Splittings of Binomial Edge Ideals

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Aniketh Sivakumar

IISER PUNE

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2024

Supervisor: Prof. Adam Van Tuyl, Prof. A.V. Jayanthan

Certificate

This is to certify that this dissertation entitled Splittings of Binomial Edge Ideals towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Aniketh Sivakumar at Indian Institute of Science Education and Research under the supervision of Prof. Adam Van Tuyl, Professor, Department of Math and Stats, McMaster University and Prof. A.V. Jayanthan, Professor, Department of Mathematics, IIT Madras during the academic year 2023-2024.

Prof. Adam Van Tuyl

Prof. A.V. Jayanthan

Committee:

Prof. Adam Van Tuyl, Prof. A.V. Jayanthan

Prof. Amit Hogadi

This thesis is dedicated to my friends and family

Declaration

I hereby declare that the matter embodied in the report entitled Splittings of Binomial Edge Ideals are the results of the work carried out by me at the Department of Mathematics , McMaster University and IIT Madras, under the supervision of Prof. Adam Van Tuyl and Prof. A.V. Jayanthan and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

Aniketh Sivakumar - 20191046

Acknowledgments

I'm incredibly grateful to the following individuals and institutions whose encouragement and support fueled my journey through my Master's thesis.

First and foremost, I would like to thank my advisors, Prof. Adam Van Tuyl and Prof. A.V. Jayanthan. Their constant support, encouragement and guidance throughout this project have been invaluable. Their insight has been paramount to the direction of this research, from the choice of the problem statement to refining and extending this work. I am grateful for their patience, understanding, and willingness to answer my countless questions throughout the research and writing process. I also want to express my gratitude to my thesis expert Prof. Amit Hogadi, for his valuable feedback during this project.

I am grateful to IIT-Madras and McMaster University for hosting me throughout this project. I would also like to thank MITACS for awarding me the MITACS Globalink Research Award for the duration of my thesis. Their financial assistance made this research project possible.

Furthermore, I want to express my appreciation to my colleagues and peers within the math program at McMaster University and IISER Pune. In particular, I would like to thank Kieran, Silas, Mike and Varun. Their support, encouragement, and stimulating discussions provided a valuable network of support throughout my studies.

Finally, I am immensely grateful to my family and friends for their unwavering love and support. My mother Leela, my father Sivakumar, and my brother Satchit have motivated me throughout this journey. Their belief in me and my abilities has been a pillar of my strength. My heartfelt thanks also go to Prerana for her unwavering companionship and encouragement these past few years. Her caring and understanding nature has kept me motivated in my times of need. My friends, Amogh Rakesh, Neev, Varun, Garvit, Amogh

Ranade, Aditya, Divyansh, Arjun, Vatsal, Vasudha, Jezer and Pallav have also helped me through challenging times. I am grateful for their friendship and for all the time we have spent over the last five years.

This thesis could not have been accomplished without the constant presence of everyone above. Thank you to all of you.

Abstract

Consider a finite simple graph G. One can associate an ideal to the edges of this graph, called its binomial edge ideal J_G. Many homological invariants, such as the Betti numbers, Castelnuovo-Mumford regularity ($\mathrm{reg}(J_G)$) and the projective dimension ($\mathrm{pd}(J_G)$) of these ideals are widely studied. For binomial edge ideals of graphs, these invariants are often intimately related to graph-theoretic notions such as connectivity, free vertices and so on. In this thesis, we study the method of Betti splittings applied to binomial edge ideals. We give some examples of Betti splittings and introduce the notion of a partial Betti splitting. We demonstrate that removing a vertex from the graph results in a partial splitting of the associated binomial edge ideal. A similar study is also done to obtain a partial splitting for the initial ideal of a binomial edge ideal. We also prove new bounds for some homological invariants of J_G and explore some of their implications.

Contents

Chapter 1

Introduction

In recent decades, combinatorial commutative algebra has become a popular field of study. With the rise of homological methods in commutative algebra, several connections have been found to geometry and combinatorics. This story begins in the field of geometric combinatorics. Here, one of the major topics of research is convex and discrete geometry and their properties.

