac{1}{G}u - m_k\delta. \tag{3.2.24}$$

Note that to apply Lemma 3.2.4 we need u_k to be nonnegative in $\mathbb{R}^d$. Therefore we work with u_k^+, the positive part of u_k. Let $u_k = u_k^+ - u_k^-$ and by the induction hypothesis,

$$u_k^+ = u_k \quad \text{and} \quad u_k^- = 0 \quad \text{in} \quad B_{R_k}. \tag{3.2.25}$$

We need a lower bound on u_k. Since $u_k \geq 0$ in B_{R_k}, we get for $x \in B_{R_k}^c$ that

$$u_k(x) = u_k(R_k x_{\mathrm{u}}) + u_k(x) - u_k(R_k x_{\mathrm{u}}) \geq -C_L|x - R_k x_{\mathrm{u}}|, \tag{3.2.26}$$

where $z_{\mathrm{u}} = \frac{1}{|z|}z$ for $z \neq 0$ and C_L denotes a Lipschitz constant of u_k which can be chosen independent of k. Using Lemma 3.1.1 we also have $|u_k| \leq G^{-1} + \mathrm{diam}(\Omega) = C_1$ for all $x \in \mathbb{R}^d$. Thus using (3.2.25) and (3.2.26) we calculate Lu_k^- in $D_{\frac{R_k}{2}}$. Let $x \in D_{R_k/2}(x_0)$. By (3.2.25), $\Delta u_k^-(x) = 0$. Denote by

$$\tilde{g}(r) = \begin{cases} |\log(r)| & \text{for } r > 0,\ \alpha = 1, \\ r^{1-\alpha} & \text{for } r > 0,\ \alpha \in (1,2). \end{cases}$$

Then

$$\begin{aligned}
0 \leq Iu_k^-(x) &= \int_{x+y \notin B_{R_k}} u_k^-(x+y)k(y)\mathrm{d}y \\
&\leq \int_{\left\{|y| \geq \frac{R_k}{2}, x+y \neq 0\right\}} u_k^-(x+y)k(y)\mathrm{d}y \\
&\leq C_L \int_{\left\{\frac{R_k}{2} \leq |y| \leq 1,\ x+y \neq 0\right\}} \Big|(x+y) - R_k(x+y)_{\mathrm{u}}\Big| \widehat{k}(y)\mathrm{d}y + C_1 \int_{|y| \geq 1} J(y)\mathrm{d}y \\
&\leq C_L \int_{\frac{R_k}{2} \leq |y| \leq 1} (|x| + R_k) \frac{1}{|y|^{d+\alpha}} \mathrm{d}y + C_L \int_{\frac{R_k}{2} \leq |y| \leq 1} \frac{1}{|y|^{d+\alpha-1}} \mathrm{d}y + \kappa_1 C_1 \\
&\leq \kappa_3((R_k)^{1-\alpha} + \tilde{g}(R_k/2) + 1) \\
&\leq \kappa_4 \tilde{g}(R_k)
\end{aligned}$$

for some constants $\kappa_1, \kappa_3, \kappa_4$, independent of k.

Now we write $u_k^+ = G^{-1}u - m_k\delta + u_k^-$ and applying the operator L, we get

$$\begin{aligned}
|Lu_k^+| &\leq G^{-1}|Lu| + m_k|L\delta| + |Lu_k^-| \\
&\leq G^{-1} + m_k C g(x) + \kappa_4 \tilde{g}(R_k),
\end{aligned} \tag{3.2.27}$$

using Lemma 3.2.5. Since $\rho_1 \geq R_k \geq \delta$ in D_{R_k}, for $\alpha \geq 1$, we have $R_k^{1-\alpha} \leq \delta^{1-\alpha}$, and hence, from (3.2.27), we have

$$|Lu_k^+| \leq \Big[G^{-1}[\tilde{g}(\rho_1)]^{-1} + C + \kappa_4\Big] g(x) := \kappa_5 g(x) \quad \text{in} \quad D_{R_k/2}.$$

Now we are ready to apply Lemma 3.2.4. Recalling that

$$u_k^+ = u_k = G^{-1}u - m_k\delta \quad \text{in} \quad D_{R_k},$$

we get from Lemmas 3.2.3 and 3.2.4 that

$$\begin{aligned}\sup_{D^+_{\kappa' R_k/2}}\Big(G^{-1}\frac{u}{\delta}-m_k\Big) &\le C\Big(\inf_{D^+_{\kappa' R_k/2}}\Big(G^{-1}\frac{u}{\delta}-m_k\Big)+\kappa_5 R_k^{\hat{\alpha}}\Big)\\ &\le C\Big(\inf_{D_{R_k/4}}\Big(G^{-1}\frac{u}{\delta}-m_k\Big)+\kappa_5 R_k^{\hat{\alpha}}\Big).\end{aligned} \tag{3.2.28}$$

Repeating a similar argument for the function $\tilde{u}_k = M_k\delta - G^{-1}u$, we obtain

$$\sup_{D^+_{\kappa' R_k/2}}\Big(M_k - G^{-1}\frac{u}{\delta}\Big) \le C\Big(\inf_{D_{R_k/4}}\Big(M_k - G^{-1}\frac{u}{\delta}\Big)+\kappa_5 R_k^{\hat{\alpha}}\Big) \tag{3.2.29}$$

Combining (3.2.28) and (3.2.29) we obtain

$$\begin{aligned}M_k - m_k &\le C\Big(\inf_{D^+_{R_k/4}}\Big(M_k - G^{-1}\frac{u}{\delta}\Big)+\inf_{D^+_{R_k/4}}\Big(G^{-1}\frac{u}{\delta}-m_k\Big)+\kappa_5 R_k^{\hat{\alpha}}\Big)\\ &= C\Big(\inf_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} - \sup_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} + M_k - m_k + \kappa_5 R_k^{\hat{\alpha}}\Big).\end{aligned} \tag{3.2.30}$$

Putting $M_k - m_k = \frac{1}{4^{\tau k}}$ in (3.2.30), we have

$$\begin{aligned}\sup_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} - \inf_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} &\le \Big(\frac{C-1}{C}\frac{1}{4^{\tau k}}+\kappa_5 R_k^{\hat{\alpha}}\Big)\\ &= \frac{1}{4^{\tau k}}\Big(\frac{C-1}{C}+\kappa_5 R_k^{\hat{\alpha}}4^{\tau k}\Big).\end{aligned} \tag{3.2.31}$$

Since $R_k = \frac{\rho_1}{4^k}$ for $\rho_1 \in (0,\rho_0)$, we can choose ρ_0 and τ small so that

$$\Big(\frac{C-1}{C}+\kappa_5 R_k^{\hat{\alpha}}4^{\tau k}\Big) \le \frac{1}{4^{\tau}}.$$

Putting in (3.2.31) we obtain

$$\sup_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} - \inf_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} \le \frac{1}{4^{\tau(k+1)}}.$$

Thus we find m_{k+1} and M_{k+1} such that (3.2.21) and (3.2.22) hold. It is easy to prove (3.2.20) from (3.2.21)-(3.2.22). $\square$

Now we are ready to establish Hölder regularity of u/δ up to the boundary.

Theorem 3.2.1. *Suppose that Assumption 3.0.1 holds. Let u be a viscosity solution to (3.0.1). Then there exists $\kappa \in (0, (2-\alpha) \wedge 1)$ such that*

$$\|u/\delta\|_{C^{\kappa}(\bar{\Omega})} \leq C_1 K, \tag{3.2.32}$$

for some constant C_1, where κ, C_1 depend on $d, C_0, \widehat{k}, \Omega$.

Proof. As mentioned before, it is enough to consider (3.2.1). Replacing u by $\frac{u}{CK}$ we may assume that $|Lu| \leq 1$ in Ω. Let $v = u/\delta$. From Lemma 3.1.1 we then have

$$\|v\|_{L^\infty(\Omega)} \leq C, \tag{3.2.33}$$

for some constant C. Also, from Theorem 3.1.2 we have

$$\|u\|_{C^{0,1}(\mathbb{R}^d)} \leq C. \tag{3.2.34}$$

It is also easily seen that for any $x \in \Omega$ with $R = \delta(x)$ we have

$$\sup_{z_1, z_2 \in B_{R/2}(x)} \frac{|\delta^{-1}(z_1) - \delta^{-1}(z_2)|}{|z_1 - z_2|} \leq C R^{-2}.$$

Combining it with (3.2.34) gives

$$\sup_{z_1, z_2 \in B_{R/2}(x)} \frac{|v(z_1) - v(z_2)|}{|z_1 - z_2|} \leq C(1 + R^{-2}). \tag{3.2.35}$$

Again, by Proposition 3.2.1, for each $x_0 \in \partial\Omega$ and for all $r > 0$ we have

$$\sup_{D_r(x_0)} v - \inf_{D_r(x_0)} v \leq C r^{\tau}. \tag{3.2.36}$$

where $D_r(x_0) = B_r(x_0) \cap \Omega$ as before. To complete the proof it is enough to show that

$$\sup_{x, y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^{\kappa}} \leq C, \tag{3.2.37}$$

for some $\eta > 0$. Consider $x, y \in \Omega$ and let $r = |x - y|$. We also suppose that

$\delta(x) \geq \delta(y)$. If $r \geq 1/2$, then

$$\frac{|v(x)-v(y)|}{|x-y|^{\upkappa}} \leq C2^{1+\upkappa},$$

by (3.2.33). So we suppose $|x-y| = r < 1/2$. Let $R = \delta(x)$ and $x_0, y_0 \in \partial\Omega$ satisfying $\delta(x) = |x-x_0|$ and $\delta(y) = |y-y_0|$. Fix $p > 2$. Set $\kappa = [2 + \operatorname{diam}\Omega]^{-p}$. If $r \leq \kappa R^p$, then $r < \frac{1}{2}R$. In this case, it follows from (3.2.35) that

$$|v(x)-v(y)| \leq C(1+R^{-2})r \leq C(r + \kappa^{2/p} r^{1-2/p}) \leq C_1 r^{1-2/p}.$$

Again, if $r \geq \kappa R^p$, we have $R \leq [r/\kappa]^{\frac{1}{p}}$. Thus, $y \in B_{\kappa_1 r^{\frac{1}{p}}}(x_0)$ for $\kappa_1 = 1 + \kappa^{-1/p}$. From (3.2.36) we then have

$$|v(x)-v(y)| \leq C_2 r^{\uptau/p}.$$

Thus (3.2.37) follows by fixing $\upkappa = \min\{\frac{\uptau}{p}, 1 - \frac{2}{p}\}$. This completes the proof. $\square$

Remark 3.2.2. The regularity of $\partial\Omega$ in Theorem 3.2.1 can be relaxed to $C^{1,1}$. In this case, δ will be a $C^{1,1}$ function. Therefore, $I\delta$ is defined classically and $L\delta$ can be interpreted in the viscosity sense (see [49]). The proof of Theorem 3.2.1 goes through due to the coupling result in Theorem 3.1.1.

3.3 Fine boundary regularity of Du

Using Theorem 3.2.1 we show that $Du \in C^\gamma(\Omega)$ for some $\gamma > 0$. Recall (3.0.1)

$$\begin{aligned} Lu + C_0|Du| &\geq -K \quad \text{in } \Omega, \\ Lu - C_0|Du| &\leq K \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c. \end{aligned} \tag{3.3.1}$$

Let $v = u/\delta$. From Theorem 3.2.1 we know that $v \in C^{\upkappa}(\Omega)$. We extend v in all of $\mathbb{R}^d$ as a $C^{\upkappa}$ function without altering its $C^{\upkappa}$ norm (cf. [144, Lemma 3.8]). Below we find the equations satisfied by v.

Lemma 3.3.1. *If $|Lu| \leq C$ in Ω and $u = 0$ in Ω^c, then we have*

$$\frac{1}{\delta}[-C - vL\delta - Z[v,\delta]] \leq Lv + 2\frac{D\delta}{\delta} \cdot Dv \leq \frac{1}{\delta}[C - vL\delta - Z[v,\delta]], \tag{3.3.2}$$

in Ω, where

$$Z[v,\delta](x) = \int_{\mathbb{R}^d} (v(y) - v(x))(\delta(y) - \delta(x))k(y-x)\mathrm{d}y.$$

Proof. First of all, since $u \in C^1(\Omega)$ Lemma 3.0.1, we have $v \in C^1(\Omega)$. Therefore, $Z[v,\delta]$ is continuous in Ω. Consider a test function $\psi \in C^2(\Omega)$ that touches v from above at $x \in \Omega$. Define

$$\psi_r(z) = \begin{cases} \psi(z) & \text{in } B_r(x), \\ v(z) & \text{in } B_r^c(x). \end{cases}$$

By our assertion, we must have $\psi_r \geq v$ for all r small. To verify (3.3.2) we must show that

$$L\psi_r(x) + 2\frac{D\delta}{\delta} \cdot D\psi_r(x) \geq \frac{1}{\delta(x)}[-C + v(x)L\delta(x) - Z[v,\delta](x)], \tag{3.3.3}$$

for some r small. We define

$$\tilde{\psi}_r(z) = \begin{cases} \delta(z)\psi(z) & \text{in } B_r(x), \\ u(z) & \text{in } B_r^c(x). \end{cases}$$

Then, $\tilde{\psi}_r \geq u$ for all r small. Since $|Lu| \leq C$ and $\delta\psi_r = \tilde{\psi}_r$, we obtain

$$-C \leq L\tilde{\psi}_r(x) = \delta(x)L\psi_r(x) + v(x)L\delta(x) + 2D\delta(x) \cdot D\psi_r(x) + Z[\psi_r,\delta](x)$$

for all r small. Rearranging the terms we have

$$-C - v(x)L\delta(x) - Z[\psi_r,\delta](x) \leq \delta(x)L\psi_r(x) + 2D\delta(x) \cdot D\psi_r(x). \tag{3.3.4}$$

Let $r_1 \leq r$. Since ψ_r is decreasing with r, we get from (3.3.4) that

$$\delta(x)L\psi_r(x) + 2D\delta(x) \cdot D\psi_r(x) \geq \delta(x)L\psi_{r_1}(x) + 2D\delta(x) \cdot D\psi_r(x)$$

$$
\begin{aligned}
&\geq \lim_{r_1\to 0}[-C - v(x)L\delta(x) - Z[\psi_{r_1},\delta](x)] \\
&= [-C - v(x)L\delta(x) - Z[v,\delta](x)],
\end{aligned}
$$

by dominated convergence theorem. This gives (3.3.3). Similarly we can verify the other side of (3.3.2). □

In order to prove the fine boundary regularity of Du, we also need the following estimate on v. Define $\Omega_\sigma = \{x \in \Omega \ : \ \mathrm{dist}(x, \Omega^c) \geq \sigma\}$. Then we have

Lemma 3.3.2. *For some constant C it holds that*

$$
\|Dv\|_{L^\infty(\Omega_\sigma)} \leq CK\sigma^{\kappa-1} \quad \textit{for all } \sigma \in (0,1). \tag{3.3.5}
$$

Furthermore, there exists $\eta \in (0,1)$ such that for any $x \in \Omega_\sigma$ and $0 < |x-y| \leq \sigma/8$ we have

$$
\frac{|Dv(y) - Dv(x)|}{|x-y|^\eta} \leq CK\sigma^{\kappa-1-\eta},
$$

for all $\sigma \in (0,1)$.

Proof. As earlier, we suppose $K > 0$. Diving u by K in (3.3.1) we may assume $K = 1$. Using Theorem 3.1.2 we can write $|Lu| \leq C_1$ in Ω, for some constant C_1. By Lemma 3.3.1 we then have

$$
\frac{1}{\delta}[-C_1 - vL\delta - Z[v,\delta]] \leq Lv + 2\frac{D\delta}{\delta}\cdot Dv \leq \frac{1}{\delta}[C_1 - vL\delta - Z[v,\delta]], \tag{3.3.6}
$$

in Ω. Fix $x_0 \in \Omega_\sigma$ and define

$$
w(x) = v(x) - v(x_0).
$$

From (3.3.6) we then obtain

$$
-\frac{1}{\delta}C_1 - \ell \leq Lw + 2\frac{D\delta}{\delta}\cdot Dw \leq \frac{1}{\delta}C_1 - \ell, \tag{3.3.7}
$$

in Ω, where

$$
\ell(x) = \frac{1}{\delta(x)}[w(x)L\delta(x) + Z[v,\delta](x) + v(x_0)L\delta(x)].
$$

Set $r = \frac{\sigma}{2}$. We claim that

$$\|\ell\|_{L^\infty(B_r(x_0)} \le \kappa_1 \sigma^{\kappa-2}, \quad \text{for all } \sigma \in (0,1), \tag{3.3.8}$$

for some constant κ_1. Let us denote by

$$\xi_1 = \frac{wL\delta}{\delta}, \quad \xi_2 = \frac{1}{\delta} Z[v,\delta] \quad \text{and} \quad \xi_3 = \frac{v(x_0)}{\delta} L\delta.$$

Recall that $\kappa \in (0, (2-\alpha) \wedge 1)$ from Theorem 3.2.1. Since

$$\|\Delta\delta\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \|I\delta\|_{L^\infty(\Omega_\sigma)} \lesssim \begin{cases} (\delta(x) \wedge 1)^{1-\alpha} & \text{for} \quad \alpha > 1, \\ \log(\frac{1}{\delta(x)\wedge 1}) + 1 & \text{for} \quad \alpha = 1, \\ 1 & \text{for} \quad \alpha \in (0,1), \end{cases}$$

(cf. Lemma 3.2.5), and

$$\|v\|_{L^\infty}(\mathbb{R}^d) < \infty, \quad \|w\|_{L^\infty(B_r(x_0))} \lesssim r^\kappa,$$

it follows that

$$\|\xi_3\|_{L^\infty(B_r(x_0))} \lesssim \left\{ \begin{matrix} \sigma^{-(\alpha\vee 1)} & \text{for } \alpha \ne 1, \\ \sigma^{-1}|\log\sigma| & \text{for } \alpha = 1, \end{matrix} \right\} \lesssim \sigma^{-2+\kappa},$$

and

$$\|\xi_1\|_{L^\infty(B_r(x_0))} \lesssim \left\{ \begin{matrix} \sigma^{\kappa-1+1-(1\vee\alpha)} & \text{for } \alpha \ne 1, \\ \sigma^{\kappa-1}|\log\sigma| & \text{for } \alpha = 1, \end{matrix} \right\} \lesssim \sigma^{-2+\kappa}.$$

So we are left to compute the bound for ξ_2. Let $x \in B_r(x_0)$. Denote by $\hat{r} = \delta(x)/4$. Note that

$$\delta(x) \ge \delta(x_0) - |x - x_0| \ge 2r - r = r \Rightarrow \hat{r} \ge r/4.$$

Thus, since $u \in C^1(\Omega)$ by Lemma 3.0.1 (as mentioned before, the proof of [133] works for inequations),

$$|Dv| \le |\frac{Du}{\delta}| + |\frac{uD\delta}{\delta^2}| \lesssim [\delta(x)]^{-1} \quad \text{in } B_{\hat{r}}(x),$$

using (3.1.1). Since δ is Lipscitz and bounded in $\mathbb{R}^d$, we obtain

$$
\begin{aligned}
|Z[v,\delta](x)| \leq \Lambda \int_{y\in B_{\hat{r}}(x)} & \frac{|\delta(x)-\delta(y)||v(x)-v(y)|}{|x-y|^{d+\alpha}}\mathrm{d}y \\
& + \int_{y\in B^c_{\hat{r}}(x)} |\delta(x)-\delta(y)||v(x)-v(y)|\widehat{k}(y-x)\mathrm{d}y \\
\lesssim{}& [\delta(x)]^{-1}\int_{y\in B_{\hat{r}}(x)} |x-y|^{2-d-\alpha}\mathrm{d}y \\
& + \int_{y\in B_1(x)\setminus B_{\hat{r}}(x)} \frac{(\delta(x)-\delta(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\mathrm{d}y \\
& + \int_{|y|>1} |\delta(x)-\delta(y+x)||v(x)-v(y+x)|J(y)\mathrm{d}y \\
\lesssim{}& [\delta(x)]^{1-\alpha} + \int_{y\in B_1(x)\setminus B^c_{\hat{r}}(x)} \frac{|\delta(x)-\delta(y)||v(x)-v(y)|}{|x-y|^{d+\alpha}}\mathrm{d}y + \kappa_2,
\end{aligned}
$$

for some constant κ_2. The second integration on the right hand side can be computed as follows: for $\alpha \leq 1$ we write

$$
\begin{aligned}
\int_{y\in B_1(x)\setminus B^c_{\hat{r}}(x)} \frac{|\delta(x)-\delta(y)||v(x)-v(y)|}{|x-y|^{d+\alpha}}\mathrm{d}y &\lesssim \int_{y\in B_1(x)\setminus B^c_{\hat{r}}(x)} |x-y|^{-d-\alpha+1+\kappa}\mathrm{d}y \\
&\lesssim (1-\hat{r}^{1-\alpha+\kappa}) \lesssim \sigma^{-1+\kappa},
\end{aligned}
$$

whereas for $\alpha \in (1,2)$ we can compute it as

$$
\begin{aligned}
\int_{y\in B_1(x)\setminus B^c_{\hat{r}}(x)} \frac{|\delta(x)-\delta(y)||v(x)-v(y)|}{|x-y|^{n+\alpha}}\mathrm{d}y &\lesssim \int_{y\in B_1(x)\setminus B^c_{\hat{r}}(x)} |x-y|^{1-n-\alpha} \\
&\lesssim \hat{r}^{-\alpha+1} \lesssim \sigma^{-1+\kappa}.
\end{aligned}
$$

Combining the above estimates we obtain

$$
\|\xi_2\|_{L^\infty B_r(x_0)} \lesssim \sigma^{-2+\kappa}.
$$

Thus we have established the claim (3.3.8).

Let us now define $\zeta(z) = w(\frac{r}{2}z + x_0)$. Letting $b(z) = 2\frac{D\delta(\frac{r}{2}z+x_0)}{\delta(\frac{r}{2}z+x_0)}$ and $r_1 = \frac{r}{2}$ it follows from (3.3.7) that

$$r_1^2\left(-\frac{C}{\delta}-\ell\right)(r_1z+x_0)\leq \Delta\zeta+I^{r_1}\zeta+r_1b(z)\cdot D\zeta\leq r_1^2\left(\frac{C}{\delta}-\ell\right)(r_1z+x_0) \quad (3.3.9)$$

in $B_2(0)$, where

$$I^{r_1}f(x)=r_1^2\frac{1}{2}\int_{\mathbb{R}^d}(f(x+y)+f(x-y)-f(x))k(r_1y)r_1^d\mathrm{d}y.$$

Consider a cut-off function φ satisfying $\varphi=1$ in $B_{3/2}(0)$ and $\varphi=0$ in $B_2^c(0)$. Defining $\tilde{\zeta}=\zeta\varphi$ we get from (3.3.9) that

$$|\Delta\tilde{\zeta}(z)+I^{r_1}\tilde{\zeta}(z)+r_1b(z)\cdot D\tilde{\zeta}|\leq r_1^2(\frac{C}{\delta}+|\ell|)(r_1z+x_0)+|I^{r_1}((\varphi-1)\zeta)|$$

in $B_1(0)$. Since

$$\|r_1b\|_{L^\infty(B_1(0))}\leq\kappa_3\quad\text{for all }\rho\in(0,1),$$

applying Lemma 3.0.1 we obtain, for some $\eta\in(0,1)$,

$$\begin{aligned}\|D\zeta\|_{C^\eta(B_{\frac{1}{2}})}\leq\kappa_6\Big(&\|\tilde{\zeta}\|_{L^\infty(\mathbb{R}^d)}+Cr_1+r_1^2\|\ell(r_1\cdot+x_0)\|_{L^\infty(B_1)}\\&+\|I^{r_1}((\varphi-1)\zeta)\|_{L^\infty(B_1)}\Big),\end{aligned} \quad (3.3.10)$$

for some constant κ_6 independent of $\rho\in(0,1)$. Since v is in $C^\kappa(\mathbb{R}^d)$, it follows that

$$\|\tilde{\zeta}\|_{L^\infty(\mathbb{R}^d)}=\|\tilde{\zeta}\|_{L^\infty(B_2)}\leq\|\zeta\|_{L^\infty(B_2)}\lesssim r^\kappa.$$

Also, by (3.3.8),

$$r_1^2\|\ell(r_1\cdot+x_0)\|_{L^\infty(B_1)}\lesssim\sigma^\kappa.$$

Note that, for $z\in B_1(0)$,

$$\begin{aligned}|I^{r_1}(\varphi-1)\zeta|&\leq r_1^{2-\alpha}\int_{|y|\geq1/2}|(\varphi(x+y)-1)\zeta(x+y)|\widehat{k}(y)\mathrm{d}y\\&\leq 2r_1^{2-\alpha}\|v\|_{L^\infty(\mathbb{R}^d)}\int_{|y|\geq1/2}\widehat{k}(y)\mathrm{d}y\\&\leq\kappa_3r_1^{2-\alpha}\end{aligned}$$

for some constant κ_3. Putting these estimates in (3.3.10) and calculating the gradient at $z = 0$ we obtain

$$|Dv(x_0)| \leq \kappa_4 \sigma^{-1+\kappa},$$

for all $\sigma \in (0, 1)$. This proves the first part.

For the second part, compute the Hölder ratio with $D\zeta(0) - D\zeta(z)$ where $z = \frac{2}{r}(y - x_0)$ for $|x_0 - y| \leq \sigma/8$. This completes the proof.

□

Now we can establish the Hölder regularity of the gradient up to the boundary (compare it with Fall-Jarohs [81]). This is the content of our next result.

Theorem 3.3.1. *Let Assumption 3.0.1 hold. There exist constants γ, C, dependent on $\Omega, C_0, d, \widehat{k}$, such that for any solution u of (3.0.1) we have*

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leq CK. \tag{3.3.11}$$

Proof. Since $u = v\delta$ it follows that

$$Du = vD\delta + \delta Dv.$$

Since $\delta \in C^2(\bar{\Omega})$, it follows from Theorem 3.2.1 that $vD\delta \in C^{\kappa}(\bar{\Omega})$. Thus, we only need to concentrate on $\vartheta = \delta Dv$. Consider η from Lemma 3.3.2 and with no loss of generality, we may fix $\eta \in (0, \kappa)$.

For $|x - y| \geq \frac{1}{8}(\delta(x) \vee \delta(y))$ it follows from (3.3.5) that

$$\frac{|\vartheta(x) - \vartheta(y)|}{|x - y|^{\eta}} \leq C(\delta^{\kappa}(x) + \delta^{\kappa}(y))(\delta(x) \vee \delta(y))^{-\eta} \leq 2C.$$

So consider the case $|x - y| < \frac{1}{8}(\delta(x) \vee \delta(y))$. Without loss of generality, we may assume that $|x - y| < \frac{1}{8}\delta(x)$. Then

$$\frac{9}{8}\delta(x) \geq |x - y| + \delta(x) \geq \delta(y) \geq \delta(x) - |x - y| \geq \frac{7}{8}\delta(x).$$

By Lemma 3.3.2, it follows

$$\begin{aligned}\frac{|\vartheta(x)-\vartheta(y)|}{|x-y|^\eta} &\leq |Dv(x)|\frac{|\delta(x)-\delta(y)|}{|x-y|^\eta}+\delta(y)\frac{|Dv(x)-Dv(y)|}{|x-y|^\eta}\\ &\lesssim \delta(x)^{-1+\kappa}(\delta(x))^{1-\eta}+\delta(y)[\delta(x)]^{\kappa-1-\eta}\\ &\leq C.\end{aligned}$$

This completes the proof by setting $\gamma=\eta$. □

Remark 3.3.1. By the dependency of the constants in Theorems 3.1.2, 3.2.1, and 3.3.1 on $\widehat{k}$ we mean the dependency on α, Λ and $\int_{|y|\geq 1} J(y)\mathrm{d}y$.

To cite a specific application of the above results, let us consider $u, v \in C(\mathbb{R}^d)$ satisfying

$$\begin{aligned} Lu + H_1(Du, x) &= 0, \quad \text{in } \Omega, \quad u=0 \quad \text{in } \Omega^c,\\ Lv + H_2(Dv, x) &= 0, \quad \text{in } \Omega, \quad v=0 \quad \text{in } \Omega^c,\end{aligned}$$

respectively. If $|H_1(p,x)-H_2(q,x)|\leq C_0|p-q|+K$ for all $p,q\in\mathbb{R}^d$ and $x\in\Omega$, then using the interior regularity of u,v from Lemma 3.0.1 and the coupling result Theorem 3.1.1 it can be easily seen that $w=u-v$ satisfies (3.0.1). Our result Theorem 3.3.1 then gives a $C^{1,\gamma}$ estimate of w up to the boundary. The above results can also be used to establish anti-maximum principle for the generalized principal eigenvalues of nonlinear operators of the form (3.0.2) (cf. [27, 34, 64]).

Remark 3.3.2. Though the above results are mentioned for viscosity solutions, Theorem 3.1.2 and Theorem 3.3.1 can also be applied for weak solutions (at least for equations). To see this, let us assume that Ω be a $C^{2,\kappa}$ domain, $\kappa\in(0,1)$. Suppose that for some given Lipschitz function $f:\mathbb{R}^d\to\mathbb{R}$, there exists a unique weak solution $u\in H_0^1(\Omega)$ to

$$Lu + f(Du) = g \quad \text{in } \Omega, \quad u=0 \quad \text{in } \Omega^c, \tag{3.3.12}$$

for every $g\in L^\infty(\Omega)$. Now consider a sequence of smooth mollifications g_ε of g such that $\sup_\Omega |g_\varepsilon - g|\to 0$, as $\varepsilon\to 0$. Let u_ε be the unique weak solution to (3.3.12) corresponding to g_ε. Since, by Sobolev embedding $u_{x_i}\in L^p(\Omega)$ for $p\in[1,\frac{2d}{d-2}]$, applying [96, Theorem 3.1.22] and a bootstrapping argument we obtain that for some $p > n$

$$\|u_\varepsilon\|_{W^{2,p}(\Omega)} \le C\left(1 + \|Du_\varepsilon\|_{L^2(\Omega)} + \|g_\varepsilon\|_{L^\infty(\Omega)}\right), \tag{3.3.13}$$

for some constant C independent of u_ε. This, of course, implies $u_\varepsilon \in C^{1,\gamma}(\bar{\Omega})$. Applying [96, Theorem 3.1.12] we have $u_\varepsilon \in C^{2,\gamma}(\bar{\Omega})$ and therefore, u_ε is a viscosity solution to (3.3.12) when g is replaced by g_ε. Hence we can Theorems 3.1.2, 3.2.1, and 3.3.1 on u_ε. In particular,

$$\sup_{\varepsilon\in(0,1)} \|Du_\varepsilon\|_{L^\infty(\Omega)} < \infty.$$

Now, using the stability estimate (3.3.13), we can pass the limit, as $\varepsilon \to 0$, to show that $u_\varepsilon \to u$ where u is the weak solution to (3.3.12) with data g and u also satisfies the estimates in Theorems 3.1.2, 3.2.1, and 3.3.1.

3.4 Overdetermined problems

Next we discuss another application of Theorems 3.2.1 and 3.3.1 to study an overdetermined problem. More precisely, we consider a solution u to the problem

$$\begin{aligned} Lu + H(|Du|) = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \Omega^c, \quad u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \mathrm{n}} = c \quad \text{on } \partial\Omega, \end{aligned} \tag{3.4.1}$$

where n is the unit inward normal and $H : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^d \to \mathbb{R}$ are locally Lipschitz. We will show that Ω must be a ball, provided the nonlocal kernel k satisfies certain conditions. Historically Overdetermined problem originates from the famous work of [152], where he answered the following problem posed by Prof. R. L. Fosdick. Suppose there exists a positive solution u to the equation

$$-\Delta u = 1 \quad \text{in } \Omega,$$

together with the boundary condition

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \mathrm{n}} = c \quad \text{on } \partial\Omega,$$

then Ω must be a ball. Serrin's work has been generalized for a vast class of operators, see for instance, [30, 34, 62, 82, 83, 85, 86, 156]. Here we follow the method of [30, 82] to establish our result on the overdetermined problem concerning (3.4.1).

Our main result of this section that we obtain in [38], is the following.

Theorem 3.4.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded set with C^2 boundary. Suppose that Assumption 3.0.1 holds, $k = k(|y|)$ and $k : (0, \infty) \to (0, \infty)$ is strictly decreasing. Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and u be a viscosity solution to*

$$\begin{aligned} Lu + H(|Du|) &= f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \quad u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathrm{n}} &= c \quad \text{on } \partial\Omega, \end{aligned} \tag{3.4.2}$$

for some fixed $c > 0$, where n is the unit inward normal on $\partial\Omega$. Then Ω must be a ball. Furthermore, u is radially symmetric and strictly decreasing in the radial direction.

Proof of Theorem 3.4.1 follows from the boundary estimates in Theorem 3.2.1 combined with the approach of [30,82]. Also, note that we have taken k to be positive valued. This is just for convenience and the proofs below can be easily modified to include kernel k that is non-increasing but strictly decreasing in a neighbourhood of 0, provided we assume Ω to be connected. We provide a sketch for the proof of Theorem 3.4.1 and the finer details can be found in [30, 82]. From Theorems 3.1.2 and 3.3.1 we see that $u \in C^{0,1}(\mathbb{R}^d) \cap C^{1,\gamma}(\bar{\Omega})$, and therefore, $u \in C^{2,\eta}(\Omega)$ by [133]. Therefore, we can assume that u is a classical solution to (3.4.2).

Given a unit vector e, let us define the half space

$$\mathcal{H} = \mathcal{H}_{\lambda,e} = \{x \in \mathbb{R}^d : x \cdot e > \lambda\},$$

and let $\bar{x} = \mathscr{R}_{\lambda,e}(x) = x - 2(x \cdot e)e + 2\lambda e$ be the reflection of x along $\partial\mathcal{H} = \{x \cdot e = \lambda\}$. We say $v : \mathbb{R}^d \to \mathbb{R}$ is anti-symmetric if $v(x) = -v(\bar{x})$ for all $x \in \mathbb{R}^d$. Now let $D \subset \mathcal{H}$

be a bounded open set and u be a bounded anti-symmetric solution to

$$\begin{aligned} Lu - \beta|Du| \le g \quad & \text{in } D \\ u \ge 0 \quad & \text{in } \mathcal{H} \setminus D, \end{aligned}$$

where $\beta > 0$ is a fixed constant. Let

$$v = \begin{cases} -u & \text{in } \{u < 0\} \cap D, \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily seen that v solves

$$Lv - \beta|Dv| \ge -g \quad \text{in} \quad \Sigma := \{u < 0\} \cap D, \tag{3.4.3}$$

in the viscosity sense. To check (3.4.3), consider $x \in \Sigma$ and test function ϕ such that $\phi(x) = v(x)$ and $\phi(y) > v(y)$ for $y \in \mathbb{R}^d \setminus \{x\}$. Define $\psi := \phi + (-u - v)$. Then $\psi(x) = -u(x)$ and $\psi(y) > -u(y)$ for $y \in \mathbb{R}^d \setminus \{x\}$. Furthermore, $\psi = \phi$ in Σ. Thus, we get $L\psi(x) + \beta|D\phi(x)| \ge -g(x)$. This implies $L\phi + \beta|D\phi| + L(-u - v)(x) + |D\phi(x)| \ge -g(x)$. Since k is radially decreasing and u is anti-symmetric, it follows that $L(-u - v)(x) \le 0$ (cf. [30, p. 11]). This gives us (3.4.3).

The following narrow domain maximum principle is a consequence of the ABP estimate in [131, Theorem A.4].

Lemma 3.4.1. *Let $\mathcal{H}$ be the half space and $D \subset \mathcal{H}$ be open and bounded. Also, assume c to be bounded. Then there exists a positive constant C, depending on* $\mathrm{diam}(D), d, k$, *such that if $u \in C_b(\mathbb{R}^d)$ is an anti-symmetric supersolution of*

$$\begin{aligned} Lu - \beta|Du| - c(x)u = 0 \quad & \textit{in } D, \\ u \ge 0 \quad & \textit{in } \mathcal{H} \setminus D, \end{aligned}$$

then we have

$$\sup_{\Omega} u^- \le C\|c^+\|_{L^\infty(D)}\|u^-\|_{L^d(D)}.$$

In particular, given $\kappa_\infty > 0$ and $c^+ \le \kappa_\infty$ on D, there is a $\delta > 0$ such that if $|D| < \delta$, we must have $u \ge 0$ in $\mathcal{H}$.

Proof. Set $\Sigma = \{u < 0\} \cap D$ and define v as above. From (3.4.3) we see that

$$Lv + \beta|Dv| + c(x)v \geq 0 \quad \text{in } \Sigma.$$

Since $v \geq 0$ in Σ and $c \leq c^+$, we get

$$Lv + \beta|Dv| + c^+ v \geq 0 \quad \text{in } \Sigma.$$

Taking $f = -c^+ v$ and using [131, Theorem A.4] we obtain, for some constant C_1, that

$$\sup_\Omega u^- = \sup_\Sigma v \leq \sup_{\Sigma^c} |v| + C_1 \|f\|_{L^d(D)} \leq C_1 \|c^+\|_{L^\infty(D)} \|v\|_{L^d(\Sigma)} = C_1 \|c^+\|_{L^\infty(D)} \|u^-\|_{L^d(D)}.$$

This completes the proof of the lemma. □

Next result is a Hopf's lemma for anti-symmetric functions.

Lemma 3.4.2. *Let $\mathcal{H}$ be a half space, $D \subset \mathcal{H}$, and $c \in L^\infty(D)$. If $u \in C_b(\mathbb{R}^d)$ is an anti-symmetric supersolution of $Lu - \beta|Du| - c(x)u = 0$ in D with $u \geq 0$ in $\mathcal{H}$, then either $u \equiv 0$ in $\mathbb{R}^d$ or $u > 0$ in D. Furthermore, if $u \not\equiv 0$ in D and there exists a $x_0 \in \partial D \setminus \partial \mathcal{H}$ with $u(x_0) = 0$ such that there is a ball $B \subset D$ with $x_0 \in \partial B$, then there exists a $C > 0$ such that*

$$\liminf_{t \to 0} \frac{u(x_0 - t\mathrm{n})}{t} \geq C,$$

where n is the inward normal at x_0.

Proof. With any loss of generality, we may assume that $c \geq 0$. Suppose that $u \not\equiv 0$ and $u \not> 0$ in D. Then there exists a compact set $K \subset D$ such that $\inf_K u = \delta > 0$ and a point $x_1 \in D$ such that $u(x_1) = 0$. For ε small enough we can choose the test function

$$\phi(y) = \begin{cases} 0 & \text{for } y \in B_\varepsilon(x_1), \\ u & \text{for } y \in B_\varepsilon^c(x_1). \end{cases}$$

Note that $\Delta\phi(x_1) = 0$, $D\phi(x_1) = 0$. Since k is radially non-increasing and positive, from the proof of [30, Theorem 3.2] it follows that $u \equiv 0$ in D. This contradicts our assertion. Thus either $u \equiv 0$ in $\mathbb{R}^d$ or $u > 0$ in D.

Now we prove the second part of the lemma. Assume that $u > 0$ in D. Let B be a ball in D that touches ∂D at x_0 and $B \Subset \mathcal{H}$. This is possible since $x_0 \in \partial D \setminus \partial\mathcal{H}$. Let ϑ be the positive solution to

$$L\vartheta - \beta|D\vartheta| = -1 \quad \text{in } B, \quad \text{and} \quad \vartheta = 0 \quad \text{in } B^c.$$

Existence of ϑ follows from Theorem 3.3.1 and Leray-Schauder fixed point theorem. Define $w = \kappa(\vartheta - \vartheta \circ \mathscr{R})$. Then we have $Lw \geq -\kappa C$ in B for some positive constant C. Now repeating the arguments of [30, Theorem 3.2] it follows that for some $\kappa > 0$ we have $u \geq w$ in B. To complete the proof we need to apply Hopf's lemma on ϑ at the point x_0 (cf. Theorem 2.2.2). $\square$

Given $\lambda \in \mathbb{R}$, $e \in \partial B_1(0)$, define

$$v(x) = v_{\lambda,e}(x) = u(x) - u(\bar{x}) \quad x \in \mathbb{R}^d, \tag{3.4.4}$$

where $\bar{x} = \mathscr{R}_{\lambda,e}(x)$ denotes the reflection of x by $T_{\lambda,e} \coloneqq \partial\mathcal{H}_{\lambda,e}$ and $\mathcal{H}_{\lambda,e} = \{x \in \mathbb{R}^d : x \cdot e > \lambda\}$. We note that $\mathbb{R}^d \setminus \overline{\mathcal{H}}_{\lambda,e} = \mathcal{H}_{-\lambda,-e}$. Moreover, let $\lambda < l \coloneqq \sup_{x\in D} x \cdot e$. Then $\mathcal{H} \cap \Omega$ is nonempty for all $\lambda < l$ and we put $D_\lambda \coloneqq \mathscr{R}_{\lambda,e}(\Omega \cap \mathcal{H})$. Then for all $\lambda < l$ the function v satisfies

$$\begin{aligned} Lv + \beta|Dv(x)| - c(x)v &\geq 0 \quad \text{in } D_\lambda, \\ Lv - \beta|Dv(x)| - c(x)v &\leq 0 \quad \text{in } D_\lambda, \\ v &\geq 0 \quad \text{in } \mathcal{H}_{-\lambda,-e} \setminus D_\lambda, \\ v(x) &= -v(\bar{x}) \quad \text{for all } x \in \mathbb{R}^d, \end{aligned} \tag{3.4.5}$$

where β is the Lipschitz constant of H on the interval $[0, \sup|Du|]$ and

$$c(x) = \frac{f(u(x)) - f(u(\bar{x}))}{u(x) - u(\bar{x})}\,.$$

In view of Lemmas 3.4.1-3.4.2 and (3.4.5), we see that $v = v_{\lambda,e}$ is either 0 in $\mathbb{R}^d$ or positive in D_λ for λ close to l. Now as we decrease λ one of the following two situations may occur.

Situation A: there is a point $p_0 \in \partial\Omega \cap \partial D_\lambda \setminus T_{\lambda,e}$,

Situation B: $T_{\lambda,e}$ is orthogonal to $\partial\Omega$ at some point $p_0 \in \partial\Omega \cap T_{\lambda,e}$.

λ_0 be the maximal value in $(-\infty, l)$ such that one of these situations occur. We show that Ω is symmetric with respect to $T_{\lambda_0,e}$. This would complete the proof of Theorem 3.4.1 since e is arbitrary. Also, note that, since $u > 0$ in Ω, to establish the symmetry of Ω with respect to $T_{\lambda_0,e}$, it is enough to show that $v = 0$ in $\mathbb{R}^d$. Suppose, to the contrary, that $v > 0$ in D_{λ_0}.

Situation A: In this case we have $v(p_0) = 0$ and therefore, by Theorem 3.3.1 and Lemma 3.4.2, we get $\frac{\partial v}{\partial \mathrm{n}}(p_0) > 0$. But, by (3.4.2), we have

$$\frac{\partial v}{\partial \mathrm{n}}(p_0) = \frac{\partial u}{\partial \mathrm{n}}(p_0) - \frac{\partial u}{\partial \mathrm{n}}(\mathscr{R}(p_0)) = 0.$$

This is a contradiction.

Situation B: This situation is a bit more complicated than the previous one. Set $T = T_{\lambda_0,e}, \mathcal{H} = \mathcal{H}_{\lambda_0,e}$ and $\mathscr{R} = \mathscr{R}_{\lambda_0,e}$. By rotation and translation, we may set $\lambda_0 = 0$, $p_0 = 0$, $e = e_1$ and $e_2 \in T$ is the interior normal at ∂D.

Next two lemmas are crucial to get a contradiction in Situation B.

Lemma 3.4.3. *We have*

$$v(t\bar{\eta}) = o(t^2), \quad \textit{as } \ t \to 0^+,$$

where $\bar{\eta} = e_2 - e_1 = (-1, 1, 0, ..., 0) \in \mathbb{R}^d$.

Proof of Lemma 3.4.3 follows from Theorem 3.2.1 and [30, Lemma 3.2].

Lemma 3.4.4. *Let* $D \subset \mathbb{R}^d, d \geq 2$, *be an open bounded domain such that* $0 \in \partial D$ *and* $\{x_1 = 0\}$ *is orthogonal and there is a ball* $B \subset D$ *with* $\bar{B} \cap \partial\bar{D} = \{0\}$. *Denote*

$$D^* := D \cap \{x_1 < 0\},$$

and assume that $w \in C_b(\mathbb{R}^d)$ *is an anti-symmetric supersolution of*

$$\begin{aligned} Lw - \beta|Dw| - c(x)w = 0 \quad & \textit{in } D^* \\ w \geq 0 \quad & \textit{in } \{x_1 < 0\} \\ w > 0 \quad & \textit{in } D^*. \end{aligned}$$

Let $\bar{\eta} = e_2 - e_1 = (-1, 1, \cdots, 0) \in \mathbb{R}^d$, then there exist positive C, t_0, dependent on D^*, d, such that

$$w(t\bar{\eta}) \geq Ct^2$$

for all $t \in (0, t_0)$.

Clearly, Lemmas 3.4.3 and 3.4.4 give a contradiction to the Situation B.

Proof of Theorem 3.4.1. The proof follows from the above discussion and the arguments in [82, p. 11]. □

In the remaining part of this section we provide a proof of Lemma 3.4.4.

