SPINORIAL REPRESENTATIONS OF LIE GROUPS

A thesis

submitted in partial fulfillment of the requirements

of the degree of

Doctor of Philosophy

by

Rohit Suhas Joshi

ID: 20093046

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

August, 2016

Dedicated to

My Parents & Teachers

Certificate

Certified that the work incorporated in the thesis entitled "*Spinorial Representations of Lie groups*", submitted by *Rohit S. Joshi* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: August 30, 2017

Dr. Steven Spallone

Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Date: August 31, 2016

Rohit Suhas Joshi

Roll Number: 20093046

Acknowledgements

To my Family: because I owe it all to you. First and foremost, I would like to express my sincere gratitude to my thesis supervisor Dr. Steven Spallone for the continuous support, for his patience, motivation, enthusiasm and encouragement. He was always ready to discuss with me. He trusted my ability and was patient enough to explain anything to me. I would like to thank Prof. Dipendra Prasad for formulating a nice question for the thesis.

Besides my supervisor, I would like to thank the rest of my research advisory committee: Prof. Raghuram and Dr. Chandrasheel Bhagawat for their insightful comments and encouragement. I had the opportunity to talk mathematics with several people. I would like to thank them for their support and encouragement. In particular, I would like to thank Prof. Dipendra Prasad, Prof. Amritanshu Prasad.

I thank Dr. Diganta Borah who was minor thesis advisor. I also thank Dr. Rama Mishra and Dr. Anupam Singh for supporting in my bad time. I owe thanks to Dr. Vivek Mallik , Dr. Kaneenika who were my teachers in my coursework. I am grateful to all of them. I am thankful to CSIR for the financial support in the form of the research fellowship. I would like to acknowledge the support of the institute and its administrative staff members for their cooperation, special thanks to Mrs. Suvarna Bharadwaj, Mr. Tushar Kurulkar and Mr. Kalpesh Pednekar. I am grateful to my teachers starting from my school days till date having faith in me and guiding me in right direction. I *thank all my friends in IISER Pune, with whom I shared good times and bad times as well. Many of you will recognize yourselves, and I hope that you will forgive me for not naming you individually. All the students of Mathematics at IISER Pune deserve a note of appreciation for being enthusiastic about discussing mathematics with me. I thank Sushil, Yasmeen, Hitesh, Rashmi, Sudhir, Manidipa, Pralhad, Prabhat, Makarand, Jatin, Neha, Gunja, Jyotirmoy, Milan, Tathagata, Debangana, Ayesha, Girish, Advait, and others for their help and discussions. I thank Tommy Tang, Sushil and Neha for correcting my English.*

Finally, I must express my deepest gratitude to my parents for providing me with unconditional support and constant encouragement throughout my xi years of study and through the process of research and write this thesis and my life in general, without whom this thesis would not have existed. I am also grateful to my other family members who have supported me along the way. It is not possible to express my gratitude towards them in words.

Contents

Abstract

We solve the question: which finite-dimensional irreducible orthogonal representations of connected reductive complex Lie groups lift to the spin group? We have found a criterion in terms of the highest weight of the representation, essentially a polynomial in the highest weight, whose value is even if and only if the corresponding representation lifts. The criterion is closely related to the Dynkin Index of the representation. We deduce that the highest weights of the lifting representations are periodic with a finite fundamental domain. Further, we calculate these periods explicitly for a few low-rank groups.

Chapter 1

Introduction

1.1 General Method

Let (ϕ, V) be an irreducible self-dual finite-dimensional complex representation of a connected reductive Lie group G. Then V admits a G-invariant non-degenerate bilinear form B, unique up to scalars. In the case, when B is symmetric, ϕ can be regarded as a homomorphism $\phi : G \to \mathrm{SO}(V)$. Such a ϕ is called an orthogonal representation. Further, if this representation of G lifts to the spin group $\mathrm{Spin}(V)$ then it is called spinorial.

Here we restrict our study to complex reductive Lie groups. Consider three basic questions:

1) Which irreducible representations of G are self-dual?

2) Which of those self-dual representations are orthogonal?

3) Which of those orthogonal representations are spinorial?

The spinoriality question is equivalent to completing the follwing diagram:

$$\begin{array}{ccc} & & \mathrm{Spin}(n,\mathbb{C}) \\ & \overset{\psi}{\nearrow} & \big\downarrow \rho \\ G & \xrightarrow{\phi} & \mathrm{SO}(n,\mathbb{C}). \end{array}$$

If we restrict our study to semi-simple complex Lie groups, we can get answers in terms of the highest weight of the representation. A good reference for highest weight theory is [Serre(2001)]. The answer to the first two questions are well-known. (See Sections 2.2 and 2.3)

Spinoriality: The third question was posed by Dipendra Prasad and Dinakar Ramakrishnan in their paper [Prasad and Ramakrishnan(1995)]. We present a solution to the third question for the case of general reductive complex Lie groups.

Any irreducible orthogonal representation of a complex reductive group G factors through $G/(Z(G)^{\circ})$. The question of spinoriality in the case of the reductive group G is the same as that for the semi-simple group $G/(Z(G)^{\circ})$ (see Section 7.1). Here we can use highest weight theory. We would like an answer to the spinoriality question in terms of highest weights.

General Strategy: We focus on a complex semi-simple Lie group G. Let $\phi : G \to \mathrm{SO}(N,\mathbb{C})$ be an orthogonal irreducible representation of G of highest weight λ. Let ϕ_* denote the map induced by ϕ between their fundamental groups. The lift exists if and only if the map ϕ_* is trivial. Thus for simply connected groups all irreducible orthogonal representations are spinorial.

Let T_G be a maximal torus of G. The map induced at the level of fundamental groups by the canonical injection of T_G into G is surjective. Thus, it is enough to check whether the map $\phi|_{T_G *}$ is trivial.

Let Q denote the additive subgroup of $X_*(T_G)$ (see 2.1) generated by co-roots of T_G. We know that $Q < X_*(T_G)$, and $\pi_1(G) \cong X_*(T_G)/Q$. The map $\phi|_{T_G *}$ takes $\nu \in X_*(T_G)$ to $\phi \circ \nu \in X_*(T_{\mathrm{SO}(N,\mathbb{C})})$. Since T_G is a complex torus, due to topological reasons, $\phi|_{T_G *}$ is trivial if and only if $(\phi \circ \nu)_*$ is trivial for every co-character ν of T_G. The map $(\phi \circ \nu)_*$ is trivial if and only if the co-character $\phi \circ \nu$ lifts to $\mathrm{Spin}(N, \mathbb{C})$.

Thus the co-character $\phi \circ \nu$ lifts to the spin group, for every ν, if the representation ϕ_λ is a spinorial. Thus we need a criterion for a given co-character ν, when the co-character $\phi \circ \nu$ lifts to the Spin group.

We have shown that $\phi \circ \nu$ is a lifting co-character if and only if

$$F_\nu(\lambda) = \sum_{\{\mu \in P(\phi) | \langle \mu, \nu \rangle > 0\}} m_\lambda(\mu) \langle \mu, \nu \rangle,$$

is an even integer, where $P(\phi)$ is the set of weights appearing in the representation ϕ, and $m_\lambda(\mu)$ is the multiplicity of μ in ϕ. This is similar to Lemma 3 in [Prasad and Ramakrishnan(1995)].

Now if we take an irreducible ϕ with highest weight λ, we have proved

$$F_\nu(\lambda) \equiv Q''_\nu(1)/2 \mod 2,$$

where

$$Q_\nu(a) = \sum_{\mu \in P_\phi} m_\lambda(\mu) a^{\langle \mu, \nu \rangle},$$

and $m_\lambda(\mu)$ is as above. It is easy to see that, for a self-dual representation ϕ_λ, we have

$$Q''_\nu(1) = \sum_{\mu \in P_\phi} m_\lambda(\mu) \langle \mu, \nu \rangle^2. \tag{1.1}$$

Now the whole task is to find out a nice expression for the RHS of Equation (1.1). For that, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. As in [Goodman and Wallach(2009)] we work with the $\mathbb{C}$-algebra $\mathbb{C}[\mathfrak{h}^*]$. A typical element of this algebra is of the form

$$\sum_{\mu \in \mathfrak{h}^*} a_\mu e^\mu,$$

where the sum is finite. We define

$$A(\mu) = \sum_{w \in W} \mathrm{sgn}(w)(e^{w(\mu)}) \in \mathbb{C}[\mathfrak{h}^*],$$

where W is the Weyl group of G. Let $\frac{\partial}{\partial \nu}$ be the derivation of this algebra defined by $\frac{\partial}{\partial \nu}(e^\mu) = (\mu, \nu)e^\mu$. Let us define the augmentation map ϵ by $\epsilon(\sum_{\beta \in S} c_\beta e^\beta) = \sum_{\beta \in S} c_\beta$, where $S \subset \mathfrak{h}^*$ is a finite set. We put $\mathrm{Ch}(V^\lambda) = \sum_{\mu \in P_\phi} m_\lambda(\mu) e^\mu$. We have

$$\sum_{\mu \in P_\phi} m_\lambda(\mu)(\mu, \nu)^2 = \epsilon\left(\frac{\partial^2}{(\partial \nu)^2} \mathrm{Ch}(V^\lambda)\right). \tag{1.2}$$

It remains to compute the RHS. To do this, we use the Weyl character formula, which in the above notation is

$$A(\lambda + \rho) = \mathrm{Ch}(V^\lambda) A(\rho).$$

Now we apply the derivation $m+2$ times to both sides of the Weyl character formula, where m is the number of positive roots of $\mathfrak{h}$ in $\mathfrak{g}$. Suppose $\mathfrak{g}$ is simple, after some work we arrive at the satisfying answer

$$Q''_\nu(1) = \frac{(\dim V^\lambda)(|\nu|^2)(|\lambda + \rho|^2 - |\rho|^2)}{\dim \mathfrak{g}}. \tag{1.3}$$

To summarize, we have :

Theorem 1.1.1. *Let G be a connected complex Lie group having a simple Lie algebra $\mathfrak{g}$. Let T_G be its maximal torus. Then ϕ_λ is spinorial if and only* *if the integer*

$$\frac{(\dim V^{\lambda})(|\nu|^2)(|\lambda+\rho|^2-|\rho|^2)}{2\cdot\dim\mathfrak{g}}$$

is even for every co-character ν *of* T_G*, where the norms correspond to the Killing form on* $\mathfrak{h}$ *and* $\mathfrak{h}^*$*.*

Observe that the terms other than $|\nu|^2$ in the theorem do not change when we keep Lie algebra same and change the group, however observe that $|\nu|^2 = |d\nu(1)|^2$, and $d\nu(1)$ which is a member of infinitesimal co-character lattice which changes with the group. For example if $m : G_1 \to G_2$ is an isogeny and suppose a maximal torus of G_1 is T_1 which maps to that of G_2 is say T_2 under m, then the set $S_1 = \{d\nu(1) \mid \nu \in X_*(T_1)\}$ is a subset of $S_2 = \{d\nu(1) \mid \nu \in X_*(T_2)\}$. Thus a representation of G_1 which factors through G_2 is spinorial for G_1 if it is spinorial for G_2, but converse is not true. For example take $G_1 = \mathrm{SL}(2,\mathbb{C})$ and $G_2 = \mathrm{PGL}(2,\mathbb{C})$ and $m(A) = A$ mod $Z(\mathrm{GL}(2,\mathbb{C}))$. Then

$$S_1 = \{\begin{bmatrix} n & 0 \\ 0 & -n \end{bmatrix}, n \in \mathbb{Z}\}$$

, while

$$S_2 = \{\begin{bmatrix} n/2 & 0 \\ 0 & -n/2 \end{bmatrix}, n \in \mathbb{Z}\}$$

and

$$\inf_{v\in S_1} ord_2(|v|^2) = 2$$

while that for S_2 is 1. According to the Theorem 1.1.1 of the thesis, if we take the highest weight j then according to section 6.2.1 of the thesis the rest part is

$$\frac{j(j+1)(2j+1)}{6} \in \mathbb{Z}.$$

So for $\mathrm{SL}(2,\mathbb{C})$ we have to check parity of

$$\frac{2^2 \cdot j(j+1)(2j+1)}{2 \cdot 6}$$

which is always even hence it is always spinorial. For $\mathrm{PGL}(2,\mathbb{C})$ the parity of

$$\frac{2^1 \cdot j(j+1)(2j+1)}{2 \cdot 6}$$

matters which can be odd for example if $j = 2$ it is 5.

With some more work for general case i.e., for connected complex semisimple Lie groups G whose Lie algebra $\mathfrak{g}$ may not be simple, we have the following theorem.

Theorem 1.1.2. *Let G be a connected complex semisimple Lie groups whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g} = \oplus \mathfrak{g}_{\mathrm{i}}$, where each $\mathfrak{g}_i$ is simple. Then the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the direct sum of the Cartan subalgebras $\mathfrak{h}_i$ of $\mathfrak{g}_i$, and we have $\mathfrak{h}^* \cong \oplus \mathfrak{h}_i^*$. Therefore we can write $\lambda = \oplus \lambda_i$, $\rho = \oplus \rho_i$ and for an infinitesimal cocharacter $\nu = \oplus \nu_i$.*

Then the representation ϕ_λ is spinorial if and only if the integer

$$\frac{Q''_\nu(1)}{2} = \dim V^\lambda \sum_{i=1}^{k} \frac{|\nu_i|^2(|\lambda_i + \rho_i|^2 - |\rho_i|^2)}{2 \dim \mathfrak{g}_i}$$

is even for every co-character ν of T_G, where the norms correspond to the Killing forms of $\mathfrak{g}_i$.

Scholium 1.1.3. *To determine the spinoriality of ϕ it is enough to determine the parity of $Q''_\nu(1)/2$ for the co-characters ν which represent the generators of $\pi_1(G) = X_*(T)/Q$ which is finite, where $X_*(T)$ is the co-character lattice and Q is the co-root lattice of G. Thus problem reduces to checking this criterion for **finite number of co-characters**.*

An important point is to note that the $Q''_\nu(1)/2$ is a polynomial in λ.

There is a nice relation with the Dynkin invariant of the representation ϕ_λ. The Dynkin invariant of a morphism between simple Lie algebras, is defined to be the ratio, $\frac{(\phi_\lambda(x),\phi_\lambda(y))_d}{(x,y)_d}$, where $(,)_d$ denotes the normalized Killing form such that $(\alpha,\alpha)_d = 2$, where α is any long root in the corresponding simple Lie algebra. In fact, the Dynkin invariant is an non-negative integer. A good reference for Dynkin Index is [Vinberg(1994)].

In our case, we assume that $\mathfrak{g}$ is simple. We have the map $\phi_\lambda : \mathfrak{g} \to \mathfrak{so}(V)$. The corresponding Dynkin invariant is

$$\operatorname{dyn}(\phi_\lambda) = \frac{\dim V^\lambda \cdot (|\lambda+\rho|^2 - |\rho|^2)}{\dim \mathfrak{g} \cdot (\alpha,\alpha)},$$

where $(,)$ is the Killing form.

Thus our criterion reduces to

Theorem 1.1.4. *In the case where $\mathfrak{g}$ is simple, the representation ϕ_λ is spinorial if and only if*

$$\frac{(\alpha,\alpha)(\nu,\nu)\operatorname{dyn}(\phi_\lambda)}{2} \equiv 0 \mod 2,$$

for every co-character ν.

Thus

$$F_\nu(\lambda) \equiv \frac{(\alpha,\alpha)(\nu,\nu)\operatorname{dyn}(\phi_\lambda)}{2} \mod 2.$$

Here we obtain an integer valued function $F_\nu(\lambda)$, whose parity determiness whether $\phi \circ \nu$ is a lifting co-character. Hence we obtain $\phi_* : \nu \mapsto (F_\nu(\lambda) \mod 2)$. Since $\pi_1(G)$ is finite, our problem reduces to a **finite problem** of determining whether the representative co-characters for the generators of $\pi_1(G)$ lift.

For n odd, it is easy to see that all of the irreducible orthogonal representations of $\mathrm{GL}(n, \mathbb{C})$ are spinorial.

For the reductive group $G = \mathrm{GL}(n, \mathbb{C})$, the associated semisimple group is $\mathrm{PGL}(n, \mathbb{C})$. We have made some explicit calculation for the expression in Equation (1.3) for $\mathrm{PGL}(n, \mathbb{C})$ for n even.

We have also calculated the expression in Equation (1.3) for the case $\mathrm{SO}(m, \mathbb{C})$, which exhausts the case of classical groups.

We prove here that the adjoint representation is spinorial if and only if half the sum of positive roots is an integral weight.

To treat compact Lie groups we simply complexify. There is a corresponding compact Spin group which is the double cover of real orthogonal group. We denote it by $\mathrm{Spin}(n, \mathbb{R})$. We have $\phi : G \to \mathrm{SO}(n, \mathbb{R})$ lifts to $\mathrm{Spin}(n, \mathbb{R})$ $\Leftrightarrow$ its complexification is spinorial (see Chapter 9).

A cone lattice is the intersection of a lattice and a cone with 0 as its vertex. The highest weights corresponding to irreducible self-dual representations and orthogonal representations form a cone lattice. We call them by $P_{\mathrm{sd}}(G)$ or $P_{\mathrm{sd}}(\mathfrak{g})$ and $P_{\mathrm{orth}}(G)$ or $P_{\mathrm{orth}}(\mathfrak{g})$ respectively depending upon the context. See Chapter 5.

It is observed that the spinorial representations in general may not form a cone lattice. But they are periodic in the sense that there are suitable vectors p such that the representation corresponding to ϖ is spinorial if and only if the representation corresponding to $\varpi + p$ is spinorial.

This leads us to define:

$P'_{\mathrm{Spin}}(G) = \{\lambda \in P_{\mathrm{orth}} \mid F_\nu(\lambda) \equiv 0 \mod 2 \forall \nu \in X_*(T)\}$

$P_{\mathrm{Spin}}(G) = \{p \in P_{\mathrm{orth}}(G) \mid \lambda \in P'_{\mathrm{Spin}}(G) \Leftrightarrow \lambda + p \in P'_{\mathrm{Spin}}(G)\}$

Thus we have a chain $P_{\mathrm{Spin}}(G) \subseteq P_{\mathrm{orth}}(G) \subseteq P_{\mathrm{sd}}(G) \subseteq P_{\mathrm{sd}}(\mathfrak{g})$.

$P_{\mathrm{Spin}}(G)$ gives a kind of periodicity of the spinorial weights inside $P_{\mathrm{orth}}(G)$.

Theorem 1.1.5. Periodicity Theorem *The index* $[P_{\mathrm{orth}}(G) : P_{\mathrm{Spin}}(G)]$ *is* *finite.*

Thus we only have to determine the spinoriality for a finite set, the fundamental domain. And by periodicity we obtain the spinoriality for all the other weights. Thus we have converted an infinite problem into a finite algorithm.

1.2 Determinantal Identity Method

Theorem (1.1.1) solves in principle our spinoriality question, but our Periodicity Theorem leads to further questions

1) Determine precisely $P_{\mathrm{Spin}}(G)$.

2) What proportion of othogonal irreducible representations of G are spinorial?

We pursue these questions for $\mathrm{PGL}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$, and have complete answers for $\mathrm{PGL}(4), \mathrm{SO}(3), \mathrm{SO}(4)$ and $\mathrm{SO}(5)$. Our method is to use determinantal identities such as the Jacobi-Trudy identity for the character of the representations.

Using the "Determinantal Identity Method", we have determined $P_{\mathrm{Spin}}(G)$ and found the proportion of spinorial weights explicitly for the groups $\mathrm{PGL}(4), \mathrm{SO}(3), \mathrm{SO}(4), \mathrm{SO}(5)$. By use of this method we have also found explicit polynomials, whose variables are parameters of highest weight. If the polynomial value at certain weight is even it is spinorial otherwise not.

For example $\mathrm{GL}(2, \mathbb{C})$ corresponds to the semi-simple group $\mathrm{SL}(2, \mathbb{C})/Z \cong \mathrm{SO}(3, \mathbb{C})$. The highest weight for $\mathrm{SO}(3, \mathbb{C})$ is parametrized by a single integer n. All of the representations of $\mathrm{SO}(3, \mathbb{C})$ are self-dual and orthogonal.

Here F turns out to be $F_\nu(n) = \dfrac{n(n-1)}{2}$, which is a polynomial in

n. Hence, F is even, which is the criterion for spinoriality is given by the equation

$$n \equiv 0 \text{ or } 3 \mod 4.$$

So, for example here, we can see that the representation with highest weight n is spinorial if and only if the representation with highest weight $n+4$ is spinorial. Thus we have $P_{\mathrm{Spin}}(\mathrm{SO}(3,\mathbb{C})) = 4P_{\mathrm{orth}}(\mathrm{SO}(3,\mathbb{C}))$.

Now we discuss the "Determinantal Identity Method" for $\mathrm{GL}(n,\mathbb{C})$.

Strategy for $\mathrm{GL}(n)$

The Weyl character formula for $\mathrm{GL}(n,\mathbb{C})$ gives

$$\mathrm{Trace}(\phi_\lambda((x_1, x_2, \ldots, x_n))) = S_\lambda(x_1, x_2, \ldots, x_n).$$

Here S_λ is the Schur polynomial and λ is the highest weight of the representation.

We have the Jacobi-Trudy identity which says

$$S_\lambda(x_1, x_2, \ldots, x_n) = |H_{\lambda_i+j-i}|,$$

where the matrix in RHS has $(i,j)^{th}$ entry H_{λ_i+j-i}. Here H_n are complete symmetric polynomials. See page 455 [Fulton and Harris(1991)].

We have used here a slightly modified version of the same identity mentioned in page 131 [Prasad(2015)].

Here are a few results which we obtain by using the "Determinantal Identity Method". The group $\mathrm{GL}(2n,\mathbb{C})$ corresponds to the semi-simple group $\mathrm{PGL}(2n,\mathbb{C})$.

Theorem 1.2.1. *We have*

$$P_{\text{Spin}}(\mathrm{PGL}(2n)) \supseteq \langle 2^k(\varpi_1 + \varpi_{2n-1}), 2^k(\varpi_2 + \varpi_{2n-2}), \ldots, \\ 2^k(\varpi_{n-1} + \varpi_{n+1}), 2^{k+1}\varpi_n \rangle,$$

where ϖ_i *are the fundamental weights of* $\mathfrak{sl}_{2n}$.

We have $P_{\text{Spin}}(\mathrm{SO}(3,\mathbb{C})) = 4P_{\text{orth}}(\mathfrak{so}(3,\mathbb{C}))$.

The representation of $\mathrm{SO}(4,\mathbb{C})$ of highest weight (x, y) is spinorial if and only if

$$(1/6)(1 + x + y)(2x + x^2 - y - xy + y^2) \equiv 0 \mod 2.$$

We have $P_{\text{Spin}}(\mathrm{SO}(4)) = \langle (4, 0), (0, 4) \rangle = 4 \cdot P_{\text{orth}}(\mathfrak{so}(4,\mathbb{C}))$.

The representation of $\mathrm{SO}(5,\mathbb{C})$ with highest weight $\lambda = (\lambda_1, \lambda_2)$ is spinorial if and only if

$$\binom{\lambda_1 + 3}{4} - \binom{\lambda_2 + 2}{4} \equiv 0 \mod 2.$$

We have $P_{\text{Spin}}(SO(5,\mathbb{C})) = \langle (4, 4), (4, -4) \rangle = 8 \cdot P_{\text{orth}}(\mathfrak{so}(5,\mathbb{C}))$.

Here is the summary of the thesis. Throughout the thesis the group under consideration is a connected reductive complex Lie group. The second chapter contains the definitions and the criteria for an irreducible representation of being self-dual and orthogonal in terms of their highest weight. The third chapter starts with few lemmas useful for the strategy. It contains the strategy for determining whether the orthogonal representation is spinorial. The fourth chapter is the heart of the thesis. In this chapter we discuss the method to obtain general criterion for the spinoriality of a representation. The criterion is clearly in terms of the highest weight of the representation. Also we discuss the relation of this criterion with the Dynkin index of the representation. Chapter five contains definitions of certain free abelian groups that we associate to the Lie group under consideration or its Lie algebra. We denote them by $P_{\mathrm{sd}}(G)$, $P_{\mathrm{orth}}(G)$, $P_{\mathrm{Spin}}(G)$ and similarly for Lie algebra of G. Then we have the periodicity theorem which tells that the highest weights of the spinorial representations are periodic in the orthogonal weights and the fundamental domain $P_{\mathrm{orth}}(G)/P_{\mathrm{Spin}}(G)$ is finite. In the sixth chapter we discuss how representations of compact groups are related to their corresponding complexification. We prove here that the question of spinoriality of the representation of compact group G and its complexification are same. Hence we have the answer for the compact groups also. The seventh chapter contains the actual expression for the criterion of spinoriality for the classical groups. Here we also relate it to the Dynkin index of the representation. This much is the first part of the thesis.

We give the name "Determinantal Identity Method" to the second part of the thesis. In the tenth and eleventh chapter we use the determinantal identities for the Weyl character formula. In this part we deduce a determinantal polynomial expression in the highest weight as the criterion for the spinoriality for the groups $\mathrm{GL}(2n,\mathbb{C}), \mathrm{SO}(3,\mathbb{C}), \mathrm{SO}(4,\mathbb{C}),\ \mathrm{SO}(5,\mathbb{C})$. Using this expression we have deduced some lower bounds for $P_{\mathrm{Spin}}(\mathrm{GL}(2n,\mathbb{C}))$, also we have calculated $P_{\mathrm{Spin}}(\mathrm{SO}(3)), P_{\mathrm{Spin}}(\mathrm{SO}(4)), P_{\mathrm{Spin}}(\mathrm{SO}(5))$ and $P_{\mathrm{Spin}}(\mathrm{GL}(4))$ exactly. The twelfth chapter is the summary of the entire thesis. The Last chapter is Appendix which has some useful combinatorial lemmas.

Chapter 2

Preliminaries

Notation Throughout the thesis we denote the $k \times k$ diagonal matrix with diagonal entries $a_1, a_2, \cdots, a_k$ by $a_1 \oplus a_2 \oplus \cdots \oplus a_k$.

2.1 Definitions

Let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$.

We write a maximal Torus and a Cartan subalgebra of $\mathfrak{g}$ by T and $\mathfrak{h}$ respectively. We write Ad for the adjoint Representation of G. Furthermore we write α or β for the roots, R_+ for the set of positive roots, and R for the root system. Next we write H_α and $R^\vee$ for the co-roots and the inverse root system. The weights of a representation and the fundamental weights we denote by μ and ϖ_i respectively. Let ϕ be the irreducible representation and λ be its highest weight. The positive Weyl chamber we denote by C_0. Let W be the Weyl group of T with respect to G. Let w_0 be its longest element with respect to C_0. We denote the characters and the co-characters of maximal torus by μ and by ν respectively and the set of characters by $X^*(T)$ and the set of co-characters by $X_*(T)$.

We insist that reader should see any standard book on Lie group representation theory for example [Goodman and Wallach(2009)] for definitions of the above mentioned concepts.

Let $\phi : G \to \mathrm{GL}(V)$ be a finite dimensional complex representation.

Definition 2.1.1. *We call ϕ an **orthogonal (symplectic represen)** representation if it preserves a non-degenerate symmetric (alternating) bilinear form B, i.e, $B(\phi(g)v, \phi(g)w) = B(v, w)$ for all $v, w \in V$ and for all $g \in G$.*

Since all the non-degenerate symmetric bilinear forms are equivalent for the complex field, the complex orthogonal Lie group $O(V)$ is unique up to isomorphism. Further if G is connected, then ϕ can be realized as a map from G to $\mathrm{SO}(V)$, i.e. , $\phi : G \to \mathrm{SO}(V)$.

Definition 2.1.2. *The universal covering Lie group of $\mathrm{SO}(N, \mathbb{C})$ $(N \geq 1)$ is called the **complex spin group**. We denote it by $\mathrm{Spin}(N, \mathbb{C})$ and the covering map by ρ, i.e., $\rho : \mathrm{Spin}(N, \mathbb{C}) \to \mathrm{SO}(N, \mathbb{C})$.*

It is well-known that the fundamental group of $\mathrm{SO}(N, \mathbb{C})$ is $\mathbb{Z}/2\mathbb{Z}$. Hence, the group $\mathrm{Spin}(N, \mathbb{C})$ is the double cover of the group $\mathrm{SO}(N, \mathbb{C})$ and its fundamental group is trivial. We can give a constructive definition of $\mathrm{Spin}(n, \mathbb{C})$ as follows.

For this definition we refer [Jacobson(1980)].

Let $\mathbb{C}$ be the complex field. Let V be a finite dimensional vector space over $\mathbb{C}$. Let $T(V)$ be the tensor algebra of V. Let Q_0 be a non-degenerate quadratic form on V. Define Clifford algebra $C(V, Q_0) = \dfrac{T(V)}{< v \otimes v - Q_0(v) >}$. Let $C^+(V, Q)$ be the sub-algebra of $C(V, Q_0)$ generated by elements of the form $u \cdot v$, where $u, v \in V$. Define $\Gamma(V) = \{x \in C(V, Q_0) \mid x$ is invertible and $x \cdot v \cdot x^{-1} \in V \forall v \in V\}$. We define $\chi : \Gamma(V) \to \mathrm{GL}(V)$ by putting $\chi(x)(v) = x \cdot v \cdot x^{-1}$. Let $\Gamma^+(V) = \Gamma(V) \cap C^+(V, Q_0)$. Let $x \in \Gamma^+(V)$ then $x = v_1 \cdot v_2 \cdots v_{2r}$, where $v_i \in V$. We define map $N : \Gamma^+ \to F^\times$ by $N(x) = \prod_{i=1}^{2r} Q_0(v_i)$. Now Spin group $\mathrm{Spin}(V, Q_0)$ is defined as kernel of N.

Take $V = \mathbb{C}^n$ with a non-degenerate quadratic form

$$Q(x_1, \ldots, x_n) = -(x_1^2 + \cdots + x_n^2)$$

on it. Since all the non-degenerate quadratic forms on $\mathbb{C}^n$ are equivalent, the Clifford algebra is unique up to isomorphism.

According to page 309 Equation (20.31) of [Fulton and Harris(1991)]

$$\mathrm{Spin}(V, Q) = \{\pm w_1 \cdot w_2 \cdots w_{2k} \mid k \in \mathbb{Z}_{\geq 0}, w_i \in \mathbb{C}^m, Q(w_i, w_i) = -1\}.$$

Definition 2.1.3. *An orthogonal representation is called **spinorial** if it lifts to* $\mathrm{Spin}(N, \mathbb{C})$*. That is, if there exists a homomorphism of complex Lie groups* $\psi : G \to \mathrm{Spin}(N, \mathbb{C})$ *such that* $\rho \circ \psi = \phi$*.*

Equivalently, the following diagram should commute :

$$\begin{array}{ccc} & & \mathrm{Spin}(N, \mathbb{C}) \\ & \overset{\psi}{\nearrow} & \big\downarrow \rho \\ G & \xrightarrow{\phi} & \mathrm{SO}(N, \mathbb{C}). \end{array}$$

Let us denote the dual complex vector space of V by V^*.

Definition 2.1.4. *Let* $\phi : G \to \mathrm{GL}(V)$ *be a complex representation of group* G*. Then* $\phi^* : G \to \mathrm{GL}(V^*)$ *defined as* $\phi^*(g)(f)(x) = f(\phi(g^{-1})x)$ *for* $f \in V^*$ *is called the **dual representation** of* ϕ*.*

Definition 2.1.5. *A representation of G which is isomorphic to its dual, is called **self-dual**.*

Definition 2.1.6. *A polynomial* $f \in \mathbb{Z}[x, x^{-1}]$ *satisfying the condition* $f(x) = f(x^{-1})$ *is called **Laurent-palindromic polynomial**. The ring of*

Laurent palindromic polynomials is denoted by $\mathbb{Z}[x, x^{-1}]^{\text{sym}}$.

Equivalently the coefficient of x^i in f is the same as the coefficient of x^{-i} for every i, for f to be a Laurent palindromic polynomial.

We define the degree of $f \in \mathbb{Z}[x, x^{-1}]^{\text{sym}}$ to be the largest of the degrees of monomials appearing in f. We denote by $\mathbb{Z}[x, x^{-1}]_d^{\text{sym}}$ the set of Laurent polynomials of degree d.

Definition 2.1.7. *Given a degree n Laurent-palindromic polynomial*

$$f(t) = a_n(t^n + t^{-n}) + \cdots + a_1(t + t^{-1}) + a_0,$$

we define an operator $\Psi : \mathbb{Z}[t, t^{-1}]^{\text{sym}} \to \mathbb{Z}/2\mathbb{Z}$, *by*

$$\Psi(f) = \sum_{i=1}^{n} i a_i \mod 2.$$

Observe that Ψ is a $\mathbb{Z}$ linear operator.

Definition 2.1.8. *A polynomial of the form*

$$a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n$$

is called ***palindromic*** *if* $a_i = a_{n-i}$ *for* $0 \leq i \leq n$. *We denote such polynomials by* $\mathbb{Z}[x]^{\text{pal}}$.

We denote the set of palindromic polynomials of degree d by $\mathbb{Z}[x]_d^{\text{pal}}$.

Definition 2.1.9. *For* $f \in \mathbb{Z}[x]$, *such that*

$$f(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

where $a_i \in \mathbb{Z}$, *we define* $\tilde{\Psi}_d : \mathbb{Z}[x] \to \mathbb{Z}/2\mathbb{Z}$ *by*

$$\tilde{\Psi}_d(f) = \sum_{i=1}^{d} i a_{d-i} \mod 2.$$

Observe that $\tilde{\Psi}_d$ is also a $\mathbb{Z}$- linear operator and the diagram :

$$\begin{array}{ccc} \mathbb{Z}[t,t^{-1}]_d^{\mathrm{sym}} & \xrightarrow{\times t^d} & \mathbb{Z}[t]_{2d}^{\mathrm{pal}} \\ \downarrow{\scriptstyle \Psi} & \swarrow{\scriptstyle \tilde{\Psi}_d} & \\ \mathbb{Z}/2\mathbb{Z} & & \end{array} \tag{2.1}$$

commutes, where the horizontal arrow is the multiplication by t^d.

2.2 Self-dual Representations

Let G be a connected complex Lie group. Let $\phi : G \to \mathrm{GL}(V)$ be a self-dual representation. If after fixing a basis, $\phi(g) = A$, where A is a matrix, then A is conjugate to ${}^tA^{-1}$.

Theorem 2.2.1. *(Criterion for self-duality of an irreducible representation) Let (ϕ, V) be an irreducible representation of a complex semi-simple Lie group G with highest weight ϖ. Let (ϕ^*, V^*) be its dual representation. Let ω_0 be the longest element of the Weyl group of G. Then ϕ^* has highest weight $-\omega_0(\varpi)$. Hence ϕ is self-dual, if and only if, $\varpi = -\omega_0(\varpi)$.*

Proof. See Page 134 Chapter VIII section 7.5 Proposition 11 [Bourbaki(2005)]. □

Theorem 2.2.2. *Let (ϕ, V) be an irreducible self-dual representation of a complex Lie group G. Then it is either orthogonal or symplectic but not both.*

Proof. See Page 135 Chapter VIII section 7.5 Proposition 12 [Bourbaki(2005)]. □

2.3 Orthogonal Representations

Let G be a connected complex semi-simple Lie group and let $\mathfrak{g}$ be its Lie algebra.

Theorem 2.3.1. *Criterion for an irreducible self-dual representation to be orthogonal : Let (ϕ, V) be a self-dual irreducible representation of G with highest weight ϖ. Let H_α denote the co-root to root α. Then let m be the integer $\sum_{\alpha \in R_+} \langle \varpi, H_\alpha \rangle$, where R_+ denotes the set of positive roots.*

1) If m is even then ϕ is orthogonal.

2) If m is odd then ϕ is symplectic.

Proof. See Page 135 Chapter VIII section 7.5 Proposition 12 [Bourbaki(2005)].

□

Chapter 3

Determining Spinoriality

For setting up a strategy to determining spinoriality we first see the general criterion for lifting any group homomorphism $\phi : H \to G$ to a cover of G. Next we observe that inclusion of maximal torus induces surjection at the level of the fundamental groups. Thirdly we quote the isomorphism of fundamental group and the quotient of co-character lattice by co-root lattice.