Consider the space $\mathbb{R}^d$. We say that $S \subset \mathbb{R}^d$ is *convex* if it has the property that for any points $x, y \in S$, the line segment $\{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$ with endpoints x, y is completely contained in S. For any set of points $P \subset \mathbf{R}^{\mathbf{d}}$, the *convex hull* of P is the smallest convex set that contains P. A *convex polytope* is defined as the convex hull of a finite set of points. The study of convex polytopes is central in the field of geometric combinatorics. Specifically, a lot of research is done on the facial structure of these polytopes. One interesting question is counting the number of faces of different dimensions in different convex polytopes.

One important type of convex polytope is the cyclic polytope $C(n, d)$, which is the convex hull of n distinct points on the moment curve, which is parametrised by $(t, t^2, \ldots, t^d), -\infty < t < \infty$. In 1970, Peter McMullen proved the Upper Bound Theorem, which states that cyclic polytopes have the largest number of faces of all convex polytopes with a given dimension and a fixed number of vertices. In 1975, Stanley extended this result to triangulations of simplicial spheres via a different method. To do this, he used what would later be known as the Stanley-Reisner ring. This ring is the quotient of a multivariable polynomial ring with a square-free monomial ideal. He related algebraic quantities like the Hilbert function to the number of faces of the polytope. His proof involved a careful study of the Stanley-Reisner ring, in the case that it was Cohen-Macaulay. This pioneered the use of commutative algebra to study questions in geometric combinatorics.

Over the next decade, square-free monomial ideals were widely studied. With the advent of Gröbner basis, computational techniques to study questions in commutative algebra became commonplace. One important object that became widely studied is the minimal graded free resolution of ideals in a polynomial ring. This object has a variety of numerical invariants associated with it, which give insight into a variety of different properties of the ideal. Some important invariants we shall see are the graded Betti numbers $\beta_{i,j}(I)$, the projective dimension $\mathrm{pd}(I)$, and the Castelnuovo-Mumford regularity $\mathrm{reg}(I)$. Many different techniques were developed to study these invariants in the case of monomial ideals.

One such technique was developed in 1990 by Elaihou and Kervaire in [5], where they study the minimal free resolution of a class of monomial ideals called Boreal fixed ideals. They defined a new concept called a *splittable ideal*, $I = J + K$, where the Betti numbers of I can be written using the Betti numbers of J, K and $J \cap K$. This idea was later generalised by Francisco, Hà, and Van Tuyl in [8], where they introduced the notion of a *Betti splitting* of a monomial ideal. An ideal I has a Betti splitting if there exist two other monomial ideals J and K such that $I = J + K$ and

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i, j \geq 0.$$

In particular, they gave several criteria for monomial ideals to have Betti splittings, which helped study the Betti numbers of edge ideals, a monomial ideal associated with a graph. This idea of splitting the Betti numbers of an ideal into the Betti numbers of 'smaller' ideals has only been briefly studied for arbitrary graded ideals.

One important class of ideals to consider are binomial ideals. A binomial belonging to $S = k[x_1, \ldots, x_n]$ is a polynomial of the form $u - v$, where u and v are monomials in S. A *binomial ideal* is an ideal of S generated by binomials. In the 1990s, the study of binomial ideals grew popular, when they were seen to have applications to algebra, combinatorics and statistics. One important type of binomial ideal that is widely studied even today are toric ideals. The toric ideal of a graph is a binomial ideal that is associated with a finite simple graph. Much like monomial ideals, these ideals were studied extensively for their homological properties. The idea of Betti splittings was also modified to study toric ideals by Favacchio, Hofscheier, Keiper, and Van Tuyl in [7].