Proof of Lemma 3.4.4. We follow the arguments of [30, Lemma 3.3]. Fix a ball $B = B_R(Re_2) \subset D$ for some $R > 0$ small enough with $\partial B \cap \partial D = \{0\}$. Denote

$$K = \{x_1 < 0\} \cap B.$$

Let $M_1 \Subset D^*$ such that $\theta := \inf_{M_1} w > 0$ and $M_2 = \mathscr{R}(M_1)$, that is, reflection of M_1 with respect to $\{x_1 = 0\}$. Furthermore, we may assume that M_1 to be an open ball and taking R smaller, we also assume that $\mathrm{dist}(K, M_1) > 0$ and $|K| < \varepsilon$ for some small $\varepsilon > 0$. Now let g be the unique positive viscosity solution to

$$\begin{aligned} Lg - \beta|Dg| &= -1 \quad \text{in } B \\ g &= 0 \quad \text{in } B^c. \end{aligned}$$

From Theorem 3.1.2, we know that $\|g\|_{C^{0,1}(\mathbb{R}^d)} \leq C$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$, $\mathrm{support}(\phi) \subset M_1$, $0 \leq \phi \leq 1$ and there exists a $U \subset M_1$ such that $\phi = 1$ in U, $|U| > 0$. Construct a barrier function h of the following form:

$$h(x) = -\kappa x_1 g(x) + \theta\phi(x) - \theta\phi(\mathscr{R}(x)).$$

Choosing $\kappa > 0$ small enough, it can be easily checked that (see [30, Lemma 3.3])

$$Lh - \beta|Dh| + c(x)h \geq 0$$

in K. It is also standard to see that g is radial about the point Re_2 (cf. Theorem 2.4.1). Thus we have : (i) $w-h$ is anti symmetric, (ii) $w-h \geq 0$ in $\{x_1 < 0\} \setminus K$, since because $\theta > 0$, and (iii) $L(w-h) - \beta|D(w-h)| - c(x)(w-h) \leq 0$. Applying Lemma 3.4.1, we obtain $w - h \geq 0$ in $\{x_1 < 0\}$. Hence

$$w(t\bar{\eta}) \geq h(t\bar{\eta}) \geq Ct^2$$

for $t \in (0, t_0)$, where we used Hopf's lemma on g. This completes the proof. $\square$

4

Regularity theory of fully nonlinear intergo-differential equation

In Chapter 3, we studied boundary regularity property of linear integro-diffrential operator. This motivates us to analyze regularity theory for a large class of fully nonlinear integro-differential operators of the form

$$\mathcal{L}u(x) := \mathcal{L}[x, u] = \sup_{\theta \in \Theta} \inf_{\nu \in \Gamma} \left\{ \mathrm{Tr}(a_{\theta\nu}(x) D^2 u(x)) + \mathcal{I}_{\theta\nu}[x, u] \right\}, \tag{4.0.1}$$

for some index sets Θ, Γ. The coefficient $a_{\theta\nu} : \Omega \to \mathbb{R}^{d \times d}$ is a matrix valued function and $\mathcal{I}_{\theta\nu}$ is a nonlocal operator defined as

$$\mathcal{I}_{\theta\nu}u(x) := \mathcal{I}_{\theta\nu}[x, u] = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_1}(y) Du(x) \cdot y) N_{\theta\nu}(x, y) \, \mathrm{d}y.$$

Let Ω be a bounded C^2 domain in $\mathbb{R}^d$. We want to study regularity up to the boundary of the solution u to the inequations

$$\begin{aligned} \mathcal{L}u + C_0 |Du| &\geq -K \quad \text{in } \Omega, \\ \mathcal{L}u - C_0 |Du| &\leq K \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \end{aligned} \tag{4.0.2}$$

where $C_0, K \geq 0$. The above inequations (4.0.2) are motivated by Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations of the form

$$Iu(x) := \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\{L_{\theta\nu}u(x) + c_{\theta\nu}(x)u(x) + f_{\theta\nu}(x)\} = 0, \tag{4.0.3}$$

where

$$L_{\theta\nu}u(x) = \operatorname{Tr} a_{\theta\nu}(x)D^2u(x) + \mathcal{I}_{\theta\nu}[x,u] + b_{\theta\nu}(x)\cdot Du(x), \tag{4.0.4}$$

$b_{\theta\nu}(\cdot), c_{\theta\nu}(\cdot)$ and $f_{\theta\nu}(\cdot)$ are bounded functions on Ω. These linear operators (4.0.4) are extended generator for a wide class of d-dimensional Feller processes (more precisely, jump diffusions) and the nonlinear operator $Iu(\cdot)$ has its connection to the stochastic control problems and differential games (see [25,31] and the references therein). The other motivation to study such operators comes from the generalization of (4.0.3). Recall that if I is any translation invariant operator that maps $C_b^2(\mathbb{R}^d)$ functions to $C_b(\mathbb{R}^d)$ functions and satisfies the degenerate ellipticity assumption then I should have the form in (4.0.3)(see Chapter 1).

We set the following assumptions on the coefficient $a_{\theta\nu}(\cdot)$ and the kernel $N_{\theta\nu}(x,y)$, throughout this chapter.

Assumption 4.0.1.

(a) $a_{\theta\nu} \in C_b(\bar{\Omega}, \mathbf{S}^d)$ are uniformly continuous with respect to the parameters $\theta \in \Theta, \nu \in \Gamma$. Furthermore, $a_{\theta\nu}(\cdot)$ satisfies the uniform ellipticity condition $\lambda\boldsymbol{I} \leq a_{\theta\nu}(\cdot) \leq \Lambda\boldsymbol{I}$ for some $0 < \lambda \leq \Lambda$ where $\boldsymbol{I}$ denotes the $d\times d$ identity matrix.

(b) For each $\theta \in \Theta, \nu \in \Gamma$, $N_{\theta\nu} : \Omega \times \mathbb{R}^d$ is a measurable function and for some $\alpha \in (0,2)$ there exists a kernel k that is measurable in $\mathbb{R}^d \setminus \{0\}$ such that for any $\theta \in \Theta, \nu \in \Gamma, x \in \Omega$, we have

$$0 \leq N_{\theta\nu}(x,y) \leq k(y)$$

and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^\alpha)k(y)\mathrm{d}y < \infty,$$

where we denote $p \wedge q := \min\{p,q\}$ for $p, q \in \mathbb{R}$.

In the context of Chapter 1 (1.2.3), $\mathcal{L}$ is a fully nonlinear integro-differential operator with respect to the class $\mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$ where $\mathfrak{A}$ is a collection of all function $a_{\theta\nu}(\cdot)$ satisfying Assumption 4.0.1(a) and $\mathfrak{B}$ is a collection of all kernel $N_{\theta\nu}(\cdot)$ satisfying Assumption 4.0.1(b). Another thing to notice here is that the linear class $\mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$ contains the linear integro-differential operators considered in Chapter 3 (see Assumption 3.0.1) to study regularity theory. In fact Assumption 4.0.1(b) is quite general as compared to Assumption 3.0.1 and includes large collection of nonlocal kernels. Thus regularity results obtained in this chapter will not only generalize the results obtained in Chapter 3 for a fully nonlinear integro-differential operator but also for a large class of linear integro-differential operator that were not considered in Chapter 3. In this context, we leverage techniques similar to those employed in Chapter 3, up to a certain extent to obtain the regularity results. However, the intricacies of the current situation necessitate a more delicate approach, primarily owing to the generality of the operators in consideration. Consequently, this refined methodology not only validates the desired regularity results but also extends its applicability to a broader class of linear integro-differential operators.

Let us briefly comment on Assumption 4.0.1. The uniform continuity of $a_{\theta\nu}(\cdot)$ is required for the stability of viscosity sub or supersolutions under appropriate limits and is useful in proving interior $C^{1,\alpha}$ regularity. The Assumption 4.0.1(b) includes a large class of kernels. We mention some of them below.

Example 4.0.1. Consider the following kernels $N_{\theta\nu}(x,y)$:

(i) $N_{\theta\nu}(x,y) = \frac{1}{|y|^{d+\sigma}}$ for $\sigma \in (0,2)$. Clearly we can take $k(y) = \frac{1}{|y|^{d+\sigma}}$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y)\mathrm{d}y$ is finite for $\alpha \in [1+\sigma/2, 2)$.

(ii) $N_{\theta\nu}(x,y) = \sum_{i=1}^{\infty} \frac{a_i}{|y|^{d+\sigma_i}}$ for $\sigma_i \in (0,2)$, $\sigma_0 = \sup_i \sigma_i < 2$ and $\sum_{i=1}^{\infty} a_i = 1$. Similarly taking $N_{\theta\nu}(x,y) = k(y)$ we can see $\int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y) < \infty$ for $\alpha \in [1+\sigma_0/2, 2)$.

(iii) $$N_{\theta\nu}(x,y) = \begin{cases} \frac{(1-\log|y|)^{\beta}}{|y|^{d+\sigma}} & \text{for } \ 0<|y|\leq 1 \\ \frac{(1+\log|y|)^{-\beta}}{|y|^{d+\sigma}} & \text{for } \ |y|\geq 1, \end{cases}$$
where $\sigma \in (0,2)$.
(a) For $2(2-\sigma) > \beta \geq 0$, taking $N_{\theta\nu}(x,y) = k(y)$ we have $\int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y)\mathrm{d}y < \infty$ for $\alpha \in [1+\frac{\sigma}{2}+\frac{\beta}{4}, 2)$.

(b) For $-\sigma < \beta < 0$, taking $N_{\theta\nu}(x, y) = k(y)$ we have $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y < \infty$ for $\alpha \in [1 + \frac{\sigma}{2}, 2)$.

Proof of (a):

$$\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y = \int_{|y|\leq 1} \frac{|y|^\alpha(1 - \log|y|)^\beta}{|y|^{d+\sigma}}\mathrm{d}y + \int_{|y|>1} \frac{(1 + \log|y|)^{-\beta}}{|y|^{d+\sigma}}\mathrm{d}y$$
$$:= I_1 + I_2.$$

Using $(1 - \log|y|) \leq \frac{1}{\sqrt{|y|}} + 1$ and the convexity of $\xi(t) = t^p$ for $p \geq 1$ we get

$$(1 - \log|y|)^\beta \leq C\left(\frac{1}{|y|^{\beta/2}} + 1\right).$$

Therefore

$$I_1 \leq \int_{|y|\leq 1} \frac{C\mathrm{d}y}{|y|^{\beta/2+d+\sigma-\alpha}} + \int_{|y|\leq 1} \frac{C\mathrm{d}y}{|y|^{d+\sigma-\alpha}} < \infty \quad \text{for} \quad \alpha \in [1 + \sigma/2 + \beta/4, 2),$$

and

$$I_2 \leq \int_{|y|>1} \frac{\mathrm{d}y}{|y|^{d+\sigma}} < \infty.$$

Proof of (b): Since $\beta < 0$ in this case, we have $(1 - \log|y|)^\beta \leq 1$ and $I_1 < \infty$ for $\alpha \in [1 + \frac{\sigma}{2}, 2)$. To estimate I_2, observe $(1 + \log|y|)^{-\beta} \leq (1 + |y|)^{-\beta}$ and

$$I_2 \leq C\int_{|y|>1} \frac{(1 + |y|^{-\beta})}{|y|^{d+\sigma}}\mathrm{d}y < \infty \quad \text{since} \quad \sigma > -\beta.$$

(iv) $N_{\theta\nu}(x, y) = \frac{\Psi(1/|y|^2)}{|y|^{d+\sigma(x,y)}}$, where $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying

$$0 < \sigma^- := \inf_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} \sigma(x, y) \leq \sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} \sigma(x, y) := \sigma^+ < 2.$$

and Ψ is a Bernstein function (for several examples of such functions, see [150]) vanishing at zero. Furthermore, Ψ is non-decreasing, concave and satisfies a *weak upper scaling property* i.e, there exists $\mu \geq 0$ and $c \in (0, 1]$ such that

$$\Psi(\lambda x) \leq c\lambda^{\mu}\Psi(x) \quad \text{for} \quad x \geq s_0 > 0, \lambda \geq 1.$$

For $\mu < 2(2 - \sigma^+)$, we can take

$$k(y) = \begin{cases} \frac{\Psi(1)}{|y|^{d+2\mu+\sigma^+}}, & \text{if} \quad 0 < |y| \leq 1, \\ \frac{\Psi(1)}{|y|^{d+\sigma^-}}, & \text{if} \quad |y| > 1 \end{cases}$$

and $\int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y)\mathrm{d}y < \infty$ for $\alpha \in [1 + \mu + \sigma^+/2, 2)$.

The main purpose of this chapter is to establish a global Lipschitz regularity and a higher boundary regularity of the solutions satisfying (4.0.2) under the Assumption 4.0.1.

4.1 Interior regularity

In this section we aim to present a proof of the interior $C^{1,\alpha}$ regularity result which mildly generalize the result of [133].

We mention here that to show $C^{1,\alpha}$ interior regularity we closely follow the approach of [133] which tailors the approach of [51] using blowup and approximation techniques. This requires us to scale the solution u of (4.0.2) by considering v of the form

$$v(x) = u(s(x - x_0) + x_0) \quad \text{for each} \quad x_0 \in \Omega\,, 0 < s \leq 1\,.$$

Since the operator $\mathcal{L}$ is not scale invariant and does not have any natural order, v will satisfy a different integro-differential equation $\mathcal{L}^s$. This requires us to find an operator that has $C^{1,\alpha}$ regularity and which is very close to the operator $\mathcal{L}^s$ with respect to some weak topology for small s so that this information can be transferred to v to prove higher regularity of u. So let us first introduce the re-scaled operator $\mathcal{L}^s$. Let $x_0 \in \Omega$ and $0 < s \leq 1$ then $\mathcal{L}^r(x_0)$ is defined as

$$\mathcal{L}^s(x_0)[x, u] = \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\{\mathrm{Tr}(a_{\theta\nu}(s(x - x_0) + x_0)D^2u(x)) + \mathcal{I}^s_{\theta\nu}(x_0)[x, u]\},$$

where $\mathcal{I}^s_{\theta\nu}(x_0)$ is the rescaled nonlocal operator defined as

$$\mathcal{I}^s_{\theta\nu}(x_0)[x,u] = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_{\frac{1}{s}}}(y) Du(x)\cdot y) N^s_{\theta\nu}(x_0,x,y)\,\mathrm{d}y.$$

Where

$$N^s_{\theta\nu}(x_0,x,y) = s^{d+2} N_{\theta\nu}(s(x-x_0)+x_0, ry).$$

When $x_0 = 0$ we denote $\mathcal{L}^s(0) \coloneqq \mathcal{L}^s$. Note that, as discussed in Chapter 1, to make sense of $\mathcal{L}^s(x_0)$ all coefficient functions $a_{\theta\nu}$ and nonlocal kernel $N_{\theta\nu}$ should be well defined on set $\Omega^r(x_0)$. However, it is not necessary to impose such assumption here due to the fact that these scaled operators will be always utilized over a small neighbourhood of point x_0, say $B_s(x_0)$, such that $B_{2s}(x_0) \subset \Omega$.

Next we define extremal Pucci operators for second order term and the nonlocal term.

$$\mathcal{P}^+ u(x) \coloneqq \sup\left\{\mathrm{Tr}(AD^2u(x)), A \in M^d, \lambda \boldsymbol{I} \le A \le \Lambda \boldsymbol{I}\right\},$$
$$\mathcal{P}^- u(x) \coloneqq \inf\left\{\mathrm{Tr}(AD^2u(x)), A \in M^d, \lambda \boldsymbol{I} \le A \le \Lambda \boldsymbol{I}\right\},$$

and

$$\mathcal{P}^+_{k,s} u(x) \coloneqq \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_{\frac{1}{s}}}(y) Du(x)\cdot y)^+ s^{d+2} k(sy)\mathrm{d}y,$$
$$\mathcal{P}^-_{k,s} u(x) \coloneqq -\int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{B_{\frac{1}{s}}}(y) Du(x)\cdot y)^- s^{d+2} k(sy)\mathrm{d}y.$$

Denote $\mathcal{P}^+_{k,1} = \mathcal{P}^+_k$ and $\mathcal{P}^-_{k,1} = \mathcal{P}^-_k$.

As we have seen in Chapter 3, we often used interior $C^{1,\alpha}$ regularity for re-scaled operators $\mathcal{L}^r$ and Assumption 3.0.1(a) was used to deal with such situation. But we do not have such assumption here, thus we need uniform $C^{1,\alpha}$ interior regularity result with respect to re-scaled operators $\mathcal{L}^r$. The interior $C^{1,\alpha}$ regularity theorem for the scaled operator $\mathcal{L}^s$ we want to prove is as follows.

Theorem 4.1.1. *Let* $0 < s \le 1$ *and* $u \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ *solves the inequations*

$$\begin{aligned} \mathcal{L}^s[x,u] + C_0 s|Du(x)| &\ge -K \quad \text{in } B_2, \\ \mathcal{L}^s[x,u] - C_0 s|Du(x)| &\le K \quad \text{in } B_2, \end{aligned} \tag{4.1.1}$$

in the viscosity sense. Then there exist constants $0 < \gamma < 1$ *and* $C > 0$ *independent of* s*, such that*

$$||u||_{C^{1,\gamma}(B_1)} \le C\Big(||u||_{L^\infty(\mathbb{R}^d)} + K\Big),$$

where γ *and* C *depend only on* d, λ, Λ, C_0 *and* $\int_{\mathbb{R}^d}(1 \wedge |y|^2)k(y)\mathrm{d}y$.

To prove Theorem 4.1.1, we first introduce the following scaled operator. Let $x_0 \in \Omega$ and $r > 0$, we define the doubly scaled operator as

$$\mathcal{L}^{r,s}(x_0)[x,u] = \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\Big\{\operatorname{Tr} a_{\theta\nu}(sr(x-x_0)+sx_0)D^2u(x) + \mathcal{I}^{r,s}_{\theta\nu}(x_0)[x,u]\Big\} \tag{4.1.2}$$

where

$$\begin{aligned} &\mathcal{I}^{r,s}_{\theta\nu}(x_0)[x,u] \\ &\quad= \int_{\mathbb{R}^d}(u(x+y)-u(x)-\mathbb{1}_{B_{\frac{1}{sr}}}(y)Du(x)\cdot y)(rs)^{d+2}N_{\theta\nu}(rs(x-x_0)+sx_0, sry)\mathrm{d}y. \end{aligned}$$

Further, we define

$$\mathcal{L}^{0,s}(x_0)[x,u] := \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\Big\{\operatorname{Tr} a_{\theta\nu}(sx_0)D^2u(x)\Big\}. \tag{4.1.3}$$

Now we give the definition of weak convergence of operators.

Definition 4.1.1. Let $\Omega \subset \mathbb{R}^d$ be open and $0 < r < 1$. A sequence of operators $\mathcal{L}^m$ is said to converge weakly to $\mathcal{L}$ in Ω, if for any test function $\varphi \in L^\infty(\mathbb{R}^d)\cap C^2(B_r(x_0))$ for some $B_r(x_0) \subset \Omega$, we have

$$\mathcal{L}^m[x,\varphi] \to \mathcal{L}[x,\varphi] \quad \text{uniformly in } B_{\frac{r}{2}}(x_0) \text{ as } m \to \infty.$$

The next lemma is a slightly modified version of [133, Lemma 4.1] which can be proved by similar arguments.

Lemma 4.1.1. *Let $x_0 \in B_1$ and $0 < s < 1$. Moreover let $\mathcal{L}^{r,s}(x_0)$ and $\mathcal{L}^{0,s}(x_0)$ be as in (4.1.2) - (4.1.3) and assume that Assumption 3.0.1 is satisfied by the corresponding coefficients on $\Omega = B_2$.*
We suppose that, for a given $M, \varepsilon > 0$ and a given modulus of continuity ρ, there exist constants $r_0, \eta > 0$, independent of x_0 and s, for which the following assertion holds: if we have

(i) $\mathcal{L}^{0,s}(x_0)[x, v] = 0$ *in* B_1,

(ii) for some $0 < r < r_0$, *we have*

$$\begin{aligned} \mathcal{L}^{r,s}(x_0)[x, u] + C_0 rs|Du(x)| &\geq -\eta \ \ \textit{in} \ \ B_1 \\ \mathcal{L}^{r,s}(x_0)[x, u] - C_0 rs|Du(x)| &\leq \eta \ \ \textit{in} \ \ B_1, \\ u &= v \ \ \textit{in} \ \ \partial B_1. \end{aligned}$$

(iii) $|u(x)| + |v(x)| \leq M$ *in* $\mathbb{R}^d$ *and* $|u(x) - u(y)| + |v(x) - v(y)| \leq \rho(|x - y|)$ *for all* $x, y \in \bar{B}_1$,

then we have

$$|u - v| \leq \varepsilon$$

in B_1.

It is worth mentioning that in [133], the authors have set a uniform continuity assumption on the nonlocal kernels $N_{\theta\nu}(x, y)$ (for the precise assumption, see Assumption (C) of [133, p. 391]) which is a standard assumption to make for the stability property of viscosity solutions. Namely, if we have a sequence of integro-differential operators $\mathcal{L}^m$ converging weakly to $\mathcal{L}$ in Ω and a sequence of subsolutions (or supersolutions) in Ω converging locally uniformly on any compact subset of Ω, then the limit is also a subsolution (or supersolution) with respect to $\mathcal{L}$. However in the case of the operator $\mathcal{L}^{r,s}$ defined in (4.1.2), the nonlocal term $\mathcal{I}^{r,s}_{\theta\nu}$ can be treated as a lower order term that converges to zero as $r \to 0$ without any kind of continuity assumptions on nonlocal kernels $N_{\theta\nu}$. This observation was exploited in [129] to obtain the above result and subsequently interior $C^{1,\alpha}$ regularity property.

Now we give the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. We will closely follow the proof of [133, Theorem 4.1]. Fix any $x_0 \in B_1$, let $\mathcal{L}^{r_k,s}(x_0)$ and $\mathcal{L}^{0,s}(x_0)$ is given by (4.1.2) and (4.1.3) respectively. Then by [133, Lemma 3.1] as $r_k \to 0$, we have

$$\mathcal{L}^{r_k,s}(x_0) \to \mathcal{L}^{0,s}(x_0),$$

in the sense of Definition 4.1.1. By interior regularity [48, Corollary 5.7], $\mathcal{L}^{0,s}(x_0)$ has $C^{1,\beta}$ estimate for an universal constant $\beta > 0$. Now without loss of any generality we may assume that $x_0 = 0$. Also dividing u by $||u||_{L^\infty(\mathbb{R}^d)} + K$ in (4.1.1) we may assume that $K = 1$ and $||u||_{L^\infty(\mathbb{R}^d)} \leq 1$.
Using the Hölder regularity [133, Lemma 2.1], we have $u \in C^\beta(B_1)$. Following [51, Theorem 52], we will show that there exists $\delta, \mu \in (0, \frac{1}{4})$, independent of s and a sequence of linear functions $l_k(x) = a_k + b_k x$ such that

$$\begin{cases} (i) \ \sup\limits_{B_{2\delta\mu^k}} |u - l_k| \leq \mu^{k(1+\gamma)} \ , \\ (ii) \ \ |a_k - a_{k-1}| \leq \mu^{(k-1)(1+\gamma)} \ , \\ (iii) \ \ \mu^{k-1}|b_k - b_{k-1}| \leq C\mu^{(k-1)(1+\gamma)} \ , \\ (iv) \ \ |u - l_k| \leq \mu^{-k(\gamma'-\gamma)}\delta^{-(1+\gamma')}|x|^{1+\gamma'} \ \text{ for } \ x \in B^c_{2\delta\mu^k} \ , \end{cases} \tag{4.1.4}$$

where $0 < \gamma < \gamma' < \beta$ do not depend on s. We plan to proceed by induction, when $k = 0$, since $||u||_{L^\infty(\mathbb{R}^d)} \leq 1$, (4.1.4) holds with $l_{-1} = l_0 = 0$. Assume (4.1.4) holds for some k and we shall show (4.1.4) for $k + 1$.

Let $\xi : \mathbb{R}^d \to [0, 1]$ be a continuous function such that

$$\xi(x) = \begin{cases} 1 \ \text{ for } \ x \in B_3, \\ 0 \ \text{ for } \ x \in B_4^c. \end{cases}$$

Let us define

$$w_k(x) = \frac{(u - \xi l_k)(\delta\mu^k x)}{\mu^{k(1+\gamma)}}.$$

We claim that there exists a universal constant $C > 0$, such that for all k, we have

$$\begin{aligned}
&\mathcal{L}^{r_k,s}[x, w_k] - C_0 r_k s|Dw_k(x)| \le C\delta^2\mu^{k(1-\gamma)} \le C\delta^2,\\
&\mathcal{L}^{r_k,s}[x, w_k] + C_0 r_k s|Dw_k(x)| \ge -C\delta^2\mu^{k(1-\gamma)} \ge -C\delta^2,
\end{aligned} \tag{4.1.5}$$

in B_2 in viscosity sense. Let $\phi \in C^2(B_2) \cap C(\mathbb{R}^d)$ which touches w_k from below at x' in B_2. Let

$$\psi(x) := \mu^{k(1+\gamma)}\phi\left(\frac{x}{\delta\mu^k}\right) + \xi l_k(x).$$

Then $\psi \in C^2(B_{2\delta\mu^k}) \cap C(\mathbb{R}^d)$ is bounded and touches u from below at $\delta\mu^k x'$. Taking $r_k = \delta\mu^k$, we have

$$\mathcal{I}^{r_k,s}_{\theta\nu}[x', \phi] = \delta^2\mu^{k(1-\gamma)}\mathcal{I}^s_{\theta\nu}[r_k x', \psi - \xi l_k].$$

Thus we get

$$\begin{aligned}
&\mathcal{L}^{r_k,s}[x', \phi] - C_0 r_k s|D\phi(x')|\\
&\quad = \delta^2\mu^{k(1-\gamma)}\Big[\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\Big\{\operatorname{Tr} a_{\theta\nu}(s r_k x')D^2\psi(r_k x') + \mathcal{I}^s_{\theta\nu}[r_k x', \psi - \xi l_k]\Big\}\\
&\qquad\qquad - sC_0|D\psi(r_k x') - b_k|\Big]\\
&\quad \le \delta^2\mu^{k(1-\gamma)}\Big[\mathcal{L}^s[r_k x', \psi] - sC_0|D\psi(r_k x')| + \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\{-\mathcal{I}^s_{\theta\nu}[r_k x', \xi l_k]\} + sC_0|b_k|\Big]\\
&\quad \le C\delta^2\mu^{k(1-\gamma)} \le C\delta^2.
\end{aligned}$$

In the second last inequality we use that

$$\mathcal{L}^s[x, u] - C_0 s|Du(x)| \le 1,$$

and $|a_k|$, $|b_k|$ are uniformly bounded and for all $x' \in B_2$, $\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\{-\mathcal{I}^s_{\theta\nu}[r_k x', \xi l_k]\}$ is bounded independent of s and k . Thus we have proved

$$\mathcal{L}^{r_k,s}[x, w_k] - C_0 r_k s|Dw_k(x)| \le C\delta^2 \ \text{ in } \ B_2,$$

in viscosity sense. Similarly the other inequality in (4.1.5) can be proven. Define

$w_k'(x) := \max\{\min\{w_k(x), 1\}, -1\}$. We see that w_k' is uniformly bounded independent of k. We claim that in $B_{\frac{3}{2}}$

$$\begin{aligned}
\mathcal{L}^{r_k,s}[x, w_k'] - C_0 r_k s |Dw_k'(x)| \leq C\delta^2 + \omega_1(\delta), \\
\mathcal{L}^{r_k,s}[x, w_k'] + C_0 r_k s |Dw_k'(x)| \geq -C\delta^2 - \omega_1(\delta)
\end{aligned} \tag{4.1.6}$$

Now take any bounded $\phi \in C^2(B_2) \cap C(\mathbb{R}^d)$ that touches w_k' from below at x' in $B_{3/2}$. By the definition of w_k', in B_2 we have $|w_k| = |w_k'| \leq 1$ and ϕ touches w_k from below at x'. Hence

$$\begin{aligned}
&\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\Big\{ \operatorname{Tr} a_{\theta\nu}(sr_k x') D^2\phi(x') \\
&\quad + \int_{B_{1/2}} (\phi(x'+z) + \phi(x') - \mathbb{1}_{B_{\frac{1}{r_k s}}}(z) D\phi(x')\cdot z)(r_k s)^{d+2} N_{\theta\nu}(r_k s x, s r_k z)\mathrm{d}z \\
&\quad - \int_{\mathbb{R}^d\setminus B_{1/2}} (w_k(x'+z) - w_k'(x'+z) \\
&\quad + w_k'(x'+z) - \phi(x') - \mathbb{1}_{B_{\frac{1}{r_k s}}}(z) D\phi(x')\cdot z))(r_k s)^{d+2} N_{\theta\nu}(r_k s x, s r_k z)\mathrm{d}z \Big\} \\
&\quad - C_0 r_k s |D\phi(x')| \leq C\delta^2
\end{aligned}$$

Therefore by Definition 1.2.1 of viscosity supersolution and using the bounds on the kernel we get the following estimate:

$$\begin{aligned}
&\mathcal{L}^{r_k,s}[x, w_k'] - C_0 r_k s |Dw_k'(x)| \\
&\qquad \leq \int_{\mathbb{R}^d\setminus B_{1/2}} |w_k(x'+z) - w_k'(x'+z)| \, (r_k s)^{d+2} k(r_k s z)\mathrm{d}z + C\delta^2.
\end{aligned}$$

in the viscosity sense. By the inductive assumptions, we have a_k and b_k uniformly bounded. Since $||u||_{L^\infty(\mathbb{R}^d)} \leq 1$ and ξl_k is uniformly bounded, $|w_k| \leq C\mu^{-k(1+\gamma)}$ in $\mathbb{R}^d$. Using (iv) from (4.1.4) we have

$$|w_k(x)| = \frac{(u - \xi l_k)(r_k x)}{\mu^{k(1+\gamma)}} \leq \left(\frac{1}{r_k}\right)^{1+\gamma'} |r_k x|^{1+\gamma'} = |x|^{1+\gamma'},$$

for any $x \in B_2^c \cap B_{\frac{2}{r_k}}$. Again for any $x \in B_{2/r_k}^c$, we find

$$|w_k(x)| \le C\mu^{-k(1+\gamma')} \cdot \mu^{-k(\gamma-\gamma')} \le C\mu^{-k(1+\gamma')} \le C\frac{\delta^{1+\gamma'}}{2}|x|^{1+\gamma'} \le C|x|^{1+\gamma'}.$$

Now, since w_k' is uniformly bounded, we have for $x \in B_2^c$,

$$|w_k| + |w_k' - w_k| \le C\min\{|x|^{1+\gamma'}, \mu^{-k(1+\gamma)}\}. \tag{4.1.7}$$

For $x' \in B_{3/2}$, using (4.1.7) we have the following estimate.

$$\begin{aligned}
&\int_{\mathbb{R}^d} |w_k(x'+z) - w_k'(x'+z)|\,(r_k s)^{d+2} k(r_k s z)\mathrm{d}z \\
&\quad\le \int_{\{z:|x'+z|\ge 2\}\cap B_{1/r_k}} |w_k - w_k'|\,(x'+z)(r_k s)^{d+2} k(r_k s z)\mathrm{d}z \\
&\qquad\qquad + \delta^2\mu^{k(1-\gamma)} \int_{B^c_{\frac{1}{r_k}}} \frac{(r_k s)^{d+2}k(r_k s z)}{(\delta\mu^k)^2}\mathrm{d}z \\
&\quad\le C\Big[\int_{B^c_{1/2}\cap B_{\frac{1}{\sqrt{r_k}}}} |z|^2 (r_k s)^{d+2} k(r_k s z)\mathrm{d}z + r_k^{\frac{(1-\gamma')}{2}} \int_{B^c_{\frac{1}{\sqrt{r_k}}}\cap B_{\frac{1}{r_k}}} |z|^2 (r_k s)^{d+2} k(r_k s z)\mathrm{d}z \\
&\qquad\qquad + \delta^2\mu^{k(1-\gamma)} \int_{B_s^c} s^2 k(z)\mathrm{d}z\Big] \\
&\quad\le C\Big[\int_{B_{\sqrt{r_k}}} |y|^2 k(y)\mathrm{d}y + (r_k^{\frac{(1-\gamma')}{2}} + \delta^2\mu^{k(1-\gamma)}) \int_{\mathbb{R}^d} (1\wedge |y|^2) k(y)\mathrm{d}y\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |w_k(x'+z) - w_k'(x'+z)|\, k(r_k s z)\mathrm{d}z \\
&\qquad\le \tilde{C}\left(\int_{B_{\sqrt{\delta}}} |y|^2 k(y)\mathrm{d}y + \delta^{\frac{1-\gamma'}{2}} + \delta^2\right) = \omega_1(\delta),
\end{aligned}$$

where $\omega_1(\delta) \to 0$ as $\delta \to 0$. Therefore we proved $\mathcal{L}^{r_k,s}[x, w_k'] - C_0 r_k s|Dw_k'(x)| \le C\delta^2 + \omega_1(\delta)$. The other inequality of (4.1.6) can be proved in a similar manner.

Since w'_k satisfies the equation (4.1.6), by [133, Lemma 2.1] we have $||w'_k||_{C^\beta(\bar{B}_1)} \leq M_1$ for some M_1 independent of k, s. Now we consider the a function h which solves

$$\begin{aligned} \mathcal{L}^{0,s}(x_0)[x,h] = 0 \qquad & in \ \ B_1 \\ h = w'_k \qquad\qquad & on \ \ \partial B_1. \end{aligned}$$

Existence of such h can be seen from [155, Theorem 1]. Moreover, using [155, Theorem 2] we have $||h||_{C^\alpha(\bar{B}_1)} \leq M_2$ where $\alpha < \frac{\beta}{2}$ and M_2 is independent of k, s. Now for any $0 < \varepsilon < 1$, let $r_0 := r_0(\varepsilon)$ and $\eta := \eta(\varepsilon)$ as given in Lemma 4.1.1. Also for $x \in B_1$ and $\delta := \delta(\varepsilon) \leq r_0$, we have

$$\begin{aligned} \mathcal{L}^{r_k,s}[x, w'_k] + C_0 r_k s |Dw'_k(x)| \geq -\eta, \\ \mathcal{L}^{r_k,s}[x, w'_k] - C_0 r_k s |Dw'_k(x)| \leq \eta. \end{aligned}$$

Therefore by Lemma 4.1.1, we conclude $|w'_k - h| \leq \varepsilon$ in B_1. Again by using [48, Corollary 5.7], we have $h \in C^{1,\beta}(B_{1/2})$ and we can take a linear part $l(x) := a + bx$ of h at the origin. By $C^{1,\beta}$ estimate of $\mathcal{L}^{0,s}(x_0)$ and $|w'_k| \leq 1$ in B_1 we obtain that the coefficients of l, i.e, a, b are bounded independent of k, s. Further for $x \in B_{1/2}$, we have

$$|h(x) - l(x)| \leq C_1 |x|^{1+\beta},$$

where C_1 is independent of k, s. Hence using the previous estimate we get

$$|w'_k(x) - l(x)| \leq \epsilon + C_1 |x|^{1+\beta} \ \ in \ B_{1/2}.$$

Again using (4.1.7) and $|w_k| \leq 1$ in B_2 we have

$$\begin{aligned} |w_k(x) - l(x)| &\leq 1 + |a| + |b| \leq C_2 \ \ in \ \ B_1, \\ |w_k(x) - \xi(\delta\mu^k x) l(x)| &\leq C|x|^{1+\gamma'} + C_3|x| \ \ in \ \ B_1^c. \end{aligned}$$

Next defining

$$\begin{aligned} l_{k+1}(x) &:= l_k(x) + \mu^{k(1+\gamma)} l\left(\delta^{-1}\mu^{-k} x\right), \\ w_{k+1}(x) &:= \frac{(u - \xi l_{k+1})(\delta\mu^{k+1}x)}{\mu^{(k+1)(1+\gamma)}}, \end{aligned}$$

and following the proof of [133, Theorem 4.1] we conclude that (4.1.4) holds for

$k + 1$. This completes the proof. □

We refer [48] for a comprehensive review on the regularity theory for fully nonlinear elliptic equations. In a seminal paper, Caffarelli and Silvestre [50] studied the regularity theory for fully nonlinear integro-differential equations of the form : $\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\mathcal{I}[x,u]$ where $\mathcal{I}[x,u]$ is given by (1.2.2). By obtaining a nonlocal ABP estimate, they established the Hölder regularity and Harnack inequality when $N_{\theta\nu}(y)$ ($N_{\theta\nu}(y)$ denotes the x-independent form of $N_{\theta\nu}(x,y)$) is positive, symmetric and comparable with the kernel of the fractional Laplacian. From a large amount of literature that extend the work of Caffarelli and Silvestre [50], we mention [111] where the authors considered integro-PDEs with regularly varying kernel, [57, 108] where regularity results are obtained for symmetric and non-symmetric stable-like operators and [109] for kernels with variable order. Also a recent paper [115] studies Hölder regularity and a scale invariant Harnack inequality under some weak scaling condition on the kernel. The regularity results and Harnack inequality for mixed fractional p-Laplace equations are recently obtained in [93, 94]. The interior regularity theory for HJBI-type integro-PDEs has been studied in [131, 133].

4.2 Global Lipschitz regularity

In this section, we will prove Lipschitz regularity of the solution u of the inequations (4.0.2) up to the boundary. The first step is to show that the distance function $\delta(x) = \mathrm{dist}(x, \Omega^c)$ can be used as a barrier to u in Ω. Once this is done, we can prove Lipschitz regularity by considering different cases depending on the distance between any two points in Ω or their distance from $\partial\Omega$ and combining $|u| \leq C\delta$ with an interior $C^{1,\gamma}$-estimate for scaled operators (cf. Theorem 4.1.1). It also requires maximum principle and coupling property. We must point out that one needs to bypass the use of comparison principle Theorem 3.1.1 in such analysis, since the mentioned theorem is for translation invariant linear operators. For non-translation invariant operators, such comparison principle is unavailable, see Remark 4.2.1 for details. Now we present a maximum principle type result similar to Lemma 2.1.1 (compare it with Theorem 3.1.1). We report the proof here for reader's convenience.

Lemma 4.2.1. *Let u be a bounded function on $\mathbb{R}^d$ which is in $USC(\bar{\Omega})$ and satisfies $\mathcal{P}^+u + \mathcal{P}_k^+u + C_0|Du| \geq 0$ in Ω. Then we have $\sup_\Omega u \leq \sup_{\Omega^c} u$.*

Proof. From [131, Lemma 5.5] we can find a non-negative function $\chi \in C^2(\bar{\Omega}) \cap C_b(\mathbb{R}^d)$ satisfying

$$\mathcal{P}^+\chi + \mathcal{P}_k^+\chi + C_0|D\chi| \leq -1 \quad \text{in } \Omega.$$

Note that, since $\chi \in C^2(\bar{\Omega})$, the above inequality holds in the classical sense. For $\varepsilon > 0$, we let ϕ_M to be

$$\phi_M(x) = M + \varepsilon\chi.$$

Then $\mathcal{P}^+\phi_M(x_0) + \mathcal{P}_k^+\phi_M + C_0|D\phi_M| \leq -\varepsilon$ in Ω.

Let M_0 be the smallest value of M for which $\phi_M \geq u$ in $\mathbb{R}^d$. We show that $M_0 \leq \sup_{\Omega^c} u$. Suppose, to the contrary, that $M_0 > \sup_{\Omega^c} u$. Then there must be a point $x_0 \in \Omega$ for which $u(x_0) = \phi_{M_0}(x_0)$. Otherwise using the upper semicontinuity of u, we get a $M_1 < M_0$ such that $\phi_{M_1} \geq u$ in $\mathbb{R}^d$, which contradicts the minimality of M_0. Now ϕ_{M_0} would touch u from above at x_0 and thus, by the definition of the viscosity subsolution, we would have that $\mathcal{P}^+\phi_{M_0}(x_0) + \mathcal{P}_k^+\phi_{M_0} + C_0|D\phi_{M_0}| \geq 0$. This leads to a contradiction. Therefore, $M_0 \leq \sup_{\Omega^c} u$ which implies that for every $x \in \mathbb{R}^d$

$$u \leq \phi_{M_0} \leq M_0 + \varepsilon \sup_{\mathbb{R}^d} \chi \leq \sup_{\Omega^c} u + \varepsilon \sup_{\mathbb{R}^d} \chi.$$

The result follows by taking $\varepsilon \to 0$. □

Remark 4.2.1. Although we have the above maximum principle, one can not simply compare two viscosity sub and supersolutions for the operator (4.0.1). More precisely, if u, v are bounded functions and $u \in USC(\bar{\Omega}), v \in LSC(\bar{\Omega})$ satisfy

$$\mathcal{L}u + C|Du| \geq f \quad \text{and} \quad \mathcal{L}v + C|Dv| \leq g \quad \text{in} \quad \Omega$$

in viscosity sense for two continuous functions f and g, and for some $C \geq 0$, then $\mathcal{L}(u - v) + C|D(u - v)| \geq f - g$ may not always holds true in Ω. However, as we have seen in Lemma 1.2.5 if one of them is C^2, then we have

$$\mathcal{P}^+(u - v) + \mathcal{P}_k^+(u - v) + C|D(u - v)| \geq f - g \quad \text{in} \quad \Omega.$$

Indeed, without loss of generality, let us assume $v \in C^2(\Omega)$ and φ be a C^2 test function that touches $u-v$ at $x \in \Omega$ from above then clearly $\varphi+v$ touches u at x from above. By definition of viscosity subsolution we have $\mathcal{L}(\varphi+v)(x)+C|D(\varphi+v)(x)| \geq f(x)$, which implies

$$\mathcal{P}^+\varphi(x) + \mathcal{P}_k^+\varphi(x) + \mathcal{L}v(x) + C|D\varphi(x)| + C|Dv(x)| \geq f(x)$$

and hence we obtain

$$\mathcal{P}^+\varphi(x) + \mathcal{P}_k^+\varphi(x) + C|D\varphi(x)| \geq f(x) - g(x).$$

We will start by showing that the distance function $\delta(x)$ is a barrier to u.

Lemma 4.2.2. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and u be a continuous function which solves (4.0.2) in the viscosity sense. Then there exists a constant C which depends only on $d, \lambda, \Lambda, C_0, \operatorname{diam}(\Omega)$, radius of exterior sphere and $\int_{\mathbb{R}^d}(1 \wedge |y|^2)k(y)\mathrm{d}y$, such that*

$$|u(x)| \leq CK\delta(x) \quad \textit{for all } x \in \Omega. \tag{4.2.1}$$

Proof. First we show that

$$|u(x)| \leq \kappa K \quad x \in \mathbb{R}^d, \tag{4.2.2}$$

for some constant κ. From [131, Lemma 5.5], there exists a non-negative function $\chi \in C^2(\bar{\Omega}) \cap C_b(\mathbb{R}^d)$, with $\inf_{\mathbb{R}^d} \chi > 0$, satisfying

$$\mathcal{P}^+\chi + \mathcal{P}_k^+\chi + C_0|D\chi| \leq -1 \quad \text{in} \quad \Omega.$$

We define $\psi = K\chi$ which gives that $\inf_{\mathbb{R}^d} \psi \geq 0$ and

$$\mathcal{P}^+\psi + \mathcal{P}_k^+\psi + C_0|D\psi| \leq -K \quad \text{in} \quad \Omega.$$

Then by using Remark 4.2.1, we get

$$\mathcal{P}^+(u-\psi) + \mathcal{P}_k^+(u-\psi) + C_0|D(u-\psi)| \geq 0.$$

Now applying Lemma 4.2.1 on $u-\psi$ we obtain

$$\sup_{\Omega}(u-\psi) \le \sup_{\Omega^c}(u-\psi) \le 0.$$

Note that in the second inequality above we used $u=0$ in Ω^c. This proves that $u \le \psi$ in $\mathbb{R}^d$. Similar calculation using $-u$ will also give us $-u \le \psi$ in $\mathbb{R}^d$. Thus

$$|u| \le \sup_{\mathbb{R}^d} |\chi|\, K \quad \text{in } \mathbb{R}^d,$$

which gives (4.2.2).

Now we shall prove (4.2.1). Since $\partial\Omega$ is $C^{1,1}$, Ω satisfies a uniform exterior sphere condition from outside. Let $r_\circ$ be a radius satisfying uniform exterior condition. From [131, Lemma 5.4] there exists a bounded, Lipschitz continuous function φ, Lipschitz constant being $r_\circ^{-1}$, satisfying

$$\begin{aligned} \varphi &= 0 \quad \text{in} \quad \bar{B}_{r_\circ}, \\ \varphi &> 0 \quad \text{in} \quad \bar{B}^c_{r_\circ}, \\ \varphi &\ge \varepsilon \quad \text{in} \quad B^c_{(1+\delta)r_\circ}, \\ \mathcal{P}^+\varphi + \mathcal{P}^+_k\varphi + C_0|D\varphi| &\le -1 \quad \text{in} \quad B_{(1+\delta)r_\circ} \setminus \bar{B}_{r_\circ}, \end{aligned}$$

for some constants ε, δ, dependent on $C_0, d, \lambda, \Lambda, d$ and $\int_{\mathbb{R}^d}(1\wedge|y|^2)k(y)\mathrm{d}y$. Furthermore, φ is C^2 in $B_{(1+\delta)r_\circ} \setminus \bar{B}_{r_\circ}$. For any point $y \in \partial\Omega$, we can find another point $z \in \Omega^c$ such that $\overline{B}_{r_\circ}(z) \subset \Omega^c$ touches $\partial\Omega$ at y. Let $w(x) = \varepsilon^{-1}\kappa K\varphi(x-z)$. Also $\mathcal{P}^+(w) + \mathcal{P}^+_k(w) + C_0|Dw| \le -K$. Then by using Remark 4.2.1 we have

$$\mathcal{P}^+(u-w) + \mathcal{P}^+_k(u-w) + C_0|D(u-w)| \ge 0 \quad \text{in } B_{(1+\delta)r_\circ}(z) \cap \Omega.$$

Since, by (4.2.2) $u - w \le 0$ in $(B_{(1+\delta)r_\circ}(z) \cap \Omega)^c$, applying Lemma 4.2.1 on $u-w$, it follows that $u(x) \le w(x)$ in $\mathbb{R}^d$. Repeating a similar calculation for $-u$, we can conclude that $|u(x)| \le w(x)$ in $\mathbb{R}^d$. Since this relation holds for any $y \in \partial\Omega$, taking $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < r_\circ$, one can find $y \in \partial\Omega$ satisfying $\operatorname{dist}(x, \partial\Omega) = |x-y| < r_\circ$. Then using the previous estimate we would obtain

$$|u(x)| \le \varepsilon^{-1}\kappa K\varphi(x-z) \le \varepsilon^{-1}\kappa K(\varphi(x-z) - \varphi(y-z)) \le \varepsilon^{-1}\kappa K\, r_\circ^{-1} \operatorname{dist}(x, \partial\Omega),$$

which gives us (4.2.1). □

Now we are ready to prove that $u \in C^{0,1}(\mathbb{R}^d)$.