3.1 Structural Lemmas

Let G be a complex Lie group. Let T be a maximal torus of G. Let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of T. Let $\exp : \mathfrak{g} \to G$ denote the exponential map. Let us denote $\mathrm{Hom}(T, \mathbb{C}^\times)$ by $X^*(T)$, and $\mathrm{Hom}(\mathbb{C}^\times, T)$ by $X_*(T)$. For a homomorphism f between two topological groups, let f_* denote the map between the fundamental groups, induced by f.

Lemma 3.1.1. *Let G, G', H be connected complex Lie groups and $\phi : H \to G$ be a homomorphism. Let $\alpha : G' \to G$ be a cover. Then ϕ can be lifted to G' if and only if the image of ϕ_* in $\pi_1(G)$ is contained in the image of α_*.*

Equivalently the following diagram should exist

$$\begin{array}{ccc} & & G' \\ & \nearrow & \big\downarrow \alpha \\ H & \xrightarrow{\phi} & G. \end{array}$$

Proof. It follows from the lifting theorem in algebraic topology, that there is a unique continuous topological lift ψ, which takes identity of H to identity of G'. We will prove that ψ is a group homomorphism. Let $*$ denote the multiplication in any group. We have $\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$. So we get $\alpha \circ \psi(g_1 * g_2) = \alpha\psi(g_1) * \alpha\psi(g_2)$. Hence $\alpha(\psi(g_1 * g_2)\psi(g_1)^{-1}\psi(g_2)^{-1}) = 1$. The image of the map $p : H \times H \to G'$ given by $(g_1, g_2) \to \psi(g_1 * g_2)\psi(g_1)^{-1}\psi(g_2)^{-1}$ is connected, since H is connected. The kernel of α is discrete, as it is a covering map. Thus, we get $(\psi(g_1 * g_2)\psi(g_1)^{-1}\psi(g_2)^{-1}) = 1$. Hence $\psi(g_1 * g_2) = \psi(g_1)\psi(g_2)$. Thus ψ is, in fact, a group homomorphism.

□

Theorem 3.1.2. *Let ι the inclusion map $\iota : T \hookrightarrow G$, then ι_* is surjective.*

Proof. See page 67 Theorem 2 c) in [Serre(2001)]. Let G be a complex reductive Lie group and T_G be a maximal complex torus of G. Let K be a maximal compact subgroup of G and T_K be its maximal compact torus which is also contained in T_G.

$$\begin{array}{ccc} T_G & \xrightarrow{i_1} & G \\ {\scriptstyle i_2}\big\uparrow & & \big\uparrow{\scriptstyle i_4} \\ T_K & \xrightarrow{i_3} & K \end{array}$$

According to page 257 Theorem 2.2 part 3 of [Helgason(1978)], G is diffeomorphic to $\mathbb{R}^d \times K$ for some positive integer d. Hence i_{4*} is surjective. According to page 223 Theorem (7.1) of [Bröcker and tom Dieck(2013)] the injection of compact maximal torus in a compact group induces a surjection between fundamental groups. Thus i_{3*} is surjective. We have $i_4 \circ i_3 = i_1 \circ i_2$, thus $i_{4*} \circ i_{3*} = i_{1*} \circ i_{2*}$. Thus i_{1*} is surjective and hence the proof.

□

Theorem 3.1.3. *We have $\pi_1(G) \cong X_*(T)/Q$, where Q is the sublattice of $X_*(T)$ generated by the co-roots of T.*

Proof. See page 67 Theorem 2 c) in [Serre(2001)]. It says that $\pi_1(G)$ is isomorphic to Γ/Q, where Γ is the kernel of the map $\exp|\mathrm{t} : \mathfrak{t} \to T$, which takes $x \in \mathfrak{t}$ to $\exp(2\pi i x) \in T$. Now $X_*(T) \cong \Gamma$ via the map $\nu \to d\nu(1)$, where $d\nu$ is the derivative of the map $\nu : \mathbb{C}^\times \to T$. From the commutativity of following diagram we get that $d\nu(1) \in \Gamma$.

$$\begin{array}{ccc} \mathbb{C}^\times & \xrightarrow{\nu} & T \\ {\scriptstyle e^{2\pi i z}}\big\uparrow & & \big\uparrow{\scriptstyle \exp|\mathrm{t}} \\ \mathbb{C} & \xrightarrow{\mathrm{d}\nu} & \mathfrak{t} \end{array}$$

□

3.2 Strategy

Let f_* denote the induced map at the level of π_1 by f. Theorem 3.1.2 states that if $i : T \hookrightarrow G$ is the inclusion of a maximal complex torus T into G then i_* is surjective. Lemma 3.1.1 says a lift exists if and only if $\text{image of } \phi_* \subseteq \text{ image of } \rho_*$.

$$\begin{array}{ccc} & & \mathrm{Spin}(N,\mathbb{C}) \ . \\ & \nearrow & \downarrow \rho \\ G & \xrightarrow{\phi} & \mathrm{SO}(N,\mathbb{C}) \end{array}$$

Since Spin(V) is simply connected, the image of ρ_* is trivial. Hence the lift exists if and only if the image of ϕ_* is trivial, which is true if and only if the map $\phi|_{T*}$ is trivial.

Now $\phi_* : \pi_1(G) \to \pi_1(\mathrm{SO}(N,\mathbb{C}))$ is trivial if and only if $\phi|_{T*} : \pi_1(T) \to \pi_1(\mathrm{SO}(N,\mathbb{C}))$ is trivial. We know that $\pi_1(G) \cong X_*(T)/Q$,by Lemma 3.1.3, where Q is the lattice generated by co-root. The map $\phi|_{T*}$ takes $\nu \in X_*(T)$ to $\phi \circ \nu \in X_*(T_{\mathrm{SO}(N,\mathbb{C})})$. Since T is a complex torus, due to topological reasons, $\phi|_{T*}$ is trivial if and only if $(\phi \circ \nu)_*$ is trivial for every co-character ν of T. $(\phi \circ \nu)_*$ is trivial if and only if the co-character $\phi \circ \nu$ lifts to a co-character of Spin($N, \mathbb{C}$) by Lemma 3.1.1.

Thus $\phi \circ \nu$ should be a lifting co-character for every ν if ϕ is a lifting representation. Thus we need a criterion for given ν, when is $\phi \circ \nu$ a lifting co-character.

Here we obtain an integer valued function $F_\nu(\lambda)$ (see 3.2.3), if whose parity is even then $\phi \circ \nu$ is lifting co-character and non-lifting otherwise. Since $\pi_1(\mathrm{SO}(N,\mathbb{C}))$ is $\mathbb{Z}/2\mathbb{Z}$, if the image $\phi \circ \nu$ is non-lifting, it corresponds to the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$. Hence we obtain ϕ_* which takes ν to $F_\nu(\lambda) \mod 2$. Since the fundamental group is finite, our problem becomes a finite problem of checking whether the representative co-characters for the generators of π_1, are lifting.

Lemma 3.2.1. *Let $V = \mathbb{C}^m$. Then take the maximal torus of* $\mathrm{SO}(V)$

$$T_V = x_1 \oplus x_2 \oplus \cdots \oplus x_k \oplus x_1^{-1} \oplus x_2^{-1} \oplus \cdots \oplus x_{k-1}^{-1} \oplus \begin{cases} x_k^{-1} & \text{if } m = 2k, \\ x_k^{-1} \oplus 1 & \text{if } m = 2k+1, \end{cases}$$

where each $x_i \in \mathbb{C}^\times$. Then $V = \oplus_{i=1}^k V_i \oplus_{i=1}^k V_i' \oplus V_0$, where V_i, V_i' and V_0 are the common eigenspaces of T_V corresponding to x_i, x_i^{-1} and 1 respectively. Let ν be a co-character of T_V. If

$$\nu(z) = z^{\theta_1} \oplus z^{\theta_2} \oplus \cdots \oplus z^{\theta_k} \oplus z^{-\theta_1} \oplus z^{-\theta_2} \oplus \cdots \oplus z^{-\theta_{k-1}} \oplus \begin{cases} z^{-\theta_k} & \text{if } m = 2k, \\ z^{-\theta_k} \oplus 1 & \text{if } m = 2k+1, \end{cases}$$

put $S_+ = \{i \mid \theta_i > 0\}$, $S_0 = \{i \mid \theta_i = 0\}$, and $S_- = \{i \mid \theta_i < 0\}$. Then

$$\sum_{i \in S_+} \theta_i + \sum_{i \in S_-} (-\theta_i) \equiv 0 \mod 2,$$

if and only if ν lifts to $\mathrm{Spin}(V)$.

Proof. Here our main reference will be Section 6.3 in the revised edition of [Goodman and Wallach(2009)] . Lemma 6.3.4 says that we can parametrize the maximal torus of $\mathrm{Spin}(V)$ by $w_1, w_2, \ldots, w_l$, where w_i are a set of coordinate functions. Then Theorem 6.3.5 says that

$$\rho(w_1, w_2, \ldots, w_l) = \begin{cases} z_1^2 \oplus z_2^2 \oplus \cdots \oplus z_k^2 \oplus z_k^{-2} \oplus z_{k-1}^{-2} \oplus \cdots \oplus z_1^{-2}, & \text{if } \dim V = 2k, \\ z_1^2 \oplus z_2^2 \oplus \cdots \oplus z_k^2 \oplus 1 \oplus z_k^{-2} \oplus z_{k-1}^{-2} \oplus \cdots \oplus z_1^{-2}, & \text{if } \dim V = 2k+1, \end{cases}$$

where

$$z_1^2 = \frac{w_1 w_2 \dots w_{l-1}}{w_l^{l-3}},$$

and

$$z_i^2 = \frac{w_l}{w_{i-1}},$$

for $2 \leq i \leq l$.

Let us assume that $m = 2l$. Now suppose

$$\nu(z) = z^{\theta_1} \oplus \cdots \oplus z^{\theta_l} \oplus z^{-\theta_l} \oplus \cdots \oplus z^{-\theta_1}.$$

Then we have to solve the system

$$z^{\theta_1} = \frac{w_1 w_2 \dots w_{l-1}}{w_l^{l-3}}, \tag{3.1}$$

and

$$z^{\theta_i} = \frac{w_l}{w_{i-1}},$$

for $2 \leq i \leq l$.

So we get

$$w_j = w_l z^{-\theta_{j+1}}, \tag{3.2}$$

for $1 \leq j \leq l-1$.

Finally putting (3.2) in the expression (3.1) for z^{θ_1} we get

$$z^{\theta_1} = w_l^2 z^{-\sum_{i=2}^{l} \theta_i}.$$

Thus we have

$$z^{\sum_{i=1}^{l} \theta_i} = w_l^2.$$

Therefore w_l is an integer power of z if and only if

$$\sum_{i=1}^{l} \theta_i$$

is even.

Now it is easy to see that

$$\sum_{i=1}^{l} \theta_i \equiv \sum_{i \in S_+} \theta_i + \sum_{i \in S_0} \theta_i + \sum_{i \in S_-} \theta_i \equiv \sum_{i \in S_+} \theta_i + \sum_{i \in S_-} (-\theta_i) \mod 2.$$

Hence

$$\sum_{i \in S_+} \theta_i + \sum_{i \in S_-} (-\theta_i) \equiv 0 \mod 2,$$

if and only if ν lifts to $\mathrm{Spin}(V)$.

A similar proof holds when m is odd. □

Lemma 3.2.2. *Let T be a maximal torus of G. Let $\nu : \mathbb{C}^\times \to T$ be a co-character. Let ϕ be an orthogonal irreducible finite-dimensional representation of G. Then*

$$\mathrm{Trace}(\phi \circ \nu(t)) = a_d(t^{-d} + t^d) + a_{d-1}(t^{-d+1} + t^{d-1}) + \cdots + a_1(t^{-1} + t) + a_0,$$

where t^i occur as the weights of the representation $\phi \circ \nu$ of $\mathbb{C}^\times$ with multiplicities a_i. We define $\Psi_\phi(\nu) = \Psi(\mathrm{Trace}(\phi \circ \nu))$ (see Definition 2.1.7). The co-character $\phi \circ \nu$ lifts to the spin group if and only if

$$\Psi_\phi(\nu) = \sum_{k=1}^{d} k \cdot a_k \equiv 0 \mod 2.$$

Proof. First we will prove that the $\mathrm{Trace}(\phi \circ \nu(t))$ has the form given above.

Let $A = \phi(t_1, t_2, \ldots, t_n) = \chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_N$ be the matrix of representation, where χ_i are the weights of T under ϕ. Since ϕ is self-dual, A is conjugate to ${}^{T}A^{-1}$. Then ${}^{T}A^{-1} = \chi_1^{-1} \oplus \chi_2^{-1} \oplus \cdots \oplus \chi_N^{-1}$. Thus if a character χ occurs k times, as one of the χ_is, then χ^{-1} should also occur k times. Observe that $\mathrm{Trace}(\phi \circ \nu(t)) = \sum_{j=1}^{N} \chi_j \circ \nu(t)$. The character $\chi_j \circ \nu(t) = t^e$ for some $e \in \mathbb{Z}$. So t^e and t^{-e} occur the same number of times in $\mathrm{trace}(\phi \circ \nu(t))$, which proves the first claim.

Now we prove the second assertion. From the above, it follows that t^j occurs the same number of times as t^{-j} in the trace. Hence it follows that

$$\mathrm{Trace}(\phi \circ \nu(t)) = a_d(t^d + t^{-d}) + a_{d-1}(t^{d-1} + t^{-d+1}) + \cdots + a_1(t + t^{-1}) + a_0,$$

where a_i is the multiplicity of t^i in the trace polynomial. Thus we have

$$\phi(\nu(t)) = (t^d \oplus t^{-d})^{\oplus a_d} \oplus (t^{d-1} \oplus t^{-(d-1)})^{\oplus a_{d-1}} \oplus \cdots \oplus (t \oplus t^{-1})^{\oplus a_1} \oplus 1^{\oplus a_0}.$$

With reference to Lemma 3.2.1,

$$\sum_{i \in S_+} \theta_i + \sum_{i \in S_-} (-\theta_i)$$

is the sum of positive numbers occurring as powers of t. Since a_i are non-negative, we have

$$\sum_{i \in S_+} \theta_i + \sum_{i \in S_-} (-\theta_i) = d \cdot a_d + (d-1) \cdot a_{d-1} + \cdots + 1 \cdot a_1.$$

Therefore by Lemma 3.2.1, ν is spinorial if and only if

$$\sum_{i=1}^{d} i \cdot a_i \equiv 0 \mod 2.$$

□

Proposition 3.2.3. *Let G be a complex semi-simple Lie group with maximal torus T. Let (ϕ, V) be a finite-dimensional holomorphic irreducible orthogonal representation of G of highest weight λ. Let ν be a co-character of T. Then $\phi \circ \nu$ lifts to the spin group if and only if*

$$F_\nu(\lambda) = \sum_{\{\mu \in P(\phi) | \langle \mu, \nu \rangle > 0\}} m(\mu, \lambda) \langle \mu, \nu \rangle,$$

is even. Here $P(\phi)$ denotes the set of weights appearing in ϕ and $m(\mu, \lambda)$ is the multiplicity of the weight μ in ϕ, and $\langle , \rangle$ is the pairing between the characters and the co-characters.

Proof. The representation ϕ is spinorial if and only if $\forall \nu : \mathbb{C}^\times \to T$, $\phi \circ \nu$ lifts to $\mathrm{Spin}(N, \mathbb{C})$. We can assume that $\phi(T) \subseteq T_V$, where T_V is the maximal torus mentioned in the Lemma 3.2.1.

Let ν be a co-character of T. Since ϕ is self-dual, $m(\mu, \lambda) = m(-\mu, \lambda)$. Let us denote the set $\{\mu \in P(\phi) \mid \langle \mu, \nu \rangle > 0\}$ by B and the set $\{\mu \in P(\phi) \mid \langle \mu, \nu \rangle = 0\}$ by B_0. Furthermore

$$\phi(\nu(t)) = \oplus_{\mu \in B} \left(\left(t^{\langle \mu, \nu \rangle}\right)^{\oplus m(\mu, \lambda)} \oplus \left(t^{\langle -\mu, \nu \rangle}\right)^{\oplus m(\mu, \lambda)} \right) \oplus_{\mu \in B_0} 1.$$

Then by Lemma 3.2.1 the sum of the positive indices of t in the co-character $\phi \circ \nu$ of T_V is

$$\sum_{i \in S_+} \theta_i - \sum_{i \in S_-} \theta_i = \sum_{\{\mu \in B\}} m(\mu, \lambda) \langle \mu, \nu \rangle.$$

Hence the proof. □

Proposition 3.2.4. *Let $F_\nu(\lambda)$ be as in Proposition 3.2.3. Then*

$$F_{w(\nu)}(\lambda) = F_\nu(\lambda).$$

Proof. Now

$$
\begin{aligned}
F_{w(\nu)}(\lambda) = & \sum_{\{\mu \in P(\phi) | \langle \mu, w(\nu) \rangle > 0\}} m(\mu, \lambda)\langle \mu, w(\nu)\rangle \\
& = \sum_{\mu \in P(\phi)} m(\mu, \lambda)\langle \mu, w(\nu)\rangle \delta_{w(\nu)}(\mu), \\
& (\text{where } \delta_\nu(\mu) = 1 \text{ if } \langle \mu, \nu \rangle > 0 \text{ and } 0 \text{ otherwise}) \\
& = \sum_{\mu \in P(\phi)} m(w^{-1}(\mu), \lambda)\langle w^{-1}(\mu), \nu\rangle \delta_{w(\nu)}\mu, (\text{since } \langle w(\mu), w(\nu)\rangle = \langle \mu, \nu \rangle) \\
& (\text{put } \mu' = w^{-1}(\mu)) \\
& = \sum_{\mu' \in P(\phi)} m(\mu', \lambda)\langle \mu', \nu\rangle \delta_{w(\nu)}(w(\mu')) \\
& = \sum_{\mu' \in P(\phi)} m(\mu', \lambda)\langle \mu', \nu\rangle \delta_{(\nu)}((\mu')), (\text{since } \langle w(\mu), w(\nu)\rangle = \langle \mu, \nu \rangle) \\
& = F_\nu(\lambda).
\end{aligned}
$$

□

Lemma 3.2.5. *Let G be a complex semisimple group. Let T be a maximal torus and ν be a co-character of T. Let ϕ be an irreducible orthogonal representation of G with highest weight λ. Let $\Psi_\phi(\nu)$ and $F_\nu(\lambda)$ be defined as in Lemma 3.2.2 and Lemma 3.2.3. Then*

$$\Psi_\phi(\nu) \equiv F_\nu(\lambda) \mod 2.$$

Proof. The co-character $\phi \circ \nu$ lifts if and only if $\Psi_\phi(\nu) \equiv 0 \mod 2$. And a similar statement for $\Psi_\phi(\nu)$ replaced with $F_\nu(\lambda)$ holds. Hence $\Psi_\phi(\nu) \equiv F_\nu(\lambda)$ mod 2. □

Chapter 4

The Main Theorem

Let G be a connected complex Lie group having a semi-simple Lie algebra $\mathfrak{g}$. Let T_G be a maximal torus of G. In this chapter, we derive a formula for determining the spinoriality of an irreducible orthogonal representation of G. That formula gives an integer which is when even, we can say that the representation is spinorial, otherwise not.

4.1 Discussion

Let ϕ_λ be an irreducible orthogonal finite-dimensional representation of G, with highest weight λ. Let T be a maximal torus of G. Let us denote the Lie algebra of T by $\mathfrak{h}$ and its dual by $\mathfrak{h}^*$. Let ν be a co-character of T.

Notation: We use $(,)$ for the Killing form and the same for the canonical pairing between characters and co-characters depending on the context.

Let $Q_\nu = \Theta \circ \nu$, where Θ is the Weyl character formula. It is clear that $Q_\nu : \mathbb{C}^\times \to \mathbb{C}$ and $Q_\nu(t) \in \mathbb{Z}[t, t^{-1}]$. The reason is that, the Weyl character formula is the sum of the characters of T and the co-domain of the pairing between the characters and the co-characters of T is $\mathbb{Z}$. More concretely $Q_\nu(a) = \sum_{\mu \in P_\lambda} m_\lambda(\mu) a^{(\mu,\nu)}$, where $m_\lambda(\mu)$ is the multiplicity of weight μ in

ϕ_λ and P_λ is the set of weights appearing in ϕ_λ. From Lemma 3.2.2, Q_ν is palindromic. Thus, $Q_\nu(t) = a_d(t^d+t^{-d})+a_{d-1}(t^{d-1}+t^{1-d})+\cdots+a_1(t+t^{-1})+a_0$. Hence $Q_\nu \in \mathbb{Z}[t,t^{-1}]^{\mathrm{sym}}$. From the discussion above the same lemma, we can conclude that ϕ_λ is spinorial if and only if $\Psi_\phi(\nu) = \sum_{k=1}^{d} k \cdot a_k \equiv 0 \mod 2$, for every ν.

Lemma 4.1.1. *The polynomial $Q_\nu(t)$ can be uniquely written in the form $P(t)+P(t^{-1})$, where $P \in \mathbb{Z}[t]$ up to a constant term which may lie in $(1/2)\mathbb{Z}$.*

Proof. Since $Q_\nu(t)$ is palindromic, it is of the form $a_d(t^d+t^{-d})+a_{d-1}(t^{d-1}+t^{-d-1}+\cdots+a_1(t+t^{-1}))+a_0$ where $a_i \in \mathbb{Z}$. We can choose $P(t)$ uniquely as $a_dt^d+a_{d-1}t^{d-1}+\cdots+a_1t+(a_0/2)$. □

Lemma 4.1.2. *We have $Q_\nu''(1)/2 \equiv \Psi_\phi(\nu) \mod 2$, where Ψ_ϕ is as in Lemma 3.2.2. Hence ϕ_λ is spinorial if and only if $Q_\nu''(1)/2 \equiv 0 \mod 2$ for every co-character ν.*

Proof. Using Lemma 4.1.1 we can write $Q_\nu(t) = P(t) + P(t^{-1})$ uniquely. Thus we are interested in $P'(1) = \Psi_{\phi_\lambda}(\nu) = \sum_{i=1}^{d} i \cdot a_i \mod 2$, where d is the degree of Q_ν(see Definition 2.1.6).

$$Q_\nu'(t) = P'(t) - P'(t^{-1})t^{-2}. \tag{4.1}$$

Differentiating again we get

$$Q_\nu''(t) = P''(t) - ((-2)t^{-3}P'(t^{-1}) - t^{-4}P''(t^{-1})).$$

Hence we have

$$Q_\nu''(1) = 2(P'(1) + P''(1)).$$

Since $P \in \mathbb{Z}[t]$ up to a constant, $P''(1)$ is always an even integer. Hence

$$Q''_\nu(1)/2 \equiv P'(1) \equiv \Psi_{\phi_\lambda}(\nu) \mod 2.$$

□

Proposition 4.1.3. *The polynomial Q''_ν attains the following value at 1:*

$$Q''_\nu(1) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu)^2,$$

where P_λ denotes the set of weights appearing in ϕ_λ and $m_\lambda(\mu)$ is the multiplicity of the weight μ in ϕ_λ.

Proof. Observe that

$$Q_\nu(t) = \sum_{\mu \in P_\lambda} m_\lambda(\mu) t^{(\mu,\nu)},$$

and

$$Q'_\nu(t) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu) t^{(\mu,\nu)-1}. \tag{4.2}$$

Moreover we have

$$Q''_\nu(t) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu)((\mu,\nu)-1) t^{(\mu,\nu)-2},$$

and

$$Q'_\nu(1) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu). \tag{4.3}$$

By Equation (4.1)

$$Q'_\nu(1) = P'(1) - P'(1) = 0, \tag{4.4}$$

thus we get

$$Q''_\nu(1) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu)^2 + Q'_\nu(1) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu)^2.$$

□

Now let $\mathbb{C}[\mathfrak{h}^*]$ denote the vector space whose basis is the set of weights, where we denote an element $\beta \in \mathfrak{h}^*$ by e^β. We define $e^{\beta_1} \cdot e^{\beta_2} = e^{\beta_1+\beta_2}$, it becomes a $\mathbb{C}$-algebra. We define an operator $\frac{\partial}{\partial\nu} : \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*]$, by defining it on e^β by $\frac{\partial}{\partial\nu}(e^\beta) = (\beta,\nu)e^\beta$, and extending it linearly on $\mathbb{C}[\mathfrak{h}^*]$. It is easy to check that $\frac{\partial}{\partial\nu}$ is a derivation on $\mathbb{C}[\mathfrak{h}^*]$.

We define

$$\mathrm{Ch}(V^\lambda) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)e^\mu.$$

We define $\epsilon : \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*]$ by

$$\epsilon\left(\sum_{\beta \in \mathfrak{h}^*} c_\beta e^\beta\right) = \sum_{\beta \in \mathfrak{h}^*} c_\beta,$$

where the sum is finite. It is easy to see that $\epsilon : \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}$ is a ring homomorphism.

Now the expression we would like to calculate is

$$Q''_\nu(1) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu)^2 = \epsilon\left(\frac{\partial^2}{(\partial\nu)^2}\,\mathrm{Ch}(V^\lambda)\right).$$

By the Weyl character formula we know that

$$A(\lambda+\rho) = \mathrm{Ch}(V^\lambda)A(\rho),$$

where

$$A(\mu) = \sum_{w \in W} \operatorname{sgn}(w)(e^{w(\mu)}),$$

and ρ is half the sum of positive roots.

It is well-known that $A(\rho) = \prod_{\beta \in R_+}(e^{\beta/2} - e^{-\beta/2})$.

We discuss a method of obtaining an expression for $Q''_\nu(1)$ which is same as

$$Q''_\nu(1) = \sum_{\mu \in P_\lambda} m_\lambda(\mu)(\mu,\nu)^2 = \epsilon\left(\frac{\partial^2}{(\partial\nu)^2}\operatorname{Ch}(V^\lambda)\right). \tag{4.5}$$

4.2 Calculation of $Q''_\nu(1)$

Let $m = |R_+|$, where R_+ denotes the set of positive roots in the root system corresponding to $\mathfrak{g}$.

Observe that

$$\frac{\partial}{\partial\nu}(f \cdot g) = \frac{\partial f}{\partial\nu} \cdot g + f \cdot \frac{\partial g}{\partial\nu},$$

for all $f, g \in \mathbb{C}[\mathfrak{h}^*]$, i.e., $\frac{\partial}{\partial\nu}$ is a derivation on $\mathbb{C}[\mathfrak{h}^*]$.

Now we have the Liebniz rule

$$\frac{\partial^n}{(\partial\nu)^n}(f \cdot g) = \sum_{i=0}^{n}\binom{n}{i}\frac{\partial^i f}{(\partial\nu)^i} \cdot \frac{\partial^{n-i} g}{(\partial\nu)^{n-i}}. \tag{4.6}$$

Theorem 4.2.1.

$$\begin{aligned} Q''_\nu(1) &= \epsilon\left(\frac{\partial^2 \operatorname{Ch}(V^\lambda)}{(\partial\nu)^2}\right) \\ &= \frac{2\sum_{w\in W}\operatorname{sgn}(w)(w(\lambda+\rho),\nu)^{m+2}}{(m+2)!(\prod_{i=1}^m(\beta_i,\nu))} - \left(\frac{1}{3}\right)(\dim(V^\lambda))\left(\sum_{i=1}^{m}\frac{(\beta_i,\nu)^2}{4}\right), \end{aligned}$$

where m is as above, and $\beta_i \in R_+$.

Proof. We have

$$A(\lambda+\rho) = \mathrm{Ch}(V^\lambda)A(\rho). \tag{4.7}$$

If we apply operator $\epsilon \circ \frac{\partial^{m+2}}{(\partial\nu)^{m+2}}$ on LHS of (4.7), we obtain

$$\sum_{w\in W} \mathrm{sgn}(w)(w(\lambda+\rho),\nu)^{m+2}.$$

On the other hand, if we apply $\epsilon \circ \frac{\partial^{m+2}}{(\partial\nu)^{m+2}}$ on the RHS of (4.7), and apply (4.6), we get

$$\sum_{i=0}^{m+2} \binom{m+2}{i} \cdot \epsilon(\frac{\partial^i A(\rho)}{(\partial\nu)^i}) \cdot \epsilon(\frac{\partial^{m+2-i}\,\mathrm{Ch}(V^\lambda)}{(\partial\nu)^{m+2-i}}). \tag{4.8}$$

Now we concentrate on finding $\epsilon(\frac{\partial^i A(\rho)}{(\partial\nu)^i})$.

Lemma 4.2.2. *Let R be a complex algebra. Let D be a derivation on R (i.e. D is linear and satisfies $D(f\cdot g) = f\cdot D(g) + D(f)\cdot g$). Let $f : R \to \mathbb{C}$ be an algebra homomorphism on R and $r_1, r_2, \ldots, r_m$ be some m elements satisfying $f(r_i) = 0$. Then*

$$f(D^i(r_1\cdot r_2\cdots r_m)) = \begin{cases} 0, & 0 \le i \le m-1 \\ m!\prod_{i=1}^m f(D(r_i)), & i = m. \end{cases}$$

Proof. We proceed by induction on m. If $m = 1$, then $f(D^0(r_1)) = f(r_1) = 0$ and the second assertion is trivial.

Let us assume the statement for m and prove it for $m+1$. Since D is a derivation, the Liebniz product rule (4.6) is true for D. Also f is an algebra homomorphism. Hence

$$f(D^i(r_1\cdot r_2\cdots r_{m+1})) = \sum_{j=0}^{i} \binom{i}{j} f(D^j(r_1\cdot r_2\cdots r_m))f(D^{i-j}(r_{m+1})).$$

If $i \leq m-1$, then $j \leq m-1$, and then all the terms in the summation of the form $f(D^j(r_1 \cdot r_2 \cdots r_m))$ are 0, by induction. Hence $f(D^i(r_1 \cdot r_2 \cdots r_{m+1})) = 0$.

If $i = m$, then for $0 \leq j < m$, $f(D^j(r_1 \cdot r_2 \cdots r_m))$, which are the terms in the summation, are 0, and when $j = m \Rightarrow i - j = 0$, thus the last term in the summation is $f(D^m(r_1 \cdot r_2 \cdots r_m))f(r_{m+1})$, which is 0, as $f(r_{m+1}) = 0$. Hence $f(D^m(r_1 \cdot r_2 \cdots r_{m+1})) = 0$. Hence we have the first assertion.

We would like to prove

$$f(D^{m+1}(r_1 \cdot r_2 \cdots r_{m+1})) = (m+1)! \prod_{i=1}^{m+1} f(D(r_i)),$$

and we have

$$f(D^{m+1}(r_1 \cdot r_2 \cdots r_{m+1})) = \sum_{i=0}^{m+1} \binom{m+1}{i} f(D^i(r_1 \cdot r_2 \cdots r_m)) f(D^{m+1-i}(r_{m+1})).$$

Thus by induction,

$f(D^i(r_1 \cdot r_2 \cdots r_m)) = 0$, for $0 \leq i \leq m-1$. If $i = m+1$ then $m+1-i = 0$ which implies $f(D^{m+1-i}(r_{m+1})) = 0$. For $i = m$, we get

$$\binom{m+1}{m} f(D^m(r_1 \cdot r_2 \cdots r_m) f(D(r_{m+1})),$$

which by induction equals

$$(m+1)! \prod_{i=1}^{m+1} f(D(r_i)).$$

Hence the proof. □

Lemma 4.2.3. *In the setting of Lemma 4.2.2, if* $D^2(r_i) = c_i \cdot r_i$*, for some*

$c_i \in \mathbb{C}$, *for each* i, *then we have the following recurrence relation*

$$f(D^{m+2}(r_1 \cdot r_2 \cdots r_m)) = \binom{m+2}{3} f(D^{m-1}(r_1 \cdot r_2 \cdots r_{m-1})) f(D^3(r_m)) + (m+2) f(D^{m+1}(r_1 \cdot r_2 \cdots r_{m-1})) f(D(r_m)).$$

Thus we get a formula $f(D^{m+2}(r_1 \cdot r_2 \cdots r_m))$ *equals*

$$\sum_{i=1}^{m} \frac{(m+2)!}{6(m-i)!} f(D^{m-i}(r_1 \cdot r_2 \cdots r_{m-i})) f(D^3(r_{m-i+1})) f(D(r_{m-i+2})) \cdots f(D(r_m)),$$

which is the same as $f(D^{m+2}(r_1 \cdot r_2 \cdots r_m))$ *equals*

$$\frac{(m+2)!}{6} \sum_{i=1}^{m} f(D(r_1)) f(D(r_2)) \cdots f(D(r_{m-i})) f(D^3(r_{m-i+1})) f(D(r_{m-i+2})) \cdots f(D(r_m)), \tag{4.9}$$

using Lemma 4.2.2.

Proof. We have

$$\begin{aligned} f(D^{m+2}(r_1 \cdot r_2 \cdots r_m)) = & f(D^{m+2}(r_1 \cdot r_2 \cdots r_{m-1})) f(r_m) \\ & + \binom{m+2}{1} f(D^{m+1}(r_1 \cdot r_2 \cdots r_{m-1})) f(D(r_m)) \\ & + \binom{m+2}{2} f(D^{m}(r_1 \cdot r_2 \cdot r_{m-1})) f(D^2(r_m)) \\ & + \binom{m+2}{3} f(D^{m-1}(r_1 \cdot r_2 \cdot r_{m-1})) f(D^3(r_m)), \end{aligned}$$

as $f(D^j(r_1 \cdot r_2 \cdots r_{m-1})) = 0$ for $0 \leq j \leq m-2$ from the lemma above. As $f(r_m) = 0$ and $f(D^2(r_m)) = f(c \cdot r_m) = c \cdot f(r_m) = 0$. Thus the first and the third sum vanish in the above summation. Hence we get our recurrence relation.

Assuming the following induction formula for $l = m - 1$
$f(D^{l+2}(r_1 \cdot r_2 \cdots r_l))$ equals

$$\sum_{i=1}^{l} \frac{(l+2)!}{6(l-i)!} f(D^{l-i}(r_1 \cdot r_2 \cdots r_{l-i})) f(D^3(r_{l-i+1})) f(D(r_{l-i+2})) \cdots f(D(r_l)). \tag{4.10}$$

We want to prove $f(D^{m+2}(r_1 \cdots r_m))$ equals

$$\sum_{i=1}^{m} \frac{(m+2)!}{6(m-i)!} f(D^{m-i}(r_1 \cdots r_{m-i})) f(D^3(r_{m-i+1})) f(D(r_{m-i+2})) \cdots f(D(r_m)).$$

Writing terms for $i = 1$ and $i = 2$ to n, we want to prove
$f(D^{m+2}(r_1 \cdots r_m))$ is equal to

$$\binom{m+2}{3} f(D^{m-1}(r_1 \cdot r_2 \cdots r_{m-1})) f(D^3(r_m)) + (m+2) f(D(r_m)) \times$$
$$\left(\sum_{i=2}^{m} \frac{(m+1)!}{6(m-i)!} f(D^{m-i}(r_1 \cdot r_2 \cdots r_{m-i})) f(D^3(r_{m-i+1})) f(D(r_{m-i+2})) \cdots f(D(r_{m-1})) \right).$$

Now taking $j = i - 1$, we want to prove $f(D^{m+2}(r_1 \cdots r_m))$ is equal to

$$\binom{m+2}{3} f(D^{m-1}(r_1 \cdot r_2 \cdots r_{m-1})) f(D^3(r_m)) + (m+2) f(D(r_m)) \times$$
$$\left(\sum_{j=1}^{m-1} \frac{(m+1)!}{6(m-j-1)!} f(D^{m-j-1}(r_1 \cdot r_2 \cdots r_{m-j-1})) f(D^3(r_{m-j}) \cdot f(D(r_{m-j+1})) \cdots f(D(r_{m-1})) \right).$$

By putting $l = m - 1$ in the Equation (4.10), we get

$$\begin{aligned} f(D^{m+2}(r_1 \cdot r_2, \cdots r_m)) = & \binom{m+2}{3} f(D^{m-1}(r_1 \cdot r_2 \cdots r_{m-1})) f(D^3(r_m)) \\ & + (m+2) f(D^{m+1}(r_1 \cdot r_2 \cdots r_{m-1})) f(D(r_m)), \end{aligned}$$

which is the recurrence relation that we got earlier.