In the 2010s, a new class of binomial ideals called *binomial edge ideals* was introduced by Herzog, Hibi, Hreinsdóttir, Kahle and Rauh in [11] and independently by Ohtani in [23], with applications to algebraic statistics. Like edge ideals and toric ideals of graphs, binomial edge ideals are also ideals associated with graphs. In the polynomial ring $S =$

$k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with k a field, we define the binomial edge ideal of the graph G to be the ideal given by

$$J_G = \langle x_i y_j - x_j y_i \mid \{i, j\} \in E(G) \rangle.$$

Over the last few years, there has been a lot of work done on the properties of homological invariants of these ideals (for some examples, see [19], [16], [27]). The main goal of this thesis is to try and modify the technique of Betti splittings designed for monomial and toric ideals and extend it to binomial edge ideals. This can be phrased as follows:

Question 1.1. *Let G be a finite simple graph with binomial edge ideal J_G. Is it possible to 'split' the graph G into two subgraphs, H and K, in a way that reveals a connection between the graded Betti numbers of J_G and those of the J_H and J_K?*

In the course of this thesis, we will answer this question and a related question for the initial ideals of binomial edge ideals. We will also touch upon new bounds for some invariants associated with the Betti numbers of the binomial edge ideals of different kinds of graphs. The following is a brief structure of the thesis:

In Chapter 2, we introduce relevant topics from both graph theory, commutative algebra, and homological algebra. We also go through important definitions and theorems that will be used throughout the later chapters.

In Chapter 3, we discuss some examples of complete Betti splittings for binomial edge ideals. We discuss a known result on the Betti splitting of graphs with a cut edge and prove a generalisation of the same. We also survey some results on the linear strand of the Betti table of any binomial edge ideal. We then apply our results to study Betti numbers of the binomial edge ideals of trees.

Chapter 4 then introduces the notion of a partial Betti splitting and describes conditions for the same. We then obtain a partial Betti splitting for the binomial edge ideal of any graph. We then discuss partial Betti splittings for the initial ideals of binomial edge ideals. In certain cases, we show that the Betti splitting for the binomial edge ideal J_G 'descends' to the initial ideal, in J_G.

In Chapter 5, we give a new bound on some homological invariants of any binomial edge ideal. Using this new bound, we can partially recover some known bounds and prove new bounds for the regularity of binomial edge ideals of certain types of graphs.

Finally, in Chapter 6, we discuss further extensions of our work. We describe some conjectures made during this project and suggest other relevant problems that can be studied.

None of the material in Chapter 2 is original. In Chapters $3, 4$ and 5, a lot of the material is original content, with some necessary results surveyed along the way.

Chapter 2

Preliminaries

In this chapter, we survey relevant definitions and theorems in graph theory and commutative algebra. In the first section, we establish some basic graph theory notation and describe some types of graphs and graph-theoretic properties that we will encounter in the rest of the thesis. In the next section, we define complexes and resolutions and describe some of their properties. We also introduce the minimal free resolution and describe some important homological invariants associated with it. In section three, we study monomial ideals and their resolutions. We describe the class of simplicial resolutions for monomial ideals and introduce the technique of Betti splittings for them. Finally, in the last section, we introduce binomial edge ideals and describe some of their algebraic properties. We identify a reduced Gröbner basis, characterise the minimal primes for the binomial edge ideal of any graph, and give some important bounds on the homological invariants of these ideals.

2.1 Graph theory

In this section, we will describe some basic graph theoretic terminology which will be used frequently in later sections. Throughout this thesis, we will only be working with finite simple graphs.

Definition 2.1. A **graph** is a pair $G = (V(G), E(G))$, where $V(G)$ is a set whose elements are called vertices and $E(G)$ is a set of paired vertices, whose elements are called edges.

A **finite simple graph** is a graph where $V(G)$ is finite and $E(G)$ is a set of distinct unordered pairs of distinct elements of $V(G)$. In other words, $E(G) \subset \{\{u, v\} \mid u, v \in V(G), u \neq v\}$. Note that this implies that these graphs cannot have edges from a vertex to itself and cannot have multiple edges between two vertices.

Graphs can also be visualised by associating the vertices of $V(G)$ with points in space and the edges of $E(G)$ with line segments between corresponding vertices.

Example 2.2. Consider the finite simple graph G, with $V(G) = \{1, 2, 3, 4, 5\}$, where $E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{4, 5\}\}$. A visual representation of G is given in Figure 2.1.