Theorem 4.2.1. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and u be a continuous function which solves the inequations (4.0.2) in viscosity sense. Then u is in $C^{0,1}(\mathbb{R}^d)$ and there exists a constant C, depending only on $d, \Omega, \lambda, \Lambda, C_0$, $\int_{\mathbb{R}^d}(1 \wedge |y|^2)k(y)\mathrm{d}y$, such that*

$$\|u\|_{C^{0,1}(\mathbb{R}^d)} \leq CK. \tag{4.2.3}$$

Proof. Let $x_0 \in \Omega$ and $s \in (0,1]$ be such that $2s = \mathrm{dist}(x_0, \partial\Omega) \wedge 1$. Without loss of any generality, we assume $x_0 = 0$. Define $v(x) = u(sx)$ in $\mathbb{R}^d$. Using Lemma 4.2.2 we already have $|u(x)| \leq C_1 K\delta(x)$, from that one can deduce

$$|v(x)| \leq C_1\, Ks(1 + |x|) \qquad \text{for all} \;\; x \in \mathbb{R}^d, \tag{4.2.4}$$

for some constant C_1 independent of s. We recall the scaled operator

$$\mathcal{I}^s_{\theta\nu}[x, f] := \int_{\mathbb{R}^d} (f(x+y) - f(x) - \mathbb{1}_{B_{\frac{1}{s}}}(y) Df(x) \cdot y) s^{d+2} N_{\theta\nu}(sx, sy)\mathrm{d}y.$$

To compute $\mathcal{L}^s[x, v] + C_0 s|Dv(x)|$ in B_2, first we observe that $D^2v(x) = s^2 D^2u(sx)$ and $Dv(x) = sDu(sx)$. Also

$$\begin{aligned}
\mathcal{I}^s_{\theta\nu}[x, v] &= s^2 \int_{\mathbb{R}^d} (v(x+y) - v(x) - \mathbb{1}_{B_{\frac{1}{s}}}(y) Dv(x) \cdot y) N_{\theta\nu}(sx, sy) s^d \mathrm{d}y \\
&= s^2 \int_{\mathbb{R}^d} (u(sx+sy) - u(sx) - \mathbb{1}_{B_1}(sy) Du(sx) \cdot sy) N_{\theta\nu}(sx, sy) s^d \mathrm{d}y \\
&= s^2 \mathcal{I}_{\theta\nu}[sx, u].
\end{aligned}$$

Thus, it follows from (4.0.2) that

$$\begin{aligned}
&\mathcal{L}^s[x, v] + C_0 s|Dv(x)| \geq -Ks^2 \quad \text{in} \quad B_2, \\
&\mathcal{L}^s[x, v] - C_0 s|Dv(x)| \leq Ks^2 \quad \text{in} \quad B_2.
\end{aligned} \tag{4.2.5}$$

Now consider a smooth cut-off function $\varphi, 0 \le \varphi \le 1$, satisfying

$$\varphi = \begin{cases} 1 & \text{in } B_{3/2}, \\ 0 & \text{in } B_2^c. \end{cases}$$

Let $w = \varphi v$. Clearly, $((\varphi-1)v)(y) = 0$ for all $y \in B_{3/2}$, which gives $D((\varphi-1)v) = 0$ and $D^2((\varphi-1)v) = 0$ in $x \in B_{3/2}$. Since $w = v + (\varphi-1)v$, from (4.2.5) we obtain

$$\begin{aligned} &\mathcal{L}^s[x, w] + C_0 s|Dw(x)| \ge -Ks^2 - |\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma} \mathcal{I}^s_{\theta\nu}[x, (\varphi-1)v)]| \quad \text{in} \quad B_1, \\ &\mathcal{L}^s[x, w] - C_0 s|Dw(x)| \le Ks^2 + |\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma} \mathcal{I}^s_{\theta\nu}[x, (\varphi-1)v)]| \quad \text{in} \quad B_1. \end{aligned} \tag{4.2.6}$$

Again, since $(\varphi-1)v = 0$ in $B_{3/2}$, we have in B_1 that

$$\begin{aligned} |\mathcal{I}^s_{\theta\nu}[x, (\varphi-1)v]| &= \Big| \int_{B^c_{\frac{1}{2}}} ((\varphi-1)v)(x+y) - ((\varphi-1)v)(x)) s^{d+2} N_{\theta\nu}(sx, sy) \mathrm{d}y \Big| \\ &\le \int_{B^c_{\frac{1}{2}}} |v(x+y)| s^{d+2} N_{\theta\nu}(sx, sy)\mathrm{d}y + |v(x)| \int_{B^c_{\frac{1}{2}}} s^{d+2} N_{\theta\nu}(sx, sy)\mathrm{d}y \\ &:= I_1 + I_2. \end{aligned}$$

Since $x \in B_1$, using $s^{d+2}N_{\theta\nu}(sx, sy) \le s^{d+2}k(sy)$ and (4.2.4) we can have the following estimate,

$$I_2 \le 2C_1 K s \int_{\mathbb{R}^d} (1 \wedge |y|^2)\mathrm{d}y.$$

Now write

$$\begin{aligned} I_1 &= \int_{1/2\le|y|\le 1/s} |v(x+y)| s^{d+2} N_{\theta\nu}(sx, sy)\mathrm{d}y + \int_{|y|\ge 1/s} |v(x+y)| s^{d+2} N_{\theta\nu}(sx, sy)\mathrm{d}y \\ &:= I_{s,1} + I_{s,2}\,. \end{aligned}$$

Let us first estimate $I_{s,1}$. Since $x \in B_1$ and $|y| \ge \frac{1}{2}$ we have $1 + |x+y| \le 5|y|$. By using this estimate and (4.2.4) we obtain

$$\begin{aligned}
I_{s,1} &= s^{d+2}\int_{\frac{1}{2}\le|y|\le\frac{1}{s}} |v(x+y)|N_{\theta\nu}(sx,sy)\mathrm{d}y \\
&\le 5C_1K\int_{\frac{1}{2}\le|y|\le\frac{1}{s}} |sy|s^{d+2}k(sy)\mathrm{d}y \le 5C_1Ks\int_{\frac{s}{2}\le|z|\le1}|sz|k(z)\mathrm{d}z \\
&\le C_2s\int_{\frac{s}{2}\le|z|\le1}|z|^2k(z)\mathrm{d}z \le C_2s\int_{\mathbb{R}^d}(1\wedge|y|^2)k(z)\mathrm{d}z \le C_3s,
\end{aligned}$$

for some constants C_3. For $I_{s,2}$, a change of variable and (4.2.2) gives

$$\begin{aligned}
I_{s,2} \le \kappa s^2K\int_{s|y|>1} s^dk(ry)\mathrm{d}y &= \kappa s^2K\int_{|y|>1}k(y)\mathrm{d}y \\
&\le \kappa s^2K\int_{\mathbb{R}^d}(1\wedge|y|^2)k(y)\mathrm{d}y \le C_4s^2K
\end{aligned}$$

for some constant C_4. Therefore, putting the estimates of I_1 and I_2 in (4.2.6) we obtain

$$\begin{aligned}
\mathcal{L}^s[x,w] + C_0s|Dw(x)| &\ge -C_5Ks \quad \text{in} \quad B_1, \\
\mathcal{L}^s[x,w] - C_0s|Dw(x)| &\ge C_5Ks \quad \text{in} \quad B_1,
\end{aligned} \tag{4.2.7}$$

for some constant C_5. Now applying Theorem 4.1.1, from (4.2.7) we have

$$\|v\|_{C^1(B_{\frac{1}{2}})} \le C_6\Big(\|v\|_{L^\infty(B_2)} + sK\Big) \tag{4.2.8}$$

for some constant C_6. From (4.2.4) and (4.2.8) we then obtain

$$\sup_{y\in B_{s/2}(x), y\neq x}\frac{|u(x)-u(y)|}{|x-y|} \le C_7K, \tag{4.2.9}$$

for some constant C_7. Now we can complete the proof. Note that if $|x-y|\ge\frac{1}{8}$, then

$$\frac{|u(x)-u(y)|}{|x-y|} \le 2\kappa K,$$

by (4.2.2). So we consider $|x-y|<\frac{1}{8}$. If $|x-y|\ge 8^{-1}(\delta(x)\vee\delta(y))$, then using

Lemma 4.2.2 we get

$$\frac{|u(x)-u(y)|}{|x-y|} \le 4CK(\delta(x)+\delta(y))(\delta(x)\vee\delta(y))^{-1} \le 8CK.$$

Now let $|x-y| < 8^{-1}\min\{\delta(x)\vee\delta(y), 1\}$. Then either $y \in B_{\frac{\delta(x)\wedge 1}{8}}(x)$ or $x \in B_{\frac{\delta(y)\wedge 1}{8}}(y)$. Without loss of generality, we suppose $y \in B_{\frac{\delta(x)\wedge 1}{8}}(x)$. From (4.2.9) we get

$$\frac{|u(x)-u(y)|}{|x-y|} \le C_7 K.$$

This completes the proof. $\square$

4.3 Sub/Super solution and a *weak version* of the Harnack inequality

Aim of this section is to construct appropriate sub and super solutions which are locally C^2 and prove a *weak version* of Harnack inequality. These results are crucial to better understanding the regularity of u/δ which will be discussed in the next section. Since u is Lipschitz, (4.0.2) can be written as

$$|\mathcal{L}u| \le CK \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{in } \Omega^c.$$

We start by constructing a C^2 subsolution on an annulus.

Lemma 4.3.1. *There exists a constant $\tilde{\kappa}$, which depends only on $d, \lambda, \Lambda, \int_{\mathbb{R}^d}(1 \wedge |y|^2)k(y)\mathrm{d}y$, such that for any $r \in (0,1]$, we have a bounded radial function ϕ_r satisfying*

$$\begin{cases} \mathcal{P}^-\phi_r + \mathcal{P}_k^-\phi_r \ge 0 & \text{in } B_{4r}\setminus \bar{B}_r, \\ 0 \le \phi_r \le \tilde{\kappa} r & \text{in } B_r, \\ \phi_r \ge \frac{1}{\tilde{\kappa}}(4r-|x|) & \text{in } B_{4r}\setminus B_r, \\ \phi_r \le 0 & \text{in } \mathbb{R}^d \setminus B_{4r}. \end{cases}$$

Moreover, $\phi_r \in C^2(B_{4r}\setminus \bar{B}_r)$.

Proof. We use the same subsolution constructed in Lemma 3.2.1 and show that it is indeed a subsolution with respect to minimal Pucci operators. Fix $r \in (0,1]$ and define $v_r(x) = \mathrm{e}^{-\eta q(x)} - \mathrm{e}^{-\eta(4r)^2}$, where $q(x) = |x|^2 \wedge 2(4r)^2$ and $\eta > 0$. Clearly, $1 \geq v_r(0) \geq v_r(x)$ for all $x \in \mathbb{R}^d$. Thus using the fact that $1 - e^{-\xi} \leq \xi$ for all $\xi \geq 0$ we have

$$v_r(x) \leq 1 - \mathrm{e}^{-\eta(4r)^2} \leq \eta(4r)^2, \tag{4.3.1}$$

Again for $x \in B_{4r} \setminus B_r$, we have that

$$\begin{aligned} v_r(x) = \mathrm{e}^{-\eta(4r)^2}(\mathrm{e}^{\eta((4r)^2 - q(x))} - 1) &\geq \eta \mathrm{e}^{-\eta(4r)^2}((4r)^2 - |x|^2) \\ &\geq 5\eta r \mathrm{e}^{-\eta(4r)^2}(4r - |x|). \end{aligned} \tag{4.3.2}$$

Fix $x \in B_{4r} \setminus \bar{B}_r$. We start by estimating the local minimal Pucci operator $\mathcal{P}^-$ of v. Using rotational symmetry we may always assume $x = (l, 0, \cdots, 0)$ Then

$$\partial_i v_r(x) = -2\eta \mathrm{e}^{-\eta|x|^2} x_i = \begin{cases} -2\eta \mathrm{e}^{-\eta|x|^2} l & i = 1, \\ 0 & i \neq 1 \end{cases}$$

and

$$\begin{aligned} \partial_{ij} v_r(x) &= \begin{cases} 4\eta^2 x_i^2 \mathrm{e}^{-\eta|x|^2} - 2\eta \mathrm{e}^{-\eta|x|^2} & i = j, \\ 4\eta^2 x_i x_j \mathrm{e}^{-\eta|x|^2} & i \neq j. \end{cases} \\ &= \begin{cases} 4\eta^2 l^2 \mathrm{e}^{-\eta|x|^2} - 2\eta \mathrm{e}^{-\eta|x|^2} & i = j = 1, \\ -2\eta \mathrm{e}^{-\eta|x|^2} & i = j \neq 1, \\ 0 & i \neq j. \end{cases} \end{aligned}$$

By the above calculation, for any $x \in B_{4r} \setminus \bar{B}_r$, choosing $\eta > \frac{1}{r^2}$ we have

$$\begin{aligned} \mathcal{P}^- v_r(x) &= \lambda 4\eta^2 l^2 \mathrm{e}^{-\eta|x|^2} - \lambda 2\eta \mathrm{e}^{-\eta|x|^2} - \Lambda(d-1)2\eta \mathrm{e}^{-\eta|x|^2} \\ &\geq \lambda 4\eta^2 l^2 \mathrm{e}^{-\eta|x|^2} - d\Lambda 2\eta \mathrm{e}^{-\eta|x|^2}. \end{aligned}$$

Now to determine nonlocal minimal Pucci operator, using the convexity of exponential map we get,

$$\begin{aligned}
&e^{-\eta|x+y|^2} - e^{-\eta|x|^2} + 2\eta \mathbb{1}_{\{|y|\leq 1\}} y\cdot x e^{-\eta|x|^2} \\
&\qquad \geq -\eta e^{-\eta|x|^2}\left(|x+y|^2 - |x|^2 - 2\mathbb{1}_{\{|y|\leq 1\}} y\cdot x\right).
\end{aligned}$$

Since $\mathcal{P}_k^- v_r = \mathcal{P}_k^-(v_r + e^{-\eta(4r)^2})$ and using above inequality we obtain

$$\begin{aligned}
\mathcal{P}_k^-(e^{-\eta q(\cdot)})(x) &= -\int_{\mathbb{R}^d}\left(e^{-\eta q(x+y)} - e^{-\eta q(x)} - \mathbb{1}_{B_1}(y) D e^{-\eta q(x)}\cdot y\right)^- k(y)\mathrm{d}y \\
&\geq -\eta e^{-\eta|x|^2}\int_{|y|\leq r}\Big(|x+y|^2 - |x|^2 - 2y\cdot x\Big) k(y)\mathrm{d}y \\
&\quad - \int_{r<|y|\leq 1}\left|e^{-\eta(|x|^2+2(4r)^2)} - e^{-\eta|x|^2} + 2\eta y\cdot x e^{-\eta|x|^2}\right| k(y)\mathrm{d}y \\
&\quad - \int_{|y|>1}\left|e^{-\eta(|x|^2+2(4r)^2)} - e^{-\eta|x|^2}\right| k(y)\mathrm{d}y \\
&\geq -\eta e^{-\eta|x|^2}\bigg[\int_{|y|<r}|y|^2 k(y)\mathrm{d}y + \int_{r<|y|\leq 1} 4^3|y|^2 k(y)\mathrm{d}y \\
&\quad + \int_{|y|>1} 2(4r)^2 k(y)\mathrm{d}y\bigg] \\
&\geq -\eta e^{-\eta|x|^2} 4^3\int_{\mathbb{R}^d}(1\wedge|y|^2)k(y)\mathrm{d}y,
\end{aligned}$$

where in the second line we used $|x+y|^2\wedge 2(4r)^2 \leq |x|^2 + 2(4r)^2$. Combining the above estimates we see that, for $x\in B_{4r}\setminus \bar{B}_r$,

$$\begin{aligned}
P^- v_r(x) + P_k^- v_r(x) &\geq \eta \mathrm{e}^{-\eta|x|^2}\left[4\eta\lambda|x|^2 - 2d\Lambda - 4^3\int_{\mathbb{R}^d}(1\wedge|y|^2)k(y)\mathrm{d}y\right] \\
&\geq \eta \mathrm{e}^{-\eta|x|^2}\left[4\eta\lambda r^2 - 2d\Lambda - 4^3\int_{\mathbb{R}^d}(1\wedge|y|^2)k(y)\mathrm{d}y\right].
\end{aligned}$$

Thus, finally letting $\eta = \frac{1}{\lambda r^2}(2d\Lambda + 4^3\int_{\mathbb{R}^d}(1\wedge|y|^2)k(y)\mathrm{d}y)$, we obtain

$$\mathcal{P}^- v_r + \mathcal{P}^- v_r > 0 \quad \text{in } B_{4r}\setminus\bar{B}_r.$$

Note that the final choice of η is admissible since $\frac{1}{\lambda r^2}(2d\Lambda + 4^3\int_{\mathbb{R}^d}(1\wedge|y|^2)k(y)\mathrm{d}y) > \frac{1}{r^2}$. Now set $\phi_r = r v_r$ and the result follows from (4.3.1)-(4.3.2). □

Next we prove a *weak version* of Harnack inequality.

Theorem 4.3.1. *Let $s \in (0,1]$, $\alpha' = 1 \wedge (2-\alpha)$ and u be a continuous non-negative function satisfying*

$$\mathcal{P}^{-}u + \mathcal{P}^{-}_{k,s}u \leq C_0 s^{1+\alpha'}, \quad \mathcal{P}^{+}u + \mathcal{P}^{+}_{k,s}u \geq -C_0 s^{1+\alpha'} \quad \text{in } B_2.$$

Furthermore if $\sup_{\mathbb{R}^d}|u| \leq M_0$ *and* $|u(x)| \leq M_0 s(1+|x|)$ *for all* $x \in \mathbb{R}^d$, *then*

$$u(x) \leq C(u(0) + (M_0 \vee C_0)s^{1+\alpha'})$$

for every $x \in B_{\frac{1}{2}}$ *and for some constant* C *which only depends on* $\lambda, \Lambda, d, \int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y)\mathrm{d}y$.

Proof. Dividing by $u(0) + (M_0 \vee C_0)s^{1+\alpha'}$, it can be easily seen that $\sup_{\mathbb{R}^d}|u| \leq s^{-(1+\alpha')}$ and $|u(x)| \leq s^{-\alpha'}(1+|x|)$ for all $x \in \mathbb{R}^d$ and u satisfies

$$\begin{aligned}
&\mathcal{P}^{-}u + \mathcal{P}^{-}_{k,s}u \leq 1,\\
&\mathcal{P}^{+}u + \mathcal{P}^{+}_{k,s}u \geq -1.
\end{aligned}$$

Fix $\varepsilon > 0$ from [131, Corollary 3.14] and let $\gamma = \frac{d}{\varepsilon}$. Let

$$t_0 := \min\left\{t \ : \ u(x) \leq h_t(x) := t(1-|x|)^{-\gamma} \ \text{ for all } x \in B_1\right\}.$$

Clearly this set is nonempty since $u(0) \leq 1$, thus t_0 exist. Let $x_0 \in B_1$ be such that $u(x_0) = h_{t_0}(x_0)$. Let $\eta = 1 - |x_0|$ be the distance of x_0 from ∂B_1. For $r = \frac{\eta}{2}$ and $x \in B_r(x_0)$, we can write

$$B_r(x_0) = \left\{u(x) \leq \frac{u(x_0)}{2}\right\} \cup \left\{u(x) > \frac{u(x_0)}{2}\right\} := A + \tilde{A}.$$

The goal is to estimate $|B_r(x_0)|$ in terms of $|A|$ and $|\tilde{A}|$. Proceeding this way, we show that $t_0 < C$ for some universal C which, in turn, implies that $u(x) < C(1-|x|)^{-\gamma}$. This would prove our result. Next, Using [131, Corollary 3.14] we obtain

$$|\tilde{A} \cap B_1| \leq C\left|\frac{2}{u(x_0)}\right|^{\varepsilon} \leq C t_0^{-\varepsilon}\eta^d\,,$$

whereas $|B_r| = \omega_d(\eta/2)^d$. In particular,

$$\left|\tilde{A} \cap B_r(x_0)\right| \leq C t_0^{-\varepsilon} |B_r|. \tag{4.3.3}$$

So if t_0 is large, $\tilde{A}$ can cover only a small portion of $B_r(x_0)$. We shall show that for some $\delta > 0$, independent of t_0 we have

$$|A \cap B_r(x_0)| \leq (1-\delta)|B_r|,$$

which will provide an upper bound on t_0 completing the proof. We start by estimating $|A \cap B_{\theta r}(x_0)|$ for $\theta > 0$ small. For every $x \in B_{\theta r}(x_0)$ we have

$$u(x) \leq\ h_{t_0}(x)\ \leq\ t_0 \left(\frac{2\eta - \theta\eta}{2}\right)^{-\gamma}\ \leq\ u(x_0)\left(1 - \frac{\theta}{2}\right)^{-\gamma},$$

with $\left(1 - \frac{\theta}{2}\right)$ close to 1. Define

$$v(x) := \left(1 - \frac{\theta}{2}\right)^{-\gamma} u(x_0) - u(x).$$

Then we get $v \geq 0$ in $B_{\theta r}(x_0)$ and also $\mathcal{P}^- v + \mathcal{P}^-_{k,s} v \leq 1$ as $\mathcal{P}^+ u + \mathcal{P}^+_{k,s} u \geq -1$. We would like to apply [131, Corollary 3.14] to v, but v need not be non-negative in the whole of $\mathbb{R}^d$. Thus we consider the positive part of v, i.e, $w = v^+$ and find an upper bound of $\mathcal{P}^- w + \mathcal{P}^-_{k,s} w$. Since v^- is C^2 in $B_{\frac{\theta r}{4}}(x_0)$, we have

$$\begin{aligned}\mathcal{P}^- w(x) + \mathcal{P}^-_{k,s} w(x) &\leq [\mathcal{P}^- v(x) + \mathcal{P}^-_{k,s} v(x)] + [\mathcal{P}^+ v^-(x) + \mathcal{P}^+_{k,s} v^-(x)] \\ &\leq 1 + \mathcal{P}^+ v^-(x) + \mathcal{P}^+_{k,s} v^-(x).\end{aligned} \tag{4.3.4}$$

Also, using $v^-(x) = Dv^-(x) = D^2 v^-(x) = 0$ for all $x \in B_{\frac{\theta r}{4}}(x_0)$, we get

$$\mathcal{P}^+ v^-(x) + \mathcal{P}^+_{k,s} v^-(x) = \int_{\{v(x+y) \leq 0\}} v^-(x+y) s^{d+2} k(sy) \mathrm{d}y. \tag{4.3.5}$$

Now plugging (4.3.5) into (4.3.4), for all $x \in B_{\frac{\theta r}{4}}(x_0)$ we obtain

$\mathcal{P}^- w(x) + \mathcal{P}^-_{k,s} w(x)$

$$
\begin{aligned}
&\le 1 + \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{2}}(x-x_0)} \left(u(x+y) - \left(1 - \frac{\theta}{2}\right)^{-\gamma} u(x_0) \right)^+ s^{d+2} k(sy) \mathrm{d}y \\
&\le 1 + \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{4}}} |u(x+y)|\, s^{d+2} k(sy) \mathrm{d}y + \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{4}}} \left(1 - \frac{\theta}{2}\right)^{-\gamma} |u(x_0)| s^{d+2} k(sy) \mathrm{d}y \\
&:= 1 + I_1 + I_2.
\end{aligned}
$$

Estimate of I_1: Let us write

$$
I_1 = \int_{\frac{\theta r}{4} \le |y| \le \frac{1}{s}} |u(x+y)|\, s^{d+2} k(sy) \mathrm{d}y + \int_{|y| \ge \frac{1}{s}} |u(x+y)|\, s^{d+2} k(sy) \mathrm{d}y := I_{11} + I_{12}.
$$

Simply using change of variable and $\sup_{\mathbb{R}^d} |u| \le s^{-(1+\alpha')}$, we obtain

$$
I_{12} \le \int_{|z| \ge 1} k(z) \mathrm{d}z.
$$

Now we estimate I_{11} using $|u(x)| \le s^{-\alpha'}(1+|x|)$ for all $x \in \mathbb{R}^d$.

$$
\begin{aligned}
I_{11} &\le \int_{\frac{\theta r}{4} \le |y| \le \frac{1}{s}} (1 + |x+y|)\, s^{d+2-\alpha'} k(sy) \mathrm{d}y \\
&\le \frac{5}{4} \int_{\frac{\theta r}{4} \le |y| \le \frac{1}{s}} s^{d+2-\alpha'} k(sy) \mathrm{d}y + \int_{\frac{\theta r}{4} \le |y| \le \frac{1}{s}} s^{d+2-\alpha'} |y| k(sy) \mathrm{d}y\,.
\end{aligned}
$$

We consider two cases. First consider the case $\alpha' = 1$ so $\alpha \le 1$. This implies

$$
I_{11} \le \frac{5}{4} \int_{\frac{\theta rs}{4} \le |z| \le 1} s k(z) \mathrm{d}z + \int_{\frac{\theta rs}{4} \le |z| \le 1} |z| k(z) \mathrm{d}z \le 6(\theta r)^{-1} \int_{\mathbb{R}^d} (1 \wedge |z|^{\alpha}) k(z) \mathrm{d}z.
$$

Now consider the case $\alpha' = 2 - \alpha$, and hence $\alpha > 1$. In this case

$$
\begin{aligned}
I_{11} &\le \frac{5}{4} \int_{\frac{\theta r}{4} \le |y| \le \frac{1}{s}} s^{\alpha} s^d k(sy) \mathrm{d}y + \int_{\frac{\theta r}{4} \le |y| \le \frac{1}{s}} s^{\alpha-1} |sy| s^d k(sy) \mathrm{d}y \\
&= \frac{5}{4} \left(\frac{\theta r}{4}\right)^{-\alpha} \int_{\frac{\theta rs}{4} \le |z| \le 1} \left(\frac{\theta r}{4} s\right)^{\alpha} k(z) \mathrm{d}z + \left(\frac{\theta r}{4}\right)^{1-\alpha} \int_{\frac{\theta rs}{4} \le |z| \le 1} \left(\frac{\theta r}{4} s\right)^{\alpha-1} |z| k(z) \mathrm{d}z \\
&\le C(\theta r)^{-2} \int_{\mathbb{R}^d} (1 \wedge |z|^{\alpha}) k(z) \mathrm{d}z\,.
\end{aligned}
$$

Combining the estimates of I_{11} and I_{12}, we get

$$I_1 \leq C(\theta r)^{-2} \int_{\mathbb{R}^d} (1 \wedge |z|^\alpha) k(z) \mathrm{d}z.$$

Estimate of I_2: If $\alpha' = 1$, then $\alpha \leq 1$ and using $|u(x_0)| \leq s^{-\alpha'}(1 + |x_0|)$ we have

$$\begin{aligned}
I_2 &:= \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{4}}} \left| \left(1 - \frac{\theta}{2}\right)^{-\gamma} u(x_0) \right| s^{d+2} k(sy) \mathrm{d}y \leq C \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{4}}} s^{d+2-\alpha'} k(sy) \mathrm{d}y \\
&= C \int_{\mathbb{R}^d \setminus B_{\frac{\theta rs}{4}}} sk(z) \mathrm{d}z \leq C \left[\int_{\frac{\theta rs}{4} \leq |z| \leq 1} sk(z) \mathrm{d}z + \int_{|z| \geq 1} sk(z) \mathrm{d}z \right] \\
&\leq C \left[\frac{4}{\theta r} \int_{\frac{\theta rs}{4} \leq |z| \leq 1} |z|^\alpha k(z) \mathrm{d}z + \int_{|z|>1} k(z) \mathrm{d}z \right] \leq C(\theta r)^{-1} \int_{\mathbb{R}^d} (1 \wedge |y|^\alpha) k(z) \mathrm{d}z.
\end{aligned}$$

If $\alpha' = 2 - \alpha$ then $\alpha > 1$. In that case, using similar calculation as above we have

$$\begin{aligned}
I_2 &:= \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{4}}} \left| \left(1 - \frac{\theta}{2}\right)^{-\gamma} u(x_0) \right| s^{d+2} k(sy) \mathrm{d}y \leq C \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{4}}} s^{d+\alpha} k(sy) \mathrm{d}y \\
&= C \int_{\mathbb{R}^d \setminus B_{\frac{\theta rs}{4}}} s^\alpha k(z) \mathrm{d}z \leq C(\theta r)^{-\alpha} \int_{\mathbb{R}^d} (1 \wedge |y|^\alpha) k(z) \mathrm{d}z.
\end{aligned}$$

Since $\alpha \in (0, 2)$, combining the above estimates we obtain

$$\mathcal{P}^- w + \mathcal{P}^-_{k,s} w \leq \frac{C}{(\theta r)^2} \quad \text{in} \quad B_{\frac{\theta r}{4}}(x_0).$$

Now using [131, Corollary 3.14] for w we get

$$\begin{aligned}
|A \cap B_{\frac{\theta r}{8}}(x_0)| &= \left| \left\{ w \geq u(x_0)((1 - \theta/2)^{-\gamma} - 1/2) \right\} \cap B_{\frac{\theta r}{8}}(x_0) \right| \\
&\leq C(\theta r)^d \left(\inf_{B_{\frac{\theta r}{8}}(x_0)} w + \frac{\theta r}{8} \cdot \frac{C}{(\theta r)^2} \right)^\varepsilon \cdot \left[u(x_0)((1 - \theta/2)^{-\gamma} - 1/2) \right]^{-\varepsilon} \\
&\leq C(\theta r)^d \left[\left((1 - \frac{\theta}{2})^{-\gamma} - \frac{1}{2} \right) + \frac{C}{8} (\theta r)^{-1} t_0^{-1} (2r)^d \right]^\varepsilon \\
&\leq C(\theta r)^d \left(\left((1 - \theta/2)^{-\gamma} - 1 \right)^\varepsilon + C_0 (\theta r)^{-\varepsilon} t_0^{-\varepsilon} r^{d\varepsilon} \right).
\end{aligned}$$

Now let us choose $\theta > 0$ small enough (independent of t_0) to satisfy

$$C(\theta r)^d \left((1 - \theta/2)^{-\gamma} - 1\right)^\varepsilon \leq \frac{1}{4}|B_{\frac{\theta r}{8}}(x_0)|\,.$$

With this choice of θ if t_0 becomes large, then we also have

$$C(\theta r)^d \theta^{-\varepsilon} r^{(n-1)\varepsilon} t_0^{-\varepsilon} \leq \frac{1}{4}|B_{\frac{\theta r}{8}}(x_0)|\,,$$

and hence

$$|A \cap B_{\frac{\theta r}{8}}(x_0)| \leq \frac{1}{2}|B_{\frac{\theta r}{8}}(x_0)|\,.$$

This estimate of course implies that

$$|\tilde{A} \cap B_{\frac{\theta r}{8}}(x_0)| \geq C_2|B_r|,$$

but this is contradicting (4.3.3). Therefore t_0 cannot be large and this completes the proof. □

Now by standard covering argument and Theorem 4.3.1 we obtain the following result.

Corollary 4.3.1. *Let u satisfies the conditions of Theorem 4.3.1, then the following holds.*

$$\sup_{B_{\frac{1}{4}}} u \leq C\left(\inf_{B_{\frac{1}{4}}} u + (M_0 \vee C_0)s^{1+\alpha'}\right).$$

Proof. Take any point $x_0 \in \bar{B}_{\frac{1}{4}}$ such that $u(x_0) = \inf_{B_{\frac{1}{4}}} u(x)$. Clearly $B_{\frac{1}{4}} \subset B_{\frac{1}{2}}(x_0)$. Defining $\tilde{u}(x) := u(x + x_0)$ and applying Theorem 4.3.1 on $\tilde{u}$ we find

$$\tilde{u}(x) \leq C\left(\tilde{u}(0) + (M_0 \vee C_0)s^{1+\alpha'}\right) \quad \text{in} \quad B_{\frac{1}{2}}.$$

This implies

$$\sup_{B_{\frac{1}{4}}} u(x) \leq \sup_{B_{\frac{1}{2}}(x_0)} u(x) \leq C\left(\inf_{B_{\frac{1}{4}}} u(x) + (M_0 \vee C_0)s^{1+\alpha'}\right).$$

This proves the claim. □

Now we will give some auxiliary lemmas which will be used to construct appropriate supersolutions.

Lemma 4.3.2. *Let Ω be a bounded C^2 domain in $\mathbb{R}^d$, then for any $0 < \epsilon < 1$, we have the following estimate*

$$\left|\mathcal{I}_{\theta\nu}(\delta^{1+\epsilon})\right| \leq C\left(1 + \mathbb{1}_{(1,2)}(\alpha)\delta^{1-\alpha}\right) \quad in \ \ \Omega, \tag{4.3.6}$$

where $C > 0$ depends only on d, Ω and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y$.

Proof. Since $\delta \in C^{0,1}(\mathbb{R}^d) \cap C^2(\bar{\Omega})$ [74, Theorem 5.4.3], using the Lipschtiz continuity of $\delta^{1+\epsilon}$ near the origin and boundedness away from the origin we can easily obtain the estimate (4.3.6) for $\alpha \in (0,1]$. Next consider the case $\alpha \in (1,2)$. For any $x \in \Omega$ we have

$$\begin{aligned}\left|\mathcal{I}_{\theta\nu}(\delta^{1+\epsilon})(x)\right| &\leq \int_{\mathbb{R}^d}\left|\delta^{1+\epsilon}(x+y) - \delta^{1+\epsilon}(x) - \mathbb{1}_{B_1}(y)y\cdot D\delta^{1+\epsilon}(x)\right| k(y)\mathrm{d}y \\ &= \int_{|y|<\frac{\delta(x)}{2}} + \int_{\frac{\delta(x)}{2}\leq|y|\leq 1} + \int_{|y|>1} := I_1 + I_2 + I_3\,.\end{aligned}$$

Since $|y| \leq \frac{\delta(x)}{2}$ and $\delta(x) < 1$, we have the following estimate on I_1.

$$\begin{aligned}\left|\delta^{1+\epsilon}(x+y) - \delta^{1+\epsilon}(x) - \mathbb{1}_{B_1}(y)y\cdot D\delta^{1+\epsilon}(x)\right| &\leq ||\delta^{1+\epsilon}||_{C^2(B_{\frac{\delta(x)}{2}}(x))}|y|^2 \\ &\leq 4C\frac{||\delta||_{C^2(\bar{\Omega})}}{\delta(x)^{1-\epsilon}}|y|^2 \leq 4C\frac{||\delta||_{C^2(\bar{\Omega})}\delta(x)^{2-\alpha}}{\delta(x)^{1-\epsilon}}|y|^\alpha.\end{aligned}$$

This implies

$$I_1 \leq 4C||\delta||_{C^2(\bar{\Omega})}\delta(x)^{1+\epsilon-\alpha}\int_{\mathbb{R}^d}|y|^\alpha k(y)\mathrm{d}y \leq 4C_0C||\delta||_{C^2(\bar{\Omega})}\delta(x)^{1+\epsilon-\alpha}. \tag{4.3.7}$$

Again for I_2 we have

$$\begin{aligned}I_2 \leq C\int_{\frac{\delta(x)}{2}\leq|y|\leq 1}|y|k(y)\mathrm{d}y &\leq \left(\frac{C\delta(x)}{2}\right)^{1-\alpha}\int_{\frac{\delta(x)}{2}\leq|y|\leq 1}|y|^\alpha k(y)\mathrm{d}y \\ &\leq \left(\frac{C\delta(x)}{2}\right)^{1-\alpha}\int_{\mathbb{R}^d}(1\wedge|y|^\alpha)\,k(y)\mathrm{d}y.\end{aligned}$$

Finally,

$$I_3 = \int_{|y|>1} |\delta^{1+\epsilon}(x+y) - \delta^{1+\epsilon}(x)| k(y) \mathrm{d}y \leq 2(\operatorname{diam}\Omega)^{1+\epsilon} \int_{\mathbb{R}^d} (1 \wedge |y|^{\alpha}) k(y) \mathrm{d}y. \tag{4.3.8}$$

Combining (4.3.7)-(4.3.8) we obtain (4.3.6). □

Next we obtain an estimate on minimal Pucci operator $\mathcal{P}^-$ applied on $\delta^{1+\epsilon}$.

Lemma 4.3.3. *Let Ω be a bounded C^2 domain in $\mathbb{R}^d$, then for any $0 < \epsilon < 1$, we have the following estimate*

$$\mathcal{P}^- \left(\delta^{1+\epsilon}\right) \geq C_1 \cdot \epsilon \delta^{\epsilon-1} - C_2 \quad \text{in} \quad \Omega,$$

where C_1, C_2 depends only on $d, \Omega, \lambda, \Lambda$.

Proof. Since $\partial\Omega$ is C^2, we have $\delta^{1+\epsilon} \in C^2(\Omega)$ and for any $x \in \Omega$

$$\frac{\partial^2}{\partial x_i \partial x_j} \delta^{1+\epsilon}(x) = (1+\epsilon)\left[\delta^{\epsilon}(x) \frac{\partial^2}{\partial x_i \partial x_j} \delta(x) + \epsilon \delta^{\epsilon-1}(x) \frac{\partial \delta(x)}{\partial x_i} \cdot \frac{\partial \delta(x)}{\partial x_j}\right] := A + B$$

where A, B are two $d \times d$ matrices given by

$$A := (a_{i,j})_{1 \leq i,j \leq d} = (1+\epsilon) \delta^{\epsilon}(x) \frac{\partial^2}{\partial x_i \partial x_j} \delta(x)$$

and

$$B := (b_{i,j})_{1 \leq i,j \leq d} = (1+\epsilon) \epsilon \delta^{\epsilon-1}(x) \frac{\partial \delta(x)}{\partial x_i} \cdot \frac{\partial \delta(x)}{\partial x_j}.$$

Note that B is a positive definite matrix and $||A|| \leq d^2(1+\epsilon)(\operatorname{diam}\Omega)^{\epsilon} ||\delta||_{C^2(\bar{\Omega})}$. Therefore we have

$$\begin{aligned}
\mathcal{P}^-(\delta^{1+\epsilon}(x)) &= \mathcal{P}^-(A+B) \geq \mathcal{P}^-(B) + \mathcal{P}^-(A) \\
&\geq \mathcal{P}^-(B) - d^2 \Lambda (1+\epsilon)(\operatorname{diam}\Omega)^{\epsilon} ||\delta||_{C^2(\bar{\Omega})} \\
&\geq \epsilon(1+\epsilon)\delta^{\epsilon-1}(x) \lambda |D\delta(x)|^2 - d^2 \Lambda (1+\epsilon)(\operatorname{diam}\Omega)^{\epsilon} ||\delta||_{C^2(\bar{\Omega})} \\
&\geq C_1 \cdot \epsilon \delta^{\epsilon-1}(x) - C_2.
\end{aligned}$$

□

Next we obtain an estimate on $\mathcal{L}\delta$ in Ω.

Lemma 4.3.4. *Let Ω be a bounded C^2 domain in $\mathbb{R}^d$. Then we have the following estimate*

$$|\mathcal{L}\delta| \leq C(1 + \mathbb{1}_{(1,2)}\delta^{1-\alpha}) \ \text{in} \ \Omega, \tag{4.3.9}$$

where constant C depends only on $d, \Omega, \lambda, \Lambda$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y$.

Proof. First of all, for all $x \in \Omega$ we have

$$|\mathcal{L}\delta(x)| \leq \sup_{\theta,\nu} |\operatorname{Tr}(a_{\theta\nu}(x)D^2\delta(x))| + \sup_{\theta,\nu} |\mathcal{I}_{\theta\nu}\delta(x)| \leq \kappa + \sup_{\theta,\nu} |\mathcal{I}_{\theta\nu}\delta(x)|, \tag{4.3.10}$$

for some constant κ, depending on Ω and uniform bound of $a_{\theta\nu}$. For $\alpha \in (0, 1]$, (4.3.9) follows from the same arguments of Lemma 4.3.2. For $\alpha \in (1, 2)$, it is enough to obtain the estimate (4.3.9) for all $x \in \Omega$ such that $\delta(x) < 1$. We follow the similar calculation as in Lemma 4.3.2 and get

$$\begin{aligned} |\mathcal{I}_{\theta\nu}\delta(x)| &\leq \int_{\mathbb{R}^d} |\delta(x+y) - \delta(x) - \mathbb{1}_{B_1}(y)y \cdot D\delta(x)|k(y)\mathrm{d}y \\ &= \int_{|y| \leq \frac{\delta(x)}{2}} + \int_{\frac{\delta(x)}{2} < |y| < 1} + \int_{|y| > 1} \end{aligned}$$

and

$$|I_{\theta\nu}\delta(x)| \leq \kappa_1 \int_{\mathbb{R}^d} (1 \wedge |y|^\alpha)k(y)\mathrm{d}y\delta^{1-\alpha}(x)$$

for some constant κ_1. Inserting these estimates in (4.3.10) we obtain

$$|\mathcal{L}\delta(x)| \leq \kappa_2\delta^{1-\alpha}(x)$$

for some constant κ_2 and (4.3.9) follows. □

Let us now recall the sets D_R that we use for our oscillation estimates (see Definition 3.2.1).

Definition 4.3.1. Let $\kappa \in (0, \frac{1}{16})$ be a fixed small constant and let $\kappa' = 1/2 + 2\kappa$. Given a point $x_0 \in \partial\Omega$ and $R > 0$, we define

$$D_R = D_R(x_0) = B_R(x_0) \cap \Omega,$$

and

$$D^+_{\kappa' R} = D^+_{\kappa' R}(x_0) = B_{\kappa' R}(x_0) \cap \{x \in \Omega : (x - x_0) \cdot \mathrm{n}(x_0) \geq 2\kappa R\},$$

where $\mathrm{n}(x_0)$ is the unit inward normal at x_0. For any bounded $C^{1,1}$-domain, we know that there exists $\rho > 0$, depending on Ω, such that the following inclusions hold for each $x_0 \in \partial\Omega$ and $R \leq \rho$:

$$B_{\kappa R}(y) \subset D_R(x_0) \qquad \text{for all } y \in D^+_{\kappa' R}(x_0), \tag{4.3.11}$$

and

$$B_{4\kappa R}(y^* + 4\kappa R\mathrm{n}(y^*)) \subset D_R(x_0), \quad \text{and} \quad B_{\kappa R}(y^* + 4\kappa R\mathrm{n}(y^*)) \subset D^+_{\kappa' R}(x_0) \tag{4.3.12}$$

for all $y \in D_{R/2}$, where $y^* \in \partial\Omega$ is the unique boundary point satisfying $|y - y^*| = \mathrm{dist}(y, \partial\Omega)$. Note that, since $R \leq \rho$, $y \in D_{R/2}$ is close enough to $\partial\Omega$ and hence the point $y^* + 4\kappa R\,\mathrm{n}(y^*)$ belongs to the line joining y and y^*.

Remark 4.3.1. We can fix $\rho > 0$, so that (4.3.11)-(4.3.12) hold whenever $R \leq \rho$ and $x_0 \in \partial\Omega$. We also fix $\sigma > 0$ small enough so that for $0 < r \leq \rho$ and $x_0 \in \partial\Omega$ we have

$$B_{\eta r}(x_0) \cap \Omega \subset B_{(1+\sigma)r}(z) \setminus \bar{B}_r(z) \quad \text{for} \quad \eta = \sigma/8,\ \sigma \in (0, \gamma),$$

for any $x' \in \partial\Omega \cap B_{\eta r}(x_0)$, where $B_r(z)$ is a ball contained in $\mathbb{R}^d \setminus \Omega$ that touches $\partial\Omega$ at point x' (see Remark 3.2.1).

In the following lemma, using Lemma 4.3.2 and Lemma 4.3.3 we construct supersolutions. We denote $\Omega_\rho := \{x \in \Omega |\, \mathrm{dist}(x, \Omega^c) < \rho\}$.

Lemma 4.3.5. *Let Ω be a bounded C^2 domain in $\mathbb{R}^d$ and $\alpha \in (1, 2)$, then there* *exist $\rho_1 > 0$ and a C^2 function ϕ_1 satisfying*

$$\begin{cases} \mathcal{P}^+\phi_1(x) + \mathcal{P}_k^+\phi_1(x) \leq -C\delta^{-\frac{\alpha}{2}}(x) & \text{in} \quad \Omega_{\rho_1}, \\ C^{-1}\delta(x) \leq \phi_1(x) \leq C\delta(x) & \text{in} \quad \Omega, \\ \phi_1(x) = 0 & \text{in} \quad \mathbb{R}^d \setminus \Omega, \end{cases}$$

where the constants ρ_1 and C depend only on $d, \alpha, \Omega, \lambda, \Lambda$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y$.

Proof. Let $\epsilon = \frac{2-\alpha}{2}$ and $c = \frac{1}{(\operatorname{diam}\Omega)^2}$, and define

$$\phi_1(x) = \delta(x) - c\delta^{1+\epsilon}(x).$$

Since both δ and $\delta^{1+\epsilon}$ are in $C^2(\Omega)$, we have $\mathcal{P}^+\phi_1(x) \leq \mathcal{P}^+\delta(x) - c\mathcal{P}^-\delta^{1+\epsilon}(x)$. Then by Lemma 4.3.3 and $\sup_{\theta\nu} |\operatorname{Tr}(a_{\theta\nu}(x)D^2\delta(x))| \leq \tilde{C}$, we get for all $x \in \Omega_\rho$

$$\mathcal{P}^+\phi_1(x) \leq \mathcal{P}^+\delta(x) - c\mathcal{P}^-\delta^{1+\epsilon}(x) \leq C - c(C_1 \cdot \epsilon\delta^{\epsilon-1}(x)).$$

Similarly for all $x \in \Omega_\rho$, using Lemma 4.3.2 and Lemma 4.3.4 we get

$$\mathcal{P}_k^+\phi_1(x) \leq |\mathcal{P}_k^+\delta(x)| + c|\mathcal{P}_k^-\delta^{1+\epsilon}(x)| \leq C_2\delta^{1-\alpha}(x).$$

Combining the above inequalities we have

$$\begin{aligned} \mathcal{P}^+\phi_1(x) + \mathcal{P}_k^+\phi_1(x) &\leq C - cC_1\epsilon\delta^{\epsilon-1}(x) + C_2\delta^{1-\alpha}(x) \\ &\leq -\delta^{\epsilon-1}(x)\left(\frac{C_1(2-\alpha)}{2(\operatorname{diam}\Omega)^2} - C\delta^{\frac{\alpha}{2}}(x) - C_2\delta^{\frac{2-\alpha}{2}}(x)\right), \end{aligned}$$

for all $x \in \Omega_\rho$. Now choose $0 < \rho_1 \leq \rho < 1$ such that

$$\left(\frac{C_1(2-\alpha)}{2(\operatorname{diam}\Omega)^2} - C\rho_1^{\frac{\alpha}{2}} - C_2\rho_1^{\frac{2-\alpha}{2}}\right) \geq \frac{C_1(2-\alpha)}{4(\operatorname{diam}\Omega)^2}.$$

Thus for all $x \in \Omega_{\rho_1}$, we have

$$\mathcal{P}^+\phi_1(x) + \mathcal{P}_k^+\phi_1(x) \leq -\frac{C_1(2-\alpha)}{4(\operatorname{diam}\Omega)^2}\delta^{-\frac{\alpha}{2}}(x).$$

Finally the construction of ϕ_1 immediately gives us that

$$C^{-1}\delta(x) \le \phi_1(x) \le C\delta(x) \quad \text{in} \quad \Omega,$$

and $\phi_1 = 0$ in Ω^c. This completes the proof of the lemma. □

4.4 Fine boundary regularity of u/δ

In this section, we investigate the regularity of u/δ near $\partial\Omega$. As we have seen in Section 3.2 a key step is the oscillation lemma. Since if we can control the oscillation of u/δ near $\partial\Omega$ appropriately then one can easily get Hölder regularity of u/δ up to the boundary.

We need following two Lemmas to prove oscillation lemma. In the first lemma we obtain a lower bound of $\inf_{D_{\frac{R}{2}}} \frac{u}{\delta}$ whereas the second lemma controls $\sup_{D^+_{\kappa' R}} \frac{u}{\delta}$ by using that lower bound.

Lemma 4.4.1. *Let $\alpha \in (0,2)$ and Ω be a bounded C^2 domain in $\mathbb{R}^d$. Also, let u be such that $u \ge 0$ in $\mathbb{R}^d$, and $|\mathcal{L}u| \le C_2(1 + \mathbb{1}_{(1,2)}(\alpha)\delta^{1-\alpha})$ in D_R, for some constant C_2. If $\hat{\alpha}$ is given by*

$$\hat{\alpha} = \begin{cases} 1 & if \quad \alpha \in (0,1], \\ \frac{2-\alpha}{2} & if \quad \alpha \in (1,2), \end{cases}$$

then there exists a positive constant C depending only on $d, \Omega, \Lambda, \lambda, \alpha, \int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y)\mathrm{d}y$, such that

$$\inf_{D^+_{\kappa' R}} \frac{u}{\delta} \le C\left(\inf_{D_{\frac{R}{2}}} \frac{u}{\delta} + C_2 R^{\hat{\alpha}}\right) \tag{4.4.1}$$

for all $R \le \rho_0$, where the constant ρ_0 depends only on $d, \Omega, \lambda, \Lambda, \alpha$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y)\mathrm{d}y$.

Proof. Suppose $R \le \eta\rho$, where ρ is given by Remark 4.3.1 and $\eta \le 1$ be some constant that will be chosen later. Define $m = \inf_{D^+_{\kappa' R}} u/\delta \ge 0$. Let us first observe that by (4.3.11) we have,

$$u \geq m\delta \geq m\,(\kappa R) \quad \text{in} \;\; D^+_{\kappa' R}. \tag{4.4.2}$$

Moreover by (4.3.12), for any $y \in D_{R/2}$, we have either $y \in D^+_{\kappa' R}$ or $\delta(y) < 4\kappa R$. If $y \in D^+_{\kappa' R}$, then by the definition of m we get $m \leq u(y)/\delta(y)$.