For $m = 1$, it is trivial to check that the formula holds. Hence by induction the proof follows. □

Lemma 4.2.4.

$$\epsilon\left(\frac{\partial^i A(\rho)}{(\partial\nu)^i}\right) = 0,$$

for $0 \le i \le m-1$, *and*

$$\epsilon\left(\frac{\partial^m A(\rho)}{(\partial\nu)^m}\right) = m!\prod_{i=1}^{m}(\beta_i,\nu),$$

where β_i *runs over the set of positive roots.*

Moreover

$$\epsilon\left(\frac{\partial^{m+2} A(\rho)}{(\partial\nu)^m}\right) = \frac{(m+2)!}{6}\prod_{i=1}^{m}(\beta_i,\nu)(\sum_{i=1}^{m}\frac{(\beta_i,\nu)^2}{4}).$$

Proof. Let us define

$$B(\beta) = e^{\beta/2} - e^{-\beta/2},$$

and define

$$C(\beta) = e^{\beta/2} + e^{-\beta/2}.$$

Observe that

$$\frac{\partial B(\beta_i)}{\partial\nu} = \frac{(\beta_i,\nu)}{2}C(\beta_i),$$

and

$$\frac{\partial C(\beta_i)}{\partial\nu} = \frac{(\beta_i,\nu)}{2}B(\beta_i),$$

also we have

$$A(\rho) = \prod_{i=1}^{m} B(\beta_i).$$

Now in the setting of Lemma 4.2.2, we put $D = \frac{\partial}{\partial\nu}$, which is a derivation, $r_i = B(\beta_i)$ and $f = \epsilon$. Thus the first assertion is obvious from Lemma 4.2.2.

The second part of Lemma 4.2.2 says

$$\begin{aligned}
\epsilon\left(\frac{\partial^m A(\rho)}{(\partial\nu)^m}\right) &= m!\prod_{i=1}^{m}\epsilon\left(\frac{\partial B(\beta_i)}{\partial\nu}\right)\\
&= m!\prod_{i=1}^{m}\frac{(\beta_i,\nu)}{2}\epsilon(C(\beta_i))\\
&= m!\prod_{i=1}^{m}(\beta_i,\nu),
\end{aligned}$$

as $\epsilon(C(\beta_i)) = 2$ for every i.

Now in the same setting of Lemma 4.2.2, we observe that $\frac{\partial^2 B(\beta_i)}{(\partial\nu)^2} = \frac{(\beta_i,\nu)^2}{4}B(\beta_i)$.

Thus we can apply Lemma 4.2.3. For applying this lemma we observe that

$$\epsilon\left(\frac{\partial^3 B(\beta_i)}{(\partial\nu)^3}\right) = \frac{(\beta_i,\nu)^3}{4},$$

and

$$\epsilon\left(\frac{\partial B(\beta_i)}{(\partial\nu)}\right) = (\beta_i,\nu).$$

By plugging in this data in Equation 4.9, we get

$$\epsilon\left(\frac{\partial^{m+2} A(\rho)}{(\partial\nu)^m}\right) = \frac{(m+2)!}{6}(\prod_{i=1}^{m}(\beta_i,\nu))(\sum_{i=1}^{m}\frac{(\beta_i,\nu)^2}{4}).$$

□

From Lemma 4.2.4, (4.8) reduces to

$$\binom{m+2}{m}\epsilon\left(\frac{\partial^m A(\rho)}{(\partial\nu)^m}\right)\epsilon\left(\frac{\partial^2\operatorname{Ch}(V^\lambda)}{(\partial\nu)^2}\right) + \binom{m+2}{m+1}\epsilon\left(\frac{\partial^{m+1} A(\rho)}{(\partial\nu)^{m+1}}\right)\epsilon\left(\frac{\partial\operatorname{Ch}(V^\lambda)}{(\partial\nu)}\right)$$

$$+\binom{m+2}{m+2}\epsilon\left(\frac{\partial^{m+2} A(\rho)}{(\partial\nu)^{m+2}}\right)\epsilon\left(\operatorname{Ch}(V^\lambda)\right).$$

Here observe that $\epsilon\left(\frac{\partial^2 \operatorname{Ch}(V^\lambda)}{(\partial\nu)^2}\right)$ is the thing we would like to know. We have proved

$$\epsilon\left(\frac{\partial^m A(\rho)}{(\partial\nu)^m}\right) = m!\prod_{i=1}^{m}(\beta_i,\nu).$$

We know that

$$\epsilon(\operatorname{Ch}(V^\lambda)) = \dim(V^\lambda).$$

We also have

$$\epsilon\left(\frac{\partial \operatorname{Ch}(V^\lambda)}{(\partial\nu)}\right) = \epsilon\left(\left(\frac{\partial}{(\partial\nu)}\right)(\sum_{\mu\in P_\lambda} m_\lambda(\mu)e^\mu)\right) = \sum_{\mu\in P_\lambda} m_\lambda(\mu)(\mu,\nu) = Q'_\nu(1) = 0,$$

from equation (4.3) and (4.4).

Thus we have

$$\sum_{w\in W}\operatorname{sgn}(w)(w(\lambda+\rho),\nu)^{m+2} = \binom{m+2}{m}\epsilon\left(\frac{\partial^2 \operatorname{Ch}(V^\lambda)}{(\partial\nu)^2}\right) m!\prod_{i=1}^{m}(\beta_i,\nu) + \frac{(m+2)!}{6}\prod_{i=1}^{m}(\beta_i,\nu)\left(\sum_{i=1}^{m}\frac{(\beta_i,\nu)^2}{4}\right)\dim(V^\lambda).$$

Therefore we have $Q''_\nu(1) = \epsilon\left(\frac{\partial^2 \operatorname{Ch}(V^\lambda)}{(\partial\nu)^2}\right) =$

$$\frac{\sum_{w\in W}\operatorname{sgn}(w)(w(\lambda+\rho),\nu)^{m+2}}{\binom{m+2}{2}m!(\prod_{i=1}^{m}(\beta_i,\nu))} - (1/3)\dim(V^\lambda)\left(\sum_{i=1}^{m}\frac{(\beta_i,\nu)^2}{4}\right).$$

Hence the proof. □

Here on-wards, we take the bilinear form on the Cartan sub-algebra $\mathfrak{h}$ to be the Killing form, which is the bilinear form $\langle X,Y\rangle = \sum_{\alpha\in R}\langle\alpha,X\rangle\cdot\langle\alpha,Y\rangle$ by Page 226 of [Bourbaki(2005)]. The form induced by the Killing form on $\mathfrak{h}^*$, is denoted by Φ_R in chapter VI section 1 no. 12 page 184 of [Bourbaki(2002)].

Proposition 4.2.5. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and Weyl group W. For each positive integer k we define*

$g_k(\mu,\nu) = \sum_{w\in W} \mathrm{sgn}(w)\langle w(\mu),\nu\rangle^k$, *for* $\mu \in \mathfrak{h}^*$ *and* $\nu \in \mathfrak{h}$.

Let $\mathfrak{g}$ *be simple. Then*

(1) $g_k(\mu,\nu) = 0$ *for* $0 \le k \le m-1$ *or* $k = m+1$

(2) $g_m(\mu,\nu) = \dfrac{m!}{\prod_{\beta\in R_+}(\beta,\rho)} \prod_{\beta\in R_+}(\beta,\mu)\prod_{\beta\in R_+}(\beta,\nu)$.

(3) $g_{m+2}(\mu,\nu) = c_{\mathfrak{g}} \prod_{\beta\in R_+}(\mu,\beta)\prod_{\beta\in R_+}(\beta,\nu)(\mu,\mu)(\sum_{\beta\in R}(\beta,\nu)^2)$,

where (,) is according to the notation introduced in the section 4.1. We calculate $c_{\mathfrak{g}}$ *in the next lemma.*

Proof. If the Weyl group W acts on a vector space V, then a function $f \in \mathrm{sym}(V^*)$ is called anti-W-invariant if $f(w(v)) = \mathrm{sgn}(w)f(v)\ \forall v \in V\ \forall w \in W$. Taking $V = \mathfrak{h}^*$ or $\mathfrak{h}$, and by Section 3.13 page 69 of [Humphreys(1990)] we know that if a function $f \in \mathrm{sym}(V^*)$ is anti-W-invariant then it is divisible by $\prod_{\beta\in R_+} B'(\beta,x)$ in $\mathrm{sym}(V^*)$, where B' is a non-zero W-invariant bilinear form. Furthermore the quotient is a W-invariant polynomial. If $\mathfrak{g}$ is simple then by Theorem 14.31 of [Fulton and Harris(1991)] h and h^* is a irreducible representation of W. Now B' is a scalar multiple of the Killing form, because the W-invariant bilinear forms corresponds to the set $\mathrm{Hom}_W(V,V^*)$. It is one dimensional if V is irreducible. Now the polynomial g_k for each positive integer k, is anti-W-invariant in both the variables. Hence it is divisible by $\prod_{\beta\in R_+}(\mu,\beta^\vee)$. As $(\mu,\beta^\vee) = 2(\mu,\beta)/(\beta,\beta)$, the expression is divisible by $\prod_{\beta\in R_+}(\mu,\beta)$, and also by $\prod_{\beta\in R_+}(\beta,\nu)$. Since $g_k(\mu,\nu)$ is anti-W-invariant in both the variables μ and ν, we get

$$g_k(\mu,\nu) = f_k(\mu,\nu)\cdot \prod_{\beta\in R_+}(\mu,\beta)\cdot\prod_{\beta\in R_+}(\beta,\nu),$$

where (,) is the Killing form or the pairing between characters and co-characters depending upon the context, and where f_k is a W-invariant polynomial in both the variables.

When $0 \leq k \leq m-1$ the degrees of both sides cannot match unless f_k is identically 0, Hence $g_k(\mu,\nu)=0$.

For $k = m+1$, since each entry $\langle w(\mu),\nu\rangle^{m+1}$ is a degree $m+1$ homogeneous polynomial in μ and ν, the degree of g_{m+1} is either $m+1$ in each varaible otherwise it is identically 0. If $g_{m+1}=0$ we are through. Otherwise we have the following argument. If we fix the variable ν, then $f_{m+1}(\mu,\nu)$ should be a degree one W-invariant polynomial in the variable μ. By the table of degrees of the basic invariants on page 59 of [Humphreys(1990)], we know that there is no degree one W-invariant polynomial for simple Lie algebras. Thus g_{m+1} is identically 0.

For (3), we have the following argument.

Lemma 4.2.6. *The polynomial g_{m+2} has degree $m+2$ in both the variables.*

Proof. The LHS is a homogeneous polynomial in both μ and ν, hence it is either of degree $m+2$ or it is identically 0. If it is 0 then the LHS in Theorem 4.2.1 is non-negative as it is sum $m(\mu,\lambda)(\mu,\nu)^2$ while the RHS is strictly negative as one of $(\beta_i,\nu)\neq 0$ as β_i spans the space and the pairing is non-degenerate. This leads to a contradiction. □

Hence f_{m+2} is a degree 2 W-invariant polynomial in both the variables. As both the polynomials are homogeneous, f_{m+2} will be homogeneous. Any quadratic invariant homogeneous polynomial is a quadratic form corresponding to a symmetric bilinear form on $\mathfrak{h}$ or $\mathfrak{h}^*$. We know that a quadratic W-invariant homogeneous polynomial corresponds to a W-invariant quadratic form which in turn corresponds to a W invariant bilinear form. Again by Theorem 14.31 of [Fulton and Harris(1991)] it should be the Killing form up to scalars. Hence it is divisible by (μ,μ) and $|\nu|^2 = (\sum_{\beta\in R}(\beta,\nu)^2)$. Therefore we conclude

$$g_{m+2}(\mu,\nu) = c_{\mathfrak{g}}\prod_{\beta\in R_+}(\mu,\beta)\prod_{\beta\in R_+}(\beta,\nu)(\mu,\mu)(\textstyle\sum_{\beta\in R}(\beta,\nu)^2).$$

For (2), by a degree argument, it is clear that f_m should be a scalar c. To find c we again use the Weyl character formula, i.e.,

$$A(\lambda+\rho) = \mathrm{Ch}(V^\lambda)A(\rho),$$

where $A(\mu) = \sum_{w\in W} \mathrm{sgn}(w)e^w(\mu)$.

This time we apply $\epsilon(\frac{\partial^m}{\partial\nu^m})$ to both sides of above equation. Again applying the Leibniz rule to the RHS and using Lemma 4.2.4 we get

$$\begin{aligned} g_m(\lambda+\rho,\nu) &= \epsilon(\mathrm{Ch}(V^\lambda))\epsilon\left(\frac{\partial^m(A(\rho))}{(\partial\nu)^m}\right) \\ &= \dim(V^\lambda)\cdot m!\cdot \prod_{\beta\in R_+}(\beta,\nu). \end{aligned}$$

By using the Weyl dimension formula we get

$$g_m(\lambda+\rho,\nu) = \frac{\prod_{\beta\in R_+}(\beta,\lambda+\rho)}{\prod_{\beta\in R_+}(\beta,\rho)}\cdot m!\cdot \prod_{\beta\in R_+}(\beta,\nu).$$

Hence we get $f_m = c = \dfrac{m!}{\prod_{\beta\in R_+}(\beta,\rho)}$ and (2) follows.
Therefore, we conclude the proof of Proposition 4.2.5. □

We calculate $c_{\mathfrak{g}}$ in the next lemma.

Lemma 4.2.7. *We have*

$$c_{\mathfrak{g}} = \frac{(m+2)!}{48(\prod_{\beta\in R_+}(\rho,\beta))\cdot(\rho,\rho)}.$$

Proof. Let us denote $\prod_{\beta\in R_+}(\mu,\beta)$ by d_μ and $\prod_{\beta\in R_+}(\beta,\nu)$ by d_ν. Let us denote (μ,μ) by $|\mu|^2$ and $\sum_{\beta\in R_+}(\beta,\nu)^2$ by $|\nu|^2$.

Now we will use Exercise 7 c) on page 257 of [Bourbaki(2005)] which says

$$\sum_{\mu\in P}((\mu,\rho))^2(m_\lambda(\mu)) = (1/24)(\dim V^\lambda)(\lambda,\lambda+2\rho), \qquad (4.11)$$

where P is the weight lattice of $\mathfrak{g}$. Let

$$F(\mu,\nu) = \sum_{w\in W}\operatorname{sgn}(w)(w(\mu),\nu)^{m+2}.$$

Since $(,)$ is non-degenerate, for every co-character ν, we can define a weight μ_ν satisfying $(\mu',\nu) = (\mu',\mu_\nu)\ \forall\mu'\in\mathfrak{h}^*$. Similarly for every weight μ, we can define a co-character ν_μ satisfying $(\mu',\nu_\mu) = (\mu',\mu)\ \forall\mu'\in\mathfrak{h}^*$. With this notation, if we substitute ν_ρ in place of ν in (4.5), we get

$$Q''_{\nu_\rho}(1) = \sum_{\mu\in P_\lambda} m_\lambda(\mu)(\mu,\nu_\rho)^2 = \sum_{\mu\in P_\lambda}(\mu,\rho)^2(m_\lambda(\mu)),$$

which equals

$$= (1/24)\dim(V^\lambda)(\lambda,\lambda+2\rho),$$

by (4.11).

On the other hand, by Theorem 4.2.1

$$Q''_{\nu_\rho}(1) = \frac{2F(\lambda+\rho,\nu_\rho)}{((m+2)!d_{\nu_\rho})} - (1/3)(\dim V^\lambda)(1/4)(\frac{|\nu_\rho|^2}{2}).$$

From the lemma above, we get

$$F(\lambda+\rho,\nu_\rho) = c_{\mathfrak{g}}d_{\lambda+\rho}d_{\nu_\rho}|\lambda+\rho|^2|\nu_\rho|^2.$$

Using the dimension formula we get

$$= c_{\mathfrak{g}}(\dim V^\lambda)d_\rho d_{\nu_\rho}|\lambda+\rho|^2|\nu_\rho|^2.$$

Here

$$\begin{aligned} d_{\nu_\rho} &= \prod_{\beta \in R_+} (\beta, \nu_\rho) \\ &= \prod_{\beta \in R_+} (\beta, \rho) \\ &= d_\rho, \end{aligned}$$

and

$$\begin{aligned} |\nu_\rho|^2 &= \sum_{\beta \in R} (\beta, \nu_\rho)^2 \\ &= \sum_{\beta \in R} (\beta, \rho)^2 \\ &= (\rho, \rho) \\ &= |\rho|^2. \end{aligned}$$

See Equation (16) on page 184, of [Bourbaki(2002)] for the third equality.

Combining all this we have

$$Q''_{\nu_\rho}(1) = \frac{2c_{\mathfrak{g}}(\dim V^\lambda) d_\rho d_\nu |\lambda + \rho|^2 |\nu_\rho|^2}{(m+2)! d_\nu} - (1/24)(\dim V^\lambda)|\nu_\rho|^2,$$

$$= (\dim V^\lambda)|\rho|^2 \left(\frac{2c_{\mathfrak{g}} d_\rho |\lambda + \rho|^2}{(m+2)!} - 1/24 \right).$$

Equating this with

$$= (1/24) \dim(V^\lambda)(\lambda, \lambda + 2\rho),$$

we get

$$c_{\mathfrak{g}} = \frac{(m+2)!}{48 d_\rho (\rho, \rho)}.$$

□

Theorem 4.2.8. *If $\mathfrak{g}$ is simple then*

$$Q''_\nu(1) = \frac{(\dim V^\lambda)(|\nu|^2)(|\lambda+\rho|^2 - |\rho|^2)}{\dim \mathfrak{g}}.$$

Proof. For the formula, plug in the value of $c_\mathfrak{g}$ in Theorem 4.2.1 and use the Weyl dimension formula. We get

$$\begin{aligned} Q''_\nu(1) &= \frac{c_\mathfrak{g} d_{\lambda+\rho} d_\nu |\lambda+\rho|^2|\nu|^2}{((m+2)!/2)d_\nu} - \frac{\dim V^\lambda |\nu|^2}{24} \\ &= (1/24)(\dim V^\lambda)(|\nu|^2)\left(\frac{|\lambda+\rho|^2}{|\rho|^2} - 1\right) \\ &= \frac{(\dim V^\lambda)|\nu|^2(|\lambda+\rho|^2 - |\rho|^2)}{24|\rho|^2}. \end{aligned}$$

See page 257 exercise 7 *d*) of [Bourbaki(2005)] which says $\dim \mathfrak{g} = 24(\rho,\rho)$ if $\mathfrak{g}$ is simple. □

Hence we get the following theorem.

Theorem 4.2.9. *Let ϕ_λ be an irreducible orthogonal finite-dimensional representation with highest weight λ of a connected Lie group G having a simple Lie algebra $\mathfrak{g}$. Then ϕ_λ is spinorial if and only if the integer*

$$\frac{Q''_\nu(1)}{2} = \frac{(\dim V^\lambda)(|\nu|^2)(|\lambda+\rho|^2 - |\rho|^2)}{2(\dim \mathfrak{g})}$$

is even for all co-characters ν of T_G , where $||$ is the killing norm and ρ is half the sum of positive roots.

Now let $\mathfrak{g}$ be semi-simple. Hence $\mathfrak{g} = \oplus_{i=1}^k \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is simple. Then $\mathfrak{h} = \oplus_{i=1}^k \mathfrak{h}_i$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$. Let ϕ_λ be a orthogonal irreducible representation of a group G with Lie algebra $\mathfrak{g}$ with highest weight λ. Let $\lambda = \oplus_{i=1}^k \lambda_i$, where

$\lambda_i \in \mathfrak{h}_i^*$. Let $\mu \in \mathfrak{h}^*$, so $\mu = \oplus_{i=1}^k \mu_i$, where $\mu_i \in \mathfrak{h}_i^*$. Let ρ be half the sum of the positive roots of $\mathfrak{g}$. Then $\rho = \oplus_{i=1}^k \rho_i$, where $\rho_i \in \mathfrak{h}_i^*$ be half the sum of positive roots of $\mathfrak{g}_i$. Let ν be a co-character of the maximal torus of G corresponding to $\mathfrak{h}$. Then $\nu = \oplus_{i=1}^k \nu_i$ where $\nu_i \in \mathfrak{h}_i$. Let m be the number of positive roots of $\mathfrak{g}$. Let m_i be the number of positive roots of $\mathfrak{g}_i$. Thus $m = \sum_{i=1}^k m_i$. Let $R_{i+} = \{\beta_{i1}, \beta_{i2}, \ldots, \beta_{im_i}\}$ be the positive roots of $\mathfrak{g}_i$ and $R_+ = \cup_{i=1}^k R_{i+}$ is the set of positive roots of $\mathfrak{g}$. Let the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ be W. Then $W = W_1 \times \cdots \times W_k$, where W_i is the Weyl group of $\mathfrak{h}_i$ in $\mathfrak{g}_i$.

Proposition 4.2.10. *Let $\mathfrak{g}, \lambda, \rho, \mu, \nu, m$ and $\mathfrak{g}_i, \lambda_i, \rho_i, \mu_i, \nu_i, m_i$ be as above. Let g_k be the functions described in Proposition 4.2.5. Then we have*

$$g_{m+2}(\mu, \nu) = \sum_{i=1}^{k} \binom{m+2}{m_1, \ldots, m_i + 2, \ldots, m_k} g_{m_1}(\mu_1, \nu_1) \cdots g_{m_i+2}(\mu_i, \nu_i) \cdots g_{m_k}(\mu_k, \nu_k).$$

Proof. By definition we have $g_{m+2}(\mu,\nu)$ is equal to

$$
\begin{aligned}
&= \sum_{w\in W} \operatorname{sgn}(w)\langle w(\mu),\nu\rangle^{m+2},\\
&= \sum_{w\in W} \operatorname{sgn}(w)\langle \oplus_{i=1}^k w_i(\mu_i), \oplus_{i=1}^k \nu_i\rangle^{m+2},\\
&= \sum_{w\in W} \operatorname{sgn}(w)(\sum_{i=1}^k \langle w_i(\mu_i),\nu_i\rangle)^{m+2},\\
&= \sum_{w\in W} \operatorname{sgn}(w)\left(\sum_{i=1}^k \sum_{\{(n_i)\in\mathbb{Z}^k_{\geq 0}|\sum n_i=m+2\}} \binom{m+2}{n_1,\ldots,n_k}\prod_{i=1}^k \langle w_i(\mu_i),\nu_i\rangle^{n_i}\right).
\end{aligned}
$$

Now interchanging summations we get

$$
\begin{aligned}
&= \sum_{i=1}^k \sum_{\{(n_i)\in\mathbb{Z}^k_{\geq 0}|\sum n_i=m+2\}} \binom{m+2}{n_1,\ldots,n_k} \sum_{w\in W}\operatorname{sgn}(w)\prod_{i=1}^k \langle w_i(\mu_i),\nu_i\rangle^{n_i},\\
&= \sum_{i=1}^k \sum_{\{(n_i)\in\mathbb{Z}^k_{\geq 0}|\sum n_i=m+2\}} \binom{m+2}{n_1,\ldots,n_k}\times\\
&\left(\sum_{w_1\in W_1}\sum_{w_2\in W_2}\cdots\sum_{w_k\in W_k}\prod_{i=1}^k(\operatorname{sgn}(w_i)\langle w_i(\mu_i),\nu_i\rangle^{n_i})\right),\\
&= \sum_{i=1}^k \sum_{\{(n_i)\in\mathbb{Z}^k_{\geq 0}|\sum n_i=m+2\}} \binom{m+2}{n_1,\ldots,n_k}\times\\
&\left(\sum_{w_1\in W_1}\operatorname{sgn}(w_1)\langle w_1(\mu_1),\nu_1\rangle^{n_1}\cdots\sum_{w_k\in W_k}\operatorname{sgn}(w_k)\langle w_k(\mu_k),\nu_k\rangle^{n_k}\right),\\
&= \sum_{i=1}^k \sum_{\{(n_i)\in\mathbb{Z}^k_{\geq 0}|\sum n_i=m+2\}} \binom{m+2}{n_1,\ldots,n_k}\prod_{i=1}^k g_{n_i}(\mu_i,\nu_i).
\end{aligned}
$$

Now since each $\mathfrak{g}_i$ is simple, we can apply Proposition 4.2.5 to say that $g_{n_i}(\mu_i,\nu_i) = 0$ if $0 \leq n_i \leq m_i - 1$ or $n_i = m_i + 1$. Thus the k-tuples $(n_i) \in \mathbb{Z}^k_{\geq 0}$ which contribute to the sum satisfy $n_i \notin \{0, 1, \ldots, m_i - 1, m_i + 1\}$ and $\sum n_i = m + 2 = (\sum m_i) + 2$. It is easy to see that this will happen if and only if exactly one of $n_i = m_i + 2$ and $n_j = m_j$ for $j \neq i$. Hence the proposition.

□

Proposition 4.2.11. *If* $\mathfrak{g}, \lambda, \rho, \nu$ *and* $\mathfrak{g}_i, \lambda_i, \rho_i, \nu_i$ *be as above, then*

$$Q''_\nu(1) = \dim(V^\lambda) \sum_{i=1}^{k} \frac{((|\nu_i|^2)(|\lambda_i + \rho_i|^2 - |\rho_i|^2)}{\dim \mathfrak{g}_i}.$$

Proof. By Theorem 4.2.1

$$\begin{aligned} Q''_\nu(1) &= \epsilon \left(\frac{\partial^2 \operatorname{Ch}(V^\lambda)}{(\partial \nu)^2} \right) \\ &= \frac{2 \sum_{w \in W} \operatorname{sgn}(w)(w(\lambda+\rho), \nu)^{m+2}}{(m+2)!(\prod_{i=1}^m (\beta_i, \nu))} - (\frac{1}{3})(\dim(V^\lambda)) \left(\sum_{i=1}^m \frac{(\beta_i, \nu)^2}{4} \right) \\ &= \frac{2 g_{m+2}(\lambda+\rho, \nu)}{(m+2)!(\prod_{i=1}^m (\beta_i, \nu))} - (\frac{1}{3})(\dim(V^\lambda)) \left(\sum_{i=1}^m \frac{(\beta_i, \nu)^2}{4} \right) \end{aligned} \tag{4.12}$$

By Proposition 4.2.10, we have

$$g_{m+2}(\mu, \nu) = \sum_{i=1}^{k} \binom{m+2}{m_1, \ldots, m_i+2, \ldots, m_k} g_{m_1}(\mu_1, \nu_1) \cdots g_{m_i+2}(\mu_i, \nu_i) \cdots g_{m_k}(\mu_k, \nu_k).$$

Since each $\mathfrak{g}_i$ is simple, we can apply Proposition 4.2.5 to obtain

$$g_{m_j}(\mu_j, \nu_j) = \frac{m_j!}{\prod_{l=1}^{m_j} (\beta_{jl}, \rho_i)} \prod_{l=1}^{m_j} (\mu_j, \beta_{jl}) \prod_{l=1}^{m_j} (\beta_{jl}, \nu_j),$$

and

$$g_{m_j+2}(\mu_j, \nu_j) = \frac{(m_j+2)!}{48 \prod_{l=1}^{m_j} (\beta_{jl}, \rho_i) \cdot (\rho_i, \rho_i)} \prod_{l=1}^{m_j} (\beta_{jl}, \mu_j) \prod_{l=1}^{m_j} (\beta_{jl}, \nu_j)(\mu_j, \mu_j)(\nu_j, \nu_j).$$

Let us define $d_j(\mu_j) = \prod_{l=1}^{m_j} (\beta_{jl}, \mu_j)$ and $d_j(\nu_j) = \prod_{l=1}^{m_j} (\beta_{jl}, \nu_j)$ and $d(\mu) = \prod_{\beta \in R_+} (\beta, \mu)$ and $d(\nu) = \prod_{\beta \in R_+} (\beta, \nu)$. Observe that $d(\mu) = \prod_{i=1}^k d_j(\mu_j)$ and

$d(\nu) = \prod_{i=1}^{k} d_j(\nu_j)$.

Then we get $g_{m+2}(\mu, \nu)$

$$= \sum_{i=1}^{k} \binom{m+2}{m_1, \dots, m_i + 2, \dots, m_k} \left(\prod_{l \neq i} m_l! \right) (m_i + 2)! \left(\prod_{j=1}^{k} \frac{d_j(\mu_j)}{d_j(\rho_j)} d_j(\nu_j) \right) \frac{|\mu_i|^2 |\nu_i|^2}{48 |\rho_i|^2}$$
$$= \sum_{i=1}^{k} \frac{(m+2)!}{2} \left(\prod_{j=1}^{k} \frac{d_j(\mu_j)}{d_j(\rho_j)} d_j(\nu_j) \right) \frac{|\mu_i|^2 |\nu_i|^2}{24 |\rho_i|^2}.$$

Now we put $\mu = \lambda + \rho$ and use the Weyl dimension formula. Thus we get $g_{m+2}(\lambda + \rho)$

$$= \sum_{i=1}^{k} \frac{(m+2)!}{2} \left(\prod_{j=1}^{k} \frac{d_j(\lambda_j + \rho_j)}{d_j(\rho_j)} d_j(\nu_j) \right) \frac{|\lambda_i + \rho_i|^2 |\nu_i|^2}{24 |\rho_i|^2}$$
$$= \sum_{i=1}^{k} \frac{(m+2)!}{2} \left(\prod_{j=1}^{k} \dim V^{\lambda_j} \right) d(\nu) \frac{|\lambda_i + \rho_i|^2 |\nu_i|^2}{24 |\rho_i|^2}.$$

The representation V^λ is the external tensor product, i.e, $V^\lambda = \boxtimes_{j=1}^{k} V^{\lambda_j}$, where V^{λ_j} is the irreducible representation of $\mathfrak{g}_i$ of highest weight λ_j. Thus $\dim V^\lambda = \prod_{j=1}^{k} \dim V^{\lambda_j}$. Hence we get

$$g_{m+2}(\lambda + \rho) = \frac{(m+2)!}{2} \dim V^\lambda d(\nu) \sum_{i=1}^{k} \frac{|\lambda_i + \rho_i|^2 |\nu_i|^2}{24 |\rho_i|^2}.$$

Putting this expression in Equation (4.12), we get $Q''_\nu(1)$

$$= \frac{2(m+2)! \dim V^\lambda d(\nu) \sum_{i=1}^k \frac{|\lambda_i+\rho_i|^2|\nu_i|^2}{24|\rho_i|^2}}{2(m+2)!d(\nu)} - \frac{\dim(V^\lambda)(\sum_{\beta\in R}(\beta,\nu)^2)}{24}$$
$$= \frac{\dim V^\lambda}{24} \cdot \left(\sum_{i=1}^k \frac{|\lambda_i+\rho_i|^2|\nu_i|^2}{|\rho_i|^2} - (\sum_{i=1}^k \sum_{\beta\in R_{i+}} (\beta,\nu)^2) \right).$$

observe that $\sum_{\beta\in R_i} (\beta,\nu)^2 = |\nu_i|^2$. Hence we get

$$Q''_\nu(1) = \frac{\dim V^\lambda}{24} \sum_{i=1}^k |\nu_i|^2 \frac{|\lambda_i+\rho_i|^2 - |\rho_i|^2}{|\rho_i|^2}$$

Since $\mathfrak{g}_i$ is simple we have. $24|\rho_i|^2 = \dim \mathfrak{g}_i$, So this equals

$$= \dim V^\lambda \sum_{i=1}^k \frac{|\nu_i|^2(|\lambda_i+\rho_i|^2 - |\rho_i|^2)}{\dim \mathfrak{g}_i}.$$

Hence the proof.

□

Theorem 4.2.12. *Let G be a connected complex semi-simple group with complex Lie algebra $\mathfrak{g}$. Let T_G be a maximal torus of G. Let ϕ_λ be an irreducible orthogonal holomorphic representation with the highest weight λ. Then let $\mathfrak{g} = \oplus \mathfrak{g}_i$, where $\mathfrak{g}_i$ is simple, $\lambda = \oplus\lambda_i$, $\rho = \oplus\rho_i$ and $\nu = \oplus\nu_i$, where $\nu \in \mathfrak{h}$. The representation ϕ_λ is spinorial if and only if the integer*

$$\frac{Q''_\nu(1)}{2} = \dim V^\lambda \sum_{i=1}^k \frac{|\nu_i|^2(|\lambda_i+\rho_i|^2 - |\rho_i|^2)}{2\dim \mathfrak{g}_i}$$

is even for every co-character ν of T_G.

Proof. It follows from the Lemma 4.1.2 and Proposition 4.2.11. □

Scholium 4.2.13. *To determine the spinoriality of ϕ, it is enough to determine the parity of $Q''_\nu(1)/2$ for the co-characters ν which represent the generators of $\pi_1(G) \cong X_*(T)/Q$ which is finite, where $X_*(T)$ is the co-character lattice and Q is the co-root lattice of G. Thus problem reduces to checking* *this criterion for* ***finite number of co-characters***.

Proof. The map $X_*(T) \to \pi_1(G) \xrightarrow{\phi_*} \pi_1(SO(V))$ is simply given by $\nu \mapsto Q''_\nu(1)/2$. Because whenever ν is such that $\phi \circ \nu$ is lifting then it maps under above map to 0 otherwise not, and the map $\nu \mapsto Q''_\nu(1)/2$ does exactly the same job. Now to check ϕ_* is trivial, we just have to check it on the representative co-characters of generators of $\pi_1(G)$. Hence the proof. □

4.3 Relation with the Dynkin Index

Let $\mathfrak{g}$ be a simple Lie algebra with a long root α. We define a bilinear form on $\mathfrak{g}$ by

$$(x,y)_d = \frac{2(x,y)}{(\alpha,\alpha)},$$

where $(,)$ is the Killing form. In other words we normalize the Killing form so that $(\alpha,\alpha)_d = 2$. Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a morphism of simple Lie algebras. Then there exists a number $\mathrm{dyn}(\phi)$ called Dynkin invariant, so that

$$(\phi(x),\phi(y))_d = \mathrm{dyn}(\phi)(x,y)_d.$$

Definition 4.3.1. *Dynkin Index: Let $\phi : \mathfrak{g} \to \mathfrak{sl}(V)$ be a representation of simple Lie algebra $\mathfrak{g}$, then the Dynkin invariant of ϕ is called the Dynkin index of representation ϕ. It is denoted by j_ϕ.*

By Proposition 2.5 Page 101 of [Vinberg(1994)] we can say that Dynkin invariant is a non-negative integer. It is 0 if and only if ϕ is trivial.

Properties of Dynkin invariant

1) For two representations ϕ_1 and ϕ_2 of the same Lie algebra $\mathfrak{g}$, we have $\mathrm{dyn}(\phi_1 \oplus \phi_2) = \mathrm{dyn}(\phi_1) + \mathrm{dyn}(\phi_2)$.

2)For three simple Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ and the morphisms $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\phi : \mathfrak{g}_2 \to \mathfrak{g}_3$, we have $\mathrm{dyn}(\phi \circ \psi) = \mathrm{dyn}(\phi) \cdot \mathrm{dyn}(\psi)$.

Lemma 4.3.2. *Let $\iota : \mathfrak{so}(V) \to \mathfrak{sl}(V)$ be the standard inclusion, then $j_\iota = 2$.*

Proof. A Cartan subalgebra of $\mathfrak{so}(V)$ is

$$\mathfrak{h}_{\mathfrak{so}} = \begin{cases} x_1 \oplus \cdots \oplus x_n \oplus (-x_1) \oplus \cdots \oplus (-x_n) & \text{if } \dim V \text{ is even }, \\ x_1 \oplus \cdots \oplus x_n \oplus (-x_1) \oplus \cdots \oplus (-x_n) \oplus 0 & \text{if } \dim V \text{ is odd,} \end{cases}$$

where $x_i \in \mathbb{C}$. Under the map ι, it maps to the same matrix in $\mathfrak{sl}(V)$. The Killing form for $\mathfrak{h}_{\mathfrak{sl}}$ is $K_{\mathfrak{sl}}(X,Y) = 2m\,\mathrm{Trace}(X \cdot Y)$. The long root is $e_1 - e_2$, so if we normalize the Killing form as $K'_{\mathfrak{sl}}(X,Y) = \mathrm{Trace}(X \cdot Y) = \sum_{j=1}^{2n} x_j y_j$, then $K'_{\mathfrak{sl}}(e_1 - e_2, e_1 - e_2) = 2$. Similarly for $\mathfrak{h}_{\mathfrak{so}}$, the Killing form is $K_{\mathfrak{so}}(X,Y) = (m-2) \cdot \mathrm{Trace}(X,Y)$. Here the long root is again $e_1 - e_2$, so if we take the normalized Killing form to be

$$\begin{aligned} K'_{\mathfrak{so}}(X,Y) &= (1/2)\,\mathrm{Trace}(X \cdot Y) \\ &= (1/2)(\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n}(-x_i)(-y_i)) \\ &= \sum_{i=1}^{n} x_i y_i, \end{aligned}$$

then $K'_{\mathfrak{so}}(e_1 - e_2, e_1 - e_2) = 2$. Since $K'_{\mathfrak{sl}}$ restricted to $\mathfrak{so}(V)$ is $2K'_{\mathfrak{so}}$, the Dynkin index of ι is 2. □

Let $\pi_\lambda : \mathfrak{g} \to \mathfrak{sl}(V)$ be an orthogonal representation. We may write $\pi_\lambda = i \circ \pi'_\lambda$, where $\pi'_\lambda : \mathfrak{g} \to \mathfrak{so}(V)$. Thus $\mathrm{dyn}(\pi) = 2\,\mathrm{dyn}(\pi')$ if $\dim V \neq 1, 2, 4$, as in these cases $\mathfrak{so}(V)$ is not simple.