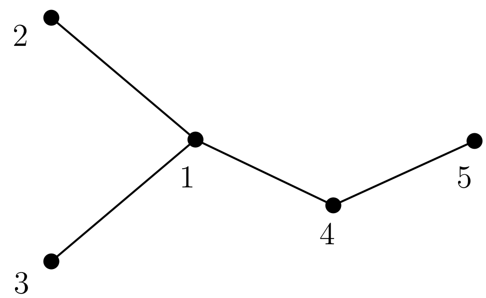


Figure 2.1: G

Definition 2.3. Consider a graph G. If $e = \{u, v\}$ is an edge in $E(G)$, then we say that u and v are **adjacent**. Furthermore, the set of all adjacent vertices to a vertex $v \in V(G)$ is called the set of **neighbours** of v and is denoted by $N_G(v)$. In other words $N_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$. The **degree** of a vertex v is the number of vertices adjacent to v. Hence, $\deg v = |N_G(v)|$.

Definition 2.4. A vertex of a graph G that is adjacent to only one other vertex is called a **pendant** vertex or a **leaf**. An edge of G that is incident to a pendant vertex is called a **pendant** edge.

Definition 2.5. A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph is said to be **induced** if for all $u, v \in V(H)$, if $\{u, v\} \in E(G)$, then $\{u, v\} \in E(H)$. The induced subgraph of G on $S \subseteq V(G)$ is denoted by $G[S]$.

Example 2.6. Consider the graph G in Example 2.2. Let G_1 be a graph with $V(G_1) = \{1, 2, 3\}$ and $E(G_1) = \{\{1, 2\}\}$ and let G_2 be a graph with $V(G_2) = \{1, 2, 3\}$ and $E(G_2) = \{\{1, 2\}, \{1, 3\}\}$. It can be seen that both G_1 and G_2 are subgraphs of G, where G_1 is not an induced subgraph, but G_2 is the induced subgraph $G(\{1, 2, 3\})$

Induced subgraphs show up while studying the Betti numbers of the binomial edge ideals of a graph, as we shall see later. Given a graph, it is possible to label the vertices in different ways. In most cases, a graph and its relabelling have identical properties.

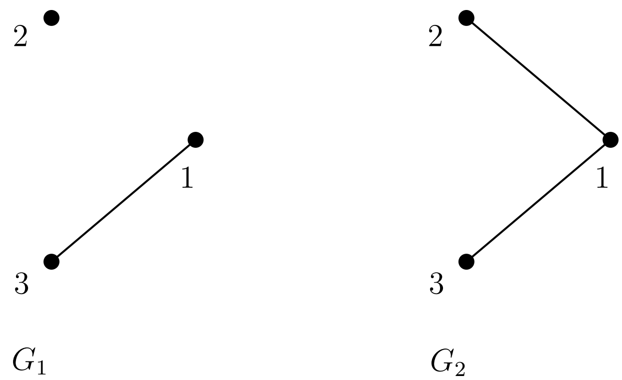


Figure 2.2:

Definition 2.7. Two graphs G_1 and G_2 are said to be **isomorphic**, if there exists a function $f : V(G_1) \to V(G_2)$ such that:

1. f is a bijection

2. $\{u, v\} \in E(G_1)$ if and only if $\{f(u), f(v)\} \in E(G_2)$.

Example 2.8. Consider the graphs G_1 and G_2, where $V(G_1) = V(G_2) = \{1, 2, 3, 4\}$. Let $E(G_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$ and $E(G_2) = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{2, 4\}\}$. Clearly the bijection $f : V(G_1) \to V(G_2)$ defined by $f(1) = 2$, $f(2) = 1$, $f(3) = 3$ and $f(4) = 4$ is an isomorphism between G_1 and G_2.

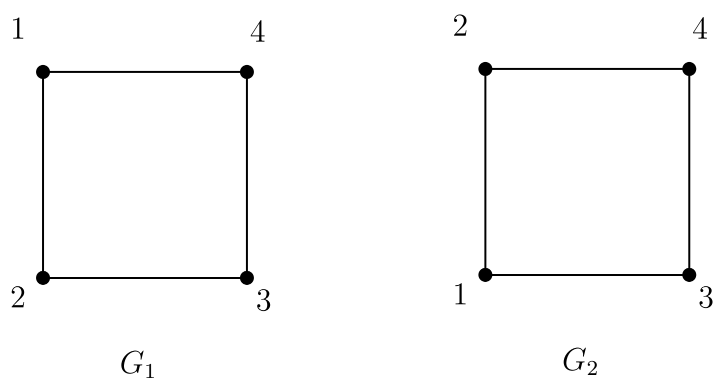


Figure 2.3: Isomorphic graphs

We now define the concept of a walk in a graph, which will come up several times in this thesis.