Next we consider $\delta(y) < 4\kappa R$. Let y^* be the nearest point to y on $\partial\Omega$, i.e, $\mathrm{dist}(y, \partial\Omega) = |y - y^*|$ and define $\tilde{y} = y^* + 4\kappa R\, \mathrm{n}(y^*)$. Again by (4.3.12), we have

$$B_{4\kappa R}(\tilde{y}) \subset D_R \;\; \text{and} \;\; B_{\kappa R}(\tilde{y}) \subset D^+_{\kappa' R}.$$

Denoting $r = \kappa R$ and using the subsolution constructed in Lemma 4.3.1, define $\tilde{\phi}_r(x) := \frac{1}{\kappa}\phi_r(x - \tilde{y})$. We will consider two cases.

Case 1: $\alpha \in (0, 1]$. Take $r' = \frac{R}{\eta}$. Since $r' \leq \rho$, points of $\partial\Omega$ can be touched by exterior ball of radius r'. In particular, for $y^* \in \partial\Omega$, we can find a point $z \in \Omega^c$ such that $\bar{B}_{r'}(z) \subset \Omega^c$ touches $\partial\Omega$ at y^*. Now from [131, Lemma 5.4] there exists a bounded, Lipschitz continuous function $\varphi_{r'}$, with Lipschitz constant $\frac{1}{r'}$, that satisfies

$$\begin{cases} \varphi_{r'} = 0, & \text{in} \quad \bar{B}_{r'}, \\ \varphi_{r'} > 0, & \text{in} \quad \bar{B}^c_{r'}, \\ \mathcal{P}^+\varphi_{r'} + \mathcal{P}^+_k\varphi_{r'} \leq -\frac{1}{(r')^2}, & \text{in} \quad B_{(1+\sigma)r'} \setminus \bar{B}_{r'}, \end{cases}$$

for some constant σ, independent of r'. Without any loss of any generality we may assume $\sigma \leq \gamma$ (see Remark 4.3.1). Then setting $\eta = \frac{\sigma}{8}$ and using Remark 4.3.1, we have

$$D_R \subset B_{(1+\sigma)r'}(z) \setminus \overline{B}_{r'}(z)$$

and by (4.3.12) we have

$$B_{4r}(\tilde{y}) \setminus \overline{B}_r(\tilde{y}) \subset D_R \subset B_{(1+\sigma)r'}(z) \setminus \overline{B}_{r'}(z).$$

We show that $v(x) = m\tilde{\phi}_r(x) - C_2(r')^2\varphi_{r'}(x - z)$ is an appropriate subsolution. Since both $\tilde{\phi}_r$ and $\varphi_{r'}$ are C^2 functions in $B_{4r}(\tilde{y}) \setminus \bar{B}_r(\tilde{y})$, we conclude that v is C^2

function in $B_{4r}(\tilde{y}) \setminus \bar{B}_r(\tilde{y})$. For $x \in B_{4r}(\tilde{y}) \setminus \bar{B}_r(\tilde{y})$,

$$\begin{aligned}&\mathcal{P}^- v(x) + \mathcal{P}_k^- v(x) \\ &\qquad \geq m\left[\mathcal{P}^- \tilde{\phi}_r(x) + \mathcal{P}_k^- \tilde{\phi}_r(x)\right] - C_2(r')^2\left[\mathcal{P}^+ \varphi_{r'}(x-z) + \mathcal{P}_k^+ \varphi_{r'}(x-z)\right] \geq C_2.\end{aligned}$$

Therefore by Remark 4.2.1 we have

$$\mathcal{P}^+(v-u) + \mathcal{P}_k^+(v-u) \geq 0 \quad \text{in} \quad B_{4r}(\tilde{y}) \setminus \bar{B}_r(\tilde{y}).$$

Furthermore, using (4.4.2) and $u \geq 0$ in $\mathbb{R}^d$ we obtain $u(x) \geq m\tilde{\phi}_r(x) - C_2(r')^2\varphi_{r'}(x-z)$ in $\left(B_{4r}(\tilde{y}) \setminus \bar{B}_r(\tilde{y})\right)^c$. Hence an application of maximum principle Lemma 4.2.1 gives $u \geq v$ in $\mathbb{R}^d$. Now for $y \in D_{R/2}$, using the Lipschitz continuity of $\varphi_{r'}$ we get

$$m\tilde{\phi}_r(y) \leq u(y) + C_2(r')^2\left[\varphi_{r'}(y-z) - \varphi_{r'}(y^*-z)\right] \leq u(y) + C_2 r' \cdot \delta(y)$$

and as y lies on the line segment joining y^* to $\tilde{y}$ we get

$$\frac{u(y)}{\delta(y)} + C_2 r' \geq \frac{m}{(\tilde{\kappa})^2}.$$

This gives

$$\inf_{D^+_{\kappa' R}} \frac{u}{\delta} \leq C\left(\inf_{D_{R/2}} \frac{u}{\delta} + C_2\frac{R}{\eta}\right)$$

and finally choosing $\rho_0 = \eta\rho$ we have (4.4.1).

Case 2: $\alpha \in (1,2)$. Let ρ_1 as in Lemma 4.3.5 and consider $R \leq \rho_1 < 1$. Here we aim to construct an appropriate subsolution using $\tilde{\phi}_r(x)$ and supersolution constructed in Lemma 4.3.5. Since $\delta(x) \leq 1$ in D_R, we have $|\mathcal{L}u(x)| \leq C_2(1+\delta^{1-\alpha}(x)) \leq 2C_2\delta^{1-\alpha}(x)$ in D_R. Also by Lemma 4.3.5, we have a bounded function ϕ_1 which is C^2 in $\Omega_{\rho_1} \supset D_R$ and satisfies

$$\mathcal{P}^+\phi_1(x) + \mathcal{P}_k^+\phi_1(x) \leq -C\delta^{-\frac{\alpha}{2}}(x) = -C\frac{1}{\delta^{\frac{2-\alpha}{2}}(x)}\delta^{1-\alpha}(x) \leq \frac{-C}{R^{\hat{\alpha}}}\delta^{1-\alpha}(x),$$

for all $x \in D_R$. Now we define the subsolutions as

$$v(x) = m\tilde{\phi}_r(x) - \mu R^{\hat{\alpha}}\phi_1(x),$$

where the constant μ is chosen suitably so that $\mathcal{P}^- v(x) + \mathcal{P}_k^- v(x) \geq 2C_2\delta^{1-\alpha}(x)$ in $B_{4r}(\tilde{y}) \setminus \bar{B}_r(\tilde{y})$ (i.e. $\mu = \frac{2C_2}{C}$). Also $u \geq v$ in $(B_{4r}(\tilde{y}) \setminus \bar{B}_r(\tilde{y}))^c$. Using the same calculation as previous case for $v-u$ and maximum principle Lemma 4.2.1 we derive that $u \geq v$ in $\mathbb{R}^d$. Again, repeating the arguments of **Case 1** we get

$$\inf_{D^+_{\kappa' R}} \frac{u}{\delta} \leq C\left(\inf_{D_{\frac{R}{2}}} \frac{u}{\delta} + 2C_2 R^{\hat{\alpha}}\right).$$

Choosing $\rho_0 = \eta\rho \wedge \rho_1$ completes the proof. □

Lemma 4.4.2. *Let $\alpha' = 1 \wedge (2-\alpha)$ and Ω be a bounded C^2 domain in $\mathbb{R}^d$. Also, let u be a bounded continuous function such that $u \geq 0$ and $u \leq M_0\delta(x)$ in $\mathbb{R}^d$, and $|\mathcal{L}u| \leq C_2(1 + \mathbb{1}_{(1,2)}(\alpha)\delta^{1-\alpha})$ in D_R, for some constant C_2. Then, there exists a positive constant C, depending only on $d, \lambda, \Lambda, \Omega$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y$, such that*

$$\sup_{D^+_{\kappa' R}} \frac{u}{\delta} \leq C\left(\inf_{D^+_{\kappa' R}} \frac{u}{\delta} + (M_0 \vee C_2)R^{\alpha'}\right) \tag{4.4.3}$$

for all $R \leq \rho$, where constant ρ is given by Remark 4.3.1.

Proof. We will use the weak Harnack inequality proved in Theorem 4.3.1 to show (4.4.3). Let $R \leq \rho$. Then for each $y \in D^+_{\kappa' R}$, we have $B_{\kappa R}(y) \subset D_R$. Hence we have $|\mathcal{L}u| \leq C_2(1 + \mathbb{1}_{(1,2)}(\alpha)\delta^{1-\alpha}(x))$ in $B_{\kappa R}(y)$. Without loss of generality, we may assume $y = 0$. Let $s = \kappa R$ and define $v(x) = u(sx)$ for all $x \in \mathbb{R}^d$. Then, it can be easily seen that

$$s^2\mathcal{L}[sx, u] = \mathcal{L}^s[x, v] := \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\left\{\mathrm{Tr}\, a_{\theta\nu}(sx)D^2v(x) + \mathcal{I}^s_{\theta\nu}[x, v]\right\} \qquad \text{for all } x \in B_2.$$

This gives

$$\begin{aligned} |\mathcal{L}^s[x, v]| &\leq C_2 s^2(1 + \mathbb{1}_{(1,2)}(\alpha)\delta^{1-\alpha}(sx)) \\ &\leq C_2\left(s^2 + \mathbb{1}_{(1,2)}(\alpha)s^2\left(\kappa R\right)^{1-\alpha}\right) \\ &\leq C_2 s^{1+\alpha'}, \end{aligned}$$

in B_2 where $\alpha' = 1 \wedge (2-\alpha)$. In second line, we used that for each $x \in B_{\kappa R}$, $|sx| < \kappa R$ and hence $\delta(sx) > \frac{\kappa R}{2} = \frac{s}{2}$. From $u \leq M_0\delta(x)$ we have $v(y) \leq M_0 \operatorname{diam}\Omega$

and $v(y) \le M_0 s(1+|y|)$ in whole $\mathbb{R}^d$. Hence by Corollary 4.3.1, we obtain

$$\sup_{B_{\frac{1}{4}}} v \le C\left(\inf_{B_{\frac{1}{4}}} v + (M_0 \vee C_2)s^{1+\alpha'}\right),$$

where constant C does not depend on s, M_0, C_2. This of course, implies

$$\sup_{B_{\frac{\kappa R}{64}}(y)} u \le C\left(\inf_{B_{\frac{\kappa R}{64}}(y)} u + (M_0 \vee C_2)R^{1+\alpha'}\right),$$

for all $y \in D^+_{\kappa' R}$. Now cover $D^+_{\kappa' R}$ by a finite number of balls $B_{\kappa R/64}(y_i)$, independent of R, to obtain

$$\sup_{D^+_{\kappa' R}} u \le C\left(\inf_{D^+_{\kappa' R}} u + (M_0 \vee C_2)R^{1+\alpha'}\right).$$

Then (4.4.3) follows since $\kappa R/2 \le \delta \le 3\kappa R/2$ in $D^+_{\kappa' R}$. □

Now we are ready to prove the oscillation lemma.

Proposition 4.4.1. *Let u be a bounded continuous function such that $|\mathcal{L}u| \le K$ in Ω, for some constant K, and $u = 0$ in Ω^c. Given any $x_0 \in \partial\Omega$, let D_R be as in the Definition 4.3.1. Then for some $\tau \in (0, \hat{\alpha})$ there exists C, dependent on $\Omega, d, \lambda, \Lambda, \alpha$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y$ but not on x_0, such that*

$$\sup_{D_R} \frac{u}{\delta} - \inf_{D_R} \frac{u}{\delta} \le CKR^\tau \tag{4.4.4}$$

for all $R \le \rho_0$, where $\rho_0 > 0$ is a constant depending only on $\Omega, d, \lambda, \Lambda, \alpha$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y$.

Proof. For the proof we follow a standard method, similar to [144], with the help of Lemmas 4.3.4, 4.4.1, and 4.4.2. Fix $x_0 \in \partial\Omega$ and consider $\rho_0 > 0$ to be chosen later. With no loss of generality, we assume $x_0 = 0$. In view of (4.2.2), we only consider the case $K > 0$. By considering u/K instead of u, we may assume that $K = 1$, that is, $|\mathcal{L}u| \le 1$ in Ω. From Theorem 4.2.1 we note that $||u||_{C^{0,1}(\mathbb{R}^d)} \le C_1$. Below, we consider two cases.

Case 1: For $\alpha \in (0, 1]$, $\mathcal{I}_{\theta\nu}u$ is classically defined and $|\mathcal{I}_{\theta\nu}u| \le \tilde{C}$ in Ω for all θ and ν. Consequently, one can combine the nonlocal term on the RHS and only deal with local nonlinear operator $\tilde{\mathcal{L}}[x,u] := \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\{\operatorname{Tr} a_{\theta\nu}(x)D^2u(x)\}$. In this case the proof is simpler and can be done following the same method as for the local case. However, the method we use below would also work with an appropriate modification.

Case 2: Now we deal with the case $\alpha\in(1,2)$. We show that there exists $\mathscr{K}>0$, $\rho_1\in(0,\rho_0)$ and $\tau\in(0,1)$, dependent only on $\Omega, d, \lambda, \Lambda, \alpha$ and $\int_{\mathbb{R}^d}(1\wedge|y|^\alpha)k(y)\mathrm{d}y$, and monotone sequences $\{M_k\}$ and $\{m_k\}$ such that, for all $k\geq 0$,

$$M_k - m_k = \frac{1}{4^{k\tau}}, \quad -1\leq m_k\leq m_{k+1} < M_{k+1}\leq M_k\leq 1, \tag{4.4.5}$$

and

$$m_k\leq \mathscr{K}^{-1}\frac{u}{\delta}\leq M_k \quad \text{in} \quad D_{R_k}, \quad \text{where} \quad R_k=\frac{\rho_1}{4^k}. \tag{4.4.6}$$

Note that (4.4.6) is equivalent to the following

$$m_k\delta\leq \mathscr{K}^{-1}u\leq M_k\delta, \quad \text{in} \quad B_{R_k}, \quad \text{where} \quad R_k=\frac{\rho_1}{4^k}. \tag{4.4.7}$$

Next we construct monotone sequences $\{M_k\}$ and $\{m_k\}$ by induction.

The existence of M_0 and m_0 such that (4.4.5) and (4.4.7) hold for $k=0$ is guaranteed by Lemma 4.2.2. Assume that we have the sequences up to M_k and m_k. We want to show the existence of M_{k+1} and m_{k+1} such that (4.4.5)-(4.4.7) hold. We set

$$u_k=\frac{1}{\mathscr{K}}u-m_k\delta.$$

Note that to apply Lemma 4.4.2 we need u_k to be nonnegative in $\mathbb{R}^d$. Therefore we shall work with u_k^+, the positive part of u_k. Let $u_k=u_k^+-u_k^-$ and by the induction hypothesis,

$$u_k^+=u_k \quad \text{and} \quad u_k^-=0 \quad \text{in} \quad B_{R_k}. \tag{4.4.8}$$

We need to find a lower bound on u_k. Since $u_k\geq 0$ in B_{R_k} and u_k is Lipschitz in

$\mathbb{R}^d$, we get for $x \in B^c_{R_k}$ that

$$u_k(x) = u_k(R_k x_{\mathrm{u}}) + u_k(x) - u_k(R_k x_{\mathrm{u}}) \geq -C_L |x - R_k x_{\mathrm{u}}|, \tag{4.4.9}$$

where $z_{\mathrm{u}} = \frac{1}{|z|} z$ for $z \neq 0$ and C_L denotes a Lipschitz constant of u_k which can be chosen independent of k. Using Lemma 4.2.2 we also have $|u_k| \leq \mathscr{K}^{-1} + \mathrm{diam}(\Omega) = C_1$ for all $x \in \mathbb{R}^d$. Thus using (4.4.8) and (4.4.9) we calculate $\mathcal{L}[x, u_k^-]$ in $D_{\frac{R_k}{2}}$. Let $x \in D_{R_k/2}$. By (4.4.8), $D^2 u_k^-(x) = 0$. Then

$$\begin{aligned}
0 &\leq \mathcal{I}_{\theta\nu}[x, u_k^-] \\
&= \int_{x+y \notin B_{R_k}} u_k^-(x+y) N_{\theta\nu}(x,y) \mathrm{d}y \\
&\leq \int_{\left\{|y| \geq \frac{R_k}{2}, x+y \neq 0\right\}} u_k^-(x+y) k(y) \mathrm{d}y \\
&\leq C_L \int_{\left\{\frac{R_k}{2} \leq |y| \leq 1,\, x+y \neq 0\right\}} \Big|(x+y) - R_k (x+y)_{\mathrm{u}}\Big| k(y) \mathrm{d}y + C_1 \int_{|y| \geq 1} k(y) \mathrm{d}y \\
&\leq C_L \int_{\frac{R_k}{2} \leq |y| \leq 1} (|x| + R_k)\, k(y)\, \mathrm{d}y + C_L \int_{\frac{R_k}{2} \leq |y| \leq 1} |y| k(y)\, \mathrm{d}y + C_1 \int_{|y| \geq 1} k(y) \mathrm{d}y \\
&\leq \kappa_3 \left[\int_{\mathbb{R}^d} (1 \wedge |y|^\alpha) k(y)\, \mathrm{d}y\right] \left(R_k^{1-\alpha} + 1\right) \\
&\leq \kappa_4 R_k^{1-\alpha},
\end{aligned} \tag{4.4.10}$$

for some constants κ_3, κ_4, independent of k.

Now we write $u_k^+ = \mathscr{K}^{-1} u - m_k \delta + u_k^-$. Since δ is C^2 and $u_k^- = 0$ in $D_{\frac{R_k}{2}}$, first note that

$$\begin{aligned}
\mathcal{L} u_k^+ &\leq \mathscr{K}^{-1} - (\mathcal{P}^- + \mathcal{P}_k^-)(m_k \delta) + (\mathcal{P}^+ + \mathcal{P}_k^+)(u_k^-), \\
\mathcal{L} u_k^+ &\geq -\mathscr{K}^{-1} - (\mathcal{P}^+ + \mathcal{P}_k^+)(m_k \delta) + (\mathcal{P}^- + \mathcal{P}_k^-)(u_k^-).
\end{aligned}$$

Using Lemma 4.3.4 and (4.4.10) in the above estimate we have

$$|\mathcal{L} u_k^+| \leq \mathscr{K}^{-1} + m_k C \delta^{1-\alpha} + \kappa_4 (R_k)^{1-\alpha} \ \text{ in } D_{\frac{R_k}{2}}. \tag{4.4.11}$$

Since $\rho_1 \geq R_k \geq \delta$ in D_{R_k}, for $\alpha > 1$, we have $R_k^{1-\alpha} \leq \delta^{1-\alpha}$, and hence, from

(4.4.11), we have

$$|\mathcal{L}u_k^+| \le \Big[\mathscr{K}^{-1}[(\rho_1)]^{\alpha-1} + C + \kappa_4\Big]\delta^{1-\alpha}(x) := \kappa_5\delta^{1-\alpha}(x) \quad \text{in} \quad D_{R_k/2}.$$

Now we are in a position to apply Lemmas 4.4.1 and 4.4.2. Recalling that

$$u_k^+ = u_k = \mathscr{K}^{-1}u - m_k\delta \quad \text{in} \quad D_{R_k},$$

and using Lemma 4.2.2 we also have $|u_k^+| \le |u_k| \le (\mathscr{K}^{-1}+1)\delta(x) = C_1\delta(x)$ for all $x \in \mathbb{R}^d$. We get from Lemmas 4.4.1 and 4.4.2 that

$$\begin{aligned}\sup_{D^+_{\kappa' R_k/2}} \Big(\mathscr{K}^{-1}\frac{u}{\delta} - m_k\Big) &\le C\Big(\inf_{D^+_{\kappa' R_k/2}} \Big(\mathscr{K}^{-1}\frac{u}{\delta} - m_k\Big) + (\kappa_5 \vee C_1)R_k^{\hat\alpha}\Big)\\ &\le C\Big(\inf_{D_{R_k/4}} \Big(\mathscr{K}^{-1}\frac{u}{\delta} - m_k\Big) + (\kappa_5 \vee C_1)R_k^{\hat\alpha}\Big).\end{aligned} \tag{4.4.12}$$

Repeating a similar argument for the function $\tilde u_k = M_k\delta - \mathscr{K}^{-1}u$, we find

$$\sup_{D^+_{\kappa' R_k/2}} \Big(M_k - \mathscr{K}^{-1}\frac{u}{\delta}\Big) \le C\Big(\inf_{D_{R_k/4}} \Big(M_k - \mathscr{K}^{-1}\frac{u}{\delta}\Big) + (\kappa_5 \vee C_1)R_k^{\hat\alpha}\Big). \tag{4.4.13}$$

Combining (4.4.12) and (4.4.13) we obtain

$$\begin{aligned}M_k - m_k &\le C\Big(\inf_{D^+_{R_k/4}} \Big(M_k - \mathscr{K}^{-1}\frac{u}{\delta}\Big) + \inf_{D^+_{R_k/4}} \Big(\mathscr{K}^{-1}\frac{u}{\delta} - m_k\Big) + (\kappa_5 \vee C_1)R_k^{\hat\alpha}\Big)\\ &= C\Big(\inf_{D_{R_{k+1}}} \mathscr{K}^{-1}\frac{u}{\delta} - \sup_{D_{R_{k+1}}} \mathscr{K}^{-1}\frac{u}{\delta} + M_k - m_k + (\kappa_5 \vee C_1)R_k^{\hat\alpha}\Big).\end{aligned} \tag{4.4.14}$$

Putting $M_k - m_k = \frac{1}{4^{\tau k}}$ in (4.4.14), we have

$$\begin{aligned}\sup_{D_{R_{k+1}}} \mathscr{K}^{-1}\frac{u}{\delta} - \inf_{D_{R_{k+1}}} \mathscr{K}^{-1}\frac{u}{\delta} &\le \Big(\frac{C-1}{C}\frac{1}{4^{\tau k}} + (\kappa_5 \vee C_1)R_k^{\hat\alpha}\Big)\\ &= \frac{1}{4^{\tau k}}\Big(\frac{C-1}{C} + (\kappa_5 \vee C_1)R_k^{\hat\alpha}4^{\tau k}\Big).\end{aligned} \tag{4.4.15}$$

Since $R_k = \frac{\rho_1}{4^k}$ for $\rho_1 \in (0, \rho_0)$, we can choose ρ_0 and τ small so that

$$\left(\frac{C-1}{C} + (\kappa_5 \vee C_1) R_k^{\hat{\alpha}} 4^{\tau k}\right) \leq \frac{1}{4^{\tau}}.$$

Putting in (4.4.15) we obtain

$$\sup_{D_{R_{k+1}}} \mathscr{K}^{-1}\frac{u}{\delta} - \inf_{D_{R_{k+1}}} \mathscr{K}^{-1}\frac{u}{\delta} \leq \frac{1}{4^{\tau(k+1)}}.$$

Thus we find m_{k+1} and M_{k+1} such that (4.4.5) and (4.4.6) hold. It is easy to prove (4.4.4) from (4.4.5)-(4.4.6). □

Next we establish Hölder regularity of u/δ up to the boundary.

Theorem 4.4.1. *Suppose that Assumption 4.0.1 holds. Let Ω be a bounded C^2 domain and u be a viscosity solution to the inequations (4.0.2). Then there exists $\kappa \in (0, \hat{\alpha})$ such that*

$$\|u/\delta\|_{C^{\kappa}(\overline{\Omega})} \leq C_1 K, \tag{4.4.16}$$

for some constant C_1, where κ, C_1 depend on $d, \Omega, C_0, \Lambda, \lambda, \alpha$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^{\alpha})k(y)\mathrm{d}y$. Here $\hat{\alpha}$ is given by

$$\hat{\alpha} = \begin{cases} 1 & if \quad \alpha \in (0,1] \\ \frac{2-\alpha}{2} & if \quad \alpha \in (1,2). \end{cases}$$

Proof. Replacing u by $\frac{u}{CK}$ we may assume that $|\mathcal{L}u| \leq 1$ in Ω. Let $v = u/\delta$. From Lemma 4.2.2 we then have

$$\|v\|_{L^{\infty}(\Omega)} \leq C,$$

for some constant C and from Theorem 4.2.1 we have

$$\|u\|_{C^{0,1}(\mathbb{R}^d)} \leq C. \tag{4.4.17}$$

Also from Proposition 4.4.1 for each $x_0 \in \partial\Omega$ and for all $r > 0$ we have

$$\sup_{D_r(x_0)} v - \inf_{D_r(x_0)} v \leq C r^{\tau}. \tag{4.4.18}$$

where $D_r(x_0) = B_r(x_0) \cap \Omega$ as before. To complete the proof we shall show that

$$\sup_{x,y\in\Omega, x\neq y} \frac{|v(x)-v(y)|}{|x-y|^{\kappa}} \leq C, \tag{4.4.19}$$

for some $\kappa > 0$. Let $r = |x-y|$ and there exists $x_0, y_0 \in \partial\Omega$ such that $\delta(x) = |x-x_0|$ and $\delta(y) = |y-y_0|$. If $r \geq \frac{1}{8}$, then

$$\frac{|v(x)-v(y)|}{|x-y|^{\kappa}} \leq 2\cdot 8^{\kappa}||v||_{L^\infty(\Omega)}.$$

If $r < \frac{1}{8}$ and $r \geq \frac{1}{8}(\delta(x)\vee\delta(y))^p$ for some $p > 2$ then clearly $y \in B_{\kappa r^{1/p}}(x_0)$ for some $\kappa > 0$. Now using (4.4.18) we obtain

$$|v(x)-v(y)| \leq \sup_{D_{\kappa r^{1/p}}(x_0)} v - \inf_{D_{\kappa r^{1/p}}(x_0)} v \leq C_\kappa r^{\tau/p}.$$

If $r < \frac{1}{8}$ and $r < \frac{1}{8}(\delta(x)\vee\delta(y))^p$, then $r < \frac{1}{8}(\delta(x)\vee\delta(y))$ and this implies $y \in B_{\frac{1}{8}(\delta(x)\vee\delta(y))}(x)$ or $x \in B_{\frac{1}{8}(\delta(x)\vee\delta(y))}(y)$. Without loss of any generality assume $\delta(x) \geq \delta(y)$ and $y \in B_{\frac{\delta(x)}{8}}(x)$. Using (4.4.17) and the Lipschitz continuity of δ, we get

$$|v(x)-v(y)| = \left|\frac{u(x)}{\delta(x)} - \frac{u(y)}{\delta(y)}\right| \leq \frac{M(K, \operatorname{diam}\Omega)r}{\delta(x)\cdot\delta(y)}.$$

Also we have $(8r)^{1/p}\delta(y) < \delta(x)\cdot\delta(y)$. This implies

$$|v(x)-v(y)| \leq \frac{M(K,\operatorname{diam}\Omega)r}{\delta(x)\cdot\delta(y)} < \frac{M(K,\operatorname{diam}\Omega)}{8^{1/p}}\cdot\frac{r^{1-1/p}}{\delta(y)}.$$

Now if $r < \frac{1}{8}(\delta(y))^p$ then one obtains

$$|v(x)-v(y)| < \frac{M(K,\operatorname{diam}\Omega)}{8^{1/p}}\cdot\frac{r^{1-1/p}}{\delta(y)} \leq Cr^{1-2/p}.$$

On the other hand, if $r \geq \frac{1}{8}(\delta(y))^p$, since $\delta(y) > \frac{1}{64}\delta(x)$ we have $r \geq \frac{1}{8}\cdot\left(\frac{1}{64}\right)^p(\delta(x))^p$ and this case can be treated as previous. Therefore choosing $\kappa = (1-\frac{2}{p})\wedge\frac{\tau}{p}$ we conclude (4.4.19). This completes the proof. $\square$

4.5 Global Hölder regularity of the gradient

In this section we prove the Hölder regularity of Du up to the boundary. First, let us recall

$$\mathcal{L}[x,u] = \sup_{\theta\in\Theta}\inf_{\nu\in\gamma}\left\{\operatorname{Tr} a_{\theta\nu}(x)D^2u(x) + \mathcal{I}_{\theta\nu}[x,u]\right\}.$$

We denote $v = \frac{u}{\delta}$. Following [38], next we obtain the inequations satisfied by v.

Lemma 4.5.1. *Let Ω be bounded C^2 domain in $\mathbb{R}^d$. If $|\mathcal{L}u| \le K$ in Ω and $u = 0$ in Ω^c, then we have*

$$\begin{aligned}
\mathcal{L}v + 2K_0 d^2\frac{|D\delta|}{\delta}|Dv| &\ge \frac{1}{\delta}\Big[-K - |v|(P^+ + P_k^+)\delta - \sup_{\theta,\nu} Z_{\theta\nu}[v,\delta]\Big],\\
\mathcal{L}v - 2K_0 d^2\frac{|D\delta|}{\delta}|Dv| &\le \frac{1}{\delta}\Big[K - |v|(P^- + P_k^-)\delta - \inf_{\theta,\nu} Z_{\theta\nu}[v,\delta]\Big]
\end{aligned} \tag{4.5.1}$$

for some K_0, where

$$Z_{\theta\nu}[v,\delta](x) = \int_{\mathbb{R}^d}(v(y)-v(x))(\delta(y)-\delta(x))N_{\theta\nu}(x,y-x)\mathrm{d}y.$$

Proof. First note that, since $u \in C^1(\Omega)$ by Theorem 4.1.1, we have $v \in C^1(\Omega)$. Therefore, $Z_{\theta\nu}[v,\delta]$ is continuous in Ω. Consider a test function $\psi \in C^2(\Omega)$ that touches v from above at $x \in \Omega$. Define

$$\psi_r(z) = \begin{cases} \psi(z) & \text{in } B_r(x),\\ v(z) & \text{in } B_r^c(x).\end{cases}$$

By our assertion, we have $\psi_r \ge v$ for all r small. To verify the first inequality in (4.5.1) we must show that

$$\begin{aligned}
&\mathcal{L}[x,\psi_r] + 2k_0 d^2\frac{|D\delta(x)|}{\delta(x)}\cdot|D\psi_r(x)|\\
&\qquad \ge \frac{1}{\delta(x)}[-K - |v(x)|(\mathcal{P}^+ + \mathcal{P}_k^+)\delta(x) - \sup_{\theta,\nu} Z_{\theta\nu}[v,\delta](x)],
\end{aligned} \tag{4.5.2}$$

for some r small. We define

$$\tilde{\psi}_r(z) = \begin{cases} \delta(z)\psi(z) & \text{in } B_r(x), \\ u(z) & \text{in } B_r^c(x). \end{cases}$$

Then, $\tilde{\psi}_r \geq u$ for all r small. Since $|\mathcal{L}u| \leq K$ and $\delta\psi_r = \tilde{\psi}_r$, we obtain at a point x

$$\begin{aligned} -K \leq& \mathcal{L}[x, \tilde{\psi}_r] \\ =& \sup_{\theta\in\Theta} \inf_{\nu\in\gamma} \Big[\delta(x) \left(\operatorname{Tr} a_{\theta\nu}(x) D^2\psi_r(x) + \mathcal{I}_{\theta\nu}\psi_r(x) \right) \\ & \quad + \psi_r(x) \left(\operatorname{Tr} a_{\theta\nu}(x) D^2\delta(x) + \mathcal{I}_{\theta\nu}\delta(x) \right) \\ & \quad + \operatorname{Tr} \left[\left(a_{\theta\nu}(x) + a_{\theta\nu}^T(x) \right) \cdot (D\delta(x) \otimes D\psi_r(x)) \right] + Z_{\theta\nu}[\psi_r, \delta](x) \Big] \\ \leq& \ \delta(x)\mathcal{L}[x, \psi_r] + \sup_{\theta,\nu} \Big[|\psi_r(x)| \left(\operatorname{Tr} a_{\theta\nu}(x) D^2\delta(x) + \mathcal{I}_{\theta\nu}\delta(x) \right) \\ & \quad + \operatorname{Tr} \left[\left(a_{\theta\nu}(x) + a_{\theta\nu}^T(x) \right) \cdot (D\delta(x) \otimes D\psi_r(x)) \right] + Z_{\theta\nu}[\psi_r, \delta](x) \Big] \\ \leq& \ \delta(x)\mathcal{L}[x, \psi_r] + |v(x)| \left(\mathcal{P}^+ + \mathcal{P}_k^+ \right) \delta(x) + 2K_0 d^2 |D\delta(x)| \cdot |D\psi_r(x)| \\ & \quad + \sup_{\theta,\nu} Z_{\theta\nu}[\psi_r, \delta](x), \end{aligned}$$

for all r small and some constant K_0, where $D\delta(x) \otimes D\psi_r(x) := \left(\frac{\partial\delta}{\partial x_i} \cdot \frac{\partial\psi_r}{\partial x_j} \right)_{i,j}$. Rearranging the terms we have

$$\begin{aligned} & -K - |v(x)|(\mathcal{P}^+ + \mathcal{P}_k^+)\delta(x) - \sup_{\theta,\nu} Z_{\theta\nu}[\psi_r, \delta](x) \\ & \qquad \leq \delta(x)\mathcal{L}[x, \psi_r] + 2K_0 d^2 |D\delta(x)| \cdot |D\psi_r(x)|. \end{aligned} \tag{4.5.3}$$

Let $r_1 \leq r$. Since ψ_r is decreasing with r, we get from (4.5.3) that

$$\begin{aligned} & \delta(x)\mathcal{L}[x, \psi_r] + 2K_0 d^2 |D\delta(x)| \cdot |D\psi_r(x)| \\ & \quad \geq \delta(x)\mathcal{L}[x, \psi_{r_1}] + 2K_0 d^2 |D\delta(x)| \cdot |D\psi_{r_1}(x)| \\ & \quad \geq \lim_{r_1\to 0} \left[-K - |v(x)| \left(\mathcal{P}^+ + \mathcal{P}_k^+ \right) \delta(x) - \sup_{\theta,\nu} Z_{\theta\nu}[\psi_{r_1}, \delta](x) \right] \end{aligned}$$

$$= \left[-K - |v(x)|\left(\mathcal{P}^+ + \mathcal{P}_k^+\right)\delta(x) - \sup_{\theta,\nu} Z_{\theta\nu}[v,\delta](x)\right],$$

by dominated convergence theorem. This gives (4.5.2). Similarly we can verify the second inequality of (4.5.1). □

Next we obtain a the following estimate on v, away from the boundary. Denote $\Omega^\sigma = \{x \in \Omega \,:\, \mathrm{dist}(x, \Omega^c) \geq \sigma\}$.

Lemma 4.5.2. *Let Ω be bounded C^2 domain in $\mathbb{R}^d$. If $|\mathcal{L}u| \leq K$ in Ω and $u = 0$ in Ω^c, then for some constant C it holds that*

$$\|Dv\|_{L^\infty(\Omega^\sigma)} \leq CK\sigma^{\kappa-1} \quad \textit{for all } \sigma \in (0,1). \tag{4.5.4}$$

Furthermore, there exists $\eta \in (0,1)$ such that for any $x \in \Omega^\sigma$ and $0 < |x-y| \leq \sigma/8$ we have

$$\frac{|Dv(y) - Dv(x)|}{|x-y|^\eta} \leq CK\sigma^{\kappa-1-\eta},$$

for all $\sigma \in (0,1)$.

Proof. Using Lemma 4.5.1 we have

$$\begin{aligned}
\mathcal{L}v + 2K_0 d^2 \frac{|D\delta|}{\delta}|Dv| &\geq \frac{1}{\delta}\Big[-K - |v|(\mathcal{P}^+ + \mathcal{P}_k^+)\delta - \sup_{\theta,\nu} Z_{\theta\nu}[v,\delta]\Big],\\
\mathcal{L}v - 2K_0 d^2 \frac{|D\delta|}{\delta}|Dv| &\leq \frac{1}{\delta}\Big[K - |v|(\mathcal{P}^- + \mathcal{P}_k^-)\delta - \inf_{\theta,\nu} Z_{\theta\nu}[v,\delta]\Big]
\end{aligned} \tag{4.5.5}$$

in Ω. Fix a point $x_0 \in \Omega^\sigma$ and define

$$w(x) = v(x) - v(x_0).$$

From (4.5.5) we then obtain

$$\begin{aligned}
\mathcal{L}w + 2K_0 d^2 \frac{|D\delta|}{\delta}|Dw| &\geq \Big[-\frac{1}{\delta}K - \ell_1\Big],\\
\mathcal{L}w - 2K_0 d^2 \frac{|D\delta|}{\delta}|Dw| &\leq \Big[\frac{1}{\delta}K + \ell_2\Big]
\end{aligned} \tag{4.5.6}$$

in Ω, where

$$\ell_1(x) = \frac{1}{\delta(x)}\left[|w(x)|(\mathcal{P}^+ + \mathcal{P}_k^+)\delta(x) + \sup_{\theta,\nu} Z_{\theta\nu}[w,\delta](x) + |v(x_0)|(\mathcal{P}^+ + \mathcal{P}_k^+)\delta(x)\right]$$

And

$$\ell_2(x) = \frac{1}{\delta(x)}\left[|w(x)|(\mathcal{P}^- + \mathcal{P}_k^-)\delta(x) - \inf_{\theta,\nu} Z_{\theta\nu}[w,\delta](x) - |v(x_0)|(\mathcal{P}^- + \mathcal{P}_k^-)\delta(x)\right].$$

We set $r = \frac{\sigma}{2}$ and claim that

$$\|\ell_i\|_{L^\infty(B_r(x_0))} \leq \kappa_1 \sigma^{\kappa-2}, \quad \text{for all } \sigma \in (0,1) \text{ and } i = 1,2, \tag{4.5.7}$$

for some constant κ_1. Let us denote by

$$\xi_1^\pm = \frac{|w(x)|(\mathcal{P}^\pm + \mathcal{P}_k^\pm)\delta}{\delta}, \qquad \xi_2 = \frac{1}{\delta}\sup_{\theta,\nu} Z_{\theta\nu}[w,\delta],$$
$$\xi_3^\pm = \frac{|v(x_0)|(\mathcal{P}^\pm + \mathcal{P}_k^\pm)\delta}{\delta}, \qquad \xi_4 = \frac{1}{\delta}\inf_{\theta,\nu} Z_{\theta\nu}[v,\delta].$$

Recall that $\kappa \in (0,\hat{\alpha})$. Since

$$\|\mathcal{P}^\pm\delta\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \|\mathcal{P}_k^\pm\delta\|_{L^\infty(\Omega_\sigma)} \lesssim \left(1 + \mathbb{1}_{(1,2)}(\alpha)\delta^{1-\alpha}\right)$$

(cf Lemma 4.3.4), and

$$\|v\|_{L^\infty(\mathbb{R}^d)} < \infty, \quad \|w\|_{L^\infty(B_r(x_0))} \lesssim r^\kappa,$$

it follows that

$$\|\xi_3^\pm\|_{L^\infty(B_r(x_0))} \lesssim \left\{\begin{array}{ll} \frac{1}{\sigma} & \text{if } \alpha \in (0,1], \\ \frac{1}{\sigma^\alpha} & \text{if } \alpha \in (1,2) \end{array}\right\} \lesssim \sigma^{\kappa-2},$$

and

$$\|\xi_1^\pm\|_{L^\infty(B_r(x_0))} \lesssim \left\{\begin{array}{ll} \frac{\sigma^\kappa}{\delta^2} & \text{if } \alpha \in (0,1], \\ \frac{\sigma^\kappa}{\delta^\alpha} & \text{if } \alpha \in (1,2) \end{array}\right\} \lesssim \sigma^{\kappa-2}.$$

Next we estimate ξ_2 and ξ_4. Let $x \in B_r(x_0)$. Denote by $\hat{r} = \delta(x)/4$. Note that

$$\delta(x) \geq \delta(x_0) - |x - x_0| \geq 2r - r = r \Rightarrow \hat{r} \geq r/4.$$

Since $u \in C^1(\Omega)$ by Theorem 4.1.1 and $|u| \leq C\delta$ in $\mathbb{R}^d$ by Lemma 4.2.2. Thus we have

$$|Dv| \leq \left|\frac{Du}{\delta}\right| + \left|\frac{uD\delta}{\delta^2}\right| \lesssim \frac{1}{\delta(x)} \quad \text{in } B_{\hat{r}}(x). \tag{4.5.8}$$

Now we calculate

$$\begin{aligned}
|Z_{\theta\nu}[w, \delta](x)| &\leq \int_{\mathbb{R}^d} |\delta(x) - \delta(y)||v(x) - v(y)|k(y - x)\mathrm{d}y \\
&= \int_{B_{\hat{r}}(x)} + \int_{B_1(x)\setminus B_{\hat{r}}(x)} + \int_{B_1^c(x)} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

To estimate I_1, first we consider $\alpha \leq 1$. Since δ is Lipschitz continuous and v bounded on $\mathbb{R}^d$, I_1 can be written as

$$\begin{aligned}
I_1 &= \int_{B_{\hat{r}}(x)} \frac{|\delta(x) - \delta(y)|}{|x - y|}|v(x) - v(y)| \cdot |x - y|k(y - x)\mathrm{d}y \\
&\lesssim \int_{B_{\hat{r}}(x)} |x - y|^\alpha k(y - x)\mathrm{d}y \leq \int_{\mathbb{R}^d} (1 \wedge |z|^\alpha)k(z)\mathrm{d}z.
\end{aligned}$$

For $\alpha \in (1, 2)$, using the Lipschitz continuity of δ and (4.5.8) we get

$$\begin{aligned}
I_1 &= \int_{B_{\hat{r}}(x)} \frac{|\delta(x) - \delta(y)|}{|x - y|} \cdot \frac{|v(x) - v(y)|}{|x - y|} \cdot |x - y|^\alpha |x - y|^{2-\alpha} k(y - x)\mathrm{d}y \\
&\lesssim \frac{\hat{r}^{2-\alpha}}{\delta(x)} \int_{B_{\hat{r}}(x)} |x - y|^\alpha k(y - x)\mathrm{d}y \lesssim \delta(x)^{1-\alpha} \int_{\mathbb{R}^d} (1 \wedge |z|^\alpha)k(z)\mathrm{d}z \lesssim \sigma^{\kappa - 1}.
\end{aligned}$$

Bounds on I_2 can be computed as follows: for $\alpha \leq 1$, we write

$$\begin{aligned}
I_2 &= \int_{B_1(x)\setminus B_{\hat{r}}(x)} |\delta(x) - \delta(y)||v(x) - v(y)|k(y - x)\mathrm{d}y \\
&\lesssim \int_{B_1(x)\setminus B_{\hat{r}}(x)} |x - y|^\alpha k(y - x)\mathrm{d}y
\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^d} (1 \wedge |z|^\alpha) k(z) \mathrm{d}z.$$

In the second line of the above inequality we used

$$|\delta(x) - \delta(y)| \lesssim |x - y| \ \text{ and } \ ||v||_{L^\infty(\mathbb{R}^d)} < \infty.$$

For $\alpha \in (1, 2)$ we can compute I_2 as

$$\begin{aligned}
\int_{B_1(x)\setminus B_{\hat{r}}(x)} & |\delta(x) - \delta(y)||v(x) - v(y)|k(y - x)\mathrm{d}y \\
& \lesssim \int_{B_1(x)\setminus B_{\hat{r}}(x)} |x - y|^{1-\alpha} \cdot |x - y|^\alpha k(y - x)\mathrm{d}y \\
& \lesssim \delta(x)^{1-\alpha} \int_{\mathbb{R}^d} (1 \wedge |z|^\alpha) k(z) \mathrm{d}z \lesssim \sigma^{\kappa-1}.
\end{aligned}$$

Moreover, since δ and v are bounded in $\mathbb{R}^d$, we get $I_3 \leq \kappa_3$. Combining the above estimates we obtain

$$\|\xi_i\|_{L^\infty B_r(x_0)} \lesssim \sigma^{\kappa-2} \ \text{ for } \ i = 2, 4.$$

Thus the claim (4.5.7) is established.

Let us now define $\zeta(z) = w(\frac{r}{2}z + x_0)$. Letting $b(z) = \frac{D\delta(\frac{r}{2}z+x_0)}{2\delta(\frac{r}{2}z+x_0)}$ it follows from (4.5.6) that

$$\begin{aligned}
\tilde{\mathcal{L}}^r\zeta + K_0 d^2 r b(z) \cdot |D\zeta| &\geq -\frac{r^2}{4}\left[\frac{1}{\delta}K + l_1\right]\left(\frac{r}{2}z + x_0\right) \\
\tilde{\mathcal{L}}^r\zeta - K_0 d^2 r b(z) \cdot |D\zeta| &\leq \frac{r^2}{4}\left[\frac{1}{\delta}K + l_2\right]\left(\frac{r}{2}z + x_0\right)
\end{aligned} \tag{4.5.9}$$

in $B_2(0)$, where

$$\tilde{\mathcal{L}}^r[x, u] := \sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\left\{\mathrm{Tr}\left(a_{\theta\nu}\left(\frac{r}{2}x + x_0\right)D^2u(x)\right) + \tilde{\mathfrak{I}}^r_{\theta\nu}[x, u]\right\}$$

and $\tilde{\mathfrak{I}}^r_{\theta\nu}$ is given by

$$\tilde{\mathfrak{I}}^r_{\theta\nu}[x, f] = \int_{\mathbb{R}^d}\left(f(x + y) - f(x) - \mathbb{1}_{B_{\frac{1}{r}}(y)}\nabla f(x) \cdot y\right)\left(\frac{r}{2}\right)^{d+2} N_{\theta\nu}\left(\frac{r}{2}x + x_0, ry\right)\mathrm{d}y.$$

Consider a cut-off function φ satisfying $\varphi = 1$ in $B_{3/2}$ and $\varphi = 0$ in B_2^c. Defining $\tilde{\zeta} = \zeta\varphi$ we get from (4.5.9) that

$$\tilde{\mathcal{L}}^r[z,\tilde{\zeta}] + K_0 d^2 r b(z)|D\tilde{\zeta}(z)| \geq -\frac{r^2}{4}\left[\frac{K}{\delta} + |l_1|\right]\left(\frac{r}{2}z + x_0\right) - \left|\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\tilde{\mathfrak{I}}^r_{\theta\nu}[z,(\varphi-1)\zeta]\right|$$

$$\tilde{\mathcal{L}}^r[z,\tilde{\zeta}] - K_0 d^2 r b(z)|D\tilde{\zeta}(z)| \leq \frac{r^2}{4}\left[\frac{K}{\delta} + |l_1|\right]\left(\frac{r}{2}z + x_0\right) - \left|\sup_{\theta\in\Theta}\inf_{\nu\in\Gamma}\tilde{\mathfrak{I}}^r_{\theta\nu}[z,(\varphi-1)\zeta]\right|$$

in B_1. Since

$$\|rb\|_{L^\infty(B_1(0))} \leq \kappa_3 \quad \text{for all } \sigma \in (0,1),$$

applying Theorem 4.1.1 we obtain, for some $\eta \in (0,1)$,

$$\|D\zeta\|_{C^\eta(B_{1/2}(0))} \leq \kappa_6\left(\|\tilde{\zeta}\|_{L^\infty(\mathbb{R}^d)} + \kappa_4\sigma + \kappa_5\sigma^{\kappa}\right), \tag{4.5.10}$$

for some constant κ_6 independent of $\sigma \in (0,1)$, where we used

$$\left|\tilde{\mathfrak{I}}^r_{\theta\nu}[z,(\varphi-1)\zeta]\right| \lesssim \sigma \quad \text{(cf. the proof of Theorem 4.2.1)} \quad \text{and} \quad |l_1|(\frac{r}{2}\cdot + x_0) \lesssim \sigma^{\kappa-2}.$$

Since v is in $C^{\kappa}(\mathbb{R}^d)$, it follows that

$$\|\tilde{\zeta}\|_{L^\infty(\mathbb{R}^d)} = \|\tilde{\zeta}\|_{L^\infty(B_2)} \leq \|\zeta\|_{L^\infty(B_2)} \lesssim r^{\kappa}.$$

Putting these estimates in (4.5.10) and calculating the gradient at $z = 0$ we obtain

$$|Dv(x_0)| \lesssim \sigma^{\kappa-1},$$

for all $\sigma \in (0,1)$. This proves the Hölder estimate (4.5.4).

