Superconformal Field Theories: A Momentum Space Voyage

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled *Superconformal Field Theories: A Momentum Space Voyage* towards the partial fulfillment of the BS-MS dual degree program at the Indian Institute of Science Education and Research, Pune, represents study/work carried out by Shivang Yadav at Indian Institute of Science Education and Research under the supervision of Sachin Jain, Department of Physics, Indian Institute of Science Education and Research, Pune, during the academic year 2023-2024.

Sachin Jain

Sachin Jain

Committee:

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Well, apparently, rock bottom has a basement.

Declaration

I hereby declare that the matter embodied in the report entitled *Superconformal Field Theories: A Momentum Space Voyage* are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Sachin Jain and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

Shivang

Shivang Yadav

List of Publications

This thesis is based on two pre-prints written during the period of MS research project. All permissions are obtained from the coauthors to use the material from pre-prints.

[1] Dhruva K.S, D.Mazumdar, and S.Yadav, "*n-point functions in Conformal Quantum Mechanics: A Momentum Space Odyssey*", arxiv: 2402.16947 [hep-th, math-ph].

[2] S.Jain, Dhruva K.S, D.Mazumdar, and S.Yadav, "*A Foray on* $SCFT_3$ *via Super Spinor Helicity and Grassmann Twistor Variables*", arxiv: 2312.03059 [hep-th].

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Abstract

This thesis aims to find correlators in conformal and superconformal field theories in momentum space. In this direction, we studied the implication of conformal invariance in Conformal Quantum Mechanics. This analysis provides the first instance of a closed form for generic momentum space conformal correlators in contrast to higher dimensions. We found that n-point functions in 1 dimensions can be mapped to Lauricella Functions, E_A, with $n-3$ undetermined parameters. We test our expressions against free theory and DFF model correlators, finding an exact agreement. Further, we show that multiple solutions to the momentum space conformal ward identity can be attributed to the Fourier transforms of the various possible time orderings. We extend our analysis to theories with $\mathcal{N}=1,2$ supersymmetry. Moreover, we develop the first momentum superspace formalism for $\mathcal{N}=1,2$ superconformal field theories in 3 dimensions. This formalism comprises new variables, "Super Spinor Helicity" and "Grassmann Twistor Variables". Using this formalism, we first compute all three point correlation functions involving conserved super-currents with arbitrary spins in $\mathcal{N}=1,2$ theories. We discover interesting double copy relations in $\mathcal{N}=1$ super-correlators. Also, we discovered super double copy relations that take us from $\mathcal{N}=1$ to $\mathcal{N}=2$ super-correlators.

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Part I

Prelude

Chapter 1

Introduction

Sorry! I don't want any adventures, thank you.
Not Today. Good morning! But please come to
tea – any time you like! Why not tomorrow?
Good bye!

Bilbo Baggins, An Unexpected Journey, Hobbit

Quantum Field Theories sometimes enjoy scale invariance other than just having Lorentz invariance, and, in most cases, they further have conformal invariance; these theories are called Conformal Field Theory (CFT). They provide a gateway to understanding properties like phase transition and critical exponents, and their applications range from string theory to condensed matter physics. The conformal symmetry has been successful in constraining functional forms of two and three-point functions, which general QFTs can't do. Since the results are based only on the symmetry of space and time, they give us non-perturbative results. The non-perturbative nature finds applications in many fields, and constraining higher point functions is one of the most important challenges. Hence, there has been a lot of research in this direction. In two dimensions, this program was initiated in the seminal paper by Belavin, Polyakov, and Zamolodchikov in 1984 [1]. In dimensions greater than two, there has also been significant progress since the initiating work [2] in 2008, see [3, 4] for a review of recent developments. There has been a lot of work on understanding CFTs in position and Mellin space, and only a decade or so ago, the momentum space approach began [5]. It is quite advantageous to formulate such theories in momentum space, as it has led to various interesting results, such as making connections to flat space scattering amplitudes in one higher dimension [6, 7] through AdS_{d+1}/CFT_d correspondence, discovering double copy structures at the level of correlation functions [5, 8–24] and even making a connection with early universe cosmology [6, 25–29] and AdS amplitude [30–35]. Despite all this success, there has been a difficulty in setting up the conformal bootstrap intrinsically in momentum space. Although momentum space conformal blocks [36–38] and even a simplex representation for generic scalar

n-point functions [39–41] have been obtained, they are quite complicated to use and obtain interesting results. Further, a special class of CFTs also possesses supersymmetry, i.e., superconformal field theories (SCFTs), see for instance [42, 43]. These theories are extremely interesting as they provide the supersymmetric extension to CFTs. Supersymmetry provides extra constraints on the correlators and can be used to find connections between different spinning correlators. While there has been a lot of development in the Fourier space approach to non-supersymmetric CFTs over the past decade, its supersymmetric counterparts have mostly been left untouched.

Some of the important approaches to constrain observables are to find variables that have fewer ambiguities (e.g., spinor helicity) or change the dimensions. For the first part of this thesis, we will focus on the latter way, i.e., on the one-dimensional case, Conformal Quantum Mechanics. Compared to its higher dimensional counterparts, it provides fewer technical hurdles to obtain exact analytical results, such as closed-form expressions of four and higher point functions. Historically, the study of conformal quantum mechanics dates back to 1976, with the work of de Alfaro, Fubini, and Furlan (DFF)[44] who analyzed a particular quantum mechanical model (A particle moving in an inverse square potential) that possesses conformal invariance. Almost five decades after this paper, there has been a great deal of development with applications ranging from the connection of CQM to M2 branes, black holes and even to molecular physics [45–62]. A major motivation to study and bootstrap CQM stems from the AdS/CFT correspondence [63]. $\mathrm{AdS}_2/\mathrm{CFT}_1$ is often referred to as the runt of the correspondence as CFT_1 does not possess a local stress tensor that generates conformal transformations which is in sharp contrast to its higher dimensional counterparts. In the context of the DFF model, this correspondence and its subtleties were first discussed and explored in [64]. Another extremely important case is that of the SYK models and their holographic bulk duals [65–67]. To investigate such dualities, having a firm foothold on the conformal theory side of things is essential. Most of the analysis of CQM has taken place in the real space aka, the time domain. This is where the one-dimensional case can serve as a toy model, which one can then attempt to emulate in higher dimensions. Due to the decreased technical difficulties, one can probe deeper into the structure of momentum space CFT and obtain illuminating results. For the second part of this thesis, we will be focusing on supersymmetric CFTs in $3d$. This development also has an application in Supersymmetric Chern-Simons matter such as [68–70] and $\mathcal{N} = 6$ ABJM theory[71] in $3d$ which is holographically dual to M-theory on $AdS_4 \times S^7$. However, such pursuits first require the development of momentum superspace and most of the literature has been in the arena of position superspace [72–79]. All things considered, in this thesis, you will find n-point functions for $1d$ CFT in momentum space, the explanation of multiple solutions in momentum space, momentum space conformal blocks, and a new formalism for finding all spinning three point function in $3d$ SCFT[80, 81] .

Outline:

In chapter 2, we will discuss CFT in position space in detail, Ward Identities, and how to find correlators. In chapter 3, we will discuss the $\mathfrak{sl}(2, \mathbb{R})$ symmetry of conformal quantum mechanics and its implications on correlation functions of primary operators. In section 3.1, we follow up by solving the constraints due to the conformal invariance in momentum space. We obtain the general form of three-point and four-point functions, which we then generalize to arbitrary n-point functions. The computation of momentum space conformal partial waves then follows it. Further, it is shown that the presence of multiple solutions to the conformal Ward identities is attributed to the correlator's various possible time orderings. Moreover, explicit checks of our formulae by free theory and DFF model computations are provided. In chapter 4 the analysis is extended to $\mathcal{N} = 1, 2$ superconformal quantum mechanics. In chapter 5, the formalism for superconformal field theory in $3d$ is given. In section 5.1 , momentum superspace formalism for $\mathcal{N} = 1$ SCFTs is provided. After working out a three-point super-correlator as an example, it is shown that rather than working in ordinary momentum superspace variables, working with super spinor helicity variables is quite advantageous. A dramatic simplification occurs if *Grassmann twistor space* is used, forming the base of this formalism. In section 5.2, we present our results for all two and three-point correlators with arbitrary (half) integer spin insertions in $\mathcal{N} = 1$ SCFTs. In section 6, this formalism is extended to theories with four supercharges as well as to $\mathcal{N} = 2$ super-correlators via a super double copy construction of their $\mathcal{N} = 1$ counterparts.

Chapter 2

Preliminaries

Elementary, my dear Watson

Sherlock Holmes on this chapter

Before delving into the world of correlators in momentum space, I would like to present the basics of Conformal Field theory in position space. The machinery will be used in momentum space by Fourier transforming the generators and then finding the correlators. You might ask why we don't Fourier transform the correlators from position space, but it turns out extremely challenging. Hence, we must start in momentum space from the first step. I assume some basic knowledge of QFT and Path Integrals for this chapter. All the works in this thesis will be in Euclidean d dimensions. Now, let us start with "Conformal Field Theory".

2.1 Conformal Field Theory

Infinitesimal Transformation of any sort for position coordinate x^μ and fields $\Phi(x)$ is given as

$$x \to x' \tag{2.1.1}$$

$$\Phi(x) \to \Phi'(x') = \mathcal{F}(\Phi(x)) \tag{2.1.2}$$

where the new coordinates and new field configuration is given as

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \tag{2.1.3}$$

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{2.1.4}$$

Now, if you keep the coordinates fixed, then we get the representation of generators of transformation G_a,

$$\delta_\omega \Phi(x) \equiv \Phi'(x) - \Phi(x) = -i\omega_a G_a \Phi(x) \tag{2.1.5}$$

Thus, comparing both equations, we will get the generator action on any field,

$$iG_a\Phi = \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\Phi - \frac{\delta\mathcal{F}}{\delta\omega_a} \tag{2.1.6}$$

I would not prove it, but the action of these transformations gives some transformation in the action $S = \int d^dx\mathcal{L}(\Phi(x), \partial_\mu\Phi(x))$, which is given as $\delta S = S' - S$, given as,

$$\delta S = \int d^dx\,\partial_\mu j^\mu_a\,\omega_a \tag{2.1.7}$$

where,

$$j^\mu_a = \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta^\mu_\nu\mathcal{L}\right\}\frac{\delta x^\nu}{\delta\omega_a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\mathcal{F}}{\delta\omega_a} \tag{2.1.8}$$

This is the conserved current for any given symmetry transformation.

2.1.1 Transformation of the Correlation Functions

Consider a theory involving a collection of fields Φ with an action $S[\Phi]$ invariant under the transformation given above. Now, using path integral, any correlation function is given as the following:

$$\langle\Phi(x_1)\ldots\Phi(x_n)\rangle = \frac{1}{Z}\int[d\Phi]\Phi(x_1)\ldots\Phi(x_n)e^{-S[\Phi]} \tag{2.1.9}$$

The consequence of symmetry of the action and the measure of the correlation functions may also be expressed via the star of this thesis, "**Ward Identities**," which we will derive now.
Any infinitesimal transformation may be written in terms of the generators as

$$\Phi'(x) = \Phi(x) - i\omega_a G_a\Phi(x) \tag{2.1.10}$$

where ω_a are the infinitesimal parameters. Let us denote $X = \Phi(x_1)\ldots\Phi(x_n)$ and $\delta_\omega X$ by its infinitesimal transformation,

$$\langle X'\rangle = \frac{1}{Z}\int[d\Phi'](X + \delta X)e^{-\{S[\Phi] + \int d^dx\,\partial_\mu j^\mu_a\,\omega_a\}} \tag{2.1.11}$$

Ignoring *anomalous* behaviour of the measure, expanding this equation upto $\mathcal{O}(\omega_a)$, we get the following:

$$\langle\delta X\rangle = \int d^dx\,\partial_\mu\langle j^\mu_a X\rangle\,\omega_a(x) \tag{2.1.12}$$

Though, if (2.1.10) done explicitly on X, we get

$$\delta X = -i\sum_{i=1}^{n}\left(\Phi(x_1)\ldots G_a\Phi(x_i)\ldots\Phi(x_n)\right)\omega_a(x_i)$$

$$= -i\sum_{i=1}^{n}\int d^dx\,\omega_a(x)\delta(x-x_i)\left\{\Phi(x_1)\dots G_a\Phi(x_i)\dots\Phi(x_n)\right\}\delta(x-x_i) \tag{2.1.13}$$

Using this expression in (2.1.12), we obtain the **Ward Identity**,

$$\partial_\mu\langle j_a^\mu(x)\Phi(x_1)\dots\Phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\Phi(x_1)\dots G_a\Phi(x_n)\dots\Phi(x_n)\rangle \tag{2.1.14}$$

Now, we integrate the (2.1.14), over the spacetime which includes all the points x_i, then the (l.h.s) becomes

$$\int_\Sigma dS_\mu\langle j_a^\mu\Phi(x_1)\dots\Phi(x_n)\rangle \tag{2.1.15}$$

If we take the surface to infinity, then this integral goes to 0, hence the (l.h.s) becomes zero. This then implies that the transformation in the correlator given by the (r.h.s) is 0, which is given as following:

$$\delta_\omega\langle\Phi(x_1)\dots\Phi(x_n)\rangle \equiv -i\omega_a\sum_{i=1}^{n}\langle\Phi(x_1)\dots G_a\Phi(x_i)\dots\Phi(x_n)\rangle = 0 \tag{2.1.16}$$

Let us say we have any field $\Phi(x)$ which transform under the action of charge Q_a with an infinitesimal parameter ω_a. Since, the fields are quantum, they get transformed as $\Phi(x) \to U(\omega)\Phi(x)U^\dagger(\omega)$, where $U(\omega) = e^{i\omega_a Q_a}$. Then, the action with the generators on the fields with their respective representation G_a is given as,

$$e^{i\omega_a Q_a}\Phi(x)e^{-i\omega_a Q_a} = e^{iG_a\omega_a}\Phi(x) \tag{2.1.17}$$

$$[Q_a, \Phi(x)] = -iG_a\Phi(x) \tag{2.1.18}$$

upto the order $\mathcal{O}(\omega_a)$. Thus, the equation (2.1.16), becomes

$$\sum_{i=1}^{n}\langle\Phi(x_1)\dots[Q_{ia},\Phi(x_i)]\dots\Phi(x_n)\rangle = 0 \tag{2.1.19}$$

For general quantum field theory, we would have generators $P_\mu, M_{\mu\nu}$, but there would be more in a conformal group, which I present below.

2.1.2 Conformal group in d dimensions

For dimension d, we have metric $g_{\mu\nu}$. Conformal transformations of the coordinates is an invertible mapping $x \to x'$, which leaves the metric tensor invariant up to a scale:

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x) \tag{2.1.20}$$

Poincare group is a subgroup where $\Lambda(x) \equiv 1$. Using these definitions we can have only certain transformations

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu \tag{2.1.21}$$

where,

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{2.1.22}$$

These correspond to the following transformations on the coordinates

$$x'^\mu = x^\mu + a^\mu \qquad \text{(Translation)} \tag{2.1.23}$$

$$x'^\mu = \alpha x^\mu \qquad \text{(Dilation)} \tag{2.1.24}$$

$$x'^\mu = M^\mu_\nu x^\nu \qquad \text{(Rigid Rotation)} \tag{2.1.25}$$

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b.x + b^2x^2} \qquad \text{(SCT)} \tag{2.1.26}$$

Action of the fields on the Fields $\Phi(x)$ is given as

$$[P_\mu, \Phi(x)] = -i\partial_\mu \Phi(x) \tag{2.1.27}$$

$$[M_{\mu\nu}, \Phi(x)] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x) \tag{2.1.28}$$

$$[D, \Phi(x)] = (-ix^\mu\partial_\mu + \Delta)\Phi(x) \tag{2.1.29}$$

$$[K_\mu, \Phi(x)] = (ix^2\partial_\mu + 2x_\mu\Delta - 2ix_\mu x^\nu\partial_\nu - x^\nu S_{\mu\nu})\Phi(x) \tag{2.1.30}$$

This group is isomorphic to $SO(d+1,1)$ for d dimension with $\frac{(d+2)(d+1)}{2}$ parameters.

2.1.3 Correlators in Conformal Field Theory in d dimensions

Any Field $\Phi_\Delta(x)$ is characterized by the scaling dimension Δ, which can be obtained by the action of D, and they can be characterized as "Primary" or "Descendants."
A "Primary" operator is defined as

$$[D, \Phi_\Delta(0)] = \Delta\Phi_\Delta(0) \tag{2.1.31}$$

$$[K_\mu, \Phi_\Delta(0)] = 0 \tag{2.1.32}$$

All the "Descendant" fields for this primary field are given by the action of the Momentum generator

$$\Phi_\Delta(0) \to P_{\mu_1}P_{\mu_2}\dots P_{\mu_n}\Phi_\Delta(0) \text{ (Descendants)} \tag{2.1.33}$$

$$\Delta \to \Delta + n \tag{2.1.34}$$

The action of the generators constrains the correlators of the primary field to be of some particular functional forms. The two, three, and four-point functions are given as

$$\langle\phi_{\Delta_1}(x_1)\phi_{\Delta_2}(x_2)\rangle = \frac{C_{12}\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \tag{2.1.35}$$

$$\langle\phi_{\Delta_1}(x_1)\phi_{\Delta_2}(x_2)\phi_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1 - x_3|^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.1.36}$$

There will be only correlators from now on. So, better appreciate field theory right now.

$$\langle \phi_{\Delta_1}(x_1)\phi_{\Delta_2}(x_2)\phi_{\Delta_3}(x_3)\phi_{\Delta_4}(x_4)\rangle = \prod_{i<j}^{4} x_{ij}^{\Delta_t/3-\Delta_i-\Delta_j}\, f(u,v) \tag{2.1.37}$$

where, C_{12}, C_{123} are constants, $\Delta_t = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$, and $u = \dfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, , $v = \dfrac{x_{12}^2 x_{34}^2}{x_{23}^2 x_{14}^2}$.
It is a wonderful achievement that we can find all the three-point functions in any CFT, independent of the theory, but this wonder gets fuzzy after three-point functions.

2.2 Summary

In this chapter, we went through the basics of Conformal field theory in d dimensions. We saw how to find the Ward Identities, what a conformal group is, and how to find the correlators until the three-point function. There has been a lot of effort in finding ways to constrain functional form for a four-point function. We will see how to find a closed form for higher point functions in a CFT in the following chapter, how to extend the analysis to supersymmetry, and a new formalism for momentum superspace.

Part II

Conformal Field Theory in $1d$

Chapter 3

Conformal Field Theory in $1d$

Do not meddle in the affairs of Wizards, for they are subtle and quick to anger.

Gildor on CFT, The Fellowship of the Ring

This chapter provides a momentum space analysis of Conformal Field Theory in $1d$. It is also called Conformal Quantum Mechanics. We saw that higher dimensions are plagued with difficulties and we can circumvent through those difficulties by working in $1d$. It provides a clear pathway to understanding many properties of CFTs and provides possible solutions for higher dimensions problems. We will see that we can not just constrain the four-point, but miraculously, all the n-point functions, we will see an explanation of multiple solutions to momentum space conformal ward identities. The algebra of the conformal quantum mechanics and then the action of these generators on the primary operator on both time and momentum space representations is presented in this section. The conformal algebra is $\mathfrak{so}(2,1)$, which is in accordance with the fact that the conformal algebra in d dimensions is $\mathfrak{so}(d+1,1)$ and hence $\mathfrak{so}(2,1)$ in $d=1$. It has three generators, which we denote as H, D, and K, respectively. H is the Hamiltonian, D is the dilatation operator and K is the generator of special conformal transformations. The algebra that they obey is given by,

$$[D,H]=-iH, \quad [D,K]=iK, \quad [K,H]=-2iD. \tag{3.0.1}$$

The action of these generators on primary operators is given by,

$$\begin{aligned}
[H,\mathcal{O}_\Delta(t)] &= i\frac{d}{dt}\mathcal{O}_\Delta(t), \\
[D,\mathcal{O}_\Delta(t)] &= i\left(t\frac{d}{dt}+\Delta\right)\mathcal{O}_\Delta(t), \\
[K,\mathcal{O}_\Delta(t)] &= i\left(t^2\frac{d}{dt}+2t\Delta\right)\mathcal{O}_\Delta(t).
\end{aligned} \tag{3.0.2}$$

The Fourier transformed expressions gives their momentum space counterparts:

$$\begin{aligned}
&[H, \mathcal{O}_\Delta(\omega)] = \omega \mathcal{O}_\Delta(\omega), \\
&[D, \mathcal{O}_\Delta(\omega)] = -i\left(\omega \frac{d}{d\omega} + (1-\Delta)\right) \mathcal{O}_\Delta(\omega), \\
&[K, \mathcal{O}_\Delta(\omega)] = -\left(\omega \frac{d^2}{d\omega^2} + 2(1-\Delta)\frac{d}{d\omega}\right) \mathcal{O}_\Delta(\omega).
\end{aligned} \tag{3.0.3}$$

We now consider the implications of this $\mathfrak{so}(2,1)$ invariance on correlation functions of primary operators. Consider an arbitrary n-point function of primary operators,

$$f_n(t_1, \cdots, t_n) = \langle O_{\Delta_1}(t_1) \cdots O_{\Delta_n}(t_n) \rangle. \tag{3.0.4}$$

The invariance of this correlator under the simultaneous action of H, D or K on all operators (2.1.16), is given by,

$$\langle [\mathcal{L}, O_{\Delta_1}(t_1)] \cdots O_{\Delta_n}(t_n) \rangle + \cdots \langle O_{\Delta_1}(t_1) \cdots [\mathcal{L}, O_{\Delta_n}(t_n)] \rangle = 0 \quad , \quad \mathcal{L} \in \{H, D, K\}. \tag{3.0.5}$$

Using the commutators provided in (3.0.2) in (3.0.5) we obtain the following Ward identities, due to H, D and K respectively:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\partial}{\partial t_i} f_n(t_1, \cdots, t_n) &= 0, \\
\sum_{i=1}^{n} \left(t_i \frac{\partial}{\partial t_i} + \Delta_i\right) f_n(t_1, \cdots, t_n) &= 0, \\
\sum_{i=1}^{n} \left(t_i^2 \frac{\partial}{\partial t_i} + 2 t_i \Delta_i\right) f_n(t_1, \cdots, t_n) &= 0.
\end{aligned} \tag{3.0.6}$$

The Fourier space counterparts of above equations are readily obtained and are given by,

$$\sum_{i=1}^{n} \omega_i f_n(\omega_1, \cdots, \omega_n) = 0, \tag{3.0.7}$$

$$\sum_{i=1}^{n} \left(\omega_i \frac{\partial}{\partial \omega_i} + (1-\Delta_i)\right) f_n(\omega_1, \cdots, \omega_n) = 0, \tag{3.0.8}$$

$$\sum_{i=1}^{n} \left(\omega_i \frac{\partial^2}{\partial \omega_i^2} + 2(1-\Delta_i)\frac{\partial}{\partial \omega_i}\right) f_n(\omega_1, \cdots, \omega_n) = 0. \tag{3.0.9}$$

With the conformal ward identities, (3.0.6), and (3.0.7), (3.0.8), (3.0.9) in hand, we now proceed to solve them in the next section.

3.1 Correlators in Conformal Quantum Mechanics

Obtaining closed-form expressions in general dimensions for momentum space four and higher point functions has been a daunting task. So far, a simplex integral representation has been achieved for arbitrary scalar n-point functions [39–41]. In this section, we shall see that we obtain a stronger result in one dimension, i.e., find closed-form analytic expressions for not only four-point functions but arbitrary n-point ones. Further, we shall compute conformal partial waves, which will be tested against several examples. Moreover, the existence of multiple solutions to the momentum space Conformal Ward identities is explained as arising due to the in-equivalent Fourier transforms of time domain correlator. The analysis is checked by performing and comparing with examples in the free bosonic, free fermionic and DFF model.

Let us first see the general solution to two, three, and four point functions to the time domain Ward identities. By solving the differential equations (3.0.6) we obtain,

$$\begin{aligned}
\langle O_{\Delta_1}(t_1)O_{\Delta_2}(t_2)\rangle &= \frac{c_{12}\delta_{\Delta_1,\Delta_2}}{|t_1-t_2|^{2\Delta}},\\
\langle O_{\Delta_1}(t_1)O_{\Delta_2}(t_2)O_{\Delta_3}(t_3)\rangle &= \frac{f_{123}}{|t_1-t_2|^{\Delta_1+\Delta_2-\Delta_3}|t_2-t_3|^{\Delta_2+\Delta_3-\Delta_1}|t_1-t_3|^{\Delta_1+\Delta_3-\Delta_2}},\\
\langle O_{\Delta_1}(t_1)O_{\Delta_2}(t_2)O_{\Delta_3}(t_3)O_{\Delta_4}(t_4)\rangle &= \prod_{1\leq i\leq j\leq 4}(|t_i-t_j|)^{\frac{\Delta_1+\Delta_2+\Delta_3+\Delta_4}{3}-\Delta_i-\Delta_j}\,G(\chi),\\
\langle O_{\Delta_1}(t_1)\cdots O_{\Delta_n}(t_n)\rangle &= \prod_{1\leq i\leq j\leq n}(|t_i-t_j|)^{2\alpha_{ij}}G_n(\chi_1,\cdots\chi_{n-3}),
\end{aligned}\tag{3.1.1}$$

where, $\chi=\dfrac{|t_1-t_2||t_3-t_4|}{|t_1-t_3||t_2-t_4|}$ is the four point cross ratio, $\chi_1,\cdots\chi_{n-3}$ are the cross ratios for n-point functions which take the form $\dfrac{|t_i-t_j||t_k-t_l|}{|t_i-t_k||t_j-t_l|}$, $i,j,k,l\in\{1,\cdots n\}$ and the α_{ij} satisfy $\Delta_i=-\sum_{j=1}^{n}\alpha_{ij}$ $,i\in\{1,\cdots,n\}$. An artefact of the fact that we are in one dimension is that we have only a single (real) cross ratio at the level of four points in contrast to the case in higher dimensions. For n -point functions, we have only $n-3$ cross ratios in contrast to the $\frac{n(n-3)}{2}$ cross ratios in higher dimensions.