Definition 2.9. Consider a graph G. A **walk** is a finite sequence of vertices $(v_1, v_2, \ldots, v_m)$ such that $\{v_i, v_{i+1}\}$ is an edge for all $1 \leq i < m$. The **length** of a walk is the number of edges in the sequence $(v_1, v_2, \ldots, v_m)$. In other words, the length of the walk $(v_1, v_2, \ldots, v_m)$ is $m-1$. A **path** is a walk $(v_1, v_2, \ldots, v_m)$, such that all v_i are distinct.

Walks which begin and end at the same vertex are also widely studied.

Definition 2.10. A walk $(v_1, v_2, \ldots, v_m)$ is said to be **closed** if $v_m = v_1$. A closed walk where, $m \geq 3$ and all vertices in the walk are pairwise distinct, except for v_1 and v_m, is called a **cycle**. In other words, the walk $(v_1, v_2, \ldots, v_m)$ is a cycle if $v_i \neq v_j$ for all $1 < i < j < m$ and $v_1 = v_m$. If the cycle is a walk of length m, it is called an $m-$ cycle.

Example 2.11. Consider the graph G with $V(G) = \{0, 1, 2, 3, 4\}$ and $E(G) = \{\{0, 1\}, \{0, 3\}$ $, \{0, 4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Consider the sequence $W = (2, 3, 0, 1, 2)$. Since $\{2, 3\}, \{3, 0\}$ $, \{0, 1\}, \{1, 2\}$ are all edges in $E(G)$ and the first and last vertex are the same, we know that W is a closed walk. All the vertices except the first and last are also pairwise distinct. Hence, it is a $4-$cycle.

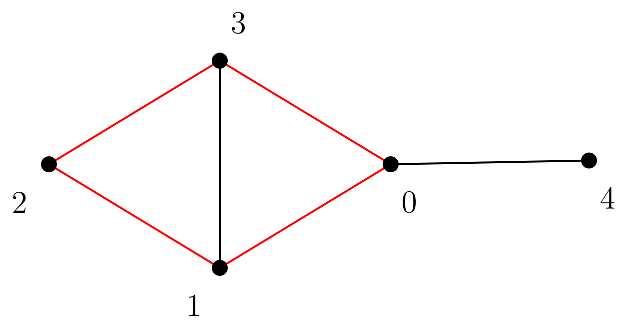


Figure 2.4: The graph G with walk $(2, 3, 1, 0)$.

Definition 2.12. Two vertices u_1 and u_2 are **connected** if there exists a walk $(v_1, \ldots, v_m)$ with $v_1 = u$ and $v_m = u_2$. A graph is said to be connected if any two vertices in the graph are connected. A **connected component** is a maximal connected subgraph of a graph. Each vertex belongs to exactly one connected component, as does each edge. A graph is connected if and only if it has exactly one connected component.

Definition 2.13. An edge e in G is a **cut edge** if its deletion from G yields a graph with more connected components than G. Let $G \setminus e$ be the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$. Hence, e is said to be a cut edge if and only if $G \setminus e$ has more connected components than G.

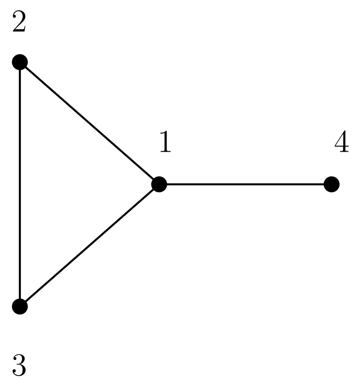


Figure 2.5: $G = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\}$.