For the second part, compute the Hölder ratio with $D\zeta(0) - D\zeta(z)$ where $z = \frac{2}{r}(y - x_0)$ for $|x_0 - y| \leq \sigma/8$. This completes the proof.

□

Now we can prove the Hölder regularity of Du up to the boundary. If u is solution of the inequations (4.0.2) then using Theorem 4.2.1 we have $|\mathcal{L}u| \leq CK$. Now the proof can be obtained by following the same lines as in Theorem 3.3.1. We present it here for the sake of completeness.

Theorem 4.5.1. *Suppose that Assumption 4.0.1 holds and Ω be a bounded C^2 domain. Then for any viscosity solution u to the inequations (4.0.2) we have*

$$||Du||_{C^\eta(\overline{\Omega})} \leq CK,$$

for some $\eta \in (0,1)$ and C, depending only on $d, \Omega, C_0, \Lambda, \lambda, \alpha$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y$. In particular, we have

$$||u||_{C^{1,\eta}(\overline{\Omega})} \leq CK$$

Proof. Since $u = v\delta$ it follows that

$$Du = vD\delta + \delta Dv.$$

Since $\delta \in C^2(\bar{\Omega})$, it follows from Theorem 4.4.1 that $vD\delta \in C^\kappa(\bar{\Omega})$. Thus, we only need to concentrate on $\vartheta = \delta Dv$. Consider η from Lemma 4.5.2 and with no loss of generality, we may fix $\eta \in (0, \kappa)$.

For $|x-y| \geq \frac{1}{8}(\delta(x) \vee \delta(y))$ it follows from (4.5.4) that

$$\frac{|\vartheta(x) - \vartheta(y)|}{|x-y|^\eta} \leq CK(\delta^\kappa(x) + \delta^\kappa(y))(\delta(x) \vee \delta(y))^{-\eta} \leq 2CK.$$

So consider the case $|x-y| < \frac{1}{8}(\delta(x) \vee \delta(y))$. Without loss of generality, we may assume that $|x-y| < \frac{1}{8}\delta(x)$. Then

$$\frac{9}{8}\delta(x) \geq |x-y| + \delta(x) \geq \delta(y) \geq \delta(x) - |x-y| \geq \frac{7}{8}\delta(x).$$

By Lemma 4.5.2, it follows

$$\begin{aligned} \frac{|\vartheta(x) - \vartheta(y)|}{|x-y|^\eta} &\leq |Dv(x)| \frac{|\delta(x) - \delta(y)|}{|x-y|^\eta} + \delta(y) \frac{|Dv(x) - Dv(y)|}{|x-y|^\eta} \\ &\lesssim \delta(x)^{\kappa-1}(\delta(x))^{1-\eta} + \delta(y)[\delta(x)]^{\kappa-1-\eta} \\ &\leq CK. \end{aligned}$$

This completes the proof. □

5

Interior regularities of perturbed stable-like operators

The goal of this chapter is to discuss the interior regularity for a certain class of integro-differential operators. More precisely, we are concerned with the nonlinear integro-differential elliptic operators of the form

$$Iu(x) = \inf_{\theta}\sup_{\gamma} \mathcal{I}_{\theta\gamma}u(x) = \inf_{\theta}\sup_{\gamma} \int_{\mathbb{R}^d} \Big(u(x+y)+u(x-y)-2u(x)\Big)\frac{k_{\theta\gamma}(y)}{|y|^d}\,\mathrm{d}y\,, \quad (5.0.1)$$

where $k_{\theta\gamma}$ is symmetric and satisfies

$$(2-\alpha)\lambda\frac{1}{|y|^{\alpha}} \ \leq\ k_{\theta\gamma}(y) \leq \Lambda\left(\frac{2-\alpha}{|y|^{\alpha}} + \varphi(1/|y|)\right), \quad 0<\lambda\leq\Lambda\,, \qquad \text{(A1)}$$

for some function $\varphi : (0,\infty) \to (0,\infty)$ satisfying a weak upper scaling property with exponent $\beta < \alpha < 2$. Which means that φ is a locally bounded function satisfying

$$\varphi(st) \leq \kappa_{\circ} s^{\beta}\varphi(t) \quad \text{for} \quad s\geq 1, t>0\,, \qquad \text{(A2)}$$

for some $\kappa_{\circ} > 0$. We also assume that

$$\int_0^1 \frac{\varphi(y)}{y}\mathrm{d}y \ <\ \infty\,. \qquad \text{(A3)}$$

Note that (A2) and (A3) give us

$$\int_{\mathbb{R}^d} (|y|^2 \wedge 1) \frac{\varphi(1/|y|)}{|y|^d} \, \mathrm{d}y \, < \, \infty.$$

The ellipticity class is defined with respect to the set of nonlocal operators $\mathscr{L}$ containing operator $\mathcal{I}$ of the form

$$\mathcal{I}u(x) = \int_{\mathbb{R}^d} \Big(u(x+y) + u(x-y) - 2u(x) \Big) \frac{k(y)}{|y|^d} \, \mathrm{d}y \,, \tag{5.0.2}$$

where it holds that $k(y) = k(-y)$ and (A1) holds for some fixed $\lambda, \Lambda, \alpha, \beta$ such that $0 < \lambda \leq \Lambda$ and $\alpha \in (\beta, 2)$. In the perspective of (1.2.3) what we essentially have is that class $\mathscr{L} = \mathscr{L}_{(\mathfrak{A},\mathfrak{B})}$, where $\mathfrak{A} = \{0\}$ and $\mathfrak{B}$ is collection of all symmetric function $k(y)/|y|^d$ where satisfies inequality (A1) with assumptions (A2) and (A3). Here we can write nonlocal operator $\mathcal{I}$ using symmetric difference of u is due to the fact that k is symmetric.

We will first obtain a nonlocal version of Aleksandrov-Bakelman-Pucci(ABP) estimates which will be useful for Harnack inequality, and Hólder estimates for viscosity solutions of I. We will end our discussion on fully nonlinear nonlocal operators I with boundary Harnak property. Since the early 2000s, the Harnack inequalities and Hólder estimates for nonlocal operators have been studied extensively. First such result for nonlocal operators has been proved via probabilistic approaches in [11–13]. In an analytic setup, a series of influential works [50–52] Caffarelli and Silvestre develop a regularity theory of nonlinear stable-like integro-differential operator with symmetric kernels. Whereas the boundary Harnack property is considered by Serra and Ros-Oton in [147]. Kriventsov in [120] studies interior $C^{1,\gamma}$ regularity for rough symmetric kernel and his result is further improved by Serra in [151] who establishes interior $C^{\alpha+\gamma}$ estimate with rough symmetric kernels.

There is also an extensive amount of work extending the results of Caffarelli and Silvestre [50–52]. In [111] the authors generalized these results to fully nonlinear integro-differential operators with regularly varying kernels. Regularity results for nonsymmetric stable-like kernels are studied in [57, 108]. Recently, [109] generalized these results for kernels with variable orders. These kernels are closely related to an important family of Lévy processes known as subordinate Brownian motions.

Subordinate Brownian motions(sBM) are obtained by time-changing the Brownian motion by an independent subordinator (i.e., nondecreasing, non-negative Lévy process). In particular, when the subordinator is α-stable we obtain a α-stable process as sBM whose generator is given by the fractional Laplacian. Let us also mention [6, 106] which also studies regularity results for similar models. We studied regularity theory of the operator of the form (5.0.1) in [35] and will be discussed in this chapter. Our work should be seen as a nonlocal counterpart of [131] where the Hölder regularity follows due to the non-degeneracy of α-stable kernel.

5.1 ABP estimates and weak Harnack inequality

In this section we obtain an Aleksandrov-Bakelman-Pucci (ABP) estimate which is the main ingredient for weak-Harnack inequality and point estimate. Let us begin by defining the extremal Pucci operators with respect to $\mathscr{L}$ which are defined to be $\mathcal{M}^+u = \sup_{L\in\mathscr{L}} Lu$ and $\mathcal{M}^-u = \inf_{L\in\mathscr{L}} Lu$. Defining $\delta(u,x,y) = u(x+y) + u(x-y) - 2u(x)$, we find from (A1) that

$$\mathcal{M}^+u(x) = \int_{\mathbb{R}^d} \frac{\Lambda\delta^+(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|)\right) - \frac{\lambda\delta^-(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}\right) \mathrm{d}y,$$
$$\mathcal{M}^-u(x) = \int_{\mathbb{R}^d} \frac{\lambda\delta^+(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}\right) - \frac{\Lambda\delta^-(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|)\right) \mathrm{d}y\,.$$

The extremal Pucci operators are of great importance in control theory and were first considered by Pucci [141] to study the principal eigenvalue problem for local nonlinear elliptic operators. Recall that a nonlinear operator I defined as in (5.0.1) is elliptic with respect to the class $\mathscr{L}$ and it holds that

$$\mathcal{M}^-(u-v) \;\le\; Iu - Iv \;\le\; \mathcal{M}^+(u-v)\,.$$

It should be observed that if a operator is elliptic with respect to a subset of $\mathscr{L}$ it is also elliptic with respect to $\mathscr{L}$. For instance, if we let $\varphi(r) = r^\beta, \beta \in (0,\alpha)$, and

$\mathscr{L}_1 \subset \mathscr{L}$ be the collection of all kernel functions k satisfying

$$(2-\alpha)\lambda\frac{1}{|y|^{\alpha}} \;\leq\; k(y) \leq \Lambda\left(\frac{2-\alpha}{|y|^{\alpha}} + \mathbb{1}_{B_1^c}(y)\frac{1}{|y|^{\beta}}\right),$$

then the results will hold for any nonlinear operator of the form (5.0.1) with respect to $\mathscr{L}_1$. We remark that this class of operators are not covered by [109, 111]. We also remark that the boundedness assumption of u assures integrability of u at infinity with respect to the jump kernel. This can be removed by assuming suitable integrability criterion.

We also need scaled extremal operators which we introduce now. Define $\varphi_i(|y|) = \frac{\kappa_\circ}{(2^i)^{(\alpha-\beta)}}\varphi(|y|)$ for $i \geq 0$. The scaled extremal Pucci operators are defined to be

$$\mathcal{M}_i^+ u(x) = \int_{\mathbb{R}^d} \frac{\Lambda\delta^+(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^{\alpha}} + \varphi_i(1/|y|)\right) - \frac{\lambda\delta^-(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^{\alpha}}\right)\mathrm{d}y\,, \tag{5.1.1}$$

$$\mathcal{M}_i^- u(x) = \int_{\mathbb{R}^d} \frac{\lambda\delta^+(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^{\alpha}}\right) - \frac{\Lambda\delta^-(u,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^{\alpha}} + \varphi_i(1/|y|)\right)\mathrm{d}y\,. \tag{5.1.2}$$

We used these scaled extremal operators in order to keep a track of viscosity solutions on every scale to find scale invariant uniform estimates because equation (5.0.1) is not scale invariant.

Let us begin by defining the concave envelope and contact set. Let u be a function that is non-positive outside B_1 (unit ball around 0). The concave envelope Γ of u in B_3 is defined as follows

$$\Gamma(x) = \begin{cases} \inf\{p(x) \;:\; p \text{ is a plane satisfying } p \geq u^+ \text{ in } B_3\} & \text{in } B_3, \\ 0 & \text{in } B_3^c. \end{cases}$$

The contact set is defined to be $\Sigma = \{\Gamma = u\} \cap B_1$. Let us first prove that there is at least one good ring near a contact point where u stays quadratically close to the tangent plane of Γ at the contact point. The following lemma follows by adapting [50, Lemma 8.1] in our setting.

Lemma 5.1.1. *Let $u \leq 0$ in $\mathbb{R}^d \setminus B_1$ and Γ be its concave envelope in B_3. Assume*

$\mathcal{M}_i^+ u(x) \geq -f(x)$ in B_1 for some $i \geq 0$. Let $\rho_0 = 1/16\sqrt{d}$, $r_k = \rho_0 2^{-1/2-\alpha} 2^{-k}$, and $R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$. Then there exists a constant C_0 independent of $i \geq 0$ and α such that for any $x \in \Sigma$ and any $M > 0$ there is a k satisfying

$$|R_k(x) \cap \{u(y) < u(x) + (y-x) \cdot \nabla\Gamma(x) - Mr_k^2\}| \leq C_0 \frac{f(x)}{M} |R_k(x)|.$$

Furthermore, C_0 depends only on (λ, d, ρ_0).

Proof. First we notice that $\mathcal{M}_0^+ u \geq M_i^+ u$ for all i. Therefore $\mathcal{M}_i^+ u(x) \geq -f(x)$ implies that $\mathcal{M}_0^+ u(x) \geq -f(x)$. Hence it is enough to prove lemma for the case $i = 0$.

Let $x \in \Sigma$ and recall that $\delta(u, x, y) = u(x+y) + u(x-y) - 2u(x)$. If both $x + y \in B_3$ and $x - y \in B_3$ then $\delta(u, x, y) \leq 0$ since $u(x) = \Gamma(x) = p(x)$ for some plane that remains above u in B_3. If either $x + y$ or $x - y \in B_3^c$ then both $x + y$ and $x - y \in B_1^c$ and since $u(x) = \Gamma(x) \geq 0$ we have $\delta(u, x, y) \leq 0$. Thus, using (5.1.1), we find

$$\begin{aligned} -f(x) \leq \mathcal{M}_0^+ u(x) &= (2-\alpha) \int_{\mathbb{R}^d} \frac{-\lambda \delta^-(u,x,y)}{|y|^d} \left(\frac{1}{|y|^\alpha}\right) \mathrm{d}y \\ &\leq (2-\alpha) \int_{B_{r_0}} \frac{-\lambda \delta^-(u,x,y)}{|y|^d} \left(\frac{1}{|y|^\alpha}\right) \mathrm{d}y \,, \end{aligned} \tag{5.1.3}$$

where $r_0 = \rho_0 2^{-\frac{1}{2-\alpha}}$. Let

$$E_k^{\pm} \coloneqq \{R_k \cap \{u(x \pm y) < u(x) \pm y \cdot \nabla\Gamma(x) - Mr_k^2\}\}.$$

Then on this set we will have $\delta^-(u, x, y) \geq 2Mr_k^2$. Also $|E_k^+| = |E_k(x)|$ where $E_k(x) \coloneqq \{R_k(x) \cap \{u(y) < u(x) + (y-x) \cdot \nabla\Gamma(x) - Mr_k^2\}\}$. Now suppose that the result does not hold for any C_0. We will arrive at contradiction for large enough C_0. Using (5.1.3) we obtain that

$$\begin{aligned} f(x) &\geq (2-\alpha)\lambda \sum_{k=0}^{\infty} \int_{R_k} \frac{\delta^-(u,x,y)}{|y|^{d+\alpha}} \mathrm{d}y \\ &\geq (2-\alpha)\lambda \sum_{k=0}^{\infty} \int_{E_k} \frac{2Mr_k^2}{|y|^{d+\alpha}} \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&\geq 2(2-\alpha)\lambda \sum_{k=0}^{\infty} M \frac{r_k^2}{r_k^{d+\alpha}} |E_k| \\
&\geq 2(2-\alpha)\lambda \sum_{k=0}^{\infty} M \frac{r_k^2}{r_k^{d+\alpha}} \frac{C_0 f(x)}{M} |R_k| \\
&\geq 2(2-\alpha)\lambda \left[\sum_{k=0}^{\infty} \frac{r_k^2}{r_k^{d+\alpha}} \omega_d (r_k^d - r_{k+1}^d) \right] C_0 f(x) \\
&= 2(2-\alpha)\lambda \omega_d \left[\sum_{k=0}^{\infty} r_k^{2-\alpha} \left(1 - \left(\frac{1}{2}\right)^d \right) \right] C_0 f(x),
\end{aligned}
$$

since $\frac{r_{k-1}}{r_k} = \frac{1}{2}$ for any k, where ω_d denotes volume of the unit ball. Now we notice that $\sum_{k=0}^{\infty} r_k^{2-\alpha}$ is a geometric series, and therefore,

$$
f(x) \geq 2(2-\alpha)\lambda \omega_d \left(1 - \left(\frac{1}{2}\right)^d \right) \left[\frac{\rho_0^{2-\alpha}}{2} \left(\frac{1}{1 - 2^{-(2-\alpha)}} \right) \right] C_0 f(x) \, .
$$

Take

$$
c = \lambda \omega_d \left(1 - \left(\frac{1}{2}\right)^d \right) \left(\frac{\rho_0^2 (2-\alpha)}{1 - 2^{-(2-\alpha)}} \right),
$$

and since $\frac{(2-\alpha)}{1-2^{-(2-\alpha)}}$ remains bounded below for all $\alpha \in (0,2)$, we have c positive for any $\alpha \in (0,2)$. Thus

$$
f(x) \geq c \, C_0 f(x).
$$

Choosing $C_0 > c^{-1}$ leads to a contradiction. Hence the proof. □

Using Lemma 5.1.1 and the arguments in [50, Section 8] we arrive at the following result which is a discrete version of ABP estimates. This is a mild extension to [50, Theorem 8.7].

Theorem 5.1.1. *Let u and Γ be same as in Lemma 5.1.1. Then there is a finite family of open cubes $\mathcal{Q}_j$ with diameters d_j such that following hold.*

(i) Any two cubes $\mathcal{Q}_i$ and $\mathcal{Q}_j$ in the family do not intersect.

(ii) $\{u = \Gamma\} \subset \bigcup_{j=1}^{m} \mathcal{Q}_j$.

(iii) $\{u = \Gamma\} \cap \overline{\mathcal{Q}}_j \neq \emptyset$ for any $\mathcal{Q}_j$.

(iv) $d_j \leq \rho_0 2^{-1/2-\alpha}$, where $\rho_0 = 1/16\sqrt{d}$.

(v) $|\nabla\Gamma(\overline{\mathcal{Q}}_j)| \leq C(\max_{\overline{\mathcal{Q}}_j} f(x))^d|\overline{\mathcal{Q}}_j|$.

(vi) $|\{y \in 8\sqrt{d}\mathcal{Q}_j : u(y) > \Gamma(y) - C(\max_{\overline{\mathcal{Q}}_j} f(x))d_j^2\}| \geq \mu|\mathcal{Q}_j|$.

The constant $C > 0$ and $\mu > 0$ depends only on (λ, d, ρ_0) but not on i and α.

Next we consider a *special function* which will play a key role in our analysis on point estimate and weak-Harnack inequality. Let $p > 0$ and δ be small positive number. Define

$$f(x) \coloneqq \min\{\delta^{-p}, \max\{|x|^{-p}, (2\sqrt{n})^{-p}\}\}\,.$$

We claim that, for a given $r \in (0, 1)$, we can choose p and δ so that

$$\mathcal{M}_0^- f(x) \geq 0 \quad \text{for } r < |x| \leq 2\sqrt{n}\,. \tag{5.1.4}$$

For any $0 < r < 1$, define

$$\hat{f}(x) = \min\left\{\left(\frac{\delta}{r}\right)^{-p}, \max\left\{|x|^{-p}, \left(\frac{2\sqrt{n}}{r}\right)^{-p}\right\}\right\}\,.$$

Then clearly, $f(rx) = r^{-p}\hat{f}(x)$ and for any $|x| \geq r$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} \delta(f, x, y)\frac{k(y)}{|y|^d}\mathrm{d}y &= \int_{\mathbb{R}^d} \left(f(x+ry) + f(x-ry) - 2f(x)\right)\frac{k(ry)}{|y|^d}\mathrm{d}y \\ &= r^{-p}\int_{\mathbb{R}^d} \delta(\hat{f}, {}^{x}\!/_{r}, y)\frac{k(ry)}{|y|^d}\mathrm{d}y\,.\end{aligned}$$

Therefore, to establish (5.1.4) it is enough tho show that for all $1 \leq |x| \leq \frac{2\sqrt{n}}{r}$,

$$r^{-p}\inf_k \int_{\mathbb{R}^d} \delta(\hat{f}, x, y)\frac{k(ry)}{|y|^d}\mathrm{d}y \geq 0\,, \tag{5.1.5}$$

where infimum is taken over all kernel k satisfying (A3). Note that $\hat{f}$ is radially non-increasing function. Fix $|x| \geq 1$ and define $\tilde{f}(y) = |x|^p\hat{f}(|x|y)$. Then it implies that $\tilde{f}(y) \geq \hat{f}(y)$, for all $y \in \mathbb{R}^d$ and $\tilde{f}(x/|x|) = \hat{f}(x)$. Thus we obtain

$$\delta(\hat{f}, x, y) = \frac{1}{|x|^p}\left[\tilde{f}(\frac{x+y}{|x|}) + \tilde{f}(\frac{x-y}{|x|}) - 2\tilde{f}(\frac{x}{|x|})\right]$$

$$\geq \frac{1}{|x|^p}\left[\hat{f}(\frac{x+y}{|x|})+\hat{f}(\frac{x-y}{|x|})-2\hat{f}(\frac{x}{|x|})\right].$$

Without any loss of generality we may assume that $x/|x| = e_1 = (1,\dots,0)$. Then

$$\begin{aligned}\int_{\mathbb{R}^d}\delta(\hat{f},x,y)\frac{k(ry)}{|y|^d}\mathrm{d}y &\geq \frac{1}{|x|^p}\int_{\mathbb{R}^d}\left[\hat{f}(\frac{x+y}{|x|})+\hat{f}(\frac{x-y}{|x|})-2\hat{f}(\frac{x}{|x|})\right]\frac{k(ry)}{|y|^d}\mathrm{d}y\\ &= \frac{1}{|x|^p}\int_{\mathbb{R}^d}\left[\hat{f}(\frac{x}{|x|}+y)+\hat{f}(\frac{x}{|x|}-y)-2\hat{f}(\frac{x}{|x|})\right]\frac{k(r|x|y)}{|y|^d}\mathrm{d}y\\ &= \frac{1}{|x|^p}\int_{\mathbb{R}^d}\left[\hat{f}(e_1+y)+\hat{f}(e_1-y)-2\hat{f}(e_1)\right]\frac{k(r|x|y)}{|y|^d}\mathrm{d}y\\ &\geq \frac{1}{|x|^p}\int_{\mathbb{R}^d}\delta(\hat{f},e_1,y)\frac{k(r|x|y)}{|y|^d}\mathrm{d}y\,.\end{aligned}$$

Hence, by (5.1.2), we get

$$\begin{aligned}\inf_k\int_{\mathbb{R}^d}\delta(\hat{f},x,y)\frac{k(ry)}{|y|^d}\mathrm{d}y \geq \;& \frac{1}{|x|^p}\int_{\mathbb{R}^d}\frac{\lambda\delta^+(\hat{f},e_1,y)}{|y|^d}\left(\frac{2-\alpha}{(r|x|)^\alpha|y|^\alpha}\right)\\ &-\frac{\Lambda\delta^-(\hat{f},e_1,y)}{|y|^d}\left(\frac{2-\alpha}{(r|x|)^\alpha|y|^\alpha}+\varphi_0(1/r|x||y|)\right)\mathrm{d}y\\ :=\;& I_1 - I_2\,.\end{aligned}$$

We now recall the following elementary relations that hold for any $a > b > 0$ and $q > 0$:

$$(a+b)^{-q} \geq a^{-q}\left(1-q\frac{b}{a}\right),$$

$$(a+b)^{-q}+(a-b)^{-q} \geq 2a^{-q}+q(q+1)b^2a^{-q-2}\,.$$

Fixing $\delta < \frac{r}{2}$, we then see that for $|y| < 1/2$,

$$\begin{aligned}\delta(\hat{f},e_1,y) &= |e_1+y|^{-p}+|e_1-y|^{-p}-2\\ &= (1+|y|^2+2y_1)^{-p/2}+(1+|y|^2-2y_1)^{-p/2}-2\\ &\geq 2(1+|y|^2)^{-p/2}+p(p+2)y_1^2(1+|y|^2)^{-p/2-2}-2\\ &\geq p\left(-|y|^2+(p+2)y_1^2-\frac{1}{2}(p+2)(p+4)y_1^2|y|^2\right)\,.\end{aligned} \tag{5.1.6}$$

Let us first calculate I_2. For any $|y| < \frac{1}{2}$ we have from (5.1.6) that

$$\delta^-(\hat{f}, e_1, y) \leq p\left(1 + \frac{1}{2}(p+2)(p+4)\right)|y|^2 .$$

Denote by $C_p = p\left(1 + \frac{1}{2}(p+2)(p+4)\right)$. Then

$$\begin{aligned}
I_2 &\leq C_p \Lambda \int_{|y|<\frac{1}{2}} \frac{|y|^2}{|y|^d}\left[\frac{2-\alpha}{(r|x|)^\alpha |y|^\alpha} + \varphi_0(1/r|x||y|)\right] \mathrm{d}y \\
&\quad + \Lambda \int_{|y|\geq \frac{1}{2}} \frac{2\hat{f}(e_1)}{|y|^d}\left[\frac{2-\alpha}{(r|x|)^\alpha |y|^\alpha} + \varphi_0(1/r|x||y|)\right] \mathrm{d}y \\
&= I_{21} + I_{22} .
\end{aligned}$$

We observe that

$$\begin{aligned}
\frac{2-\alpha}{(r|x|)^\alpha}\int_{|y|<\frac{1}{2}} \frac{|y|^2}{|y|^d}\frac{1}{|y|^\alpha}\mathrm{d}y &= \frac{\omega_d}{(r|x|)^\alpha}\left(\frac{1}{2}\right)^{2-\alpha}, \\
\frac{2-\alpha}{(r|x|)^\alpha}\int_{|y|\geq\frac{1}{2}} \frac{2}{|y|^d}\frac{1}{|y|^\alpha}\mathrm{d}y &= \frac{\omega_d}{(r|x|)^\alpha}\frac{2-\alpha}{\alpha}2^{\alpha+1} .
\end{aligned}$$

On the other hand, for $r|x| \leq 2\sqrt{n}$,

$$\begin{aligned}
\int_{|y|<\frac{1}{2}} \frac{|y|^2}{|y|^d}\varphi_0(1/r|x||y|)\mathrm{d}y &\leq \kappa_\circ\left(\frac{2\sqrt{n}}{r|x|}\right)^\beta \int_{|y|<\frac{1}{2}} \frac{|y|^2}{|y|^d}\frac{1}{|y|^\beta}\varphi(\frac{1}{2\sqrt{n}})\,\mathrm{d}y \\
&= \kappa_\circ\left(\frac{2\sqrt{n}}{r|x|}\right)^\beta \varphi(\frac{1}{2\sqrt{n}})\frac{\omega_d}{2-\beta}\left(\frac{1}{2}\right)^{2-\beta},
\end{aligned}$$

using the fact $|y| < \frac{1}{2}$ and (A1). Again using (A1)-(A2)

$$\begin{aligned}
\int_{|y|\geq\frac{1}{2}} \frac{2}{|y|^d}\varphi_0(\frac{1}{r|x||y|})\mathrm{d}y &= 2\omega_d \int_{1/2}^\infty \frac{1}{t}\varphi(\frac{1}{r|x|t})\mathrm{d}t \\
&\leq 2\kappa_\circ\omega_d\left(\frac{2\sqrt{n}}{r|x|}\right)^\beta \int_{1/2}^\infty \frac{1}{t}\varphi(\frac{1}{2\sqrt{n}t})\mathrm{d}t \\
&= 2\kappa_\circ\omega_d\left(\frac{2\sqrt{n}}{r|x|}\right)^\beta \int_{2\sqrt{n}/2}^\infty \frac{1}{t}\varphi(\frac{1}{t})\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= 2\kappa_\circ \omega_d \left(\frac{2\sqrt{n}}{r|x|}\right)^{\beta} \int_0^{1/\sqrt{n}} \frac{1}{t}\varphi(t)\mathrm{d}t \\
&\leq 2\kappa_\circ \omega_d \left(\frac{2\sqrt{n}}{r|x|}\right)^{\beta} \kappa_1,
\end{aligned}
$$

for some constant κ_1 depending only on φ. Thus combining we obtain for $1 \leq |x| \leq \frac{2\sqrt{n}}{r}$

$$
\begin{aligned}
I_2 \leq\ & C_p \Lambda \left[\frac{\omega_d}{r^\alpha}\left(\frac{1}{2}\right)^{2-\alpha}\right] + \Lambda\left[\frac{\omega_d}{r^\alpha}\frac{2-\alpha}{\alpha}2^{\alpha+1}\right] \\
&+ C_p \kappa_\circ \Lambda \left[\left(\frac{2\sqrt{n}}{r}\right)^{\beta} \varphi(\frac{1}{2\sqrt{n}})\frac{\omega_d}{2-\beta}\left(\frac{1}{2}\right)^{2-\beta}\right] + \kappa_\circ \Lambda \left[2\omega_d \left(\frac{2\sqrt{n}}{r}\right)^{\beta}\kappa_1\right].
\end{aligned} \tag{5.1.7}
$$

Next we calculate I_1. Notice that if $\delta < \frac{r}{16}$, then $\delta(\hat{f}, e_1, y) \geq (\delta/r)^{-p}$ for all $y \in B_{\delta/4r}(e_1)$. Hence

$$
I_1 = \int_{\mathbb{R}^d} \frac{\lambda\delta^+(\hat{f}, e_1, y)}{|y|^d} \frac{2-\alpha}{(r|x|)^\alpha |y|^\alpha}\,\mathrm{d}y \geq \frac{\lambda(2-\alpha)}{(r|x|)^\alpha}\kappa_2 (\delta/r)^{d-p} \geq \kappa_2 \frac{\lambda(2-\alpha)}{(2\sqrt{n})^\alpha}(\delta/r)^{d-p},
$$

for some constant κ_2. Thus choosing $p > d$ and δ small enough (5.1.5) follows from (5.1.7). This leads to the following

Lemma 5.1.2. *Given any $r, n > 0$ there are positive p and δ such that the function*

$$
f(x) = \min\{\delta^{-p}, \max\{|x|^{-p}, (2\sqrt{n})^{-p}\}\},
$$

is a solution to

$$
\mathcal{M}_0^- f(x) \geq 0,
$$

for every $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 2$ and $|x| > r$.

Proof. For any $|x| \geq 2\sqrt{n}$, by the definition of f, we get that $\delta(f, x, y) = f(x+y) + f(x-y) - 2f(x) \geq 0$ for all $y \in \mathbb{R}^d$. Therefore, $\mathcal{M}_0^- f(x) \geq 0$ for all $|x| \geq 2\sqrt{n}$. Hence the proof follows from (5.1.4). □

Applying Lemma 5.1.2 we obtain the following corollary. The proof would be same as [50, Corollary 9.3] and thus omitted.

Corollary 5.1.1. *Given any $\alpha \in [\alpha_0, \alpha_1]$ and $i \geq 0$ there is a function Φ such that*

(i) Φ *is continuous in* $\mathbb{R}^d$,

(ii) $\Phi(x) = 0$ *for x outside* $B_{2\sqrt{d}}$,

(iii) $\Phi > 2$ *for* $x \in \mathcal{Q}_3$, *and*

(iv) $\mathcal{M}_i^- \Phi > -\psi(x)$ *in* $\mathbb{R}^d$ *for some positive function ψ supported in* $\bar{B}_{1/4}$.

Using Theorem 5.1.1 and Corollary 5.1.1 and repeating the arguments of [50, Lemma 10.1] we arrive at the following result.

Lemma 5.1.3. *There exist constants $\varepsilon_0 > 0$, $0 < \mu < 1$, and $M > 1$ (depending only on $d, \lambda, \Lambda, \alpha, \varphi$), such that if*

(i) $u \geq 0$ *in* $\mathbb{R}^d$,

(ii) $\inf_{\mathcal{Q}_3} u \leq 1$, *and*

(iii) $\mathcal{M}_i^- u \leq \varepsilon_0$ *in* $\mathcal{Q}_{4\sqrt{d}}$ *for some* $i \geq 0$,

then $|\{u \leq M\} \cap \mathcal{Q}_1| > \mu$.

The above lemma is a key tool in obtaining weak-Harnack estimate. Combining a Calderó-Zygmund type argument with Lemma 5.1.3 we obtain the following

Theorem 5.1.2. *There exist constant $\tilde{\varepsilon} > 0$, $0 < \tilde{\mu} < 1$, and $\tilde{M} > 1$ (depending only on $d, \lambda, \Lambda, \alpha, \varphi$), such that if*

(i) $u \geq 0$ *in* $\mathbb{R}^d$

(ii) $\inf_{\mathcal{Q}_3} u \leq 1$, *and*

(iii) $\mathcal{M}^- u \leq \tilde{\varepsilon}$ *in* $\mathcal{Q}_{4\sqrt{d}}$,

then

$$|\{u \geq \tilde{M}^k\} \cap \mathcal{Q}_1| \leq (1 - \tilde{\mu})^k$$

for $k \in \mathbb{N}$. As a consequence, we have that

$$|\{u \geq t\} \cap \mathcal{Q}_1| \leq \kappa\, t^{-\varepsilon} \quad \forall\, t > 0\,,$$

for some universal constant κ, ε.

Proof. The case $k = 1$ follows from Lemma 5.1.3. Note that $\kappa_\circ = 1$ implies $\mathcal{M}_0^- = \mathcal{M}^-$ and thus from the arguments of Lemma 5.1.3 we can obtain the constants $(\varepsilon_1, M_1, \mu_1)$ satisfying Lemma 5.1.3 with operator $\mathcal{M}^-$. We set $\tilde{\varepsilon} = \varepsilon_0 \wedge \varepsilon_1$, $\tilde{M} = M \vee M_1$ and $\tilde{\mu} = \mu \wedge \mu_1$. Now using induction hypothesis assume that theorem is true for $m = 1, \ldots, k-1$, and denote by

$$A = \{u > \tilde{M}^k\} \cap \mathcal{Q}_1, \qquad B = \{u > \tilde{M}^{k-1}\} \cap \mathcal{Q}_1 .$$

Thus we only need to show that

$$|A| \leq (1 - \tilde{\mu})|B| . \tag{5.1.8}$$

Clearly, $A \subset B \subset \mathcal{Q}_1$ and $|A| \leq |\{u > \tilde{M}\} \cap \mathcal{Q}_1| \leq 1 - \tilde{\mu}$. We show that if $\mathcal{Q} := \mathcal{Q}_{1/2^i}(x_0)$ is a dyadic cube such that

$$|A \cap \mathcal{Q}| > (1 - \tilde{\mu})|\mathcal{Q}|, \tag{5.1.9}$$

then $\widetilde{\mathcal{Q}} \subset B$. where $\widetilde{\mathcal{Q}}$ is a predecessor of $\mathcal{Q}$. Then (5.1.8) follows from [48, Lemma 4.2]. Suppose that $\widetilde{\mathcal{Q}} \not\subset B$. Take

$$\tilde{x} \in \widetilde{\mathcal{Q}} \quad \text{such that} \quad u(\tilde{x}) \leq \tilde{M}^{k-1} .$$

We consider the function

$$v(y) := \frac{u(x_0 + \frac{1}{2^i} y)}{\tilde{M}^{k-1}} .$$

Clearly, $v \geq 0$ in $\mathbb{R}^d$ and $\inf_{\mathcal{Q}_3} v \leq 1$. We claim that $\mathcal{M}_i^- v \leq \varepsilon_0$ in $\mathcal{Q}_{4\sqrt{d}}$ where ε_0 is given by Lemma 5.1.3. Let $\hat{x} = x_0 + \frac{1}{2^i} x$ for some $x \in \mathcal{Q}_{4\sqrt{n}}$. Then simple calculation shows that

$$\frac{1}{\tilde{M}^{k-1}} \delta(u, \hat{x}, \frac{y}{2^i}) = \delta(v, x, y) ,$$

and using (A1), we obtain

$$\begin{aligned} \frac{1}{(2^i)^\alpha \tilde{M}^{k-1}} \mathcal{M}^- u(\hat{x}) = \frac{1}{(2^i)^\alpha \tilde{M}^{k-1}} \int_{\mathbb{R}^d} \Big[&\frac{\lambda \delta^+(u, \hat{x}, \frac{y}{2^i})}{|y|^d} \left(\frac{(2-\alpha)(2^i)^\alpha}{|y|^\alpha} \right) \\ &- \frac{\Lambda \delta^-(u, \hat{x}, \frac{y}{2^i})}{|y|^d} \left(\frac{(2-\alpha)(2^i)^\alpha}{|y|^\alpha} + \varphi(2^i/|y|) \right) \Big] \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\tilde{M}^{k-1}} \int_{\mathbb{R}^d} \frac{\lambda\delta^+(u,\hat{x},\frac{y}{2^i})}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}\right) \\
&\qquad - \frac{\Lambda\delta^-(u,\hat{x},\frac{y}{2^i})}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha} + \frac{\kappa_\circ}{2^{i(\alpha-\beta)}}\varphi(\frac{1}{|y|})\right)\mathrm{d}y \\
&\geq \mathcal{M}_i^- v(x)\,.
\end{aligned}
$$

Thus we have the claim $\mathcal{M}_i^- v \leq \varepsilon_0$ in $\mathcal{Q}_{4\sqrt{d}}$. Therefore, we can apply Lemma 5.1.3 to obtain

$$\tilde{\mu} < |\{v(x) \leq M\} \cap \mathcal{Q}_1| = 2^{id}|\{u(x) \leq M^k\} \cap \mathcal{Q}|$$

implying

$$|\{u(x) \leq M^k\} \cap \mathcal{Q}| \;>\; \tilde{\mu}|\mathcal{Q}|.$$

This gives us (5.1.9). This completes the proof. □

Remark 5.1.1. Note that the constants $(\tilde{\varepsilon}, \tilde{M}, \tilde{\mu})$ in Theorem 5.1.2 also work if we replace $\mathcal{M}^-$ by $\mathcal{M}_i^-$ for all $i \geq 0$.

By a standard covering argument we obtain following result

Theorem 5.1.3. *Let $u \geq 0$ in $\mathbb{R}^d$, $u(0) \leq 1$, and $\mathcal{M}_i^- u \leq \tilde{\varepsilon}_0$ in B_2. Then*

$$|\{u \geq t\} \cap B_1| \leq C\, t^{-\varepsilon} \;\; \textit{for every}\; t > 0\,,$$

where the constant C and ε depend only on $(d, \lambda, \Lambda, \alpha, \varphi)$.

We conclude the section by proving a weak-Harnack estimate. It is referred in the literature as L^ϵ-estimate.

Theorem 5.1.4. *Let $u \geq 0$ in $\mathbb{R}^d$ and $\mathcal{M}^- u \leq C_0$ in B_{2r}, $0 < r \leq 1$. Then*

$$|\{u \geq t\} \cap B_r| \leq Cr^d(u(0) + C_0 r^\alpha)^\varepsilon t^{-\varepsilon} \;\; \textit{for every}\; t > 0\,,$$

for some constants C, ε as in Theorem 5.1.3. In particular,

$$\|u\|_{L^{\varepsilon/2}(B_r)} \leq C(u(0) + C_0 r^\alpha).$$

Proof. Choose $k \in \mathbb{N} \cup \{0\}$ satisfying $2^{-k} < r \leq \frac{3}{2}2^{-k}$. Let $v(x) = u(\frac{1}{2^k}x)$. Then from the calculation of Theorem 5.1.2 it follows that

$$\mathcal{M}_k^- v(x) \le \frac{1}{2^{k\alpha}} \mathcal{M}^- u(\frac{1}{2^k} x) \le r^\alpha C_0 \quad \text{in } B_2.$$

Multiplying v with $\frac{\tilde{\varepsilon}_0}{v(0)+r^\alpha C_0}$, it follows from Theorem 5.1.3 (modifying the argument a bit)

$$|\{v \ge t\} \cap B_{\frac{3}{2}}| \le C\, t^{-\varepsilon} \quad t > 0\,. \tag{5.1.10}$$

Hence, by our choice of k, we get

$$|\{u \ge t\} \cap B_r| \le C r^d (u(0) + C_0 r^\alpha)^\varepsilon t^{-\varepsilon}.$$

The second conclusion follows by integrating both sides of (5.1.10) with respect to t. □

5.2 Hölder regularity

Using the results developed in Section 5.1, in this section we establish an interior Hölder regularity. The main theorem of this section is the following.

Theorem 5.2.1. *Let u be a bounded continuous function defined on $\mathbb{R}^d$ and satisfy*

$$\mathcal{M}^+ u \ \ge -C_0, \quad \mathcal{M}^- u \le \ C_0 \quad \text{in } B_1,$$

for some constant C_0. Then $u \in C^\gamma(B_{\frac{1}{2}})$ and

$$\|u\|_{C^\gamma(B_{\frac{1}{2}})} \ \le \ C(\|u\|_{L^\infty(\mathbb{R}^d)} + C_0),$$

where γ, C depend only on $d, \lambda, \Lambda, \alpha, \varphi$.

We follow the approach of [50] to prove Theorem 5.2.1. Theorem 5.2.1 would follow from the following result and a covering argument.

Lemma 5.2.1. *Let u be a continuous function satisfying*

$$-\frac{1}{2} \le u \le \frac{1}{2} \quad \text{in } \mathbb{R}^d, \quad \mathcal{M}^+ u \ge -\tilde{\varepsilon}, \quad \mathcal{M}^- u \le \tilde{\varepsilon} \quad \text{in } B_1\,.$$

Then there is a $\gamma > 0$ (depending on $\Lambda, \lambda, \alpha, \varphi$) such that $u \in C^\gamma$ at the origin. In particular,

$$|u(x) - u(0)| \leq C|x|^\gamma$$

for some constant C.

Proof. Following [50] We show that there exists sequences m_k and M_k satisfying $m_k \leq u \leq M_k$ in $B_{8^{-k}}$ and

$$M_k - m_k = 8^{-\gamma k}.$$

Then the result follows by choosing $C = 8^\gamma$.

For $k = 0$ we choose $m_0 = -\frac{1}{2}$ and $M_0 = \frac{1}{2}$. By assumption we have $m_0 \leq u \leq M_0$ in the whole space $\mathbb{R}^d$. We proceed to construct the sequences M_k and m_k by induction. So by induction hypothesis we assume the construction of m_j, M_j for $j = 0, \ldots, k$. We want to show that we can continue the sequences by finding m_{k+1} and M_{k+1}.

Consider the ball $B_{\frac{1}{8^{k+1}}}$. Then one of the following holds

$$|\{u \geq \frac{M_k + m_k}{2}\} \cap B_{\frac{1}{8^{k+1}}}| \ \geq \ \frac{1}{2}|B_{\frac{1}{8^{k+1}}}|; \tag{5.2.1}$$

$$|\{u \leq \frac{M_k + m_k}{2}\} \cap B_{\frac{1}{8^{k+1}}}| \ \geq \ \frac{1}{2}|B_{\frac{1}{8^{k+1}}}|. \tag{5.2.2}$$

Suppose that (5.2.1) holds. Define

$$v(x) := \frac{u(8^{-k}x) - m_k}{{(M_k - m_k)}/{2}}.$$

Then that $v(x) \geq 0$ in B_1 and $|\{v \geq 1\} \cap B_{1/8}| \geq {|B_{1/8}|}/{2}$. Moreover, since $\mathcal{M}^- u \leq \tilde{\varepsilon}$ in B_1, we get from the calculation in Theorem 5.1.2

$$\mathcal{M}^-_{3k} v \leq \frac{8^{-k\alpha}\tilde{\varepsilon}}{{(M_k - m_k)}/{2}} = 2\tilde{\varepsilon} 8^{-k(\alpha - \gamma)} \leq 2\tilde{\varepsilon} \quad \text{in } B_{8^k},$$

provided we set $\gamma \leq \alpha$. From the induction hypothesis, for any $j \geq 1$, we have

$$v \geq \frac{(m_{k-j} - m_k)}{{(M_k - m_k)}/{2}} \geq \frac{(m_{k-j} - M_{k-j} + M_k - m_k)}{{(M_k - m_k)}/{2}} \geq -2 \cdot 8^{\gamma j} + 2 \geq 2(1 - 8^{\gamma j}),$$

in B_{8^j}. Thus $v(x) \geq \max\{-2(|8x|^\gamma - 1), -2(8^{(k+1)\gamma} - 1)\} := -g(x)$ outside B_1. Letting $w(x) = \max(v, 0)$ we also see that

$$\mathcal{M}_{3k}^- w \leq \mathcal{M}_{3k}^- v + \mathcal{M}_{3k}^+ v^-. \tag{5.2.3}$$

We claim that $\mathcal{M}_{3k}^+ v^- \leq 2\tilde{\varepsilon}$ in $B_{3/4}$, for all k, if we choose γ small enough. For $x \in B_{3/4}$, since $v^-(x) = 0$, we have $\delta(v^-, x, y) = \delta^+(v^-, x, y) = v^-(x+y) + v^-(x-y)$ for all $y \in \mathbb{R}^d$, and by (5.1.1),

$$\mathcal{M}_{3k}^+ v^-(x) = \int_{\mathbb{R}^d} \frac{\Lambda \delta^+(v^-, x, y)}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + 8^{(\beta-\alpha)k} \varphi_0(1/|y|) \right) \mathrm{d}y \,.$$

If $|y| < \frac{1}{4}$, then both $x+y$ and $x-y$ is in B_1 so $v^-(x+y) = v^-(x-y) = 0$. This gives us

$$\begin{aligned}
\mathcal{M}_0^+ v^-(x) &= \Lambda \int_{\{|y| \geq \frac{1}{4}\}} \frac{v^-(x+y) + v^-(x-y)}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + 8^{(\beta-\alpha)k} \varphi_0(1/|y|) \right) \mathrm{d}y \\
&\leq \Lambda \int_{\{|y| \geq \frac{1}{4}\}} \frac{g^+(x+y) + g^+(x-y)}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + 8^{(\beta-\alpha)k} \kappa_\circ \varphi(1/|y|) \right) \mathrm{d}y \\
&\leq 2\Lambda \int_{\{|y| \geq \frac{1}{4}\}} \frac{g^+(x+y)}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + 8^{(\beta-\alpha)k} \kappa_\circ \varphi(\frac{1}{|y|}) \right) \mathrm{d}y \\
&\leq 4\Lambda \int_{\{|y| \geq \frac{1}{4}\}} \frac{(32^\gamma |y|^\gamma - 1)^+}{|y|^d} \frac{2-\alpha}{|y|^\alpha} \,\mathrm{d}y \\
&\qquad + 4\Lambda \int_{\{|y| \geq \frac{1}{4}\}} (8^{\gamma(k+1)} - 1) \frac{8^{(\beta-\alpha)k} \kappa_\circ}{|y|^d} \varphi(\frac{1}{|y|}) \mathrm{d}y \\
&= I_1 + I_2.
\end{aligned}$$

Notice that the function $f_\gamma = (32^\gamma |y|^\gamma - 1)^+ \mathbb{1}_{\{|y| \geq \frac{1}{4}\}}$ decreases to 0 as $\gamma \to 0$. Also, the function becomes integrable if we choose $\gamma < \alpha$. Thus for a small γ we have $I_1 \leq \tilde{\varepsilon}$. So we calculate I_2. We fix $\gamma < \alpha - \beta$. Define the function $h(t) = \log[(8^{\gamma t} - 1) 8^{(\beta-\alpha)t}]$ for $t > 0$. Note that

$$h'(t) = \log 8[-(\alpha - \beta) + \gamma \frac{8^{\gamma t}}{8^{\gamma t} - 1}] < 0,$$

for large t. So $h(t)$ attends its maximum and

$$h'(t) = 0 \;\Rightarrow 8^{\gamma t} - 1 = \gamma \frac{8^{\gamma t}}{\alpha - \beta}.$$

Thus,

$$\max_{t \geq 1} e^{h(t)} \leq \; \gamma \sup_{t \geq 1} \frac{8^{(\gamma - \alpha + \beta)t}}{\alpha - \beta} \to 0,$$

as $\gamma \to 0$. Thus using (A2) we can choose γ small enough to satisfy $I_2 < \tilde{\varepsilon}$, uniformly in k. This gives us the claim.