Let us now move on to solve the momentum space conformal Ward identities (3.0.7), (3.0.8), (3.0.9) to obtain the analog of (3.1.1).

3.1.1 Two Point Functions

The translation ward identity (3.0.7) gives the following form for the two point correlator:

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)\rangle=\delta(\omega_1+\omega_2)\tilde{G}(\omega_1).\tag{3.1.2}$$

The dilatation ward identity (3.0.8) yields,

$$\left(\omega_1\frac{\partial}{\partial\omega_1}+\omega_2\frac{\partial}{\partial\omega_2}+2-\Delta_1-\Delta_2\right)\left(\delta(\omega_1+\omega_2)\tilde{G}(\omega_1)\right)=0.\tag{3.1.3}$$

We integrate both sides of this equation with respect to ω_2 to obtain an equation to solve for $\tilde{G}(\omega_1)$:

$$\omega_1\frac{d\tilde{G}(\omega_1)}{d\omega_1}=(\Delta_1+\Delta_2-1)\tilde{G}(\omega_1)\implies\tilde{G}(\omega_1)=\tilde{C}_{12}\omega_1^{\Delta_1+\Delta_2-1}.\tag{3.1.4}$$

Finally, the special conformal ward identity (3.0.9) reads,

$$\left(\omega_1\frac{\partial^2}{\partial\omega_1^2}+\omega_2\frac{\partial^2}{\partial\omega_2^2}+2(1-\Delta_1)\frac{\partial}{\partial\omega_1}+2(1-\Delta_2)\frac{\partial}{\partial\omega_2}\right)\big(\delta(\omega_1+\omega_2)\tilde{G}(\omega_1)\big)=0$$
$$\implies (\Delta_2-\Delta_1)(\Delta_1+\Delta_2-1)\tilde{C}_{12}=0. \tag{3.1.5}$$

Therefore, we see that for operators with generic scaling dimensions, we require $\Delta_1=\Delta_2$ thus yielding,

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)\rangle=\tilde{C}_{12}\delta_{\Delta_1,\Delta_2}\omega_1^{2\Delta_1-1}\delta(\omega_1+\omega_2), \tag{3.1.6}$$

which is indeed the Fourier transform of the time domain two point function provided in (3.1.1) as can be easily verified. Let us now move on to the three point level.

3.1.2 Three Point Functions

Translation invariance (3.0.7) constrains the three point function to take the following form:

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)\rangle=\delta(\omega_1+\omega_2+\omega_3)\tilde{G}(\omega_1,\omega_2). \tag{3.1.7}$$

The dilatation ward identity (3.0.8) yields,

$$\left(2-\Delta_t+\omega_1\frac{\partial}{\partial\omega_1}+\omega_2\frac{\partial}{\partial\omega_2}\right)\tilde{G}(\omega_1,\omega_2)=0. \tag{3.1.8}$$

where we defined $\Delta_t=\Delta_1+\Delta_2+\Delta_3$.
The special conformal ward identity (3.0.9) demands,

$$\left(\omega_1\frac{\partial^2}{\partial\omega_1^2}+\omega_2\frac{\partial^2}{\partial\omega_2^2}+2(1-\Delta_1)\frac{\partial}{\partial\omega_1}+2(1-\Delta_2)\frac{\partial}{\partial\omega_2}\right)\tilde{G}(\omega_1,\omega_2)=0. \tag{3.1.9}$$

Therefore, our task is to simultaneously solve the PDEs in (3.1.8) and (3.1.9) for the function $\tilde{G}(\omega_1,\omega_2)$. The solution to the dilatation ward identity (3.1.8) is easily obtained:

$$\tilde{G}(\omega_1,\omega_2)=\omega_1^{\Delta_t-2}\tilde{g}(x),\quad x=\frac{\omega_2}{\omega_1}. \tag{3.1.10}$$

Plugging (3.1.10) into (3.1.9) results in an ordinary differential equation for $\tilde{g}(x)$:

$$(x^2+x)\frac{d^2\tilde{g}(x)}{dx^2}-2\big(\Delta_2-1+x(\Delta_t-\Delta_1-2)\big)\frac{d\tilde{g}(x)}{dx}+(\Delta_t-2)(\Delta_t-2\Delta_1-1)\tilde{g}(x)=0, \tag{3.1.11}$$

which can be recognized as the hypergeometric differential equation. Solving (3.1.11) gives the most general solution:

$$\tilde{g}(x)=c_1\ {}_2F_1(2-\Delta_t,1+2\Delta_1-\Delta_t,2-2\Delta_2;-x)$$
$$+c_2\,x^{2\Delta_2-1}\ {}_2F_1(1+2\Delta_2-\Delta_t,\Delta_t-2\Delta_3,2\Delta_2;-x),\quad c_1,c_2\in\mathbb{R}. \tag{3.1.12}$$

Therefore, the three point function in ω space is given by,

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)\rangle = \omega_1^{\Delta_1+\Delta_2+\Delta_3-2}\Big[c_1\ {}_2F_1(2-\Delta_t, 1+2\Delta_1-\Delta_t, 2-2\Delta_2; -x) + c_2\, x^{2\Delta_2-1}\ {}_2F_1(1+2\Delta_2-\Delta_t, \Delta_t-2\Delta_3, 2\Delta_2; -x)\Big]\delta(\omega_1+\omega_2+\omega_3). \tag{3.1.13}$$

In contrast to the unique three point correlator in the time domain provided in (3.1.1), we have here, two linearly independent solutions. The interpretation of the same will be made clear in subsection 3.3 where it will be shown that the two solutions correspond to the Fourier transform of the various possible "time orderings" of the time domain three point function.

3.1.3 Four Point Functions

Translation invariance (3.0.7) instills the following form for the correlator:

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4)\tilde{G}(\omega_1,\omega_2,\omega_3). \tag{3.1.14}$$

The dilatation ward identity (3.0.8) yields,

$$\begin{aligned}\Big(\omega_1\frac{\partial}{\partial\omega_1}+\omega_2\frac{\partial}{\partial\omega_2}+\omega_3\frac{\partial}{\partial\omega_3}+(3-\Delta_t)\Big)\tilde{G}(\omega_1,\omega_2,\omega_3) = 0 \\ \implies \tilde{G}(\omega_1,\omega_2,\omega_3) = \omega_1^{\Delta_t-3}\tilde{g}(x,y),\end{aligned} \tag{3.1.15}$$

where $\Delta_t = \Delta_1+\Delta_2+\Delta_3+\Delta_4$ and the ratios $x = \dfrac{\omega_2}{\omega_1}$ and $y = \dfrac{\omega_3}{\omega_1}$.
The ward identity due to special conformal transformations (3.0.9) gives us the following partial differential that the function $\tilde{g}(x,y)$ has to satisfy:

$$\begin{aligned}\Big(x(x+1)\frac{\partial^2}{\partial x^2}+y(y+1)\frac{\partial^2}{\partial y^2}+2xy\frac{\partial^2}{\partial x\partial y}-2\big(\Delta_2-1+x(\Delta_t-\Delta_1-4)\big)\frac{\partial}{\partial x} \\ -2\big(\Delta_3-1+y(\Delta_t-\Delta_1-4)\big)\frac{\partial}{\partial y}+(\Delta_t-3)(\Delta_t-2\Delta_1-2)\Big)\tilde{g}(x,y) = 0.\end{aligned} \tag{3.1.16}$$

Very interestingly, (3.1.16) is an equation obeyed by a unheard-of function, Appell's generalized hypergeometric function F_2 [82]. The system of differential equations that $F_2(a,b_1,b_2,c_1,c_2;x,y)$ obeys is the following:

$$\begin{aligned}x(1-x)\frac{\partial^2 F_2}{\partial x^2}-xy\frac{\partial^2 F_2}{\partial x\partial y}+(c_1-(a+b_1+1)x)\frac{\partial F_2}{\partial x}-b_1y\frac{\partial F_2}{\partial y}-ab_1F_2 = 0, \\ y(1-y)\frac{\partial^2 F_2}{\partial y^2}-xy\frac{\partial^2 F_2}{\partial x\partial y}+(c_2-(a+b_2+1)y)\frac{\partial F_2}{\partial y}-b_2x\frac{\partial F_2}{\partial x}-ab_2F_2 = 0.\end{aligned} \tag{3.1.17}$$

If we add these two equations we obtain,

$$x(1-x)\frac{\partial^2 F_2}{\partial x^2}+(c_1-(a+b_1+b_2+1)x)\frac{\partial F_2}{\partial x}+y(1-y)\frac{\partial^2 F_2}{\partial y^2}+(c_2-(a+b_1+b_2+1)y)\frac{\partial F_2}{\partial y} -2xy\frac{\partial^2 F_2}{\partial x\partial y}-a(b_1+b_2)F_2 = 0. \tag{3.1.18}$$

Let us now perform the following re-labeling and mapping:

$$
\begin{aligned}
&x \to -x, \quad y \to -y, \qquad c_1 = 2(1-\Delta_2), \quad c_2 = 2(1-\Delta_3),\\
&\big\{a = (3-\Delta_t), \quad \sum_{i=1}^{2} b_i = 2 + 2\Delta_1 - \Delta_t\big\} \text{ or } \big\{a = 2 + 2\Delta_1 - \Delta_t, \quad \sum_{i=1}^{2} b_i = 3 - \Delta_t\big\}.
\end{aligned}
\tag{3.1.19}
$$

Thus, (3.1.18) becomes identical to the equation for the four point function (3.1.16)! It can be seen that (3.1.19) fixes $a, c_1, c_2', b_1 + b_2$ in terms of the scaling dimensions of the external operators. The key point here is that the combination $b_1 + b_2$ is fixed in terms of the external operator scaling dimensions but not b_1 and b_2 individually. If we fix b_2 in terms of b_1 using (3.1.19), then b_1 is left completely undetermined. Thus, our solution to the four-point function is,

$$
\begin{aligned}
\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)\rangle = \sum_{b_1} \omega_1^{\Delta_t - 3}\delta(\omega_1+\omega_2+\omega_3+\omega_4)\Big[&k_1 F_2(a, b_1, b_2, c_1, c_2; -x, -y)\\
&+ k_2(-x)^{1-c_1} F_2(a - c_1 + 1, b_1 - c_1 + 1, b_2, 2 - c_1, c_2; -x, -y)\\
&+ k_3(-y)^{1-c_2} F_2(a - c_2 + 1, b_1, b_2 - c_2 + 1, c_1, 2 - c_2; -x, -y)\\
&+ k_4(-x)^{1-c_1}(-y)^{1-c_2} F_2(a - c_1 - c_2 + 2, b_1 - c_1 + 1, b_2 - c_2 + 1, 2 - c_1, 2 - c_2; -x, -y)\Big],
\end{aligned}
\tag{3.1.20}
$$

with the parameters given in (3.1.19). A series expansion for F_2 is provided in appendix B.1. Contrasting this expression with the time domain four-point function in (3.1.1) where we had a single undetermined function of χ (the cross-ratio), here we have an Appell F_2 function[1] with one undetermined parameter b_1 which shows that b_1 is the momentum space counterpart to the cross-ratio χ. It should be noted that, although (3.1.20) solves the conformal ward identities, it may not be the unique solution. The reason being that the Appell F_2 functions are the unique solutions to the two equations in (3.1.18), whereas, our equation (3.1.18) is the sum of these two equations, and obviously, while the Appell F_2 functions are solutions, they may not be the unique ones. However, every explicit computation performed to compute a four-point function is in accordance to (3.1.20). Let us move on to the even more complicated case of five-point functions.

3.1.4 Five Point Functions

By solving the translation and dilatation Ward identities ((3.0.7) and (3.0.8) respectively) we see that the correlator takes the following form:

$$
\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)O_{\Delta_5}(\omega_5)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5)\omega_1^{\Delta_t - 4}\tilde{g}(x, y, z),
\tag{3.1.21}
$$

where,

$$
\Delta_t = \sum_{i=1}^{5} \Delta_i \ , \ x = \frac{\omega_2}{\omega_1}, y = \frac{\omega_3}{\omega_1}, z = \frac{\omega_4}{\omega_1}.
\tag{3.1.22}
$$

[1]Strictly speaking, we have four solutions corresponding to the four solutions of Appell's F_2 differential equation. The existence of multiple solutions rather than a unique one is also what we found at the three point level (3.1.13). As mentioned earlier, we will elaborate on this in subsection 3.3.

The special conformal Ward identity (3.0.9) implies that the function $\tilde{g}(x,y,z)$ satisfies the following differential equation:

$$\Big(x(x+1)\frac{\partial^2}{\partial x^2}+y(y+1)\frac{\partial^2}{\partial y^2}+z(z+1)\frac{\partial^2}{\partial z^2}+2xy\frac{\partial^2}{\partial x\partial y}+2yz\frac{\partial^2}{\partial y\partial z}+2zx\frac{\partial^2}{\partial z\partial x}-2\big(\Delta_2-1+x(\Delta_t-\Delta_1-4)\big)\frac{\partial}{\partial x}$$
$$-2\big(\Delta_3-1+y(\Delta_t-\Delta_1-4)\big)\frac{\partial}{\partial y}-2\big(\Delta_4-1+z(\Delta_t-\Delta_1-4)\big)\frac{\partial}{\partial z}+(\Delta_t-4)(\Delta_t-2\Delta_1-3)\Big)\tilde{g}(x,y,z)=0. \tag{3.1.23}$$

Miraculously, just like we found Appell Functions for the four-point function special conformal Ward identity, we found an even more unfamiliar function known as "Lauricella function" $E_A^{(3)}$ [83].This three variable Lauricella function[2] is the solution to the following system of three partial differential equations:

$$\begin{aligned}&\mathcal{L}_i^{(3)}E_A^{(3)}(a,b_1,b_2,b_3,c_1,c_2,c_3,x_1,x_2,x_3)=0,\;\; i\in\{1,2,3\}\;\;\text{where,}\\ &\mathcal{L}_i^{(3)}=x_i(1-x_i)\frac{\partial^2}{\partial x_i^2}-x_i\sum_{j\neq i}^{3}x_j\frac{\partial^2}{\partial x_i\partial x_j}+(c_i-(a+b_i+1)x_i)\frac{\partial}{\partial x_i}-b_i\sum_{j\neq i}^{3}x_j\frac{\partial}{\partial x_j}.\end{aligned} \tag{3.1.24}$$

This obviously implies,

$$\sum_{i=1}^{3}\mathcal{L}_i^{(3)}E_A^{(3)}(a,b_1,b_2,b_3,c_1,c_2,c_3,x_1,x_2,x_3)=0. \tag{3.1.25}$$

If we choose,

$$\begin{aligned}&x_1=-x,\quad x_2=-y,\quad x_3=-z,\quad c_1=2(1-\Delta_2),\quad c_2=2(1-\Delta_3),\quad c_3=2(1-\Delta_4),\\ &\big\{a=(4-\Delta_t),\;\sum_{i=1}^{3}b_i=3+2\Delta_1-\Delta_t\big\}\text{ or }\big\{a=3+2\Delta_1-\Delta_t,\;\sum_{i=1}^{3}b_i=4-\Delta_t\big\}.\end{aligned} \tag{3.1.26}$$

Thus, (3.1.25) becomes identical to the five point conformal Ward identity (3.1.23)! Note that the map (3.1.26) specifies c_1, c_2, c_3 and a in terms of the scaling dimensions of the external operators. As for the b_i, it fixes just one of them in terms of the other too, say, b_3 in terms of b_1, b_2. The number of five point cross ratios is two and the number of undetermined parameters we have is also two. Thus, we have found a momentum space analogue for these two cross ratios. Putting everything together, our solution for five point functions reads,

$$\begin{aligned}\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)O_{\Delta_5}(\omega_5)\rangle&=\delta(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6)\omega_1^{\Delta_t-4}\\ &\qquad\big(k_1E_A^{(3)}(a,b_1,b_2,b_3,c_1,c_2,c_3,-x,-y,-z)+\cdots,\end{aligned} \tag{3.1.27}$$

with the parameters provided in (3.1.26). The dots indicate the seven other linearly independent solutions to the Lauricella $E_A^{(3)}$ PDE. A possible sum over different values of b_1 and b_2 is also suppressed as they are not fixed by conformal invariance and a single correlator can be a linear combination of structures with different

[2]A series expansion for $E_A^{(3)}$ is provided in appendix B.1.

b_1 and b_2.

3.1.5 Six Point Functions

By solving the translation and dilatation Ward identities ((3.0.7) and (3.0.8) respectively), we see that the correlator takes the following form:

$$\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)O_{\Delta_5}(\omega_5)O_{\Delta_6}(\omega_6)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6)\omega_1^{\Delta_t-5}\tilde{g}(x,y,z,u), \tag{3.1.28}$$

where we have defined,

$$\Delta_t = \sum_{i=1}^{6}\Delta_i\ ,\ x = \frac{\omega_2}{\omega_1}, y = \frac{\omega_3}{\omega_1}, z = \frac{\omega_4}{\omega_1}, u = \frac{\omega_5}{\omega_1}. \tag{3.1.29}$$

The special conformal Ward identity (3.0.9) implies that the function $\tilde{g}(x,y,z,u)$ satisfies the following differential equation:

$$\begin{aligned}&\Bigg(x(x+1)\frac{\partial^2}{\partial x^2} + y(y+1)\frac{\partial^2}{\partial y^2} + z(z+1)\frac{\partial^2}{\partial z^2} + u(u+1)\frac{\partial^2}{\partial u^2} + 2xy\frac{\partial^2}{\partial x\partial y} + 2yz\frac{\partial^2}{\partial y\partial z} + 2zx\frac{\partial^2}{\partial z\partial x}\\ &+ 2xu\frac{\partial^2}{\partial x\partial u} + 2yu\frac{\partial^2}{\partial y\partial u} + 2zu\frac{\partial^2}{\partial z\partial u} - 2\big(\Delta_2 - 1 + x(\Delta_t-\Delta_1-5)\big)\frac{\partial}{\partial x} - 2\big(\Delta_3 - 1 + y(\Delta_t-\Delta_1-5)\big)\frac{\partial}{\partial y}\\ &- 2\big(\Delta_4 - 1 + z(\Delta_t-\Delta_1-5)\big)\frac{\partial}{\partial z} - 2\big(\Delta_5 - 1 + u(\Delta_t-\Delta_1-5)\big)\frac{\partial}{\partial u} + (\Delta_t-5)(\Delta_t-2\Delta_1-4)\Bigg)\tilde{g}(x,y,z,u) = 0.\end{aligned} \tag{3.1.30}$$

A solution to (3.1.30) turns out to be none other than the four variable Lauricella function $E_A^{(4)}$ [84]! This four variable Lauricella function is the solution to the following system of four partial differential equations:

$$\begin{aligned}&\mathcal{L}_i^{(4)}E_A^{(4)}(a,b_1,b_2,b_3,b_4,c_1,c_2,c_3,c_4,x_1,x_2,x_3,x_4) = 0,\ \ i\in\{1,2,3\}\ \ \text{where,}\\ &\mathcal{L}_i^{(4)} = x_i(1-x_i)\frac{\partial^2}{\partial x_i^2} - x_i\sum_{j\neq i}^{4}x_j\frac{\partial^2}{\partial x_i\partial x_j} + (c_i-(a+b_i+1)x_i)\frac{\partial}{\partial x_i} - b_i\sum_{j\neq i}^{4}x_j\frac{\partial}{\partial x_j}.\end{aligned} \tag{3.1.31}$$

This obviously implies,

$$\sum_{i=1}^{3}\mathcal{L}_i^{(4)}E_A^{(4)}(a,b_1,b_2,b_3,b_4,c_1,c_2,c_3,c_4,x_1,x_2,x_3,x_4) = 0. \tag{3.1.32}$$

If we choose,

$$\begin{aligned}&x_1=-x,\ \ x_2=-y,\ \ x_3=-z,\ \ x_4=-u,\ \ c_1=2(1-\Delta_2),\ \ c_2=2(1-\Delta_3),\ \ c_3=2(1-\Delta_4),\ \ c_4=2(1-\Delta_5),\\ &\big\{a=(5-\Delta_t),\sum_{i=1}^{3}b_i = 4+2\Delta_1-\Delta_t\big\}\ \text{or}\ \big\{a=4+2\Delta_1-\Delta_t,\sum_{i=1}^{3}b_i=5-\Delta_t\big\},\end{aligned} \tag{3.1.33}$$

(3.1.32) becomes identical to (3.1.30)! Thus, our solution for six point functions is,

$$\begin{aligned}\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)O_{\Delta_5}(\omega_5)O_{\Delta_6}(\omega_6)\rangle = \Big(&\delta(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6)\omega_1^{\Delta_t-5} \\ &k_1 E_A^{(4)}(a,b_1,b_2,b_3,b_4,c_1,c_2,c_3,c_4;-x,-y,-z,-u)+\cdots\Big),\end{aligned} \tag{3.1.34}$$

where parameters are given in (3.1.33) and the $\cdots$ stand for the 15 other linearly independent solutions to the Lauricella $E_A^{(4)}$ PDE [3]. Note that (3.1.33) fixes a and the c_i but leaves say, b_1, b_2 and b_3 undetermined. Thus the general six point function can be a sum over such solutions with different values of b_1, b_2 and b_3. Note that three is also the number of independent cross ratios for six point functions, indicating that these parameters are their momentum space analogue. Motivated by our success at the four, five, and six-point levels, let us attempt to generalize our results to arbitrary n-point functions.

3.1.6 Generalization to n point functions

By solving the translation and dilatation Ward identities ((3.0.7) and (3.0.8) respectively) we see that any n point correlator takes the following form:

$$\langle O_{\Delta_1}(\omega_1)\cdots O_{\Delta_n}(\omega_n)\rangle = \delta(\omega_1+\cdots+\omega_n)\omega_1^{\Delta_t-(n-1)}\tilde{g}(y_1,\cdots,y_{n-2}), \tag{3.1.35}$$

where we have defined,

$$\Delta_t = \sum_{i=1}^{n}\Delta_i\ ,\ \ y_i = \frac{\omega_{i+1}}{\omega_1}. \tag{3.1.36}$$

The special conformal Ward identity (3.0.9) implies that the function $\tilde{g}(y_1,\cdots,y_{n-2})$ satisfies the following differential equation:

$$\begin{aligned}\Bigg(\sum_{i=1}^{n-2} y_i(y_i+1)\frac{\partial^2}{\partial y_i^2} + 2\sum_{1\le j<i\le n-2} y_i y_j \frac{\partial^2}{\partial y_i \partial y_j} - 2\sum_{i=1}^{n-2}\big(\Delta_{i+1}-1+y_i(\Delta_t-\Delta_1-n+1)\big)\frac{\partial}{\partial y_i} \\ +(\Delta_t-n+1)(\Delta_t-2\Delta_1-n+2)\Bigg)\tilde{g}(y_1,\cdots,y_{n-2}) = 0.\end{aligned} \tag{3.1.37}$$

The generalized Lauricella function of $n-2$ variables, $E_A^{(n-2)}(a,b_1,\cdots,b_{n-2},c_1,\cdots,c_{n-2};x_1,\cdots,x_{n-2})$ obeys the following system of PDEs [84]:

$$\begin{aligned}&\mathcal{L}_i^{(n-2)} E_A^{(n-2)}(a,b_1,\cdots,b_{n-2},c_1,\cdots,c_{n-2};x_1,\cdots,x_{n-2}) = 0,\ i\in\{1,\cdots,n-2\},\ \text{where,} \\ &\mathcal{L}_i^{(n-2)} = x_i(1-x_i)\frac{\partial^2}{\partial x_i^2} - x_i\sum_{j\neq i}^{n-2} x_j\frac{\partial^2}{\partial x_i\partial x_j} + (c_i-(a+b_i+1)x_i)\frac{\partial}{\partial x_i} - b_i\sum_{j\neq i}^{n-2} x_j\frac{\partial}{\partial x_j}.\end{aligned} \tag{3.1.38}$$

[3] A series expansion for $E_A^{(4)}$ is provided in appendix B.1.

This obviously implies,

$$\sum_{i=1}^{n-2}\mathcal{L}_i^{(n-2)}E_A^{(n-2)}(a,b_1,\cdots,b_{n-2},c_1,\cdots,c_{n-2};x_1,\cdots,x_{n-2})=0. \tag{3.1.39}$$

If we choose,

$$\begin{gathered} x_i=-y_i,\quad c_i=2(1-\Delta_{i+1}),\\ \Big\{a=n-1-\Delta_t,\ \sum_{i=1}^{n-2}b_i=n-2+2\Delta_1-\Delta_t\Big\}\ \text{or}\ \Big\{a=n-2+2\Delta_1-\Delta_t,\ \sum_{i=1}^{n-2}b_i=n-1-\Delta_t\Big\}. \end{gathered} \tag{3.1.40}$$

Thus, (3.1.39) becomes identical to the n point conformal Ward identity (3.1.37)! The map (3.1.40) fixes a and the $n-2$ c_i but leaves $n-3$ out of the $n-2$ b_i undetermined. This is exactly the number of independent conformal cross ratios in the time domain and thus we have obtained their momentum space analogues for any n point correlation function. Our result for the n point function reads,

$$\langle O_{\Delta_1}(\omega_1)\cdots O_{\Delta_n}(\omega_n)\rangle=\delta(\omega_1+\cdots+\omega_n)\omega_1^{\Delta_t-(n-1)}\left(k_1E_A^{(n-2)}(a,b_1,\cdots,b_{n-2},c_1,\cdots,c_{n-2};-x_1,\cdots,-x_{n-2})+\cdots\right), \tag{3.1.41}$$

where the parameters are given in (3.1.40) and the $\cdots$ stand for the remaining 2^{n-2} linearly independent solutions to the Lauricella $E_A^{(n-2)}$ PDEs[4](We provide a series expansion for $E_A^{(n-2)}$ in appendix B.1). As in the previous cases, we have suppressed a possible sum over the b_i that are not determined by conformal invariance. The result (3.1.41) provides the first closed-form expression for a conformal n-point function in momentum space[5], as mentioned in the Preliminary.