Example 2.14. Consider G with $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\}$ from Figure 2.5. Then for $e = \{1, 4\}$, $G \setminus e$ will have two connected components. Hence e is a cut edge. It can also be seen that no other edge in G is a cut edge.

It is possible to define a notion of connectedness in terms of vertex removal.

Definition 2.15. The **connectivity** (or vertex connectivity) of a connected graph G is the minimum number of vertices whose removal makes G disconnected or reduces it to a trivial graph. This number is denoted by $K(G)$. The graph is said to be $k-$**vertex connected** for all $k \leq K(G)$.

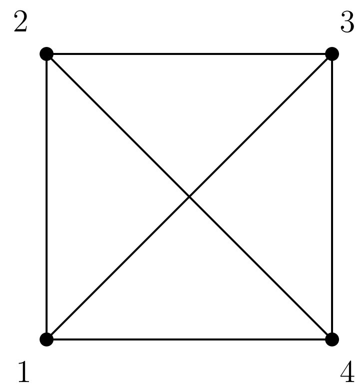
Example 2.16. Consider the graph G in Example 2.11. It can be seen that removing the vertex $\{0\}$ disconnects the graph. Hence, it is $1-$vertex connected.

2.1.1 Types of graphs

In this thesis, we will study the binomial edge ideals of many different kinds of graphs. We will list some important types of graphs which we will use in later chapters. One important type of graph we will need is the complete graph,

Definition 2.17. A **complete** graph is a graph in which each vertex is adjacent to every other vertex. In other words, G is complete if and only if $\{u, v\} \in E(G)$ for all $u, v \in V(G)$. The complete graph on n vertices is denoted by K_n.

Example 2.18. The graph K_4 has $V(K_4) = \{1, 2, 3, 4\}$ and $E(K_4) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{1, 3\}, \{2, 4\}\}$. It is described in Figure 2.6.



K_4

Figure 2.6:

Definition 2.19. A **clique** of a graph G, is a subset of vertices S of $V(G)$ such that $G[S]$, the induced subgraph on S, is a complete graph.

Given a vertex in a graph, it is always part of a clique $G[\{v\}]$, which is the graph with one vertex and is trivially a complete graph. Hence, a natural extension is to talk about the largest possible clique in G that contains the vertex $v \in V(G)$. To that end, we make the following definition.

Definition 2.20. A clique $G[S]$ is said to be a **maximal clique** if for all $S \subsetneq S' \subseteq V(G)$, $G[S]$ is a clique and $G[S']$ is not a clique.

The above definition tells us that every vertex in a finite simple graph is part of a maximal clique. A given vertex can be a part of several maximal cliques. An example is given below.

Example 2.21. Consider the graph G with $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\}$ from Example 2.14. We can see that the vertex $\{1\}$ is a part of two maximal cliques M_1 and M_2 with $G[M_1] = G[\{1, 2, 3\}] = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ and $G[M_2] = G[\{1, 4\}] = \{\{1, 4\}\}$.

Definition 2.22. A **free vertex** of a graph G is a vertex $v \in V(G)$ such that it is contained in only one maximal clique.

Example 2.23. Consider the graph G from Example 2.21 in Figure 2.5. We can see that the vertex $\{2\}$ is contained in only 1 maximal clique M_1 where $G[M_1] = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. The same is true for the vertex $\{3\}$. Hence, they are both free vertices.

Another important class of graphs are cyclic graphs.

Definition 2.24. A cycle graph is a graph G, such that there is a cycle $C = (v_1, \ldots, v_m)$ containing all vertices in $V(G)$, with $e = \{a, b\} \in E(G)$ if and only if $a = v_i$ and $b = v_{i+1}$ for some $1 \leq i \leq m - 1$. A cycle graph with n vertices is denoted by C_n.

Example 2.25. Consider the graph G with $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}\}$. Then G is the cycle graph on 6 vertices, C_6.