Using (5.2.3) we obtain $M^-_{3k} w \leq 4\tilde{\varepsilon}$ in $B_{\frac{3}{4}}$, provided γ is small enough. We also have

$$|\{w \geq 1\} \cap B_{\frac{1}{8}}| \geq \frac{|B_{\frac{1}{8}}|}{2}.$$

Given any point $x \in B_{\frac{1}{4}}$, we can apply Theorem 5.1.4 in $B_{\frac{2}{4}}(x)$ to obtain

$$C(w(x) + 4\tilde{\varepsilon})^{\varepsilon} \geq |\{w > 1\} \cap B_{\frac{1}{4}}(x)| \geq \frac{1}{2} |B_{\frac{1}{8}}| \,.$$

If we have chosen $\tilde{\varepsilon}$ small, this implies that $w > \theta$ in $B_{\frac{1}{8}}$ for some $\theta > 0$. Thus if we let $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta \frac{(M_k - m_k)}{2}$, we have $m_{k+1} \leq u \leq M_{k+1}$ in $B_{8^{-k-1}}$. Moreover, $M_{k+1} - m_{k+1} = (1 - {}^{\theta}\!/_2) 8^{-\gamma k}$. So we must choose γ and θ small and so that $(1 - {}^{\theta}\!/_2) = 8^{-\gamma}$ and we obtain $M_{k+1} - m_{k+1} = 8^{-\gamma(k+1)}$.
On the other hand, if (5.2.2) holds, we define

$$v(x) = \frac{M_k - u(8^{-k} x)}{{}^{(M_k - m_k)}\!/_2}$$

and continue in the same way using that $\mathcal{M}^+ u \geq -\tilde{\varepsilon}$.

□

5.3 Harnack inequality

In this section and the next section we discuss Harnack's inequality and boundary Harnack inequality. This will be done for a smaller class of operators. Let $\tilde{\mathscr{L}} \subset \mathscr{L}$

be the set of operators containing kernel function k satisfying

$$\lambda\left(\frac{2-\alpha}{|y|^{\alpha}}+\varphi(\frac{1}{|y|})\right)\ \leq\ k(y)\leq\Lambda\left(\frac{2-\alpha}{|y|^{\alpha}}+\varphi(\frac{1}{|y|})\right)\quad\text{for some }\ \alpha\in(\beta,2),\qquad\text{(A4)}$$

and φ is non-decreasing. The associated extremal operators are denoted by $\tilde{\mathcal{M}}^{\pm}$. In particular,

$$\begin{aligned}\tilde{\mathcal{M}}^{+}u(x)&=\int_{\mathbb{R}^d}\frac{\Lambda\delta^{+}(u,x,y)}{|y|^{d}}\left(\frac{2-\alpha}{|y|^{\alpha}}+\varphi(\frac{1}{|y|})\right)-\frac{\lambda\delta^{-}(u,x,y)}{|y|^{d}}\left(\frac{2-\alpha}{|y|^{\alpha}}+\varphi(\frac{1}{|y|})\right)\mathrm{d}y\,,\\ \tilde{\mathcal{M}}^{-}u(x)&=\int_{\mathbb{R}^d}\frac{\lambda\delta^{+}(u,x,y)}{|y|^{d}}\left(\frac{2-\alpha}{|y|^{\alpha}}+\varphi(\frac{1}{|y|})\right)-\frac{\Lambda\delta^{-}(u,x,y)}{|y|^{d}}\left(\frac{2-\alpha}{|y|^{\alpha}}+\varphi(\frac{1}{|y|})\right)\mathrm{d}y\,.\end{aligned}$$

It is also evident that $\mathcal{M}^{-}u\leq\tilde{\mathcal{M}}^{-}u\leq\tilde{\mathcal{M}}^{+}u\leq\mathcal{M}^{+}u$. It should be observed that we do not require weak lower scaling property on φ (compare with [109]). For instance, $\varphi(r)=\log(1+r^{\beta})$ does not satisfy weak lower scaling i.e. there is no $\mu>0$ so that $\varphi(sr)\gtrsim s^{\mu}\varphi(r)$ for $s\geq 1, r>0$. But it does satisfy a weak upper scaling property since for every $s\geq 1$,

$$1+s^{\beta}r^{\beta}\leq(1+r^{\beta})^{s^{\beta}}\ \Rightarrow\ \varphi(sr)\leq s^{\beta}\varphi(r)\,.$$

Since $\log(1+r^{\beta})$ does not satisfy assumption (A2), consider

$$\varphi(r)=\min\left\{r^{\beta},\log(1+r^{\beta})\right\}.$$

Then it satisfies assumption (A2) and for $s\geq 1, r>0$ we have

$$\begin{aligned}\varphi(sr)&=\min\left\{s^{\beta}r^{\beta},\log(1+(sr)^{\beta})\right\}\\&=s^{\beta}\min\left\{r^{\beta},\frac{1}{s^{\beta}}\log(1+(sr)^{\beta})\right\}\\&\leq s^{\beta}\min\left\{r^{\beta},\log(1+r^{\beta})\right\}=s^{\beta}\varphi(r).\end{aligned}$$

Here the last inequality follows from the fact that if $g\leq h$, then $\min\{f,g\}\leq\min\{f,h\}$, since $\log(1+(sr)^{\beta})\leq s^{\beta}\log(1+r^{\beta})$ for $s\geq 1, r>0$, as previously calculated.

Our main result of this section is the following

Theorem 5.3.1. *Let u be a non-negative function satisfying*

$$\tilde{\mathcal{M}}^+ u \geq -C_0, \quad \text{and} \quad \tilde{\mathcal{M}}^- u \leq C_0 \quad \text{in } B_2.$$

Then $u(x) \leq C(u(0) + C_0)$ for every $x \in B_{\frac{1}{2}}$, for some constant C dependent only on $\lambda, \Lambda, \alpha, \varphi$.

Proof. We again follow the idea of [50]. Dividing by $u(0) + C_0$, it is enough to consider $u(0) \leq 1$ and $C_0 = 1$. Fix $\varepsilon > 0$ from Theorem 5.1.4 and let $\gamma = \frac{d}{\varepsilon}$. Let

$$t := \min\{s \ : \ u(x) \leq h_s(x) := s(1 - |x|)^{-\gamma} \ \text{ for all } x \in B_1\}.$$

Let $x_0 \in B_1$ be such that $u(x_0) = h_t(x_0)$. Let $\eta = 1 - |x_0|$ be the distance of x_0 from ∂B_1. We show that $t < C$ for some universal C which in turn, implies that $u(x) < C(1 - |x|)^{-\gamma}$. This would prove our result.

For $r = \frac{\eta}{2}$, we estimate the portion of the ball $B_r(x_0)$ covered by $\{u < \frac{u(x_0)}{2}\}$ and $\{u > \frac{u(x_0)}{2}\}$. Define $A := \{u > \frac{u(x_0)}{2}\}$. Using Theorem 5.1.4 we then obtain

$$|A \cap B_1| \leq C \left| \frac{2}{u(x_0)} \right|^{\varepsilon} \leq C t^{-\varepsilon} \eta^d \,,$$

whereas $|B_r| = \omega_d (\eta/2)^d$. In particular,

$$\left| \left\{ u > \frac{u(x_0)}{2} \right\} \cap B_r(x_0) \right| \leq C t^{-\varepsilon} |B_r|. \tag{5.3.1}$$

So if t is large, A can cover only a small portion of $B_r(x_0)$. We shall show that for some $\delta > 0$, independent of t we have

$$|\{u \leq \frac{u(x_0)}{2}\} \cap B_r(x_0)| \leq (1 - \delta)|B_r|$$

which will provide an upper bound on t completing the proof.

We start by estimating $|\{u \leq \frac{u(x_0)}{2}\} \cap B_{\theta r}(x_0)|$ for $\theta > 0$ small. For every $x \in B_{\theta r}(x_0)$ we have

$$u(x) \leq h_t(x) \leq t \left(\frac{2\eta - \theta\eta}{2} \right)^{-\gamma} \leq u(x_0) \left(1 - \frac{\theta}{2} \right)^{-\gamma} ,$$

with $\left(1-\frac{\theta}{2}\right)$ close to 1. Define

$$v(x) := \left(1-\frac{\theta}{2}\right)^{-\gamma} u(x_0) - u(x)\,.$$

Then that $v \geq 0$ in $B_{\theta r}(x_0)$, and also $\tilde{\mathcal{M}}^- v \leq 1$ as $\tilde{\mathcal{M}}^+ u \geq 1$. We would like to apply Theorem 5.1.4 to v but v need not be non-negative in the whole space $\mathbb{R}^d$. Thus we consider $w = v^+$ and find an upper bound of $\tilde{\mathcal{M}}^- w$.

We already know that

$$\begin{aligned}\tilde{\mathcal{M}}^- v(x) &= \int_{\mathbb{R}^d} \frac{\lambda\delta^+(v,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(\frac{1}{|y|})\right) - \frac{\Lambda\delta^-(v,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(\frac{1}{|y|})\right)\mathrm{d}y\\ &\leq 1.\end{aligned}$$

Therefore, for $x \in B_{\frac{\theta r}{4}}(x_0)$

$$\begin{aligned}\tilde{\mathcal{M}}^- w(x) &= \int_{\mathbb{R}^d} \frac{\lambda\delta^+(w,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(\frac{1}{|y|})\right) - \frac{\Lambda\delta^-(w,x,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(\frac{1}{|y|})\right)\mathrm{d}y\\ &\leq 1 + 2\int_{\mathbb{R}^d\cap\{v(x+y)<0\}} -\Lambda\frac{v(x+y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(\frac{1}{|y|})\right)\mathrm{d}y\\ &\leq 1 + 2\int_{B^c_{\frac{\theta r}{2}}(x_0-x)} \Lambda\frac{(u(x+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(\frac{1}{|y|})\right)\mathrm{d}y.\end{aligned} \tag{5.3.2}$$

So to find an upper bound we must compute the second expression. Let us consider the largest value $\tau > 0$ such that $u(x) \geq g_\tau := \tau\,(1-|4x|^2)$. There must be a point $x_1 \in B_{\frac{1}{4}}$ such that $u(x_1) = \tau(1-|4x|^2)$. The value of τ cannot be larger than 1 since $u(0) \leq 1$. Also truncate g_τ and define $\hat{g}_\tau := g_\tau \mathbb{1}_{B_{\frac{1}{3}}}$ which implies $u(x) \geq \hat{g}_\tau(x) \geq g_\tau(x)$ for all $x \in \mathbb{R}^d$ and $u(x_1) = \hat{g}_\tau(x_1) = g_\tau(x_1)$. Thus we have the upper bound

$$\begin{aligned}&\int_{\mathbb{R}^d} \frac{\delta^-(u,x_1,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(1/|y|)\right)\mathrm{d}y\\ &\leq \int_{\mathbb{R}^d} \frac{\delta^-(\hat{g}_\tau,x_1,y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha}+\varphi(1/|y|)\right)\mathrm{d}y\end{aligned}$$

$$\leq \int_{B_1} \frac{\delta^-(g_\tau, x_1, y)}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \mathrm{d}y + \int_{\mathbb{R}^d \setminus B_1} \frac{32}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + \varphi(\frac{1}{|y|}) \right) \mathrm{d}y$$
$$\leq \int_{B_1} \frac{32|y|^2}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + \frac{\kappa_\circ}{|y|^\beta} \varphi(1) \right) \mathrm{d}y + C_1 \leq C_2 \,,$$

for some constants C_1, C_2 dependent only on (d, α, φ), where we used following inequality

$$\hat{g}_\tau(x_1+y) + \hat{g}_\tau(x_1-y) - 2\hat{g}_\tau(x_1) \geq \tau \left(2|4x_1|^2 - |4(x_1+y)|^2 - |4(x_1-y)|^2 \right) = 32|y|^2,$$

for $y \in B_1$. Since $\tilde{\mathcal{M}}^- u(x_1) \leq 1$, we get using the above estimate that

$$\int_{\mathbb{R}^d} \frac{\delta^+(u, x_1, y)}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \mathrm{d}y \leq \ C \,.$$

In particular, since $u(x_1) \leq 1$ and $u(x_1 - y) \geq 0$,

$$\int_{\mathbb{R}^d} \frac{(u(x_1+y) - 2)^+}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \mathrm{d}y \ \leq \ C \,.$$

We use this estimate to compute the RHS of (5.3.2). Without any loss of generality we may assume that $u(x_0) > 2$, since otherwise t would not be large. We can use the inequality above to get following estimate

$$2(2-\alpha) \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{2}}(x_0 - x)} \Lambda \frac{(u(x+y) - (1-\frac{\theta}{2})^{-\gamma} u(x_0))^+}{|y|^{d+\alpha}} \mathrm{d}y$$
$$\leq 2(2-\alpha) \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{2}}(x_0 - x)} \Lambda \frac{(u(x_1 + x + y - x_1) - 2)^+}{|y + x - x_1|^{d+\alpha}} \quad \frac{|y + x - x_1|^{d+\alpha}}{|y|^{d+\alpha}} \mathrm{d}y$$
$$\leq C(\theta r)^{-d-\alpha},$$

here we used the fact that $y \notin B_{\frac{\theta r}{2}}(x_0 - x)$ implies $y \notin B_{\frac{\theta r}{4}}$. Again, a simple calculation gives

$$\frac{|y + x - x_1|}{|y|} \leq \frac{|y| + |x - x_1|}{|y|} \leq 12(\theta r)^{-1}$$

and using the monotonicity property of φ,

$$\frac{\varphi\left(1/|y|\right)}{\varphi\left(1/|y+x-x_1|\right)} \leq \frac{\varphi\left(\frac{|y+x-x_1|}{|y+x-x_1|}\frac{1}{|y|}\right)}{\varphi\left(\frac{1}{|y+x-x_1|}\right)} \leq \kappa_\circ\left(12(\theta r)^{-1}\right)^{\beta},$$

by (A1). This gives us

$$\begin{aligned}
&2\Lambda\int_{\mathbb{R}^d\setminus B_{\frac{\theta r}{2}}(x_0-x)} \frac{(u(x+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^{+}}{|y|^d}\varphi\left(\frac{1}{|y|}\right)\mathrm{d}y \\
&\leq 2\Lambda\int_{\mathbb{R}^d\setminus B_{\frac{\theta r}{2}}(x_0-x)} \frac{(u(x_1+x+y-x_1)-2)^{+}}{|y+x-x_1|^d}\quad\varphi\left(1/|y+x-x_1|\right) \\
&\qquad\qquad\frac{|y+x-x_1|^n}{|y|^n}\quad\frac{\varphi\left(1/|y|\right)}{\varphi\left(1/|y+x-x_1|\right)}\mathrm{d}y \\
&\leq C\kappa_\circ(\theta r)^{-d-\beta} \leq C(\theta r)^{-d-\alpha}.
\end{aligned}$$

Thus we obtain from (5.3.2)

$$\tilde{\mathcal{M}}^{-}w \;\leq\; C_1(\theta r)^{-d-\alpha} \quad \text{in } B_{\frac{\theta r}{4}}(x_0).$$

We apply Theorem 5.1.4 to w in $B_{\theta r/4}(x_0)$. Recalling that $w(x_0)=((1-\theta/2)^{-\gamma}-1)u(x_0)$, we have

$$\begin{aligned}
&\left|\left\{u\leq\frac{u(x_0)}{2}\right\}\cap B_{\frac{\theta r}{8}}(x_0)\right| \\
&=\left|\left\{w\geq u(x_0)((1-\frac{\theta}{2})^{-\gamma}-\frac{1}{2})\right\}\cap B_{\frac{\theta r}{8}}(x_0)\right| \\
&\leq C(\theta r)^d\left(\left((1-\frac{\theta}{2})^{-\gamma}-1\right)u(x_0)+C_1(\theta r)^{-d}\right)^{\varepsilon}\left[u(x_0)((1-\frac{\theta}{2})^{-\gamma}-\frac{1}{2})\right]^{-\varepsilon} \\
&\leq C(\theta r)^d\left(\left((1-\frac{\theta}{2})^{-\gamma}-1\right)^{\varepsilon}+\theta^{-d\varepsilon}t^{-\varepsilon}\right).
\end{aligned}$$

Now let us choose $\theta>0$ small enough (independent of t) to satisfy

$$C(\theta r)^d\left((1-\theta/2)^{-\gamma}-1\right)^{\varepsilon} \leq \frac{1}{4}|B_{\frac{\theta r}{8}}(x_0)|.$$

With this choice of θ if t becomes large, then we also have

$$C(\theta r)^d \theta^{-d\varepsilon} t^{-\varepsilon} \leq \frac{1}{4}|B_{\frac{\theta r}{8}}(x_0)|\,,$$

and hence,

$$\left|\left\{u \leq \frac{u(x_0)}{2}\right\} \cap B_{\frac{\theta r}{8}}(x_0)\right| \leq \frac{1}{2}|B_{\frac{\theta r}{8}}(x_0)|\,.$$

This of course, implies that

$$\left|\left\{u > \frac{u(x_0)}{2}\right\} \cap B_{\frac{\theta r}{8}}(x_0)\right| \geq C_2|B_r|,$$

but this is contradicting to (5.3.1). Therefore t cannot be large and we finish the proof. □

We remark here that in the case of local operators, the Hölder estimate is a trivial consequence of the Harnack inequality. The Hölder estimate does not follow immediately from the Harnack inequality for a nonlocal operator. This is due to the fact that the Harnack inequality requires u to be nonnegative in the whole space $\mathbb{R}^d$, not in a ball. Thus needed an investigation of u outside the balls.

Mimicking Theorem 5.3.1 we also obtain the following result which will be useful to establish a boundary Harnack property. The following also known as the *half Harnack* inequality for subsolutions.

Theorem 5.3.2. *Let u be a function continuous in $\overline{B}_1$, and satisfy*

$$\int_{\mathbb{R}^d} \frac{|u(y)|}{1+|y|^{d+\alpha}}\,\mathrm{d}y + \int_{\mathbb{R}^d} \frac{|u(y)|}{1+|y|^d(\varphi(1/|y|))^{-1}}\,\mathrm{d}y \leq C_0,$$

and

$$\tilde{\mathcal{M}}^+ u \geq -C_0 \quad \textit{in } B_1\,.$$

Then

$$u(x) \leq C\,C_0\,,$$

for every $x \in B_{\frac{1}{2}}$, where the constant $C > 0$ depends only on $(d, \lambda, \Lambda, \alpha, \varphi)$.

Proof. We follow the approach of [52] and Theorem 5.3.1. Dividing u by C_0, it is enough to consider $C_0 = 1$. Also, without any loss of generality we may assume that

u is positive somewhere in B_1. Otherwise, there is nothing to prove. As before we consider the minimum value of t such that

$$u(x) \leq h_t(x) := t(1-|x|)^{-d} \quad \textit{for every } x \in B_1,$$

and find $x_0 \in B_1$ with $u(x_0) = h_t(x_0)$. Denote by $\eta = 1 - |x_0|$, $r = \eta/2$ and $A = \{u > u(x_0)/2\}$. As shown in Theorem 5.3.1, we need to find an upper bound of t.

By assumption, we have $u \in L^1(B_1)$ and thus

$$|A \cap B_1| \leq C\left|\frac{2}{u(x_0)}\right| \leq Ct^{-1}\eta^d\,,$$

whereas $|B_r| = C\eta^d$, so if t is large, A can cover only a small portion of $B_r(x_0)$ at most. In particular,

$$\left|\left\{u > \frac{u(x_0)}{2}\right\} \cap B_r(x_0)\right| \leq Ct^{-1}|B_r|. \tag{5.3.3}$$

We define

$$v(x) = \left(1 - \frac{\theta}{2}\right)^{-d} u(x_0) - u(x)$$

for small $\theta > 0$, and observe that $v \geq 0$ in $B_{\theta r}(x_0)$. Let $w = v^+$. Repeating the arguments of Theorem 5.3.1 we find, for $x \in B_{\frac{\theta r}{4}}(x_0)$, that

$$\tilde{\mathcal{M}}^- w(x) \leq 1 + 2\int_{\mathbb{R}^d \cap \{v(x+y)<0\}} -\Lambda \frac{v(x+y)}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|)\right) \mathrm{d}y\,,$$

since $v \geq 0$ in $B_{\theta r}(x_0)$ and $x \in B_{\frac{\theta r}{4}}(x_0)$, we will have $x+y$ and $x-y$ both in $B_{\theta r}(x_0)$ for all $y \in B_{\frac{\theta r}{2}}$. Now we need to estimate the integral on the RHS of the above. Note that u need to be non-negative here and thus we can not apply the technique of cut-off function as done in Theorem 5.3.1. So we use the integral condition imposed on u.

$$\tilde{\mathcal{M}}^- w(x) \leq 1 + 2\int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{2}}} \Lambda \frac{(u(x+y) - (1-\frac{\theta}{2})^{-d} u(x_0))^+}{|y|^d}\left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|)\right) \mathrm{d}y$$

$$
\begin{aligned}
&\le 1 + 2\Lambda \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{2}}} \frac{u^+(x+y)}{|y|^d} \left(\frac{2-\alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \mathrm{d}y \\
&\le 1 + 2\Lambda \int_{\mathbb{R}^d \setminus B_{\frac{\theta r}{2}}(x)} \frac{|u(x)|}{|x-y|^d} \left(\frac{2-\alpha}{|x-y|^\alpha} + \varphi(1/|x-y|) \right) \mathrm{d}y \,. \qquad (5.3.4)
\end{aligned}
$$

Using $|x-y| \ge \frac{\theta r}{2}$ and $|x| < 1$ we obtain the following estimates

$$
\begin{aligned}
\frac{1}{|x-y|^{d+\alpha}} &= \frac{1}{1+|y|^{d+\alpha}} \cdot \frac{1+|y|^{d+\alpha}}{|x-y|^{d+\alpha}} \\
&\le \frac{1}{1+|y|^{d+\alpha}} \cdot \left[\frac{1}{|x-y|^{d+\alpha}} + \left(\frac{|x|+|x-y|}{|x-y|} \right)^{d+\alpha} \right] \\
&\le \frac{1}{1+|y|^{d+\alpha}} \left(\frac{\theta r}{2} \right)^{-d-\alpha} \left[1 + 2^{d+\alpha} \right] \le C(\theta r)^{-d-\alpha} \frac{1}{1+|y|^{d+\alpha}} \,,
\end{aligned}
$$

and, since $\frac{|y|}{|x-y|} \le 1 + \frac{|x|}{|x-y|}$,

$$
\begin{aligned}
&\varphi(1/|x-y|) \le \kappa_\circ \left(1 + \frac{|x|}{|x-y|} \right)^\beta \varphi(1/|y|), \\
&\varphi(1/|x-y|) \le \varphi(2/r\theta) \le \kappa_\circ (2/r\theta)^\beta \varphi(1),
\end{aligned}
$$

giving us

$$
\begin{aligned}
&\frac{1}{|x-y|^d (\varphi(1/|x-y|))^{-1}} \\
&\qquad = \frac{1}{1+|y|^d(\varphi(1/|y|))^{-1}} \left[\frac{\varphi(1/|x-y|)}{|x-y|^d} + \frac{|y|^d}{|x-y|^d} \frac{\varphi(1/|x-y|)}{\varphi(1/|y|)} \right] \\
&\qquad \le \frac{1}{1+|y|^d(\varphi(1/|y|))^{-1}} \left[\frac{\kappa_\circ (2/r\theta)^\beta \varphi(1)}{|x-y|^d} + \kappa_\circ \left(1 + \frac{|x|}{|x-y|} \right)^{d+\beta} \right] \\
&\qquad \le C_1 \frac{1}{1+|y|^d(\varphi(1/|y|))^{-1}} (\theta r)^{-d-\beta} \\
&\qquad \le C_1 \frac{1}{1+|y|^d(\varphi(1/|y|))^{-1}} (\theta r)^{-d-\alpha},
\end{aligned}
$$

for some constant C_1 dependent on d, φ. Using these estimates in (5.3.4) we thus obtain

$$\tilde{\mathcal{M}}^{-} w(x) \leq C_3 (\theta r)^{-d-\alpha} \quad \text{in } B_{\frac{r\theta}{4}}(x_0)\,.$$

Now we repeat the arguments of Theorem 5.3.1 and get a contradiction to (5.3.3) if t is large. This completes the proof. □

5.4 Boundary Harnack estimate

We prove a boundary Harnack property in this section for operators in $\tilde{\mathscr{L}}$ introduced in Section 5.3. Being inspired from [147] we prove the following result

Theorem 5.4.1. *Let $\Omega \subset \mathbb{R}^d$ be any open set. Assume that there is $x_0 \in B_{\frac{1}{2}}$ and $\varrho > 0$ such that $B_{2\varrho}(x_0) \subset \Omega \cap B_{\frac{1}{2}}$. Then there exists $\delta > 0$, dependent only on $(d, \alpha, \varrho, \varphi, \lambda, \Lambda)$, such that the following statement holds.*

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions of

$$\begin{aligned} \tilde{\mathcal{M}}^{+}(au_1 + bu_2) &\geq -\delta(|a| + |b|) \quad \textit{in} \quad B_1 \cap \Omega, \\ u_1 = u_2 &= 0 \quad \textit{in} \quad B_1 \setminus \Omega\,, \end{aligned} \tag{5.4.1}$$

for all $a, b \in \mathbb{R}$, and such that

$$u_i \geq 0 \quad \textit{in} \quad \mathbb{R}^d, \qquad \int_{\mathbb{R}^d} \frac{u_i(y)}{1 + |y|^{d+\alpha}} \mathrm{d}y + \int_{\mathbb{R}^d} \frac{|u_i(y)|}{1 + |y|^d (\varphi(1/|y|))^{-1}} \,\mathrm{d}y = 1\,. \tag{5.4.2}$$

Then

$$C^{-1} u_2 \leq u_1 \leq C u_2 \qquad \textit{in} \quad B_{\frac{1}{2}},$$

where the constant C depends only on $(d, \alpha, \varrho, \varphi, \lambda, \Lambda)$.

Theorem 5.4.1 is bit stronger than the boundary Harnack principle. To see it suppose that for some $L \in \tilde{\mathscr{L}}$ we have $Lu_i = 0$ in $B_1 \cap \Omega$, in viscosity sense, and $u_i = 0$ in $B_1 \setminus \Omega$. Then clearly (5.4.1) holds for all $a, b \in \mathbb{R}$ (cf. [50, Theorem 5.9]). Furthermore, if (5.4.2) holds, then Theorem 5.4.1 gives us

$$C^{-1} u_2 \leq u_1 \leq C u_2 \qquad \textit{in} \quad B_{\frac{1}{2}}\,.$$

To prove Theorem 5.4.1 we need Lemmas 5.4.1 and 5.4.2 below.

Lemma 5.4.1. *Assume that $u \in C(B_1)$ and satisfies $\tilde{\mathcal{M}}^- u \le C_0$ in B_1 in viscosity sense. In addition, assume that $u \ge 0$ in $\mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d} \frac{u(y)}{1+|y|^{d+\alpha}} \mathrm{d}y + \int_{\mathbb{R}^d} \frac{u(y)}{1+|y|^d(\varphi(1/|y|))^{-1}} \mathrm{d}y \le C \left(\inf_{B_{\frac{1}{2}}} u + C_0 \right),$$

where the constant C depends only on $(d, \lambda, \Lambda, \alpha, \varphi)$.

Proof. We need few basic estimates. We show that there exists a constant $\kappa > 0$ such that for any $x_0 \in B_{\frac{3}{4}}$ and $z \in \mathbb{R}^d$ we have

$$|x_0 - z|^{d+\alpha} \le \kappa(1+|z|^{d+\alpha}), \tag{5.4.3}$$

$$|x_0 - z|^d (\varphi(1/|x_0 - z|))^{-1} \le \kappa(1+|z|^d(\varphi(1/|z|))^{-1}) . \tag{5.4.4}$$

(5.4.3) is trivial since $|x_0 + z| \le 1 + |z|$ implies $|x_0 - z|^{d+\alpha} \le 2^{d+\alpha}(1+|z|^{d+\alpha})$. On the other hand

$$\frac{1}{|z|} \le \left(\frac{1+|z|}{|z|}\right) \frac{1}{|x_0 - z|},$$

implies

$$\begin{aligned}\varphi\left(\frac{1}{|z|}\right) &\le \varphi\left(\left(\frac{1+|z|}{|z|}\right)\frac{1}{|x_0-z|}\right)\\ &\le \kappa_\circ \left(\frac{1+|z|}{|z|}\right)^\beta \varphi\left(\frac{1}{|x_0-z|}\right).\end{aligned}$$

Thus

$$(\varphi(1/|x_0 - z|))^{-1} \le \kappa_\circ \left(\frac{1+|z|}{|z|}\right)^\beta (\varphi(1/|z|))^{-1} . \tag{5.4.5}$$

Let $|z| \le 1$. Then using (5.4.5) we get

$$\begin{aligned}|x_0 - z|^d (\varphi(1/|x_0 - z|))^{-1} &\le (1+|z|)^{d+\beta} (\varphi(1/|x_0 - z|))^{-1}\\ &\le 2^{d+\beta}(1+|z|^{d+\beta}) (\varphi(1/|x_0 - z|))^{-1}\\ &\le 2^{d+\beta} \left((\varphi(1/|x_0 - z|))^{-1} + |z|^{d+\beta} (\varphi(1/|x_0 - z|))^{-1}\right)\\ &\le 2^{d+\beta} \left(\kappa + |z|^{d+\beta} (\varphi(1/|x_0 - z|))^{-1}\right)\\ &\le 2^{d+\beta} \left(\kappa + \kappa_\circ 2^\beta |z|^d (\varphi(1/|z|))^{-1}\right),\end{aligned}$$

where $\kappa = \max\{(\varphi(1/2))^{-1}, 1\}$. Here we use $(\varphi(1/|x_0 - z|))^{-1} \leq (\varphi(1/2))^{-1}$, since $|x_0 - z| < 2$. Again, for $|z| > 1$, we use (5.4.5) to obtain

$$
\begin{aligned}
|x_0 - z|^d (\varphi(1/|x_0 - z|))^{-1} &\leq (1 + |z|)^d \kappa_\circ \left(\frac{1 + |z|}{|z|}\right)^\beta (\varphi(1/|z|))^{-1} \\
&\leq \kappa_\circ 2^\beta (1 + |z|^d)(\varphi(1/|z|))^{-1} \\
&\leq \kappa_\circ 2^{\beta+1} |z|^d (\varphi(1/|z|))^{-1} \\
&\leq \kappa_\circ 2^{\beta+1} (1 + |z|^d (\varphi(1/|z|))^{-1}) \,.
\end{aligned}
$$

This gives us (5.4.4).

Let $\chi \in C_c^\infty(B_{\frac{3}{4}})$ be such that $0 \leq \chi \leq 1$ and $\chi = 1$ in $B_{\frac{1}{2}}$. Let $t > 0$ be the maximum value for which $u \geq t\chi$. It is easily seen that $t \leq \inf_{B_{\frac{1}{2}}} u$. Since u and χ are continuous in B_1 there exists $x_0 \in B_{\frac{3}{4}}$ such that $u(x_0) = t\chi(x_0)$. We also get

$$
\tilde{\mathcal{M}}^-(u - t\chi)(x_0) \leq \tilde{\mathcal{M}}^- u(x_0) - t\tilde{\mathcal{M}}^- \chi \leq C_0 + Ct \quad \text{in } B_1 \,.
$$

On the other hand, since $u - t\chi \geq 0$ in $\mathbb{R}^d$ and $(u - t\chi)(x_0) = 0$, we also obtain from (5.4.3)-(5.4.4)

$$
\begin{aligned}
\tilde{\mathcal{M}}^-(u - t\chi)(x_0) &= 2\lambda \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{|x_0 - z|^d} \left(\frac{2 - \alpha}{|x_0 - z|^\alpha} + \varphi(1/|x_0 - z|)\right) \mathrm{d}z \\
&\geq 2(2 - \alpha)\lambda \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{|x_0 - z|^{d+\alpha}} \mathrm{d}z \\
&\qquad + 2\lambda \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{|x_0 - z|^d (\varphi(1/|x_0 - z|))^{-1}} \mathrm{d}z \\
&\geq 2(2 - \alpha)\lambda\kappa \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{1 + |z|^{d+\alpha}} \mathrm{d}z + 2\lambda\kappa \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{1 + |z|^d (\varphi(1/|z|))^{-1}} \mathrm{d}z \\
&\geq C_1 \left(\int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^{d+\alpha}} \mathrm{d}z + \int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^d (\varphi(1/|z|))^{-1}} \mathrm{d}z\right) - C_2 t \,,
\end{aligned}
$$

for some constants C_1, C_2. Combining we get

$$
\begin{aligned}
(C + C_2) \inf_{B_{\frac{1}{2}}} u &\geq (C + C_2)t \\
&\geq -C_0 + C_1 \left(\int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^{d+\alpha}} \mathrm{d}z + \int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^d (\varphi(1/|z|))^{-1}} \mathrm{d}z\right) \,,
\end{aligned}
$$

and the result follows. □

Using Theorem 5.3.2 and Lemma 5.4.1 we obtain the following

Lemma 5.4.2. *Let $\Omega \subset \mathbb{R}^d$ be any open set. Suppose there exists $x_0 \in B_{\frac{1}{2}}$ and $\varrho > 0$ such that $B_{2\varrho}(x_0) \subset \Omega \cap B_{\frac{1}{2}}$. Denote by $D = B_\varrho(x_0)$. Let $u \in C(B_1)$ be a viscosity solution of*

$$\begin{aligned} \tilde{\mathcal{M}}^+ u \geq -C_0 \quad \text{and} \quad \tilde{\mathcal{M}}^- u \leq C_0 \quad &\text{in} \quad B_1 \cap \Omega, \\ u = 0 \quad &\text{in} \quad B_1 \setminus \Omega\,. \end{aligned}$$

Assume in addition, that $u \geq 0$ in $\mathbb{R}^d$. Then

$$\sup_{B_{\frac{3}{4}}} u \leq C \left(\inf_D u + C_0 \right),$$

where constant C depends only on $(d, \lambda, \Lambda, \alpha, \varphi, \varrho)$.

Proof. Since $u \geq 0$ in B_1 and $\tilde{\mathcal{M}}^+ u \geq -C_0$ in $B_1 \cap \{u > 0\}$, we have $\tilde{\mathcal{M}}^+ u \geq -C_0$ in all of B_1. Thus, by Theorem 5.3.2 and a standard covering argument, we have

$$\sup_{B_{\frac{3}{4}}} u \leq C \left(\int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^{d+\alpha}} \mathrm{d}y + \int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^d (\varphi(1/|y|))^{-1}} \mathrm{d}y + C_0 \right).$$

Again, using Lemma 5.4.1 in the ball $B_{2\varrho}(x_0)$, we get

$$\int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^{d+\alpha}} \mathrm{d}y + \int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^d (\varphi(1/|y|))^{-1}} \mathrm{d}y \leq C \left(\inf_D u + C_0 \right),$$

where $D = B_\varrho(x_0)$. Combining the previous estimates, the result follows. □

Finally, we prove Theorem 5.4.1

Proof of Theorem 5.4.1. Proof follows by following the arguments of [147, Theorem 1.2] and using Lemmas 5.4.1 and 5.4.2. □

6

Nonlocal Fisher-KPP model

One of the most celebrated reaction-diffusion models was introduced by Fisher [89] and Kolmogorov, Petrovsky and Piskunov [117] in 1937 (popularly known as Fisher-KPP model). Since then, it has been widely used to model spatial propagation or spreading of biological species into homogeneous environments (see books [134, 138] for a review). The corresponding equation is given by

$$(\partial_t - \nu\Delta)u(x,t) = au(1 - \frac{u}{N}) \quad \text{in } \Omega \times (0,T), \quad u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,T],$$

where $u = u(x,t)$ represents the population density at the space-time point (x,t), ν is the diffusion parameter, $N > 0$ is the carrying capacity of the environment. Imposing the solution to vanish outside the domain Ω corresponds to a confinement situation, for instance in a hostile environment. Various generalizations to the above model have been studied both in bounded and unbounded domains.

However, it is recently observed that the heat operator may be too restrictive to describe the spreading of species and for this reason a nonlocal operator may be more useful than a local one, see for instance Berestycki-Coville-Vo [14], Humphries et al. [103], Huston et al. [104], Massaccesi-Valdinoci [126], Viswanathan et al. [161]. On the other hand, starting from the seminal work of Caffarelli-Silvestre [50] the theory of fractional Laplacian has significantly expanded in many directions and there is a large existing literature for this operator. The fractional Laplacian operators have been extensively used for mathematical modelling, for instance anomalous diffusion [47, 160], crystal dislocation [76], water waves [45]. However, there are other types of nonlocal operators that are also of importance. For instance, relativistic operators appearing in quantum mechanics [3, 80], sum of fractional Laplacians of different order appearing in the modelling of acoustic wave propagation in attenuating media [163]. This calls for consideration of a general family of Lévy operators (including the above mentioned nonlocal operators) for which a unified theory can be developed. This motivates us to study positive solutions to the following nonlocal logistic equation

$$\begin{aligned} \Psi(-\Delta)\, u &= au - f(x,u) - ch(x,u) \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{in } \Omega^c\,, \end{aligned}$$

where $a, c \in \mathbb{R}$, h represents the harvesting term and $\Psi(-\Delta)$ denotes the class of non-local operators which are generators of a large family of Lévy processes, known as subordinate Brownian motions. These processes are obtained by a time change of a Brownian motion by independent subordinators. We briefly recall the essentials of the subordinate process which will be particularly used in this chapter.

Subordinate Brownian motion : A Bernstein function is a non-negative completely monotone function, that is, an element of the set

$$\mathcal{B} = \left\{ f \in C^\infty((0,\infty)) : \, f \geq 0 \;\text{ and }\; (-1)^n \frac{\mathrm{d}^n f}{\mathrm{d}x^n} \leq 0, \text{ for all } n \in \mathbb{N} \right\}.$$

In particular, Bernstein functions are increasing and concave. We will consider the following subset

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \, \lim_{x \downarrow 0} f(x) = 0 \right\}.$$

For a detailed discussion of Bernstein functions we refer to the monograph [150]. Bernstein functions are closely related to subordinators. Recall that a subordinator $\{S_t\}_{t\geq 0}$ is a one-dimensional, non-decreasing Lévy process defined on some probability space $(\Omega_S, \mathcal{F}_S, \mathbb{P}_S)$. The Laplace transform of a subordinator is given by a

Bernstein function, i.e.,

$$\mathbb{E}_{\mathbb{P}_S}[e^{-xS_t}] = e^{-t\Psi(x)}, \quad t, x \geq 0,$$

where $\Psi \in \mathcal{B}_0$. In particular, there is a bijection between the set of subordinators on a given probability space and Bernstein functions with vanishing right limits at zero.

Let B be an $\mathbb{R}^d$-valued Brownian motion on the Wiener space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$, running twice as fast as standard d-dimensional Brownian motion, and let S be an independent subordinator with characteristic exponent Ψ. The random process

$$\Omega_W \times \Omega_S \ni (\omega_1, \omega_2) \mapsto B_{S_t(\omega_2)}(\omega_1) \in \mathbb{R}^d\,,$$

is called subordinate Brownian motion under S. For simplicity, we will denote a subordinate Brownian motion by $\{X_t\}_{t\geq 0}$, its probability measure for the process starting at $x \in \mathbb{R}^d$ by $\mathbb{P}_x$, and expectation with respect to this measure by $\mathbb{E}_x$. Note that the characteristic exponent of a pure jump process $\{X_t\}_{t\geq 0}$ is given by

$$\Psi(|z|^2) = \int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos(y\cdot z)) j(|y|)\, \mathrm{d}y,$$

where the Lévy measure of $\{X_t\}_{t\geq 0}$ has a density $y \mapsto j(|y|)$, $j : (0,\infty) \to (0,\infty)$, with respect to the Lebesgue measure, given by

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}}\, \mathfrak{m}(\mathrm{d}t),$$

where $\mathfrak{m}$ is the unique measure on $(0,\infty)$ satisfying [150, Theorem 3.2]

$$\Psi(s) = \int_{(0,\infty)} (1 - e^{-st})\mathfrak{m}(\mathrm{d}t).$$

In particular, we have

$$\int_{\mathbb{R}^d} (|y|^2 \wedge 1)\, j(|y|)\, \mathrm{d}y < \infty.$$

The operator $-\Psi(-\Delta)$ is defined by

$$
\begin{aligned}
-\Psi(-\Delta)\, f(x) &= \frac{1}{2}\int_{\mathbb{R}^d} \left(f(x+y)+f(x-y)-2f(x)\right) j(|y|)\,\mathrm{d}y \qquad (6.0.1)\\
&= \int_{\mathbb{R}^d} (f(x+y)-f(x)-\mathbb{1}_{\{|y|\le 1\}} y\cdot Df(x)) j(|y|)\,\mathrm{d}y,
\end{aligned}
$$

which is classically defined for $f\in C_b^2(\mathbb{R}^d)$. Also, $-\Psi(-\Delta)$ is the generator of the strong Markov process $\{X_t\}_{t\ge 0}$ we introduced above.

In this chapter, we impose the following *weak scaling condition* on the subordinators.

$$
\begin{gathered}
\text{There are } 0<\kappa_1\le\kappa_2<1\le b_1 \text{ such that}\\
\frac{1}{b_1}\left(\frac{R}{r}\right)^{\kappa_1} \le \frac{\Psi(R)}{\Psi(r)} \le b_1\left(\frac{R}{r}\right)^{\kappa_2} \quad \text{for } 1\le r\le R<\infty,
\end{gathered} \tag{A1}
$$

and,

$$
\text{there is } b_2>1 \text{ such that} \quad j(r)\le b_2\, j(r+1) \quad \text{for } r\ge 1. \tag{A2}
$$

There is large family of subordinators that satisfy (A1) (see [32, 112]). Moreover, any complete Bernstein function (see [150, Definition 6.1]) satisfying (A1) also satisfies (A2) ([113, Theorem 13.3.5], [114]). The conditions (A1)-(A2) are imposed throughout this chapter without any further mention. It is also helpful to keep in mind that for any $c>0$ we have

$$
j(r)\asymp \frac{\Psi(r^{-2})}{r^d} \quad \text{for } 0<r<c\,,
$$

where the comparison constants might depend on c and whenever (A1) holds for all $R\ge r>0$ then we may take $c=\infty$ (see [42]).

Example 6.0.1. Some important examples of complete Bernstein functions Ψ satisfying (A1) are given by

(i) $\Psi(x)=x^{\alpha/2}$, $\alpha\in(0,2]$, with $\kappa_1=\kappa_2=\frac{\alpha}{2}$;

(ii) $\Psi(x)=(x+m^{2/\alpha})^{\alpha/2}-m$, $m>0$, $\alpha\in(0,2)$, with $\kappa_1=\kappa_2=\frac{\alpha}{2}$;

(iii) $\Psi(x)=x^{\alpha/2}+x^{\beta/2}$, $\alpha,\beta\in(0,2]$, with $\kappa_1=\frac{\alpha}{2}\wedge\frac{\beta}{2}$, and $\kappa_2=\frac{\alpha}{2}\vee\frac{\beta}{2}$;

(iv) $\Psi(x) = x^{\alpha/2}(\log(1+x))^{-\beta/2}$, $\alpha \in (0,2]$, $\beta \in [0,\alpha)$ with $\kappa_1 = \frac{\alpha-\beta}{2}$ and $\kappa_2 = \frac{\alpha}{2}$;

(v) $\Psi(x) = x^{\alpha/2}(\log(1+x))^{\beta/2}$, $\alpha \in (0,2)$, $\beta \in (0,2-\alpha)$, with $\kappa_1 = \frac{\alpha}{2}$ and $\kappa_2 = \frac{\alpha+\beta}{2}$.

Corresponding to the examples above, the related processes are (i) $\frac{\alpha}{2}$-stable subordinator, (ii) relativistic $\frac{\alpha}{2}$-stable subordinator, (iii) sums of independent subordinators of different indices, etc.

In connection to the examples above, the related $-\Psi(-\Delta)$ operators are (i) $\frac{\alpha}{2}$-fractional Laplacian, (ii) $\frac{\alpha}{2}$-relativistic operator, (iii) sum of fractional Laplacians etc.

One of our main goals is to study existence and multiplicity of solutions for different values of a and c. For $\Psi(-\Delta) = -\Delta$ similar problems have been studied widely in literature (cf. [55, 56, 66, 71, 99, 118, 139, 153]). But for nonlocal situation there are only few results and to the best of our knowledge, all of them consider the case $\Psi(-\Delta) = (-\Delta)^{\alpha/2}$, the fractional Laplacian (cf. [17, 47, 61, 125, 143]). Our results not only generalizes the existing works but also introduces several new methods. Recently, there have been quite a few works studying pde involving $\Psi(-\Delta)$ (cf. [28, 30, 32, 33, 40, 109–112]). We also mention the recent work Biswas-Lőrinczi [34] where several maximum principles and generalized eigenvalue problems for $\Psi(-\Delta)$ have been studied. Our novelty in this work also comes from the study of the long time asymptotic of the parabolic pde

$$\begin{aligned}
(\partial_t - \Psi(-\Delta))u + au - f(x,u) &= 0 \quad \text{in } \Omega \times [0,T), \\
u(x,T) = u_0(x) \ \text{ and } u(x,t) &= 0 \quad \text{in } \Omega^c \times [0,T].
\end{aligned}$$

We use several potential theoretic tools to establish this long time behaviour.

Before we conclude this section let also also mention another type of nonlocal kernel, known dispersal nonlocal kernel, widely used to model nonlocal reaction-diffusion equations (cf. [14, 53, 95, 104] and references therein). It should be noted that dispersal nonlocal kernels are quite different from the nonlocal kernels of $\Psi(-\Delta)$ and therefore, the proof techniques involved in these models are different from ours.

6.1 Preliminaries

In this section we introduce the relation between viscosity solution and the Green function representation. We also gather few results which will later be used to prove our main results.

The viscosity solution of

$$\Psi(-\Delta)\, u = f \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{in } \Omega^c\,, \tag{6.1.1}$$

can be represented using Green function and this representation is going to play a key role in this chapter. Let us recall few notations from Chapter 1 to introduce this representation. Let τ be the first exit time of X from Ω i.e.,

$$\tau = \inf\{t > 0 \;:\; X_t \notin \Omega\}.$$

We define the killed process $\{X_t^\Omega\}$ by

$$X_t^\Omega = X_t \quad \text{if } t < \tau, \quad \text{and} \quad X_t^\Omega = \partial \quad \text{if } t \geq \tau,$$

where ∂ denotes a cemetery point. X_t^Ω has transition density $p_\Omega(t,x,y)$ and its transition semigroup $\{P_t^\Omega\}_{t\geq 0}$ is given by

$$P_t^\Omega f(x) = \mathbb{E}_x[f(X_t)\mathbb{1}_{\{t<\tau\}}] = \int_\Omega f(y) p_\Omega(t,x,y)\, \mathrm{d}y. \tag{6.1.2}$$

The Green function of X^Ω is defined by

$$G^\Omega(x,y) = \int_0^\infty p_\Omega(t,x,y)\, \mathrm{d}t\,.$$

Then the solution of (6.1.1) can be represented as (see [26, Section 3.1],[112])

$$u(x) = \mathcal{G} f(x) := \int_\Omega G^\Omega(x,y) f(y)\, \mathrm{d}y = \mathbb{E}_x\left[\int_0^\tau f(X_t)\, \mathrm{d}t\right], \tag{6.1.3}$$

where the last equality follows from (6.1.2).