3.2 Momentum space Conformal Partial Waves

In this section, we will compute momentum space conformal partial waves, which provides another way to find correlators by putting the complete states in between the correlator. Conformal partial waves are eigenvectors of the quadratic conformal Casimir C_2 [87]. The expression for C_2 is as follows:

$$C_2=\frac{1}{2}(HK+KH)-D^2. \tag{3.2.1}$$

It is easy to see that C_2 commutes with H,K and D using the $\mathfrak{sl}(2,\mathbb{R})$ conformal algebra (3.0.1). The conformal partial wave $W_\Delta(t_1,t_2,t_3,t_4)$ satisfies the following differential equation:

$$C_{12}W_\Delta(t_1,t_2,t_3,t_4)=\Delta(\Delta-1)W_\Delta(t_1,t_2,t_3,t_4). \tag{3.2.2}$$

[4]It is interesting to note that certain Lauricella functions also pop out as solutions of conformal integrals [85] as well as in solutions to the conformal Ward identity in special kinematics [86].

[5]Previously, a simplex representation for n point functions in momentum space was obtained [39]. Their result, in contrast to out algebraic formulae (3.1.41) is a simplex integral representation for the correlator.

The conformal partial wave (in the s channel) is given by the following integral [88]:

$$W_{\Delta}^{(s)}(t_1,t_2,t_3,t_4)=\int dt\langle O_{\Delta_1}(t_1)O_{\Delta_2}(t_2)O_{\Delta}(t)\rangle\langle\tilde{O}_{\Delta}(t)O_{\Delta_3}(t_3)O_{\Delta_4}(t_4)\rangle, \tag{3.2.3}$$

where we have introduced the shadow operator, $\tilde{O}$, which is defined as,

$$\tilde{O}(t)=\int_{-\infty}^{\infty}\frac{dx}{|x-t|^{2-2\Delta}}O_{\Delta}(t). \tag{3.2.4}$$

Using (3.2.4) in (3.2.3) we obtain,

$$W_{\Delta}^{(s)}(t_1,t_2,t_3,t_4)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dtdx}{|x-t|^{2-2\Delta}}\langle O_{\Delta_1}(t_1)O_{\Delta_2}(t_2)O_{\Delta}(t)\rangle\langle O_{\Delta}(t)O_{\Delta_3}(t_3)O_{\Delta_4}(t_4)\rangle. \tag{3.2.5}$$

Fourier transforming with respect to t_1, t_2, t_3 and t_4 we obtain,

$$W_{\Delta}^{(s)}(\omega_1,\omega_2,\omega_3,\omega_4)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dtdx}{|x-t|^{2-2\Delta}}\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta}(t)\rangle\langle O_{\Delta}(t)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)\rangle. \tag{3.2.6}$$

Performing the integrals over x and t, we obtain,

$$W_{\Delta}^{(s)}(\omega_1,\omega_2,\omega_3,\omega_4)=\frac{\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{(\omega_1+\omega_2)^{2\Delta-1}}\langle\langle O_{\Delta_1}(\omega_1)O_{\Delta_2}(\omega_2)O_{\Delta}(-\omega_1-\omega_2)\rangle\rangle\langle\langle O_{\Delta}(\omega_1+\omega_2)O_{\Delta_3}(\omega_3)O_{\Delta_4}(\omega_4)\rangle\rangle, \tag{3.2.7}$$

where the double bracket notation is defined as,

$$\langle . \rangle=\delta(\omega_1+\cdots)\langle\langle\ .\ \rangle\rangle. \tag{3.2.8}$$

Having obtained the conformal partial wave in momentum space (3.2.7), we can now extract the conformal block. We then use the expression for generic three point functions that we obtained in (3.1.13) in (3.2.7). Recall however, that we have two, rather than a unique solution to the three point function. As we explained in the beginning of this section and as we shall elaborate in subsection 3.3, this fact owes itself to the possibility of the various possible time orderings yielding different expressions when Fourier transformed. Let us now write (3.2.7) explicitly. First, we define,

$$\begin{aligned} f_{\Delta_i,\Delta_j,\Delta_k}(\omega_i,\omega_j)=\omega_i^{\Delta_i+\Delta_j+\Delta_k-2}\Bigg(&c_{1,ijk}\ {}_2F_1\left(2-\Delta_i-\Delta_j-\Delta_k,1+\Delta_i-\Delta_j-\Delta_k,2(1-\Delta_j);\frac{-\omega_j}{\omega_i}\right)\\ &+\left(\frac{\omega_j}{\omega_i}\right)^{2\Delta_j-1}c_{2,ijk}\ {}_2F_1\left(1-\Delta_i+\Delta_j-\Delta_k,\Delta_i+\Delta_j-\Delta_k,2\Delta_j;-\frac{\omega_j}{\omega_i}\right)\Bigg). \end{aligned} \tag{3.2.9}$$

We can then re-write (3.2.7) as,

$$W_{\Delta}^{(s)}(\omega_1,\omega_2,\omega_3,\omega_4)=\frac{1}{(\omega_1+\omega_2)^{2\Delta-1}}f_{\Delta_1,\Delta_2,\Delta}(\omega_1,\omega_2)f_{\Delta,\Delta_3,\Delta_4}(\omega_1+\omega_2,\omega_3)\delta(\omega_1+\omega_2+\omega_3+\omega_4), \tag{3.2.10}$$

which is our final expression for the s channel momentum space conformal partial wave. These (3.2.10) will be put to the test in subsection (3.4).

As we have seen throughout this subsection, we have found multiple solutions for the momentum space three and four-point functions ((3.1.13) and (3.1.20)) and now for the conformal partial waves (3.2.10). Through an example, we shall now explain why there exist multiple solutions in momentum space in contrast to the unique expressions in the time domain.

3.3 Time Ordering and the Existence of Multiple Solutions

To illustrate the existence of multiple momentum space solutions to the conformal Ward identities, let us compute a three point function via three different routes. First, we directly compute it in momentum space, then we Fourier transform the time domain correlators for the various possible time orderings, and finally we do so by using our solution (3.1.13) to the conformal ward identity.

Direct Momentum Space Computation

Consider a three point function of a $\Delta = -1$ scalar operator. Its three point function can be expressed as the following one loop integral[6]:

$$\begin{aligned} \langle O(\omega_1)O(\omega_2)O(\omega_3)\rangle &= \delta(\omega_1+\omega_2+\omega_3)\int_{-\infty}^{\infty} dl\,\frac{1}{l^2(l-\omega_1)^2(l+\omega_2)^2} \\ &= \delta(\omega_1+\omega_2+\omega_3)\tilde{G}(\omega_1,\omega_2). \end{aligned} \tag{3.3.1}$$

We now employ an appropriate $i\epsilon$ prescription and use the residue theorem to evaluate (3.3.1). Enclosing all the poles, the result identically vanishes. Enclosing just one out of three poles yields three possible results:

$$\langle O(\omega_1)O(\omega_2)O(\omega_3)\rangle = 4\pi i\delta(\omega_1+\omega_2+\omega_3)\begin{cases} \dfrac{2(\omega_1+2\omega_2)}{\omega_2^3(\omega_1+\omega_2)^3}, \text{ Enclosing the pole at } -\omega_2. \\[2ex] -\dfrac{2(\omega_1-\omega_2)}{\omega_1^3\omega_2^3}, \text{ Enclosing the pole at } 0. \\[2ex] -\dfrac{2(2\omega_1+\omega_2)}{\omega_1^3(\omega_1+\omega_2)^3}, \text{ Enclosing the pole at } \omega_1. \end{cases} \tag{3.3.2}$$

The other three cases are the inclusion of two poles which give the same result (upto an overall negative sign) as the three above.

[6] This is the one dimensional cousin of the triple K integral [5].

Direct Fourier Transform

We now directly Fourier transform the time domain $\langle OOO\rangle$ correlator. It is given by the following expression[7]:

$$\langle O(t_1)O(t_2)O(t_3)\rangle = \frac{C_{123}}{(t_1-t_2)^{-1}(t_1-t_3)^{-1}(t_2-t_3)^{-1}} = C_{123}(t_1-t_2)(t_1-t_3)(t_2-t_3). \tag{3.3.3}$$

Let us now Fourier transform (3.3.3) for the various time orderings that are possible.

$\mathbf{t_1 > t_2 > t_3}$

In this time ordering, the Fourier transform reads,

$$\begin{aligned}\langle O(\omega_1)O(\omega_2)O(\omega_3)\rangle &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \; \langle O(t_1)O(t_2)O(t_3)\rangle \, e^{i\omega_1 t_1} e^{i\omega_2 t_2} e^{i\omega_3 t_3}, \\ &= \delta(\omega_1+\omega_2+\omega_3)\, \frac{2i(2\omega_1+\omega_2)}{\omega_1^3(\omega_1+\omega_2)^3}. \end{aligned} \tag{3.3.4}$$

$\mathbf{t_1 > t_3 > t_2}$

Here we have,

$$\begin{aligned}\langle O(\omega_1)O(\omega_2)O(\omega_3)\rangle &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_3 \int_{-\infty}^{t_3} dt_2 \; \langle O(t_1)O(t_2)O(t_3)\rangle \, e^{i\omega_1 t_1} e^{i\omega_2 t_2} e^{i\omega_3 t_3}, \\ &= \delta(\omega_1+\omega_2+\omega_3)\, \frac{2i(\omega_1-\omega_2)}{\omega_1^3\omega_2^3}. \end{aligned} \tag{3.3.5}$$

Repeating this procedure for the remaining time orderings, we obtain their Fourier transforms as well. We summarize our results in table 3.1. Notice that just like in (3.3.2), three possible expressions for the momentum space correlator are obtained. However, note that only two of them are linearly independent. Therefore, the Fourier transform of every possible time ordering corresponds to a linear combination of these two expressions. Let us now see how the results of these various pole enclosures/ time orderings fit in with the general solution (3.1.13) that we obtained by solving the conformal Ward identities.

Comparison with the General Solution

Our general solution for three point functions (3.1.13) for $\Delta_1 = \Delta_2 = \Delta_3 = -1$ (recall that our operator O has scaling dimension -1) is given by:

$$\tilde{G}(\omega_1,\omega_2) = \omega_1^{-5}\left[c_1\,\frac{\omega_1^2(2\omega_1+\omega_2)}{2(\omega_1+\omega_2)^3} + c_2\,\frac{\omega_1^2(\omega_1-\omega_2)}{\omega_2^3}\right] \tag{3.3.6}$$

$$= c_1\,\frac{2\omega_1+\omega_2}{\omega_1^3(\omega_1+\omega_2)^3} + c_2\,\frac{(\omega_1-\omega_2)}{\omega_1^3\omega_2^3}. \tag{3.3.7}$$

---

[7]This expression can be obtained by setting $\Delta_1 = \Delta_2 = \Delta_3 = -1$ in (3.1.1).

Time Ordering	**Correlator**
$t_1 > t_2 > t_3$	$\frac{2i(2\omega_1+\omega_2)}{\omega_1^3(\omega_1+\omega_2)^3}$
$t_2 > t_1 > t_3$	$-\frac{2i(\omega_1+2\omega_2)}{\omega_2^3(\omega_1+\omega_2)^3}$
$t_3 > t_2 > t_1$	$\frac{2i(2\omega_1+\omega_2)}{\omega_1^3(\omega_1+\omega_2)^3}$
$t_2 > t_3 > t_1$	$\frac{2i(\omega_1-\omega_2)}{\omega_1^3\omega_2^3}$
$t_1 > t_3 > t_2$	$\frac{2i(\omega_1-\omega_2)}{\omega_1^3\omega_2^3}$
$t_3 > t_1 > t_2$	$-\frac{2i(\omega_1+2\omega_2)}{\omega_2^3(\omega_1+\omega_2)^3}$

Table 3.1: Correlators in momentum space obtained via Fourier Transform with different time orderings

The expression (3.3.6) covers all the cases we find in table 3.1 by appropriately choosing the constants c_1 and c_2. This section provides one of the first explanations of the existence of the two independent solutions in (3.1.13). They correspond to the fact that the different "time orderings" give rise to two possible Fourier space expressions (see table 3.1). A similar analysis was performed in higher dimensions in [89]. The advantage of this analysis in $d = 1$ is that the physical meaning of the existence of these multiple solutions is not obscured by technical complications.

A similar preliminary analysis for the Four-Point function provides the same explanation. One would suspect that these solutions should correspond to the Fourier transform of the various possible time orderings, and this is what we see from a preliminary analysis. We also expect the same conclusions to hold for the 2^{n-2} independent solutions we obtained for the n point functions (3.1.41).

To verify our expressions for three four-point functions ((3.1.13), (3.1.20)) and conformal partial waves (3.2.10), we will reproduce correlators in the free theories and the DFF model.

3.4 Examples

Let us begin by showing that three and four point correlators in the free bosonic theory are captured by our general formulae for correlation functions.

3.4.1 Free Bosonic Theory

The action for the $U(1)$ free bosonic theory is given by,

$$S_{FB} = \int_{-\infty}^{\infty} dt\ \partial_t\bar{\phi}\partial_t\phi. \tag{3.4.1}$$

ϕ and $\bar{\phi}$ are primary operators with scaling dimension $-\frac{1}{2}$. We also consider the following composite primary operators:

$$O(t) = \bar{\phi}(t)\phi(t),\ \ J_B(t) = i(\bar{\phi}\partial_t\phi - \partial_t\bar{\phi}\ \phi), \tag{3.4.2}$$

which have the following momentum space counterparts:

$$O(\omega) = \int_{-\infty}^{\infty} dl\ \bar{\phi}(l)\phi(\omega - l),\ \ J_B(\omega) = \int_{-\infty}^{\infty} dl\ (2l - \omega)\bar{\phi}(l)\phi(\omega - l). \tag{3.4.3}$$

The first is a $\Delta = -1$ scalar, while the second is the conserved $U(1)$ current. Let us now compute several three and four-point correlators involving $\phi, \bar{\phi}$ and these composite operators.

<u>$\langle\bar{\phi}(\omega_1)\phi(\omega_2)J_B(\omega_3)\rangle$</u>

The Wick contractions yield,

$$\langle\bar{\phi}(\omega_1)\phi(\omega_2)J_B(\omega_3)\rangle = \frac{\omega_1 - \omega_2}{\omega_1^2\omega_2^2}\delta(\omega_1 + \omega_2 + \omega_3). \tag{3.4.4}$$

In terms of the general solution (3.1.13) we see that,

$$\langle\bar{\phi}(\omega_1)\phi(\omega_2)J(\omega_3)\rangle = (\omega_1)^{-3}\left(\frac{\omega_2}{\omega_1}\right)^{-2}\ {}_2F_1\left(1, -1, -1, -\frac{\omega_2}{\omega_1}\right), \tag{3.4.5}$$

providing another check of our result. Let us now move on to a four-point function.

<u>$\langle\bar{\phi}(\omega_1)O(\omega_2)O(\omega_3)\phi(\omega_4)\rangle$</u>

Using the definitions of the operators provided in (3.4.3) we obtain,

$$\begin{aligned}
&\langle\phi(\omega_1)O(\omega_2)O(\omega_3)\bar{\phi}(\omega_4)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dldk\langle\phi(\omega_1)\bar{\phi}(l)\phi(\omega_2 - l)\bar{\phi}(k)\phi(\omega_3 - k)\bar{\phi}(\omega_4)\rangle \\
&= \frac{1}{(2\pi)^2\omega_1^2\omega_4^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dldk\left[\frac{1}{k^2}\delta(\omega_1 + l)\delta(\omega_2 - l + k)\delta(\omega_3 - k + \omega_4) + \frac{1}{l^2}\delta(l + \omega_3 + k)\delta(\omega_2 - l + \omega_4)\delta(\omega_1 + k)\right] \\
&= \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4)\omega_1^{-6}\frac{1}{(1 + x + y)^2}\left(\frac{1}{(1 + x)^2} + \frac{1}{(1 + y)^2}\right),
\end{aligned} \tag{3.4.6}$$

where $x = \frac{\omega_2}{\omega_1}$ and $y = \frac{\omega_3}{\omega_1}$ as we defined earlier.
We now define for convenience,

$$\langle\phi(\omega_1)O(\omega_2)O(\omega_3)\bar{\phi}(\omega_4)\rangle = \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4)\omega_1^{-6}\psi_{\phi OO\bar{\phi}}(x, y), \tag{3.4.7}$$

where,

$$\psi_{\phi OO\bar{\phi}}(x, y) = \frac{1}{(1 + x + y)^2}\left(\frac{1}{(1 + x)^2} + \frac{1}{(1 + y)^2}\right). \tag{3.4.8}$$

As we saw in (3.1.20), the ward identities do not fix the form of the 4 point function and lead to an undetermined parameter b_1. Our aim now is to find value(s) of b_1 that reproduce this correlator.
We have $\Delta_1 = \Delta_4 = -\frac{1}{2}$ and $\Delta_2 = \Delta_3 = -1$. One of the solutions to the Ward identities (3.1.20) is,

$$\omega_1^{-6} f_{b_1}(x, y) = \omega_1^{-6} F_2(4, b_1, 6 - b_1, 4, 4; -x, -y). \tag{3.4.9}$$

Using the definition of Appell functions, we find that,

$$\begin{aligned} F_2(4, 2, 4, 4, 4; -x, -y) &= \frac{1}{(1 + x + y)^2(1 + y)^2}, \\ F_2(4, 4, 2, 4, 4; -x, -y) &= \frac{1}{(1 + x + y)^2(1 + x)^2}. \end{aligned} \tag{3.4.10}$$

which correspond to the choices $b_1 = 2$ and $b_2 = 4$ respectively. This gives us,

$$\omega_1^{-6}(f_2(x, y) + f_4(x, y)) = \omega_1^{-6} \frac{1}{(1 + x + y)^2} \left(\frac{1}{(1 + x)^2} + \frac{1}{(1 + y)^2} \right). \tag{3.4.11}$$

Comparing with (3.4.7) we find,

$$\psi_{\phi OO\phi} = \Big(F_2(4, 2, 4, 4, 4; -x, -y) + F_2(4, 4, 2, 4, 4; -x, -y) \Big). \tag{3.4.12}$$

Therefore, the correlator (3.4.6) can be written as,

$$\langle \phi(\omega_1) O(\omega_2) O(\omega_3) \bar{\phi}(\omega_4) \rangle = \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) \omega_1^{-6} \left(F_2 \left(4, 2, 4, 4, 4; -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1} \right) + F_2 \left(4, 4, 2, 4, 4; -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1} \right) \right), \tag{3.4.13}$$

which provides a check of our general result (3.1.20). Let us now also reproduce this correlator using conformal partial waves. Consider the s channel conformal partial wave (3.2.10), set $c_{2,ijk} = 0$ in (3.2.9) and $\Delta_1 = \Delta_4 = -\frac{1}{2}, \Delta_2 = \Delta_3 = -1$. For the exchanged operator having $\Delta = -\frac{1}{2}$ we find,

$$W^{(s)}_{\Delta=-\frac{1}{2}} = \frac{1}{(1 + x)^2(1 + x + y)^2 \omega_1^6} \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4). \tag{3.4.14}$$

We can also obtain the u channel conformal partial wave by a $(2 \leftrightarrow 3)$ exchange and add it with the s channel result which yields,

$$W^{(s)}_{\Delta=-\frac{1}{2}} + W^{(u)}_{\Delta=-\frac{1}{2}} = \omega_1^{-6} \frac{1}{(1 + x + y)^2} \left(\frac{1}{(1 + x)^2} + \frac{1}{(1 + y)^2} \right) \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) = \langle \phi(\omega_1) O(\omega_2) O(\omega_3) \bar{\phi}(\omega_4) \rangle. \tag{3.4.15}$$

To summarize, we have obtained two distinct representations for this correlator: One in terms of the momentum space conformal partial waves and the other, in terms of the Appell $F2$ function.

$$\langle \phi(\omega_1) O(\omega_2) O(\omega_3) \bar{\phi}(\omega_4) \rangle = W^{(s)}_{\Delta=-\frac{1}{2}} + W^{(u)}_{\Delta=-\frac{1}{2}}$$

$$= \delta(\omega_1+\omega_2+\omega_3+\omega_4)\omega_1^{-6}\left(F_2\left(4,2,4,4,4;-\frac{\omega_2}{\omega_1},-\frac{\omega_3}{\omega_1}\right)+F_2\left(4,4,2,4,4;-\frac{\omega_2}{\omega_1},-\frac{\omega_3}{\omega_1}\right)\right). \tag{3.4.16}$$

Let us now provide two more examples of correlators in the free bosonic theory. Since the details of the calculation are similar to the above example, we just provide the final results.

$\underline{\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle}$

The Fourier space expression for this correlator with the time ordering $t_1 > t_2 > t_3 > t_4$ reads[8],

$$\langle\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle\rangle = \frac{2(1+x)^2(41+7x(4+x))+2(1+x)(54+x(31+7x))y+4(10+x(5+x))y^2}{(1+x)^5(1+x+y)^3\omega_1^7}. \tag{3.4.17}$$

We find that this correlator receives contributions from *three* different conformal blocks corresponding to exchanges of operators with $\Delta = -2, -1$, and 0. Setting $c_{2,ijk} = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = -1$ in (3.2.10), we see that,

$$\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle = 16W^{(s)}_{\Delta=-1} - \frac{2}{3}W^{(s)}_{\Delta=0} + \frac{200}{3}W^{(s)}_{\Delta=-2}. \tag{3.4.18}$$

We also find that this result can be expressed in terms of our general four point function (3.1.20). For $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = -1$ and setting $k_2 = k_3 = k_4 = 0$ in (3.1.20) we obtain the following two possible expressions (corresponding to the two choices of $a, b_1 + b_2$ in (3.1.19)):

$$f_1(b_1) = \frac{1}{\omega_1^7}F_2(7, b_1, 4-b_1, 4, 4, -x, -y) \text{ and } f_2(b_1) = \frac{1}{\omega_1^7}F_2(4, b_1, 7-b_1, 4, 4, -x, -y). \tag{3.4.19}$$

It turns out that the second solution in (3.4.19) is the required one for this example. Consider the following linear combination of the $b_1 = 4$, $b_1 = 5$ and $b_1 = 6$ solutions:

$$\begin{aligned}\langle O_1(\omega_1)O_1(\omega_2)O_1(\omega_3)O_1(\omega_4)\rangle &= \delta(\omega_1+\omega_2+\omega_3+\omega_4)\big(14f_2(4)+28f_2(5)+40f_2(6)\big)\\ &= \frac{\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{\omega_1^7}\big(14F_2(4,4,3,4,4,-x,-y)+28F_2(4,5,2,4,4,-x,-y)+40F_2(4,6,1,4,4,-x,-y)\big).\end{aligned} \tag{3.4.20}$$

Let us now compare the conformal partial wave representation (3.4.18) and the Appell F_2 representation of the correlator (3.4.20). In (3.4.18), it takes the sum of three different conformal blocks to reproduce the correlator. In (3.4.20), it takes three different "b_1 exchanges" to reproduce the correlator. In lieu of this, one might think that b_1 is somehow related to the dimension of the exchanged operators. This, however, is not quite true as the number of conformal blocks does not always tally up with the number of "b_1 exchanges".

[8]One way in which this expression can be obtained is by performing the Wick contractions in the time domain and then Fourier transforming with this time ordering with the appropriate $i\epsilon$ prescriptions.

3.4.2 Free Fermionic Theory

The action for the $U(1)$ free massless Dirac fermion theory is,

$$S_{FF} = i \int dt\ \psi^\dagger \partial_t \psi. \tag{3.4.21}$$

ψ and $\psi^\dagger$ are dimensionless operators. We also consider the conserved $U(1)$ current,

$$J_F(t) = \psi^\dagger(t)\psi(t), \tag{3.4.22}$$

which in momentum space is given by,

$$J_F(\omega) = \int dl\ \psi^\dagger(l)\psi(\omega - l). \tag{3.4.23}$$

Let us now consider the following correlator:

$\underline{\langle\bar{\psi}(\omega_1)\psi(\omega_2)J_F(\omega_3)\rangle}$

Performing the Wick contractions gives,

$$\langle\bar{\psi}(\omega_1)\psi(\omega_2)O(\omega_3)\rangle = \frac{1}{\omega_1\omega_2}\delta(\omega_1+\omega_2+\omega_3). \tag{3.4.24}$$

This is reproduced by the general solution (3.1.13) in the following way:

$$\langle\langle\bar{\psi}(\omega_1)\psi(\omega_2)O(\omega_3)\rangle\rangle = (\omega_1)^{-2}\left(\frac{\omega_2}{\omega_1}\right)^{-1}\ {}_2F_1\left(1,0,0,\frac{-\omega_2}{\omega_1}\right). \tag{3.4.25}$$

We now move on to a four-point example.

$\underline{\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle}$

Via Wick contractions, we obtain the following expression for this correlator:

$$\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle = \frac{\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{\omega_1^3}\frac{2+x+y}{(1+x)(1+y)(1+x+y)}. \tag{3.4.26}$$

This correlator is reproduced by the sum of a single conformal block in the s and u channels (obtained by setting $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = -1, c_{2,ijk} = 0$ in (3.2.10) and the u channel expression obtained via a $(2 \leftrightarrow 3)$ exchange.):

$$\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle = W^{(s)}_{\Delta=0} + W^{(u)}_{\Delta=0}. \tag{3.4.27}$$

As for the Appell function representation, set $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0, k_2 = k_3 = k_4 = 0$ in (3.1.20). We obtain,

$$f_1(b_1) = \frac{1}{\omega_1^3}F_2(3, b_1, 2-b_1, 2, 2; -x, -y) \text{ and } f_2(b_1) = F_2(2, b_1, 3-b_1, 2, 2; -x, -y). \tag{3.4.28}$$

We find,

$$\langle\psi^\dagger(\omega)J_F(\omega_2)J_F(\omega_3)\psi(\omega_4)\rangle=\delta(\omega_1+\omega_2+\omega_3+\omega_4)f_1(1)=\delta(\omega_1+\omega_2+\omega_3+\omega_4)F_2(3,1,1,2,2;-x,-y),\tag{3.4.29}$$

providing another test of our results. Let us now consider and reproduce correlation functions in the DFF model.