Theorem 4.3.3. *Let ϕ_λ be the irreducible representation of $\mathfrak{g}$ with highest weight λ with $\mathfrak{g}$ simple. Then*

$$j_{\phi_\lambda} = \frac{\dim V^\lambda \cdot 2 \cdot (\lambda, \lambda + 2\rho)}{\dim \mathfrak{g} \cdot (\alpha, \alpha)},$$

where $(,)$ is the Killing form and α is the longest root of $\mathfrak{g}$.

Proof. By Theorem 2.19 page 101 of [Vinberg(1994)], we have

$$j_{\phi_\lambda} = \frac{\dim V^\lambda (\lambda, \lambda + 2\rho)_d}{\dim \mathfrak{g}}.$$

Thus by definition of $(,)_d$ the theorem follows. □

Corollary 4.3.4. *In the setting of the theorem above we have*

$$\frac{Q_\nu''(1)}{2} = \frac{(\alpha,\alpha)(\nu,\nu)\,\mathrm{dyn}(\phi_\lambda')}{2},$$

where $(,)$ *denotes the Killing form and* α *is the longest root.*

Proof. By Theorem 4.2.9, we have

$$\frac{Q_\nu''(1)}{2} = \frac{(\dim V^\lambda)|\nu|^2(|\lambda+\rho|^2 - |\rho|^2)}{2\dim\mathfrak{g}}.$$

By the theorem above, we have

$$\frac{\dim V^\lambda(\lambda,\lambda+2\rho)}{\dim\mathfrak{g}} = \frac{(\alpha,\alpha)\cdot j_{\phi_\lambda}}{2}.$$

Note that $(\lambda,\lambda+2\rho) = |\lambda+\rho|^2 - |\rho|^2$. Furthermore $\mathrm{dyn}(\phi_\lambda') = (1/2)j_{\phi_\lambda}$. Combining all the three statements we get the above result. ☐

4.4 Spinoriality of the Adjoint Representation

Let G be a complex Lie group having a simple Lie algebra $\mathfrak{g}$. Let Ad be the adjoint representation of G. Let T be a maximal torus of G and $X^*(T)$ be the character group of T. Let ρ be half the sum of roots. We are proving it for complex groups having a simple Lie algebra using our method in this section.

Theorem 4.4.1. Ad *is spinorial if and only if* $\rho \in X^*(T)$.

Proof. From Theorem 4.2.9, it is clear that Ad is spinorial if and only if

$$\frac{\dim V^\lambda (|\lambda+\rho|^2 - |\rho|^2)|\nu|^2}{2 \dim \mathfrak{g}} \equiv 0 \mod 2,$$

for every ν, where λ is the highest weight of Ad, i.e, the highest root. Since for the adjoint representation $V^\lambda = \mathfrak{g}$, we deduce that Ad is spinorial if and only if

$$\frac{(|\lambda+\rho|^2 - |\rho|^2)|\nu|^2}{2} \equiv 0 \mod 2.$$

By Exercise 8.7 (4) on page 195 of [Kirillov(2009)], we deduce that $(\lambda, \lambda + 2\rho) = |\lambda+\rho|^2 - |\rho|^2 = 1$, since λ is the highest root.

Thus the condition for the spinoriality becomes $\frac{|\nu|^2}{2} \equiv 0 \mod 2$ for every ν.

Lemma 4.4.2. $\frac{|\nu|^2}{2}$ *is even for every co-character* ν *if and only if* $\rho \in X^*(T)$.

Proof.

$$\begin{aligned}
\frac{|\nu|^2}{2} &= (1/2)\sum_{\beta\in R}(\beta,\nu)^2 \\
&= \sum_{\beta\in R_+}(\beta,\nu)^2 \\
&\equiv \sum_{\beta\in R_+}(\beta,\nu)^2 + 2\sum_{\{\beta_i,\beta_j\}\subset R_+}(\beta_i,\nu)(\beta_j,\nu) \mod 2 \\
&\equiv (\sum_{\beta\in R_+}\beta,\nu)^2 \mod 2 \\
&\equiv (2\rho,\nu)^2 \mod 2 \\
&\equiv 4(\rho,\nu)^2 \mod 2
\end{aligned} \tag{4.13}$$

If $\rho \in X^*(T)$ then $(\rho,\nu) \in \mathbb{Z}$ for all co-characters ν, which means $\frac{|\nu|^2}{2} \in 2\mathbb{Z}$.

For the converse, we suppose that $\frac{|\nu|^2}{2}$ is even for all co-characters of T.

Since $(2\rho,\nu) \in \mathbb{Z}$ then either $(\rho,\nu) \in \mathbb{Z}$ or $(\rho,\nu) = \frac{a}{2}$, where a is an odd integer. If $(\rho,\nu) \in \mathbb{Z}$ for every co-character ν of T, which means $\rho \in X^*(T)$ due to perfect pairing of characters and co-characters. If $(\rho,\nu) \notin \mathbb{Z}$ for some ν, then $(\rho,\nu) = \frac{a}{2}$ where a is an odd integer. Due to Equation (4.13) $\frac{|\nu|^2}{2} \equiv 4(a/2)^2 \mod 2$. Hence we get $\frac{|\nu|^2}{2} \equiv a^2 \mod 2$, which is a contradiction since a is odd and we assumed that the LHS is even. Therefore the second case cannot occur and we get $\rho \in X^*(T)$. □

Hence we conclude the proof of the Theorem 4.4.1.

□

This reproves page 416 exercise 7*b* in [Bourbaki(2005)] for real compact Lie groups after complexifying.

Chapter 5

Highest weight Lattices

In this section, we will study lattices of highest weights associated with the representations of a semisimple Lie group or a Lie algebra. After definition we study the relation amongst them. Let $\mathfrak{g}$ be a semisimple Lie algebra. Now fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Definition 5.0.1. *We define the* ***weight lattice*** *of a semisimple Lie algebra* $\mathfrak{g}$ *as*

$$P(\mathfrak{g}) = \{p \in \mathfrak{h}^* \mid \langle p, H_\alpha \rangle \in \mathbb{Z}, \forall \alpha \in R\},$$

where R is the set of roots and $\{H_\alpha, \alpha \in R\}$ is the set of co-roots.

Let G be a group having Lie algebra $\mathfrak{g}$.

Definition 5.0.2. *We define*

$$P(G) = \{p \in P(\mathfrak{g}) \mid p = d\lambda \text{ for some } \lambda \in X^*(T)\},$$

where T is the maximal torus corresponding to $\mathfrak{h}$, $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ and $d\lambda \in \mathfrak{h}^*$ is the derivative of λ.

The weight lattice for a semi-simple Lie algebra $\mathfrak{g}$ is the set of weights whose intersection with with each Weyl chamber gives the highest weights of irreducible finite dimensional representation of $\mathfrak{g}$. After fixing a positive Weyl chamber C_0, we can define three lattices.

Definition 5.0.3. *For a semisimple Lie algebra* $\mathfrak{g}$ *we define*

$$P_{\mathrm{sd}}(\mathfrak{g}) = \{p \in P \mid w_0(p) = -p\},$$

where w_0 *is the longest element of the Weyl group. Similarly for* G *we put*

$$P_{\mathrm{sd}}(G) = P_{\mathrm{sd}}(\mathfrak{g}) \cap P(G)$$

We know from Lemma 2.2.1 that an irreducible representation of a semisimple Lie group G or a semisimple Lie algebra $\mathfrak{g}$ of highest weight ϖ is self-dual if and only if $\varpi = -\omega_0(\varpi)$ where ω_0 is the longest element of the Weyl group of G or $\mathfrak{g}$. This means that P_{sd} $(P_{\mathrm{sd}}(G))$ is the set of ϖ in P $(P(G))$ satisfying the linear equation $(\omega_0 + I)(\varpi) = 0$. So $P_{\mathrm{sd}}(P_{\mathrm{sd}}(G))$ is a free abelian group because it is the intersection of the weight lattice and the kernel of $(\omega_0 + I)$. Note that $P_{\mathrm{sd}}(G)$ depends on the choice of w_0, which depends on the choice of a positive Weyl chamber.

Definition 5.0.4. *For a semisimple Lie algebra* $\mathfrak{g}$ *we define*

$$P_{\mathrm{orth}}(\mathfrak{g}) = \left\{ \varpi \in P_{\mathrm{sd}} \mid \sum_{\alpha \in R_+} \langle \varpi, H_\alpha \rangle \equiv 0 \mod 2 \right\},$$

where R_+ *denotes the set of positive roots. For a semisimple group* G*, put*

$$P_{\mathrm{orth}}(G) = P_{\mathrm{orth}}(\mathfrak{g}) \cap P(G).$$

From Theorem 2.3.1 we know that a self-dual irreducible representation with highest weight ϖ is real if and only if $\sum_{\alpha \in R_+} \langle \varpi, H_\alpha \rangle$ is even, where R_+ is the set of positive roots. We can think of $\varpi \mapsto (\sum_{\alpha \in R_+} \langle \varpi, H_\alpha \rangle \mod 2)$ as a group homomorphism from $P_{\mathrm{sd}}(\mathfrak{g})$ or $P_{\mathrm{sd}}(G)$ to $\mathbb{Z}/2\mathbb{Z}$. The kernel of this map is denoted by $P_{\mathrm{orth}}(\mathfrak{g})$ or $P_{\mathrm{orth}}(G)$. The intersection of P_{sd} ($P_{\mathrm{orth}}(G)$) with C_0 are precisely the highest weights of the irreducible orthogonal representations of $\mathfrak{g}$ or G.

Definition 5.0.5. *We define*

$$P'_{\mathrm{Spin}}(G) = \{\lambda \in P_{\mathrm{orth}} \mid F_\nu(\lambda) \equiv 0 \mod 2 \forall \nu \in X_*(T)\}.$$

Note that $\lambda \in (P'_{\mathrm{Spin}}(G) \cap$ Weyl chamber $\Leftrightarrow$ the representation with highest weight λ is spinorial. Unlike the highest weights of self-dual representations or orthogonal representations, highest weights of spinorial representations need not form a subgroup. By which we mean $P'_{\mathrm{Spin}}(G)$ may not a subgroup of $P_{\mathrm{orth}}(G)$. Although the spinorial weights are in some sense periodic.

Definition 5.0.6. *We define*

$$P_{\mathrm{Spin}}(G) = \{p \in P_{\mathrm{orth}}(G) \mid \lambda \in P'_{\mathrm{Spin}}(G) \Leftrightarrow \lambda + p \in P'_{\mathrm{Spin}}(G).\}$$

By this definition we mean that we collect the set of vectors $p \in P_{\mathrm{orth}}(G)$ such that whenever $\lambda \in P'_{\mathrm{Spin}}(G)$ then $\lambda + p \in P'_{\mathrm{Spin}}(G)$. This means that if p is such a vector and if λ and $\lambda + p$ are elements of $P_{\mathrm{orth}}(G) \cap$ Weyl chamber, then whenever λ is spinorial, $\lambda + p$ is also spinorial. We show further that such non-zero vectors p always exist. By definition, $P_{\mathrm{Spin}}(G)$ is a lattice and the Theorem 5.0.13 shows that the index $[P_{\mathrm{orth}}(G) : P_{\mathrm{Spin}}(G)]$ is finite. In this sense $P_{\mathrm{Spin}}(G)$ captures the periodicity of the spinorial weights.

For any complex variable x and natural number i we define a polynomial

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}.$$

Let $S = P_{\text{orth}}(G) \otimes \mathbb{Q}$. Let $P_{\text{orth}}^{\vee}(G) \subset S^*$ be its dual lattice.

Write $\binom{P_{\text{orth}}^{\vee}(G)}{\mathbb{Z}}$ for the $\mathbb{Z}$-algebra of polynomial maps from S to $\mathbb{Q}$ that take integer values on $P_{\text{orth}}(G)$. For example, let $\alpha \in P_{\text{orth}}^{\vee}(G)$, and define $\binom{\alpha}{n} \in \binom{P_{\text{orth}}^{\vee}(G)}{\mathbb{Z}}$ by the prescription $\binom{\alpha}{n} : \lambda \to \binom{\langle \alpha, \lambda \rangle}{n}$ for $\lambda \in S$.

Proposition 5.0.7. *The $\mathbb{Z}$-algebra $\binom{P_{\text{orth}}^{\vee}(G)}{\mathbb{Z}}$ is generated as an algebra by $\binom{\alpha_i}{n}$, where α_i is a set of generators of $P_{\text{orth}}^{\vee}(G)$ and $n \geq 1$.*

Proof. See Proposition 2 in [Bourbaki(2005)] page 177. □

Proposition 5.0.8. *Let $f : S \to \mathbb{Q}$ be a polynomial map in $\binom{P_{\text{orth}}^{\vee}(G)}{\mathbb{Z}}$. Then there is a subgroup $L < P_{\text{orth}}(G)$ of finite index so that for all $\lambda \in P_{\text{orth}}(G)$ and all $\ell \in L$ we have $f(\lambda + \ell) \equiv f(\lambda) \mod 2$.*

Proof. From the proposition above, we have generators $\alpha_1, \alpha_2, \ldots, \alpha_r$ of $P_{\text{orth}}^{\vee}(G)$ and integers $n_1, n_2, \ldots, n_r$, and a polynomial $g \in \mathbb{Z}[x_1, x_2, \cdots, x_r]$ so that $f = g\left(\binom{\alpha_1}{n_1}, \ldots, \binom{\alpha_r}{n_r}\right)$. Put $k = \max(k_1, \ldots, k_r)$ where $k_i = [log_2 n_i] + 1$. Note that $\binom{\alpha_i}{n_i}(\lambda + 2^k \mu) \equiv \binom{\alpha_i}{n_i}(\lambda) \mod 2$ for all $\lambda, \mu \in P_{\text{orth}}(G)$ by Lemma 12.1.2. Thus we can take $L = 2^k \cdot P_{\text{orth}}(G)$. □

Lemma 5.0.9. *Let $A(x)$ be a polynomial in one variable with real coefficients. if $A(\mathbb{Z}_{\geq 0}) \subseteq \mathbb{Z}$, then $A(\mathbb{Z}) \subseteq \mathbb{Z}$.*

Proof. Observe that $\binom{x}{i}$ forms a basis of vector space of real polynomials. Hence

$$A(x) = \sum_{i=0}^{m} c_i \binom{x}{i},$$

where $c_i \in \mathbb{R}$ and m is some natural number. Observe that $\binom{x}{i}$ takes integer values on integer input. Hence if we prove $c_i \in \mathbb{Z}$ then we are done.

Lemma 5.0.10. *The numbers c_i are integers.*

Proof. We will proceed by induction. The base case is easy, i.e., $A(0) = c_0$ is an integer. Let us assume $c_0, c_1, \cdots, c_{j-1}$ are integers. Observe that $A(j) = \sum_{i=0}^{j-1} c_i \binom{j}{i} + c_j \binom{j}{j}$ is an integer. Thus $c_j = A(j) - \sum_{i=0}^{j-1} c_i \binom{j}{i}$ is an integer. Hence the proof. □

Therefore we conclude the proof of Lemma 5.0.9. □

Lemma 5.0.11. *Let C be a polyhedral cone in $\mathbb{R}^n$ with vertex 0, such that $\mathrm{Span}(C) = \mathbb{R}^n$. Let $L \subset \mathbb{R}^n$ be a lattice and let $P \in L$. Then the intersection of C and $P + C$ is again a cone of the form $P' + C$ for some point P'. Since $\mathrm{Span}(C) = \mathbb{R}^n$, $(P' + C) \cap L$ is nonempty. We join P to a point P'' in $(P' + C) \cap L$. Then $P'' - P$ and $P + \mathbb{Z}_{\geq 1} \cdot (P'' - P) \in C \cap L$.*

Proof. A polyhedral cone C, such that $\mathrm{Span}(C) = \mathbb{R}^n$ can be defined in two ways. Either $C = \{x = (x_1, x_2, \ldots, x_n) \mid B \cdot x \geq d\}$, where B is a rank-n $m \times n$ real matrix, where $m \geq n$ and d is a vector in $\mathbb{R}^n$, or $C = \{v_0 + \sum_{i=1}^{r} \alpha_i v_i \mid \alpha_i \geq 0\}$, where the set $\{v_i \mid 1 \leq i \leq r\}$ spans $\mathbb{R}^n$.

To prove that C and $P + C$ have non-empty intersection we use the second version of the definition. Let $C = \{\sum_{i=1}^{r} \alpha_i v_i \mid \alpha_i \in \mathbb{R}_{\geq 0}\}$, then $P + C = \{P + \sum_{i=1}^{r} \alpha_i v_i \mid \alpha_i \in \mathbb{R}_{\geq 0}\}$. We can write $P = \sum_{i=1}^{r} \beta_i v_i$. Choose $\alpha_i > -\beta_i$ so that $\sum_{i=1}^{r} (\alpha_i + \beta_i) v_i \in C \cap (P + C)$. Thus the intersection is nonempty.

To prove that the intersection is a cone, we use the first definition. For both C and $P + C$ the matrix B is the same. For C, $d = 0$, while for $P + C$, $d = B \cdot P$. Let $Y = B \cdot P = (y_1, y_2, \ldots, y_n)$. Let $Y' = (\max\{0, y_1\}, \max\{0, y_2\}, \ldots, \max\{0, y_n\})$. Then the intersection is given by the equation $B \cdot x \geq Y'$. Hence it is a cone. Therefore it is equal to $P' + C'$ for some point P', and $C' = C$ since B is the same.

Since $\mathrm{Span}(C \cap (P + C)) = \mathbb{R}^n$, the intersection of L with $C \cap (P + C)$ is non empty.

Take a point P'' in $L \cap (P' + C)$. Here we will use the second version of the definition. Since $P'' \in P' + C$, we have $P'' \in C$ and $P'' - P \in C$ Thus both P'' and $P'' - P$ are of the form $\sum_{i=1}^{r} \alpha_i v_i$, where $\alpha_i \in \mathbb{R}_{\geq 0}$. Hence $P + \mathbb{Z}_{\geq 0}(P'' - P) \in C$. Therefore $P + \mathbb{Z}_{\geq 1} \cdot (P'' - P) \in C \cap (P + C) \cap L$. □

Proposition 5.0.12. *Let C be a polyhedral cone such that* $\mathrm{Span}(C) = \mathbb{R}^n$*, and L be a lattice in $\mathbb{R}^n$. If $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ takes integer values on $C \cap L$, then f takes integer values on the whole of L.*

Proof. Let $P \in L$. We will show that $f(P) \in \mathbb{Z}$. By the lemma above, there exists a point P'' in $C \cap L$ such that $P'' - P$ and $P + \mathbb{Z}_{\geq 1} \cdot (P'' - P) \in C \cap L$. Now we define a polynomial $g \in \mathbb{R}[t]$ by $g(t) = f(P'' + t \cdot (P'' - P))$. Since P'' and $P'' - P$ both belong to $C \cap L$, it is clear that $g(\mathbb{Z}_{\geq 0}) \subseteq \mathbb{Z}$. Hence by Lemma 5.0.9 we have $g(\mathbb{Z}) \subseteq \mathbb{Z}$. Thus $g(-1) = f(P) \in \mathbb{Z}$. Since the choice of P was arbitrary, we conclude that f takes integer values on the whole of L. □

Theorem 5.0.13. ***[Periodicity Theorem]*** *The index $[P_{\mathrm{orth}}(G) : P_{\mathrm{Spin}}(G)]$ is finite.*

Proof. Now observe that the expression

$$f_\nu(\lambda) = \frac{\dim V^\lambda |\nu|^2 (|\lambda + \rho|^2 - |\rho|^2)}{2 \dim \mathfrak{g}},$$

is a real polynomial in the variable λ using the Weyl dimension formula. From Theorem 4.2.9 we can say that this polynomial takes integer values on $P_{\mathrm{orth}}(G) \cap C_0$, where C_0 is the Weyl chamber.

We have $|\nu|^2/2 = \sum_{\beta \in R_+} (\beta, \nu)^2$. Thus $|\nu|^2/2$ is an integer valued function on $X_*(T)$. Since we are interested in only parity of f_ν, we may only concentrate on $\min_{\nu \in X_*(T)} \mathrm{ord}_2(|\nu|^2/2)$. The function $\mathrm{ord}_2(|\nu|^2/2)$ achieves its minimum for some ν let us say for ν_0. So we have ϕ_λ is spinorial if and only if $f_{\nu_0}(\lambda)$ is even.

Applying the proposition above we deduce that the polynomial f_{ν_0} takes integer values on the whole of $P_{\text{orth}}(G)$. We can identify $S \otimes \mathbb{R}$ with a real subspace say K of $\mathfrak{h}^*$ such that $P_{\text{orth}}(G)$ maps identically to $P_{\text{orth}}(G)$. Note that since f_{ν_0} is a polynomial on $\mathfrak{h}^*$, $f_{\nu_0}|_K$ is a polynomial in co-ordinates of K and $P_{\text{orth}}(G)$ is a lattice in K. Thus $f|_K \in \binom{P^\vee_{\text{orth}}(G)}{\mathbb{Z}}$. Thus by Proposition 5.0.8 there exists a subgroup L of $P_{\text{orth}}(G)$ of finite index such that $f(\lambda+\ell) \equiv f(\lambda) \mod 2$ for every $\lambda \in P_{\text{orth}}(G)$ and $\ell \in L$. Thus $L \subseteq P_{\text{Spin}}(G)$. Hence $P_{\text{orth}}(G)/P_{\text{Spin}}(G)$ is finite. □

Thus in order to determine the spinoriality of the representation of highest weight λ we only need to check it on the coset representatives of $P_{\text{Spin}}(G)$ in $P_{\text{orth}}(G)$, which are finite in number.

Chapter 6

Applications

6.1 Preliminaries

6.1.1 Orthogonal Representations of GL(n)

In this section, we determine, which of the irreducible representations of $\mathrm{GL}(n, \mathbb{C})$ are orthogonal. From page 268 Section (5.2) (vii) of [Bröcker and tom Dieck(2013)] we deduce that the self dual irreducible representations of $U(n)$ are real or orthogonal. Therefore all of the irreducible self-dual representations of $\mathrm{GL}(n, \mathbb{C})$ are orthogonal, by complexification. Let Z denote the centre of $\mathrm{GL}(n, \mathbb{C})$. Let T be the maximal torus of invertible diagonal matrices in $\mathrm{GL}(n, \mathbb{C})$. Let $T' = T \cap \mathrm{SL}(n, \mathbb{C})$. Let W be the Weyl group of T in $\mathrm{GL}(n, \mathbb{C})$. Now for the rest of the discussion we will use the conventions and results of Section 5.5.4 of [Goodman and Wallach(2009)].

For $a = a_1 \oplus a_2 \oplus \cdots \oplus a_n \in T$, define $\epsilon_i \in X^*(T)$ as $\epsilon_i(a) = a_i$.

We start from the irreducible representation (π, V) of highest weight

$$\mu = \sum_{i=1}^{n} m_i \epsilon_i,$$

where $m_i \in \mathbb{Z}, m_i \geq m_{i+1}$.

Then

$$\varpi_i = \sum_{j=1}^{i} \epsilon_j,$$

are the fundamental weights of $\mathfrak{sl}(n, \mathbb{C})$ for $1 \leq i \leq n-1$.

First let us restrict π to Z. We have $\pi(zI) = z^{\sum_{i=1}^{n} m_i} I$ by page 274 Theorem 5.5.22 [Goodman and Wallach(2009)]. As the representation is self-dual, $\pi(zI)$ should be conjugate to ${}^{T}(\pi(zI))^{-1}$. So Z acts trivially on V. This forces

$$\sum_{i=1}^{n} m_i = 0.$$

Now we restrict π to $\mathrm{SL}(n, \mathbb{C})$. It is still irreducible because $\mathrm{GL}(n, \mathbb{C}) = Z \cdot \mathrm{SL}(n, \mathbb{C})$, and Z acts trivially. By Section 5.5.4 of [Goodman and Wallach(2009)], the corresponding highest weight is

$$\mu_0 = \sum_{i=1}^{n-1} (m_i - m_{i+1}) \varpi_i.$$

Recall from Theorem 2.2.1 π is self dual if its highest weight ϖ satisfies $\varpi = -w_0(\varpi)$, where w_0 is the longest element of its Weyl group. Now the Weyl group of $\mathrm{SL}(n, \mathbb{C})$ is the symmetric group S_n and the longest element in cycle notation is

$$w_0 = \begin{cases} (1, n)(2, n-1)(3, n-2) \cdots (\frac{n}{2}, \frac{n}{2} + 1), & \text{if } n \text{ is even,} \\ (1, n)(2, n-1) \cdots (\frac{n+1}{2} - 1, \frac{n+1}{2} + 1), & \text{if } n \text{ is odd.} \end{cases}$$

See [Bourbaki(2002)] Chapter VI section 4 no. 7 XI. Hence $w_0(\epsilon_j) = \epsilon_{n+1-j}$.

Observe that

$$\begin{aligned}
w_0(\varpi_j) &= w_0\left(\sum_{i=1}^{j}\epsilon_i\right)\\
&= w_0(\epsilon_1) + w_0(\epsilon_2) + \cdots + w_0(\epsilon_j)\\
&= \epsilon_n + \epsilon_{n-1} + \cdots + \epsilon_{n+1-j}\\
&= -(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1}) + \epsilon_{n-1} + \epsilon_{n-2} + \cdots + \epsilon_{n+1-j}\\
&= -\epsilon_1 - \epsilon_2 - \cdots - \epsilon_{n-j}\\
&= -\varpi_{n-j},
\end{aligned}$$

since $\sum_{i=1}^{n}\epsilon_i = 0$, as we are now in $\mathrm{SL}(n,\mathbb{C})$.

Now if π is self-dual then μ_0 should satisfy $\mu_0 = -w_0(\mu_0)$. Thus we get

$$\mu_0 = \sum_{i=1}^{n-1}(m_i - m_{i+1})\varpi_i = -w_0(\mu_0) = \sum_{j=1}^{n-1}(m_j - m_{j+1})\varpi_{n-j}.$$

Since ϖ_i are linearly independant, we get

$$m_i - m_{i+1} = m_{n-i} - m_{n-i+1}\ \forall\ i < n.$$

Let $n = 2k$. Thus we put

$$\begin{aligned}
&m_1 - m_2 = m_{n-1} - m_n = r_k,\\
&m_2 - m_3 = m_{n-2} - m_{n-1} = r_{k-1},\\
&\vdots\\
&m_{k-1} - m_k = m_{k+1} - m_{k+2} = r_2.
\end{aligned}$$

Note the condition $\sum_i^n m_i = 0$ forces $m_k + m_{k+1} = 0$. Hence we set $m_k = -m_{k+1} = r_1$. Thus the highest weight corresponding to our orthogonal representation of $\mathrm{GL}(n,\mathbb{C})$ is of the form

$$(r_k + r_{k-1} + \cdots + r_1, r_{k-1} + r_{k-2} + \cdots + r_1, \ldots, r_1, -r_1, -r_1 - r_2, \ldots, -r_1 - r_2 - \cdots - r_k),$$

where $r_1, r_2, \ldots, r_k$ are nonnegative integer parameters.

6.1.2 Orthogonal Representations of $\mathrm{SO}(2n+1)$

Theorem 6.1.1. *All of the irreducible representations of* $\mathrm{SO}(2n+1,\mathbb{C})$ *are orthogonal.*

Proof. From Proposition 26.26 of [Fulton and Harris(1991)] which says the representation of highest weight $a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1} + a_n\varpi_n/2$, where $a_i \in \mathbb{Z}$ and ϖ_j are fundamental weights of $\mathfrak{so}(2n+1,\mathbb{C})$, is orthogonal if either a_n is even or $n \equiv 0$ or $3 \mod 4$. Here $\varpi_j = \sum_{i=1}^{j} \epsilon_i$ for $1 \leq j \leq n-1$ and $\varpi_n = (1/2)(\sum_{i=1}^{n} \epsilon_i)$. Moreover 23.13 of [Fulton and Harris(1991)] says that for representation of Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$ to be the differential of a representation of the group $\mathrm{SO}(2n+1)$, the highest weight vector should be integral i.e. of the form $\sum b_i\epsilon_i$, where $b_i \in \mathbb{Z}$. For that to happen 4 should divide a_n, then the representation is orthogonal since a_n is even. Thus all the irreducible representations of $\mathrm{SO}(2n+1,\mathbb{C})$ are orthogonal. □

6.1.3 Orthogonal Representations of $\mathrm{SO}(2n)$

The fundamental representations here are $\varpi_k = \sum_{i=1}^{k} \epsilon_i$, for $1 \leq k \leq n-2$. The fundamental weight $\varpi_{n-1} = (1/2)(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)$, furthermore $\varpi_n = (1/2)(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)$.

Theorem 6.1.2. *A representation of* $\mathrm{SO}(2n,\mathbb{C})$ *of highest weight* $\varpi = \sum_{i=1}^{n-2} a_i\varpi_i + a_{n-1}\varpi_{n-1} + a_n\varpi_n$ *is orthogonal if and only if either* n *is odd and* $a_{n-1} = a_n$ *or* n *is even and* $a_{n-1} + a_n$ *is even.*

Proof. See Proposition 26.27 and Proposition 23.13 (iii) of [Fulton and Harris(1991)]. □

6.2 Examples

6.2.1 Case PGL(2)

In this section we will determine the spinoriality of irreducible orthogonal representations of $G = \mathrm{PGL}(2, \mathbb{C})$. The Lie algebra of $\mathrm{PGL}(2, \mathbb{C})$ is isomorphic to $\mathfrak{gl}(2, \mathbb{C})/(Scalars) \cong \mathfrak{sl}(2, \mathbb{C})$. It is simple, and we denote it by $\mathfrak{g}$. Hence cartan subalgebra of $\mathfrak{g}$ is $\dfrac{a \oplus b}{x \oplus x}$, where $a, b, x \in \mathbb{C}$. We denote it by $\mathfrak{h}$. Define $\alpha \in h^*$ by $\alpha\left(\dfrac{a \oplus b}{x \oplus x}\right) = a - b$, note that it is well defined. In fact α is the single positive root as it is weight of the adjoint representation with eigenvector $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Hence $\rho = (1/2)\alpha$. Any integral weight of $\mathfrak{h}$ will be of the form $n\alpha$, where $n \in \mathbb{Z}$. Now we choose the co-character $\nu : y \to \dfrac{y \oplus 0}{x \oplus x}$.

Since $\mathfrak{g}$ is simple, our formula is

$$\frac{Q''_\nu(1)}{2} = \frac{(\dim E^\lambda)(|\nu|^2)(|\lambda + \rho|^2 - |\rho|^2)}{2 \dim \mathfrak{g}}.$$

For $\mathfrak{sl}(2, \mathbb{C})$ the Killing form on $\mathfrak{h}$ is $4\,\mathrm{Trace}(X \cdot Y) = 4(x_1 y_1 + x_2 y_2)$ where $X = x_1 \oplus x_2, Y = y_1 \oplus y_2$. Hence on $\mathfrak{h}^*$ it is $(1/4)(x_1^* y_1^* + x_2^* y_2^*)$, where $X^* = x_1^* \epsilon_1 + x_2^* \epsilon_2$, $Y^* = y_1^* \epsilon_1 + y_2^* \epsilon_2$. Here ϵ_i are projections on i'th factor. Therefore we get

$$\begin{aligned}
|\lambda+\rho|^2-|\rho|^2 &= (n\alpha+(1/2)\alpha, n\alpha+(1/2)\alpha)-((1/2)\alpha,(1/2)\alpha) \\
&= ((n+(1/2))^2-(1/2)^2)(\alpha,\alpha) \\
&= (n^2+n)(\alpha,\alpha) \\
&= (n^2+n)(\epsilon_1-\epsilon_2,\epsilon_1-\epsilon_2) \\
&= (n^2+n)(1/4)(1^2+(-1)^2) \\
&= (1/2)(n^2+n).
\end{aligned}$$

Using the dimension formula,

$$\begin{aligned}
\dim(E) &= \frac{(n\alpha+(1/2)\alpha,\alpha)}{((1/2)\alpha,\alpha)} \\
&= 2n+1.
\end{aligned}$$

By definition of the Killing form, we get,

$$\begin{aligned}
|\nu|^2 &= (\alpha,\nu)^2+(-\alpha,\nu)^2 \\
&= ((1,-1)\cdot(1,0))^2+((-1,1)\cdot(1,0))^2 \\
&= 2.
\end{aligned}$$

Substituting in the formula we get

$$\frac{Q''_{(1,0)}(1)}{2} = \frac{(2n+1)\cdot 2\cdot(1/2)(n^2+n)}{2\cdot\dim(\mathfrak{pgl}_2(\mathbb{C}))} = \frac{n(n+1)(2n+1)}{6}.$$

Since $2n+1$ is odd, ϕ is spinorial if and only if

$$\frac{n(n+1)}{2}\equiv 0 \mod 2.$$

6.2.2 Case PGL(n)

The group SL($n, \mathbb{C}$) is a finite cover of PGL($n, \mathbb{C}$), hence it has the same Lie algebra as SL($n, \mathbb{C}$), which is simple. Therefore we can apply Theorem 4.2.9 here. Let ϕ_λ be an orthogonal irreducible finite-dimensional representation of PGL($n, \mathbb{C}$) of highest weight λ. For PGL($n, \mathbb{C}$), the Cartan subalgebra $\mathfrak{h}$ is $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ such that $\sum x_i = 0$. This is a subspace of $\mathfrak{h}_{\mathfrak{gl}(n,\mathbb{C})}$.

Hence $\mathfrak{h}^*$ is $\frac{\mathfrak{h}_{\mathfrak{gl}(n,\mathbb{C}))}}{\mathbb{C}\cdot(1,1,\ldots,1)}$. Recall that we want to calculate

$$\frac{(\dim V^\lambda)(|\nu|^2)(|\lambda+\rho|^2 - |\rho|^2)}{2\dim \mathfrak{g}}.$$

Lemma 6.2.1. *If n is an odd natural number, then all of the irreducible orthogonal finite-dimensional representations of* PGL($n, \mathbb{C}$) *are spinorial.*

Proof. As $\pi_1(\mathrm{PGL}(n, \mathbb{C})))$ is $\mathbb{Z}/n\mathbb{Z}$, and $\pi_1(\mathrm{SO}(N, \mathbb{C}))$ is $\mathbb{Z}/2\mathbb{Z}$. The map at the level of π_1 is trivial. Hence by Lemma 3.1.1 all the corresponding representations are spinorial. □

So the groups of interest are PGL($2n, \mathbb{C}$). The highest weight of an orthogonal irreducible representation is of the form

$$\lambda = (\sum_{i=1}^{n} r_i, \sum_{i=1}^{n-1} r_i, \ldots, r_1 + r_2, r_1, -r_1, -r_1 - r_2, \ldots, -\sum_{i=1}^{n} r_i) \tag{6.1}$$

from Subsection 6.1.1.

From page 218 of (vi) of proposition 6.2 of [Bröcker and tom Dieck(2013)] we get that half of sum of roots for PGL($2n, \mathbb{C}$) is

$$\rho = \Big(\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, \frac{1}{2}, \frac{-1}{2}, \ldots, \frac{1-2n}{2}\Big).$$

Now we calculate

$$|\lambda+\rho|^2-|\rho|^2.$$

It is well-known that the Killing form for $\mathfrak{h}_{\mathfrak{sl}_{2n}}$ is $2(2n)\operatorname{Trace}(X,Y)$, i.e., $4n\sum_{i=1}^{n} x_i y_i$. We induce the Killing form on $\mathfrak{h}^*$. It is easy to see that if we choose representatives $\mu=(\mu_1,\mu_2,\ldots,\mu_{2n})+\mathbb{C}\cdot(1,1,\ldots,1)$ and $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_{2n})+\mathbb{C}\cdot(1,1,\ldots,1)$ of the space $\mathfrak{h}^*$, such that $\sum\mu_i=0$ and $\sum\gamma_i=0$, then the induced Killing form will be just $1/(4n)\sum\mu_i\gamma_i$.