For some of our proofs below we will use some information on the normalized ascending ladder-height process of $\{X_t^1\}_{t\geq 0}$, where X_t^1 denotes the first coordinate of X_t. Recall that the ascending ladder-height process of a Lévy process $\{Z_t\}_{t\geq 0}$ is the process of the right inverse $\{Z_{L_t^{-1}}\}_{t\geq 0}$, where L_t is the local time of Z_t reflected at its supremum (for details and further information we refer to [18, Chapter 6]). Also, we note that the ladder-height process of $\{X_t^1\}_{t\geq 0}$ is a subordinator with Laplace exponent

$$\tilde{\Psi}(x) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log \Psi(x^2y^2)}{1+y^2}\,\mathrm{d}y\right), \quad x \geq 0.$$

Consider the potential measure $V(x)$ of this process on the half-line $(-\infty, x)$. Its Laplace transform is given by

$$\int_0^\infty V(x)e^{-sx}\,\mathrm{d}x = \frac{1}{s\tilde{\Psi}(s)}, \quad s > 0.$$

It is also known that $V = 0$ for $x \leq 0$, the function V is continuous and strictly increasing in $(0, \infty)$ and $V(\infty) = \infty$ (see [92] for more details). As shown in [41, Lemma 1.2] and [42, Corollary 3], there exists a constant $C = C(d)$ such that

$$\frac{1}{C}\,\Psi(r^{-2}) \leq \frac{1}{V^2(r)} \leq C\,\Psi(r^{-2}), \quad r > 0. \tag{6.1.4}$$

This function V will appear in several places of this chapter. Let us recall the following up to the boundary regularity result from [112, Theorem 1.1 and 1.2]

Theorem 6.1.1. *Assume* (A1)-(A2) *and $f \in C(\Omega)$. Let u be the solution of* (6.1.1). *Then for some constant C, dependent on d, Ω, Ψ, we have*

$$\|u\|_{C^\phi(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}, \tag{6.1.5}$$

where $\phi = \Psi(r^{-2})^{-\frac{1}{2}}$ and

$$\|u\|_{C^\phi(\Omega)} \coloneqq \|u\|_{C(\Omega)} + \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{\phi(|x-y|)}.$$

Furthermore, there exists α, dependent on d, Ω, Ψ, satisfying

$$\left\| \frac{u}{V(\delta_\Omega)} \right\|_{C^\alpha(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}, \tag{6.1.6}$$

where δ_Ω denotes the distance function from $\partial\Omega$.

Using (A1), $\phi(r) \leq \kappa r^{\kappa_1}$ for $r \leq 1$, for some constant κ, and thus, it follows from (6.1.5) that u is κ_1-Hölder continuous upto the boundary. (6.1.6) provides a fine boundary decay estimate and this should be compared with the results in [144]. Our next result is the Hopf's lemma which we borrow from [34, Theorem 3.3].

Theorem 6.1.2. *Let $u \in C_b(\mathbb{R}^d)$ be a non-negative viscosity solution of*

$$-\Psi(-\Delta)\, u + c(x)u \leq 0 \quad \textit{in } \Omega,$$

where c is a bounded function. Then either $u \equiv 0$ in $\mathbb{R}^d$ or $u > 0$ in Ω. Furthermore, if $u > 0$ in Ω, then there exists $\eta > 0$ satisfying

$$\frac{u(x)}{V(\delta_\Omega(x))} > \eta \quad \textit{for } x \in \Omega\,. \tag{6.1.7}$$

To introduce our next results we required the principal eigenvalue for the operator $-\Psi(-\Delta) + c$ where c is a continuous and bounded function in Ω. The principal eigenvalue is defined in the same fashion as in [16] and given by

$$\lambda(c) \;=\; \sup\{\lambda \;:\; \exists\, \psi \in C_{b,+}(\Omega) \text{ such that } -\Psi(-\Delta)\,\psi + (c(x)+\lambda)\psi \leq 0 \text{ in } \Omega\}. \tag{6.1.8}$$

Note that for $c = 0$ we have $\lambda(0) = \lambda_1$. Next we recall the following refined maximum principle from [34, Theorem 3.4 and Lemma 3.1].

Theorem 6.1.3. *Suppose that $\lambda(c) > 0$ and $v \in C_b(\mathbb{R}^d)$ be a solution to*

$$-\Psi(-\Delta)\, v + cv \geq 0 \quad \textit{in } \Omega, \qquad v \leq 0 \quad \textit{in } \Omega^c.$$

Then we have $v \leq 0$.

Again, if $w \in C_b(\mathbb{R}^d)$ is a solution to

$$-\Psi(-\Delta)\,w + (c(x)+\lambda(c))w \geq 0 \quad \textit{in}\ \Omega, \quad w \leq 0 \quad \textit{in}\ \Omega^c, \quad w(x_0) > 0,$$

for an $x_0 \in \Omega$, then $w = t\varphi^*$ for some $t > 0$, where φ^* denotes the positive principal eigenfunction corresponding to $\lambda(c)$.

The next result is an anti-maximum principle which is slightly stronger than [34, Theorem 3.5].

Theorem 6.1.4. *Let $f \in C(\bar{\Omega})$ and $f \lneq 0$. Then there exists a $\delta > 0$ such that for every $\lambda \in (\lambda(c), \lambda(c)+\delta)$ if u is a solution of*

$$-\Psi(-\Delta)\,u + (c(x)+\lambda)u = f \quad \textit{in}\ \Omega, \quad \textit{and} \quad u = 0 \quad \textit{in}\ \Omega^c\,, \tag{6.1.9}$$

then $\sup_{\Omega} \frac{u(x)}{V(\delta_\Omega(x))} < 0$.

Proof. Using [34, Theorem 3.5] we have a $\delta_1 > 0$ such that for any $\lambda \in (\lambda(c), \lambda(c) + \delta_1)$ if u is a solution to (6.1.9) then $u < 0$ in Ω. Now suppose, on the contrary, that the conclusion of the theorem does not hold. Then we find a sequence of $\delta_n \to 0$ and solution $u_n < 0$ satisfying

$$\max_{\partial\Omega} \frac{u_n(x)}{V(\delta_\Omega(x))} = 0. \tag{6.1.10}$$

First we observe that $\|u_n\|_{L^\infty} \to \infty$ as $n \to \infty$. Otherwise, using the argument of Step 1 in [34, Theorem 3.5] we obtain a solution $u \lneq 0$ of

$$-\Psi(-\Delta)\,u + (c(x)+\lambda^*)u = f(x) \quad \text{in}\ \Omega, \quad u = 0 \quad \text{in}\ \Omega^c.$$

In view of Theorem 6.1.3, we must have $u = t\varphi^*$ for some $t < 0$, where φ^* is the positive Dirichlet principal eigenfunction of $-\Psi(-\Delta) + c$ in Ω. This is not possible since $f \neq 0$. Thus we must have $\|u_n\|_{L^\infty} \to \infty$. Define $v_n = \frac{u_n}{\|u_n\|_{L^\infty}}$. Then the argument of Step 2 in [34, Theorem 3.5] gives us

$$\max_{\bar{\Omega}} \left| \frac{v_n}{V(\delta_\Omega(x))} - \frac{t\varphi^*}{V(\delta_\Omega(x))} \right| \to 0, \quad \text{as}\ n \to \infty,$$

for some $t < 0$. Combining with (6.1.10) we must find a point $x_0 \in \partial\Omega$ such that $\frac{\varphi^*(x_0)}{V(\delta_\Omega(x_0))} = 0$. But $\frac{\varphi^*}{V(\delta_\Omega)}$ can be continuously extended in $\bar\Omega$ (by Theorem 6.1.1) and the extension is positive in $\bar\Omega$, by Theorem 6.1.2. Thus we arrive at a contradiction. Hence we have a $\delta > 0$ as claimed by the theorem. $\square$

Before we conclude this section let us also mention the following implicit function theorem from [70, Appendix]. In the following theorem $\mathcal{X}$, $\mathcal{Y}$ denote Banach spaces.

Theorem 6.1.5. *Let $(s_0, u_0) \in \mathbb{R}\times\mathcal{X}$ and $F : \mathbb{R}\times\mathcal{X} \to \mathcal{Y}$ be continuously differentiable in some some neighbourhood of (s_0, u_0). Assume that $F(s_0, u_0) = 0$. Suppose that $F_u(s_0, u_0)$ is a linear homeomorphism of $\mathcal{X}$ onto $\mathcal{Y}$. Then there is exactly one C^1 function $z : (s_0 - \varepsilon, s_0 + \varepsilon) \to \mathcal{X}$ with $z(s_0) = 0$ satisfying $F(s, u_0 + z(s)) = 0$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ where ε is some positive number.*

6.2 Logistic equation with harvesting

Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ domain. For positive constants a, c we consider the following nonlocal logistic equation with a harvesting term

$$\begin{aligned} \Psi(-\Delta)\, u &= au - f(x,u) - ch(x,u) \quad \text{in } \Omega\,, \\ u &> 0 \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{in } \Omega^c\,, \end{aligned} \tag{6.2.1}$$

where $f : \bar\Omega\times[0,\infty) \to [0,\infty)$, $h : \bar\Omega\times[0,\infty) \to [0,\infty)$ are given continuous functions satisfying

$$\begin{gathered} s \mapsto f(x,s),\ h(x,s) \text{ are continuously differentiable}, f(x,0) = f_s(x,0) = 0, \\ \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{f(x,s)}{s}\right] > 0 \text{ for } s > 0,\ \lim_{s\to\infty}\inf_{x\in\Omega}\frac{f(x,s)}{s} = \infty\,, \\ \text{and } h \text{ is bounded with } \max_{\bar\Omega} h(x,0) > 0\,. \end{gathered} \tag{A3}$$

The goal of this section is to study the positive solutions of (6.2.1). A typical example for f is $b(x)u^2$ where b in a positive continuous function. By a solution of (6.2.1) we mean viscosity solution. As well known, existence of solutions to (6.2.1) is closely connected with the principal eigenvalue of the operator $-\Psi(-\Delta)$. It is also known that there are only countably many eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \to \infty$ satisfying (see [33])

$$-\Psi(-\Delta)\,\varphi_n + \lambda_n\varphi_n = 0 \quad \text{in } \Omega, \quad \text{and} \quad \varphi_n = 0 \quad \text{in } \Omega^c.$$

The first eigenvalue λ_1 is simple and $\varphi_1 > 0$ in Ω. The principal eigenvalue λ_1 also satisfies a Berestycki-Nirenbarg-Varadhan [16] type characterization, that is,

$$\lambda_1 = \sup\{\lambda \ : \ \exists\, \psi \in C_{b,+}(\Omega) \text{ such that } -\Psi(-\Delta)\,\psi + \lambda\psi \leq 0 \text{ in } \Omega\}, \tag{6.2.2}$$

where $C_{b,+}(\Omega)$ denotes the collection of all bounded, non-negative continuous functions on $\mathbb{R}^d$ that are positive inside Ω. Let us start with the main comparison principle required in this section.

Lemma 6.2.1. *Suppose that $g : \bar{\Omega} \times [0,\infty) \to \mathbb{R}$ is a continuous function, locally Lipschitz in its second argument uniformly with respect to the first, such that*

$$\frac{g(x,s)}{s} \text{ is strictly decreasing for } s > 0$$

at each $x \in \Omega$. In addition, also assume that $g(x,0) = 0$ and $g_s(x,0)$ is continuous in $\bar{\Omega}$. Let $u, v \in C_b(\mathbb{R}^d)$ be such that

1. *$-\Psi(-\Delta)\,v + g(x,v) \leq 0 \leq \tilde{g}(x) = -\Psi(-\Delta)\,u + g(x,u)$ in Ω, where $\tilde{g}$ is a continuous function.*

2. *$v > 0$, $u \gneq 0$ in Ω and $v \geq u = 0$ in Ω^c.*

Then we have $v \geq u$ in $\mathbb{R}^d$.

Proof. Let $\varrho = \sup\{t : \ tu < v \text{ in } \Omega\}$. Clearly, $\varrho < \infty$. Also, $\varrho > 0$. Note that by Hopf's lemma, Theorem 6.1.2, we have

$$\inf_{\Omega} \frac{v(x)}{V(\delta_\Omega(x))} \geq \eta > 0,$$

and by (6.1.6)

$$\sup_{x\in\Omega} \left|\frac{u(x)}{V(\delta_\Omega(x))}\right| \leq \eta_1 \tag{6.2.3}$$

for some $\eta_1 > 0$. Thus for some small $t_0 > 0$ we would have $v > t_0 u$ in Ω, giving us $\varrho \geq t_0 > 0$. To complete the proof it is enough to show that $\varrho \geq 1$. On the contrary, we suppose that $\varrho < 1$. Let $w = \varrho u$. Since $\frac{g(x,s)}{s}$ is strictly decreasing for $s > 0$ we have

$$\begin{aligned} -\Psi(-\Delta)\, w + g(x,w) &= -\Psi(-\Delta)\, w + \frac{g(x,\varrho u)}{\varrho}\varrho \\ &\gneq \varrho[-\Psi(-\Delta)\, u + g(x,u)] \geq 0 \quad \text{in } \Omega\,, \end{aligned} \tag{6.2.4}$$

Applying [50, Lemma 5.8] we then have

$$-\Psi(-\Delta)(v-w) + g(x,v) - g(x,w) \leq 0 \quad \text{in } \Omega,$$

which in turn, gives

$$-\Psi(-\Delta)(v-w) + \left(\frac{g(x,v)-g(x,w)}{v-w}\right)(v-w) \leq 0 \quad \text{in } \Omega.$$

Applying Hopf's lemma, Theorem 6.1.2, we have either $v - w = 0$ in $\mathbb{R}^d$ or $\inf_\Omega \frac{v(x)-w(x)}{V(\delta_\Omega(x))} > \eta$. The first option is not possible due to (6.2.4). Again, if the second option holds, then using (6.2.3) we can find $t_1 > 0$ satisfying $u - w > t_1 u$ in Ω implying $v > (\varrho + t_1)u$ in Ω. This contradicts the definition of ϱ. Hence we must have $\varrho \geq 1$. □

Before we state our first main result we recall the notion of stability for a solution u to the boundary value problem

$$\begin{aligned} -\Psi(-\Delta)\, u + g(x,u) &= 0 \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{in } \Omega^c\,. \end{aligned} \tag{6.2.5}$$

A solution u of (6.2.5) is said to be a *stable solution* if the Dirichlet principal eigenvalue of the operator $-\Psi(-\Delta) + g_s(x,u)$ is positive, otherwise we say u is an *unstable solution*. Let us now state our first main result that is obtained in [36] which is about the logistic equation (i.e., $h = 0$).

Theorem 6.2.1. *The logistic equation*

$$\begin{aligned} \Psi(-\Delta)\, u &= au - f(x,u) \quad \textit{in}\ \Omega\,, \\ u &> 0 \quad \textit{in}\ \Omega\,, \\ u &= 0 \quad \textit{in}\ \Omega^c\,, \end{aligned} \tag{6.2.6}$$

has no solution for $a \leq \lambda_1$ and has exactly one solution v_a for $a > \lambda_1$. Furthermore, the function $(\lambda_1, \infty) \ni a \mapsto v_a$ is continuous, increasing and v_a is stable.

Proof. Recall that (λ_1, φ_1) is the Dirichlet principal eigenpair, that is,

$$\begin{aligned} -\Psi(-\Delta)\, \varphi_1 + \lambda_1 \varphi_1 &= 0 \text{ in } \Omega, \\ \varphi_1 &= 0 \text{ in } \Omega^c\,. \end{aligned} \tag{6.2.7}$$

Suppose that $a < \lambda_1$ and v is a positive solution of (6.2.6). Then

$$-\Psi(-\Delta)\, v + av = \frac{f(x,v)}{v} v \geq 0 \quad \text{in } \Omega$$

since $\frac{f(x,s)}{s} \geq 0$ for $s \geq 0$. Applying the refined maximum principle Theorem 6.1.3 we get $v \leq 0$ in $\mathbb{R}^d$ which is a contradiction.

Similarly, if v is a positive solution with $a = \lambda_1$, we obtain $-\Psi(-\Delta)\, v + \lambda_1 v = \frac{f(x,v)}{v} v \geq 0$ in Ω. Applying second part of Theorem 6.1.3 we have $v = t\varphi_1$ for some $t > 0$ which would imply

$$-\Psi(-\Delta)\, \varphi_1 + \lambda_1 \varphi_1 - t^{-1} f(x, t\varphi_1) = 0 \quad \text{in } \Omega,$$

giving us

$$-f(x, t\varphi_1) = 0 \quad \text{in } \Omega\,.$$

This is not possible since $t\varphi_1 > 0$ in Ω. Thus we have established that no positive solution is possible for $a \leq \lambda_1$.

Next we consider the case where $a > \lambda_1$. Existence of solution would be proved using a standard monotone iteration method. To do so we need to construct a subsolution and supersolution. Let $\underline{u} = k\varphi_1$ where $k \in (0,1)$. Then we obtain from

(6.2.7) that

$$
\begin{aligned}
-\Psi(-\Delta)\,\underline{u} + a\underline{u} - f(x,\underline{u}) &= (a-\lambda_1)\underline{u} - f(x,\underline{u}) \\
&= \underline{u}\left((a-\lambda_1) - \frac{f(x,k\varphi_1)}{k\varphi_1}\right) \quad \text{in } \Omega\,.
\end{aligned}
$$

Since by mean value theorem $\frac{f(x,q)}{q} = f_s(x,r)$ for some $r \in (0,q)$ and $f_s(x,0) = 0$, by choosing k small we would easily have

$$
\left((a-\lambda_1) - \frac{f(x,k\varphi_1)}{k\varphi_1}\right) > 0 \quad \text{in } \Omega.
$$

Thus we obtain a subsolution $\underline{u}$. Again, since

$$
\lim_{s\to\infty} \inf_{x\in\Omega} \frac{f(x,s)}{s} = \infty\,,
$$

there exist large $M > \|\underline{u}\|_{C(\Omega)}$ satisfying $\frac{f(x,M)}{M} \geq a$ for all x in Ω. Fixing $v = M$ we get

$$
-\Psi(-\Delta)\,v + av - f(x,v) \leq 0 \quad \text{in } \Omega\,.
$$

Thus v is a super-solution. Now the existence of a solution is standard using monotone iteration method. Let us just sketch the argument. Define $H(x,u) = au - f(x,u)$ and let $\theta > 0$ be a Lipschitz constant for $H(x,\cdot)$ on the interval $[0,M]$, i.e.,

$$
|H(x,q_1) - H(x,q_2)| \leq \theta |q_1 - q_2| \quad \text{for } q_1, q_2 \in [0,M]\,,\ x \in \Omega\,.
$$

Now consider the solutions of the following family of problems:

$$
\begin{aligned}
-\Psi(-\Delta)\,u^{n+1} - \theta u^{n+1} &= -H(x,u^n) - \theta u^n \quad x \in \Omega, \\
u^{n+1} &= 0 \quad x \in \Omega^c\,,
\end{aligned}
$$

with $u^0 = \underline{u}$. It is standard to check that $u^0 \leq u^1 \leq u^2 \leq \cdots \leq v$. Applying Theorem 6.1.1 and Arzelà-Ascoli thereom it can be shown that the sequenece converges uniformly in $\mathbb{R}^d$ to a limit $v_a \geq \underline{u}$ and v_a is a viscosity solution to (6.2.6).

See [28, Lemma 3.3] for more details. Uniqueness of solution to (6.2.6) follows from Lemma 6.2.1.

Next we prove stability of the solution v_a for $a > \lambda_1$. Note that given $a_2 \geq a_1 > \lambda_1$ we have

$$\Psi(-\Delta)\, v_{a_2} \geq a_1 v_{a_2} - f(x, v_{a_2}) \quad \text{in } \Omega\,.$$

Therefore, by Lemma 6.2.1, we have $v_{a_1} \leq v_{a_2}$. Again, due to Theorem 6.1.1, it can easily be shown that $a \mapsto v_a$ is continuous.

Fix $a > \lambda_1$ and define $w = (1+h)v_a$ for $h > 0$. Since

$$(1+h)f(x,s) < f(x,(1+h)s) \quad \text{for } s \geq 0,\ x \in \Omega\,,$$

we have

$$-\Psi(-\Delta)\, w + aw - f(x,w) \leq 0 \quad \text{in } \Omega.$$

Using [50, Lemma 5.8] we then obtain

$$\begin{aligned} &-\Psi(-\Delta)(hv_a) + a(hv_a) - f(x,w) + f(x,v_a) \\ &\quad = -\Psi(-\Delta)(w - v_a) - a(w - v_a) - f(x,w) + f(x,v_a) \leq 0 \quad \text{in } \Omega. \end{aligned}$$

Dividing by h on both sides we get

$$-\Psi(-\Delta)\, v_a + av_a - \left[\frac{f(x,w) - f(x,v_a)}{hv_a}\right] v_a \leq 0 \quad \text{in } \Omega.$$

Letting $h \to 0$ and using the stability property of viscosity solutions [50, Lemma 4.5] we obtain

$$-\Psi(-\Delta)\, v_a + av_a - f_s(x,v_a)v_a \leq 0 \quad \text{in } \Omega.$$

Then it follows from (6.1.8) that the principal eigenvalue λ^* of the operator $-\Psi(-\Delta) + a - f_s(x,v_a)$ is non-negative. Now suppose $\lambda^* = 0$. Then from the proof of [34, Theorem 3.2] (see the last part of the proof) we get that v_a is a principal eigenfunction i.e.,

$$-\Psi(-\Delta)\, v_a + av_a - f_s(x,v_a)v_a = 0 \quad \text{in } \Omega.$$

Combining with (6.2.6) we have $f_s(x, v_a)v_a = f(x, v_a)$ for all $x \in \Omega$. But by (A3) we have $sf_s(x, s) - f(x, s) > 0$ for all $s > 0$. Thus we have a contradiction, giving us $\lambda^* > 0$. This completes the proof. □

When $\Psi(-\Delta) = -\Delta$, Theorem 6.2.1 is well known. See for instance Oruganti, Shi and Shivaji [139, Theorem 2.5]. For $\Psi(-\Delta) = (-\Delta)^{\alpha/2}$ (i.e., the fractional Laplacian), similar result (without stability analysis of solutions) is obtained recently by Marinelli-Mugani [125, Proposition 4.2] using a variational technique (see also Chhetri-Girg-Hollifield [61, Theorem 2.8]). We also refer to the work of Berestycki, Roquejoffre and Rossi [17, Theorem 1.2] which establishes a similar result for the fractional Laplacian for a periodic patch model in $\mathbb{R}^d$. We not only obtain uniqueness of solutions but also establish the result for a large class of Lévy operators. It should also be noted that we work in the framework of viscosity solution and therefore, the standard variational technique (as used in [17,61,139]) does not work here. Also, our approach is quite robust in the sense that it can also be applied to non-translation invariant operators and non self-adjoint operators.

For the remaining part of this section we consider the equation with the harvesting term h:

$$\begin{aligned} \Psi(-\Delta)\, u &= au - f(x,u) - ch(x,u) \quad \text{in } \Omega\,, \\ u &> 0 \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{in } \Omega^c\,, \end{aligned} \tag{6.2.8}$$

where h satisfies the conditions in (A3). We will study the existence of positive solutions. Note that we allow h to depend on u. One such popular example is the predation function $h(x, s) = \frac{s}{1+s}$, although our approach does not cover this particular function. The case $h(x, s) = h(x)$ is known as constant yield harvesting. Letting $F(x, u) = au - f(x, u) - ch(x, u)$ in (6.2.8) we see that $F(x, 0) \le 0$. Such problems are known as semipositone problems, see [55, 56, 71, 153] and references therein. When $\Psi(-\Delta) = -\Delta$, existence and multiplicity of solutions to (6.2.8) have been widely studied; see for instance, Korman-Shi [118], Oruganti-Shi-Shivaji [139], Costa-Drábek-Tehrani [66], Girão-Tehrani [99] and references therein.

We start with the following lemma about non-existence.

Lemma 6.2.2. *The following hold.*

(i) If $a \leq \lambda_1$ and $c \geq 0$ then equation (6.2.8) has no non negative solution except $u = 0$ when $c = 0$.

(ii) Suppose that $\inf_{s \in [0,K]} h(\cdot, s) \gneq 0$ for any $K > 0$. Then for $a > \lambda_1$, there exists $M > 0$ such that equation (6.2.8) has no nonzero non-negative solution when $c > M$.

Proof. First we consider (i). Note that

$$-\Psi(-\Delta)\, u + au = f(x,u) + ch(x,u) \geq 0 \quad \text{in } \Omega.$$

Then the arguments of Theorem 6.2.1 shows that there is no non-negative u satisfying above equation when $a \leq \lambda_1$.

(ii) Fix $a > \lambda_1$. We will prove theorem by contradiction. Assume that there exists positive increasing sequence $c_n \to \infty$ and solution $u_n \gneq 0$ to (6.2.8). We claim that for any non-negative solution u to (6.2.8), we have

$$\|u\|_{L^\infty} \leq K, \tag{6.2.9}$$

for some K and all $c \geq 0$. Since $\lim_{s \to \infty} \inf_{x \in \Omega} \frac{f(x,s)}{s} = \infty$ there exist large $K > 0$ such that $\frac{f(x,K)}{K} \geq a$ for all x in Ω. Taking $v = K$ we get

$$\begin{aligned} -\Psi(-\Delta)\, v + av - f(x,v) &= aK - f(x,K) \\ &= K\left(a - \frac{f(x,K)}{K}\right) \leq 0 \quad \text{in } \Omega\,. \end{aligned}$$

So v is a super-solution. Thus

$$-\Psi(-\Delta)\, v + av - f(x,v) \leq 0 \leq ch(x,u) = -\Psi(-\Delta)\, u + au - f(x,u) \quad \text{in } \Omega\,.$$

Using Lemma 6.2.1 we obtain (6.2.9). Now dividing both sides of (6.2.8) by c_n we have

$$-\Psi(-\Delta)\left(\frac{u_n}{c_n}\right) + a\frac{u_n}{c_n} - \frac{f(x,u_n)}{c_n} + \min_{s \in [0,K]} h(x,s)$$

$$\leq -\Psi(-\Delta)\left(\frac{u_n}{c_n}\right) + a\frac{u_n}{c_n} - \frac{f(x,u_n)}{c_n} + h(x,u_n) = 0 \quad \text{in } \Omega\,.$$

Since $\frac{u_n}{c_n}$ and $\frac{f(x,u_n)}{c_n}$ converges to 0 as $c_n \to \infty$ we get from above that $\min_{s\in[0,K]} h(x,s) = 0$ which is a contradiction. Hence the result. □

Next we prove existence of solution for small values of c.

Lemma 6.2.3. *Fix $a > \lambda_1$. Then there exist c_1 such that for $c \in (0, c_1)$ equation (6.2.8) has a solution u satisfying $u \geq m\beta\varphi_1$ where m, β are independent of $c \in (0, c_1)$.*

Proof. We will prove existence of a positive solution using a monotone iteration method. Let v be the unique solution of

$$\begin{aligned} \Psi(-\Delta)\, v &= 1 \quad \text{in } \Omega\,,\\ v &= 0 \quad \text{in } \Omega^c. \end{aligned}$$

From maximum principle it is evident that $v > 0$ in Ω. Also, recall the principal eigenfunction φ_1 from (6.2.7). Using Theorems 6.1.1 and 6.1.2 we obtain that

$$\left|\frac{v(x)}{V(\delta_\Omega(x))}\right| \leq \eta_1, \quad \eta_2 \leq \frac{\varphi_1(x)}{V(\delta_\Omega(x))} \leq \eta_3,\ x\in\Omega\,, \quad \eta_1,\,\eta_2,\,\eta_3 > 0\,. \tag{6.2.10}$$

Thus

$$\varphi_1(x) \geq \frac{\eta_2}{\eta_1} v(x) \quad x \in \Omega\,.$$

Taking $\varepsilon(\beta) = (1-\beta)\frac{\eta_2}{\eta_1}$ we get $\varphi_1 - \varepsilon v \geq \beta\varphi_1$. Define $\phi = m(\varphi_1 - \varepsilon v)$. Note that $\phi \geq m\beta\varphi_1$. Now

$$\begin{aligned} -\Psi(-\Delta)\,\phi + a\phi - f(x,\phi) - ch(x,\phi) &= -\lambda_1 m\varphi_1 + m\varepsilon + a\phi - f(x,\phi) - ch(x,\phi)\\ &\geq -\frac{\lambda_1}{\beta}\phi + a\phi - f(x,\phi) + m\varepsilon - c\|h\|_{L^\infty}\\ &\geq \left(a - \frac{\lambda_1}{\beta} - \frac{f(x,\phi)}{\phi}\right)\phi + m\varepsilon - c\|h\|_{L^\infty}\,. \end{aligned}$$

Now choose $\beta \in (\frac{\lambda_1}{a}, 1)$ and then choose m small so that

$$\frac{f(x,\phi)}{\phi} \leq a - \frac{\lambda_1}{\beta} \quad \text{in } \Omega.$$

Then for any $c \leq m\|h\|_{L^\infty}^{-1}\varepsilon = \|h\|_{L^\infty}^{-1}(1-\beta)\frac{\eta_2}{\eta_1}m := c_1$ we have

$$-\Psi(-\Delta)\,\phi + a\phi - f(x,\phi) - ch(x,\phi) \geq 0 \quad \text{in } \Omega\,.$$

Thus we have a subsolution for all $c \leq c_1$. Again, as shown in Lemma 6.2.2, we can choose a K to serve a supersolution. Then using a standard monotone iteration method (same as in Theorem 6.2.1) we can obtain a solution u to (6.2.8) satisfying $u \geq \phi$. □

Using Lemmas 6.2.2 and 6.2.3 we obtain the following.

Theorem 6.2.2. *Suppose that $a > \lambda_1$ and $\inf_{s\in[0,K]} h(\cdot, s) \gneq 0$ in Ω for every $K > 0$. Then there exists $c_\circ \geq c_1$ such that*

(i) for $0 < c < c_\circ$, (6.2.8) has a maximal positive solution $u_1(x,c)$ such that for any solution $v(x,c)$ of (6.2.8) we have $u_1 \geq v$. Furthermore,

$$\lim_{c\to 0+} \|u_1(\cdot,c) - v_a\|_{C(\Omega)} = 0; \tag{6.2.11}$$

(ii) for $c > c_\circ$, (6.2.8) has no positive solution.

Proof. (i) From Theorem 6.2.1, we know that (6.2.6) has a unique positive solution v_a when $a > \lambda_1$. Let u be any nonnegative solution of (6.2.8). Then

$$-\Psi(-\Delta)\,v_a + av_a - f(x,v_a) = 0 < ch(x,u) = -\Psi(-\Delta)\,u + au - f(x,u) \quad \text{in } \Omega\,.$$

Since $u = v_a = 0$ in Ω^c, using Lemma 6.2.1 we have that $u \leq v_a$ in $\mathbb{R}^d$. Thus whenever (6.2.8) has a nonnegative solution for some c, we can construct maximal solution of $u_1(\cdot, c)$ for the same parameter c as follows: we take v_a as a supersolution of (6.2.8), any solution u as a subsolution, and start the monotone iteration sequence starting from v_a. Then we obtain a solution u_1 in between v_a and u; in particular,

$u_1 \geq u$. Since u can be any solution, the limit of the iterated sequence starting from v_a is the maximal solution.

$$c_\circ = \sup\{c > 0 : (6.2.8) \text{ has a solution with this } c\}.$$

From Lemma 6.2.3 it is clear that $c_\circ \geq c_1$. Now we show that for any $c \in (0, c_\circ)$, (6.2.8) has a solution. Then from previous argument we can construct maximal solution for any $c \in (0, c_\circ)$. Fix $c \in (0, c_\circ)$. By definition of $c_\circ$ we can find $c' > c$ such that (6.2.8) has a solution u for c'. This also implies

$$-\Psi(-\Delta)\, u + au - f(x, u) - ch(x, u) \geq 0 \quad \text{in } \Omega\,.$$

Since $v_a \geq u$ using a monotone iteration argument we can find a solution for the parameter c. Now to show (6.2.11) we observe from Lemma 6.2.3 that for $c \in (0, c_1)$

$$m\beta\varphi_1 \leq u_1(x, c) \leq v_a \quad \text{in } \Omega. \tag{6.2.12}$$

Applying Theorem 6.1.1 we see that the family $\{u_1(\cdot, c)\}_{c \leq c_1}$ is equicontinuous and any limit point $\xi \in C(\mathbb{R}^d)$,as $c \to 0+$, would solve

$$\Psi(-\Delta)\, \xi = a\xi - f(x, \xi) \quad \text{in } \Omega.$$

From (6.2.12) it follows that $\xi > 0$ in Ω. Thus, by Theorem 6.2.1, $\xi = v_a$. This gives us (6.2.11).

(ii) follows from the definition of $c_\circ$. □

We obtain the following bifurcation result for equation (6.2.8).

Theorem 6.2.3. *Suppose that $a > \lambda_1$ and $\inf_{s \in [0,K]} h(\cdot, s) \gneq 0$ in Ω for every $K > 0$. Then the following hold.*

(i) There exists a positive constant $c_\circ$ such that (6.2.8) has a maximal solution $u_1(x, c)$ for $c < c_\circ$.

(ii) There is no solution for $c > c_\circ$.

(iii) *There exist positive* $\delta, \tilde{c}$ *such that for every* $a \in (\lambda_1, \lambda_1 + \delta)$ *there exists a solution* $u_2(x, c)$ *to* (6.2.8) *for each* $c \in (0, \tilde{c})$ *and* $u_2 \lneq u_1$. *Furthermore,* $\lim_{c \to 0+} \|u_2(\cdot, c)\|_{C(\Omega)} = 0$.

(iv) *There exists* $\hat{c} \in (0, \tilde{c})$ *so that for any* $a \in (\lambda_1, \lambda_1 + \delta)$, u_1, u_2 *are the only solutions to* (6.2.8) *for* $0 < c \leq \hat{c}$.

Proof. (i) and (ii) follows from Theorem 6.2.2. So we consider (iii). The main idea of this proof is to use Theorem 6.1.5 but due to lack of appropriate Schauder type estimate we can not apply the theorem on the forward operator. Recall the Green operator $\mathcal{G}$ associated to the Dirichlet problem (6.1.1), that is,

$$\mathcal{G}f(x) := \int_{\Omega} G^{\Omega}(x, y) f(y) \, \mathrm{d}y = \mathbb{E}_x \left[\int_0^{\tau} f(X_t) \mathrm{d}t \right]. \tag{6.2.13}$$

In view of (6.1.5), $\mathcal{G} : C_0(\Omega) \to C_0(\Omega)$ is a compact, bounded linear operator. By $C_0(\Omega)$ we denote the space of all continuous functions in $\bar{\Omega}$ vanishing on the boundary. Now extend h on $\bar{\Omega} \times \mathbb{R}$ by defining $h(x, s) = h(x, 0) + s h_s(x, 0)$. Then $s \mapsto h(x, s)$ is C^1. We define $F : \mathbb{R} \times C_0(\Omega) \to C_0(\Omega)$ by

$$F(c, u) = \mathcal{G}(au - f(x, u) - ch(x, u)) - u.$$

Since $\mathcal{G}$ is linear, it is clear that F is continuously differentiable in a neighbourhood of $(0, 0)$. In particular,

$$DF(c, u)(c_1, w) = \mathcal{G}(aw - f_s(x, u)w - c_1 h(x, u) - c h_s(x, u)w) - w.$$

Also, $F(0, 0) = 0$. Define $Tw := F_u(0, 0)w = \mathcal{G}(aw) - w$. It is clear that T is a bounded linear operator. Furthermore, $Tw = 0$ implies $\mathcal{G}(aw) = w$ giving us $w \in C_0(\Omega)$ and $-\Psi(-\Delta)\, w + aw = 0$. Since a is not an eigenvalue, we must have $w = 0$. Thus T is injective. Since $\mathcal{G}$ is compact, by Fredholm alternative on Banach spaces T is also surjective and T^{-1} is also bounded linear. Therefore, we can apply the implicit theorem Theorem 6.1.5 to obtain a C^1 curve $(c, z(c))$ in $(-\varepsilon, \varepsilon)$, with

$z(0) = 0$ and $F(c, z(c)) = 0$. In other words,

$$\begin{aligned} \Psi(-\Delta)\, z(c) &= az(c) - f(x, z(c)) - ch(x, z(c)) \quad \text{in } \Omega\,, \\ z(c) &= 0 \quad \text{in } \Omega^c\,. \end{aligned} \tag{6.2.14}$$

To complete the proof we only need to show that there exists $\tilde{c}$ such that $z(c) > 0$ in Ω. Considering $c = 0$ and $f(x) = -h(x, 0)$ in Theorem 6.1.4 we choose the corresponding δ from Theorem 6.1.4. Fix $a \in (\lambda_1, \lambda_1 + \delta)$. Since $c \mapsto z(c)$ is C^1 we have $\frac{1}{|c|}\|z(c)\|_{C(\Omega)} \leq K$ for some K and all small c. Defining $U_c = \frac{z_c}{c}$ we obtain from (6.2.14) that

$$\begin{aligned} \Psi(-\Delta)\, U_c &= aU_c - F_c(x)U_c - h(x, z(c)) \quad \text{in } \Omega\,, \\ U_c &= 0 \quad \text{in } \Omega^c\,, \end{aligned} \tag{6.2.15}$$

where $F_c(x) = \frac{f(x,z(c))}{z(c)}$. Note that the rhs of (6.2.15) is uniformly bounded for all c small. Thus applying Theorem 6.1.1 we find that $\{U_c\}, \{\frac{U_c}{V(\delta_\Omega)}\}$ are uniformly Hölder continuous in Ω. In particular, the sequences are pre-compact. Now suppose that there exists $c_n \to 0$ such that $z(c_n) \not> 0$ in Ω. Then we can extract a subsequence n_k satisfying

$$\sup_{x\in\Omega}\left|\frac{U_{c_{n_k}}(x)}{V(\delta_\Omega(x))} - \frac{W(x)}{V(\delta_\Omega(x))}\right| \to 0, \text{as } n_k \to 0, \tag{6.2.16}$$

for some $W \in C_0(\Omega)$. Furthermore, from the stability property of viscosity solution [50, Corollary 4.7] we obtain

$$\Psi(-\Delta)\, W = aW - h(x, 0) \quad \text{in } \Omega\,, \qquad W = 0 \quad \text{in } \Omega^c.$$

Using Theorem 6.1.4 we have $W > 0$ in Ω and $\inf_\Omega \frac{W}{V(\delta_\Omega)} > 0$. From (6.2.16) we then have $U_{c_{n_k}} > 0$ in Ω for all large n_k which contradicts the fact $z(c_n) \not> 0$ in Ω for all n. Hence we can find $\tilde{c} > 0$ such that $u_2(c) \coloneqq z(c) > 0$ in Ω. Moreover,

$$\lim_{c\to 0+}\|u_2(c)\|_{C(\Omega)} = 0.$$

(iv) Suppose, on the contrary, that there exists a sequence $c_n \to 0$ and solutions $v(\cdot, c_n)$ of (6.2.8) corresponding to c_n and $v(\cdot, c_n) \neq u_1(\cdot, c_n)$ and $v(\cdot, c_n) \neq u_2(\cdot, c_n)$.

To simplify the notation we denote by $v^n = v(\cdot, c_n), u_1^n = u_1(\cdot, c_n), u_2^n = u_2(\cdot, c_n)$. Since, by Theorem 6.1.1, $\{v^n\}$ is equi-conitnuous, from Theorem 6.2.1 one of the following hold.

(a) There exists a subsequence $\{n_k\}$ satisfying $\|v^{n_k} - v_a\|_{C(\Omega)} = 0$, as $n_k \to \infty$.

(b) There exists a subsequence $\{n_k\}$ satisfying $\|v^{n_k}\|_{C(\Omega)} = 0$, as $n_k \to \infty$.

We arrive a contradiction below in each of the cases. Consider (a) first. Since u_1^n is the maximal solution we have $v^n \leq u_1^n \leq v_a$. Thus, by Theorem 6.2.2, we have

$$\lim_{n_k \to \infty} \|u_1^{n_k} - v^{n_k}\|_{C(\Omega)} = 0.$$

Defining $w^n = u_1^n - v^n$ and using (6.2.8) we get

$$\Psi(-\Delta)\, w^{n_k} = a w^{n_k} - \frac{f(x, u_1^{n_k}) - f(x, v^{n_k})}{w^{n_k}} w^{n_k} - c_{n_k} \frac{h(x, u_1^{n_k}) - h(x, v^{n_k})}{w^{n_k}} w^{n_k} \quad \text{in } \Omega\,. \tag{6.2.17}$$

Since $w^{n_k} \gneq 0$ in Ω, by Theorem 6.1.2, we have $w^{n_k} > 0$ in Ω. Normalize w^{n_k} by defining $\xi^{n_k} = \frac{1}{\|w^{n_k}\|_{C(\Omega)}} w^{n_k}$. From (6.2.17) we then have

$$\begin{aligned} \Psi(-\Delta)\, \xi^{n_k} &= a \xi^{n_k} - \frac{f(x, u_1^{n_k}) - f(x, v^{n_k})}{w^{n_k}} \xi^{n_k} - c_{n_k} \frac{h(x, u_1^{n_k}) - h(x, v^{n_k})}{w^{n_k}} \xi^{n_k} \quad \text{in } \Omega\,, \\ \xi^{n_k} &= 0 \quad \text{in } \Omega^c\,, \\ \xi^{n_k} &> 0 \quad \text{in } \Omega\,. \end{aligned} \tag{6.2.18}$$

Using Theorem 6.1.1, we see that $\{\xi^{n_k}\}$ is equicontinuous and then passing to the limit along some subsequence and using stability property of the viscosity solution in (6.2.18), we find a solution $\xi \in C(\mathbb{R}^d)$ with $\xi > 0$ in Ω (due to Theorem 6.1.2) satisfying

$$\begin{aligned} \Psi(-\Delta)\, \xi &= a\xi - f_u(x, v_a)\xi \quad \text{in } \Omega\,, \\ \xi &= 0 \quad \text{in } \Omega^c\,, \\ \xi &> 0 \quad \text{in } \Omega\,. \end{aligned}$$

But this contradicts the fact v_a is a stable solution (see Theorem 6.2.1). Thus (a) is not possible. So we consider (b). Defining $w^n = u_2^n - v^n \neq 0$ and $\xi^n = \frac{1}{\|w^n\|_{C(\Omega)}} w^n$ and repeating a similar argument as above, we get a non-zero ξ satisfying

$$\Psi(-\Delta)\,\xi = a\xi \quad \text{in } \Omega\,, \quad \xi = 0 \quad \text{in } \Omega^c\,,$$

which is a contradiction since a is not an eigenvalue of $\Psi(-\Delta)$. Thus (b) is also not possible. This completes the proof of the theorem. □

Remark 6.2.1. The condition $\inf_{s\in[0,K]} h(\cdot, s) \gneq 0$ is used to prove nonexistence of solution for large values of c. This condition does not have any influence on Theorem 6.2.3(iii) and (iv).

The above result should be compared with [139, Theorem 3.2 and 3.3] which establish a similar result for $\Psi(-\Delta) = -\Delta$ and $h(x, u) = h(x)$. To our best knowledge, there are no similar existing results for nonlocal operators. For the fractional Laplacian operators only existence of a solution is obtained for $c > 0$ and $a > \lambda_1$ in [61, Theorem 2.9]. The main idea in obtaining Theorem 6.2.3(iii) is to apply the implicit function theorem of Crandall and Rabinowitz [70]. In case of the Laplacian this is applied on the forward operator [139, Theorem 3.3]. But the same method can not applied for nonlocal operators due to lack of appropriate Schauder estimates. We instead consider the inverse operator (see (6.1.3) above) and establish appropriate estimates so that the implicit function theorem can be applied.

As a corollary to the proof of Theorem 6.2.3 we get the following uniqueness result which generalizes [139, Theorem 3.4]. In the following result V denotes the potential measure function of ladder-height process corresponding to X^1 (see Section 6.1).

Corollary 6.2.1. *Suppose that*

$$\sup_{s\in[0,k]} \sup_{\Omega} \left| \frac{h(x, s)}{V(\delta_\Omega(x))} \right| < \infty, \tag{6.2.19}$$

for every finite k. Then for every $a > 2\lambda_1$, there exists a $\check{c} \in (0, c_\circ)$ so that for every $c \in (0, \check{c})$, there exists a unique solution u to (6.2.8) satisfying

$$\lambda_1\, u(x) \geq c\, h(x, u(x)), \quad x \in \mathbb{R}^d. \tag{6.2.20}$$

Proof. First we show existence. Recall from Lemma 6.2.3 and Theorem 6.2.3(i) that for any $c < c_1$ there exists a maximal solution $u_1(c) = u_1(\cdot, c)$ of

$$\begin{aligned} \Psi(-\Delta)\, u &= au - f(x,u) - ch(x,u) \quad \text{in } \Omega\,, \\ u &> 0 \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{in } \Omega^c\,, \end{aligned} \tag{6.2.21}$$

satisfying $m\beta\varphi_1 \leq u_1(c) \leq v_a$. Using Theorem 6.1.1 we see that $\{u_1(c)\}_{c<c_1}$, $\{\frac{u_1(c)}{V(\delta_\Omega)}\}_{c<c_1}$ are equi-continuous family of positive functions. Since any subsequential limit of $\{u_1(c)\}_{c<c_1}$, as $c \to 0$, would be a positive solution to (6.2.6) (by stability property of viscosity solutions), from Theorem 6.2.1 we obtain that

$$\lim_{c\to 0+} \sup_{\Omega} \left| \frac{u_1(x,c)}{V(\delta_\Omega)} - \frac{v_a(x)}{V(\delta_\Omega)} \right| = 0. \tag{6.2.22}$$

Since $\inf_\Omega \frac{v_a(x)}{V(\delta_\Omega(x))} > 0$ by Theorem 6.1.2, using (6.2.22) and (6.2.19) we can find $c_2 > 0$ so that for every $c \in (0, c_2)$ we have

$$\lambda_1 u_1(c) \geq ch(x, u_1), \quad \text{in } x \in \mathbb{R}^d.$$

Next we show uniqueness. We claim that if $w(c) = w(\cdot, c)$ be any solution to (6.2.21) satisfying (6.2.20), then there exists $c_3 > 0$ and $\delta > 0$ satisfying

$$\inf_{c\in(0,c_3)} \sup_{\mathbb{R}^d} w(c) > 0\,. \tag{6.2.23}$$

If this does not hold, then for a sequence $\{c_n\}, c_n \to 0$, we would have $\sup_{\mathbb{R}^d} w(c_n) = \sup_\Omega w(c_n) \to 0$ as $n \to \infty$. Denote by $2\eta = a - 2\lambda_1$. Using (A3), we obtain

$$\eta w(c_n) - \frac{f(x, w(c_n))}{w(c_n)} w(c_n) > 0, \quad \text{in } \Omega\,,$$

for all large n. Hence, using (6.2.20), we obtain for all large n that

$$\begin{aligned} &- \Psi(-\Delta)\, w(c_n) + (\lambda_1 + \eta) w(c_n) \\ &\quad \leq - \Psi(-\Delta)\, w(c_n) + (\eta + 2\lambda_1) w(c_n) - c_n h(x, w(c_n)) \end{aligned}$$

$$\leq -\Psi(-\Delta)\, w(c_n) + aw(c_n) - f(x, w(c_n)) - c_n h(x, w(c_n)) = 0 \quad \text{in } \Omega.$$

But this contradicts the definition of λ_1 in (6.2.2). This gives us (6.2.23). This also confirms that $\lim_{c\to 0} w(c) = v_a$. Then uniqueness follows from the argument of Theorem 6.2.3(iv) (see situation (a) there). □

6.3 Parabolic logistic equation

Next we discuss the long time behaviour of the parabolic nonlocal equation. Consider the terminal value problem

$$\begin{aligned} (\partial_t - \Psi(-\Delta))u + au - f(x,u) &= 0 \quad \text{in } \Omega \times [0,T), \\ u(x,T) = u_0(x) \;\text{ and } u(x,t) &= 0 \quad \text{in } \Omega^c \times [0,T]. \end{aligned} \tag{6.3.1}$$

By a solution of (6.3.1) we mean a potential theoretic solution. More precisely, we say $u \in C(\mathbb{R}^d \times [0,T])$ is a solution to

$$\begin{aligned} (\partial_t - \Psi(-\Delta))u + \ell(x) &= 0 \quad \text{in } \Omega \times [0,T), \\ u(x,T) = g(x) \;\text{ and } u(x,t) &= 0 \quad \text{in } \Omega^c \times [0,T], \end{aligned} \tag{6.3.2}$$

if

$$u(x,t) = \mathbb{E}_x[g(X_{(T-t)\wedge\tau})] + \mathbb{E}_x\left[\int_0^{(T-t)\wedge\tau} \ell(X_s, t+s)\mathrm{d}s\right], \quad (x,t) \in \Omega \times [0,T], \tag{6.3.3}$$

where τ denotes the first exit time of X from Ω. It can be shown that potential theoretic solutions are same as viscosity solution of (6.3.2) (see Lemma 6.3.1 below). The benefit of working with (6.3.3) is that it allows us to make use of the underlying probabilistic structure of the model.