3.4.3 The DFF model

In this subsection, we use our general formulae for correlation functions to reproduce correlators in the DFF model. For details on the model, please refer to the original paper [44]. The four point function in the DFF model was computed in [52] and reads,

$$\langle O^\dagger_{r_0}(t_1)\phi_\delta(t_2)\phi_\delta(t_3)O_{r_0}(t_4)\rangle=\frac{1}{(t_{24}t_{13})^{\delta-r_0}(t_{12}t_{34})^{\delta+r_0}t_{14}^{2(r_0-\delta)}}\chi^{r_0}\ {}_2F_1(\delta,\delta,2r_0;\chi),\tag{3.4.30}$$

where we have used the shorthand $t_{ij}=t_i-t_j$, $\chi=\dfrac{t_{12}t_{34}}{t_{13}t_{24}}$ is the cross ratio. The above expression is also in the time ordering $t_1>t_2>t_3>t_4$.

Consider the specific case where $r_0=\delta=-1$. The Fourier space expression of the correlator reads,

$$\frac{(2+x)(2+2x+y)}{(1+x)^3(1+x+y)^3\omega_1^7}\delta(\omega_1+\omega_2+\omega_3+\omega_4).\tag{3.4.31}$$

This correlator (for arbitrary δ,r_0) was found to receive contribution from just a single conformal block. Indeed, we see that just a single conformal block with $\Delta=-1$ exchange (obtained be setting $\Delta_1=\Delta_2=\Delta_3=\Delta_4=-1$, $c_{2,ijk}=0$ in (3.2.10)) suffices to reproduce this correlator:

$$W^{(s)}_{\Delta=-1}=\frac{(2+x)(2+2x+y)}{(1+x)^3(1+x+y)^3\omega_1^7}\delta(\omega_1+\omega_2+\omega_3+\omega_4).\tag{3.4.32}$$

In terms of the Appell F_2 representation, set $\Delta_1=\Delta_2=\Delta_3=\Delta_4=-1,k_2=k_3=k_4=0$ in (3.1.20). We obtain two solutions:

$$f_1(b_1)=\frac{1}{\omega_1^7}F_2(7,b_1,4-b_1,4,4;-x,-y)\text{ and }f_2(b_1)=\frac{1}{\omega_1^7}F_2(4,b_1,7-b_1,4,4;-x,-y).\tag{3.4.33}$$

We see that,

$$\begin{aligned}\delta(\omega_1+\omega_2+\omega_3+\omega_4)\frac{(2+x)(2+2x+y)}{(1+x)^3(1+x+y)^3\omega_1^7}&=2\delta(\omega_1+\omega_2+\omega_3+\omega_4)\big(f_2(4)+f_2(5)\big)\\&=\frac{2\delta(\omega_1+\omega_2+\omega_3+\omega_4)}{\omega_1^7}\big(F_2(4,4,3,4,4;-x,-y)+F_2(4,5,2,4,4;-x,-y)\big).\end{aligned}\tag{3.4.34}$$

We can also repeat the same analysis for other values of δ and r_0. We find that the momentum space correlator for any δ,r_0 is given in terms of exactly one conformal block in accordance with the results of [52].

Chapter 4

Superconformal Field theory in $1d$

I don't know half of you half as well as I should like; and I like less than half of you half as well as you deserve.

Bilbo on Supersymmetry, A Long-expected Party, The Fellowship of the Ring

In this chapter, we shall extend our analysis to theories that have $\mathcal{N} = 1, 2$ supersymmetry, in addition to the $\mathfrak{sl}(2, \mathbb{R})$ conformal symmetry. We will discuss the formalism of finding the connections between different fermionic and bosonic correlators and the exact expressions for two, three, and four-point functions in superconformal quantum mechanics.

4.1 Correlators in $\mathcal{N} = 1$ Super Conformal Quantum Mechanics

This section will discuss the $\mathcal{N} = 1$ supersymmetry. We first discuss the superspace formalism that we employ, which we then follow by solving the superconformal Ward identities for two, three, and four-point functions.

4.1.1 The Superspace Formalism

The position space or momentum space is extended to $\mathcal{N} = 1$ superspace. A point in this superspace is described by the pair (t, θ) where t is the usual time coordinate and θ is Grassmann valued. The generators of the $\mathcal{N} = 1$ superconformal algebra consists of the usual conformal generators H, K and D, the supersymmetry generator Q and the special superconformal generator S. The algebra as well as the action of the generators on primary operators are given in appendix A.1.

In terms of component fields, the superfield can be expanded as follows:

$$\mathbf{O}_\Delta(\omega,\theta) = \Phi_\Delta(\omega) + \theta\,\Psi_{\Delta+\frac{1}{2}}(\omega). \tag{4.1.1}$$

The aim is to constrain the correlation functions of these superfields by solving the superconformal ward identities. These identities read,

$$\langle[\mathcal{L},\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)]\cdots\mathbf{O}_{\Delta_n}(\omega_n,\theta_n)\rangle + \cdots\langle\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\cdots[\mathcal{L},\mathbf{O}_{\Delta_n}(\omega_n,\theta_n)]\rangle = 0,\ \mathcal{L}\in\{H,K,D,Q,S\}. \tag{4.1.2}$$

Using the action of the generators on primary operators provided in (A.1.2) and (4.1.2) yields the following equations:

$$\sum_{i=1}^{n}\omega_i f_n(\omega_1,\theta_1;\ldots;\omega_n,\theta_n) = 0, \tag{4.1.3}$$

$$\sum_{i=1}^{n}\left(\omega_i\frac{\partial}{\partial\omega_i} + (1-\Delta_i) - \frac{1}{2}\theta_i\frac{\partial}{\partial\theta_i}\right) f_n(\omega_1,\theta_1;\ldots;\omega_n,\theta_n) = 0, \tag{4.1.4}$$

$$\sum_{i=1}^{n}\left(\omega_i\frac{\partial^2}{\partial\omega_i^2} + 2(1-\Delta_i)\frac{\partial}{\partial\omega_i} - \theta_i\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\omega_i}\right) f_n(\omega_1,\theta_1;\ldots;\omega_n,\theta_n) = 0, \tag{4.1.5}$$

$$\sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta_i} + \frac{\theta_i}{2}\omega_i\right) f_n(\omega_1,\theta_1;\ldots;\omega_n,\theta_n) = 0, \tag{4.1.6}$$

$$\sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\omega_i} + \left(\frac{1}{2}-\Delta_i\right)\theta_i + \frac{\theta_i\omega_i}{2}\frac{\partial}{\partial\omega_i}\right) f_n(\omega_1,\theta_1;\ldots;\omega_n,\theta_n) = 0, \tag{4.1.7}$$

where, $f_n(\omega_1,\theta_1;\ldots;\omega_n,\theta_n)$ is a general $\mathcal{N}=1$, n-point function. We will now proceed to investigate its implications for correlation functions.

4.1.2 Correlation Functions

We begin with the two-point case.

4.1.2.1 Two Point Functions

Consider an arbitrary two-point function:

$$\langle\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\rangle = \langle\Phi_{\Delta_1}(\omega_1)\Phi_{\Delta_2}(\omega_2)\rangle - \theta_1\theta_2\langle\Psi_{\Delta_1}(\omega_1)\Psi_{\Delta_2}(\omega_2)\rangle, \tag{4.1.8}$$

where we used the superfield expansion [1] (4.1.1).
Translation, dilatation and special conformal invariance (equations (4.1.3), (4.1.4) and (4.1.5)) constrain the correlator to take the following form:

$$\langle\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\rangle = \delta(\omega_1+\omega_2)\delta_{\Delta_1,\Delta_2}\omega_1^{2\Delta_1-1}\big(c_0 - c_1\omega_1\theta_1\theta_2\big). \tag{4.1.9}$$

[1] Note that we did not retain any terms with an odd number of the θ_i as they multiply component correlators which are grassmann odd and hence zero.

The Q supersymmetric Ward identity (4.1.6) then fixes $c_1 = -\frac{c_0}{2}$. Therefore, the final result for the two point correlator reads,

$$\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\rangle = c_0\delta(\omega_1+\omega_2)\delta_{\Delta_1,\Delta_2}\omega_1^{2\Delta_1-1}\left(1+\frac{\omega_1}{2}\theta_1\theta_2\right). \tag{4.1.10}$$

Note that the result is quite reminiscent of what was obtained in three dimensions in [80].

4.1.2.2 Three Point Functions

We now move to the three point case. A generic correlator reads:

$$\begin{aligned}\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\rangle = \delta(\omega_1+\omega_2+\omega_3)\big(c_0(\omega_1,\omega_2)+c_4(\omega_1,\omega_2)\theta_1\theta_2 \\ + c_5(\omega_1,\omega_2)\theta_2\theta_3 + c_6(\omega_1,\omega_2)\theta_1\theta_3\big).\end{aligned} \tag{4.1.11}$$

This correlator is constrained by the Q Ward identity (4.1.6) which yields the following constraints:

$$c_5(\omega_1,\omega_2) = \left(c_4(\omega_1,\omega_2)+\frac{1}{2}c_0(\omega_1,\omega_2)\omega_2\right), \quad c_6(\omega_1,\omega_2) = \left(-c_4(\omega_1,\omega_2)+\frac{1}{2}c_0(\omega_1,\omega_2)\omega_1\right). \tag{4.1.12}$$

Our correlator thus takes the form:

$$\begin{aligned}\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\rangle = \delta(\omega_1+\omega_2+\omega_3)\Bigg(c_0(\omega_1,\omega_2)+c_4(\omega_1,\omega_2)\theta_1\theta_2 \\ + \left(-c_4(\omega_1,\omega_2)+\frac{1}{2}c_0(\omega_1,\omega_2)\omega_1\right)\theta_1\theta_3 + \left(c_4(\omega_1,\omega_2)+\frac{1}{2}c_0(\omega_1,\omega_2)\omega_2\right)\theta_2\theta_3\Bigg).\end{aligned} \tag{4.1.13}$$

Dilatation invariance (4.1.4) then fixes the overall scaling of the correlator thus leading to,

$$\begin{aligned}\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\rangle = \delta(\omega_1+\omega_2+\omega_3)\omega_1^{\Delta_t-2}\Big(c_0(x)\,(2+\omega_1\,\theta_1\theta_3+\omega_1\,x\,\theta_2\theta_3) \\ + 2\omega_1\,c_4(x)(\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3)\Big),\end{aligned} \tag{4.1.14}$$

where $\Delta_t = \Delta_1+\Delta_2+\Delta_3, \quad x = \dfrac{\omega_2}{\omega_1}$.

The final step is to obtain constraints from the special conformal Ward identity (4.1.5) (The S Ward identity will then trivially follows as $[K,Q] = -S$). This results in four differential equations out of which only three are independent, viz,

$$x(1+x)\frac{d^2c_0(x)}{dx^2} + 2\Big(1-\Delta_2-x(-2+\Delta_2+\Delta_3)\Big)\frac{dc_0(x)}{dx} - (1+2\Delta_1-\Delta_t)(-2+\Delta_t)c_0(x) = 0,$$

$$x(1+x)\frac{d^2c_4(x)}{dx^2} + \Big(1-2\Delta_2+x(3-2\Delta_2-2\Delta_3)\Big)\frac{dc_4(x)}{dx} - (1+2\Delta_1-\Delta_t)(-1+\Delta_t)c_4(x) = 0,$$

$$x^2(1+x)\frac{d^2c_0(x)}{dx^2} + x\Big(3-2\Delta_2-2x(-2+\Delta_2+\Delta_3)\Big)\frac{dc_0(x)}{dx} + \Big(1-2\Delta_2-x(1+2\Delta_1-\Delta_t)(-2+\Delta_t)\Big)c_0(x),$$

$$+2x(1+x)\frac{d^2c_4(x)}{dx^2}+\Big(2-4\Delta_2-4x(-1+\Delta_2+\Delta_3)\Big)\frac{dc_4(x)}{dx}-2(2\Delta_1-\Delta_t)(-1+\Delta_t)c_4(x)=0.$$

Solving the first equation gives the solution for $c_0(x)$

$$\begin{aligned}c_0(x)=c_{01}\,{}_2F_1\Big(2-\Delta_t,1+2\Delta_1-\Delta_t,2-2\Delta_2;-x\Big)\\ +c_{02}\,x^{2\Delta_2-1}\,{}_2F_1\Big(1+2\Delta_2-\Delta_t,\Delta_t-2\Delta_3,2\Delta_2;-x\Big),\end{aligned} \tag{4.1.15}$$

which is exactly the non supersymmetric correlator (3.1.13) as expected. Similarly, solving the second equation gives,

$$\begin{aligned}c_4(x)=c_{41}\,{}_2F_1\Big(1-\Delta_t,1+2\Delta_1-\Delta_t,1-2\Delta_2;-x\Big)\\ +c_{42}\,x^{2\Delta_2}\,{}_2F_1\Big(1+2\Delta_2-\Delta_t,1+\Delta_t-2\Delta_3,1+2\Delta_2;-x\Big).\end{aligned} \tag{4.1.16}$$

The third equation mixes the coefficients giving the following constraints,

$$c_{41}=c_{01}\frac{2\Delta_2-1}{2(-1+\Delta_t)},\qquad c_{42}=-c_{02}\frac{\Delta_t-2\Delta_3}{4\Delta_2}. \tag{4.1.17}$$

Using the constraints obtained by Q action (4.1.12), and Hypergeometric function Identities given in appendix B.2, we obtain the following form for $c_5(x)$:

$$\begin{aligned}c_5(x)=c_{51}\,{}_2F_1\Big(1-\Delta_t,2\Delta_1-\Delta_t,1-2\Delta_2;-x\Big)\\ +c_{52}\,x^{2\Delta_2}\,{}_2F_1\Big(1+2\Delta_2-\Delta_t,2\Delta_1+2\Delta_2-\Delta_t,1+2\Delta_2;-x\Big),\end{aligned} \tag{4.1.18}$$

where the coefficients are given by,

$$c_{51}=c_{01}\frac{2\Delta_2-1}{2(-1+\Delta_t)},\quad c_{52}=c_{02}\frac{\Delta_t-2\Delta_1}{4\Delta_2}. \tag{4.1.19}$$

Similarly, we can repeat the same procedure to obtain $c_6(x)$:

$$\begin{aligned}c_6(x)=c_{61}\,{}_2F_1\Big(1-\Delta_t,1+2\Delta_1-\Delta_t,2-2\Delta_2;-x\Big)\\ +c_{62}\,x^{2\Delta_2-1}\,{}_2F_1\Big(2\Delta_2-\Delta_t,2\Delta_1+2\Delta_2-\Delta_t,2\Delta_2;-x\Big),\end{aligned} \tag{4.1.20}$$

with coefficients c_{61} and c_{62} given by,

$$c_{61}=c_{01}\frac{\Delta_t-2\Delta_2}{2(-1+\Delta_t)},\quad c_{62}=\frac{c_{02}}{2}. \tag{4.1.21}$$

Therefore, our final expression for the three-point function in $\mathcal{N}=1$ SCQM reads,

$$\begin{aligned}&\langle\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\rangle=\delta(\omega_1+\omega_2+\omega_3)\omega_1^{\Delta_t-2}\\ &\qquad\Bigg(c_{01}\Big({}_2F_1\left(2-\Delta_t,1+2\Delta_1-\Delta_t,2-2\Delta_2;-x\right)\left(2+\omega_1\,\theta_1\theta_3+\omega_1\,x\,\theta_2\theta_3\right)\end{aligned}$$

$$
\begin{aligned}
&+\omega_1 \frac{2\Delta_2-1}{\Delta_t-1}\, {}_2F_1\left(1-\Delta_t, 1+2\Delta_1-\Delta_t, 1-2\Delta_2; -x\right)\left(\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3\right)\Big) \\
&+c_{02}\, x^{2\Delta_2-1}\Big({}_2F_1\left(1+2\Delta_2-\Delta_t, \Delta_t-2\Delta_3, 2\Delta_2; -x\right)\left(2+\omega_1\,\theta_1\theta_3+\omega_1\, x\, \theta_2\theta_3\right) \\
&-\omega_1\, x \frac{\Delta_t-2\Delta_3}{2\Delta_2}\, {}_2F_1\left(1+2\Delta_2-\Delta_t, 1+\Delta_t-2\Delta_3, 1+2\Delta_2; -x\right)\left(\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3\right)\Big)\Bigg).
\end{aligned} \tag{4.1.22}
$$

where $x = \dfrac{\omega_2}{\omega_1}$.

4.1.2.3 Four Point Functions

We now move to the four-point case. A generic correlator reads:

$$
\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\mathbf{O}_{\Delta_4}(\omega_4,\theta_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4) \tag{4.1.23}
$$

$$
\begin{aligned}
&\big(c_0(\omega_1,\omega_2,\omega_3)+c_{12}(\omega_1,\omega_2,\omega_3)\theta_1\theta_2 \\
&+c_{13}(\omega_1,\omega_2,\omega_3)\theta_1\theta_3+c_{14}(\omega_1,\omega_2,\omega_3)\theta_1\theta_4 \\
&+c_{23}(\omega_1,\omega_2,\omega_3)\theta_2\theta_3+c_{24}(\omega_1,\omega_2,\omega_3)\theta_2\theta_4 \\
&+c_{34}(\omega_1,\omega_2,\omega_3)\theta_3\theta_4+c_{1234}(\omega_1,\omega_2,\omega_3)\theta_1\theta_2\theta_3\theta_4\big).
\end{aligned} \tag{4.1.24}
$$

Constraining the correlator by the Q Ward Identity (4.1.6), yields the following constraints:

$$
\begin{aligned}
c_{1234} &= \frac{1}{2}\left(c_{23}\omega_1 - c_{13}\omega_2 + c_{12}\omega_3\right), \quad & c_{24} &= c_{12}-c_{23}+\frac{c_0\omega_2}{2}, \\
c_{14} &= -c_{12}-c_{13}+\frac{c_0\omega_1}{2}, & c_{34} &= c_{13}+c_{23}+\frac{c_0\omega_3}{2}.
\end{aligned} \tag{4.1.25}
$$

Dilatation invariance (4.1.4) then fixes the overall scaling of the correlator, thus leading to,

$$
\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3)\mathbf{O}_{\Delta_4}(\omega_4,\theta_4)\rangle = \delta(\omega_1+\omega_2+\omega_3+\omega_4)\omega_1^{\Delta_t-3} \tag{4.1.26}
$$

$$
\begin{aligned}
&\Bigg(c_0(x,y)\left(2+\omega_1\,\theta_1\theta_4+\omega_2\;\theta_2\theta_4+\omega_3\,\theta_3\theta_4\right) \\
&+\omega_1\, c_{23}(x,y)\big(2(\theta_2\theta_3-\theta_2\theta_4+\theta_3\theta_4)+\omega_1\theta_1\theta_2\theta_3\theta_4\big) \\
&+\omega_1\, c_{13}(x,y)\big(2(\theta_1\theta_3-\theta_1\theta_4+\theta_3\theta_4)-\omega_2\theta_1\theta_2\theta_3\theta_4\big) \\
&+\omega_1\, c_{12}(x,y)\big(2(\theta_1\theta_2-\theta_1\theta_4+\theta_2\theta_4)+\omega_3\theta_1\theta_2\theta_3\theta_4\big)\Bigg),
\end{aligned} \tag{4.1.27}
$$

where $\Delta_t = \Delta_1+\Delta_2+\Delta_3+\Delta_4, x = \dfrac{\omega_2}{\omega_1}, y = \dfrac{\omega_3}{\omega_1}$.

From the component expansion, it can be seen that the components $c_{12}, \ldots$ have the same functional form as $c_0(x,y)$, i.e., Appell functions (3.1.20), but differ only by the scaling dimensions. However, we can follow the same routine as we followed for the three point function and apply K to get the constraints on coefficients. We choose not to do that and instead apply S as it has a first-order action and leads to simpler equations. If we fix the component correlators by their Appell function representation (3.1.20) and leave the coefficients undetermined, then apply S, we obtain constraints connecting these coefficients. For instance at $\mathcal{O}(\theta_1)$ and $\mathcal{O}(\theta_2)$ we find,

$$2\frac{dc_{12}}{dx}+2\frac{dc_{13}}{dy}-x\frac{dc_0}{dx}-y\frac{dc_0}{dy}-(2+\Delta_1-\Delta_2-\Delta_3-\Delta_4)c_0=0, \tag{4.1.28}$$

$$-2x\frac{dc_{12}}{dx}-2y\frac{dc_{12}}{dy}+2\frac{dc_{23}}{dy}+x\frac{dc_0}{dx}-2(2-\Delta_t)c_{12}+(1-2\Delta_2)c_0=0. \tag{4.1.29}$$

Proceeding to higher orders in the Grassmann expansion, we obtain similar constraints. We now proceed to the $\mathcal{N}=2$ case.

4.2 Correlators in $\mathcal{N}=2$ Super Conformal Quantum Mechanics

In this section, we extend our results to conformal theories that also possess $\mathcal{N}=2$ supersymmetry. First, the superspace formalism is introduced in which we work, and then the superconformal Ward identities for two and three-point functions are solved.

4.2.1 The Superspace Formalism

The arena in which we work is the $\mathcal{N}=2$ superspace. A point in this superspace is described by the triplet $(t,\theta,\bar{\theta})$ where t is the usual time coordinate and θ is a complex Grassmann variable. The momentum (or rather, frequency) superspace is spanned by the triplet $(\omega,\theta,\bar{\theta})$. The generators of the $\mathcal{N}=2$ superconformal algebra consists of the usual conformal generators H,K and D, the supersymmetry generators $Q,\bar{Q}$, the special superconformal generators $S,\bar{S}$ and the $U(1)$, R-symmetry generator R. Their algebra can be found for instance in [53]. The action of all generators on primary (bosonic) superfields $\mathbf{O}_\Delta(\omega,\theta)$ are given in appendix A.2. We bring the reader's attention to the R symmetry generator (last equation of (A.2.8)). The θ_i have R charge $+1$ while the $\bar{\theta}_i$ have R charge -1. Therefore, every single term in a correlation function should contain an equal number of θ_i and $\bar{\theta}_i$.

In the $\mathcal{N}=2$ superspace formalism, (bosonic) superfields can be expanded as follows:

$$\mathbf{O}_\Delta(\omega,\theta)=\Phi_\Delta(\omega)+\theta\,\Psi_{\Delta+\frac{1}{2}}(\omega)+\bar{\theta}\,\bar{\Psi}_{\Delta+\frac{1}{2}}(\omega)+\theta\bar{\theta}F_{\Delta+1}(\omega). \tag{4.2.1}$$

The aim is to constrain the correlation functions of these superfields by solving the superconformal ward identities. These identities read,

$$\langle[\mathcal{L},\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)]\cdots\mathbf{O}_{\Delta_n}(\omega_n,\theta_n)\rangle+\cdots\langle\mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\cdots[\mathcal{L},\mathbf{O}_{\Delta_n}(\omega_n,\theta_n)]\rangle=0,\ \mathcal{L}\in\{H,K,D,Q,S,\bar{Q},\bar{S},R\}. \tag{4.2.2}$$

The explicit form of these identities can be found in appendix A.2. We now proceed to investigate its implications for correlation functions.

4.2.2 Correlation functions

In this subsection, we obtain two and three-point functions that were obtained by solving the $\mathcal{N}=2$ superconformal Ward identities. Since the procedure of obtaining the solutions is identical to that of the $\mathcal{N}=1$ case, we shall abstain from providing details and instead provide the final results.

4.2.2.1 Two Point Functions

The two point function of a generic $\mathcal{N}=2$ primary superfield takes the following form:

$$\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1,\bar{\theta}_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2,\bar{\theta}_2)\rangle = \delta_{\Delta_1,\Delta_2}\delta(\omega_1+\omega_2)\omega_1^{2\Delta_1-1}\bigg(4+2\omega_1(\theta_1\bar{\theta}_2-\theta_2\bar{\theta}_1)-\omega_1^2\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2\bigg). \quad (4.2.3)$$

It can easily be verified that this expression satisfies the superconformal Ward identities. We also note an interesting relation between (4.2.3) and it's $\mathcal{N}=1$ counterpart (4.1.10):

$$\langle \mathbf{O}_{2\Delta_1}(\omega_1,\theta_1,\bar{\theta}_1)\mathbf{O}_{2\Delta_2}(\omega_2,\theta_2,\bar{\theta}_2)\rangle_{\mathcal{N}=2} = \omega_1\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1)\mathbf{O}_{\Delta_2}(\omega_2,\bar{\theta}_2)\rangle_{\mathcal{N}=1}\ \langle \mathbf{O}_{\Delta_1}(\omega_1,\bar{\theta}_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2)\rangle_{\mathcal{N}=1}. \quad (4.2.4)$$

The above form is the unique product of $\mathcal{N}=1$ two-point functions that possess the required R symmetry. Indeed, this is reminiscent of the super double copy obtained in three-dimensional conformal field theories in [80]. Also, super double copies at the three-point level was obtained in [80] also obtained a . Let us also thus move on to the three-point case.