Hence we have,

$$|\lambda+\rho|^2-|\rho|^2=(1/(4n))\cdot 2\cdot\left(\sum_{i=1}^{n}\left(\left(\left(\sum_{j=1}^{i} r_j\right)+\frac{2i-1}{2}\right)^2-\left(\frac{2i-1}{2}\right)^2\right)\right).$$

The following is the calculation for $\dim(V^\lambda)$. The positive roots for $\mathrm{PGL}(n,\mathbb{C})$ are $\epsilon_i-\epsilon_j$ where $i<j$. We will use the Weyl dimension formula. Suppose $\mu=(\mu_1,\mu_2,\ldots,\mu_{2n})\in\mathfrak{h}^*$.

We refer to page 337 section 7.1.2 examples [Goodman and Wallach(2009)].

$$\dim V^\lambda=\prod_{1\le i<j\le n}\frac{\mu_i-\mu_j+j-i}{j-i}.$$

For calculating $|\nu|^2$, we choose ν to be,

$$\nu: z\mapsto(z\oplus 0\oplus 0\oplus\cdots\oplus 0)+\mathbb{C}\cdot(1,1,\cdots,1).$$

So basically $\nu = (1, 0, 0, \ldots, 0)$, hence

$$\begin{aligned}|\nu|^2 &= \sum_{\alpha \in R} (\alpha, \nu)^2 \\ &= \sum_{1 \le i \ne j \le 2n} (\epsilon_i - \epsilon_j, \nu)^2 \\ &= 2(2n-1).\end{aligned}$$

It is easy to see that the lattice spanned by the Weyl conjugates of ν form a spanning set for the co-character lattice. At the level of π_1 the image of a co-character is trivial if and only if the image of its Weyl conjugates are trivial. Hence it is enough to check the formula for this particular co-character ν.

Of course $\dim(\mathfrak{sl}_{2n}(\mathbb{C})) = (2n)^2 - 1$.

Recall Equation 6.1 for definition of λ_i.

Theorem 6.2.2. *ϕ is spinorial, if and only if,*

$$\begin{aligned}\frac{Q''_\nu(1)}{2} &= \frac{1}{2} \cdot \prod_{1 \le i < j \le n} \left(\frac{\lambda_i - \lambda_j + j - i}{j - i} \right) \cdot \frac{2(2n-1)}{1} \cdot \frac{1}{4n^2 - 1} \cdot \frac{\sum_{i=1}^n (((\sum_{j=1}^i r_j) + \frac{2i-1}{2})^2 - (\frac{2i-1}{2})^2)}{2n} \\ &= \prod_{1 \le i < j \le n} \left(\frac{\lambda_i - \lambda_j + j - i}{j - i} \right) \frac{\sum_{i=1}^n (((\sum_{j=1}^i r_j) + \frac{2i-1}{2})^2 - (\frac{2i-1}{2})^2)}{2n(2n+1)}\end{aligned}$$

is even.

Relation with the Dynkin index

Since we are considering A_{m-1}, we have that $\pi_1(G) \cong \mathbb{Z}/m\mathbb{Z}$.

Here all the roots have same length. Let us denote the Killing form on $\mathfrak{h}$ or $\mathfrak{h}^*$ by $(,)$. Let

$$n_G = (1/2)(\alpha, \alpha)(\nu, \nu),$$

where α is the longest root and ν is a co-character that corresponds to a generator of $\pi_1(G)$. Here both the lattices, the co-character lattice and the co-root lattice, are free abelian groups. Let $\beta_i^\vee$, $1 \leq i \leq n-1$ be the simple co-roots. Then $\nu = (1/m)\sum_{i=1}^{n-1} a_i\beta_i^\vee$, for some $a_i \in \mathbb{Z}$ and where $\gcd(a_1, a_2, , \cdots, a_{n-1}, m) = 1$. Note that, here we are not saying that we can choose a_i arbitary. The choice of a_i depends on G. Let $\beta_i^\vee$ be the co-root corresponding to the root β_i. By the usual identification

$$(\beta_i^\vee, \beta_j^\vee) = \left(\frac{2\beta_i}{(\beta_i, \beta_i)}, \frac{2\beta_j}{(\beta_j, \beta_j)}\right) = \frac{4(\beta_i, \beta_j)}{(\beta_i, \beta_i)(\beta_j, \beta_j)}. \tag{6.2}$$

Note that $(\alpha, \alpha) = (\beta_i, \beta_i)$ for $1 \leq i \leq n-1$.

Now

$$\begin{aligned}(\nu, \nu) =&((1/m)\sum_{i=1}^{n-1} a_i\beta_i^\vee, (1/m)\sum_{i=1}^{n-1} a_i\beta_i^\vee)\\ =&(\frac{1}{m^2})\left(\sum_{i=1}^{n-1} a_i^2(\beta_i^\vee, \beta_i^\vee) + 2\left(\sum_{1\leq i<j\leq n-1} a_ia_j(\beta_i^\vee, \beta_j^\vee)\right)\right)\\ =&(\frac{1}{m^2})\left(\left(\sum_{i=1}^{n-1} a_i^2\frac{4}{(\beta_i, \beta_i)}\right) + 2\left(\sum_{1\leq i<j\leq n-1} a_ia_j\frac{4(\beta_i, \beta_j)}{(\beta_i, \beta_i)(\beta_j, \beta_j)}\right)\right) \text{(by equation 6.2)).}\end{aligned}$$

Hence,

$$n_G = \frac{1}{m^2}\left(\left(\sum_{i=1}^{n-1} 2a_i^2\right) + 2\left(\sum_{1\leq i<j\leq n-1} a_ia_j\frac{2(\beta_i, \beta_j)}{(\beta_j, \beta_j)}\right)\right), \tag{6.3}$$

as $(\alpha, \alpha) = (\beta_i, \beta_i)$ for $1 \leq i \leq n-1$.

Since $\frac{2(\beta_i,\beta_j)}{(\beta_j,\beta_j)}$ is the $(i, j)^{th}$ entry of the Cartan matrix, clearly

$$n_G = \frac{1}{m^2}a^tCa,$$

where a is the $(n-1) \times (1)$ column matrix with i^{th} entry a_i and C is the Cartan matrix of the root system A_{n-1}.

Calculation of n_G for $\mathrm{PGL}(n,\mathbb{C})$

Here we calculate n_G for $\mathrm{PGL}(n,\mathbb{C})$ and $\pi_1(\mathrm{PGL}(n,\mathbb{C})) \cong \mathbb{Z}/n\mathbb{Z}$ and $m = n$. The Cartan subalgebra for $\mathrm{PGL}(n,\mathbb{C})$ is

$$\frac{\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}}{\mathbb{C}(1 \oplus 1 \oplus \cdots \oplus 1)}.$$

Here the co-character lattice is

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} + (\mathbb{C}(1 \oplus 1 \oplus \cdots \oplus 1)).$$

Let

$$\epsilon_i = 0 \oplus 0 \oplus \cdots \oplus \underbrace{1}_{i^{th}\text{ place}} \oplus \cdots \oplus 0 + \mathbb{C}(1 \oplus 1 \oplus \cdots \oplus 1).$$

The co-roots are $\epsilon_i - \epsilon_j$. We claim that ϵ_1 is the co-character corresponding to a generator $\pi_1(\mathrm{PGL}(n,\mathbb{C})))$. It is easy to see that

$$\epsilon_1 = \frac{n-1}{n}(\epsilon_1 - \epsilon_2) + \frac{n-2}{n}(\epsilon_2 - \epsilon_3) + \cdots + \frac{1}{n}(\epsilon_{n-1} - \epsilon_n).$$

Thus $a_i = n - i$. Since $\gcd(1, 2, \ldots, n-1, n) = 1$, ϵ_1 corresponds to a generator of $\pi_1(PGL(n,\mathbb{C}))$. If C is the Cartan matrix for A_{n-1}, then it is well known that $C_{(i,i)} = 2$ and $C_{(i,i+1)} = C_{(i+1,i)} = -1$ and the rest of the entries are 0. Hence from Equation (6.3) the calculation of n_G is as follows :

$$
\begin{aligned}
n_G =&\frac{2}{n^2}\left(\sum_{i=1}^{n-1}(n-i)^2-\sum_{i=1}^{n-2}(n-i-1)(n-i)\right)\\
=&\frac{2}{n^2}\left(\sum_{i=1}^{n-1}i^2-\sum_{i=1}^{n-2}i(i+1)\right)\\
=&\frac{2}{n^2}\left(\sum_{i=1}^{n-1}i^2-\sum_{i=1}^{n-2}(i^2+i)\right)\\
=&\frac{2}{n^2}\left(\sum_{i=1}^{n-1}i^2-\sum_{i=1}^{n-2}i^2+\sum_{i=1}^{n-2}i\right)\\
=&\frac{2}{n^2}\left((n-1)^2-(1/2)(n-2)(n-1)\right)\\
=&\frac{2(n-1)}{n^2}\left((n-1)-(1/2)(n-2)\right)\\
=&\frac{n-1}{n}.
\end{aligned}
$$

Thus $n_G = (n-1)/n$ for $\mathrm{PGL}(n, \mathbb{C})$). Thus we have prooved the following theorem.

Theorem 6.2.3. *For the group* $\mathrm{PGL}(n, \mathbb{C})$, *the irreducible representation* ϕ *is spinorial if and only if*

$$\frac{n-1}{n} \cdot \mathrm{dyn}(\phi) \equiv 0 \mod 2.$$

6.2.3 Case $\mathrm{SO}(2n+1)$

Here we discuss only cases $n \geq 2$. Let ϕ_λ be the representation $\mathrm{SO}(2n+1, \mathbb{C})$ of highest weight λ. From Subsection 6.1.2, we know that all the irreducible finite-dimensional representations are orthogonal. The Cartan subalgebra $\mathfrak{h}$ is the subalgebra $x_1 \oplus x_2 \oplus \cdots \oplus x_n \oplus (-x_1) \oplus (-x_2) \oplus \cdots \oplus (-x_n) \oplus 0$. We refer to page 337, Section 7.1.2, examples [Goodman and Wallach(2009)] for the formula of $\dim V^\lambda$. Here $\rho = \sum_{i=1}^n \rho_i \epsilon_i$, where $\rho_i = n - i + (1/2)$. Let $\lambda = \sum \lambda_i \epsilon_i$, where $\lambda_i \geq \lambda_{i+1} \geq 0$. Then

$$\dim V^\lambda = \prod_{1\leq i\leq j\leq n} \left(\frac{(\lambda_i+\rho_i)^2 - (\lambda_j+\rho_j)^2}{\rho_i^2 - \rho_j^2} \right) \prod_{1\leq i\leq n} \left(\frac{\lambda_i + \rho_i}{\rho_i} \right).$$

The Killing form on the Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$ is

$$\begin{aligned} K(X,Y) &= (2n+1-2)\,\mathrm{Trace}(XY) \\ &= (2n-1)\,\mathrm{Trace}(XY), \end{aligned}$$

where $X, Y \in \mathfrak{so}(2n+1,\mathbb{C})$. So on $\mathfrak{h}$,

$$(X,X) = 2(2n-1)(\textstyle\sum x_i^2).$$

Hence the gcd of all $|\nu|^2$ is $2(2n-1)$.

$$\begin{aligned} \dim(\mathfrak{so}(2n+1,\mathbb{C})) &= (2n+1)^2 - (2n+1) \\ &= (2n)(2n+1). \end{aligned}$$

Let $\mu \in \mathfrak{h}^*$ such that $\mu = \sum \mu_i \epsilon_i$. Then the induced Killing norm is exactly

$$\frac{1}{2(2n-1)} \sum \mu_i^2.$$

Theorem 6.2.4. *ϕ is spinorial if and only if*

$$\frac{Q_\nu''(1)}{2} = \left(\prod_{1\leq i\leq j\leq n} \left(\frac{(\lambda_i+\rho_i)^2 - (\lambda_j+\rho_j)^2}{\rho_i^2 - \rho_j^2} \right) \prod_{1\leq i\leq n} \left(\frac{\lambda_i+\rho_i}{\rho_i} \right) \right) \left(\frac{(\sum_{i=1}^n (\lambda_i+\rho_i)^2 - (\rho_i)^2)}{2\cdot(2n)(2n+1)} \right),$$

is even.

<u>Relation with the Dynkin index</u>

From Corollary 4.3.4 we know that

$$\frac{Q''_\nu(1)}{2} = \frac{(\alpha,\alpha)(\nu,\nu)\,\mathrm{dyn}(\phi)}{2},$$

where $(,)$ is the Killing form. If

$$d\nu(t) = (x_1 t) \oplus \cdots \oplus (x_n t) \oplus (-x_1 t) \oplus \cdots \oplus (-x_n t) \oplus 0,$$

where $x_i \in \mathbb{Z}$, then

$$\begin{aligned}(\nu,\nu) &= (d\nu(1), d\nu(1)) \\ &= 2(2n-1)\left(\sum_{i=1}^{n} x_i^2\right),\end{aligned}$$

and

$$\begin{aligned}(\alpha,\alpha) &= (\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_2) \\ &= \frac{1}{2(2n-1)}(1^2 + (-1)^2) \\ &= \frac{1}{(2n-1)}.\end{aligned}$$

It is easy to see now that

$$\frac{Q''_\nu(1)}{2} = \mathrm{dyn}(\phi)\left(\sum_{i=1}^{n} x_i^2\right),$$

where $\mathrm{dyn}(\phi)$ is the Dynkin invariant of $\phi : \mathfrak{g} \to \mathfrak{so}(\dim V, \mathbb{C})$.

This expression is even for all ν if and only if $\mathrm{dyn}(\phi)$ is even. Thus

Theorem 6.2.5. *The irreducible representation ϕ of $\mathrm{SO}(2n+1,\mathbb{C})$ is spinorial if and only if $\mathrm{dyn}(\phi)$ is even.*

6.2.4 Case SO$(2n)$

We know from subsection 6.1.3 that the irreducible representation of SO$(2n,\mathbb{C})$ of highest weight $\varpi = \sum_{i=1}^{n-2} a_i\varpi_i + a_{n-1}\varpi_{n-1} + a_n\varpi_n$ (where $a_i \in \mathbb{Z}$) is orthogonal if and only if either n is odd and $a_{n-1} = a_n$ or n is even and $a_{n-1} + a_n$ is even. The Cartan subalgebra $\mathfrak{h}$ is the subalgebra $x_1 \oplus x_2 \oplus \cdots \oplus x_n \oplus (-x_1) \oplus (-x_2) \oplus \cdots \oplus (-x_n)$. Here the fundamental weights are $\varpi_k = \sum_{i=1}^{k} \epsilon_i$, for $1 \le k \le n-2$. The fundamental weight $\varpi_{n-1} = (1/2)(\epsilon_1+\cdots+\epsilon_{n-1}-\epsilon_n)$, furthermore $\varpi_n = (1/2)(\epsilon_1+\cdots+\epsilon_{n-1}+\epsilon_n)$. We refer to page 337 section 7.1.2 examples [Goodman and Wallach(2009)] for dim V^λ. Here $\rho = \sum_{i=1}^{n} \rho_i\epsilon_i$ with $\rho_i = n - i$. Let

$$\begin{aligned}\lambda =&(\lambda_1, \lambda_2, \ldots, \lambda_n)\\ =&(a_1 + a_2 + \cdots a_{n-2} + (1/2)(a_{n-1} + a_n), a_2 + a_3 + \cdots + a_{n-2} + (1/2)(a_{n-1} + a_n),\\ &\cdots, a_{n-2} + (1/2)(a_{n-1} + a_n), (1/2)(a_{n-1} + a_n), (1/2)(a_n - a_{n-1})).\end{aligned}$$

Then

$$\dim V^\lambda = \prod_{1\le i<j\le n} \frac{(\lambda_i + \rho_i)^2 - (\lambda_j + \rho_j)^2}{\rho_i^2 - \rho_j^2}.$$

It is well-known that the Killing norm on the Lie algebra $\mathfrak{so}(2n,\mathbb{C})$ is $(2n-2)\,\mathrm{Trace}(XY)$, for $X, Y \in \mathfrak{so}(2n,\mathbb{C})$. Thus on $\mathfrak{h}$,

$$(X, X) = 2(2n-2)\left(\sum_{i=1}^{n} x_i^2\right).$$

Hence the gcd of all $|\nu|^2$ is $2(2n-2)$.

$$\begin{aligned}\dim(\mathfrak{so}(2n,\mathbb{C})) &= (2n)^2 - 2n\\ &= (2n)(2n-1).\end{aligned}$$

Let $\mu \in \mathfrak{h}^*$ such that $\mu = \sum_{i=1}^n \mu_i \epsilon_i$. Then the Killing form is exactly $\frac{1}{2(2n-2)} \sum_{i=1}^n \mu_i^2$.

Theorem 6.2.6. *ϕ is spinorial, if and only if,*

$$\frac{Q_\nu''(1)}{2} = \left(\prod_{1 \leq i < j \leq n} \frac{(\lambda_i + \rho_i)^2 - (\lambda_j + \rho_j)^2}{\rho_i^2 - \rho_j^2} \right) \left(\frac{\sum_{i=1}^n ((\lambda_i + \rho_i)^2 - \rho_i^2)}{2(2n)(2n-1)} \right)$$

is even.

Relation with the Dynkin index

From Corollary 4.3.4 we know that

$$\frac{Q_\nu''(1)}{2} = \frac{(\alpha, \alpha)(\nu, \nu)\,\mathrm{dyn}(\phi)}{2},$$

where $(,)$ is the Killing form. If

$$d\nu(t) = (x_1 t) \oplus \cdots \oplus (x_n t) \oplus (-x_1 t) \oplus \cdots \oplus (-x_n t),$$

where $x_i \in \mathbb{Z}$, then

$$\begin{aligned}(\nu, \nu) &= (d\nu(1), d\nu(1)) \\ &= 2(2n-2) \left(\sum_{i=1}^n x_i^2 \right),\end{aligned}$$

and

$$\begin{aligned}(\alpha, \alpha) &= (\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_2) \\ &= \frac{1}{2(2n-2)} (1^2 + (-1)^2) \\ &= \frac{1}{(2n-2)}.\end{aligned}$$

It is easy to see now that

$$\frac{Q''_\nu(1)}{2} = \mathrm{dyn}(\phi)\left(\sum_{i=1}^{n} x_i^2\right),$$

where $\mathrm{dyn}(\phi)$ is the Dynkin invariant for $\phi : \mathfrak{g} \to \mathfrak{so}(\dim V, \mathbb{C})$. This expression is even for all ν if and only if $\mathrm{dyn}(\phi)$ is even.

Theorem 6.2.7. *The irreducible representation* ϕ *of* $\mathrm{SO}(2n, \mathbb{C})$ *is spinorial if and only if* $\mathrm{dyn}(\phi)$ *is even.*

Chapter 7

Reductive Lie Groups

7.1 Reductive case

Lemma 7.1.1. *Let G be a complex reductive Lie group and let ϕ be an orthogonal irreducible representation of G i.e. $\phi : G \to \mathrm{SO}(N, \mathbb{C})$. Then ϕ factors through $G/Z(G)^\circ$(which is semi-simple), where $Z(G)^\circ$ is the connected component of the center of G containing the identity. Let us denote this representation of $G/Z(G)^\circ$ by ϕ'. Then ϕ is spinorial if and only if ϕ' is spinorial. The following diagram sheds more light on this lemma.*

$$\begin{array}{ccccc} & & & & \mathrm{Spin}(N, \mathbb{C}) \\ & & & \overset{\psi}{\nearrow} & \downarrow \rho \\ G & \longrightarrow & G/Z(G)^\circ & \xrightarrow{\phi} & \mathrm{SO}(N, \mathbb{C}) \end{array}$$

Proof. Since the representation is irreducible, by Schur's lemma, the center of G maps to the scalar matrices in $\mathrm{SO}(N)$, which are $\pm I$. Since $Z(G)^\circ$ is connected, its image is the identity, so ϕ factors through $G/Z(G)^\circ$.

For the second statement, denote the natural map $G \to G/Z(G)^\circ$ by m. One direction is clear: if ϕ' lifts to ψ', then ϕ lifts to $\psi' \circ m$. For the converse, if ϕ

lifts to ψ then observe that if $a \in Z(G)^\circ$, then $\phi(a) = 1$. Thus $\psi(a)$ maps to the fiber above 1 which is 2 points, because the spin group is a double cover of $\mathrm{SO}(N, \mathbb{C})$. Since this is true for every $a \in Z(G)^\circ$ and $Z(G)^\circ$ is connected, $\psi(a) = 1$ for $\forall a \in Z(G)^\circ$. Hence ψ factors through $G/Z(G)^\circ$. □

If we have a reductive Lie group G then $G/Z(G)^\circ$ is semisimple hence the problem reduces to solving semisimple case.

For example determining spinorial representations of $\mathrm{GL}(n, \mathbb{C})$ is equivalent to determining spinorial representations of $\mathrm{PGL}(n, \mathbb{C})$.

Chapter 8

Determinantal Identity Method

Theorem (1.1.1) solves in principle our spinoriality question, but our Periodicity Theorem leads to further questions :

1) Determine precisely $P_{\mathrm{Spin}}(G)$.

2) What proportion of othogonal irreducible representations of G are spinorial?

We pursue these questions for $\mathrm{PGL}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$, and have complete answers for $\mathrm{PGL}(4), \mathrm{SO}(3), \mathrm{SO}(4), \mathrm{SO}(5)$. Our method is to apply determinantal identities such as the Jacobi-Trudy identity to the character of the representations. This method is useful because the expressions appearing in the determinantal formula are binomial coefficients, and so the periodicity of their parities is well-known.

In this section we describe our alternative method which is especially useful for finding $P_{\mathrm{Spin}}(G)$. Here we use the determinantal identities for calculating the Weyl character formula which involve complete symmetric polynomials.

Definition 8.0.1. *The polynomial of the form* $\sum_{(w_1,w_2,\ldots w_k)|\sum w_i=n,} x_1^{w_1}x_2^{w_2}\cdots x_k^{w_k}$ *where the* w_i *are non-negative integers is called the complete symmetric polynomial of degree* n *in* k *variables . We denote it by* $H_n(x_1,\ldots,x_k)$.

We use the fact that the Weyl character formula is the Schur Polynomial for the series of groups $\mathrm{GL}(n,\mathbb{C})$. The Schur polynomial can be expressed as a determinant of a matrix whose entries are complete symmetric polynomials in the variable λ, i.e., the highest weight. This determinantal identity is called the Jacobi-Trudy identity. We have made extensive use of this identity to arrive at some polynomial expressions which establish the relation between $P_{\mathrm{Spin}}(\mathrm{PGL}(n))$ and $P_{\mathrm{orth}}(\mathrm{PGL}(n))$.

For $\mathrm{SO}(3), \mathrm{SO}(5)$ we use similar determinantal Weyl character formulas for doing the same job. For $\mathrm{SO}(4)$ the method is different.

8.1 Case $\mathrm{GL}(n, \mathbb{C})$

8.1.1 Notation

Recall that we choose the maximal torus of $\mathrm{GL}(n, \mathbb{C})$ to be the set of diagonal matrices in $\mathrm{GL}(n, \mathbb{C})$ i.e. $t = t_1 \oplus t_2 \oplus \cdots \oplus t_n$, where $t_i \in \mathbb{C}^\times$. We denote it by $T_{\mathrm{GL}(n,\mathbb{C})}$.

Let ν_i be the co-character of $T_{\mathrm{GL}(n,\mathbb{C})}$, defined as

$$\nu_i(t) = 1 \oplus \cdots 1 \oplus \underbrace{t}_{\text{i'th place}} \oplus 1 \oplus \cdots \oplus 1.$$

8.1.2 Preliminaries

From Section 6.1.1 the highest weight of an orthogonal irreducible representation of $\mathrm{GL}(2n, \mathbb{C})$ is of the form

$$\begin{aligned}\lambda_0 =&(r_n + r_{n-1} + \cdots + r_1, r_{n-1} + r_{n-2} + \cdots + r_1, \ldots, r_1, -r_1, -r_1 - r_2,\\ &\ldots, -r_1 - r_2 - \cdots - r_n),\end{aligned}$$

where $r_1, r_2, \ldots, r_n$ are non negative integer parameters. Put $d = \sum_{i=1}^n r_i$.

From the Weyl character formula we find that, the trace of the image of the diagonal matrix $t = t_1 \oplus t_2 \oplus \cdots \oplus t_{2n}$ in $\mathrm{GL}(2n, \mathbb{C})$ of the representation having highest weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n})$ is the Schur polynomial:

$$S_\lambda(t_1, t_2, \ldots, t_{2n}) = \frac{|t_j^{\lambda_i + 2n - i}|}{|t_j^{2n-i}|} \in \mathbb{C}[t_1, \ldots, t_{2n}], \tag{8.1}$$

where $|a(i,j)|$ denotes the determinant of the $2n \times 2n$ matrix having entry $a(i,j)$ at the $(i,j)^{th}$ place. See page 399 and page 77 Theorem 6.3(3) of [Fulton and Harris(1991)].

The j^{th} column of the matrix in the numerator of Equation (8.1) is

$$\begin{pmatrix} t_j^{\lambda_1+2n-1} \\ t_j^{\lambda_2+2n-2} \\ \cdots \\ t_j^{\lambda_{2n}} \end{pmatrix}.$$

By factoring t_j^{-d} common from the j^{th} column for each j, we obtain

$$S_\lambda(t_1, t_2, \ldots, t_{2n}) = (t_1 t_2 \cdots t_{2n})^{-d} \cdot S_{\lambda_{\text{new}}}(t_1, t_2, \ldots, t_{2n}),$$

where

$$\begin{aligned}\lambda_{\text{new}} = (&2(r_n + r_{n-1} + \cdots + r_1), r_n + 2(r_{n-1} + \cdots + r_1), \\ &r_n + r_{n-1} + 2(r_{n-2} + \cdots + r_1), \ldots, r_n + r_{n-1} + \cdots + r_2 + 2r_1, \\ &r_n + r_{n-1} + \cdots + r_2, r_n + r_{n-1} + \cdots + r_3, \ldots, 0).\end{aligned}$$

Henceforth we will write $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ for the components of λ_{new}. Note that

$$\lambda_{\text{new}} = r_n(\varpi_{2n-1} + \varpi_1) + r_{n-1}(\varpi_{2n-2} + \varpi_2) + \cdots + r_2(\varpi_{n+1} + \varpi_{n-1}) + 2r_1(\varpi_n),$$

where the ϖ_i's are the fundamental weights of $\mathfrak{sl}_{2n}$ i.e.,

$$\varpi_i = (\underbrace{1, 1, \ldots, 1}_{i \text{ times}}, \underbrace{0, 0, \ldots, 0}_{2n-i \text{ times}}),$$

for $1 \leq i \leq 2n-1$.

8.1.3 Case $\mathrm{GL}(4,\mathbb{C})$ and $\mathrm{GL}(2n,\mathbb{C})$

We will state the theorems first and give their proofs later.

Theorem 8.1.1. *The orthogonal irreducible representation ϕ of $\mathrm{GL}(2n,\mathbb{C})$ (or $\mathrm{PGL}(2n,\mathbb{C})$) of highest weight*

$$(r_n + r_{n-1} + \cdots + r_1, r_{n-1} + r_{n-2} + \cdots + r_1, \ldots, r_1, -r_1, -r_1 - r_2, \ldots, -r_1 - r_2 - \cdots - r_n),$$

is spinorial if and only if

$$\det\begin{pmatrix} d_1 & \binom{\lambda_1+2n-1}{2n-2} & \binom{\lambda_1+2n-1}{2n-3} & \cdots & 1 \\ d_2 & \binom{\lambda_2+2n-2}{2n-2} & \binom{\lambda_2+2n-2}{2n-3} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{2n} & \binom{\lambda_{2n}}{2n-2} & \binom{\lambda_{2n}}{2n-3} & \cdots & 1 \end{pmatrix} \equiv 0 \mod 2.$$

Here

$$d_i = (d+1)\binom{\lambda_i - i + 2n}{2n-1} + \binom{(\lambda_i - d) - i + 2n}{2n} - \binom{\lambda_i - i + 2n + 1}{2n},$$

where $d = \sum_{i=1}^{n} r_i$ and recall that λ_i for $1 \le i \le 2n$ are components of λ_{new}.

Theorem 8.1.2. *The spinoriality of the representation of $\mathrm{GL}(2n,\mathbb{C})$ corresponding to the above mentioned highest weight λ is periodic in each r_i of period 2^k, where $k = [\log_2 2n] + 1$. Hence,*

$$\langle 2^k(\varpi_1+\varpi_{2n-1}), 2^k(\varpi_2+\varpi_{2n-2}), \ldots, 2^k(\varpi_{n-1}+\varpi_{n+1}), 2^{k+1}\varpi_n\rangle \subseteq P_{\text{Spin}}(\mathrm{PGL}(2n,\mathbb{C})).$$

Theorem 8.1.3. *The irreducible representation of $\mathrm{GL}(4,\mathbb{C})$ with the highest weight $(r+s, r, -r, -r-s)$ (where r and s are non-negative integers) is* *spinorial if and only if*

$$c_1 \cdot b_1 + c_2 \cdot b_2 + c_3 \cdot b_3 \equiv 0 \mod 2.$$

Here

$$c_1 = \frac{(2r+s+2)(s+1)(2r+1)}{2},$$

$$c_2 = c_3 = \frac{(2r+s+2)(s+1)(2r+2s+3)}{2},$$

and

$$b_1 = (r+s+1)\binom{2r+2s+3}{3} + \binom{r+s+3}{4} - \binom{2r+2s+4}{4},$$

$$b_2 = (r+s+1)\binom{2r+s+2}{3} + \binom{r+2}{4} - \binom{2r+s+3}{4},$$

$$b_3 = (r+s+1)\binom{s+1}{3} + \binom{-r+1}{4} - \binom{s+2}{4}.$$

Theorem 8.1.4. *The spinoriality of the irreducible representation of* GL$(4,\mathbb{C})$ *of highest weight* $(r+s, r, -r, -r-s)$ *is periodic in* r *with period* 4 *and periodic in* s *with period* 8*, where* $r, s \in \mathbb{Z}_{\geq 0}$*. Thus*

$$P_{\mathrm{Spin}}(\mathrm{PGL}(4,\mathbb{C})) = \langle 8\varpi_2, 8(\varpi_1 + \varpi_3)\rangle = 8P_{\mathrm{orth}}(\mathfrak{sl}(4,\mathbb{C})).$$

The proportion of non-spinorial weights is $1/4$.

Proof of Theorem 8.1.1 : The co-characters ν_i represent the generators of $\pi_1(\mathrm{GL}(2n,\mathbb{C}))$. Let ϕ_λ be the representation with highest weight λ. Now by Scholium 4.2.13, in order to determine the spinoriality of the representation ϕ_λ it is enough to check the parity of $Q''_{\nu_i}(1)/2$ for each i. Moreover

$Q''_\nu(1)/2 \equiv \Psi_\phi(\nu) \equiv F_\nu(\lambda) \mod 2$, from Lemma 3.2.5 and Lemma 4.1.2. Since $F_{w(\nu)}(\lambda) = F_\nu(\lambda)$ by Proposition 3.2.4 and all of the co-characters ν_i are Weyl conjugate, it is enough to check the parity for only one of the co-characters ν_i, let us say the parity of $\Psi_\phi(\nu_{2n})$.

Let

$$B_\lambda(t) = S_\lambda(\underbrace{1, 1, \ldots, 1}_{2n-1 \text{ times}}, t) = \text{Trace}(\phi(\nu_{2n}(t))). \tag{8.2}$$

and

$$A_\lambda(t) = S_{\lambda_{\text{new}}}(\underbrace{1, 1, \ldots, 1}_{2n-1 \text{ times}}, t) = t^d S_\lambda(\underbrace{1, 1, \ldots, 1}_{2n-1 \text{ times}}, t) = t^d B_\lambda(t). \tag{8.3}$$

We will make use of the Jacobi-Trudy identity page 455 [Fulton and Harris(1991)], which says that in the $2n \times 2n$ case

$$S_\lambda(x_1, x_2, \ldots, x_{2n}) = \det \begin{pmatrix} H_{\lambda_1}(\mathbf{x_{2n}}) & H_{\lambda_1+1}(\mathbf{x_{2n}}) & \cdots & H_{\lambda_1+2n-1}(\mathbf{x_{2n}}) \\ H_{\lambda_2-1}(\mathbf{x_{2n}}) & H_{\lambda_2}(\mathbf{x_{2n}}) & \cdots & H_{\lambda_2+2n-2}(\mathbf{x_{2n}}) \\ \cdots & \cdots & \cdots & \cdots \\ H_{\lambda_{2n}-2n+1}(\mathbf{x_{2n}}) & H_{\lambda_{2n}-2n+2}(\mathbf{x_{2n}}) & \cdots & H_{\lambda_{2n}}(\mathbf{x_{2n}}) \end{pmatrix},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n})$ and $\mathbf{x_i} = (x_1, x_2, \ldots, x_i)$.

We will use a slightly modified identity, see page 131 of [Prasad(2015)]. The polynomial $S_\lambda(x_1, x_2, \ldots, x_{2n})$ equals

$$\det \begin{pmatrix} H_{\lambda_1}(\mathbf{x_{2n}}) & H_{\lambda_1+1}(\mathbf{x_{2n-1}}) & \cdots & H_{\lambda_1+2n-1}(\mathbf{x_1}) \\ H_{\lambda_2-1}(\mathbf{x_{2n}}) & H_{\lambda_2}(\mathbf{x_{2n-1}}) & \cdots & H_{\lambda_2+2n-2}(\mathbf{x_1}) \\ \cdots & \cdots & \cdots & \cdots \\ H_{\lambda_{2n}-2n+1}(\mathbf{x_{2n}}) & H_{\lambda_{2n}-2n+2}(\mathbf{x_{2n-1}}) & \cdots & H_{\lambda_{2n}}(\mathbf{x_1}) \end{pmatrix}. \tag{8.4}$$

Since $A_\lambda(t) = S_{\lambda_{\text{new}}}(1,1,\ldots,1,t)$, we set $(x_1,x_2,\ldots,x_{2n-1},x_{2n}) = (1,1,\ldots,1,t)$ in Equation (8.4).

$$A_\lambda(t) = \det\begin{pmatrix} H_{\lambda_1}(1,1,\ldots,1,t) & H_{\lambda_1+1}(1,1,\ldots,1) & \cdots & H_{\lambda_1+2n-1}(1) \\ H_{\lambda_2-1}(1,1,\ldots,1,t) & H_{\lambda_2}(1,1,\ldots,1) & \cdots & H_{\lambda_2+2n-2}(1) \\ \cdots & \cdots & \cdots & \cdots \\ H_{\lambda_{2n}-2n+1}(1,1,\ldots,1,t) & H_{\lambda_{2n}-2n+2}(1,1,\ldots,1) & \cdots & H_{\lambda_{2n}}(1) \end{pmatrix}$$

$H_n = 0$ for $n < 0$.

$H_0 = 1$.

The proof of the following is elementary combinatorics.

Lemma 8.1.5. *(1)* $H_n(\underbrace{1,1,\ldots,1}_{p-1\ \textit{times}},t) = \sum_{a=0}^{n}\binom{n-a+p-2}{p-2}t^a$.

(2) $H_n(\underbrace{1,1,\ldots,1,1}_{p\ \textit{times}}) = \binom{n+p-1}{p-1}$.