Throughout this section we assume that $g \in C_0(\Omega)$. It is important to observe that (6.3.3) is not different from a viscosity solution. We recall the definition of viscosity solution. By $C_b^{2,1}(x,t)$ we denote the space of all bounded continuous functions in $\mathbb{R}^d \times [0,T]$ that are in $C^{2,1}$ class in some neighbourhood of (x,t). The following definition of viscosity solution can be found in [58, 154].

Definition 6.3.1. An upper (lower) semicontinuous function u is said to be a viscosity subsolution (supersolution) of (6.3.2) if for every $(x,t) \in \Omega \times [0,T)$ and $\varphi \in C_b^{2,1}(x,t)$ satisfying

$$\varphi(x,t) = u(x,t), \quad \varphi(y,s) \geq u(y,s) \quad \text{for } y \in \mathbb{R}^d,\ t \leq s < t+\delta,$$

$$\Big(\varphi(x,t) = u(x,t), \quad \varphi(y,s) \leq u(y,s) \quad \text{for } y \in \mathbb{R}^d,\ t \leq s < t+\delta, \textit{respectively,}\Big)$$

for some $\delta > 0$, we have

$$(\partial_t - \Psi(-\Delta))\varphi(x,t) + \ell(x,t) \geq 0,$$

$$((\partial_t - \Psi(-\Delta))\varphi(x,t) + \ell(x,t) \leq 0,\ \textit{respectively}).$$

The time derivative ∂_t can also be replaced by the derivative in parabolic topology i.e.,

$$\partial_{t+}\varphi(x,t) = \lim_{h\to 0+} \frac{\varphi(x,t+h) - \varphi(x,t)}{h}.$$

Let us first show that potential theoretic solution is also a viscosity solution.

Lemma 6.3.1. *Let $u \in C_b(\mathbb{R}^d \times [0,T])$ satisfy (6.3.3). Assume that ℓ, g are continuous. Then u is the unique viscosity solution of (6.3.2).*

Proof. Let $x \in B \subset \Omega$. By τ_B we denotes the exit time from B i.e.,

$$\tau_B = \inf\{t > 0 \ :\ X_t \notin B\}.$$

It is evident that $\tau_B \leq \tau$. First we show that for any $\delta < T - t$

$$u(x,t) = \mathbb{E}_x[u(X_{\delta\wedge\tau_B}, t + \delta \wedge \tau_B)] + \mathbb{E}_x\left[\int_0^{\delta\wedge\tau_B} \ell(X_s, t+s)\mathrm{d}s\right]. \tag{6.3.4}$$

Using (6.3.3) we write

$$u(x,t) = \mathbb{E}_x[g(X_{T-t})\mathbb{1}_{\{T-t<\tau\}}] + \mathbb{E}_x\left[\int_0^{T-t} \ell(X_s, t+s)\mathbb{1}_{\{s<\tau\}}\mathrm{d}s\right]$$

$$
\begin{aligned}
&= \mathbb{E}_x[g(X_{T-t})\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}\mathbb{1}_{\{(T-t)<\tau\}}] + \mathbb{E}_x\left[\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}\int_{\delta\wedge\tau_B}^{T-t}\ell(X_s,t+s)\mathbb{1}_{\{s<\tau\}}\mathrm{d}s\right] \\
&\quad + \mathbb{E}_x\left[\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}\int_0^{\delta\wedge\tau_B}\ell(X_s,t+s)\mathbb{1}_{\{s<\tau\}}\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}\,\mathbb{E}_{X_{\delta\wedge\tau_B}}\left[g(X_{(T-t-\delta\wedge\tau_B)})\mathbb{1}_{\{T-t-\delta\wedge\tau_B<\tau\}}\right]\right] \\
&\quad + \mathbb{E}_x\left[\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}\,\mathbb{E}_{X_{\delta\wedge\tau_B}}\left[\int_0^{(T-t-\delta\wedge\tau_B)}\ell(X_s,t+\delta\wedge\tau_B+s)\mathbb{1}_{\{s<\tau\}}\mathrm{d}s\right]\right] \\
&\quad + \mathbb{E}_x\left[\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}\int_0^{\delta\wedge\tau_B}\ell(X_s,t+s)\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}u(X_{\delta\wedge\tau_B},t+\delta\wedge\tau_B)\right] + \mathbb{E}_x\left[\mathbb{1}_{\{\delta\wedge\tau_B\le\tau\}}\int_0^{\delta\wedge\tau_B}\ell(X_s,t+s)\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[u(X_{\delta\wedge\tau_B},t+\delta\wedge\tau_B)\right] + \mathbb{E}_x\left[\int_0^{\delta\wedge\tau_B}\ell(X_s,t+s)\mathrm{d}s\right],
\end{aligned}
$$

where in the third line we use strong Markov property and in the last line we use the fact that $\mathbb{P}_x(\delta\wedge\tau_B\le\tau)=1$. This proves (6.3.4). This relation is key to show that u is also a viscosity solution. We only check that u is a viscosity subsolution and the other part would be analogous. Consider $(x,t)\in\Omega\times[0,T)$ and $\varphi\in C_b^{2,1}(x,t)$ satisfying

$$
\varphi(x,t)=u(x,t),\quad \varphi(y,s)\ge u(y,s)\quad \text{for } y\in\mathbb{R}^d,\ t\le s<t+\delta.
$$

Choose a ball B, centered at x, small enough so that φ is $C^{2,1}$ in $\bar{B}\times[t,t+\delta]$. Let $\delta_1<\delta$. Then applying Dynkin-Itô formula we know that

$$
\begin{aligned}
\mathbb{E}_x\left[\int_0^{\delta_1\wedge\tau_B}(\partial_t-\Psi(-\Delta))\varphi(X_s,t+s)\mathrm{d}s\right] &= \mathbb{E}_x[\varphi(X_{\delta_1\wedge\tau_B},t+\delta_1\wedge\tau_B)]-\varphi(x,t) \\
&\ge \mathbb{E}_x[u(X_{\delta_1\wedge\tau_B},t+\delta_1\wedge\tau_B)]-u(x,t) \\
&= -\,\mathbb{E}_x\left[\int_0^{\delta_1\wedge\tau_B}\ell(X_s,t+s)\mathrm{d}s\right],
\end{aligned}
$$

using (6.3.4). Since $\mathbb{P}_x(\tau_B>0)=1$, dividing both sides by δ_1 and letting $\delta_1\to 0$ to obtain

$$
(\partial_t-\Psi(-\Delta))\varphi(x,t)+\ell(x,t)\ge 0.
$$

Similarly, we can show that u is a supersolution.

The uniqueness part follows using a similar argument as in [154, Lemma 3.3] (Note that the proof of [154, Lemma 3.2] is based on the ideas from [50] which works for general nonlocal operators). $\square$

Our next lemma concerns representation of Schrödinger equation.

Lemma 6.3.2. *Suppose that ℓ, V are continuous and bounded in Ω and $g \in C_0(\Omega)$. Define*

$$\varphi(x,t) = \mathbb{E}_x\left[e^{\int_0^{(T-t)\wedge\uptau} V(X_s,t+s)\,\mathrm{d}s} g(X_{(T-t)\wedge\uptau})\right] + \mathbb{E}_x\left[\int_0^{(T-t)\wedge\uptau} e^{\int_0^s V(X_k,t+k)\,\mathrm{d}k}\ell(X_s,t+s)\,\mathrm{d}s\right].$$

Then φ solves

$$\begin{aligned} (\partial_t - \Psi(-\Delta))\varphi + \ell + V\varphi &= 0 \quad \text{in } \Omega\times[0,T), \\ \varphi(x,T) &= g(x) \text{ and } \varphi(x,t) = 0 \quad \text{in } \Omega^c\times[0,T]. \end{aligned} \tag{6.3.5}$$

Proof. It is routine to check φ is continuous (cf. [33, Lemma 3.1]) and $\varphi(\cdot,t) = 0$ in Ω^c. It also follows from the definition $\varphi(x,T) = g(x)$. Now fix any $t\in[0,T)$ and $\delta < T-t$. Since g and φ vanish outside Ω we obtain that

$$\begin{aligned}
&\varphi(x,t) \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{T-t<\uptau\}} e^{\int_0^{T-t} V(X_s,t+s)\,\mathrm{d}s} g(X_{T-t})\right] \\
&\qquad + \mathbb{E}_x\left[\int_0^{(T-t)} \mathbb{1}_{\{s<\uptau\}} e^{\int_0^s V(X_k,t+k)\,\mathrm{d}k}\ell(X_s,t+s)\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\{\delta<\uptau\}} e^{\int_0^\delta V(X_s,t+s)\mathrm{d}s}\,\mathbb{E}_{X_\delta}\left[\mathbb{1}_{\{T-t-\delta<\uptau\}} e^{\int_0^{T-t-\delta} V(X_s,t+\delta+s)\mathrm{d}s} g(X_{T-t-\delta})\right]\right] \\
&\qquad + \mathbb{E}_x\left[\int_0^\delta \mathbb{1}_{\{s<\uptau\}} e^{\int_0^s V(X_k,t+k)\mathrm{d}k}\ell(X_s,t+s)\mathrm{d}s\right] \\
&\qquad + \mathbb{E}_x\left[\mathbb{1}_{\{\delta<\uptau\}} e^{\int_0^\delta V(X_s,t+s)\mathrm{d}s}\,\mathbb{E}_{X_\delta}\left[\int_0^{T-t-\delta} \mathbb{1}_{\{s<\uptau\}} e^{\int_0^s V(X_k,t+\delta+k)\,\mathrm{d}k}\ell(X_s,t+s)\mathrm{d}s\right]\right]
\end{aligned}$$

$$= \mathbb{E}_x\left[\mathbb{1}_{\{\delta<\tau\}} e^{\int_0^\delta V(X_s,t+s)\mathrm{d}s}\varphi(X_\delta, t+\delta)\right] + \mathbb{E}_x\left[\int_0^\delta \mathbb{1}_{\{s<\tau\}} e^{\int_0^s V(X_k,t+k)\mathrm{d}k}\ell(X_s,t+s)\mathrm{d}s\right] \tag{6.3.6}$$

where the second equality follows from the strong Markov property. Now fix $x \in \Omega$ and define

$$\xi(p) = \mathbb{E}_x[\varphi(X_{p\wedge\tau}, t+p\wedge\tau)] = \mathbb{E}_x[\varphi(X_p, t+p)\mathbb{1}_{\{p<\tau\}}].$$

Then, using (6.3.6) we note that

$$\begin{aligned}
&\xi(p) - \xi(p-\delta)\\
&= \xi(p) - \mathbb{E}_x\left[\mathbb{1}_{\{p-\delta<\tau\}}\,\mathbb{E}_{X_{p-\delta}}\left[\mathbb{1}_{\{\delta<\tau\}} e^{\int_0^\delta V(X_s,t+p-\delta+s)\mathrm{d}s}\varphi(X_\delta,t+p)\right]\right]\\
&- \mathbb{E}_x\left[\mathbb{1}_{\{p-\delta<\tau\}}\,\mathbb{E}_{X_{p-\delta}}\left[\int_0^\delta \mathbb{1}_{\{s<\tau\}} e^{\int_0^s V(X_k,t+p-\delta+k)\mathrm{d}k}\ell(X_s,t+p-\delta+s)\mathrm{d}s\right]\right]\\
&= \xi(p) - \mathbb{E}_x\left[\mathbb{1}_{\{p-\delta<\tau\}}\mathbb{E}_x\left[\mathbb{1}_{\{p<\tau\}} e^{\int_0^\delta V(X_{s+p-\delta},t+p-\delta+s)\mathrm{d}s}\varphi(X_p,t+p)\Big|F\right]\right]\\
&- \mathbb{E}_x\left[\mathbb{1}_{\{p-\delta<\tau\}}\mathbb{E}_x\left[\int_0^\delta \mathbb{1}_{\{s+p-\delta<\tau\}} e^{\int_0^s V(X_{p-\delta+k},t+p-\delta+k)\mathrm{d}k}\ell(X_{p-\delta+s},t+p-\delta+s)\mathrm{d}s\Big|F\right]\right]\\
&= \mathbb{E}_x[\varphi(X_p,t+p)\mathbb{1}_{\{p<\tau\}}] - \mathbb{E}_x\left[\mathbb{1}_{\{p<\tau\}} e^{\int_0^\delta V(X_{s+p-\delta},t+p-\delta+s)\mathrm{d}s}\varphi(X_p,t+p)\right]\\
&\quad - \mathbb{E}_x\left[\int_0^\delta \mathbb{1}_{\{\tau>s+p-\delta\}} e^{\int_0^s V(X_{p-\delta+k},t+p-\delta+k)\mathrm{d}k}\ell(X_{p-\delta+s},t+p-\delta+s)\mathrm{d}s\right],
\end{aligned}$$

where $F = \mathcal{F}_{\tau\wedge(p-\delta)}$. Then, using the quasi-continuity property of X, we obtain

$$\begin{aligned}
&\lim_{\delta\to 0+}\frac{1}{\delta}(\xi(p)-\xi(p-\delta))\\
&\quad = -\,\mathbb{E}_x[V(X_p,t+p)\varphi(X_p,t+p)\mathbb{1}_{\{\tau>p\}}] - \mathbb{E}_x[\mathbb{1}_{\{\tau\geq p\}}\ell(X_p,t+p)]\\
&\quad = -\,\mathbb{E}_x[V(X_p,t+p)\varphi(X_p,t+p)\mathbb{1}_{\{\tau>p\}}] - \mathbb{E}_x[\mathbb{1}_{\{\tau>p\}}\ell(X_p,t+p)] := -\zeta(p),
\end{aligned}$$

where the last line follows since $\mathbb{P}_x(\tau = p) = 0$. Hence the left derivative of ξ exists and given by ζ which is continuous. Therefore, ξ is a C^1 function. Now using the fundamental theorem of calculus we obtain

$$\begin{aligned}\varphi(x,t) &= \xi(T-t) + \int_0^{T-t} \zeta(s)\,\mathrm{d}s \\ &= \mathbb{E}_x[\varphi(X_{T-t})\mathbb{1}_{\{T-t<\uptau\}}] \\ &\qquad + \mathbb{E}_s\left[\int_0^{T-t} (V(X_s,t+s)\varphi(X_s,t+s) + \ell(X_s,t+s))\mathbb{1}_{\{s<\uptau\}}\mathrm{d}s\right].\end{aligned}$$

Thus φ solves (6.3.5). □

Next we get a parabolic comparison principle. Let $q : \bar{\Omega} \times [0,\infty) \to [0,\infty)$ be a continuous function, C^1 in its second variable and $q_s : \bar{\Omega} \times [0,\infty) \to \mathbb{R}$ is also continuous. Also, assume that $q(x,0) = 0$ and

$$s \mapsto \frac{q(x,s)}{s} \quad \text{is decreasing.}$$

Lemma 6.3.3. *Let u, v be two positive solutions of*

$$(\partial_t - \Psi(-\Delta))w + q(x,w) = 0 \quad \textit{in } \Omega \times [0,T), \quad w = 0 \textit{ in } \Omega^c \times [0,T].$$

If $u(x,T) \le v(x,T)$ in $\mathbb{R}^d$, then we also have $u \le v$ in $\mathbb{R}^d \times [0,T]$.

Proof. Let $G(x,t) = \frac{q(x,u(x,t))}{u(x,t)}$ and $H(x,t) = \frac{q(x,v(x,t))}{v(x,t)}$. Then using Lemma 6.3.2 we obtain

$$u(x,t) = \mathbb{E}_x\left[e^{\int_0^{T-t} G(X_s,t+s)\mathrm{d}s} u(X_{T-t},T)\mathbb{1}_{\{T-t<\uptau\}}\right], \quad (x,t) \in D \times [0,T], \tag{6.3.7}$$

$$v(x,t) = \mathbb{E}_x\left[e^{\int_0^{T-t} H(X_s,t+s)\mathrm{d}s} v(X_{T-t},T)\mathbb{1}_{\{T-t<\uptau\}}\right], \quad (x,t) \in \Omega \times [0,T]. \tag{6.3.8}$$

Note that without loss of generality we may assume $u(\cdot,T) \gneq 0$, otherwise from above we get $u = 0$ and then, there is nothing to prove. Let $K = \max_{\bar{\Omega}\times[0,T]}(|G| + |H|)$. Then it is evident from above that

$$v(x,t) \ge e^{-KT} u(x,t) \quad \text{for all } x, t. \tag{6.3.9}$$

Define

$$\beta = \sup\{t \;:\; tu \le v \quad \text{in } \Omega \times [0,T]\}.$$

Using (6.3.9) we get that $\beta \geq e^{-KT}$. To complete the proof we need to show that $\beta \geq 1$. Suppose, on the contrary, that $\beta < 1$. Denote by $u_1 = \beta u$. Then for $w = v - u_1 \geq 0$ we have

$$\begin{aligned} w(x,t) = \mathbb{E}_x &\left[w(X_{T-t}, T) \mathbb{1}_{\{T-t<\tau\}} \right] \\ &+ \mathbb{E}_x \left[\int_0^{(T-t)\wedge\tau} \Big(q(X_s, v(X_s, t+s)) - \beta q(X_s, u(X_s, t+s)) \Big) \mathrm{d}s \right]. \end{aligned}$$

For any $\delta \in (0, T-t)$, we can repeat the calculation of (6.3.6) with $V = 0$ to arrive at

$$\begin{aligned} w(x,t) = \mathbb{E}_x &\left[\mathbb{1}_{\{\delta<\tau\}} w(X_\delta, t+\delta) \right] \\ &+ \mathbb{E}_x \left[\int_0^{\delta} \mathbb{1}_{\{s<\tau\}} \Big(q(X_s, v(X_s, t+s)) - \beta q(X_s, u(X_s, t+s)) \Big) \, \mathrm{d}s \right]. \end{aligned}$$

By our assumption on q, $q(x,v) - \beta q(x,u) \geq q(x,v) - q(x,\beta u) \geq -Mw$, for some constant M. Thus defining $\xi(s) = \mathbb{E}_x[\mathbb{1}_{\{\tau>s\}} w(X_s, t+s)]$ we obtain

$$\xi(\delta) \leq \xi(0) + M \int_0^{\delta} \xi(s) \mathrm{d}s.$$

Applying Gronwall's inequality we then have

$$\xi(T-t) \leq Cw(x,t),$$

for some constant C, independent of $(x,t) \in \Omega \times [0,T]$. Since $w(x,T) \gneq 0$, we must have $w(x,t) > 0$ for all $t < T$. Furthermore, $w(x,T) \geq (1-\beta)u(x,T)$, implying

$$Cw(x,t) \geq \xi(T-t) \geq (1-\beta)\, \mathbb{E}_x[\mathbb{1}_{\{T-t<\tau\}} u(X_s, T)],$$

which combined with (6.3.7) gives $\kappa u(x,t) \leq w(x,t)$ for $(x,t) \in \Omega \times [0,T]$ and for some $\kappa > 0$. This certainly contradicts the definition of β. Hence $\beta \geq 1$, completing the proof. $\square$

Next we establish a regularity property in space up to the boundary.

Lemma 6.3.4. *Suppose that g, ℓ be such that $\|g\|_{L^\infty}, \|\ell\|_\infty \leq K$. Then for any u satisfying*

$$u(x,t) = \mathbb{E}_x[g(X_{T-t})\mathbb{1}_{\{T-t<\tau\}}]+\mathbb{E}_x\left[\int_0^{T-t} \ell(X_s, t+s)\mathbb{1}_{\{s<\tau\}}\mathrm{d}s\right], \quad (x,t) \in \Omega\times[0,T],$$

we have, for $t < T$

$$|u(x,t) - u(y,t)| \leq C\, V(|x-y|), \quad x, y \in \Omega,$$

for a constant C dependent on t, T, K where V is the potential measure introduced in Section 6.1. We can also choose the constant C uniformly in t varying in a compact subset of $[0, T)$.

Proof. Denote by

$$\begin{aligned}
\mathscr{R}_1(x) &= \mathbb{E}_x[g(X_{T-t})\mathbb{1}_{\{T-t<\tau\}}] = \int_\Omega g(y) p_\Omega(T-t, x, z)\,\mathrm{d}z, \\
\mathscr{R}_2(x) &= \mathbb{E}_x\left[\int_0^{T-t} \ell(X_s, t+s)\mathbb{1}_{\{s<\tau\}}\mathrm{d}s\right] = \int_0^{T-t} \ell(x, t+s) p_\Omega(s, x, z)\,\mathrm{d}z,
\end{aligned}$$

where $p_\Omega(t, x, z)$ denotes the transition density of the killed process X^Ω (see (6.1.2)). Then from the arguments of [112, Proposition 3.5] (see (3.13) of that paper) one can find a constant C_1, independent of t, T, satisfying

$$|\mathscr{R}_2(x) - \mathscr{R}_2(y)| \leq C_1 V(|x-y|) \quad x, y \in \Omega\,. \tag{6.3.10}$$

To calculate $\mathscr{R}_1$ we recall the following result from [123, Theorem 1.1] and [101, Theorem 1.3] (see also [59],[112, Theorem 3.1])

$$|\nabla_x p_\Omega(t,x,y)| \leq C_2 \left(\frac{1}{\delta_\Omega(x) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{t})}\right) p_\Omega(t,x,y) \quad x, y \in \Omega, \tag{6.3.11}$$

$$p_\Omega(t,x,y) \leq C_3 \left(1 \wedge \frac{V(\delta_\Omega(x))}{\sqrt{t}}\right)\left(1 \wedge \frac{V(\delta_\Omega(y))}{\sqrt{t}}\right) p(t, |x-y|) \quad x, y \in \Omega\,, \tag{6.3.12}$$

for $t \in (0, T]$ and some constants C_2, C_3, dependent on T, where p denotes the transition density of X. Using (A1) and (6.1.4) we also have, for any $\kappa > 0$, that

$$C_4^{-1}\left(\frac{R}{r}\right)^{\kappa_1} \le \frac{V(R)}{V(r)} \le C_4\left(\frac{R}{r}\right)^{\kappa_2} \qquad 0 < r \le R \le \kappa\,, \tag{6.3.13}$$

where C_4 depends on κ. Take $x, y \in \Omega$. Suppose $2|x-y| \le \max\{\delta_\Omega(x), \delta_\Omega(y)\}$. With no loss of generality we may assume that $y \in B(x, \frac{1}{2}\delta_\Omega(x))$. Note that for any point z on the line joining x and y we get from (6.3.11)-(6.3.12)

$$\begin{aligned}
|x-y||\nabla_x p_\Omega(T-t,z,y)| &\le C_2|x-y|\left(\frac{1}{\delta_\Omega(z)\wedge 1} \vee \frac{1}{V^{-1}(\sqrt{T-t})}\right) p_\Omega(T-t,z,y)\\
&\le C_2\,|x-y|\left(\frac{1}{\delta_\Omega(z)\wedge 1} \vee \frac{1}{V^{-1}(\sqrt{T-t})}\right)\\
&\quad \cdot C_3\left(1 \wedge \frac{V(\delta_\Omega(z))}{\sqrt{T-t}}\right)\left(1 \wedge \frac{V(\delta_\Omega(y))}{\sqrt{T-t}}\right) p(T-t,|z-y|)\\
&\le C_5\frac{|x-y|}{\delta_\Omega(z)} V(\delta_\Omega(z)),
\end{aligned}$$

for some $C_5 > 0$. Since $|x-y| \le \frac{1}{2}\delta_\Omega(x) \le \delta_\Omega(z)$, using (6.3.13) with $\kappa = \mathrm{diam}(\Omega)$ we then obtain

$$\frac{|x-y|}{V(|x-y|)}|\nabla_x p_\Omega(T-t,z,y) \le C_5\frac{|x-y|}{\delta_\Omega(z)}\frac{V(\delta_\Omega(z))}{V(|x-y|)} \le C_4C_5\left(\frac{|x-y|}{\delta_\Omega(z)}\right)^{1-\kappa_2} \le C_4C_5.$$

Thus

$$\begin{aligned}
|\mathscr{R}_1(x) - \mathscr{R}_1(y)| &\le \int_\Omega |g(y)||p_\Omega(T-t,x,z) - p_\Omega(T-t,y,z)|\,\mathrm{d}z\\
&\le C_4C_5V(|x-y|)\|g\|_{L^\infty}|\Omega|.
\end{aligned} \tag{6.3.14}$$

Now we consider the situation $2|x-y| \ge \max\{\delta_\Omega(x), \delta_\Omega(y)\}$. Then using (6.3.12)-(6.3.13)

$$\begin{aligned}
|\mathscr{R}_1(x) - \mathscr{R}_1(y)| &\le C_7(V(\delta_\Omega(x)) + V(\delta_\Omega(y)))\\
&\le C_7(V(2|x-y|) + V(2|x-y|)) \le C_8V(|x-y|),
\end{aligned} \tag{6.3.15}$$

for some constants C_7, C_8. Combining (6.3.10), (6.3.14) and (6.3.15) we get the result. $\square$

Now we are ready to prove an existence result.

Lemma 6.3.5. *Let q be same as in Lemma 6.3.3. Also, assume that $q(x,\cdot)$ is C^1 , uniformly with respect to x. Let $\ell_1 : \bar{\Omega} \to (-\infty, 0]$, $\ell_2 : \bar{\Omega} \to [0,\infty)$ be be two continuous functions. Let $v_i, i = 1,2,$ be a non-negative solution satisfying*

$$(\partial_t - \Psi(-\Delta))v_i + q(x, v_i) + \ell_i(x) = 0 \quad \text{in}\ \Omega \times [0,T), \quad v(x,t) = 0\ \text{in}\ \Omega^c \times [0,T],$$

and let g be such that $v_1(x,T) \le g(x) \le v_2(x,T)$. Then there exists a unique solution $u\,(v_1 \le u \le v_2)$ to

$$\begin{aligned}(\partial_t - \Psi(-\Delta))u + q(x,u) = 0 \quad & \text{in}\ \Omega \times [0,T),\\ u(x,T) \ =\ g(x)\ \text{and}\ u(x,t) \ =\ 0 \quad & \text{in}\ \Omega^c \times [0,T].\end{aligned} \tag{6.3.16}$$

Proof. The idea is similar to the elliptic case where we use monotone iteration method. Let m be Lipschitz constant of $s \mapsto q(x,s)$ in $[0, \|v\|]$, that is,

$$|q(x,s_1) - q(x,s_2)| \le m|s_1 - s_2| \quad \text{for}\ s_1, s_2 \in [0, \|v\|],\ x \in \bar{\Omega}.$$

Let $F(x,s) = q(x,s) + ms$ and $u_0 = v_2$. Define u_1 to be the solution of

$$\begin{aligned}(\partial_t - \Psi(-\Delta))u_1 - mu_1 + F(x,u_0) = 0 \quad & \text{in}\ \Omega \times [0,T),\\ u_1(x,T) \ =\ g(x)\ \text{and}\ u_1(x,t) \ =\ 0 \quad & \text{in}\ \Omega^c \times [0,T].\end{aligned}$$

By Lemma 6.3.2 we then have

$$\begin{aligned}u_1(x,t) = {} & \mathbb{E}_x\left[e^{-m(T-t)} g(X_{T-t}) \mathbb{1}_{\{T-t<\uptau\}}\right]\\ & + \mathbb{E}_x\left[\int_0^{T-t} e^{-ms} F(X_s, u_0(X_s, t+s)) \mathbb{1}_{\{s<\uptau\}} \mathrm{d}s\right].\end{aligned} \tag{6.3.17}$$

Another use of Lemma 6.3.2 gives

$$\begin{aligned}v_i(x,t) = {} & \mathbb{E}_x\left[e^{-m(T-t)} v_i(X_{T-t}, T) \mathbb{1}_{\{T-t<\uptau\}}\right]\\ & + \mathbb{E}_x\left[\int_0^{T-t} e^{-ms} \tilde{F}_i(X_s, v_i(X_s, t+s)) \mathbb{1}_{\{s<\uptau\}} \mathrm{d}s\right],\end{aligned} \tag{6.3.18}$$

where $\tilde{F}_i(x,s) = F(x,s) + \ell_i(x)$. Since F is non-decreasing in s, we have

$$F(x,v_1) + \ell_1(x) \le F(x,v_2) \le F(x,v_2) + \ell_2(x)\,.$$

Therefore, comparing (6.3.17) and (6.3.18) we have $v_1 \le u_1 \le u_0 = v_2$ in $\mathbb{R}^d \times [0,T]$. Now we find an iterative sequence of solutions as follows: u_{k+1} is a solution to

$$(\partial_t - \Psi(-\Delta))u - mu + F(x,u_k) = 0 \quad \text{in } \Omega\times[0,T), \quad u = 0 \ \text{ in } \Omega^c\times[0,T], \quad u(x,T) = g(x).$$

In other words,

$$\begin{aligned} u_{k+1}(x,t) = \mathbb{E}_x\left[e^{-m(T-t)} g(X_{T-t}) \mathbb{1}_{\{T-t<\uptau\}}\right] \\ + \mathbb{E}_x\left[\int_0^{T-t} e^{-ms} F(X_s, u_k(X_s, t+s)) \mathbb{1}_{\{s<\uptau\}} \mathrm{d}s\right]. \end{aligned} \tag{6.3.19}$$

The above argument shows that

$$v_1 \le u_{k+1} \le u_k \le \cdots \le v_2 \quad \text{in } \mathbb{R}^d \times [0,T]\,.$$

Furthermore, applying Lemma 6.3.4, we see that $\lim_{k\to\infty} u_k(\cdot,t) = u(\cdot,t)$ uniformly in x, for each $t \in [0,T]$. Thus, using dominated convergence theorem, we can pass to the limit in (6.3.19) to obtain

$$u(x,t) = \mathbb{E}_x\left[e^{-m(T-t)} g(X_{T-t}) \mathbb{1}_{\{T-t<\uptau\}}\right] + \mathbb{E}_x\left[\int_0^{T-t} e^{-ms} F(X_s, u(X_s, t+s)) \mathbb{1}_{\{s<\uptau\}} \mathrm{d}s\right]. \tag{6.3.20}$$

From (6.3.20) it is easy to show that u is continuous in $\mathbb{R}^d \times [0,T]$ (cf. [33, Lemma 3.1]). Indeed, since $x \mapsto u(x,t)$ is continuous uniformly for t in compact subsets of $[0,T)$ and $t \mapsto p_\Omega(t,x,y)$ is continuous in $(0,\infty)$, $(x,t) \mapsto u(x,t)$ is continuous in $[0,T)\times\mathbb{R}^d$. To examine the continuity at T consider a sequence $(x_n,t_n) \to (x,T)$. Note that the second term in the above display goes to 0. Again, if $x \in \partial\Omega$ then

$$\mathbb{E}_{x_n}[|g(X_{T-t_n})| \mathbb{1}_{\{T-t_n<\uptau\}}] \le \mathbb{E}_{x_n}[|g(X_{T-t_n})|] \to 0,$$

as $n \to \infty$, we get $u(x_n,t_n) \to 0$. Also, if $x \in \Omega$, since $p_\Omega(T-t_n, x_n, y)\mathrm{d}y \to \delta_x$, we get $u(x_n,t_n) \to g(x)$. This gives continuity. Applying Lemma 6.3.2 we see that

u is a solution to (6.3.16). Uniqueness of solution follows from Lemma 6.3.3. This completes the proof. □

Next we prove a sharp boundary behaviour for the solution of the parabolic equation.

Lemma 6.3.6. *Consider q from Lemma 6.3.3. Let u be a bounded solution of*

$$(\partial_t - \Psi(-\Delta))u + q(x, u) = 0 \quad \text{in } \Omega \times [0, T), \quad u = 0 \text{ in } \Omega^c \times [0, T],$$

where $u(x, T) \gneq 0$. Then for every $t < T$ there exists a constant C, dependent on t, T and $u|_{\mathbb{R}^d \times [t,T]}$, satisfying

$$\frac{1}{C} V(\delta_\Omega(x)) \le u(x, t) \le C\, V(\delta_\Omega(x)) \quad x \in \Omega\,.$$

Proof. Denote by

$$H(x, t) = \frac{q(u(x, t))}{u(x, t)}.$$

Then H is a bounded, continuous function. Using Lemma 6.3.2 we then have

$$u(x, t) = \mathbb{E}_x \left[e^{\int_0^{T-t} H(X_s, t+s)\, \mathrm{d}s} u(X_{T-t}, T) \mathbb{1}_{\{T-t<\tau\}} \right].$$

Thus, for some constant C_1, get

$$e^{-C_1 T}\, \mathbb{E}_x \left[u(X_{T-t}, T) \mathbb{1}_{\{T-t<\tau\}} \right] \le u(x, t) \le e^{C_1 T}\, \mathbb{E}_x \left[u(X_{T-t}, T) \mathbb{1}_{\{T-t<\tau\}} \right]. \quad (6.3.21)$$

Using (6.3.12) and (6.3.21) we obtain

$$u(x, t) \le C\, V(\delta_\Omega(x)),$$

which gives the upper bound. Now from [41, Theorem 4.5] we know that

$$p_\Omega(t, x, y) \ge \kappa\, \mathbb{P}_x(\tau > t/2)\ \mathbb{P}_y(\tau > t/2)\, p(t \wedge V^2(r), |x - y|)$$

and

$$\mathbb{P}_x(\tau > t/2) \ge \kappa \left(\frac{V(\delta_\Omega(x))}{\sqrt{t \wedge V(r)}} \wedge 1 \right),$$

where Ω satisfies the inner and outer ball condition with radius r. Now let $\mathcal{K} \Subset \Omega$ be such that $\min_{\mathcal{K}} u(x,T) \geq \kappa_2 > 0$. Using the lower bound in (6.3.21) and estimates above we then find

$$\begin{aligned}
u(x,t) &\geq e^{-C_1 T} \int_{\Omega} u(y,T)\, p_{\Omega}(T-t,x,y)\mathrm{d}y \\
&\geq C_2 \kappa\, \mathbb{P}_x\left(\tau > \frac{T-t}{2}\right) \int_{\mathcal{K}} u(y,T) p((T-t)\wedge V^2(r), |x-y|)\, \mathbb{P}_y\left(\tau > \frac{T-t}{2}\right) \mathrm{d}y \\
&\geq C_3 \kappa^2 \kappa_2\, V(\delta_{\Omega}(x))\, p((T-t)\wedge V^2(r), \mathrm{diam}(\Omega)) \int_{\mathcal{K}} \mathbb{P}_y\left(\tau > \frac{T-t}{2}\right) \mathrm{d}y \\
&\geq C^{-1}\, V(\delta_{\Omega}(x)),
\end{aligned}$$

for some constants C_2, C_3, C. This gives the lower bound. Hence the proof. □

Our next result establishes long time behavior of the solution of the parabolic logistic equation. Recall that given an interval $[0,T]$, u_T solves

$$\begin{aligned}
(\partial_t - \Psi(-\Delta))u_T + a u_T - f(x,u_T) &= 0 \quad \text{in } \Omega \times [0,T), \\
u_T(x,T) = u_0(x) \text{ and } u_T(x,t) &= 0 \quad \text{in } \Omega^c \times [0,T],
\end{aligned} \tag{6.3.22}$$

where $0 \lneq u_0 \in C_0(\Omega)$.

Theorem 6.3.1. *Let u_T be the positive and bounded solutions of (6.3.22) in $[0,T]$. Then the following hold.*

(a) For $a > \lambda_1$, we have $\lim_{T\to\infty} u_T(x,0) \to v_a$, uniformly in Ω, where v_a is the unique solution of (6.2.6).

(b) For $a \leq \lambda_1$, we have $\lim_{T\to\infty} u_T(x,0) \to 0$, uniformly in Ω.

Proof. First consider (i). We divide the proof in two steps.

Step 1. First we note that

$$u_T(x,T-1) = \mathbb{E}_x\left[\mathbb{1}_{\{1<\tau\}} u_0(X_1)\right] + \mathbb{E}_x\left[\int_0^1 \mathbb{1}_{\{s<\tau\}} F(X_s, u_T(X_s, T-1+s))\,\mathrm{d}s\right],$$

where $F(x,s) = as - f(x,s)$. Thus $u_T(x,T-1)$ is independent of T (by Lemma 6.3.3). In fact, it is same as $v(x,0)$ where v solves (6.3.22) in $[0,1]$. Also, from (6.3.6) (taking $\delta = T - 1 - t$) we note that for $t \leq T - 1$ we have

$$u_T(x,t) = \mathbb{E}_x\left[\mathbb{1}_{\{T-1-t<\tau\}} u_T(X_{T-1-t}, T-1)\right] + \mathbb{E}_x\left[\int_0^{T-1-t} \mathbb{1}_{\{s<\tau\}} F(X_s, u_T(X_s, t+s)) \mathrm{d}s\right].$$

Thus without any less of generality we may assume $u_0 = u_T(x, T-1)$. In particular, by Lemma 6.3.6, we obtain

$$C^{-1}\, V(\delta_\Omega(x)) \leq u_0(x) \leq C\, V(\delta_\Omega(x)), \quad \text{for } x \in \Omega. \tag{6.3.23}$$

Step 2. Let v_a be the unique positive solution (see Theorem 6.2.1) to

$$-\Psi(-\Delta)\, v_a + a v_a - f(x, v_a) = 0 \quad \text{in } \Omega, \quad v_a = 0 \quad \text{in } \Omega^c, \quad v_a > 0 \quad \text{in } \Omega\,. \tag{6.3.24}$$

Using (6.3.23), Theorems 6.1.1 and 6.1.2, we choose $\kappa > 1$ large enough so that

$$\breve{\varphi}(x) \coloneqq \kappa^{-1} v_a(x) \leq u_0(x) \leq \kappa v_a(x) \coloneqq \hat{\varphi}(x), \quad x \in \Omega\,.$$

Note that $\breve{\varphi}$ is subsolution to (6.3.24) and $\hat{\varphi}$ is a supersolution to (6.3.24). Setting $\hat{\varphi}$ as the terminal condition at time T we construct a solution $\hat{w}_T$ in $[0, T]$ with $\hat{w}_T \leq \hat{\varphi}$. This can be done using Lemma 6.3.5. Next we observe that $\hat{w}$ is increases with t. For instance, take $t_1 \leq t_2 \leq T$ with $T - t_2 = t_2 - t_1$. Observe that $\xi(x,t) = \hat{w}_T(x, t - t_2 + t_1)$ is a solution to

$$\begin{aligned}
(\partial_t - \Psi(-\Delta))u + au - f(x,u) &= 0 \quad \text{in } \Omega \times [t_2, T), \\
u(x,T) = u_T(x, t_2) \text{ and } u(x,t) &= 0 \quad \text{in } \Omega^c \times [t_2, T].
\end{aligned}$$

Using the uniqueness of solutions and comparison principle (Lemma 6.3.3) we see that $\hat{w}_T(x, t_1) \leq \hat{w}_T(x, t_2)$. For any pair $t_1 \leq t_2 \leq T$ the same comparison holds due to continuity with respect to t and a density argument. Another application of Lemma 6.3.3 gives that $u_T(x, 0) \leq \hat{w}_T(x, 0) \leq \hat{\varphi}(x)$. Now apply Lemma 6.3.4 to invoke equi-continuity and show that $\hat{w}_T(x, 0) \to \hat{w}$ as $T \to \infty$. Then passing limit in

$$\hat{w}_T(x,0) = \mathbb{E}_x\left[\mathbb{1}_{\{T<\tau\}}\hat{\varphi}(X_T)\right] + \mathbb{E}_x\left[\int_0^T \mathbb{1}_{\{s<\tau\}}F(X_s,\hat{w}_T(X_s,s))\mathrm{d}s\right],$$

as $T\to\infty$, we obtain

$$\hat{w}(x) = \mathbb{E}_x\left[\int_0^\tau F(X_s,\hat{w}(X_s,s))\mathrm{d}s\right] = \mathcal{G}F(\cdot,\hat{w})(x).$$

This, in particular, implies

$$-\Psi(-\Delta)\,\hat{w} + a\hat{w} - f(\hat{w}) = 0 \quad \text{in } \Omega, \qquad \hat{w} = 0 \quad \text{in } \Omega^c.$$

From uniqueness we must have $\hat{w} = v_a$.

Follow a similar argument to construct a sequence of solution $\breve{w}$ (decreasing in t) satisfying

$$\breve{\varphi} \le \breve{w}_T(x,0) \le u_T(x,0).$$

Argument similar to above shows that

$$\lim_{T\to\infty}\sup_{\Omega}|\breve{w}_T(x,0) - v_a| = 0.$$

Combining these two observations we complete the proof of (i).

(ii) Proof is similar to (i). For $a \le \lambda_1$, we take φ_1 as the super-solution to (6.3.24). Then repeating a same argument we can conclude the proof. □

To the best of our knowledge, there are no available results similar to Theorem 6.3.1 in nonlocal setting. However, there are quite a few works on the fractional Fisher-KPP equation in $\mathbb{R}^d$; see for instance, Berestycki-Roquejoffre-Rossi [17], Cabré-Roquejoffre [47], Léculier [124] and references therein. For nonlocal dispersal operators in $\mathbb{R}^d$ large time behaviour has been studied by Berestycki-Coville-Vo [14], Cao-Du-Li-Li [53], Su-Li-Lou-Yang [158] and references therein. The method used in these works are not applicable for our model. Since our nonlocal operator is quite general in nature there are no existing parabolic pde estimate (other than fractional Laplacian) that can be used to obtain our result. So we relied on the heat-kernel estimates of the underlying stochastic process X, and hence the reason to use probabilistic representation of the solution.

Epilogue

In this section, we will briefly discuss some open problems that arise from the thesis.

Problem 1: In Chapter 2, we proved Faber-Krahn inequality for a class of integro-differential operators characterizing that balls minimize the principal eigenvalue among sets of a given volume. Next, we could study a stability result for sets that *almost* attain the minimum principal eigenvalue. That is, if the principal eigenvalue on some set Ω is very close to the principal eigenvalue of the ball of same volume, then how different Ω is from a ball in the measure theoretic sense. Whether there exist two balls B_1 and B_2 such that $B_1 \subset \Omega \subset B_2$ and the volume is these balls are very close to the volume of Ω (see [19, Theorem 1.2]). This will require a careful analysis of the super level sets of the principal eigenfunction on a given set Ω.

Problem 2: In Chapter 4, we have seen the global $C^{1,\alpha}$-regularity of the solution of inequalities (4.0.2) over a bounded C^2 domain (see Theorem 4.5.1) with certain integrability assumption on the nonlocal kernels $N_{\theta\nu}$. More precisely, the kernels are uniformly bounded above by some kernel $k(y)$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^\alpha)k(y)\mathrm{d}y < \infty$ for some $0 \leq \alpha < 2$ (see Assumption 4.0.1). Both these assumptions arise due to some technical reasons. It will be interesting to investigate the global regularity when these conditions are relaxed. In short, we could study the boundary regularity properties of the solution u to the following inequations:

$$\begin{aligned} \mathcal{L}u + C_0|Du| &\geq -K \quad \text{in } \Omega, \\ \mathcal{L}u - C_0|Du| &\leq K \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \end{aligned}$$

where $C_0, K \geq 0$, $\mathcal{L}$ is an integro-differential operator defined in (4.0.1) and Ω is a bounded $C^{1,1}$ domain. Assume the nonlocal kernel k to satisfy Assumption 4.0.1 with $\alpha = 2$. Although most of the technical lemmas obtained in Chapter 4 fail in this scenario, the global Lipschitz regularity and higher interior regularity still hold with this assumption, as we have seen in Section 4.2 and Section 4.1.

Problem 3: Another interesting problem to consider is the symmetry result for the fully nonlinear nonlocal operators, especially with respect to the extremal Pucci operators for fraction Laplacian type. The model problem that may be considered is the following equation

$$\mathcal{M}^+_{\text{fL}}(u) + f(u) = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ in } \Omega^c, \tag{P1}$$

where f is a Lipschitz function and $\mathcal{M}^+_{\text{fL}}$ is the maximum Pucci operator defined as in Section 5.1 with $\varphi = 0$. It is important to study the symmetry of the solution of the equation (P1) provided the domain Ω is symmetric, as we did in Theorem 2.4.1 for the linear integro-differential operators. We mention here that Theorem 2.4.1 is proved via the standard moving plane method and the proof relies on the fact that the nonlocal kernel is radially decreasing. But $\mathcal{M}^+_{\text{fL}}$ does not enjoy such properties which makes it hard to utilize the moving plane method, and thus further investigation is needed to study this problem.

Problem 4: In Chapter 6, we studied the nonlocal Fisher-KPP model and showed a local bifurcation result for steady state logistic equation with harvesting term (6.2.1), establishing the existence of two distinct positive solutions u_1, u_2 in Theorem 6.2.3. It will be interesting to investigate the global bifurcation for positive solutions. For instance, in case of Laplacian operator, this problem was considered in [139] where they proved that when a is slightly larger than λ_1, the branch of large solution, that is u_1, connects to other the branch of small solution, that is u_2, after bifurcating from 0. To be precise, they showed that there exists a $\delta_1 > 0$ such that for $a \in (\lambda_1, \lambda_1 + \delta_1)$ equation (6.2.1) with the Laplacian has exactly two positive solutions $u_1(\cdot, c)$ and $u_2(\cdot, c)$ for $c \in (0, c_2)$, exactly one positive solution $u_1(\cdot, c)$ for $c = c_2$ and no positive solution for $c > c_2$. Furthermore, set (c, u) lies on some smooth curve. It would be interesting to investigate similar phenomenon for the nonlocal Fisher-KPP model with operator $\Psi(-\Delta)$. The challenging part is to determine a way to use classical bifurcation theory and develop appropriate tools for this purpose. It appears to us that most of the variational techniques may not be useful when working with a general nonlocal operator $\Psi(-\Delta)$.

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