4.2.2.2 Three Point Functions

Let us first define,

$$\begin{aligned} C_i(\Delta_1,\Delta_2,\Delta_3;-x) &= c_{i1}\ {}_2F_1(2-\Delta_1-\Delta_2-\Delta_3,1+\Delta_1-\Delta_2-\Delta_3,2(1-\Delta_2);-x)\\ &\quad + x^{2\Delta_2-1}\ c_{i2}\ {}_2F_1(1-\Delta_1+\Delta_2-\Delta_3,\Delta_1+\Delta_2-\Delta_3,2\Delta_2,-x), \end{aligned} \quad (4.2.5)$$

and,

$$\begin{aligned} c_0(x) = C_0\left(\Delta_1,\Delta_2,\Delta_3;-x\right), &\quad c_6(x) = C_6\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2},\Delta_3;-x\right),\\ c_8(x) = C_8\left(\Delta_1+\frac{1}{2},\Delta_2,\Delta_3+\frac{1}{2};-x\right), &\quad c_9(x) = C_9\left(\Delta_1,\Delta_2+\frac{1}{2},\Delta_3+\frac{1}{2};-x\right),\\ c_{11}(x) = C_{11}\left(\Delta_1,\Delta_2+1,\Delta_3;-x\right), &\quad c_{12}(x) = C_{12}\left(\Delta_1+1,\Delta_2+1,\Delta_3;-x\right). \end{aligned} \quad (4.2.6)$$

Our result for the $\mathcal{N}=2$ SCQM three-point correlator after solving the superconformal Ward identities (A.2.8) is the following expression:

$$\langle \mathbf{O}_{\Delta_1}(\omega_1,\theta_1,\bar{\theta}_1)\mathbf{O}_{\Delta_2}(\omega_2,\theta_2,\bar{\theta}_2)\mathbf{O}_{\Delta_3}(\omega_3,\theta_3,\bar{\theta}_3)\rangle = \delta(\omega_1+\omega_2+\omega_3)\omega_1^{\Delta_t-2}$$
$$\bigg[c_0(x)\bigg(1+\frac{\omega_1}{4}\Big(-2\theta_1\bar{\theta}_1-2x\theta_1\bar{\theta}_2+2(x+2)\theta_1\bar{\theta}_3+2x\theta_2\bar{\theta}_3-2(1+x)\theta_3\bar{\theta}_3$$

$$
\begin{aligned}
&+ x\,\omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3 + x^2\omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3 - \omega_1(1+x)\theta_1\theta_3\bar\theta_1\bar\theta_3 - x(1+x)\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3\Big)\Big)\\
&+ c_6(x)\Big(\frac{\omega_1}{2}\Big(- 2\theta_1\bar\theta_1 + 2\theta_1\bar\theta_3 + 2\theta_2\bar\theta_1 - 2\theta_2\bar\theta_3 + \omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3 + x\omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3\\
&\quad - x\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2 - \omega_1\theta_1\theta_3\bar\theta_1\bar\theta_3 - x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3 + x\omega_1\theta_2\theta_3\bar\theta_1\bar\theta_2\\
&\quad + \omega_1\theta_2\theta_3\bar\theta_1\bar\theta_3 + x\omega_1\theta_2\theta_3\bar\theta_2\bar\theta_3\Big) + \frac{x(x+1)}{4}\omega_1^3\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Big)\\
&+ c_8(x)\Big(\frac{\omega_1}{2}\Big(- 2\theta_1\bar\theta_1 + 2\theta_1\bar\theta_3 + 2\theta_3\bar\theta_1 - 2\theta_3\bar\theta_3 + x\omega_1\theta_1\theta_2\bar\theta_1\bar\theta_3 - x\omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2 - x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3\\
&\quad + x\omega_1\theta_2\theta_3\bar\theta_1\bar\theta_3\Big) + \frac{x^2\omega_1^3}{4}\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Big)\\
&+ c_9(x)\Big(\frac{\omega_1}{2}\Big(- 2\theta_1\bar\theta_2 + 2\theta_1\bar\theta_3 + 2\theta_3\bar\theta_2 - 2\theta_3\bar\theta_3 + x\omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3 + \omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2 - \omega_1\theta_1\theta_3\bar\theta_1\bar\theta_3\\
&\quad + \omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3 - x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3 + x\omega_1\theta_2\theta_3\bar\theta_2\bar\theta_3\Big) - \frac{x}{4}\omega_1^3\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Big)\\
&+ c_{11}(x)\Big(\frac{\omega_1}{2}\Big(- 2\theta_1\bar\theta_2 + 2\theta_1\bar\theta_3 + 2\theta_2\bar\theta_2 - 2\theta_2\bar\theta_3 + \omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3 + x\omega_1\theta_1\theta_2\bar\theta_2\bar\theta_3\\
&\quad + \omega_1\theta_1\theta_3\bar\theta_1\bar\theta_2 - \omega_1\theta_1\theta_3\bar\theta_1\bar\theta_3 - x\omega_1\theta_1\theta_3\bar\theta_2\bar\theta_3 - \omega_1\theta_2\theta_3\bar\theta_1\bar\theta_2\\
&\quad + \omega_1\theta_2\theta_3\bar\theta_1\bar\theta_3 + x\omega_1\theta_2\theta_3\bar\theta_2\bar\theta_3\Big) - \frac{(x+1)}{4}\omega_1^3\theta_1\theta_2\theta_3\bar\theta_1\bar\theta_2\bar\theta_3\Big)\\
&+ c_{12}(x)\omega_1^2\Big((\theta_1\theta_2\bar\theta_1\bar\theta_2 - \theta_1\theta_2\bar\theta_1\bar\theta_3 + \theta_1\theta_2\bar\theta_2\bar\theta_3 - \theta_1\theta_3\bar\theta_1\bar\theta_2 + \theta_1\theta_3\bar\theta_1\bar\theta_3 - \theta_1\theta_3\bar\theta_2\bar\theta_3 + \theta_2\theta_3\bar\theta_1\bar\theta_2\\
&\quad - \theta_2\theta_3\bar\theta_1\bar\theta_3) + \omega_1^2\theta_2\theta_3\bar\theta_2\bar\theta_3\Big)\Big].
\end{aligned}
\tag{4.2.7}
$$

where $x = \dfrac{\omega_2}{\omega_1}$ and the coefficients of the c_i are related as follows:

$$
\begin{aligned}
c_{81} &= -c_{01}\frac{\Delta_1}{\Delta_t - 1} - c_{61}\frac{\Delta_1 + \Delta_2 - \Delta_3}{2\Delta_2 - 1}, & c_{82} &= -\frac{c_{02}\Delta_1 + 2c_{62}\Delta_2}{\Delta_1 - \Delta_2 + \Delta_3},\\
c_{91} &= -\frac{c_{01}(2\Delta_2 - 1)(\Delta_1 - \Delta_3)}{(2\Delta_1 - \Delta_t)(\Delta_t - 1)} + c_{61}\frac{(2\Delta_3 - \Delta_t)}{(2\Delta_1 - \Delta_t)}, & c_{92} &= \frac{c_{02}(\Delta_1 - \Delta_3) - 2c_{62}\Delta_2}{2\Delta_2},\\
c_{121} &= c_{01}\frac{\Delta_2(1 - 2\Delta_2)}{2(\Delta_3 - 1)\Delta_t}, & c_{122} &= -c_{02}\frac{(\Delta_1 + \Delta_2 - \Delta_3)(1 + \Delta_1 + \Delta_2 - \Delta_3)}{8\Delta_2(1 + 2\Delta_2)},\\
c_{111} &= \Delta_2\frac{c_{01}(2\Delta_2 - 1) + 2c_{61}(\Delta_t - 1)}{(2\Delta_1 - \Delta_t)(\Delta_t - 1)}, & c_{112} &= (\Delta_t - 2\Delta_2 - 1)\frac{c_{02}(\Delta_t - 2\Delta_3) - 4c_{62}\Delta_2}{4\Delta_2(1 + 2\Delta_2)}.
\end{aligned}
\tag{4.2.8}
$$

Notice that in contrast with the $\mathcal{N} = 1$ three point function (4.1.22) which had two free parameters, its $\mathcal{N} = 2$ counterpart has four $(c_{01}, c_{02}, c_{61}, c_{62})$.

4.3 Summary

This part of the thesis provided an exhaustive approach to Conformal and Superconformal Quantum Mechanics. We provide all the n-point functions and a formalism to find the same in the superconformal case.

We found conformal partial waves for a four-point function and an Appell F_2 representation. It was shown that they need not to be in correspondence to each other, so we can choose to use anyone. Further, an explanation for multiple solutions to momentum space conformal ward identities was provided. Moreover, we saw that we can constrain superconformal correlators and find connections between different conformal correlators enforced by supersymmetry. Now, it is known that supersymmetry is extremely rich in higher dimensions, where it can connect different spin operators, and it provides insights into Cosmology, Superamplitude in $4d$, and String theory. Hence, in the next part of the thesis we will shift to $3d$ and invent a new formalism which will make our life way easier and give us previously unknown connections.

Part III

Superconformal Field Theory in $3d$

Chapter 5

$\mathcal{N} = 1$ Superconformal Field Theory in $3d$

It does not do to leave a live dragon out of your calculations, if you live near him.

J.R.R. Tolkien on supersymmetry, The Hobbit

This chapter will be focusing on superconformal theories in $3d$. There was no existing literature on momentum space SCFTs, and all the literature was on position space SCFTs, whose formalisms are extremely difficult to handle. Further, momentum space SCFTs are important for finding amplitudes in higher dimensions, connections between different spinning correlators, and extended supersymmetric theories like ABJM theories are extremely interesting to study. To fill in the gap in the literature, this chapter will provide the first formalism for momentum space SCFT in $3d$. We will see that this formalism consists of new variables called "Super Spinor Helicity" and "Grassmann Twistor Variables," which allow for universal structures independent of spins and thus easily generalizable to arbitrary spins. We will find exact forms of all spinning two and three-point functions in any SCFT in $3d$. We will see double copy relation connection different spinning correalators, and Super Ward Takahashi Identity.

5.1 Setting the stage: $\mathcal{N} = 1$ SCFT

The coordinates of $\mathcal{N} = 1$ superspace consist of a pair (x^μ, θ^a) where $x^\mu, \mu = 1, 2, 3$ are the usual position space coordinates and $\theta^a, a = 1, 2$ are the Grassmann coordinates of the superspace [90]. The generators of the $\mathcal{N} = 1$ superconformal algebra that act on the superspace consist of the usual conformal generators $P_\mu, M_{\mu\nu}, D, K_\mu$ along with the supersymmetry generator Q_a and the special superconformal generator S_a. The Lie super-algebra these generators obey and their action on primary superfields are provided in appendix

D and notation in appendix C.

Performing a Fourier transform with respect to the x^μ, gives momentum superspace, which is described by the coordinates (p_μ, θ^a), where p_μ is the three-momentum and θ^a are the same Grassmann coordinates as in the position superspace.

Let us now study SCFT in this superspace formalism. The quantities that we are interested in obtaining are correlation functions involving primary superfields. In particular the superfields that we are interested in are symmetric traceless conserved super-currents, which we shall henceforth just refer to as a supercurrent for brevity. A spin s supercurrent has the following component expansion in the superspace [72]:

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(\theta,\mathbf{x}) = J_s^{a_1\cdots a_{2s}}(\mathbf{x}) + \theta_m J_{s+\frac{1}{2}}^{(a_1\cdots a_{2s_1}m)}(\mathbf{x}) - i\frac{\theta^2}{4}(\not{\partial})^{a_1}_m J_s^{a_2\cdots a_{2s}m}(\mathbf{x}). \tag{5.1.1}$$

The indices $a_1\cdots a_{2s}$ are symmetrized. J_s and $J_{s+\frac{1}{2}}$ are component currents that are conserved. The supercurrent (5.1.1) satisfies the following conservation equation:

$$D_{a_1}\mathbf{J}_s^{a_1\cdots a_{2s}}(\theta,\mathbf{x}) = 0, \tag{5.1.2}$$

where $D_{a_1} = \frac{\partial}{\partial\theta^{a_1}} - \frac{i}{2}\theta_b(\sigma^\mu)^b_{a_1}\partial_\mu$, is the supercovariant derivative.
Performing a fourier transform with respect $\mathbf{x}$, we obtain the momentum superspace counterpart to (5.1.1):

$$\mathbf{J}_s^{a_1\cdots a_{2s}}(\theta,\mathbf{p}) = J_s^{a_1\cdots a_{2s}}(\mathbf{p}) + \theta_m J_{s+\frac{1}{2}}^{(a_1\cdots a_{2s_1}m)}(\mathbf{p}) + \frac{\theta^2}{4}(\not{p})^{a_1}_m J_s^{a_2\cdots a_{2s}m}(\mathbf{p}). \tag{5.1.3}$$

We can now construct correlation functions of these operators using the definition (5.1.3). The super-correlators obey the $\mathcal{N}=1$ super ward identities. For instance, consider the supersymmetry generator Q_a. It has the following action on superfields:

$$Q_{ia} = \left(\frac{\partial}{\partial\theta_i^a} - \frac{(\not{p}_i)_a^b}{2}\theta_{ib}\right). \tag{5.1.4}$$

For an n-point correlation function, invariance under the Q supersymmetry demands that,

$$\sum_{i=1}^n Q_{ia}\langle\mathbf{J}_{s_1}^{a_1\cdots a_{2s_1}}(\theta_1,\mathbf{p}_1)\ldots\mathbf{J}_{s_n}^{c_1\cdots c_{2s_n}}(\theta_n,\mathbf{p}_n)\rangle = 0. \tag{5.1.5}$$

This implies that n-point correlators take the following form [68, 70]:

$$\langle\mathbf{J}_{s_1}^{a_1\cdots a_{2s_1}}(\theta_1,\mathbf{p}_1)\ldots\mathbf{J}_{s_n}^{c_1\cdots c_{2s_n}}(\theta_n,\mathbf{p}_n)\rangle = e^{-\frac{1}{2n}(\theta_1^a+\cdots+\theta_n^a)\left(\theta_{1b}(\not{p}_1)_{ab}+\cdots+\theta_{nb}(\not{p}_n)_{ab}\right)} F^{a_1\cdots c_{2s_n}}(\{\theta_i-\theta_j\},\{\mathbf{p}_i\}), \tag{5.1.6}$$

where,

$$F^{a_1\cdots c_{2s_n}}(\{\theta_i-\theta_j\},\{\mathbf{p}_i\}) = F_1^{a_1\cdots c_{2s_n}}(\{\mathbf{p}_i\}) + (\theta_{1m}-\theta_{2m})F_2^{a_1\cdots c_{2s_n}m}(\{\mathbf{p}_i\}) + \cdots, \tag{5.1.7}$$

contains undetermined functions $F_i(\{\mathbf{p}_i\})$ packaged together via a Grassmann spinor expansion. One can now impose the Ward identities due to the other superconformal generators and constrain the form of these functions. These constraints, however, take the form of coupled partial differential equations involving the $F_i(\{\mathbf{p}_i\})$ and hence quite difficult to solve. There is, however, an alternative method to proceed. We expand the superfields in the correlator using (5.1.3). The result is a sum of component correlators arranged in a Grassmann spinor expansion. The invariance of the individual component correlators guarantees invariance under the conformal transformations. Please see appendix of [80] for proof. We then substitute this component correlator expansion into the LHS of (5.1.6) and equate it to the RHS order by order in the Grassmann spinor expansion. Once the resulting *algebraic* equations involving the component correlators and the $F_i(\{\mathbf{p}_i\})$ are solved, the resulting quantity is invariant under the action of the entire superconformal algebra.

5.1.1 Constraining super correlators in momentum super space

Let us now see the methodology just outlined via a three-point example. Consider the spins $s_1=\frac{3}{2}, s_2 = s_3 = \frac{1}{2}$. Using the superfield expansions provided in (5.1.3) we obtain,

$$\begin{aligned}\langle \mathbf{J}_{\frac{3}{2}}{}^{(efg)}\mathbf{J}_{\frac{1}{2}}{}^{a}\mathbf{J}_{\frac{1}{2}}{}^{b}\rangle &= \theta_{1h}\langle T^{(efgh)}O^a_{\frac{1}{2}}O^b_{\frac{1}{2}}\rangle - \theta_{2c}\langle J^{(efg)}_{\frac{3}{2}}J^{(ac)}O^b_{\frac{1}{2}}\rangle + \theta_{3d}\langle J^{(efg)}_{\frac{3}{2}}O^a_{\frac{1}{2}}J^{(bd)}\rangle\\ &+\frac{1}{4}\theta_{1h}\theta_2^2(\not{p}_2)^a_c\langle T^{(efgh)}O^c_{\frac{1}{2}}O^b_{\frac{1}{2}}\rangle + \frac{1}{4}\theta_{1h}\theta_3^2(\not{p}_3)^b_d\langle T^{(efgh)}O^a_{\frac{1}{2}}O^d_{\frac{1}{2}}\rangle - \frac{1}{4}\theta_{2c}\theta_1^2\langle J^{(hfg)}_{\frac{3}{2}}J^{(ac)}O^b_{\frac{1}{2}}\rangle\\ &-\frac{1}{4}\theta_{2c}\theta_3^2(\not{p}_3)^b_d\langle J^{(efg)}_{\frac{3}{2}}J^{(ac)}O^d_{\frac{1}{2}}\rangle + \frac{1}{4}\theta_{3d}\theta_1^2(\not{p}_1)^e_h\langle J^{(hfg)}_{\frac{3}{2}}O^a_{\frac{1}{2}}J^{(bd)}\rangle + \frac{1}{4}\theta_{3d}\theta_2^2(\not{p}_2)^a_c\langle J^{(efg)}_{\frac{3}{2}}O^c_{\frac{1}{2}}J^{bd)}\rangle\\ &+\theta_{1h}\theta_{2c}\theta_{3d}\langle T^{(efgh)}J^{(ac)}J^{(bd)}\rangle + \frac{1}{16}\theta_1^2\theta_2^2\theta_{3d}(\not{p}_1)^e_h(\not{p}_2)^a_c\langle J^{(hfg)}_{\frac{3}{2}}O^c_{\frac{1}{2}}J^{(bd)}\rangle\\ &+\frac{1}{16}\theta_1^2\theta_3^2\theta_{2c}(\not{p}_1)^e_h(\not{p}_3)^b_d\langle J^{(hfg)}_{\frac{3}{2}}J^{(ac)}O^d_{\frac{1}{2}}\rangle + \frac{1}{16}\theta_2^2\theta_3^2\theta_{1h}(\not{p}_2)^a_c(\not{p}_3)^b_d\langle T^{(efgh)}O^c_{\frac{1}{2}}O^d_{\frac{1}{2}}\rangle,\end{aligned} \tag{5.1.8}$$

while the analogue of (5.1.6) is given by,

$$\begin{aligned}\langle \mathbf{J}_{\frac{3}{2}}{}^{(efg)}\mathbf{J}_{\frac{1}{2}}{}^{a}\mathbf{J}_{\frac{1}{2}}{}^{b}\rangle &= e^{\frac{1}{6}(\theta_1^m+\theta_2^m+\theta_3^m)\left((\theta_1-\theta_2)^n(\not{p}_2)_{mn}+(\theta_1-\theta_3)^n(\not{p}_3)_{mn}\right)}\Big(A_1^{(efg)abl}(\theta_1-\theta_2)_l\\ &+A_2^{(efg)abl}(\theta_1-\theta_3)_l + B_1^{(efg)abl}(\theta_1-\theta_2)_l(\theta_1-\theta_3)^2 + B_2^{(efg)abl}(\theta_1-\theta_3)_l(\theta_1-\theta_2)^2\Big).\end{aligned} \tag{5.1.9}$$

We now equate equations (5.1.8) and (5.1.9) order by order in the Grassmann spinor expansion. For instance, we obtain at the lowest order,

$$\begin{aligned}A_1^{(efg)abl} &= \langle J^{(efg)}_{3/2}J^{(al)}O^b_{1/2}\rangle,\\ A_2^{(efg)abl} &= -\langle J^{(efg)}_{3/2}O^a_{1/2}J^{(bl)}\rangle,\\ A_1^{(efg)abl}+A_2^{(efg)abl} &= \langle T^{(efgl)}O^a_{1/2}O^b_{1/2}\rangle,\end{aligned} \tag{5.1.10}$$

which forces the relation,

$$\langle T^{(efgl)}O^a_{1/2}O^b_{1/2}\rangle = \langle J^{(efg)}_{3/2}J^{(al)}O^b_{1/2}\rangle - \langle J^{(efg)}_{3/2}O^a_{1/2}J^{(bl)}\rangle. \tag{5.1.11}$$

One can independently check that this holds true by computing LHS and RHS seperately. Going to higher orders in the Grasmann expansion yields similar but albeit more complicated and not very insightful relations also involving the component $\langle TJJ\rangle$ correlator.

In order to proceed further, we then use the fact that in a generic 3d CFT, a three point (component) correlator can have at most three independent structures [91]:

$$\langle J_{s_1} J_{s_2} J_{s_3}\rangle = n_b \langle J_{s_1} J_{s_2} J_{s_3}\rangle_b + n_f \langle J_{s_1} J_{s_2} J_{s_3}\rangle_f + n_{odd}\langle J_{s_1} J_{s_2} J_{s_3}\rangle_{odd}, \tag{5.1.12}$$

where the first and the second term are the free bosonic and free fermionic correlators whilst the third term is a parity odd piece. Using (5.1.12) for each of the component correlators appearing in the superfield expansion and demanding that relations such as (5.1.11) are satisfied, reduces the number of independent coeffcients in (5.1.12) for every component correlator. In fact, for the correlator in (5.1.8), we find $n_f = n_b$ and $n_{odd} = 0$ for all the component correlators thereby obtaining a single parity even solution:

$$\begin{aligned}
\langle TO_{1/2}O_{1/2}\rangle &= n_b\big(\langle TO_{1/2}O_{1/2}\rangle_{FB} + \langle TO_{1/2}O_{1/2}\rangle_{FF}\big),\\
\langle J_{3/2}JO_{1/2}\rangle &= n_b\big(\langle J_{3/2}JO_{1/2}\rangle_{FB} + \langle J_{3/2}JO_{1/2}\rangle_{FF}\big),\\
\langle TJJ\rangle &= n_b\big(\langle TJJ\rangle_{FB} + \langle TJJ\rangle_{FF}\big),
\end{aligned} \tag{5.1.13}$$

a result that is consistent with [78], that is, the existence of a single parity even solution. The final expression for the correlator is obtained by substituting (5.1.13) back into (5.1.8) or equivalently (5.1.9). By construction this correlator is invariant under the action of all the generators of the superconformal algebra.

While it is an excellent technique to find all the components in terms of just correlators, the procedure is quite complicated. Other than dealing with various degeneracy identities[1], there is no simple generalization to arbitrary spin correlators, no obvious connection to the flat space scattering amplitudes in four dimensions, or the existence of double copy relations between various super-correlators. Now, we can define our new variables in the next section to surpass this tedious process. A worked-out example of the same correlator and getting the same constraints on the coefficients using our new formalism is provided in appendix of [80], circumventing all the complexities of momentum super-space variables in one go.

5.1.2 En route to Spinor Helicity and Grassmann Twistor Variables

Rather than attempting brute force, we can take inspiration from works in supersymmetric scattering amplitude. We can break down Grassmann spinors into Grassmann numbers. Using this approach, we discovered a different set of variables that simplify the analysis significantly. As a first step, let us use the spinor helicity variable defined as

$$p_\mu = \frac{1}{2}(\sigma_\mu)^a_b \lambda_a \overline{\lambda}^b, \tag{5.1.14}$$

where λ and $\overline{\lambda}$ are two component commuting spinors. The momentum p_μ is invariant under the little group transformation, $\lambda \to r\lambda, \overline{\lambda} \to r^{-1}\overline{\lambda}$, $r \in \mathbb{C}$. As in the case of non-supersymmetruc correlators, the introduction of spinor helicity variables greatly helps us in dealing with degeneracies which leads to simplification of algebra as well as expressions. However, to truly exploit the power of spinor helicity variables,

[1]An example of such an identity is $(\not{p}_1)^l_f(\not{p}_2)^{mb} = (\not{p}_1)^{bl}(\not{p}_2)^m_f - (\not{p}_1)^{kl}(\not{p}_2)^m_k\delta^b_f$.

we also need to express the Grassmann spinors θ^a in the basis of λ and $\overline{\lambda}$. We define,

$$\theta^a = \frac{\overline{\eta}\lambda^a + \eta\,\overline{\lambda}^a}{2p}, \tag{5.1.15}$$

where η is a complex Grassmann variable and $\overline{\eta}$ is it's complex conjugate. This definition is consistent with the fact that the dimensionality of θ is $-\frac{1}{2}$. Further, η and $\overline{\eta}$ must transform as $\eta \to r\eta$, $\overline{\eta} \to r^{-1}\,\overline{\eta}$ under little group scalings so that θ remains unaffected.