Thus, $A_\lambda(t)$ is equal to

$$\det\begin{pmatrix} H_{\lambda_1}(1,1,\ldots,1,t) & \binom{\lambda_1+1+2n-2}{2n-2} & \binom{\lambda_1+2+2n-3}{2n-3} & \cdots & 1 \\ H_{\lambda_2-1}(1,1,\ldots,1,t) & \binom{\lambda_2+2n-2}{2n-2} & \binom{\lambda_2+1+2n-3}{2n-3} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{\lambda_{2n}-2n+1}(1,1,\ldots,1,t) & \binom{\lambda_{2n}-2n+2+2n-2}{2n-2} & \binom{\lambda_{2n}-2n+3+2n-3}{2n-3} & \cdots & 1 \end{pmatrix}. \tag{8.5}$$

Our convention is

$$\binom{a}{b} = 0 \quad \text{for} \quad a < b \quad \text{and} \quad \binom{0}{0} = 1. \tag{8.6}$$

Lemma 8.1.6. *The degree of* $A_\lambda(t)$ *is* $2(\sum_{i=1}^{n} r_i) = 2d$.

Proof. Observe here that among all $\lambda_1, \lambda_2-1, \lambda_3-2, \ldots, \lambda_{2n}-2n+1$, λ_1 is the largest integer as λ_i is a non-increasing sequence. Hence $H_{\lambda_1}(1,1,\ldots,1,t)$

has the highest degree which is $\lambda_1 = 2(\sum r_i) = 2d$ among the first column entries, if we expand this determinant along the first column. Therefore if the coefficient of $H_{\lambda_1}(1, 1, \dots, 1, t)$ is nonzero then the degree of $A_\lambda(t)$ is $2d$. The coefficient of H_{λ_1} is of the form mentioned in Lemma 12.1.3 in the Appendix. By using this Lemma, the coefficient equals

$$\frac{1}{\prod_{i=1}^{2n-1} i!}\left(\prod_{j=1}^{2n-1} a_j\right) \prod_{1 \leq i < j \leq 2n-1} (a_i - a_j)$$

with $a_i = \lambda_{i+1} - i + 2n - 1$ for $1 \leq i \leq 2n-1$. Since the sequence λ_i is non-increasing, the sequence a_i is strictly decreasing, hence all a_i are distinct. Thus, the determinant which is the coefficient of H_{λ_1} is nonzero. Therefore, the degree of $A_\lambda(t)$ is $2d$. □

Let $P_{2d} = \mathbb{Z}[t]_{2d}$ be the abelian group of polynomials of degree $\leq 2d$ with integer coefficients. For $f(t) = a_0 + a_1 t + \cdots + a_n t^n$, we define $\tilde{\Psi}_d(f) = \sum_{i=1}^{d} i a_{d-i} \mod 2$. Observe that $\tilde{\Psi}_d : P_{2d} \to \mathbb{Z}/2\mathbb{Z}$ is $\mathbb{Z}$-linear. Recall that we have commutative digaram.

$$\begin{array}{ccc} \mathbb{Z}[t, t^{-1}]_d^{\text{sym}} & \xrightarrow{\times t^d} & \mathbb{Z}[t]_{2d}^{\text{pal}} \\ \downarrow \Psi & \swarrow \tilde{\Psi}_d & \\ \mathbb{Z}/2\mathbb{Z} & & \end{array} \tag{8.7}$$

see Definition 2.1.6 and 2.1.8.

Proposition 8.1.7. *The polynomial $A_\lambda(t)$ is palindromic with degree $2d$. Further ϕ_λ is spinorial if and only if*

$$\tilde{\Psi}_d(A_\lambda) \equiv 0 \mod 2.$$

Proof. From Equation (8.3) and Lemma 8.1.6 it is clear that degree of B_λ is d.

By Lemma 3.2.2 and Equation (8.2) it is clear that

$$B_\lambda(t) = b_d(t^d + t^{-d}) + b_{d-1}(t^{d-1} + t^{-(d-1)}) + \cdots + b_0,$$

where b_j is the multiplicity of weight t^j in the representation $\phi \circ \nu_{2n}$ of $\mathbb{C}^\times$.

Hence

$$A_\lambda(t) = t^d B_\lambda(t) = b_d + b_{d-1}t + \cdots + b_1 t^{d-1} + b_0 t^d + b_1 t^{d+1} + \cdots + b_d t^{2d},$$

is palindromic. Suppose $A_\lambda(t) = a_0 + a_1 t + \cdots + a_{2d}t^{2d}$, then

$$a_i = b_{d-i} \text{ for } 0 \le i \le d, \tag{8.8}$$

$$a_{d+i} = b_i \text{ for } 0 \le i \le d. \tag{8.9}$$

From Lemma 3.2.2, $\Psi_\phi(\nu_{2n}) = \Psi(B_\lambda) = \sum_{i=1}^d i \cdot b_i$. From Equation (8.8) we conclude that $\Psi_\phi(\nu_{2n}) = \Psi(B_\lambda) = \sum_{i=1}^d i \cdot a_{d-i} = \tilde{\Psi}_d(A_\lambda)$. The diagram (8.7) sheds more light on the last equation, because $B_\lambda \in \mathbb{Z}[t, t^{-1}]_d^{\mathrm{sym}}$ and $A_\lambda \in \mathbb{Z}[t]_{2d}^{\mathrm{pal}}$ and $A_\lambda(t) = t^d \cdot B_\lambda(t)$.

We saw earlier that ϕ is spinorial if and only if $\Psi_\phi(\nu_{2n})$ is even. In this case $\Psi_\phi(\nu_{2n}) = \tilde{\Psi}_d(A_\lambda(t))$. Hence ϕ is spinorial if and only if $\tilde{\Psi}_d(A_\lambda(t))$ is even. □

Lemma 8.1.8. *We have*

$$\tilde{\Psi}_d\left(\det\begin{pmatrix} f_1(t) & a_{12} & \cdots & a_{1n} \\ f_2(t) & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_n(t) & a_{2n} & \cdots & a_{nn} \end{pmatrix}\right) = \det\begin{pmatrix} \tilde{\Psi}_d(f_1) & a_{12} & \cdots & a_{1n} \\ \tilde{\Psi}_d(f_2) & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{\Psi}_d(f_n) & a_{2n} & \cdots & a_{nn} \end{pmatrix} \tag{8.10}$$

for polynomials $f_i \in P_{2d}$ and $a_{ij} \in \mathbb{Z}$.

Proof. Since $\tilde{\Psi}_d$ is $\mathbb{Z}$-linear the lemma follows. □

By Lemma 8.1.8 $\tilde{\Psi}_d(A(t))$ is equal to

$$\det\begin{pmatrix} \tilde{\Psi}_d(H_{\lambda_1}(1,1,\ldots,1,t)) & \binom{\lambda_1+2n-1}{2n-2} & \binom{\lambda_1++2n-1}{2n-3} & \cdots & 1 \\ \tilde{\Psi}_d(H_{\lambda_2-1}(1,1,\ldots,1,t)) & \binom{\lambda_2+2n-2}{2n-2} & \binom{\lambda_2+2n-2}{2n-3} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{\Psi}_d(H_{\lambda_{2n}-2n+1}(1,1,\ldots,1,t)) & \binom{\lambda_{2n}}{2n-2} & \binom{\lambda_{2n}}{2n-3} & \cdots & 1 \end{pmatrix}.$$

Lemma 8.1.9. *We have*

$$\tilde{\Psi}_d(H_q(\underbrace{1,1,\ldots,1}_{p-1\ times},t)) = (d+1)\binom{q+p-1}{p-1}+\binom{q-d+p-1}{p}-\binom{q+p}{p} \in \mathbb{Z}/2\mathbb{Z},$$

for $d,q\in\mathbb{Z}_{\geq 0}$ and for $p\in\mathbb{N}$.

Proof. This is clear from 8.1.5 (1), 12.1.4 in the Appendix, and the definition of $\tilde{\Psi}_d$. □

Thus,

$$\tilde{\Psi}_d(H_{\lambda_i-i+1}(1,1,\ldots,1,t)) = (d+1)\binom{\lambda_i-i+2n}{2n-1}+\binom{(\lambda_i-d)-i+2n}{2n}-\binom{\lambda_i-i+2n+1}{2n}.$$

Hence we conclude the proof of 8.1.1

□

Proof of Theorem 8.1.2. It is enough to prove that $c_i \mod 2$, where c_i are cofactors corresponding to H_{λ_i-i+1} in the matrix in Equation (8.5) and $d_i \mod 2$, where d_i in the Theorem 8.1.1, are periodic in r_j for $1\leq j\leq n$ with period 2^k where k is the least integer such that $2n<2^k$, i.e., $k=[\log_2 n]+2$.

Periodicity of $c_i \mod 2$.

Recall that c_i is the determinant of the $(n-1)\times(n-1)$ matrix whose entries are of the form $\binom{\lambda_i-i+2n}{2n-j}$ with suitable $1\leq i,j\leq 2n$. Consider them as functions of the r_l. Observe that each λ_i is a linear function in r_l with a non-negative integer coefficient. Thus by Lemma 12.1.2 in the Appendix, we can say that parity of each of the matrix entries is periodic in each r_l with period 2^k. Hence each c_i is periodic with period 2^k.

Periodicity of $d_i \mod 2$.

$d_i = (d+1)\binom{\lambda_i - i + 2n}{2n-1} + \binom{(\lambda_i - d) - i + 2n}{2n} - \binom{\lambda_i - i + 2n + 1}{2n} \mod 2$

The first term involves $(d+1) = \sum r_i + 1$ whose parity is periodic in each r_i of period 2^k.

For the binomial coefficient, the upper index is $\lambda_i + (2n - i)$. Since $2n - i \geq 0$, and since the r_i take non-negative values, the upper index in the first term is a natural number. Hence we can apply Lemma 12.1.2 in the Appendix to the first term and see that it is periodic in each r_i of period 2^k.

Let us now consider the second term. Observe here that if $i > n$ then $\lambda_i - d$ is negative and is a degree one polynomial in each r_i with coefficient -1 or 0. The upper index in the second term here is $(\lambda_i - d) - i + 2n$ which is always strictly less than $2n$. Hence the second term $\binom{(\lambda_i - d) - i + 2n}{2n}$ is 0 when $i > n$. It is obviously periodic in each r_i with period 2^k.

If instead $i \leq n$ then $\lambda_i - d$ will be a degree one polynomial in each r_i with coefficient either 0 or 1. Since $2n - i \geq 0$, and since the r_i take non-negative values, the upper index in the second term is a nonnegative. Hence we can apply the Lucas theorem to the second term and say that it is periodic in each r_i of period 2^k.

For the third term, we can argue exactly the same way as in the first term. Here we draw the same conclusion as in the case of the first term. $\square$

Case GL$(4,\mathbb{C})$

We take $r_1 = r$ and $r_2 = s$. Thus here $d = r+s$. Here $A_\lambda(t) = S_{\lambda_{\text{new}}}(1,1,1,t)$, where $\lambda = (r+s, r, -r, -r-s)$, and hence $\lambda_{\text{new}} = (2r+2s, 2r+s, s, 0)$. Here we prove Theorem 8.1.3 and 8.1.4.

Proof of Theorem 8.1.3. We have

$$\begin{aligned}\lambda_{\text{new}} &= (2r+2s, 2r+s, s, 0)\\ &= s(\varpi_1+\varpi_3) + 2r\varpi_2,\end{aligned}$$

where ϖ_i are fundamental weights of $\mathfrak{sl}(4,\mathbb{C})$ and $d = r+s$.

By Theorem 8.1.1, putting in the values our determinant becomes

$$\det \begin{bmatrix} d_1 & \binom{2r+2s+3}{2} & 2r+2s+3 & 1\\ d_2 & \binom{2r+s+2}{2} & 2r+s+2 & 1\\ d_3 & \binom{s+1}{2} & s+1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

So the determinant equals

$$\Psi_{r+s}(A_\lambda) = c_1d_1 - c_2d_2 + c_3d_3,$$

where

$$\begin{aligned}c_1 &= (s+1)\binom{2r+s+2}{2} - (2r+s+2)\binom{s+1}{2}\\ &= \frac{(s+1)(2r+s+2)(2r+s+1)}{2} - \frac{(2r+s+2)(s+1)s}{2}\\ &= \frac{(2r+s+2)(s+1)(2r+1)}{2},\end{aligned}$$

similarly

$$c_2 = c_3 = \frac{(2r+s+2)(s+1)(2r+2s+3)}{2}.$$

By Theorem 8.1.1

$$d_1 = (r+s+1)\binom{2r+2s+3}{3} + \binom{r+s+3}{4} - \binom{2r+2s+4}{4},$$

$$d_2 = (r+s+1)\binom{2r+s+2}{3} + \binom{r+2}{4} - \binom{2r+s+3}{4},$$

and

$$d_3 = (r+s+1)\binom{s+1}{3} + \binom{-r+1}{4} - \binom{s+2}{4} \mod 2.$$

So the final expression becomes

$$\begin{aligned} h(r,s) = \\ c_1 &\left((r+s+1)\binom{2r+2s+3}{3} + \binom{r+s+3}{4} - \binom{2r+2s+4}{4}\right) \\ -c_2 &\left((r+s+1)\binom{2r+s+2}{3} + \binom{r+2}{4} - \binom{2r+s+3}{4}\right) \\ +c_3 &\left((r+s+1)\binom{s+1}{3} + \binom{-r+1}{4} - \binom{s+2}{4}\right) \mod 2. \end{aligned}$$

□

Proof of Theorem *8.1.4.* Let

$$\begin{aligned} h(r,s) = c_1 &\left((r+s+1)\binom{2r+2s+3}{3} + \binom{r+s+3}{4} - \binom{2r+2s+4}{4}\right) \\ -c_2 &\left((r+s+1)\binom{2r+s+2}{3} + \binom{r+2}{4} - \binom{2r+s+3}{4}\right) \end{aligned}$$

$$+c_3\left((r+s+1)\binom{s+1}{3}+\binom{-r+1}{4}-\binom{s+2}{4}\right).$$

where

$$c_1=\frac{(2r+s+2)(s+1)(2r+1)}{2},$$

and

$$c_2=c_3=\frac{(2r+s+2)(s+1)(2r+2s+3)}{2}.$$

Proposition 8.1.10. *The spinoriality of an orthogonal irreducible representation of* $\mathrm{GL}(4,\mathbb{C})$ *of highest weight* $(r+s,r,-r,-r-s)$ *is periodic in* r *with period* 4 *and periodic in* s *with period* 8*, i.e.,*

1) $h(r+4,s)=h(r,s) \mod 2$*,*

2) $h(r,s+8)=h(r,s) \mod 2$*.*

Proof. By Lemma 12.1.2 in the Appendix and the lemma above, we deduce that the expression $h(r,s)$ is periodic in both s and r of period 8 since each single term is such.

Some extra work is required in order to prove that it is periodic in r with exact period 4.

To do that observe that the only problematic terms are $\binom{r+s+3}{4}$, $\binom{r+2}{4}$ and $\binom{-r+1}{4}$. Other terms have $2r$ in the upper index so if we increase r by 4 then $2r$ increases by 8 which gives the same number mod 2 by Lemma 12.1.2 in the Appendix. Observe that $\binom{-r+1}{4}$ is 0 as r is a non-negative number. So the term we should concentrate on is $c_1\binom{r+s+3}{4}+-c_2\binom{r+2}{4} \mod 2$. Since it is easy to observe that $c_1=c_2=c_3 \mod 2$, We can concentrate on $\binom{r+s+3}{4}-\binom{r+2}{4} \mod 2$.

By Lemma 12.1.5 in the Appendix,

$$\binom{(r+4)+s+3}{4}-\binom{(r+4)+2}{4} \mod 2 = 1+\binom{r+s+3}{4}-1-\binom{r+2}{4} \mod 2,$$

which equals

$$\binom{r+s+3}{4} - \binom{r+2}{4} \mod 2.$$

Hence the expression $h(r,s)$ is periodic in r with period 4 and periodic in s with period 8. Hence the spinoriality of the representation of $\mathrm{GL}(4,\mathbb{C})$ (or $\mathrm{PGL}(4,\mathbb{C})$) with highest weight $(r+s, r, -r, -r-s)$ is periodic in r with period 4 and periodic in s with period 8, due to Theorem 8.1.3. □

Thus now we have proved $P_{\mathrm{Spin}}(\mathrm{PGL}(4,\mathbb{C})) \supseteq \langle 8\varpi_2, 8(\varpi_1+\varpi_3)\rangle$. Now we will prove that they are equal.

Lemma 8.1.11. *We have* $P_{\mathrm{Spin}}(\mathrm{PGL}(4,\mathbb{C})) = \langle 8\varpi_2, 8(\varpi_1+\varpi_3)\rangle$.

Proof. Recall that $\lambda_{\mathrm{new}} = (2r+2s, 2r+s, s, 0) = 2r\varpi_2 + s(\varpi_1+\varpi_3)$. Thus we get

$$P_{\mathrm{sd}}(\mathrm{PGL}(4,\mathbb{C})) \supseteq P_{\mathrm{Spin}}(\mathrm{PGL}(4,\mathbb{C})) \supseteq \langle 8\varpi_2, 8(\varpi_1+\varpi_3)\rangle.$$

The group $P_{\mathrm{sd}}(\mathrm{PGL}(4,\mathbb{C})) = \langle (\varpi_1+\varpi_3), 2\varpi_2\rangle$. Hence, $P_{\mathrm{Spin}}(\mathrm{PGL}(4,\mathbb{C}))/\langle 8\varpi_2, 8(\varpi_1+\varpi_3)\rangle$ is a subgroup of $P_{\mathrm{sd}}(\mathrm{PGL}(4,\mathbb{C}))/\langle 8\varpi_2, 8(\varpi_1+\varpi_3)\rangle$, with the second group isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

Now we concentrate on the group $A = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. In this group, $(4,0) = 4x$ or $2x$ or x where x can be one of the elements in S_1, where

$$\begin{aligned} S_1 = \{&(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(1,1),(3,1),(5,1),\\ &(7,1),(1,2),(3,2),(5,2),(7,2),(2,2),(6,2),(1,3),(3,3),(5,3),(7,3)\}. \end{aligned}$$

Furthermore $(0,2) = x$ or $2x$ for every $x \in S_2$, where

$$S_2 = \{(0,1),(4,1),(0,2),(0,3),(4,3)\}.$$

Moreover $(4,2) = x$ or $2x$ for every $x \in S_3$, where

$$S_3 = \{(2,1), (6,1), (4,2), (2,3), (6,3)\}.$$

Now suppose H is a subgroup of A. Then it is easy to see that if $(4,0) \notin H$ then none of the members of S_1 belong to H. A similar argument holds for $(0,2)$ and S_2 and $(4,2)$ and S_3. Thus, if none of the members $(4,0)$, $(0,2)$ and $(4,2)$ belong to H then $H \cap (S_1 \cup S_2 \cup S_3) = \emptyset$. But observe that $A - (S_1 \cup S_2 \cup S_3) = \{(0,0)\}$. Hence H will be the trivial subgroup.

Thus r corresponds to $\mathbb{Z}/4\mathbb{Z}$ and s corresponds to $\mathbb{Z}/8\mathbb{Z}$ in A which is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. So in order to prove

$$P_{\text{Spin}}(\text{PGL}(4,\mathbb{C}))/\langle 8\varpi_2, 8(\varpi_1 + \varpi_3)\rangle$$

is trivial, we only need to show that none of $(4,0), (0,2), (4,2)$ belong to this group.

That is the same as proving $(r,s) = (0,4), (2,0)$ and $(2,4)$ do not belong to $P_{\text{Spin}}(\text{PGL}(4,\mathbb{C}))$. Let $h(r,s)$ be the expression which gives the spinoriality of the representation of $\text{PGL}(4,\mathbb{C})$ of highest weight $(2r+2s, 2r+s, s, 0)$. Observe that $h(1,1)$ is odd, while $h(1,5)$ is even, hence $(0,4)$ does not belong to $P_{\text{Spin}}(\text{PGL}(4,\mathbb{C}))$. Also $h(1,1)$ is odd, while $h(3,1)$ is even, therefore $(2,0)$ does not belong to $P_{\text{Spin}}(\text{PGL}(4,\mathbb{C}))$. Moreover $h(1,2)$ is odd, while $h(3,6)$ is even, thus $(2,4)$ does not belong to $P_{\text{Spin}}(\text{PGL}(4,\mathbb{C}))$.

Hence we conclude the proof of the lemma.

□

Therefore we conclude the proof of Theorem 8.1.4. □

8.1.4 The Adjoint Representation of $\mathrm{GL}(2n, \mathbb{C})$

Introduction

In this subsection we prove that the adjoint representation of $\mathrm{GL}(2n, \mathbb{C})$ is not spinorial for every n.

Discussion

By adjoint representation we mean the action of $\mathrm{GL}(2n, \mathbb{C})$ on the Lie algebra $\mathfrak{sl}(2n, \mathbb{C})$ by conjugation. This representation is irreducible as $\mathrm{GL}(2n, \mathbb{C}) = Z(GL(2n, \mathbb{C})) \cdot \mathrm{SL}(2n, \mathbb{C})$ and scalars act trivially on $\mathfrak{sl}(2n, \mathbb{C})$ and the adjoint representation of $\mathrm{SL}(2n, \mathbb{C})$ on $\mathfrak{sl}(2n, \mathbb{C})$ is irreducible. Since the Killing form is invariant under this action, the representation is orthogonal. From Lemma 7.1.1 it follows that all self-dual representations, hence this representation factors through $\mathrm{PGL}(2n, \mathbb{C})$.

A Short Proof

We would like to put a short proof of the fact that the adjoint representation of $\mathrm{GL}(n, \mathbb{C})$ is spinorial if and only if n is odd.

Theorem 8.1.12. *The adjoint representation of* $\mathrm{PGL}(n, \mathbb{C})$ *is spinorial if and only if* n *is odd.*

Proof. Let us denote the diagonal torus of $\mathrm{SL}(n, \mathbb{C})$ by $\tilde{T}_n$. Let us denote the maximal torus of $\mathrm{PGL}(n, \mathbb{C})$ by T_n which is isomorphic to $\tilde{T}_n/\mu_n$, where by μ_n we denote the group of n^{th} roots of unity.

By Theorem 4.4.1 for spinoriality of adjoint representation of a semi-simple group, we have to just check whether ρ, which is half of sum of positive roots belongs to $\mathrm{Hom}(T_n, \mathbb{C}^\times)$. If it belongs, then the adjoint representation is spinorial otherwise not. The roots of $\mathrm{SL}(n,\mathbb{C})$ belong to $\mathrm{Hom}(\tilde{T}_n,\mathbb{C}^\times)$, while the roots of $\mathrm{PGL}(n,\mathbb{C})$ belong to $\mathrm{Hom}(T_n,\mathbb{C}^\times)$.

We have an exact sequence of abelian algebraic groups

$$1 \to \mu_n \to \tilde{T}_n \to T_n \to 1.$$

Applying the contra-variant functor $\mathrm{Hom}(-,\mathbb{C}^\times)$, we get an exact sequence

$$1 \to \mathrm{Hom}(T_n,\mathbb{C}^\times) \to \mathrm{Hom}(\tilde{T}_n,\mathbb{C}^\times) \to \mathrm{Hom}(\mu_n,\mathbb{C}^\times) \to 1. \tag{8.11}$$

Exactness of this sequence is easy to see. It is obviously exact at $\mathrm{Hom}(T_n,\mathbb{C}^\times)$. To see the exactness at $\mathrm{Hom}(\tilde{T}_n,\mathbb{C}^\times)$, observe that $f \in \mathrm{Hom}(T_n,\mathbb{C}^\times)$ if and only if $f \in \tilde{T}_n$ and f vanishes on μ_n by definition of T_n. Let us denote the projection on the first coordinate of $\tilde{T}_n$ by e_1. The map $e_1 : \tilde{T}_n \to \mathbb{C}^\times$ induces the identity on μ_n which is a generator of $\mathrm{Hom}(\mu_n,\mathbb{C}^\times)$ and hence the second map is surjective. Hence the sequence (8.11) is exact at $\mathrm{Hom}(\mu_n,\mathbb{C}^\times)$.

The Lie algebra for $\mathrm{SL}(n,\mathbb{C})$ is the same as the Lie algebra for $\mathrm{PGL}(n,\mathbb{C})$. The roots of the $\mathrm{PGL}(n,\mathbb{C})$ are obtained by applying the map $m : \mathrm{Hom}(T_n,\mathbb{C}^\times) \to \mathrm{Hom}(\tilde{T}_n,\mathbb{C}^\times)$ to the roots of $\mathrm{SL}(n,\mathbb{C})$. The map m is obtained by applying the functor $\mathrm{Hom}(-,\mathbb{C}^\times)$ to the sequence $\tilde{T}_n \to T_n$.

Let us denote the root $x_1 \oplus \cdots \oplus x_n \to x_i x_j^{-1}$ by $\tilde{\alpha}_{ij}$. Since this root is trivial on μ_n, it descends to a root of $\mathrm{PGL}(n,\mathbb{C})$. We denote it by α_{ij}.

By page 218 Proposition 6.2 vi of [Bröcker and tom Dieck(2013)] the sum (here product) of positive roots is

$$\begin{aligned} \tilde{\rho}^2 &= \prod_{i=1}^{n} x_i^{n-2i+1} \\ &= \prod_{i=1}^{n-1} x_i^{2n-2i} \left(\text{Since } \prod_{i=1}^{n} x_i = 1 \right). \end{aligned}$$

Hence $\tilde{\rho} = \prod_{i=1}^{n-1} x_i^{n-i}$, and so $\tilde{\rho} \in \mathrm{Hom}(\tilde{T}_n, \mathbb{C}^\times)$.

Since the sequence 8.11 is exact, we only need to check whether the image of $\tilde{\rho}$ in $\mathrm{Hom}(\mu_n, \mathbb{C}^\times)$ is trivial.

It is easy to see that

$$\begin{aligned} \text{The image of } \tilde{\rho} \text{ is trivial} &\Leftrightarrow \zeta_n^{(\sum_{i=1}^{n-1}(n-i))} = 1 \\ &\Leftrightarrow n \text{ divides } (\sum_{i=1}^{n-1}(n-i)) \\ &\Leftrightarrow n \text{ divides } (n(n-1)/2) \\ &\Leftrightarrow n \text{ is odd.} \end{aligned}$$

□

Jacobi-Trudy proof

Here is another proof using the Jacobi-Trudy identity.

We may view the above given representation of PGL($n, \mathbb{C}$) as a representation of SL($2n, \mathbb{C}$). Thus the weights for this representation are simply the roots of SL($n, \mathbb{C}$) namely $\epsilon_i - \epsilon_j$. The simple roots are $\epsilon_i - \epsilon_{i+1}$. Hence the highest weight is $\epsilon_1 - \epsilon_{2n}$.
But since $\sum_{i=1}^{2n} \epsilon_i = 0$ we get

$$\epsilon_1 - \epsilon_{2n} = 2\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2n-1}.$$

So the corresponding highest weight denoted by $\lambda_{\text{new}} = \varpi = (2, \underbrace{1, 1, \ldots, 1}_{2n-2 \text{ times}}, 0)$.

Here $\sum_{i=1}^{1} i \cdot a_{1-i} = a_0 \mod 2$, where the Schur polynomial with highest weight ϖ is $S_\varpi(\underbrace{1, 1, \ldots, 1}_{2n-1 \text{ times}}, t) = a_0 + a_1 t + a_2 t^2$.

Theorem 8.1.13.

$$S_\varpi(\underbrace{1, 1, 1, \ldots, 1}_{2n-1 \ times}, t) = (2n-1) + (2n-1)^2 t + (2n-1)t^2.$$

Proof. Using the following data

1)the Jacobi-Trudy identity ([Fulton and Harris(1991)] page 455)

2)a slightly modified version of the Jacobi-Trudy identity ([Prasad(2015)] page 131)

3) the identity $H_n(\underbrace{1, 1, \ldots, 1}_{p \text{ times}}) = \binom{n+p-1}{p-1}$

we see that $S_\varpi(1, 1, \ldots, 1, t)$ is the determinant of the $2n \times 2n$ matrix:

$$\begin{pmatrix} H_2(1,1,\ldots,1,t) & \binom{2n+1}{2n-2} & \binom{2n+1}{2n-3} & \cdots & \binom{2n+1}{0} \\ 1 & \binom{2n-1}{2n-2} & \binom{2n-1}{2n-3} & \cdots & \binom{2n-1}{0} \\ 0 & \binom{2n-2}{2n-2} & \binom{2n-2}{2n-3} & \cdots & \binom{2n-2}{0} \\ 0 & \binom{2n-3}{2n-2} & \binom{2n-3}{2n-3} & \cdots & \binom{2n-3}{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \binom{2}{2n-2} & \binom{2}{2n-3} & \cdots & \binom{2}{0} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{8.12}$$

Let

$$A = \det \begin{pmatrix} \binom{2n-1}{2n-2} & \binom{2n-1}{2n-3} & \cdots & \binom{2n-1}{1} \\ \binom{2n-2}{2n-2} & \binom{2n-2}{2n-3} & \cdots & \binom{2n-2}{1} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{2}{2n-2} & \binom{2}{2n-2} & \cdots & \binom{2}{1} \end{pmatrix},$$

and let

$$B = \det \begin{pmatrix} \binom{2n+1}{2n-2} & \binom{2n+1}{2n-3} & \cdots & \binom{2n+1}{1} \\ \binom{2n-2}{2n-2} & \binom{2n-2}{2n-3} & \cdots & \binom{2n-2}{1} \\ \binom{2n-3}{2n-2} & \binom{2n-3}{2n-3} & \cdots & \binom{2n-3}{1} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{2}{2n-2} & \binom{2}{2n-3} & \cdots & \binom{2}{1} \end{pmatrix}.$$

Then $S_{\varpi}(1,1,\ldots,1,t) = H_2(1,1,\ldots,1,t)A - B$.

We use Lemma 12.1.3 to calculate the A and B by putting appropriate values in a_i.

$$A = \frac{[(2n-1)(2n-2)\cdots 2][(2n-1-2)(2n-1-3)\cdots 1](2n-4)!(2n-5)!\cdots 1!}{(2n-2)!(2n-3)!(2n-4)!\cdots 1!}$$

$$= \frac{(2n-1)!(2n-3)!(2n-4)!\cdots 1!}{(2n-2)!(2n-3)!(2n-4)!\cdots 1!}$$

$$= 2n-1,$$

and $B = C/D$, where

$$C = [(2n+1)(2n-2)(2n-3)\cdots 1][(2n+1-2)(2n+1-3)\cdots 3]$$
$$[(2n-2-2)(2n-2-3)\cdots 1](2n-5)!\cdots 1!$$
$$D = (2n-2)!(2n-3)!\cdots 1!,$$

and

$$B = C/D = \frac{[(2n+1)(2n-2)!][(2n-1)(2n-2)\cdots 3](2n-4)!(2n-5)!\cdots 2!}{(2n-2)!(2n-3)!(2n-4)!\cdots 2!}$$

$$= \frac{(2n+1)(2n-1)(2n-2)}{2}.$$

Using the formula $H_n(\underbrace{1,1,\cdots,1}_{p-1 \text{ times}},t) = \sum_{a=0}^{n} \binom{n-a+p-2}{p-2} t^a$

We get

$$H_2(1,1\ldots,1,t) = \binom{2n}{2n-2} + \binom{2n-1}{2n-2}t + \binom{2n-2}{2n-2}t^2,$$

where H_2 takes $2n$ variables.

Thus

$$S_\varpi(1,1,\ldots,1,t)$$

$$= A(H_2(1,1,\ldots 1,t)) - B$$

$$= (2n-1)\left(\frac{2n(2n-2)}{2} + (2n-1)t + t^2\right) - \frac{(2n+1)(2n-1)(2n-2)}{2}$$

$$= (2n-1)\left(\frac{4n^2-2n-(4n^2-2n-2)}{2}\right) + (2n-1)^2 t + (2n-1)t^2$$

$$= (2n-1) + (2n-1)^2 t + (2n-1)t^2.$$

□

Now we want $a_0 \mod 2$ where $S_\varpi = a_0 + a_1 t + a_2 t^2$.

Clearly $a_0 = 2n - 1 \equiv 1 \mod 2$. Hence the adjoint representation of $\mathrm{GL}(2n, \mathbb{C})$ is not spinorial for any n.

8.2 Orthogonal Groups of Low Rank

8.2.1 Case SO(3)

From Section 6.1.2 we know that all the irreducible representations of $\mathrm{SO}(3,\mathbb{C})$ are orthogonal.

The Main Proof

Theorem 8.2.1. *An orthogonal irreducible representation of* $\mathrm{SO}(3,\mathbb{C})$ *of highest weight* n *is spinorial if and only if* $n \equiv 3$ *or* $0 \mod 4$. *Hence the proportion of non-spinorial weights is* $1/2$.

Proof. The highest weight of a finite dimensional irreducible representation of the group $\mathrm{SO}(3,\mathbb{C})$ is given by a single non-negative integer n. Let ϕ_n be the irreducible representation with highest weight n. Let Θ_n be its character. We select the maximal torus in $\mathrm{SO}(3,\mathbb{C})$ to be $x \oplus x^{-1} \oplus 1$, where $x \in \mathbb{C}^\times$. We take the cocharacter $\nu : x \to x \oplus x^{-1} \oplus 1$.

By page 409 Proposition 24.33 in [Fulton and Harris(1991)], we get $\Theta_n(\nu(x)) = K_n(x, x^{-1}, 1)$, where

$$K_n(x, x^{-1}, 1) = H_n(x, x^{-1}, 1) - H_{n-2}(x, x^{-1}, 1).$$

Observe that

$$\begin{aligned}
K_n(x, x^{-1}, 1) &= \sum_{e_1+e_2 \leq n} x^{e_1} x^{-e_2} - \sum_{e_1+e_2 \leq n-2} x^{e_1} x^{-e_2} \\
&= \sum_{e_1+e_2=n} x^{e_1-e_2} + \sum_{e_1+e_2=n-1} x^{e_1-e_2} \\
&= \sum_{(e_1,e_2)=(0,n),(1,n-1),\ldots,(n,0)} x^{e_1-e_2} + \sum_{(e_1,e_2)=(0,n-1),(1,n-2),\ldots,(n-1,0)} x^{e_1-e_2} \\
&= (x^{-n} + x^{-n+2} + \cdots + x^n) + (x^{-n+1} + x^{-n+3} + \cdots + x^{n-1}) \\
&= x^{-n} + x^{-n+1} + \cdots + x^{n-1} + x^n.
\end{aligned}$$

Now we apply Lemma 3.2.2 to ν. So the ϕ_n is spinorial if and only if

$$\Psi_{\phi_n}(\nu) = \sum_{i=1}^{n} i \cdot 1 \equiv 0 \mod 2.$$

We deduce that ϕ_n is spinorial if and only if

$$\frac{(n+1)n}{2} \equiv 0 \mod 2,$$

Equivalently

$$n \equiv 0 \text{ or } 3 \mod 4.$$

□

8.2.2 Case SO(4)

Notation

We write $\tilde{\phi}_k$ for the representation of $\mathrm{SL}(2, \mathbb{C})$ which is on the space $\mathrm{Sym}^k V$, where we have the standard representation on the space $V \cong \mathbb{C}^2$. We write $\tilde{\phi}(m, n)$ for the representation $\tilde{\phi}_m \otimes \tilde{\phi}_n$ of the group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. It is a standard fact that $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ is a double cover of $\mathrm{SO}(4,\mathbb{C})$. Let B denote the non-degenerate bilinear form on $M(2,\mathbb{C})$ defined as $B(X,Y) = \mathrm{Trace}(X\cdot w\cdot Y^t\cdot w^{-1})$, where $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The map m from $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SO}(4,\mathbb{C})$, given by action on $M(2,\mathbb{C})$,

$$m(g,h) : X \to g\cdot X\cdot h^{-1}$$

preserves B. It is easy to see that the representations $\tilde{\phi}(m,n)$ which factor through $\mathrm{SO}(4,\mathbb{C})$ have to satisfy the condition that $m+n$ is even. We denote the factored representation by $\phi(m,n)$.

We take the maximal torus $\tilde{T}$ in $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ to be

$$\tilde{T} = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \times \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix},$$

where $a,b\in\mathbb{C}^\times$. We take the maximal torus T in $\mathrm{SO}(4,\mathbb{C})$ to be

$$T = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_2^{-1} & 0 \\ 0 & 0 & 0 & x_1^{-1} \end{bmatrix},$$

where $x_i\in\mathbb{C}^\times$ with basis of $M(2,\mathbb{C})$ as

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The Main Theorems

From Section 6.1.3 and Proposition 23.13 (iii) of [Fulton and Harris(1991)] we infer that all the irreducible representations of $\mathrm{SO}(4,\mathbb{C})$ are orthogonal.