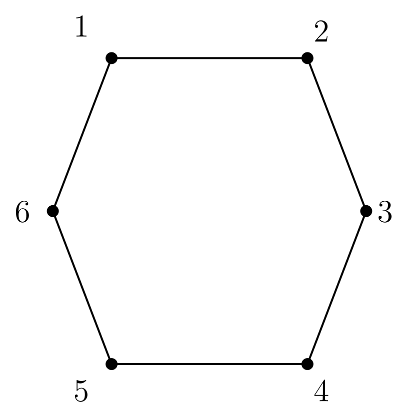


Figure 2.7: C_6

Definition 2.26. A **chordal graph** G is a graph where for any $S \subseteq V(G)$, the induced graph $G[S]$ cannot be a cycle with more than three vertices. In other words, G is a graph where any cycle with four or more vertices, has an edge between two non-consecutive vertices.

Example 2.27. Consider the graph G with $V(G) = \{0, 1, 2, 3, 4\}$ and $E(G) = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{1, 2\}, \{2, 3\}, \{2, 4\}\}$. It can be seen that this is a chordal graph.

Chordal graphs have been widely studied in a variety of contexts. There are many equivalent definitions of these graphs. One useful and important characterisation is the following.

Definition 2.28. A **perfect elimination ordering** in a graph is an ordering of the vertices of the graph such that, for each vertex v, v and the neighbours of v that occur after v in the ordering form a clique.

Example 2.29. Consider the graph G in Example 2.27 and Figure 2.8. Consider the ordering $1 < 3 < 4 < 2 < 0$. Since $1, 3$ and 4 are all free vertices with their neighbours occurring after them, the induced subgraphs with all neighbours greater than them form a clique. The

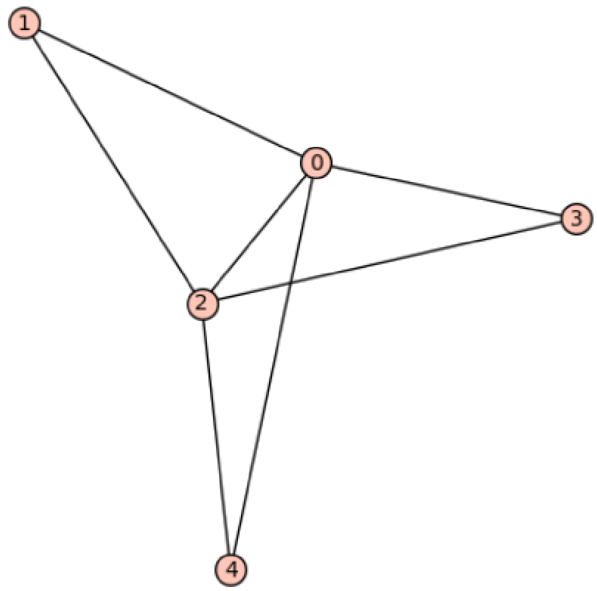


Figure 2.8: The chordal graph G from Example 2.27.

vertex 2 has only 0 greater than it. Thus since there is an edge $\{0, 2\}$ in $E(G)$, the induced graph $G[\{0, 2\}]$ is the clique on two vertices. Since there are no vertices greater than 0 in the ordering, $G[\{0\}]$ is trivially a clique.

Thus, the given ordering is a perfect elimination ordering on G.

Theorem 2.30. *A graph is chordal if and only if it has a perfect elimination ordering*

Proof. See the proof in [4]. □

The binomial edge ideals of chordal graphs have been widely studied. There are many different kinds of chordal graphs which simplify the study of the homological properties of their ideals.

Definition 2.31. A vertex of a graph G is called a **cut vertex** if its removal increases the number of connected components in G. A connected subgraph of G that has no cut vertex and is maximal with respect to this property is called a **block**.

Definition 2.32. A graph G is called a **block graph** if every block is a clique in G.

Example 2.33. Consider the graph G with $V(G) = \{0, 1, 2, 3, 4, 5, 6\}$ and $E(G) = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 5\}, \{0, 6\}, \{1, 2\}, \{3, 4\}, \{5, 6\}\}$. Here, the blocks are the induced subgraphs on the vertices $S_1 = \{0, 1, 2\}$, $S_2 = \{0, 3, 4\}$ and $S_3 = \{0, 5, 6\}$. Every block in G is isomorphic to the clique K_3.