We then contract the superfield (5.1.3) with the polarization spinors which are given by,

$$\zeta_a^- = \frac{\lambda_a}{\sqrt{p}}, \qquad \zeta_a^+ = \frac{\overline{\lambda}_a}{\sqrt{p}}, \tag{5.1.16}$$

and express the grassmann spinor θ^a in terms of the $\eta, \overline{\eta}$ variables defined in (5.1.15). We obtain the super-current in the $\pm s$ helicities[2] ,

$$\begin{aligned}\mathbf{J}_s^- &= e^{-\frac{\eta\overline{\eta}}{4}} J_s^- + \frac{\overline{\eta}}{2\sqrt{p}} J^-_{s+\frac{1}{2}},\\ \mathbf{J}_s^+ &= e^{\frac{\eta\overline{\eta}}{4}} J_s^+ + \frac{\eta}{2\sqrt{p}} J^+_{s+\frac{1}{2}},\end{aligned} \tag{5.1.17}$$

where, $\mathbf{J}_s^\pm := \zeta^\pm_{a_1}\cdots\zeta^\pm_{a_{2s}}\mathbf{J}_s^{a_1\cdots a_{2s}}$. The Super-correlator (5.1.8) in these variables becomes extremely simple. For instance in the $(--+)$ helicity configuration we obtain[3],

$$\begin{aligned}\langle \mathbf{J}^-_{\frac{3}{2}}\mathbf{J}^-_{\frac{1}{2}}\mathbf{J}^+_{\frac{1}{2}}\rangle = \frac{1}{2\sqrt{p_3}}\langle J^-_{3/2}O^-_{1/2}T^+\rangle\Big(& e^{-\frac{\eta_1\bar{\eta}_1}{4}}e^{-\frac{\eta_2\bar{\eta}_2}{4}}\eta_3\\ &-\frac{1}{E}\left(e^{-\frac{\eta_1\bar{\eta}_1}{4}}e^{\frac{\eta_3\bar{\eta}_3}{4}}\bar{\eta}_2\langle 23\rangle - e^{-\frac{\eta_2\bar{\eta}_2}{4}}e^{\frac{\eta_3\bar{\eta}_3}{4}}\bar{\eta}_1\langle 31\rangle - \frac{\bar{\eta}_1\bar{\eta}_2\eta_3}{2}\langle 12\rangle\right)\Big).\end{aligned} \tag{5.1.18}$$

These exponential of the Grassmann bilinears that appear in this expression suggest performing a Grassmann "half" Fourier transform analogous to the twistor transform [92, 93]. Given a function $F(\eta,\overline{\eta})$ a Grassmann Twistor transform can be defined as follows:

$$\tilde{F}(\eta,\chi) := \int d\overline{\eta}\; e^{-\frac{\chi\overline{\eta}}{4}} F(\eta,\overline{\eta}). \tag{5.1.19}$$

Making a variable change from (η,χ) to (ξ_+,ξ_-) which are defined as,

$$\xi_\pm = \chi \pm \eta. \tag{5.1.20}$$

In these new Grassmann "Twistor" Variables, the correlator takes the following magnificient form[4]:

$$\langle \tilde{\mathbf{J}}^-_{\frac{3}{2}}\tilde{\mathbf{J}}^-_{\frac{1}{2}}\tilde{\mathbf{J}}^+_{\frac{1}{2}}\rangle = -\frac{\xi_{3-}}{256\sqrt{p_3}}\langle J^-_{\frac{3}{2}}J^-_{\frac{1}{2}}J^+_1\rangle\left[\xi_{1+}\xi_{2+}\xi_{3+} - \frac{8}{E}\big(\xi_{1+}\langle 23\rangle + \xi_{2+}\langle 31\rangle + \xi_{3+}\langle 12\rangle\big)\right]. \tag{5.1.21}$$

[2]We also use the fact that the supercurrent (5.1.3) is symmetric, traceless and conserved and hence it has only two independent components which are the $h=\pm s$ ones.

[3]Similar expressions can be obtained for all other helicities.

[4]We can then proceed to obtain similar expressions in all the other helicity configurations as well.

Contrasted with (5.1.8), the new representation (5.1.21) is not only simpler but also appears in a language that is analogous to the four-dimensional flat space scattering amplitudes [93]. Further, we shall see in the next section that the structure appearing inside the brackets in (5.1.21) is one of two universal structures that appear in three-point super-correlators with arbitrary (half) integer spin insertions! Let us, from first principles, develop our formalism in these new variables.

Grassmann Twistor Variables

The superfield expansion in the grassmann twistor variable can be obtained by performing the transformation (5.1.19) on (5.1.17). The result is [5],

$$\begin{aligned}\tilde{\mathbf{J}}_s^- &= \frac{1}{4}\bigg(\xi_+ J_s^- + \frac{2}{\sqrt{p}}J^-_{s+\frac{1}{2}}\bigg),\\ \tilde{\mathbf{J}}_s^+ &= \frac{\xi_-}{4}\bigg(J_s^+ + \frac{\xi_+}{4\sqrt{p}}J^+_{s+\frac{1}{2}}\bigg).\end{aligned} \tag{5.1.22}$$

We then construct correlation functions of the super-currents in these new variables using the superfield expansion (5.1.22). The only Ward identity that we need to impose is the one due to the Q supersymmetry. In these new variables we have,

$$Q_{ia} = 2\lambda_{ia}\frac{\partial}{\partial\xi_{i+}} + \frac{\bar{\lambda}_{ia}}{4}\xi_{i+}. \tag{5.1.23}$$

The associated Ward identity reads,

$$\sum_{i=1}^{n} Q_{ia}\langle\tilde{\mathbf{J}}^{\pm}_{s_1}\cdots\tilde{\mathbf{J}}^{\pm}_{s_n}\rangle = 0. \tag{5.1.24}$$

As we can see from (5.1.23), $Q_a = \sum_{i=1}^{n} Q_{ia}$ is a two component spinor operator. Therefore, it can be written in the following way[6]:

$$Q_a = \lambda_{1a}q + \bar{\lambda}_{1a}\,\bar{q}, \tag{5.1.25}$$

for some q, $\bar{q}$. It is easy to show (using the Schouten identity, see appendix C) that we have,

$$\begin{aligned}q &= 2\frac{\partial}{\partial\xi_{1+}} + \frac{1}{p_1}\sum_{i=2}^{n}\bigg(\langle\bar{1}i\rangle\frac{\partial}{\partial\xi_{i+}} + \frac{\langle\bar{1}\,\bar{i}\rangle}{8}\xi_{i+}\bigg),\\ \bar{q} &= \frac{\xi_{1+}}{4} - \frac{1}{p_1}\sum_{i=2}^{n}\bigg(\langle 1i\rangle\frac{\partial}{\partial\xi_{i+}} + \frac{\langle 1\,\bar{i}\rangle}{8}\xi_{i+}\bigg).\end{aligned} \tag{5.1.26}$$

[5] We note the similarity between (5.1.22) and the $\mathcal{N}=1$ superfields in the four dimensional flat space literature, see for instance [94].

[6] We choose to work in the $(\lambda_{1a},\,\bar{\lambda}_{1a})$ basis. One can also choose to work in other basis but the results that we obtain will be basis independent.

Thus, the Q Ward identity (5.1.24), splits into two simpler Ward identities viz,

$$\begin{aligned}
&q\langle\tilde{\mathbf{J}}^{\pm}_{s_1}\cdots\tilde{\mathbf{J}}^{\pm}_{s_n}\rangle=0,\\
&\bar{q}\langle\tilde{\mathbf{J}}^{\pm}_{s_1}\cdots\tilde{\mathbf{J}}^{\pm}_{s_n}\rangle=0.
\end{aligned}\tag{5.1.27}$$

We need not impose the Ward identities at the level of the Super-correlator for the same reason that we mentioned just below (5.1.7), i.e., imposing conformal invariance at the component level suffices to have conformal invariance at the level of the Super-correlator. Further, the different coefficients that appear in the component correlators (5.1.12) get related to each other due to the Q_a ward identities (5.1.27).

5.2 Correlation functions in $\mathcal{N}=1$ SCFTs

In this section, we shall present our final results for two and three point correlators in the $\xi_\pm$ variables (5.1.20).

5.2.1 Two Point Functions

5.2.1.1 $\langle\tilde{\mathbf{J}}_s\tilde{\mathbf{J}}_s\rangle$, $s\in\mathbb{Z}_{>0}$

We obtain the following expressions for the correlator in the $(--)$ and $(++)$ helicity configurations:

$$\begin{aligned}
\langle\tilde{\mathbf{J}}^-_s\tilde{\mathbf{J}}^-_s\rangle&=\frac{\langle 12\rangle^{2s}}{16p_1}\left(\xi_{1+}\xi_{2+}-\frac{4\langle 12\rangle}{p_1}\right),\\
\langle\tilde{\mathbf{J}}^+_s\tilde{\mathbf{J}}^+_s\rangle&=-\xi_{1-}\xi_{2-}\frac{\langle\bar{1}\,\bar{2}\rangle^{2s+1}}{256p_1^2}\left(\xi_{1+}\xi_{2+}-\frac{4\langle 12\rangle}{p_1}\right).
\end{aligned}\tag{5.2.1}$$

5.2.1.2 $\langle\tilde{\mathbf{J}}_s\tilde{\mathbf{J}}_s\rangle$, $s=k+\frac{1}{2}$, $k\in\mathbb{Z}_{\geq 0}$

The expressions of the correlator in the $(--)$ and $(++)$ helicity configurations are given by,

$$\begin{aligned}
\langle\tilde{\mathbf{J}}^-_s\tilde{\mathbf{J}}^-_s\rangle&=-\frac{\langle 12\rangle^{2s}}{16p_1}\left(\xi_{1+}\xi_{2+}-\frac{4\langle 12\rangle}{p_1}\right),\\
\langle\tilde{\mathbf{J}}^+_s\tilde{\mathbf{J}}^+_s\rangle&=\xi_{1-}\xi_{2-}\frac{\langle\bar{1}\,\bar{2}\rangle^{2s+1}}{256p_1^2}\left(\xi_{1+}\xi_{2+}-\frac{4\langle 12\rangle}{p_1}\right).
\end{aligned}\tag{5.2.2}$$

5.2.1.3 Summary

Based on our results, (5.2.1) and (5.2.2), we see that general two point functions of any (half) integer spin s conserved super-currents take the following form:

$$\langle\tilde{\mathbf{J}}^-_s\tilde{\mathbf{J}}^-_s\rangle=(-1)^{2s}\frac{\langle 12\rangle^{2s}}{16p_1}\Xi_2,\quad\langle\tilde{\mathbf{J}}^+_s\tilde{\mathbf{J}}^+_s\rangle=\xi_{1-}\xi_{2-}(-1)^{2s+1}\frac{\langle\bar{1}\,\bar{2}\rangle^{2s+1}}{256p_1^2}\Xi_2,\tag{5.2.3}$$

where we have defined the two point building block,

$$\Xi_2 = \left(\xi_{1+}\xi_{2+} - \frac{4\langle 12\rangle}{p_1}\right). \tag{5.2.4}$$

Thus we see that for any spin s, the two point correlators are given by a simple kinematic factor times a universal factor Ξ_2. We will find a similar structure at the level of three points as well.

5.2.2 Three Point Correlation Functions

In this subsection, we present our results for three point correlators with arbitrary (half) integer spin insertions. In contrast to the previous analysis in position space [72, 78] etc..., we find a universal form for any super-correlator independent of the spins of the operators.

5.2.2.1 $\langle \tilde{\mathbf{J}}_{s_1}\tilde{\mathbf{J}}_{s_2}\tilde{\mathbf{J}}_{s_3}\rangle$, $s_1, s_2, s_3 \in \mathbb{Z}_{>0}$

The correlator of three integer spin conserved super-currents is given by the following expressions in the various helicity configurations:

$$\begin{aligned}
\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= \frac{1}{64}\langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle\Gamma_3, & \langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= \frac{\xi_{1-}\xi_{2-}\xi_{3-}}{512}\langle J^+_{s_1}J^+_{s_2}J^+_{s_3}\rangle\,\Xi_3,\\
\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= -\frac{\xi_{1-}\xi_{2-}}{512}\frac{\langle J^+_{s_1}J^+_{s_2}J^-_{s_3}\rangle E}{\langle 12\rangle}\Gamma_3, & \langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= -\frac{\xi_{3-}}{64}\frac{\langle J^-_{s_1}J^-_{s_2}J^+_{s_3}\rangle E}{\langle \bar{1}\,\bar{2}\rangle}\Xi_3,\\
\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= -\frac{\xi_{1-}\xi_{3-}}{512}\frac{\langle J^+_{s_1}J^-_{s_2}J^+_{s_3}\rangle E}{\langle 13\rangle}\Gamma_3, & \langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= -\frac{\xi_{2-}}{64}\frac{\langle J^-_{s_1}J^+_{s_2}J^-_{s_3}\rangle E}{\langle \bar{3}\,\bar{1}\rangle}\Xi_3,\\
\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= -\frac{\xi_{2-}\xi_{3-}}{512}\frac{\langle J^-_{s_1}J^+_{s_2}J^+_{s_3}\rangle E}{\langle 23\rangle}\Gamma_3, & \langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= -\frac{\xi_{1-}}{64}\frac{\langle J^+_{s_1}J^-_{s_2}J^-_{s_3}\rangle E}{\langle \bar{2}\,\bar{3}\rangle}\Xi_3,
\end{aligned} \tag{5.2.5}$$

where Γ_3 and Ξ_3 are given by,

$$\begin{aligned}
\Gamma_3 &= \left[\xi_{1+}\xi_{2+}\xi_{3+} - \frac{8}{E}\big(\xi_{1+}\langle 23\rangle + \xi_{2+}\langle 31\rangle + \xi_{3+}\langle 12\rangle\big)\right],\\
\Xi_3 &= \left[8 - \frac{1}{E}\big(\xi_{1+}\xi_{2+}\langle \bar{1}\,\bar{2}\rangle + \xi_{2+}\xi_{3+}\langle \bar{2}\,\bar{3}\rangle + \xi_{3+}\xi_{1+}\langle \bar{3}\,\bar{1}\rangle\big)\right],
\end{aligned} \tag{5.2.6}$$

which we note is similar to the two point case (5.2.4) where super-correlators are constructed by multiplying a component correlator and a universal spin independent factor, except that in the three point case, we have two such universal factors. Further, we find that for super-correlators that obey the triangle inequality, the only non zero components are the $(- - -)$ and $(+ + +)$ helicity configurations, i.e, the Super-correlator is homogeneous and has both a parity even and a parity odd solution. In fact we have the universal formulae [21].

$$\begin{aligned}
\langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle &= (c_{even} + ic_{odd})\frac{\langle 12\rangle^{s_1+s_2-s_3}\langle 23\rangle^{s_2+s_3-s_1}\langle 31\rangle^{s_1+s_3-s_2}}{E^{s_1+s_2+s_3}}p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1},\\
\langle J^+_{s_1}J^+_{s_2}J^+_{s_3}\rangle &= (c_{even} - ic_{odd})\frac{\langle \bar{1}\bar{2}\rangle^{s_1+s_2-s_3}\langle \bar{2}\bar{3}\rangle^{s_2+s_3-s_1}\langle \bar{3}\bar{1}\rangle^{s_1+s_3-s_2}}{E^{s_1+s_2+s_3}}p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1},
\end{aligned} \tag{5.2.7}$$

where c_{even} and c_{odd} are the OPE coefficients corresponding to the parity even and parity odd structures respectively. The relations between the various component correlators as in (5.2.5) relate the various component correlators to each other, thereby fixing their OPE coeffcients just in terms of c_{even} and c_{odd}.

For super-correlators that violate the triangle inequality, their is no parity odd solution as the component correlator on the RHS of (5.2.5) has no parity odd piece in this case. There exists however, an even piece. In terms of the free bosonic and free fermionic correlators it is given by,

$$\langle J^{\pm}_{s_1} J^{\pm}_{s_2} J^{\pm}_{s_3}\rangle = n_b\Big(\langle J^{\pm}_{s_1} J^{\pm}_{s_2} J^{\pm}_{s_3}\rangle_{FB} - \langle J^{\pm}_{s_1} J^{\pm}_{s_2} J^{\pm}_{s_3}\rangle_{FF}\Big). \tag{5.2.8}$$

Substituting (5.2.8) in (5.2.5) and reading off the component correlators using (5.1.22), the other component correlators can be obtained.

5.2.2.2 $\langle \tilde{\mathbf{J}}_{s_1}\tilde{\mathbf{J}}_{s_2}\tilde{\mathbf{J}}_{s_3}\rangle$, $s_1, s_2 \in \mathbb{Z}_{>0}, s_3 = k_3 + \frac{1}{2},\ k_3 \in \mathbb{Z}_{\geq 0}$

We obtained the following expressions for the various helicity configurations for this half integer spin correlator:

$$\begin{aligned}
\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= -\frac{1}{32}\frac{\langle J^-_{s_1}J^-_{s_2}J^-_{k_3+1}\rangle E}{\langle \bar{1}\,\bar{2}\rangle\sqrt{p_3}}\,\Xi_3, & \langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= -\frac{\xi_{1-}\xi_{2-}\xi_{3-}}{2048}\frac{\langle J^+_{s_1}J^+_{s_2}J^+_{k_3+1}\rangle E}{\langle 12\rangle\sqrt{p_3}}\,\Gamma_3,\\
\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= \frac{\xi_{1-}\xi_{2-}}{256}\frac{\langle J^+_{s_1}J^+_{s_2}J^-_{k_3+1}\rangle}{\sqrt{p_3}}\,\Xi_3, & \langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= \frac{\xi_{3-}}{256}\frac{\langle J^-_{s_1}J^-_{s_2}J^+_{k_3+1}\rangle}{\sqrt{p_3}}\,\Gamma_3,\\
\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= \frac{\xi_{1-}\xi_{3-}}{256}\frac{\langle J^+_{s_1}J^-_{s_2}J^+_{k_3+1}\rangle E}{\langle \bar{2}\,\bar{3}\rangle\sqrt{p_3}}\,\Xi_3, & \langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= \frac{\xi_{2-}}{256}\frac{\langle J^-_{s_1}J^+_{s_2}J^-_{k_3+1}\rangle E}{\langle 23\rangle\sqrt{p_3}}\,\Gamma_3,\\
\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= \frac{\xi_{2-}\xi_{3-}}{256}\frac{\langle J^-_{s_1}J^+_{s_2}J^+_{k_3+1}\rangle E}{\langle \bar{3}\,\bar{1}\rangle\sqrt{p_3}}\,\Xi_3, & \langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= -\frac{\xi_{1-}}{256}\frac{\langle J^+_{s_1}J^-_{s_2}J^-_{k_3+1}\rangle E}{\langle 31\rangle\sqrt{p_3}}\,\Gamma_3.
\end{aligned} \tag{5.2.9}$$

In contrast to the previous case with all integer spins, this class of correlators is non-homogeneous, that is, the component correlators appear as a sum of the bosonic and fermionic correlators $\langle\,\rangle_B + \langle\,\rangle_F$. The parity odd structure does not exist for these correlators.

5.2.2.3 $\langle \tilde{\mathbf{J}}_{s_1}\tilde{\mathbf{J}}_{s_2}\tilde{\mathbf{J}}_{s_3}\rangle$, $s_1 \in \mathbb{Z}_{>0}, s_i = k_i + \frac{1}{2}, i = 2,3$ and $k_i \in \mathbb{Z}_{\geq 0}$

This integer spin super-correlator has the following components:

$$\begin{aligned}
\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= \frac{1}{64}\langle J^-_{s_1}J^-_{k_2+\frac{1}{2}}J^-_{k_3+\frac{1}{2}}\rangle\Gamma_3, & \langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= -\frac{\xi_{1-}\xi_{2-}\xi_{3-}}{512}\langle J^+_{s_1}J^+_{k_2+\frac{1}{2}}J^+_{k_3+\frac{1}{2}}\rangle\,\Xi_3,\\
\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= \frac{\xi_{1-}\xi_{2-}}{512}\frac{\langle J^+_{s_1}J^+_{k_2+\frac{1}{2}}J^-_{k_3+\frac{1}{2}}\rangle E}{\langle 12\rangle}\,\Gamma_3, & \langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= \frac{\xi_{3-}}{64}\frac{\langle J^-_{s_1}J^-_{k_2+\frac{1}{2}}J^+_{k_3+\frac{1}{2}}\rangle E}{\langle \bar{1}\,\bar{2}\rangle}\,\Xi_3,\\
\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= \frac{\xi_{1-}\xi_{3-}}{512}\frac{\langle J^+_{s_1}J^-_{k_2+\frac{1}{2}}J^+_{k_3+\frac{1}{2}}\rangle E}{\langle 13\rangle}\,\Gamma_3, & \langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= \frac{\xi_{2-}}{64}\frac{\langle J^-_{s_1}J^+_{k_2+\frac{1}{2}}J^-_{k_3+\frac{1}{2}}\rangle E}{\langle \bar{3}\,\bar{1}\rangle}\,\Xi_3,\\
\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle &= \frac{\xi_{2-}\xi_{3-}}{512}\frac{\langle J^-_{s_1}J^+_{k_2+\frac{1}{2}}J^+_{k_3+\frac{1}{2}}\rangle E}{\langle 23\rangle}\,\Gamma_3, & \langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle &= \frac{\xi_{1-}}{64}\frac{\langle J^+_{s_1}J^-_{k_2+\frac{1}{2}}J^-_{k_3+\frac{1}{2}}\rangle E}{\langle \bar{2}\,\bar{3}\rangle}\,\Xi_3.
\end{aligned} \tag{5.2.10}$$

Similar to the all integer spin case (5.2.5), this correlator also has components that appear as a difference between the bosonic and fermionic structures. In addition, there also exists a parity odd structure when the triangle inequality is satisfied.

5.2.2.4 $\langle \tilde{\mathbf{J}}_{s_1}\tilde{\mathbf{J}}_{s_2}\tilde{\mathbf{J}}_{s_3}\rangle$, $s_i = k_i + \frac{1}{2}, i = 1,2,3$ and $k_i \in \mathbb{Z}_{\geq 0}$

This correlator involving three half integer insertions has the following expressions in the various helicities:

$$\begin{aligned}
&\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle = \frac{1}{32}\frac{\langle J^-_{k_1+\frac{1}{2}}J^-_{k_2+\frac{1}{2}}J^-_{k_3+1}\rangle E}{\langle \bar{1}\,\bar{2}\rangle\sqrt{p_3}}\,\Xi_3, &&\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle = \frac{\xi_{1-}\xi_{2-}\xi_{3-}}{2048}\frac{\langle J^+_{k_1+\frac{1}{2}}J^+_{k_2+\frac{1}{2}}J^+_{k_3+1}\rangle E}{\langle 12\rangle\sqrt{p_3}}\,\Gamma_3,\\
&\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle = -\frac{\xi_{1-}\xi_{2-}}{256}\frac{\langle J^+_{k_1+\frac{1}{2}}J^+_{k_2+\frac{1}{2}}J^-_{k_3+1}\rangle}{\sqrt{p_3}}\,\Xi_3, &&\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle = -\frac{\xi_{3-}}{256}\frac{\langle J^-_{k_1+\frac{1}{2}}J^-_{k_2+\frac{1}{2}}J^+_{k_3+1}\rangle}{\sqrt{p_3}}\,\Gamma_3,\\
&\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle = -\frac{\xi_{1-}\xi_{3-}}{256}\frac{\langle J^+_{k_1+\frac{1}{2}}J^-_{k_2+\frac{1}{2}}J^+_{k_3+1}\rangle E}{\langle \bar{2}\,\bar{3}\rangle\sqrt{p_3}}\,\Xi_3, &&\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle = -\frac{\xi_{2-}}{256}\frac{\langle J^-_{k_1+\frac{1}{2}}J^+_{k_2+\frac{1}{2}}J^-_{k_3+1}\rangle E}{\langle 23\rangle\sqrt{p_3}}\,\Gamma_3,\\
&\langle \tilde{\mathbf{J}}^-_{s_1}\tilde{\mathbf{J}}^+_{s_2}\tilde{\mathbf{J}}^+_{s_3}\rangle = -\frac{\xi_{2-}\xi_{3-}}{256}\frac{\langle J^-_{k_1+\frac{1}{2}}J^+_{k_2+\frac{1}{2}}J^+_{k_3+1}\rangle E}{\langle \bar{3}\,\bar{1}\rangle\sqrt{p_3}}\,\Xi_3, &&\langle \tilde{\mathbf{J}}^+_{s_1}\tilde{\mathbf{J}}^-_{s_2}\tilde{\mathbf{J}}^-_{s_3}\rangle = \frac{\xi_{1-}}{256}\frac{\langle J^+_{k_1+\frac{1}{2}}J^-_{k_2+\frac{1}{2}}J^-_{k_3+1}\rangle E}{\langle 31\rangle\sqrt{p_3}}\,\Gamma_3.
\end{aligned} \tag{5.2.11}$$

This class of correlators are all nonhomogeneous; that is, their components appear as a sum of bosonic and fermionic structures. We provide a detailed example of one such correlator in appendix **??**.