Theorem 8.2.2. *The representation $\phi(m,n)$ of $\mathrm{SO}(4)$ i.e., of highest weight (x,y) such that $x = \frac{m+n}{2}$, and $y = \frac{n-m}{2}$, where $m+n$ is even, as referred above is spinorial if and only if*

$$(1/24)(n+1)(3m^2+6m+n^2+2n) \equiv 0 \mod 2,$$

i.e.,

$$(1/6)(1+x+y)(2x+x^2-y-xy+y^2) \equiv 0 \mod 2$$

Theorem 8.2.3. *$P_{\mathrm{Spin}}(\mathrm{SO}(4)) = \langle (4,4),(4,-4)\rangle$ in variables (m,n), and $\langle (4,0),(0,-4)\rangle$ in variables (x,y). Hence $P_{\mathrm{Spin}}(\mathrm{SO}(4)) = 4 \cdot P_{\mathrm{orth}}(\mathfrak{so}(4))$. The proportion of non-spinorial representations is $3/8$.*

proof of Theorem 8.2.2. The action of $\tilde{T}$ on $M(2,\mathbb{C})$ w.r.t. X_1, X_2, X_3, X_4 is

$$\begin{bmatrix} ab^{-1} & 0 & 0 & 0 \\ 0 & ab & 0 & 0 \\ 0 & 0 & a^{-1}b^{-1} & 0 \\ 0 & 0 & 0 & ba^{-1} \end{bmatrix}.$$

Thus $m(\tilde{T})$ lies in T.

Let ν_1 the co-character of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ defined as

$$\nu_1(a) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \times \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix}.$$

Put $L_1 = m \circ \nu_1$, i.e.,

$$L_1(a) = \begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-2} \end{bmatrix}.$$

Observe that L_1 is a co-character of T.

Similarly for the co-character

$$\nu_2(a) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \times \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix},$$

put $L_2 = m \circ \nu_2$, i.e.,

$$L_2(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Observe that L_2 is a co-character of T.

Let us also define the co-characters of T

$$L'_1(a) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1} \end{bmatrix},$$

and

$$L'_2(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let ϕ be an irreducible orthogonal representation of $\mathrm{SO}(4,\mathbb{C})$. The representation $\tilde{\phi} = \phi \circ m$ is an irreducible representation of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Observe that L'_i for $i = 1, 2$ are representatives of generators of $\pi_1(\mathrm{SO}(4,\mathbb{C}))$. By Scholium 4.2.13 it is enough to check $\Psi_\phi(L'_i)$ is even for $i = 1$ and 2. But L'_1 and L'_2 are Weyl conjugate. Also $\Psi_\phi(\nu) \equiv F_\nu(\lambda) \mod 2$ by Lemma 3.2.5. Moreover by Proposition 3.2.4, $F_{w(\nu)}(\lambda) = F_\nu(\lambda)$, it is enough to check the parity of one of them say $\Psi_\phi(L'_2)$. Observe that $L_i(a) = L'_i(a^2)$ for $i = 1, 2$. Thus we have $\phi \circ L_i(a) = \phi \circ L'_i(a^2)$. Therefore if we put $z = a^2$, then $\phi \circ L_i(a) = \phi \circ L'_i(z)$. Observe that $\phi(L_2) = \phi(m(\nu_2)) = \tilde{\phi}(\nu_2)$. So we are interested in $\tilde{\phi}(\nu_2)$. In $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, $\nu_2(a) = (a \oplus a^{-1}) \times (a \oplus a^{-1})$. Let the underlying space for the standard representation of $\mathrm{SL}(2,\mathbb{C})$ be $V = \langle u, v \rangle$. Then a basis for $\mathrm{Sym}^k(V)$ is $(u^k, u^{k-1}v, u^{k-2}v^2, \ldots, uv^{k-1}, v^k)$. The basis for $\mathrm{Sym}^m V \otimes \mathrm{Sym}^n$ is $\{u^i v^j \otimes u^r v^s | 1 \le i, j \le m, 1 \le r, s \le n\}$. We write the basis of $\mathrm{Sym}^m V \otimes \mathrm{Sym}^n$ in a array or matrix form

$$\begin{bmatrix} u^m \otimes u^n & u^{m-1}v \otimes u^n & u^{m-2}v^2 \otimes u^n & \cdots & v^m \otimes u^n \\ u^m \otimes u^{n-1}v & u^{m-1}v \otimes u^{n-1}v & \cdots & \cdots & v^m \otimes u^{n-1}v \\ u^m \otimes u^{n-2}v^2 & \cdots & \cdots & \cdots & v^m \otimes u^{n-2}v^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u^m \otimes v^n & u^{m-1}v \otimes v^n & \cdots & \cdots & v^m \otimes v^n \end{bmatrix}.$$

The vectors u and v are eigenvectors of the maximal torus which is

$$\nu_0(a) = a \oplus a^{-1},$$

for the standard representation of $\mathrm{SL}(2, \mathbb{C})$. Their actions are given by

$$(a \oplus a^{-1}) \cdot u = au,$$

and

$$(a \oplus a^{-1}) \cdot v = a^{-1}v.$$

It is clear from above that $u^i v^j$, where $i + j = k$, is a basis of eigenvectors for the maximal torus $a \oplus a^{-1}$ for the representation $\mathrm{Sym}^k(V)$, where action of $a \oplus a^{-1}$ is given by

$$(a \oplus a^{-1}) \cdot u^i v^j = a^{i-j} u^i v^j.$$

The maximal torus of $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ is $(a \oplus a^{-1}) \times (b \oplus b^{-1})$. Here a basis of common eigenvectors is $u^i v^j \otimes u^r v^s$, where $i + j = m$ and $r + s = n$. The action of element of maximal torus of the form

$$\nu(a, b) = (a \oplus a^{-1}) \times (b \oplus b^{-1}),$$

is given by

$$\nu(a, b) \cdot u^i v^j \otimes u^r v^s = a^{i-j} b^{r-s} u^i v^j \otimes u^r v^s.$$

Consider $\tilde{\phi}(m,n)(\nu(a,a))$ i.e $\tilde{\phi}(m,n)(\nu_2(a))$. We can see its action on the basis of common eigenvectors in array form as follows.

$$\begin{bmatrix} a^{m+n}u^m \otimes u^n & a^{m+n-2}u^{m-1}v \otimes u^n & \cdots & a^{n-m}v^m \otimes u^n \\ a^{m+n-2}u^m \otimes u^{n-1}v & a^{m+n-4}u^{m-1}v \otimes u^{n-1}v & \cdots & a^{n-2-m}v^m \otimes u^{n-1}v \\ a^{m+n-4}u^m \otimes u^{n-2}v^2 & \cdots & \cdots & a^{n-4-m}v^m \otimes u^{n-2}v^2 \\ \cdots & \cdots & \cdots & \cdots \\ a^{m-n}u^m \otimes v^n & a^{m-2-n}u^{m-1}v \otimes v^n & \cdots & a^{-m-n}v^m \otimes v^n \end{bmatrix} \tag{8.13}$$

Now assume $n \leq m$. It is clear from the above that

$$\begin{aligned} \phi(m,n)(L_2(a)) =& \tilde{\phi}(m,n)(\nu_2(a)) \\ =& a^{m+n} \oplus \{a^{m+n-2}\}^{\oplus 2} \oplus \{a^{m+n-4}\}^{\oplus 3} \oplus \{a^{m+n-6}\}^{\oplus 4} \oplus \cdots \oplus \\ & \{a^{m-n}\}^{\oplus n+1} \oplus \{a^{m-n-2}\}^{\oplus n+1} \oplus \cdots \oplus \{a^{n-m}\}^{\oplus n+1} \oplus \{a^{n-m-2}\}^{\oplus n} \\ & \oplus \{a^{n-m-4}\}^{\oplus n-1} \oplus \cdots \oplus a^{-n-m}. \end{aligned}$$

Here $A^{\oplus k} = A \oplus A \oplus \cdots \oplus A$, where A appears k times on the diagonal. If we put $z = a^2$, we get

$$\begin{aligned} \phi(m,n)(L_2'(z)) =& z^{(m+n)/2} \oplus (z^{(m+n)/2-1})^{\oplus 2} \oplus (z^{(m+n)/2-2})^{\oplus 3} \oplus (z^{(m+n)/2-3})^{\oplus 4} \\ & \oplus \cdots \oplus (z^{(m-n)/2})^{\oplus n+1} \oplus (z^{(m-n)/2-1})^{\oplus n+1} \oplus \cdots \oplus (z^{(n-m)/2})^{\oplus n+1} \\ & \oplus (z^{(n-m)/2-1})^{\oplus n} \oplus (z^{(n-m)/2-2})^{\oplus n-1} \oplus \cdots \oplus z^{(-n-m)/2}. \end{aligned}$$

$$\begin{aligned} \Theta_{\phi(m,n)}(L_2'(z)) =& z^{(m+n)/2} + 2z^{(m+n)/2-1} + 3z^{(m+n)/2-2} + \cdots + (n)z^{(m-n)/2+1} \\ & + (n+1)\left[z^{(m-n)/2} + z^{(m-n)/2-1} + \cdots z\right] + (n+1) \\ & + (n+1)\left[z^{-1} + z^{-2} + \cdots + z^{(n-m)/2}\right] \\ & + nz^{(n-m)/2-1} + \cdots + 3z^{-(m+n)/2+2} + 2z^{-(m+n)/2+1} + z^{-(m+n)/2}. \end{aligned}$$

This is a Laurent-palindromic polynomial (see Definition 2.1.6). Recall Definition 2.1.7 for the operator denoted by $\Psi : \mathbb{Z}[t]^{\mathrm{pal}} \to \mathbb{Z}$ on Laurent-palindromic polynomials.

By Lemma 3.2.2 we know that $\phi(m, n)$ is spinorial if and only if $\Psi_{\phi(m,n)}(L_2') = \Psi(\Theta_{\phi(m,n)}) \equiv 0 \mod 2$.

We now calculate $\Psi(\Theta_{\phi(m,n)})$.

Lemma 8.2.4. *Let $n \leq m$, then we have*

$$\Psi(\mathrm{Ch}_{\phi(m,n)}) = (1/24)(1+n)(6m + 3m^2 + n^2 + 2n).$$

Proof.

$$\begin{aligned}
\mathrm{Ch}_{\phi(m,n)}(z) =& z^{(m+n)/2} + 2z^{(m+n)/2-1} + 3z^{(m+n)/2-2} + \cdots + (n)z^{(m-n)/2+1} \\
&+ (n+1)\left[z^{(m-n)/2} + z^{(m-n)/2-1} + \cdots + z\right] + (n+1) \\
&+ (n+1)\left[z^{-1} + z^{-2} + \cdots + z^{(n-m)/2}\right] + nz^{(n-m)/2-1} + \cdots \\
&+ 3z^{-(m+n)/2+2} + z^{-(m+n)/2+1} + z^{-(m+n)/2}.
\end{aligned}$$

$$\begin{aligned}
\Psi(\mathrm{Ch}_{\phi(m,n)}) &= (m+n)/2 + 2((m+n)/2-1) + 3((m+n)/2-2) + \cdots \\
&\quad + n((m+n)/2-(n-1)) + \\
&\quad (n+1)\left[(m-n)/2 + ((m+n)/2-1) + \cdots + 1\right] \\
&= \left(\sum_{i=1}^{n} i((m+n)/2-(i-1))\right) + \\
&\quad (n+1)(1/2)((m-n)/2+1)((m-n)/2) \\
&= ((m+n)/2)(1/2)(n)(n+1) - 2\sum_{i=1}^{n}\binom{i}{2} + \\
&\quad (n+1)(1/2)((m-n)/2+1)((m-n)/2) \\
&\qquad \left(\text{since } \sum_{i=1}^{n} i = (1/2)n(n+1)\right) \\
&= ((m+n)/2)(1/2)(n)(n+1) - 2\binom{n+1}{3} + \\
&\quad (n+1)(1/2)((m-n)/2+1)((m-n)/2) \\
&\qquad \left(\text{since } \sum_{i=1}^{n}\binom{i}{2} = \binom{n+1}{3}\right) \\
&= (1/24)(n+1)(3m^2+6m+n^2+2n).
\end{aligned}$$

Hence the proof. □

For the case $m < n$, by symmetry, we just have to swap n and m in the lemma above. So we deduce that $\phi(m,n)$ (where $m+n$ is even) is spinorial if and only if

$$(1/24)(n+1)(3m^2+6m+n^2+2n) \equiv 0 \mod 2 \quad \text{if } n \leq m,$$

$$(1/24)(m+1)(3n^2+6n+m^2+2m) \equiv 0 \mod 2 \quad \text{otherwise.}$$

Observe that

$$\begin{aligned} g(n,m) &= (1/24)(n+1)(3m^2+6m+n^2+2n) - (1/24)(m+1)(3n^2+6n+m^2+2m) \\ &= (1/24)((n-m)^2-4)(n-m). \end{aligned}$$

Since $n+m$ is even , $n-m$ will also be even. Let $n-m=2k$ for some integer k. Then

$$\begin{aligned} g(n,m) &= (1/24)(4k^2-4)(2k) \\ &= (1/24)8(k-1)k(k+1) \\ &= ((k-1)k(k+1))/3, \end{aligned}$$

which is obviously an integer, moreover it is even since one of $k-1$, k and $k+1$ has to be even. Hence $g(m,n) \equiv 0 \mod 2$, therefore

$$(1/24)(n+1)(3m^2+6m+n^2+2n) \equiv (1/24)(m+1)(3n^2+6n+m^2+2m) \mod 2,$$

for all (m,n) such that $m+n$ is even. Thus it is enough to use the equation in Lemma 8.2.4.

In fact $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. The highest weight (x,y) in the Weyl chamber $\langle (1,1),(1,-1) \rangle$ of $\mathfrak{so}(4)$ is related to (m,n) by relations $x=\frac{m+n}{2}$ and $y=\frac{n-m}{2}$. Since $m+n$ is even, both x and y are integers. Hence $m=(x-y)$, and $n=(x+y)$. Thus our formula becomes

$$(1/6)(1 + x + y)(2x + x^2 - y - xy + y^2).$$

This concludes the proof of Theorem 8.2.2. □

Proof of Theorem 8.2.3. For the calculation of P_{Spin}, we show that

Lemma 8.2.5. *The spinoriality of the representation of* SO(4) *is periodic in its highest weight* (m, n) *with periods* $(4, 4)$ *and* $(4, -4)$.

Proof. Let $f(m, n) = (1/24)(n + 1)(3m^2 + 6m + n^2 + 2n)$. Observe that $f(m+4, n+4) - f(m, n) = (1/2)((m+n)^2 + 12(m+n) + 40)$. Since $(m+n)$ is even, this is an even number and therefore $f(m + 4, n + 4) \equiv f(m, n) \mod 2$. Therefore, it is $(4, 4)$ periodic.

Also observe that $f(m+4, n-4) - f(m, n) = -(m-n)^2/2 - 4(m-n) - 10$. Since $(m - n)$ is even, this is an even number and thus $f(m + 4, n - 4) \equiv f(m, n) \mod 2$. Hence, it is (4, -4) periodic.

□

Lemma 8.2.6. *We have* $P_{\text{Spin}}(\mathrm{SO}(4)) = \langle (4, 4), (4, -4) \rangle$.

Proof. It is clear from the above lemma that $\langle (4, 4), (4, -4) \rangle \leq P_{\text{Spin}}(\mathrm{SO}(4))$. We would like to prove equality in this case. Let P be the weight lattice of SO(4), which is the same as the (m, n) such that $m + n$ is even. It is the same as $\langle (1, 1), (1, -1) \rangle$. So we have $\langle (4, 4), (4, -4) \rangle \leq P_{\text{Spin}}(\mathrm{SO}(4)) \leq P$. Let $A = P/\langle (4, 4), (4, -4) \rangle$. Let $A' = P_{\text{Spin}}(\mathrm{SO}(4))/\langle (4, 4), (4, -4) \rangle$. Observe that $A' \leq A$. We are done if we prove A' is trivial. Observe that $A = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Suppose A' is not trivial in A, it will contain some element $(a, b) \neq (0, 0)$. Since it is a subgroup, it will contain $(2a, 2b)$, which can be either $(0, 2)$ or $(2, 0)$ or $(2, 2)$ or $(0, 0)$. If $(2a, 2b) = (0, 0)$ then (a, b) has to be one of the four options above. Therefore if we prove that $(0, 2), (2, 0)$ and $(2, 2)$ does not belong to A' then we are done.

The elements $(2,0),(0,2),(2,2)$ correspond to $(2,2),(2,-2)$ and $(2,0)$ in P. Observe that $f(1,1) \equiv 1 \mod 2$ and $f(3,3) \equiv 0 \mod 2$. Thus $(2,2) \notin P_{\text{Spin}}(\text{SO}(4))$. Furthermore $f(3,3) \equiv 0 \mod 2$ and $f(5,1) \equiv 1 \mod 2$, Thus $(2,-2) \notin P_{\text{Spin}}(\text{SO}(4))$. Similarly $f(1,1) \equiv 1 \mod 2$ and $f(3,1) \equiv 0 \mod 2$, therefore $(2,0) \notin P_{\text{Spin}}(\text{SO}(4))$. Hence the proof. □

Thus we conclude the proof of Theorem 8.2.3.

□

8.2.3 Case SO(5)

In this subsection we determine the spinorial irreducible orthogonal representations of $\text{SO}(5,\mathbb{C})$ in terms of their highest weight. From Section 6.1.2 we know that all the irreducible representations of $\text{SO}(5,\mathbb{C})$ are orthogonal.

Theorem 8.2.7. *The representation with highest weight $\lambda = (\lambda_1, \lambda_2)$ is spinorial if and only if*

$$\binom{\lambda_1+3}{4} - \binom{\lambda_2+2}{4}$$

is even.

Theorem 8.2.8. $P_{\text{Spin}}(\text{SO}(5,\mathbb{C})) = \langle (4,4),(4,-4) \rangle = 8P_{\text{orth}}(\mathfrak{so}(5,\mathbb{C}))$. *The proportion of non-spinorial weights is* $1/2$.

Proof of Theorem 8.2.7. Here we apply the general strategy to group $\text{SO}(5,\mathbb{C})$. We take the bilinear form to be $\left[\begin{array}{c|c|c} 0 & I_2 & 0 \\ \hline I_2 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$, where I_2 denotes the 2×2 identity matrix.

We fix our maximal torus of $\text{SO}(5,\mathbb{C})$ to be $x_1 \oplus x_2 \oplus x_1^{-1} \oplus x_2^{-1} \oplus 1$, where $x_i \in \mathbb{C}^\times$. Let us denote this torus by T.

Let ν_1 be the co-character $\nu_1 : \mathbb{C}^\times \to T$ defined as $\nu_1(t) = t\oplus 1\oplus t^{-1}\oplus 1\oplus 1$. Let ν_2 be the co-character $\nu_2 : \mathbb{C}^\times \to T$ defined as $\nu_2(t) = 1 \oplus t \oplus 1 \oplus t^{-1} \oplus 1$. The co-characters ν_1 and ν_2 are representatives of generators of $\pi_1(\mathrm{SO}(5, \mathbb{C}))$. By Scholium 4.2.13 to determine spinoriality of ϕ, it is enough to check parity of $Q''_\nu(1)/2$ for $\nu = \nu_1$ and ν_2. Moreover $Q''_\nu(1)/2 \equiv \Psi_\phi(\nu) \equiv F_\nu(\lambda) \mod 2$, from Lemma 3.2.5 and Lemma 4.1.2. Since ν_1 and ν_2 are Weyl conjugate and $F_\nu(\lambda) = F_{w(\nu)}(\lambda)$ by Lemma 3.2.4, we just want the parity of one of them, let us say the parity of $\Psi_\phi(\nu_1)$.

Let us denote the irreducible representation with highest weight (λ_1, λ_2) by ϕ. We use the determinantal Weyl character formula here (see page 409 Proposition 24.33 of [Fulton and Harris(1991)]). Let Θ denote the character of the representation. Then

$$\Theta(x) = f(x_1, x_2) = \det \begin{bmatrix} K_{\lambda_1}(\mathbf{x}) & K_{\lambda_1+1}(\mathbf{x}) + K_{\lambda_1-1}(\mathbf{x}) \\ K_{\lambda_2-1}(\mathbf{x}) & K_{\lambda_2}(\mathbf{x}) + K_{\lambda_2-2}(\mathbf{x}) \end{bmatrix},$$

where $\mathbf{x} = (x_1, x_2, x_1^{-1}, x_2^{-1}, 1)$ and $K_d(\mathbf{x}) = H_d(\mathbf{x}) - H_{d-2}(\mathbf{x})$. (See Definition 8.0.1.)

Since we are only interested in $\Psi_\phi(\nu_1)$ which equals $\Psi(\Theta(\nu_1(t)))$. We would like an expression for $\Theta(\nu_1(t)) = (t, 1)$ which equals

$$\det \begin{bmatrix} K_{\lambda_1}(t,1,t^{-1},1,1) & K_{\lambda_1+1}(t,1,t^{-1},1,1) + K_{\lambda_1-1}(t,1,t^{-1},1,1) \\ K_{\lambda_2-1}(t,1,t^{-1},1,1) & K_{\lambda_2}(t,1,t^{-1},1,1) + K_{\lambda_2-2}(t,1,t^{-1},1,1) \end{bmatrix}$$

$$= (K_{\lambda_1})(K_{\lambda_2} + K_{\lambda_2-2}) - (K_{\lambda_2-1})(K_{\lambda_1+1} + K_{\lambda_1-1}).$$

Lemma 8.2.9. *We have*

$$K_d(t,1,t^{-1},1,1) = \binom{2}{2}(t^d + t^{-d}) + \binom{3}{2}(t^{d-1} + t^{1-d}) + \cdots + \binom{d+1}{2}(t + t^{-1}) + \binom{d+2}{2}.$$

Proof. By definition we have

$$K_d(t,1,t^{-1},1,1) = H_d(t,1,t^{-1},1,1) - H_{d-2}(t,1,t^{-1},1,1).$$

Also by definition we have

$$H_d(x_1,x_2,x_3,x_4,x_5) = \sum_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} x_5^{\alpha_5},$$

where α runs over the set $= \{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \in \mathbb{Z}^5_{\geq 0} \mid \sum \alpha_i = d\}$. Let $\alpha_1 = r$ and $\alpha_1 + \alpha_3 = s$ then $\alpha_2 + \alpha_4 + \alpha_5 = d - s$. The number of ways to solve $\alpha_2 + \alpha_4 + \alpha_5 = d - s$ for ordered triple of non-negative integers $\alpha_2, \alpha_4, \alpha_5$ is $\binom{d-s+2}{2}$. Thus

$$\begin{aligned} H_d(t,1,t^{-1},1,1) &= \sum_{s=0}^{d} \sum_{r=0}^{s} t^r t^{-(s-r)} \binom{d-s+2}{2} \\ &= \sum_{s=0}^{d} \sum_{r=0}^{s} t^{2r-s} \binom{d-s+2}{2}. \end{aligned}$$

Similarly

$$H_{d-2}(t,1,t^{-1},1,1) = \sum_{s=0}^{d-2} \sum_{r=0}^{s} t^{2r-s} \binom{d-s}{2}.$$

So,

$$K_d(t,1,t^{-1},1,1) = \sum_{s=0}^{d-2}\sum_{r=0}^{s} t^{2r-s}\left(\binom{d-s+2}{2}-\binom{d-s}{2}\right) + \sum_{r=0}^{d-1} t^{2r-d+1}\binom{3}{2} + \sum_{r=0}^{d} t^{2r-d}\binom{2}{2}.$$

This equals the sum

$$= \binom{2}{2}(t^{-d}+t^{-d+2}+\cdots+t^{d-2}+t^{d})$$

$$+ \binom{3}{2}(t^{-d+1}+t^{-d+3}+\cdots+t^{d-3}+t^{d-1})$$

$$+ \left(\binom{4}{2}-\binom{2}{2}\right)(t^{-d+2}+t^{-d+4}+\cdots+t^{d-4}+t^{d-2})$$

$$+ \left(\binom{5}{2}-\binom{3}{2}\right)(t^{-d+3}+t^{-d+5}+\cdots+t^{d-5}+t^{d-3})$$

$$+ \left(\binom{6}{2}-\binom{4}{2}\right)(t^{-d+4}+t^{-d+6}+\cdots+t^{d-6}+t^{d-4})$$

$$+ \left(\binom{7}{2}-\binom{5}{2}\right)(t^{-d+5}+t^{-d+7}+\cdots+t^{d-7}+t^{d-5})$$

$$\cdots$$

$$+ \left(\binom{d+1}{2}-\binom{d-1}{2}\right)(t+t^{-1})$$

$$+ \left(\binom{d+2}{2}-\binom{d}{2}\right).$$

The terms in between cancel, and it follows that

$$K_d(t,1,t^{-1},1,1) = \binom{2}{2}(t^{-d}+t^{d}) + \binom{3}{2}(t^{-d+1}+t^{d-1}) + \binom{4}{2}(t^{-d+2}+t^{d-2}) + \cdots + \binom{d+1}{2}(t^{-1}+t) + \binom{d+2}{2},$$

as desired. □

Lemma 8.2.10. *We have*

$$K_d(t,1,t^{-1},1,1)+K_{d-2}(t,1,t^{-1},1,1) \equiv t^d+t^{d-1}+\cdots+t^{-d+1}+t^{-d} \in (\mathbb{Z}/2\mathbb{Z})[t].$$

Proof. From the lemma above, we have

$$K_d(t,1,t^{-1},1,1) = \binom{2}{2}t^{-d} + \binom{3}{2}t^{-d+1} + \cdots + \binom{3}{2}t^{d-1} + \binom{2}{2}t^{d},$$

and

$$K_{d-2}(t,1,t^{-1},1,1) = \binom{2}{2}t^{-d+2} + \binom{3}{2}t^{-d+3} + \cdots + \binom{3}{2}t^{d-3} + \binom{2}{2}t^{d-2}.$$

Adding we get

$$t^{-d} + 3t^{-d+1} + \left(\binom{4}{2}+\binom{2}{2}\right)t^{-d+2} + \left(\binom{5}{2}+\binom{3}{2}\right)t^{-d+3} + \cdots \left(\binom{5}{2}+\binom{3}{2}\right)t^{d-3} + \left(\binom{4}{2}+\binom{2}{2}\right)t^{d-2} + 3t^{d-1} + t^{d}.$$

From the fact that $\binom{l+2}{2}+\binom{l}{2} = l^2+l+1$ is odd, the lemma follows. □

Write $\Theta(\nu_1(t))) = f(t,1) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 + a_1 t^{-1} + \cdots + a_{n-1}t^{-n+1} + a_n t^{-n}$ for $a_i \in \mathbb{Z}_{\geq 0}$. From Lemma 3.2.2, we can conclude that the representation ϕ with highest weight $\lambda = (\lambda_1, \lambda_2)$ is spinorial if and only if $\Psi_\phi(\nu_1) = \sum_{i=1}^{n} a_i \cdot i \equiv 0 \mod 2$.

Recall from Definitions 2.1.6 and 2.1.7, the definition of $\Psi : \mathbb{Z}[x, x^{-1}]^{\mathrm{sym}} \to \mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}$-linear operator on the abelian group of Laurent-palindromic polynomials.

Now we are interested in calculating $\Psi_\phi(\nu_1) = \Psi(f(t,1))$, where

$$f(t,1) = (K_{\lambda_1})(K_{\lambda_2} + K_{\lambda_2-2}) - (K_{\lambda_2-1})(K_{\lambda_1+1} + K_{\lambda_1-1}). \tag{8.14}$$

Lemma 8.2.11. *We get*

$$\Psi((t^n + t^{-n})(t^k + t^{k-1} + \cdots + t^{1-k} + t^{-k})) = n \in \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $E(t) = (t^n + t^{-n})(t^k + t^{k-1} + \cdots + t^{1-k} + t^{-k})$. We will make cases. In the first case, let $n > k$. Then we have

$$E = t^{n+k} + t^{n+k-1} + \cdots + t^{n-k+1} + t^{n-k} + t^{-n+k} + t^{-n+k-1} + \cdots + t^{-n-k}.$$

$$\begin{aligned}\Psi(E) = \sum_{i=n-k}^{n+k} i = \sum_{i=1}^{n+k} i - \sum_{i=1}^{n-k-1} i \\ &= (1/2)(n+k)(n+k+1) - (1/2)(n-k)(n-k-1) \\ &= n(2k+1) \\ &= n \in \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

In second case, let $n = k$. Then

$$E(t) = t^{2n} + t^{2n-1} + \cdots + t + 2 + t^{-1} + \cdots + t^{1-2n} + t^{-2n}.$$

$$\Psi(E) = \sum_{i=1}^{2n} i = (1/2)(2n)(2n+1) = n(2n+1) = n(2k+1) = n \in \mathbb{Z}/2\mathbb{Z}.$$

In the third case, let $n < k$. Then

$$E = t^{n+k} + t^{n+k-1} + \cdots + t^{k-n+1} + 2t^{k-n} + 2t^{k-n-1} + \cdots + 2t^{n-k} + t^{n-k-1} + \cdots + t^{-n-k}.$$

Moreover

$$\begin{aligned}\Psi(E) &= \sum_{i=k-n+1}^{n+k} i \mod 2 \\ &= \sum_{i=1}^{n+k} i - \sum_{i=1}^{k-n} i \\ &= (1/2)(n+k+1)(n+k) - (1/2)(k-n)(k-n+1) \\ &= n(2k+1) \in \mathbb{Z}/2\mathbb{Z} \\ &= n \in \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

□

Lemma 8.2.12. *For $d \in \mathbb{Z}_{\geq 0}$ and $d' \in \mathbb{Z}_{\geq 2}$, we have*

$$\Psi(K_d(t,1,t^{-1},1,1) \cdot (K_{d'}(t,1,t^{-1},1,1) + K_{d'-2}(t,1,t^{-1},1,1)))$$
$$= \binom{d+3}{4} + \binom{d+2}{2}\binom{d'+1}{2} \in \mathbb{Z}/2\mathbb{Z}.$$

Proof. From Lemma 8.2.10

$$K_{d'}(t,1,t^{-1},1,1) + K_{d'-2}(t,1,t^{-1},1,1) = t^{d'} + t^{d'-1} + \cdots + t^{1-d'} + t^{-d'} \in (\mathbb{Z}/2\mathbb{Z})[t].$$

From Lemma 8.2.9

$$\begin{aligned}K_d(t,1,t^{-1},1,1) =& \binom{2}{2}(t^d + t^-d) + \binom{3}{2}(t^{d-1} + t^{1-d}) + \cdots \\ &+ \binom{d+1}{2}(t + t^{-1}) + \binom{d+2}{2}.\end{aligned}$$

So

$$\Psi((K_d(t,1,t^{-1},1,1))(K_{d'}(t,1,t^{-1},1,1)+K_{d'-2}(t,1,t^{-1},1,1)))$$

$$= \binom{2}{2}\Psi((t^d+t^{-d})(t^{d'}+t^{d'-1}+\cdots+t^{1-d'}+t^{-d'}))$$

$$+ \binom{3}{2}\Psi((t^{d-1}+t^{1-d})(t^{d'}+t^{d'-1}+\cdots+t^{1-d'}+t^{-d'}))$$

$$+ \binom{4}{2}\Psi((t^{d-2}+t^{2-d})(t^{d'}+t^{d'-1}+\cdots+t^{1-d'}+t^{-d'}))$$

$$\vdots$$

$$+ \binom{d+1}{2}\Psi((t+t^{-1})(t^{d'}+t^{d'-1}+\cdots+t^{1-d'}+t^{-d'}))$$

$$+ \binom{d+2}{2}\Psi((1)(t^{d'}+t^{d'-1}+\cdots+t^{1-d'}+t^{-d'})).$$

By using Lemma 8.2.11 we get

$$= \binom{2}{2}d + \binom{3}{2}(d-1) + \cdots + \binom{d+1}{2}(1) + \binom{d+2}{2}\binom{d'+1}{2}$$

$$= \left(\sum_{i=2}^{d+1}\binom{i}{2}(d+2-i)\right) + \binom{d+2}{2}\binom{d'+1}{2}$$

$$= \left((d+3)\sum_{i=2}^{d+1}\binom{i}{2} - \sum_{i=2}^{d+1}(i+1)\binom{i}{2}\right) + \binom{d+2}{2}\binom{d'+1}{2}$$

$$= \left((d+3)\binom{d+2}{3} - 3\sum_{i=2}^{d+1}\binom{i+1}{3}\right) + \binom{d+2}{2}\binom{d'+1}{2}$$

$$= \left(4\binom{d+3}{4} - 3\binom{d+3}{4}\right) + \binom{d+2}{2}\binom{d'+1}{2}$$

$$= \binom{d+3}{4} + \binom{d+2}{2}\binom{d'+1}{2}$$

$$= \binom{d+3}{4} + \binom{d+2}{2}\binom{d'+1}{2} \in \mathbb{Z}/2\mathbb{Z}.$$

□

Lemma 8.2.13. *We have*

$$\Psi(f(t,1)) \equiv \binom{\lambda_1+3}{4} - \binom{\lambda_2+2}{4} \in \mathbb{Z}/2\mathbb{Z}.$$

Proof. Apply Ψ to both sides of Equation (8.14), to obtain

$$\Psi(f(t,1)) = \Psi(K_{\lambda_1}(t,1,t^{-1},1,1)(K_{\lambda_2}(t,1,t^{-1},1,1) + K_{\lambda_2-2}(t,1,t^{-1},1,1)))$$

$$-\Psi(K_{\lambda_2-1}(t,1,t^{-1},1,1)(K_{\lambda_1+1}(t,1,t^{-1},1,1) + K_{\lambda_1-1}(t,1,t^{-1},1,1))).$$

Using Lemma 8.2.12 we get

$$\begin{aligned}\Psi(f(t,1)) &= \left(\binom{\lambda_1+3}{4} + \binom{\lambda_2+1}{2}\binom{\lambda_1+2}{2}\right) \\ &\quad - \binom{\lambda_2+2}{4} - \binom{\lambda_2+1}{2}\binom{\lambda_1+2}{2} \\ &= \binom{\lambda_1+3}{4} - \binom{\lambda_2+2}{4} \mod 2.\end{aligned}$$

□

We already know that

$$\Psi(f(t,1)) = 0 \in \mathbb{Z}/2\mathbb{Z}$$

if and only if the representation with the highest weight λ is spinorial. Hence the representation with highest weight $\lambda = (\lambda_1, \lambda_2)$ is spinorial if and only if

$$\binom{\lambda_1+3}{4} - \binom{\lambda_2+2}{4} \equiv 0 \in \mathbb{Z}/2\mathbb{Z}.$$

Hence we conclude the proof of Theorem 8.2.7. □

Proof of Theorem 8.2.8.

Lemma 8.2.14. *We have* $\langle (4,4),(4,-4)\rangle \leq P_{\mathrm{Spin}}(\mathrm{SO}(5,\mathbb{C}))$.

Proof. This is an easy application of Lemma 12.1.5 in the Appendix. □

Let us denote the weight lattice of $\mathrm{SO}(5,\mathbb{C})$ by P. We know from the lemma above that $\langle (4,4),(4,-4)\rangle \leq P_{\mathrm{Spin}}(\mathrm{SO}(5,\mathbb{C})) \leq P$. Here $P \cong \mathbb{Z}\times\mathbb{Z}$ since it has rank 2. Let us denote the quotient group $P/\langle (4,4),(4,-4)\rangle$ by A and $P_{\mathrm{Spin}}(\mathrm{SO}(5,\mathbb{C}))/\langle (4,4),(4,-4)\rangle$ by A'. Note that A' is a subgroup of A. If we prove that A' is trivial we are done. Observe that A is isomorphic to $\mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ by identifying a representative as $a(1,0)+b(1,1)$, where

$a \in \mathbb{Z}/8\mathbb{Z}$ and $b \in \mathbb{Z}/4\mathbb{Z}$. Now we concentrate on the group $A = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. In this group, $(4,0) = 4x$ or $2x$ or x, where x can be one of the elements in S_1, where

$$S_1 = \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(1,1),(3,1),(5,1), (7,1),(1,2),(3,2),(5,2),(7,2),(2,2),(6,2),(1,3),(3,3),(5,3),(7,3)\}.$$

Furthermore $(0,2) = x$ or $2x$ for every $x \in S_2$, where

$$S_2 = \{(0,1),(4,1),(0,2),(0,3),(4,3)\}.$$

Moreover $(4,2) = x$ or $2x$ for every $x \in S_3$, where

$$S_3 = \{(2,1),(6,1),(4,2),(2,3),(6,3)\}.$$

If H is a subgroup of A then $(4,0) \notin H$ which implies none of the members of S_1 belong to H. A similar argument holds for $(0,2)$ and S_2 and $(4,2)$ and S_3. Thus, if none of the members $(4,0),(0,2)$ and $(4,2)$ belong to H then $H \cap (S_1 \cup S_2 \cup S_3) = \emptyset$. But observe that $A \setminus (S_1 \cup S_2 \cup S_3) = \{(0,0)\}$. Hence H will be the trivial subgroup.