5.2.3 Double-copy relations

Double-copy relations have played an important role in (supersymmetric) scattering amplitudes in the recent decades [95]. Similar structures were later discovered in the non-supersymmetric conformal field theory [11, 14, 18]. This new formalism provides a natural extension to the mentioned double copy for supersymmetric case as well. Indeed, in the $\xi_\pm$ variables that we have been using in this paper, such a structure becomes apparent. Let us illustrate this with an example: Consider a three point super-correlator with $s_1 = s_2 = s_3 = 2$, that is, the $\langle \tilde{\mathbf{T}}\tilde{\mathbf{T}}\tilde{\mathbf{T}}\rangle$ correlator. As we discussed below equation (5.2.5), this correlator is purely homogeneous as it satisfies the triangle inequality. Therefore, it suffices to consider its $(- - -)$ and $(+ + +)$ components. The $(- - -)$ component correlator is,

$$\langle \tilde{\mathbf{T}}^-\tilde{\mathbf{T}}^-\tilde{\mathbf{T}}^-\rangle = \frac{1}{64}\langle T^-T^-T^-\rangle\,\Gamma_3. \tag{5.2.12}$$

Consider now,

$$\langle \tilde{\mathbf{J}}_4{}^-\tilde{\mathbf{J}}_4{}^-\tilde{\mathbf{J}}_4{}^-\rangle = \frac{1}{64}\langle J_4^-J_4^-J_4^-\rangle\,\Gamma_3. \tag{5.2.13}$$

If we naively square (5.2.12) expecting to obtain (5.2.13), we will obtain zero as $\Gamma_3^2 = 0$ (evident from the definition (5.2.6)). We note, however, that the relation at the level of *component* correlators,

$$\langle J_4^-J_4^-J_4^-\rangle \propto p_1p_2p_3\langle T^-T^-T^-\rangle^2, \tag{5.2.14}$$

implies at the superfield level (5.2.13) that,

$$\langle \tilde{\mathbf{J}_4}^- \tilde{\mathbf{J}_4}^- \tilde{\mathbf{J}_4}^- \rangle \propto p_1 p_2 p_3 \langle T^- T^- T^- \rangle^2 \Gamma_3. \tag{5.2.15}$$

Equations (5.2.13) and (5.2.15) implies a double copy relation between $\langle \tilde{\mathbf{T}}^- \tilde{\mathbf{T}}^- \tilde{\mathbf{T}}^- \rangle$ and $\langle \tilde{\mathbf{J}}_4^- \tilde{\mathbf{J}}_4^- \tilde{\mathbf{J}}_4^- \rangle$. This double copy relation can be extended for arbitrary spin as follows

$$\langle \tilde{\mathbf{J}}_{s_1}^- \tilde{\mathbf{J}}_{s_2}^- \tilde{\mathbf{J}}_{s_3}^- \rangle \propto p_1 p_2 p_3 \langle J_{s_1'}^- J_{s_2'}^- J_{s_3'}^- \rangle \langle J_{s_1''}^- J_{s_2''}^- J_{s_3''}^- \rangle \Gamma_3, \tag{5.2.16}$$

$$\langle \tilde{\mathbf{J}}_{s_1}^+ \tilde{\mathbf{J}}_{s_2}^+ \tilde{\mathbf{J}}_{s_3}^+ \rangle \propto p_1 p_2 p_3 \langle J_{s_1'}^+ J_{s_2'}^+ J_{s_3'}^+ \rangle \langle J_{s_1''}^+ J_{s_2''}^+ J_{s_3''}^+ \rangle \Xi_3, \tag{5.2.17}$$

where $s_1' + s_1'' = s_1$, $s_2' + s_2'' = s_2$, and $s_3' + s_3'' = s_3$. Let us also note the spins $s_1, s_2, s_3 \in \mathbb{Z}_{>0}$ and obeys triangle inequality. We shall later see in section 6.5, an interesting double copy relation between $\mathcal{N}=1$ and $\mathcal{N}=2$ super correlators.

5.2.4 The super Ward-Takahashi identity

In this subsection the super Ward-Takahashi identity for our non homogeneous correlators is presented. Further, it is favorable to work with θ variables as they suffice for our purposes. The aim is to show that their is a stark distinction between the Grassmann even and Grassmann odd cases.
Consider a Grassmann odd correlator $\langle \mathbf{J}_{\frac{3}{2}} \mathbf{J}_0 \mathbf{J}_0 \rangle$. The corresponding super Ward-Takahashi identity that we obtain is given by:

$$D_{1a} \langle \mathbf{J}_{\frac{3}{2}}^{abc} \mathbf{J}_0 \mathbf{J}_0 \rangle = a_1 \, \theta_1^2 \bigg(\not{p}_2^{bc} \langle \mathbf{J}_0(p_2) \mathbf{J}_0(-p_2) \rangle + \not{p}_3^{bc} \langle \mathbf{J}_0(-p_3) \mathbf{J}_0(p_3) \rangle \bigg). \tag{5.2.18}$$

If we consider a Grassmann even correlator such as $\langle \mathbf{T} \mathbf{J}_0 \mathbf{J}_0 \rangle$, we find the following super Ward-Takahashi identity:

$$D_{1a} \langle \mathbf{T}^{abcd} \mathbf{J}_0 \mathbf{J}_0 \rangle = a_2 \, \theta_1^2 \bigg(\not{p}_2^{bc} \, D_2^d \langle \mathbf{J}_0(p_2) \mathbf{J}_0(-p_2) \rangle + \not{p}_3^{bc} \, D_3^d \langle \mathbf{J}_0(-p_3) \mathbf{J}_0(p_3) \rangle \bigg), \tag{5.2.19}$$

where a_1 and a_2 are proportional to the normalization of the $\langle \mathbf{J}_0 \mathbf{J}_0 \rangle$ two-point super-correlator, and the action of covariant derivative is given below equation (5.1.2).

5.3 Summary

This chapter provided the formalism on finding the super correlators of two and three point functions for any spinning operator through new variables called “Grassmann Twistor Variable”. We found that in these new variables, every spinning correlator has a universal structure, and all the nonsupersymmetric CFT correlators in superfield expansion are connected to each other because of these universal structures. One might ask what about the higher point function. In fact, preliminary analysis generalizes the above structures on four-point functions, yet it must still be done properly. Further, a double copy is provided, and how to find Super Ward Takahashi Identity for supercorrelators. In the next chapter, we will see what happens if you increase supersymmetry (Extended Supersymmetry).

Chapter 6

$\mathcal{N}=2$ Superconformal Field Theory in $3d$

I feel all thin, sort of stretched: like butter that has been scraped over too much bread.

Bilbo Baggins on extended supersymmetry, The Fellowship of the Ring

The previous chapter included the formalism and variables to find correlators in $\mathcal{N}=1$ Superconformal Field theory in $3d$. One obvious question is to ask what will happen for extended supersymmetry, and why is it even interesting to study them? There are some interesting theories living in extended supersymmetry like ABJM theory, which is $\mathcal{N}=6$ SCFT in $3d$ and it is holographically dual to M-theory in $AdS_4 \times S^7$ through AdS/CFT correspondence. The study of extended supersymmetry and double copy in this chapter will provide a pathway to go to $\mathcal{N}=6$ theories.

6.1 The superspace formulation

The $\mathcal{N}=2$ superspace can be described by the coordinates $(x^\mu, \theta^a, \bar{\theta}^a)$ where x^μ are the usual position space coordinates and θ^a and $\bar{\theta}^a$ are two component Grassmann spinors [77, 90]. The generators of the $\mathcal{N}=2$ superconformal algebra include the usual conformal generators $P_\mu, M_{\mu\nu}, D, K_\mu$ along with the supersymmetry generators $Q_a, \bar{Q}_a$ and the special superconformal generators S_a and $\bar{S}_a$. More information on this formalism as well as the action of the generators on primary superfields can be found for instance in [72, 77].

We now, parallel our analysis of the $\mathcal{N}=1$ case in section 5.1, first convert x^μ to p_μ by a Fourier transform. We then express p_μ in spinor helicity variables as in (5.1.14). The key change that arises in the $\mathcal{N}=2$ is the fact the θ^a is no longer Hermitian. Therefore, it has to be described by two complex Grassmann variables in contrast to the single complex Grassmann variables η (5.1.15). The $\mathcal{N}=2$ Grassmann spinors can be expressed as follows: For complex Grassmann variables η, μ, we define,

$$\theta^a = \frac{\bar{\eta}\lambda^a + \mu\bar{\lambda}^a}{2p},$$
$$\bar{\theta}^a = \frac{\eta\bar{\lambda}^a + \bar{\mu}\lambda^a}{2p}. \tag{6.1.1}$$

Further, just as we performed a Grassmann twistor transform in the $\mathcal{N}=1$ case (5.1.19), we do so in the $\mathcal{N}=2$ case as follows. We define,

$$\tilde{F}(\eta,\chi,\mu,\nu) = \int d\bar{\eta}\; d\bar{\mu}\; e^{-\frac{\chi\bar{\eta}}{4}-\frac{\nu\bar{\mu}}{4}} F(\eta,\bar{\eta},\mu,\bar{\mu}). \tag{6.1.2}$$

We then define the coordinates,

$$\xi_{\pm} = \chi \pm \eta, \quad \omega_{\pm} = \nu \pm \mu. \tag{6.1.3}$$

In terms of these coordinates, the supersymmetry generators of the $\mathcal{N}=2$ superconformal algebra take the following simple forms:

$$Q_a = 2\lambda_a \frac{\partial}{\partial\omega_+} + \frac{\bar{\lambda}_a}{4}\xi_+, \qquad \bar{Q}_a = 2\lambda_a \frac{\partial}{\partial\xi_+} + \frac{\bar{\lambda}_a}{4}\omega_+. \tag{6.1.4}$$

The remaining generators are identical to their $\mathcal{N}=1$ counterparts given in (D.1.4). However, there is one very important distinction between the $\mathcal{N}=1$ and $\mathcal{N}=2$ cases, that is, the presence of a $U(1)$ R-symmetry. The Q and $\bar{Q}$ operators defined in (6.1.4) have R charges -1 and $+1$ respectively. This implies the following commutation relations

$$[R, Q_a] = -Q_a, \quad [R, \bar{Q}_a] = +\bar{Q}_a. \tag{6.1.5}$$

The representation of R in the $\xi_\pm, \omega_\pm$, acting on primary super fields is the following:

$$R = \omega_+ \frac{\partial}{\partial\omega_+} - \xi_+ \frac{\partial}{\partial\xi_+}, \tag{6.1.6}$$

where R is the symmetry generator. It can easily be checked using (6.1.4) and (6.1.6) that the commutators (6.1.5) hold.

6.2 Super-currents and Ward identities

We are interested in constraining correlators involving symmetric traceless conserved super-currents. The general form of such currents in $\mathcal{N}=2$ SCFTs can be found for instance in [72]. Here, we present its momentum superspace avatar:

$$\begin{aligned}\mathbf{J}_s^{a_1a_2\ldots a_{2s}} &= J_s^{a_1a_2\ldots a_{2s}} + \theta_m J_{s+\frac{1}{2}}^{a_1a_2\ldots a_{2s}m} + \bar{\theta}_m \bar{J}_{s+\frac{1}{2}}^{a_1a_2\ldots a_{2s}m} + \theta_m\bar{\theta}_n J_{s+1}^{a_1a_2\ldots a_{2s}mn} + \frac{\theta_m\bar{\theta}^m}{2}\not{p}_n^{a_1} J_s^{a_2\ldots a_{2s}} \\ &- \frac{\theta^2\bar{\theta}_m}{4}\not{p}_n^m J_{s+\frac{1}{2}}^{a_1a_2\ldots a_{2s}n} - \frac{\bar{\theta}^2\theta_m}{4}\not{p}_n^m \bar{J}_{s+\frac{1}{2}}^{a_1a_2\ldots a_{2s}n} - \frac{\theta^2\bar{\theta}^2}{16}p^2 J_s^{a_1a_2\ldots a_{2s}}.\end{aligned} \tag{6.2.1}$$

We then use the definition (6.1.1) and perform the Grassmann twistor transform (6.1.2). Their is a drastic simplification compared to (6.2.1) when the result is expressed in the $(\xi_\pm, \omega_\pm)$ variables (6.1.3). We have,

$$\begin{aligned}
\tilde{\mathbf{J}}_{\mathbf{s}}^- &= \frac{1}{4}\left[\frac{\xi_+\omega_+}{4}J_s^- - \frac{1}{2\sqrt{p}}\left(\omega_+ J^-_{s+\frac{1}{2}} - \xi_+ \bar{J}^-_{s+\frac{1}{2}}\right) - \frac{J^-_{s+1}}{p}\right],\\
\tilde{\mathbf{J}}_{\mathbf{s}}^+ &= \frac{\xi_-\omega_-}{16}\left[J_s^+ + \frac{1}{4\sqrt{p}}\left(\omega_+ J^+_{s+\frac{1}{2}} + \xi_+ \bar{J}^+_{s+\frac{1}{2}}\right) - \frac{\xi_+\omega_+}{16p}J^+_{s+1}\right].
\end{aligned} \tag{6.2.2}$$

We can then construct super-correlators using (6.2.2), as we did for the $\mathcal{N}=1$ case. The component correlators can then be constrained using the action of Q_a and $\bar{Q}_a$, as given in (6.1.4). The R symmetry is going to play an important role here as correlators have to be R symmetric. Before proceeding further, we note that, ξ_{i+} has R charge -1, while ω_{i+} has R charge $+1$. Then, correlators are forced to be made of elements of form $\xi_{i+}\omega_{i+}$, where i is the operator label. This is exactly what we got and two and three point functions are presented below.

6.3 Two point functions

The $\mathcal{N}=2$ two point functions for any integer spin s take the following forms in the two independent helicity configurations:

$$\begin{aligned}
\langle \tilde{\mathbf{J}}_{\mathbf{s}}^- \tilde{\mathbf{J}}_{\mathbf{s}}^- \rangle &= \frac{\langle 12\rangle^{2s}}{16\,p_1}\left(\frac{\xi_{1+}\omega_{1+}\xi_{2+}\omega_{2+}}{16} - \frac{\langle 12\rangle}{4p_1}(\omega_{1+}\xi_{2+} + \xi_{1+}\omega_{2+}) + \frac{\langle 12\rangle^2}{p_1^2}\right),\\
\langle \tilde{\mathbf{J}}_{\mathbf{s}}^+ \tilde{\mathbf{J}}_{\mathbf{s}}^+ \rangle &= \xi_{1-}\omega_{1-}\xi_{2-}\omega_{2-}\frac{\langle \bar{1}\bar{2}\rangle^{2s+3}}{65536\,p_1^4}\left(\frac{\xi_{1+}\omega_{1+}\xi_{2+}\omega_{2+}}{16} - \frac{\langle 12\rangle}{4p_1}(\omega_{1+}\xi_{2+} + \xi_{1+}\omega_{2+}) + \frac{\langle 12\rangle^2}{p_1^2}\right).
\end{aligned} \tag{6.3.1}$$

Further, the building block of $\mathcal{N}=2$ two-point functions can be written as a product of two individual building blocks. Then, the correlators can be written as,

$$\begin{aligned}
\langle \tilde{\mathbf{J}}_{\mathbf{s}}^- \tilde{\mathbf{J}}_{\mathbf{s}}^- \rangle &= \frac{\langle 12\rangle^{2s}}{16\,p_1}\left(\frac{\xi_{1+}\omega_{1+}}{4} - \frac{\langle 12\rangle}{p_1}\right)\left(\frac{\xi_{2+}\omega_{2+}}{4} - \frac{\langle 12\rangle}{p_1}\right),\\
\langle \tilde{\mathbf{J}}_{\mathbf{s}}^+ \tilde{\mathbf{J}}_{\mathbf{s}}^+ \rangle &= \xi_{1-}\omega_{1-}\xi_{2-}\omega_{2-}\frac{\langle \bar{1}\bar{2}\rangle^{2s+3}}{65536\,p_1^4}\left(\frac{\xi_{1+}\omega_{1+}}{4} - \frac{\langle 12\rangle}{p_1}\right)\left(\frac{\xi_{2+}\omega_{2+}}{4} - \frac{\langle 12\rangle}{p_1}\right).
\end{aligned} \tag{6.3.2}$$

It can be clearly seen after the discussion in the above section, that the correlators are formed by the building blocks which are inherently R symmetric as they are made of elements like $\xi_{i+}\omega_{i+}$.

6.4 Three point function

The structure of the three point function in the $\mathcal{N}=2$ theories are much more complicated than their $\mathcal{N}=1$ counterpart. However, by a careful analysis, we were able still able to obtain the three point correlators in the $\mathcal{N}=2$ theories for any arbitrary integer spin in terms of the following building blocks which is reminiscent of the $\mathcal{N}=1$ case (5.2.6).

$$\begin{aligned}
\Omega_1 = \Big(&\langle 23\rangle^2\xi_{1+}\omega_{1+} + \langle 23\rangle\langle 31\rangle\xi_{1+}\omega_{2+} + \langle 12\rangle\langle 23\rangle\xi_{1+}\omega_{3+} + \langle 23\rangle\langle 31\rangle\xi_{2+}\omega_{1+} + \langle 31\rangle^2\xi_{2+}\omega_{2+}\\
&+ \langle 12\rangle\langle 31\rangle\xi_{2+}\omega_{3+} + \langle 12\rangle\langle 23\rangle\xi_{3+}\omega_{1+} + \langle 12\rangle\langle 31\rangle\xi_{3+}\omega_{2+} + \langle 12\rangle^2\xi_{3+}\omega_{3+}\Big),
\end{aligned}$$

$$\begin{aligned}
\Omega_2 &= \Big(-\langle 23\rangle \xi_{1+}\omega_{1+}\big(\xi_{2+}\omega_{3+}-\xi_{3+}\omega_{2+}\big)+\langle 31\rangle \xi_{2+}\omega_{2+}\big(\xi_{1+}\omega_{3+}-\xi_{3+}\omega_{1+}\big)\\
&\qquad -\langle 12\rangle \xi_{3+}\omega_{3+}\big(\xi_{1+}\omega_{2+}-\xi_{2+}\omega_{1+}\big)\Big),\\
\Omega_3 &= \xi_{1+}\xi_{2+}\xi_{3+}\omega_{1+}\omega_{2+}\omega_{3+}.
\end{aligned} \tag{6.4.1}$$

For example, in the $(---)$ helicity we obtain,

$$\langle \tilde{\mathbf{J}}^-_{\mathbf{s_1}}\tilde{\mathbf{J}}^-_{\mathbf{s_2}}\tilde{\mathbf{J}}^-_{\mathbf{s_3}}\rangle = \langle J^-_{s_1}J^-_{s_2}J^-_{s_3}\rangle\Big(\frac{\Omega_1}{E^2}+\frac{\Omega_2}{8E}-\frac{\Omega_3}{64}\Big). \tag{6.4.2}$$

For spins that satisfy the triangle inequality, (6.4.2) is homogeneous similar to it's $\mathcal{N}=1$ (5.2.5). It is straightforward to obtain analogous formulae in the remaining helicity configurations as well.

6.5 Double copy: $\mathcal{N}=1\otimes\mathcal{N}=1\rightarrow\mathcal{N}=2$

Two-Point function

Consider the following integer two point functions in the $\mathcal{N}=1$ theory in the $(--)$ helicity configuration (5.2.1),

$$\begin{aligned}
\langle \tilde{\mathbf{J}}^-_s(\xi_{1+})\tilde{\mathbf{J}}^-_s(\omega_{2+})\rangle_{\mathcal{N}=1} &= \frac{\langle 12\rangle^{2s}}{16p_1}\Big(\xi_{1+}\omega_{2+}-\frac{4\langle 12\rangle}{p_1}\Big),\\
\langle \tilde{\mathbf{J}}^-_s(\omega_{1+})\tilde{\mathbf{J}}^-_s(\xi_{2+})\rangle_{\mathcal{N}=1} &= \frac{\langle 12\rangle^{2s}}{16p_1}\Big(\omega_{1+}\xi_{2+}-\frac{4\langle 12\rangle}{p_1}\Big),
\end{aligned} \tag{6.5.1}$$

where we have made the dependence on the Grassmann twistor variables explicit. The reason for the strange choice of the Grassmann twistor variable dependence of the super currents in both lines of (6.5.1) will become clear shortly. We now take a product between the first and second line of (6.5.1). The result is,

$$\langle \tilde{\mathbf{J}}^-_s(\xi_{1+})\tilde{\mathbf{J}}^-_s(\omega_{2+})\rangle_{\mathcal{N}=1}\langle \tilde{\mathbf{J}}^-_s(\omega_{1+})\tilde{\mathbf{J}}^-_s(\xi_{2+})\rangle_{\mathcal{N}=1} = \frac{\langle 12\rangle^{4s}}{16p_1^2}\Big(\frac{\xi_{1+}\omega_{1+}\xi_{2+}\omega_{2+}}{16}-\frac{\langle 12\rangle}{4p_1}(\omega_{1+}\xi_{2+}+\xi_{1+}\omega_{2+})+\frac{\langle 12\rangle^2}{p_1}\Big). \tag{6.5.2}$$

By comparing the RHS of the above equation with the $\mathcal{N}=2$ two point function (6.3.1), we obtain a super double copy relation!

$$\langle \tilde{\mathbf{J}}^-_s(\xi_{1+})\tilde{\mathbf{J}}^-_s(\omega_{2+})\rangle_{\mathcal{N}=1}\langle \tilde{\mathbf{J}}^-_s(\omega_{1+})\tilde{\mathbf{J}}^-_s(\xi_{2+})\rangle_{\mathcal{N}=1} = \frac{1}{p_1}\langle \tilde{\mathbf{J}}^-_{2s}(\xi_{1+},\omega_{1+})\tilde{\mathbf{J}}^-_{2s}(\xi_{2+},\omega_{2+})\rangle_{\mathcal{N}=2}. \tag{6.5.3}$$

The reason for the weird choices of Grassmann twistor variable dependence in (6.5.1) is now clear. This and only this particular choice will ensure that the RHS of (6.5.3), a $\mathcal{N}=2$ two-point function will be invariant under the action of the R symmetry generator (6.1.6). Similar results in the other helicity configuration as well as double copy relations involving half integer $\mathcal{N}=1$ supercorrelators can also easily be obtained, due to the simplicity of the Grassmann twistor variables.

Three-Point function

Let us now proceed to the three point case. For simplicity, we consider the cases that satisfy the triangle inequality $s_i + s_j \geq s_k \, \forall i,j,k \in 1,2,3$. Our prescription for obtaining $\mathcal{N}=2$ super correlators from products of $\mathcal{N}=1$ super correlators is constructing the product such that the resulting object possesses the required R symmetry property. Our goal is to take products of $\mathcal{N}=1$ three-point functions (5.2.5) to reproduce the $\mathcal{N}=2$ answer (6.4.2). From the required $U(1)$ R symmetry of the result, we know that the total number of ξ_{i+} and ω_{i+} variables appearing in each term must be equal. This then restricts our ansatz for the $\mathcal{N}=1$ product. Let us concentrate on the $(---)$ helicity configuration[1]. We take,

$$\begin{aligned}\textbf{Ansatz} =\Big[& a_1\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\xi_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\omega_{1+})\tilde{\mathbf{J}}_s^-(\omega_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\\ &+a_2\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\\ &+a_3\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\omega_{2+})\tilde{\mathbf{J}}_s^-(\xi_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\omega_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\\ &+a_4\langle\tilde{\mathbf{J}}_s^-(\omega_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\xi_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\omega_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\Big].\end{aligned} \tag{6.5.4}$$

We then demand that the ansatz (6.5.4) is invariant under the simultaneous action of the R symmetry generator (6.1.6) on all the insertions. This yields the constraint $a_1 = a_2 = a_3 = a_4$. In fact (6.5.4) becomes,

$$\langle J_s^- J_s^- J_s^-\rangle^2\left(\frac{\Omega_1}{E^2}+\frac{\Omega_2}{8E}-\frac{\Omega_3}{64}\right), \tag{6.5.5}$$

where the Ω_i are the $\mathcal{N}=2$ building blocks (6.4.1). Therefore, we obtain a super double copy,

$$\begin{aligned}&\langle\tilde{\mathbf{J}}_{2s_1}^-(\xi_{1+},\omega_{1+})\tilde{\mathbf{J}}_{2s_2}^-(\xi_{2+},\omega_{2+})\tilde{\mathbf{J}}_{2s_3}^-(\xi_{3+},\omega_{3+})\rangle_{\mathcal{N}=2}\\ &\quad=p_1p_2p_3\Big[\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\xi_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\omega_{1+})\tilde{\mathbf{J}}_s^-(\omega_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\\ &\qquad+\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\\ &\qquad+\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\omega_{2+})\tilde{\mathbf{J}}_s^-(\xi_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\omega_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\\ &\qquad+\langle\tilde{\mathbf{J}}_s^-(\omega_{1+})\tilde{\mathbf{J}}_s^-(\xi_{2+})\tilde{\mathbf{J}}_s^-(\xi_{3+})\rangle_{\mathcal{N}=1}\langle\tilde{\mathbf{J}}_s^-(\xi_{1+})\tilde{\mathbf{J}}_s^-(\omega_{2+})\tilde{\mathbf{J}}_s^-(\omega_{3+})\rangle_{\mathcal{N}=1}\Big].\end{aligned} \tag{6.5.6}$$

Notice that the result (6.5.6) is true for any integer spins s_1, s_2 and s_3 that obey the triangle inequality. The double copy at the level of homogeneous component correlators was essential for this result. For non homogeneous correlators, there exist some double copy relations albeit more complicated ones. Our prescription to obtain $\mathcal{N}=2$ super-correlators from R symmetry preserving products of $\mathcal{N}=1$ super-correlators could potentially generalize to higher points.

6.6 Summary

This chapter provided the analysis for extended supersymmetry. One of the most important results is finding a super double copy, which can take you from $\mathcal{N}=1$ correlator to $\mathcal{N}=2$ correlator. This enticing clue hints at the existence of a deeper, more advanced double-copy relation that could unveil the $\mathcal{N}=6$ case.

[1]One can carry out an analogous analysis in the other helicities.

Part IV

Conclusions

Chapter 7

Conclusion

> Well, here at last, dear friends, on the shores of the sea comes the end of our fellowship in Middle-earth. Go in peace! I will not say: do not weep, for not all tears are evil.
>
> *Gandalf, The Return of the King*

Many ingenious ways have been found to constrain OPE data of CFT, and functional forms of correlators like simplex representation, numerical methods using ML, etc. We found some of the new ways to analytically solve two problems. As mentioned in the Introduction, most of the time, we can solve a problem in physics by two approaches: either you can change the variables or the dimensions. We planned to find ways of constraining higher point functions, which we did in the two above-mentioned ways. In part II, we saw that decreasing the dimension to $1d$ can constrain the correlators to a large extent, and we were able to find closed-form expressions for n-point functions in terms of only the scaling dimensions of the external operators with some freedom in arguments. This freedom in arguments corresponds to different theories that exist in this dimension. Using our general answers for the correlators at the level of three and four-point functions, we were able to reproduce correlators from free theory and an interacting theory, which is the DFF model. Further, there has always been a problem with the existence of multiple solutions to conformal ward identity in momentum space, whereas the position space doesn't have this problem. We explained this phenomenon as the inequivalent nature of different time orderings and showed that different time ordering will correspond to the different solutions of conformal ward identities. Moreover, we extended this analysis to the superconformal field theories, where we were able to find connections between different components of super correlators and exactly solve all the three and four-point functions in $\mathcal{N} = 1, 2$ superconformal field theories in $1d$. In part III, we kept a higher dimension, i.e., $3d$, but we went to supersymmetry cases, where we found that some new variables, i.e., "Grassmann Twistor Variable" is far more efficient in writing the correlators in general, form than commonly expected spinor form. This way, we found all the spinning two and three-point functions, and this formalism gave connections between different non-susy cft correlators. Extending this to extended supersymmetry, we found that there exists a super double copy that can take you from $\mathcal{N}=1 \to \mathcal{N}=2$. This allows us to find double-copy relations to get the correlators in maximally supersymmetric theories in $3d$ like ABJM theories.