Hence, for proving A' is trivial it is enough to prove that

$$\{(4,0),(0,2),(4,2)\} \cap A' = \emptyset$$

.

The element $(4,0)$ corresponds to $4(1,0) + 0(1,1) = (4,0)$ in P. The element $(0,2)$ corresponds to $0(1,0) + 2(1,1) = (2,2)$ in P. The element $(4,2)$ corresponds to $4(1,0) + 2(1,1) = (6,2)$ in P. Let $g(x,y) = \binom{x+3}{4} - \binom{y+2}{4}$ mod 2. Observe that $g(2,1) = 1$, while $g(6,1) = 0$, hence $(4,0) \notin P_{\mathrm{Spin}}(\mathrm{SO}(5,\mathbb{C}))$. Moreover $g(8,2) = 1$, while $g(10,4) = 0$, hence $(2,2) \notin$

$P_{\mathrm{Spin}}(\mathrm{SO}(5,\mathbb{C}))$. In addition $g(35,5)=0$, while $g(41,7)=1$, hence $(6,2)\notin P_{\mathrm{Spin}}(\mathrm{SO}(5,\mathbb{C}))$.

Hence $P_{\mathrm{Spin}}(\mathrm{SO}(5,\mathbb{C}))=\langle(4,4),(4,-4)\rangle$.

$\square$

Chapter 9

Complexification of Compact Lie Groups

Definition 9.0.1. ***complexification of a real Lie algebra*** $\mathfrak{g}$, *is the complex Lie algebra* $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 9.0.2. *A continuous function* $f : G \to \mathbb{C}$ *is called a* ***representative function*** *if* f *generates a finite dimensional* G*-subspace of continous complex valued functions under the action* $(g \cdot f)(x) = f(g^{-1}x)$. *Representative functions form a ring and it is denoted by* $A(G, \mathbb{C})$.

(see chapter 3 section 1 of [Bröcker and tom Dieck(2013)]).

Definition 9.0.3. *Let* G *be a real Lie group. Then consider the group* $G_{\mathbb{C}}$ *of all* $\mathbb{C}$*-algebra homomorphisms* $A(G, \mathbb{C}) \to \mathbb{C}$ *and let* $i : G \to G_{\mathbb{C}}$ *be the evaluation map* $(i(g))(f) = f(g)$. *Then* $G_{\mathbb{C}}$ *is a complex analytic Lie group and is called the* ***complexification of*** G. *It has a universal property that, given any complex finite dimensional representation* ϕ, *there exists a unique holomorphic representation* ψ *of* $G_{\mathbb{C}}$ *such that* $\phi = \psi \circ i$.

(see chapter 3 section 8 of [Bröcker and tom Dieck(2013)]).

It is well-known that the complexification of the unitary group $\mathrm{U}(n)$ is $\mathrm{GL}(n, \mathbb{C})$, and that of $\mathrm{SO}(N)$ is $\mathrm{SO}(N, \mathbb{C})$, and that of $\mathrm{Spin}(N)$ is $\mathrm{Spin}(N, \mathbb{C})$. The group $\mathrm{Spin}(N, \mathbb{C})$ is also a double cover of $\mathrm{SO}(N, \mathbb{C})$. Hence, for complex analytic groups an analogous question can be raised as follows : Which finite dimensional irreducible complex orthogonal representations of a complex group G lift to $\mathrm{Spin}(N, \mathbb{C})$?

Lemma 9.0.4. *Suppose we have a commutative diagram of semi-simple complex groups* C_1, C_2, C_3

$$\begin{array}{ccc} & & C_3 \\ & \overset{\beta}{\nearrow} & \downarrow \gamma \\ C_1 & \xrightarrow{\alpha} & C_2 \end{array}$$

which are complexifications of real compact semisimple groups K_1, K_2, K_3 *with the maps* $i_j : K_j \to C_j$ *for* $j \in \{1, 2, 3\}$*. Then we have a corresponding commutative diagram*

$$\begin{array}{ccc} & & K_3' \\ & \overset{\beta_1}{\nearrow} & \downarrow \gamma_1 \\ K_1' & \xrightarrow{\alpha_1} & K_2' \end{array}$$

such that $K_i' \subset C_i$*,* $K_i' \cong K_i$*, and* $\alpha_{1\mathbb{C}} = \alpha$*,* $\beta_{1\mathbb{C}} = \beta$*,* $\gamma_{1\mathbb{C}} = \gamma$*.*

Proof. Since C_j is the complexification of K_j, $i_j(K_j)$ is a maximal compact subgroup of C_j (See Chapter 3 section 8 of [Bröcker and tom Dieck(2013)]). All maximal compact subgroups of a semisimple complex Lie group are conjugate. Take $K_1' = i_1(K_1)$. Since the image of K_1' under map α is a compact subgroup of C_2, it will land inside some conjugate of $i_2(K_2)$, which is a maximal compact subgroup of C_2. Call that conjugate subgroup K_2'. Similarly we can define K_3' inside C_3. Define $\gamma_1 = \gamma|_{K_3'}$. Observe that, since the C_i are semisimple, they are the complexifications of the K_i'. Furthermore α_1, β_1, γ_1 are restrictions of α, β, γ. Hence $\alpha_{1\mathbb{C}} = \alpha$, $\beta_{1\mathbb{C}} = \beta$, $\gamma_{1\mathbb{C}} = \gamma$. □

Lemma 9.0.5. *Let K be a connected real compact Lie group. Let ϕ be an orthogonal representation of K. Let $K_{\mathbb{C}}$ be the complexification of K. As complexification is a functor, let $\phi_{\mathbb{C}}$ be the induced map between $K_{\mathbb{C}}$ and* $\mathrm{SO}(N, \mathbb{C})$*. Then, ϕ is spinorial if and only if $\phi_{\mathbb{C}}$ is spinorial, i.e., lifts to* $\mathrm{Spin}(n, \mathbb{C})$*.*

Proof. If ψ is a lift of ϕ to $\mathrm{Spin}(n, \mathbb{R})$, i.e., if we have the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Spin}(n,\mathbb{R}) \\ & \overset{\psi}{\nearrow} & \big\downarrow \rho \\ K & \xrightarrow{\phi} & \mathrm{SO}(n,\mathbb{R}) \end{array}$$

then we have the commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Spin}(n,\mathbb{C}) \\ & \overset{\psi_{\mathbb{C}}}{\nearrow} & \big\downarrow \rho_{\mathbb{C}} \\ K_{\mathbb{C}} & \xrightarrow{\phi_{\mathbb{C}}} & \mathrm{SO}(n,\mathbb{C}). \end{array}$$

Hence we have $\psi_{\mathbb{C}}$, which is a lift of $\phi_{\mathbb{C}}$.

For the other way around, let $\psi_{\mathbb{C}}$ be the lift of $\phi_{\mathbb{C}}$. Let ρ and its complexification $\rho_{\mathbb{C}}$ be the covering maps for $\mathrm{SO}(N)$ and $\mathrm{SO}(N, \mathbb{C})$ respectively. Consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}(N,\mathbb{R}) & \xrightarrow{i_1} & \mathrm{Spin}(N,\mathbb{C}) \\ \big\downarrow \rho & & \big\downarrow \rho_{\mathbb{C}} \\ \mathrm{SO}(N,\mathbb{R}) & \xrightarrow{i_2} & \mathrm{SO}(N,\mathbb{C}). \end{array}$$

The map i_1 is the standard injection of the real spin group into the complex spin group. The map i_2 is the standard injection of real special orthogonal group into the complex special orthogonal group. By the definition of $\rho_{\mathbb{C}}$, the above diagram shows that the restriction of $\rho_{\mathbb{C}}$ to the real spin group is the map ρ.

Observe that $\rho_{\mathbb{C}}$ and ρ, both are double covers. Since one is the restriction of the other, their kernels are the same, i.e., Ker $(\rho_{\mathbb{C}}) = $ Ker $(\rho) \subset \mathrm{Spin}(N, \mathbb{R})$. Hence, $\rho_{\mathbb{C}}^{-1}(\mathrm{SO}(N, \mathbb{R})) = \mathrm{Spin}(N, \mathbb{R})$.
Since $\rho_{\mathbb{C}} \circ \psi_{\mathbb{C}}|_K = \phi_{\mathbb{C}}|_K = \phi$, the map $\psi_{\mathbb{C}}|_K$ is a lift of ϕ. □

There is a close relation between the finite-dimensional holomorphic representations of compact groups and finite-dimensional representations of their complexification. From Lemma 9.0.4 it is clear that the complexification is a functor from category of compact Lie groups to category of complex reductive Lie groups. Furthermore taking maximal compact subgroup is functor in the reverse direction.

In fact the complexification of Lie algebra of compact group in the Lie algebra sense is the complex Lie algebra of its complexification in the Lie group sense. There is a one-one correspondence between finite-dimensional representations of real Lie algebras and their complexification. Moreover it takes irreducible representations to irreducible representations.

$\mathrm{GL}(n, \mathbb{C})$ is the complexification $\mathrm{U}(n)$. Thus from Lemma 9.0.5 there is one-one correspondence between the spinorial representations of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{U}(n)$. Similarly there is a one-one correspondence between spinorial representations of $\mathrm{SO}(n, \mathbb{R})$ and $\mathrm{SO}(n, \mathbb{C})$. Furthermore self-dual representations of compact groups correspond to self-dual representations of their complexification. Moreover there is a one-one correspondence between real representations of compact groups and orthogonal representations of their complexification.

We have basically figured out the spinoriality for complex groups, which in turn gives the spinoriality for compact groups.

However we have some alternative methods to figure out the spinoriality for $\mathrm{U}(n)$ and $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SO}(4,\mathbb{R})$, $\mathrm{SO}(5,\mathbb{R})$, which we will present in the next chapters.

Chapter 10

Miscellaneous

10.1 Spinoriality of Representations restricted to $S_n < \mathrm{GL}(n, \mathbb{C})$

In this chapter, we discuss the spinoriality of an orthogonal representation of $\mathrm{GL}(n, \mathbb{C})$ restricted to the subgroup of permutation matrices, i.e. to S_n.

$$\begin{array}{ccccc} & & & & \mathrm{Spin}(m, \mathbb{C}) \\ & & & \overset{\psi}{\nearrow} & \downarrow \rho \\ S_n & \longrightarrow & \mathrm{GL}(n, \mathbb{C}) & \xrightarrow{\phi} & \mathrm{SO}(m, \mathbb{C}) \end{array}$$

Theorem 10.1.1. *An orthogonal representation ϕ of $\mathrm{GL}(n, \mathbb{C})$ is spinorial if and only if its restriction to S_n is spinorial.*

Proof. It is trivial to see that if ϕ is spinorial then its restriction is spinorial.

Conversely suppose ϕ is non-spinorial. We shall prove that restriction of ϕ to S_n is non-spinorial.

Let Q be the quadratic form defined as $Q(x_1, \ldots, x_m) = -(x_1^2 + \cdots + x_m^2)$.

See Chapter 2 for the definition of

$$\mathrm{Spin}(m,\mathbb{C}) = \{\pm w_1 \cdot w_2 \cdots w_{2k} \mid w_i \in \mathbb{C}^m, Q(w_i,w_i) = -1\}.$$

We choose an orthogonal basis $(e_1,\ldots,e_m)$ of $\mathbb{C}^m$ w.r.t. Q. Thus we get relations in $\mathrm{Spin}(m,\mathbb{C})$ as $e_i \cdot e_j = -e_j \cdot e_i$ and $e_i^2 = -1$.

From these relations it is easy to see that the image of $e_i \cdot e_j \in \mathrm{Spin}(m,\mathbb{C})$ in $\mathrm{SO}(m,\mathbb{C})$ is

$$1 \oplus 1 \oplus \cdots \oplus \underbrace{-1}_{\text{i'th place}} \oplus \cdots \oplus \underbrace{-1}_{\text{j'th place}} \oplus \cdots \oplus 1.$$

Now choose the transposition (1,2) in S_n. It corresponds to the matrix

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & I_{n-2} \end{array}\right]$$

where I_{n-2} is the $(n-2) \times (n-2)$ identity matrix. Note that A is conjugate to the matrix $1 \oplus \cdots \oplus 1 \oplus -1$.

Let us define a co-character $\nu(t) = 1 \oplus \cdots \oplus 1 \oplus t$. Let $B(t) = \mathrm{Trace}(\phi(\nu(t)))$. From arguments in Section 8.1.3 it is easy to see that ϕ is spinorial if and only if $\Psi(B(t)) \equiv 0 \mod 2$.

By Lemma 3.2.2 $B(t) = \sum_{i=1}^{d} a_i(t^i + t^{-i}) + a_0$ and $\Psi(B) = \sum_{i=1}^{d} a_i \cdot i$, where a_i is the multiplicity of t^i as weight of representation $\phi \circ \nu$ of $\mathbb{C}^\times$.

Since A is conjugate to $\nu(-1)$, it is easy to see that the multiplicity of -1 as an eigenvalue in $\phi(\nu(-1))$ is $m = 2a_1 + 2a_3 + \cdots$. Thus $\frac{m}{2} \equiv \Psi(B) \mod 2$. Since ϕ is non-spinorial, $\Psi(B)$ is odd. Hence $m/2$ is odd.

Let the lift of $\phi(\nu(-1))$ in $\mathrm{Spin}(m,\mathbb{C})$ be $y = \pm e_{i_1} \cdots e_{i_m}$, because the multiplicity of -1 is m. For the lift to be a homomorphism we require $y^2 = 1$, but by relations in the spin group it is easy to observe that $y^2 = (-1)^{m(m+1)/2}$. Since $m/2$ is odd and m is even $y^2 = -1$ which is a contradiction. Hence the image of the transposition $(1, 2)$ cannot be lifted to the spin group under the representation π.

Hence $\phi \mid_{S_n} = \pi$ is also non-spinorial. $\square$

Chapter 11

Summary

In this section we summarize all the results.

(1) Let G be a connected complex semi-simple group with complex lie algebra $\mathfrak{g}$. Let ϕ_λ be an irreducible orthogonal holomorphic representation with highest weight λ. Then with the notation that $\mathfrak{g} = \oplus \mathfrak{g}_i$, where $\mathfrak{g}_i$ is simple, $\lambda = \oplus \lambda_i$, $\rho = \oplus \rho_i$ and for an infinitesimal cocharacter $\nu = \oplus \nu_i$ of the maximal torus of G, the representation ϕ_λ is spinorial if and only if

$$\frac{Q_\nu''(1)}{2} = \dim V^\lambda \sum_{i=1}^{k} \frac{|\nu_i|^2(|\lambda_i + \rho_i|^2 - |\rho_i|^2)}{2\dim \mathfrak{g}_i} \equiv 0 \mod 2,$$

for a set of co-characters ν, which represent the generators of $\pi_1(G)$. (See Theorem 4.2.12 and Scholium 4.2.13.)

(2) As a special case of above theorem in the above setting if $\mathfrak{g}$ is simple then ϕ_λ is spinorial if and only if

$$\frac{(\dim V^\lambda)(|\nu|^2)(|\lambda + \rho|^2 - |\rho|^2)}{2(\dim \mathfrak{g})} \equiv 0 \mod 2,$$

for a set of co-characters ν which represent generators of $\pi_1(G)$. (See Theorem 4.2.9 and Scholium 4.2.13.)

(3) The adjoint representation is spinorial if and only if half the sum of positive roots is integral.(See Theorem 4.4.1.)

(4) When n is odd, all of the irreducible orthogonal finite-dimensional representations of $\mathrm{PGL}(n,\mathbb{C})$ are spinorial. (See Lemma 6.2.1.)

(5) For $G=\mathrm{PGL}(2n,\mathbb{C})$, the orthogonal irreducible finite-dimensional representation of highest weight

$$\lambda=(\sum_{i=1}^{n} r_i, \sum_{i=1}^{n-1} r_i, \dots, r_1+r_2, r_1, -r_1, -r_1-r_2, \dots, -\sum_{i=1}^{n} r_i),$$

is spinorial if and only if

$$\prod_{1\leq i<j\leq n}\left(\frac{\lambda_i-\lambda_j+j-i}{j-i}\right)\cdot\frac{(\sum_{i=1}^{n}(((\sum_{j=1}^{i} r_j)+\frac{2i-1}{2})^2-(\frac{2i-1}{2})^2))}{2n(2n+1)}\equiv 0 \mod 2,$$

(see Theorem 6.2.2)

or

$$\frac{\mathrm{dyn}(\phi)}{2n}\equiv 0 \mod 2.$$

(See Theorem 6.2.3.)

(6) For $G=\mathrm{SO}(2n+1,\mathbb{C})$, ϕ_λ is spinorial if and only if

$$\left(\prod_{1\leq i\leq j\leq n}(\frac{(\lambda_i+\rho_i)^2-(\lambda_j+\rho_j)^2}{\rho_i^2-\rho_j^2})\prod_{1\leq i\leq n}(\frac{\lambda_i+\rho_i}{\rho_i})\right)\left(\frac{(\sum_{i=1}^{n}((\lambda_i+\rho_i)^2-(\rho_i)^2))}{2\cdot(2n)(2n+1)}\right)\equiv 0 \mod 2,$$

(see Theorem 6.2.4) or,

$$\mathrm{dyn}(\phi) \text{ is even.}$$

(See Theorem 6.2.5.)

(7) For $G = \mathrm{SO}(2n, \mathbb{C})$, ϕ_λ is spinorial if and only if

$$\left(\prod_{1\le i<j\le n} \frac{(\lambda_i+\rho_i)^2-(\lambda_j+\rho_j)^2}{\rho_i^2-\rho_j^2}\right)\left(\frac{\sum_{i=1}^n((\lambda_i+\rho_i)^2-\rho_i^2)}{2(2n)(2n-1)}\right) \equiv 0 \mod 2,$$

(see Theorem 6.2.6), or

$$\mathrm{dyn}(\phi) \text{ is even},$$

(see Theorem 6.2.7), where

$$\begin{aligned}\lambda =&(\lambda_1, \lambda_2, \ldots, \lambda_n)\\ =&(a_1+a_2+\cdots a_{n-2}+(1/2)(a_{n-1}+a_n), a_2+a_3+\cdots+a_{n-2}+(1/2)(a_{n-1}+a_n),\\ &\cdots, a_{n-2}+(1/2)(a_{n-1}+a_n), (1/2)(a_{n-1}+a_n), (1/2)(a_n-a_{n-1})),\end{aligned}$$

and the condition is $a_{n-1} = a_n$ when n is odd and $a_{n-1}+a_n$ is even when n is even.

(8) For any connected complex reductive group, $|P_{\mathrm{orth}}(G)/P_{\mathrm{Spin}}(G)|$ is finite. (See Theorem 5.0.13.)

This is the summary of Determinantal identity method

(1) The representation of $\mathrm{PGL}(2n, \mathbb{C})$ of highest weight

$$\lambda = r_n(\varpi_{2n-1}+\varpi_1)+r_{n-1}(\varpi_{2n-2}+\varpi_2)+\cdots+r_2(\varpi_{n+1}+\varpi_{n-1})+2r_1(\varpi_n)$$

, where $\varpi_i = (\underbrace{1, 1, \ldots, 1}_{i\text{times}}, 0, 0, \ldots, 0)$ are fundamental weights of its Lie algebra $\mathfrak{sl}(2n, \mathbb{C})$(where r_i are non negative integers) is spinorial if and only if

$$\det \begin{pmatrix} d_1 & \binom{\lambda_1+2n-1}{2n-2} & \binom{\lambda_1+2n-1}{2n-3} & \cdots & 1 \\ d_2 & \binom{\lambda_2+2n-2}{2n-2} & \binom{\lambda_2+2n-2}{2n-3} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{2n} & \binom{\lambda_{2n}}{2n-2} & \binom{\lambda_{2n}}{2n-3} & \cdots & 1 \end{pmatrix} \text{ is even.}$$

Here

$$d_i = (d{+}1)\binom{\lambda_i - i + 2n}{2n-1} + \binom{(\lambda_i - d) - i + 2n}{2n} - \binom{\lambda_i - i + 2n + 1}{2n} \mod 2,$$

where $d = \sum_{i=1}^{n} r_i$. (See Theorem 8.1.1.)

(2) The lattice $P_{\mathrm{Spin}}(\mathrm{PGL}(2n, \mathbb{C}))$ contains

$$\langle 2^k(\varpi_1 + \varpi_{2n-1}), 2^k(\varpi_2 + \varpi_{2n-2}), \ldots,$$
$$2^k(\varpi_{n-1} + \varpi_{n+1}), 2^{k+1}\varpi_n\rangle,$$

where k is the smallest non-negative integer such that $2^k > 2n$ (see Theorem 8.1.2).

(3) An orthogonal irreducible representation of $\mathrm{SO}(3, \mathbb{C})$ landing in $\mathrm{SO}(2n+1, \mathbb{C})$ is spinorial if and only if $n \equiv 3$ or $0 \mod 4$ (see Subsection 8.2.1).

(4) We have $P_{\mathrm{Spin}}(\mathrm{SO}(3, \mathbb{C})) = 4P_{\mathrm{orth}}(\mathfrak{so}(3, \mathbb{C}))$ (clear from the above theorem).

(5) The representation of $\mathrm{SO}(4, \mathbb{C}))$ having highest weight (x, y) is spinorial if and only if $(1/6)(1 + x + y)(2x + x^2 - y - xy + y^2) \equiv 0 \mod 2$ (see Theorem 8.2.2).

(6) We have $P_{\mathrm{Spin}}(\mathrm{SO}(4)) = \langle (4, 0), (0, -4)\rangle = 4P_{\mathrm{orth}}(\mathfrak{so}(4, \mathbb{C}))$ (see Theorem 8.2.3).

(7) The representation of SO$(5,\mathbb{C})$ with the highest weight $\lambda = (\lambda_1, \lambda_2)$ is spinorial if and only if

$$\binom{\lambda_1+3}{4} - \binom{\lambda_2+2}{4} \equiv 0 \mod 2,$$

(see Theorem8.2.7).

(8) $P_{\text{Spin}}(SO(5,\mathbb{C})) = \langle (4,4),(4,-4)\rangle = 8\ P_{\text{orth}}(\mathfrak{so}(5,\mathbb{C}))$ (see Theorem 8.2.8).

Group	$\mathfrak{g}$	$P(\mathfrak{g})$	w_0	$P_{\text{sd}}(\mathfrak{g})$
PGL(2)	$\mathfrak{sl}_2(\mathbb{C})$	$\langle \varpi_1 = \epsilon_1 \rangle$	(1, 2)	$w_0(\epsilon_1) = \epsilon_2 = -\epsilon_1$ so $\langle \epsilon_1 \rangle$
PGL(4)	$\mathfrak{sl}_4(\mathbb{C})$	$\langle \varpi_1 = \epsilon_1,$ $\varpi_2 = \epsilon_1 + \epsilon_2$ $, \varpi_3 = \epsilon_1 + \epsilon_2 + \epsilon_3\rangle$ $= \langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$	(1, 4)(2, 3)	$\langle \epsilon_1 + \epsilon_2,$ $\epsilon_3 + \epsilon_1 \rangle =$ $\langle \varpi_2, \varpi_1 + \varpi_3 \rangle$
$\text{SO}_4(\mathbb{C})$	$\mathfrak{so}_4(\mathbb{C})$	$\langle \varpi_1 = (1/2)(\epsilon_1 - \epsilon_2)$ $\varpi_2 = (1/2)(\epsilon_1 + \epsilon_2)\rangle$	-I	same as weight lattice
$\text{SO}_5(\mathbb{C})$	$\mathfrak{so}_5(\mathbb{C})$	$\langle \varpi_1 = \epsilon_1$ $\varpi_2 = (1/2)(\epsilon_1 + \epsilon_2)\rangle$	-I	same as weight lattice

$P_{\text{orth}}(\mathfrak{g})$	P'_{Spin}	P_{Spin}
$\langle 2\epsilon_1 \rangle$	$2n\epsilon_1$ $n \equiv 0$ or $3 \mod 4$	$\langle 8\varpi_1 \rangle$
same as self-dual lattice	See the expression above	$\langle 8(\varpi_1 + \varpi_3),$ $8\varpi_2 \rangle$
$a\epsilon_1 + b\epsilon_2$	$\{x \cdot \epsilon_1 + y \cdot \epsilon_2 \mid$ $(1/6)(1+x+y)(2x+x^2-y-xy+y^2) \equiv 0 \mod 2\}$	$\langle 4\epsilon_1,$ $4\epsilon_2 \rangle$
same as weight lattice	$\{(\lambda_1, \lambda_2) \mid \binom{\lambda_1+3}{4} - \binom{\lambda_2+2}{4} \equiv 0 \mod 2\}$	$\langle 8\varpi_1,$ $8\varpi_2 \rangle$

relation
$P_{\text{Spin}}(G) = 4P_{\text{orth}}(\mathfrak{g})$
$P_{\text{Spin}}(G) = 8P_{\text{orth}}(\mathfrak{g})$
$P_{\text{Spin}}(G) = 4P_{\text{orth}}(\mathfrak{g})$
$P_{\text{Spin}}(G) = 8P_{\text{orth}}(\mathfrak{g})$

Here the notations are as follows.

Group = The group under the consideration.

$\mathfrak{g}$ = The Lie algebra of the group.

P($\mathfrak{g}$) = The weight lattice corresponding to $\mathfrak{g}$.

w_0 = The longest element of the Weyl group.

$P_{\text{sd}}(\mathfrak{g})$ = The lattice of highest weights corresponding to self-dual irreducible representation of $\mathfrak{g}$.

$P_{\text{orth}}(\mathfrak{g})$ = The lattice of highest weights corresponding to orthogonal irreducible representations of $\mathfrak{g}$.

$P'_{\text{Spin}}(G) = \{\lambda \in P_{\text{orth}} \mid F_\nu(\lambda) \equiv 0 \mod 2 \forall \nu\}$

$P_{\text{Spin}}(G) = \{p \in P_{\text{orth}}(G) \mid \lambda \in P'_{\text{Spin}}(G) \Leftrightarrow \lambda + p \in P'_{\text{Spin}}(G)\}$

relation = The relation between $P_{\text{Spin}}(G)$ and $P_{\text{orth}}(\mathfrak{g})$.

Chapter 12

Appendix

12.1 Combinatorial Lemmas

Lemma 12.1.1. *(Lucas Theorem)(see [Fine(1947)]) Let* $n = a_0 + 2a_1 + 2^2a_2 + \cdots + 2^m a_m$ *and* $r = b_0 + 2b_1 + \cdots + 2^m b_m$*, then*

$$\binom{n}{r} \equiv \binom{a_m}{b_m} \cdot \binom{a_{m-1}}{b_{m-1}} \cdot \binom{a_{m-2}}{b_{m-2}} \cdots \binom{a_0}{b_0} \mod 2,$$

where $0 \leq a_i, b_i \leq 1$.

Observe that Lucas theorem is valid also when $0 \leq n < r$.

Lemma 12.1.2. *Fix a natural number* r.

1) Then $\binom{n+2^k}{r} \equiv \binom{n}{r} \mod 2$ *for every natural number* n*, where* k *is the least integer such that* $2^k > r$.

2) If for all n, $\binom{n+p}{r} \equiv \binom{n}{r} \mod 2$*, then* 2^k *divides* p*, where* k *is as above.*

Hence if we fix r*, then the sequence* $\binom{n}{r} \mod 2$ *is periodic in* n *with exact period* 2^k*, where* k *is as above.*

Proof. The first part follows from the Lucas theorem. Let n and r be the natural numbers and let a_i and b_i be as stated in the Lucas theorem. Let m be the largest integer such that $b_m = 1$. Now observe that if we add 2^j to n where j is the smallest integer such that 2^j is strictly greater than r, then it will not affect $a_0, a_1, \ldots, a_m$. That is, the first $(m+1)$ digits in the binary code of n will be exactly the same as the last $(m+1)$ digits of $2^j + n$. Since $b_l = 0$ for $m+1 \leq l \leq k$ adding 2^j will not affect the answer. Hence the answer is periodic in n with period 2^j where j is as above.

Furthermore for proving (2) using Lucas theorem it is easy to see that $\binom{r}{r} \neq \binom{r+2^{j-1}}{r}$ mod 2. Hence 2^{j-1} can not be a period of the sequence $\binom{n}{r}$ mod 2. Hence 2^j is the exact period of the sequence $\binom{n}{r}$ mod 2. □

Lemma 12.1.3. *Let $a_1, \ldots, a_n$ be non-negative integers. Then we have*

$$\det \begin{pmatrix} \binom{a_1}{n} & \binom{a_1}{n-1} & \cdots & \binom{a_1}{1} \\ \binom{a_2}{n} & \binom{a_2}{n-1} & \cdots & \binom{a_2}{1} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{a_n}{n} & \binom{a_n}{n-1} & \cdots & \binom{a_n}{1} \end{pmatrix} = \frac{(\prod_{j=1}^{n} a_j)(\prod_{1\leq i<j\leq n}(a_i - a_j))}{\prod_{i=1}^{n} i!}.$$

Proof. The left hand side is a polynomial $D = D(a_1, \ldots, a_n) \in \mathbb{Q}[a_1, \ldots, a_n]$ of degree of $\dfrac{n(n+1)}{2}$. Since $D(a_1, \ldots a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0$ each $a_i | D$. Similarly, if we put $a_i = a_j$ in D we get 0, so $(a_i - a_j)$ divides D. Therefore the product

$$P = \left(\prod_{j=1}^{n} a_j\right) \prod_{1\leq i<j\leq n} (a_i - a_j),$$

divides the determinant. Its degree is also $\dfrac{n(n+1)}{2}$.

Thus $D = c \cdot P$ for some $c \in \mathbb{Q}$. It only remains to calculate the ratio c.

Furthermore we get $D(n, n-1, \ldots, 1) = 1$. On the other side

$$P(n, n-1, \ldots, 1) = n!(n-1)! \cdots 1!,$$

hence

$$c = \frac{1}{\prod_{i=1}^{n} i!}$$

.

□

Lemma 12.1.4. *Let n and k be positive integers and c be any integer then*

$$\sum_{i=0}^{n} i \cdot \binom{i+c}{k} = (n+1)\binom{n+c+1}{k+1} + \binom{c+1}{k+2} - \binom{n+c+2}{k+2}.$$

Proof. The reader may verify this by induction, Otherwise we have the following proof:

$$\begin{aligned}
LHS &= \sum_{i=0}^{n}(i+c+1)\binom{i+c}{k} - \sum_{i=0}^{n}(c+1)\binom{i+c}{k} \\
&= \sum_{i=0}^{n}(k+1)\binom{i+c+1}{k+1} - (c+1)\sum_{i=0}^{n}\binom{i+c}{k} \\
&= (k+1)\sum_{l=c+1}^{n+c+1}\binom{l}{k+1} - (c+1)\sum_{l'=c}^{n+c}\binom{l'}{k} \\
&= (k+1)\left[\sum_{l=0}^{n+c+1}\binom{l}{k+1} - \sum_{l=0}^{c}\binom{l}{k+1}\right] - (c+1)\left[\sum_{l=0}^{n+c}\binom{l}{k} - \sum_{l=0}^{c-1}\binom{l}{k}\right] \\
&= (k+1)\left[\binom{n+c+2}{k+2} - \binom{c+1}{k+2}\right] - (c+1)\left[\binom{n+c+1}{k+1} - \binom{c}{k+1}\right] \\
&\quad \left(\text{since } \sum_{i=0}^{n}\binom{i}{k} = \binom{n+1}{k+1}\right)
\end{aligned}$$

$$= (k+1)\binom{n+c+2}{k+2} - (c+1)\binom{n+c+1}{k+1} - [(k+1)-(k+2)]\binom{c+1}{k+2}$$

$$\left(\text{since } (c+1)\binom{c}{k+1} = (k+2)\binom{c+1}{k+2}\right)$$

$$= [(k+1)-(k+2)]\binom{n+c+2}{k+2} + \binom{c+1}{k+2} + (n+1)\binom{n+c+1}{k+1}$$

$$\left(\text{since } -(c+1)\binom{n+c+1}{k+1} = -(n+c+2)\binom{n+c+1}{k+1} + (n+1)\binom{n+c+1}{k+1}\right)$$

$$\left(\text{and since } (n+c+2)\binom{n+c+1}{k+1} = (k+2)\binom{n+c+2}{k+2}\right)$$

$$= RHS.$$

□

Lemma 12.1.5. *Let t be a natural number then*

$$\binom{t+4}{4} \equiv \binom{t}{4} + 1 \mod 2.$$

Proof. We apply Chu-Vandermonde's identity (see page 156 Ex. 25 of [Brualdi(1977)]) which is

$$\binom{x+y}{r} = \sum_{k=0}^{r} \binom{x}{k}\binom{y}{r-k}.$$

Here x, y, r are positive integers.

On applying this we get

$$\binom{t+4}{4} \mod 2 = \binom{t}{0}\binom{4}{4} + \binom{t}{1}\binom{4}{3} + \binom{t}{2}\binom{4}{2} + \binom{t}{3}\binom{4}{1} + \binom{t}{4}\binom{4}{0} \mod 2.$$

Since $\binom{4}{3}, \binom{4}{2}, \binom{4}{1}$ are even, the lemma follows. $\square$

Bibliography

[Serre(2001)] J.-P. Serre, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics (Springer-Verlag, Berlin, 2001) pp. x+74, translated from the French by G. A. Jones, Reprint of the 1987 edition.

[Prasad and Ramakrishnan(1995)] D. Prasad and D. Ramakrishnan, Proc. Indian Acad. Sci. Math. Sci. **105**, 259 (1995).

[Goodman and Wallach(2009)] R. Goodman and N. R. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, Vol. 255 (Springer, Dordrecht, 2009) pp. xx+716.

[Vinberg(1994)] E. B. Vinberg, ed., *Lie groups and Lie algebras, III*, Encyclopaedia of Mathematical Sciences, Vol. 41 (Springer-Verlag, Berlin, 1994) pp. iv+248, structure of Lie groups and Lie algebras, A translation of ıt Current problems in mathematics. Fundamental directions. Vol. 41 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990 [MR1056485 (91b:22001)], Translation by V. Minachin [V. V. Minakhin], Translation edited by A. L. Onishchik and È. B. Vinberg.

[Fulton and Harris(1991)] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, Vol. 129 (Springer-Verlag, New York, 1991) pp. xvi+551, a first course, Readings in Mathematics.

[Prasad(2015)] A. Prasad, *Representation theory*, Cambridge Studies in Advanced Mathematics, Vol. 147 (Cambridge University Press, Delhi, 2015) pp. xii+191, a combinatorial viewpoint.

[Jacobson(1980)] N. Jacobson, San Francisco (1980).

[Bourbaki(2005)] N. Bourbaki, *Lie groups and Lie algebras. Chapters 7–9*, Elements of Mathematics (Berlin) (Springer-Verlag, Berlin, 2005) pp. xii+434, translated from the 1975 and 1982 French originals by Andrew Pressley.

[Helgason(1978)] S. Helgason, "Differential geometry, lie groups and symmetric spaces (1978)," (1978).

[Bröcker and tom Dieck(2013)] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, Vol. 98 (Springer-Verlag, New York, 2013) pp. x+313, translated from the German manuscript, Corrected reprint of the 1985 translation.

[Bourbaki(2002)] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin) (Springer-Verlag, Berlin, 2002) pp. xii+300, translated from the 1968 French original by Andrew Pressley.

[Humphreys(1990)] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, Vol. 29 (Cambridge University Press, Cambridge, 1990) pp. xii+204.

[Kirillov(2009)] A. Kirillov, Jr., *An introduction to Lie groups and Lie algebras*, Cambridge Studies in Advanced Mathematics, Vol. 113 (Cambridge University Press, Cambridge, 2009) pp. xii+222.

[Fine(1947)] N. J. Fine, The American Mathematical Monthly **54**, 589 (1947).

[Brualdi(1977)] R. A. Brualdi, *Introductory combinatorics* (Pearson Education India, 1977).