Appendix A

The Super Conformal Algebra and Ward Identities for $1d$

The algebra, the action on primary fields, and ward identities for $\mathcal{N} = 1, 2$ Superconformal Quantum Mechanics in $1d$ are provided.

A.1 $\mathcal{N} = 1$ Superconformal Algebra and Ward Identities

The generators of the $\mathcal{N} = 1$ superconformal algebra obey the following (anti)commutation relations:

$$\begin{gathered}
[D, H] = -iH, \quad [D, K] = iK, \quad [K, H] = -2iD, \\
\{Q, Q\} = H, \quad \{S, S\} = -K, \quad \{Q, S\} = iD, \\
[D, Q] = -\frac{i}{2}Q, \quad [D, S] = \frac{i}{2}S, \quad [K, Q] = -S, \quad [H, S] = -Q.
\end{gathered} \tag{A.1.1}$$

Their action on primary operators is as follows:

$$\begin{aligned}
&[H, \mathbf{O}_\Delta] = \omega \mathbf{O}_\Delta, \\
&[D, \mathbf{O}_\Delta] = -i\left(\omega\frac{\partial}{\partial\omega} + (1-\Delta) - \frac{1}{2}\theta\frac{\partial}{\partial\theta}\right)\mathbf{O}_\Delta, \\
&[K, \mathbf{O}_\Delta] = -\left(\omega\frac{\partial^2}{\partial\omega^2} + 2(1-\Delta)\frac{\partial}{\partial\omega} - \theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega}\right)\mathbf{O}_\Delta, \\
&[Q, \mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\theta} + \frac{\theta}{2}\omega\right)\mathbf{O}_\Delta, \\
&[S, \mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega} + \left(\frac{1}{2} - \Delta\right)\theta + \frac{\theta\omega}{2}\frac{\partial}{\partial\omega}\right)\mathbf{O}_\Delta.
\end{aligned} \tag{A.1.2}$$

A.2 $\mathcal{N} = 2$ Superconformal Algebra and Ward Identities

The super lie algebra obeyed by the $\mathcal{N} = 2$ superconformal generators can be found, for instance, in [53]. The action of the $\mathcal{N} = 2$ superconformal algebra generators on primary operators is as follows:

$$[H, \mathbf{O}_\Delta] = \omega\, \mathbf{O}_\Delta, \tag{A.2.1}$$

$$[D, \mathbf{O}_\Delta] = -i\left(\omega\frac{\partial}{\partial\omega} + (1-\Delta) - \frac{1}{2}\theta\frac{\partial}{\partial\theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right)\mathbf{O}_\Delta, \tag{A.2.2}$$

$$[K, \mathbf{O}_\Delta] = -\left(\omega\frac{\partial^2}{\partial\omega^2} + 2(1-\Delta)\frac{\partial}{\partial\omega} - \theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega} - \bar{\theta}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\omega}\right)\mathbf{O}_\Delta, \tag{A.2.3}$$

$$[Q, \mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\theta} + \frac{\bar{\theta}}{2}\omega\right)\mathbf{O}_\Delta, \quad [\bar{Q}, \mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\bar{\theta}} + \frac{\theta}{2}\omega\right)\mathbf{O}_\Delta, \tag{A.2.4}$$

$$[S, \mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\omega} + \left(\frac{1}{2} - \Delta\right)\bar{\theta} + \frac{\bar{\theta}\omega}{2}\frac{\partial}{\partial\omega} - \frac{\bar{\theta}\theta}{2}\frac{\partial}{\partial\theta}\right)\mathbf{O}_\Delta, \tag{A.2.5}$$

$$[\bar{S}, \mathbf{O}_\Delta] = \left(\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\omega} + \left(\frac{1}{2} - \Delta\right)\theta + \frac{\theta\omega}{2}\frac{\partial}{\partial\omega} - \frac{\theta\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}}\right)\mathbf{O}_\Delta, \tag{A.2.6}$$

$$[R, \mathbf{O}_\Delta] = \left(\theta\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right)\mathbf{O}_\Delta. \tag{A.2.7}$$

These imply the following Ward identities for the correlation functions:

$$\begin{aligned}
\sum_{i=1}^{n}\omega_i\, f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\omega_i\frac{\partial}{\partial\omega_i} + (1-\Delta_i) - \frac{1}{2}\theta_i\frac{\partial}{\partial\theta_i} - \frac{1}{2}\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\omega_i\frac{\partial^2}{\partial\omega_i^2} + 2(1-\Delta_i)\frac{\partial}{\partial\omega_i} - \theta_i\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\omega_i} - \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\frac{\partial}{\partial\omega_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta_i} + \frac{\bar{\theta}_i}{2}\omega_i\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\bar{\theta}_i} + \frac{\theta_i}{2}\omega_i\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\omega_i} + \left(\frac{1}{2} - \Delta_i\right)\bar{\theta}_i + \frac{\bar{\theta}_i\omega_i}{2}\frac{\partial}{\partial\omega_i} - \frac{\bar{\theta}_i\theta_i}{2}\frac{\partial}{\partial\theta_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\frac{\partial}{\partial\bar{\theta}_i}\frac{\partial}{\partial\omega_i} + \left(\frac{1}{2} - \Delta_i\right)\theta_i + \frac{\theta_i\omega_i}{2}\frac{\partial}{\partial\omega_i} - \frac{\theta_i\bar{\theta}_i}{2}\frac{\partial}{\partial\bar{\theta}_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0,\\
\sum_{i=1}^{n}\left(\theta_i\frac{\partial}{\partial\theta_i} - \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right) f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n) &= 0.
\end{aligned} \tag{A.2.8}$$

where, $f_n(\omega_1,\theta_1,\bar{\theta}_1;\ldots;\omega_n,\theta_n,\bar{\theta}_n)$ is a general $\mathcal{N} = 2$, n-point function.

Appendix B

Lauricella Functions Properties

B.1 Series expansions for the Lauricella functions

A series expansion for the general Lauricella function of m variables is the following [84]:

$$E_A^{(m)}(a, b_1, \cdots, b_m, c_1, \cdots, c_m; -x_1, \cdots, -x_m) = \sum_{m_i \in \mathbb{N}_0} \frac{(a)_{m_1+\cdots+m_n} \prod_{i=1}^{m}(b_i)_{m_i}}{\prod_{i=1}^{m}(c_i)_{m_i} \prod_{i=1}^{m} m_i!} \prod_{i=1}^{m} x_i^{m_i}. \tag{B.1.1}$$

$\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ is the rising Pochammer symbol. The above series converges when $\sum_{i=1}^{m} |x_i| < 1$. As a special case we obtain ${}_2F_1$, when $m = 1$; Appell F_2, when $m = 2$; $E_A^{(3)}$, when $m = 3$ and so on.

B.2 Useful identities involving hypergeometric functions

Some useful identities involving the hypergeometric functions that were used in the main text are:

$$\begin{aligned}\frac{2\Delta_2 - 1}{\Delta_t - 1}\ {}_2F_1(1 - \Delta_t, 1 - \Delta_t + 2\Delta_1, 1 - 2\Delta_2; -x) + x\ {}_2F_1(2 - \Delta_t, 1 - \Delta_t + 2\Delta_1, 2 - 2\Delta_2; -x)& \\ = \frac{2\Delta_2 - 1}{\Delta_t - 1}\ {}_2F_1(1 - \Delta_t, 2\Delta_1 - \Delta_t, 1 - 2\Delta_2; -x).&\end{aligned} \tag{B.2.1}$$

$$\begin{aligned}\frac{2\Delta_2}{2\Delta_1 + 2\Delta_2 - \Delta_t}\ {}_2F_1(1 + 2\Delta_2 - \Delta_t, 2\Delta_1 + 2\Delta_2 - \Delta_t, 2\Delta_2; -x)& \\ -\ {}_2F_1(1 + 2\Delta_2 - \Delta_t, 1 + 2\Delta_1 + 2\Delta_2 - \Delta_t, 1 + 2\Delta_2; -x)& \\ = \frac{\Delta_t - 2\Delta_1}{2\Delta_1 + 2\Delta_2 - \Delta_t}\ {}_2F_1(1 + 2\Delta_2 - \Delta_t, 2\Delta_1 + \Delta_2 - \Delta_t, 1 + 2\Delta_2; -x).&\end{aligned} \tag{B.2.2}$$

Appendix C

Notation, Conventions and some useful formulae for $3d$

In this appendix, the conventions and notations, as well as some formulas, are provided.

C.1 Notations

Momentum conservation in all our (super) correlation functions is implicit, i.e., we do not explicitly write the momentum-conserving Dirac-delta functions. The component fields are in italics, whereas superfields are always in bold typeface. For instance, J_s is a component field whereas $\mathbf{J}_s$ is a super field. Superfields, when expressed in the Grassmann twistor variables, are denoted in bold and with a tilde such as $\tilde{\mathbf{J}}_s$. The subscripts b or B and f or F for component correlators refer to the free bosonic and free fermionic theory correlators, respectively.

C.2 Conventions

We work with the usual flat Euclidean metric,

$$\delta_{\mu\nu} = \mathbf{diag}(1,1,1), \tag{C.2.1}$$

with which vector indices are raised and lowered. Since upper and lower indices are identical in this case, we do not need to distinguish between them. Our convention for the three dimensional Levi-Civita symbol is,

$$\begin{aligned} \epsilon_{123} &= \epsilon^{123} = 1, \\ \epsilon_{\mu\nu\rho}\epsilon_{\alpha\beta\rho} &= \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}. \end{aligned} \tag{C.2.2}$$

Spinor indices on the other hand are raised and lowered using the two dimensional Levi-Civita symbol ϵ_{ab} which is given by,

$$\begin{aligned}\epsilon_{12} &= \epsilon^{12} = 1,\\ \epsilon_{ab}\epsilon^{ac} &= \delta^c_b.\end{aligned} \tag{C.2.3}$$

Our convention for raising and lowering indices is as follows: Given a spinor A_a we have,

$$A^a = \epsilon^{ab}A_b \iff A_a = \epsilon_{ba}A^b. \tag{C.2.4}$$

Further, spinorial derivatives in our conventions are as follows:

$$\frac{\partial A^a}{\partial A^b} = \delta^a_b. \tag{C.2.5}$$

However, in contrast to the index raising and lowering for spinors (C.2.4), we have the following conventions for the derivatives:

$$\epsilon^{ab}\frac{\partial}{\partial A^b} = -\frac{\partial}{\partial A_a}. \tag{C.2.6}$$

We choose the following representations of the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{C.2.7}$$

which satisfy,

$$(\sigma^\mu)^a_b(\sigma^\nu)^c_a = \delta^{\mu\nu}\delta^c_b + i\epsilon^{\mu\nu\rho}(\sigma^\rho)^c_b. \tag{C.2.8}$$

C.3 Three dimensional spinor-helicity variables

Given a three-vector p_μ, we can trade it for a matrix $(\not{p})^a_b$ in the following way:

$$(\not{p})^a_b = p_\mu(\sigma^\mu)^a_b = \lambda_b\bar{\lambda}^a + p\delta^a_b. \tag{C.3.1}$$

We can extract the magnitude of the momentum (energy) through the following bracket:

$$p = -\frac{1}{2}\langle\lambda\bar{\lambda}\rangle. \tag{C.3.2}$$

We can also contract form spinor dot products belonging to different momenta in the following way:

$$\langle ij\rangle = \lambda_{ia}\lambda^a_j. \tag{C.3.3}$$

Since we work with spinning operators, we require the use of polarization vectors. Our conventions for the same are,

$$z_\mu^-(\sigma^\mu)^a_b = (\not{z}^-)^a_b = \frac{\lambda_b\lambda^a}{p}, \tag{C.3.4}$$

$$z_\mu^+(\sigma^\mu)^a_b = (\not{z}^+)^a_b = \frac{\bar{\lambda}_b\bar{\lambda}^a}{p}. \tag{C.3.5}$$

For a spin half operator we define the polarization spinor to be,

$$\zeta_a^- = \frac{\lambda_a}{\sqrt{p}}, \qquad \zeta_a^+ = \frac{\bar{\lambda}_a}{\sqrt{p}}. \tag{C.3.6}$$

In a correlation function involving n operator insertions, momentum conservation reads,

$$\sum_{i=1}^n p_i^\mu = 0. \tag{C.3.7}$$

Contracting this equation with $(\sigma_\mu)^a_b$ yields momentum conservation in terms of the spinor variables.

$$\sum_{i=1}^n \lambda_{ib}\bar{\lambda}_i^a = -E\delta^a_b. \tag{C.3.8}$$

The three dimensional dot product of two three vectors x and y can be written in spinor notation using,

$$x\cdot y = \frac{1}{2}(\not{x})^a_b(\not{y})^b_a. \tag{C.3.9}$$

Since we work with parity odd correlation functions as well, we will require the following formula:

$$\epsilon^{\mu\nu\rho} = \frac{1}{2i}(\sigma^\mu)^a_b(\sigma^\nu)^c_a(\sigma^\rho)^b_c. \tag{C.3.10}$$

For any three vectors v_1, v_2 and v_3 we define for convenience,

$$\epsilon^{v_1v_2v_3} = v_{1\mu}v_{2\nu}v_{3\rho}\epsilon^{\mu\nu\rho}. \tag{C.3.11}$$

C.4 Some useful spinor-helicity variables identities

Contracting the momentum conservation equation (C.3.8) with different combinations of spinors, we get

$$\langle ji\rangle\langle\bar{i}\bar{k}\rangle = E\langle j\bar{k}\rangle, \tag{C.4.1}$$

$$\langle ji\rangle\langle\bar{i}k\rangle = (E-2p_k)\langle jk\rangle, \tag{C.4.2}$$

$$\langle\bar{j}i\rangle\langle\bar{i}\bar{k}\rangle = (E-2p_j)\langle\bar{j}\bar{k}\rangle, \tag{C.4.3}$$

$$\langle\bar{j}i\rangle\langle\bar{i}k\rangle = (E-2p_j-2p_k)\langle\bar{j}k\rangle, \tag{C.4.4}$$

$$\langle ji\rangle\langle\bar{i}j\rangle + \langle jk\rangle\langle\bar{k}j\rangle = 0, \tag{C.4.5}$$

$$\langle ij\rangle\langle\bar{i}\bar{j}\rangle = E(E-2p_k), \tag{C.4.6}$$

where i, j and k are all distinct labels.
The dot products of the polarizations with the momenta are given by,

$$\begin{aligned} p_i\cdot z_j^- &= -\frac{\langle ij\rangle\langle\bar{i}j\rangle}{2p_j}, \\ p_i\cdot z_j^+ &= -\frac{\langle i\bar{j}\langle\bar{i}\bar{j}\rangle}{2p_j}, \end{aligned} \qquad \begin{aligned} z_i^-\cdot z_j^- &= -\frac{\langle ij\rangle^2}{2p_ip_j}, \\ z_i^-\cdot z_j^+ &= -\frac{\langle i\bar{j}\rangle^2}{2p_ip_j}, \\ z_i^+\cdot z_j^+ &= -\frac{\langle\bar{i}\bar{j}\rangle^2}{2p_ip_j}. \end{aligned} \tag{C.4.7}$$

For the contractions of momentum and polarization vectors with the three dimensional Levi-Civita symbol we have,

$$\begin{aligned} \epsilon^{z_i^- z_j^- p_k} &= i\frac{\langle ij\rangle\left(\langle i\bar{k}\rangle\langle kj\rangle + \langle ij\rangle p_k\right)}{2p_ip_j}, \\ \epsilon^{z_i^- z_j^+ p_k} &= i\frac{\langle i\bar{j}\rangle\left(\langle i\bar{k}\rangle\langle k\bar{j}\rangle + \langle i\bar{j}\rangle p_k\right)}{2p_ip_j}, \\ \epsilon^{z_i^+ z_j^+ p_k} &= i\frac{\langle\bar{i}\bar{j}\rangle\left(\langle\bar{i}\bar{k}\rangle\langle k\bar{j}\rangle + \langle\bar{i}\bar{j}\rangle p_k\right)}{2p_ip_j}, \end{aligned} \qquad \begin{aligned} \epsilon^{z_i^- p_j p_k} &= i\frac{\langle i\bar{k}\rangle\left(\langle ij\rangle\langle k\bar{j}\rangle - \langle ki\rangle p_j\right) + \langle i\bar{j}\rangle\langle ij\rangle p_k}{2p_i}, \\ \epsilon^{z_i^+ p_j p_k} &= \bar{i}\frac{\langle i\bar{k}\rangle\left(\langle\bar{i}j\rangle\langle k\bar{j}\rangle - \langle k\bar{i}\rangle p_j\right) + \langle i\bar{j}\rangle\langle\bar{i}j\rangle p_k}{2p_i}. \end{aligned} \tag{C.4.8}$$

We also used the two dimensional Schouten identity on many occasions which reads

$$\delta^a_f\epsilon^{bc} + \delta^b_f\epsilon^{ac} + \delta^c_f\epsilon^{ba} = 0. \tag{C.4.9}$$

Another very important manifestation of the Schouten identity is the following. For any two-component spinor λ_{ia} we decompose it as a linear combination of λ_{1a} and $\bar{\lambda}_{1a}$.

$$\lambda_i = -\frac{\langle i\bar{j}\rangle}{2p_j}\lambda_j + \frac{\langle ij\rangle}{2p_j}\bar{\lambda}_j, \tag{C.4.10}$$

$$\bar{\lambda}_i = -\frac{\langle\bar{i}\bar{j}\rangle}{2p_j}\lambda_j + \frac{\langle\bar{i}j\rangle}{2p_j}\bar{\lambda}_j. \tag{C.4.11}$$

Appendix D

The Super Conformal Algebra and Ward Identities for $3d$

D.1 $\mathcal{N}=1$ Superconformal algebra and Ward Identities

The $\mathcal{N}=1$ superconformal algebra in three dimensions consists of the following generators: The generator of translations, P_μ, the generator of rotations, $M_{\mu\nu}$, the generator of dilatations and the generator of special conformal transformations which are respectively denoted as D and K_μ.

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\lambda}] = i\left(\delta_{\mu\rho}M_{\nu\lambda} - \delta_{\nu\rho}M_{\mu\lambda} - \delta_{\mu\lambda}M_{\nu\rho} + \delta_{\nu\lambda}M_{\mu\rho}\right),\\
&[M_{\mu\nu}, P_\alpha] = i\left(\delta_{\mu\alpha}P_\nu - \delta_{\nu\alpha}P_\mu\right), \qquad [M_{\mu\nu}, K_\alpha] = i\left(\delta_{\mu\alpha}K_\nu - \delta_{\nu\alpha}K_\mu\right),\\
&[D, P_\mu] = iP_\mu, \qquad [P_\mu, K_\nu] = 2i\left(\delta_{\mu\nu}D - M_{\mu\nu}\right),\\
&\{Q_a, Q_b\} = i(\sigma^\mu)_{ab}P^\mu, \qquad \{S_a, S_b\} = i(\sigma^\mu)_{ab}K^\mu,\\
&[D, Q_a] = \frac{i}{2}Q_a, \qquad [D, K_\mu] = -iK_\mu,\\
&[K_\mu, Q_a] = i(\sigma_\mu)_a^b S_b, \qquad [D, S_a] = -\frac{i}{2}S_a,\\
&[M_{\mu\nu}, Q_a] = \frac{i}{2}\epsilon_{\mu\nu\rho}(\sigma^\rho)_a^b Q_b, \qquad [P_\mu, S_a] = i(\sigma_\mu)_b^a Q_b,\\
&[M_{\mu\nu}, S_a] = \frac{i}{2}\epsilon_{\mu\nu\rho}(\sigma^\rho)_a^b S_b, \qquad \{Q_a, S_b\} = \epsilon_{ab}D - \frac{i}{2}\epsilon_{\mu\nu\rho}(\sigma^\rho)_{ab}M^{\mu\nu}.
\end{aligned} \tag{D.1.1}$$

The (anti) commutators not listed above are zero. The representation of these generators acting on primary superfields are as follows:
We have,

$$P_\mu = -i\partial_\mu,$$

$$\begin{aligned}
M_{\mu\nu} &= -i\bigg(x_\mu\partial_\nu - x_\nu\partial_\mu + \frac{i}{2}\epsilon_{\mu\nu\rho}(\sigma^\rho)^a_b\theta^b\frac{\partial}{\partial\theta^a}\bigg) + \mathcal{M}_{\mu\nu},\\
D &= -i\bigg(x^\nu\partial_\nu + \frac{1}{2}\theta^a\frac{\partial}{\partial\theta^a} + \Delta\bigg),\\
K_\mu &= i\bigg(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2\Delta x_\mu\bigg) - 2x^\nu\mathcal{M}_{\mu\nu} + i(\sigma^\mu)^c_b(\sigma^\nu)^a_c x_\nu\theta^b\frac{\partial}{\partial\theta^a},\\
Q_a &= \frac{\partial}{\partial\theta^a} + \frac{i}{2}\theta_b(\sigma^\mu)^b_a\partial_\mu,
\end{aligned} \tag{D.1.2}$$

where Δ is the scaling dimension of the primary and $\mathcal{M}_{\mu\nu}$ encodes the non trivial transformation of operators with spin. The expression for S_a is obtained by computing $[K_\mu, Q_a]$ as in (D.1.1). Given the position superspace generators in (D.1.2), it is straightforward to obtain the momentum superspace generators by performing a Fourier transform to go from the x^μ to the p_μ variables.

$$\begin{aligned}
P_\mu &= p_\mu,\\
M_{\mu\nu} &= -i\bigg(p_\mu\frac{\partial}{\partial p_\nu} - p_\nu\frac{\partial}{\partial p_\mu} + \frac{i}{2}\epsilon_{\mu\nu\rho}(\sigma^\rho)^a_b\theta^b\frac{\partial}{\partial\theta^a}\bigg) + \mathcal{M}_{\mu\nu},\\
D &= i\bigg(p_\nu\frac{\partial}{\partial p_\nu} + (3-\Delta) - \frac{1}{2}\theta^a\frac{\partial}{\partial\theta^a}\bigg),\\
K_\mu &= -\bigg(p_\mu\frac{\partial^2}{\partial p_\nu\partial p^\nu} + 2(\Delta-3)\frac{\partial}{\partial p_\mu} - 2p_\nu\frac{\partial^2}{\partial p_\nu\partial p_\mu}\bigg) - 2i\frac{\partial}{\partial p^\nu}\mathcal{M}_{\mu\nu} - (\sigma^\mu)^c_b(\sigma^\nu)^a_c\theta^b\frac{\partial}{\partial\theta^a}\frac{\partial}{\partial p^\nu},\\
Q_a &= \frac{\partial}{\partial\theta^a} - \frac{1}{2}\theta_b(\sigma^\mu)^b_a p_\mu,
\end{aligned} \tag{D.1.3}$$

with S_a obtained via (D.1.1).

The generators in the super spinor helicity grassmann twistor variables (5.1.20) are given by,

$$\begin{aligned}
P_\mu &= \frac{1}{2}(\sigma^\mu)^a_b\lambda_a\overline{\lambda}^b,\\
M_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho}(\sigma^\rho)^a_b\bigg(\overline{\lambda}^b\frac{\partial}{\partial\overline{\lambda}^a} + \lambda^b\frac{\partial}{\partial\lambda^a}\bigg),\\
D &= \frac{i}{2}\bigg(\overline{\lambda}^a\frac{\partial}{\partial\overline{\lambda}^a} + \lambda^a\frac{\partial}{\partial\lambda^a} + 2\bigg),\\
K^\mu &= 2(\sigma^\mu)^{ab}\frac{\partial^2}{\partial\lambda^a\,\overline{\lambda}^b},\\
Q_a &= \bigg(2\lambda_a\frac{\partial}{\partial\xi_+} + \frac{\overline{\lambda}_a}{4}\xi_+\bigg),\\
S_a &= -2i\bigg(2\frac{\partial}{\partial\xi_+}\frac{\partial}{\partial\overline{\lambda}^a} + \frac{\xi_+}{4}\frac{\partial}{\partial\lambda^a}\bigg).
\end{aligned} \tag{D.1.4}$$

D.2 $\mathcal{N} = 2$ Superconformal algebra and Ward Identities

Similarly we get the $\mathcal{N} = 2$ algebra, all the generators will remain the same except the Q_a and S_a and we will have two more generators $\bar{Q}_a$ and $\bar{S}_a$.
Q_a and $\bar{Q}_a$ were given in the Main text (6.1.4), but we give them here agin with S_a and $\bar{S}_a$.

$$Q_a = 2\lambda_a \frac{\partial}{\partial \omega_+} + \frac{\bar{\lambda}_a}{4} \xi_+, \qquad \bar{Q}_a = 2\lambda_a \frac{\partial}{\partial \xi_+} + \frac{\bar{\lambda}_a}{4} \omega_+, \tag{D.2.1}$$

$$S_a = -2i \left(2 \frac{\partial}{\partial \omega_+} \frac{\partial}{\partial \bar{\lambda}^a} + \frac{\xi_+}{4} \frac{\partial}{\partial \lambda^a} \right), \quad \bar{S}_a = -2i \left(2 \frac{\partial}{\partial \xi_+} \frac{\partial}{\partial \bar{\lambda}^a} + \frac{\omega_+}{4} \frac{\partial}{\partial \lambda^a} \right). \tag{D.2.2}$$

We write the R-symmetry generator (6.1.6)

$$R = \omega_+ \frac{\partial}{\partial \omega_+} - \xi_+ \frac{\partial}{\partial \xi_+}. \tag{D.2.3}$$

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