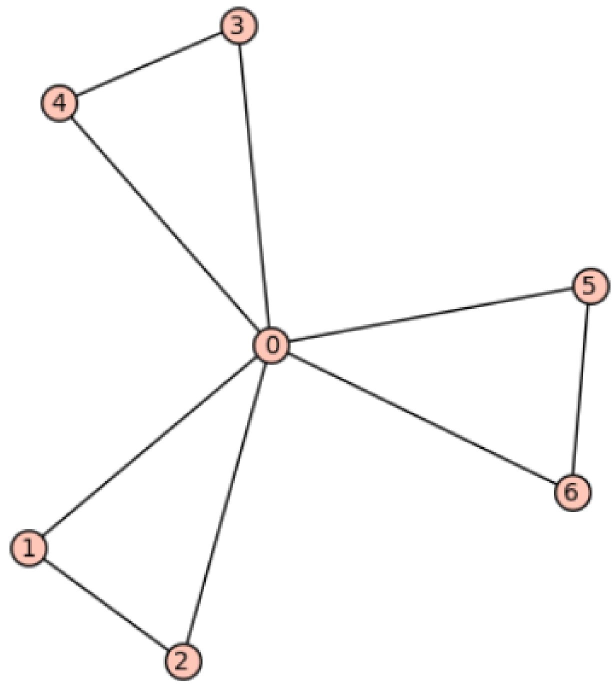


Figure 2.9: The block graph G from Example 2.33.

One common type of block graphs are trees.

Definition 2.34. A graph G where no subgraphs of G are cycles is called a **forest**. If the forest is connected, then it is called a **tree**. Every connected component of a forest is a tree.

There are many equivalent formulations for trees.

Theorem 2.35. *Consider a finite graph G where $V(G) = [n] := \{1, 2, \ldots, n\}$. Then the following are equivalent.*

- *G is a tree.*
- *G is connected and has $n - 1$ edges.*
- *Every edge in G is a cut edge.*

Proof. Refer to Theorem 1.5.1 in [3]. □

Example 2.36. Consider the graph G in Figure 2.10 with $V(G) = \{0, 1, 2, 3, 4, 5\}$ and $E(G) = \{\{0, 1\}, \{0, 4\}, \{1, 2\}, \{1, 3\}, \{1, 5\}\}$. It can be seen that G is a tree.

Remark 2.37. Since every edge in a tree T is a cut edge, that means that every vertex that has a degree greater than one is a cut point. Any connected subgraph of T is a tree. Hence,

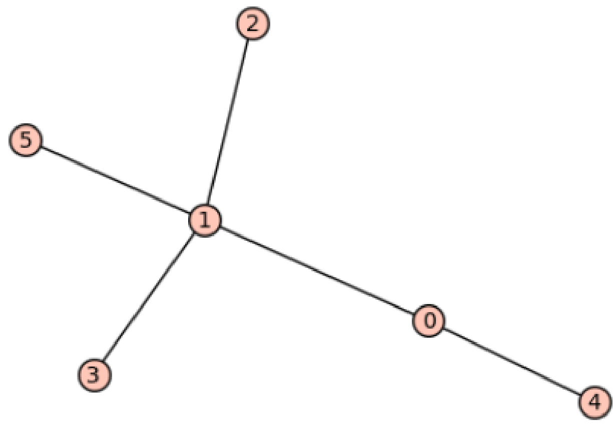


Figure 2.10: The tree G from Example 2.36.

if that subgraph has more than 2 vertices, then there will be a cut point. Thus, the maximal connected subgraph of T with no cut points can only be two vertices with an edge between them. Thus, every block is isomorphic to K_2. Hence, T is a block graph.

The star graph is one important type of tree we will study in this thesis.

Definition 2.38. Consider the graph G on $V(G) = [n]$, with $E(G) = \{\{1, i\} \mid i \in \{2, \ldots, n\}\}$. If H is a graph isomorphic to G, then H is called the **star graph** on n vertices. It is denoted by S_n.

Example 2.39. Consider the graph G on $V(G) = \{0, 1, 2, 3, 4, 5\}$ with $E(G) = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 5\}\}$. Then, G is the star graph on 6 vertices, S_6.

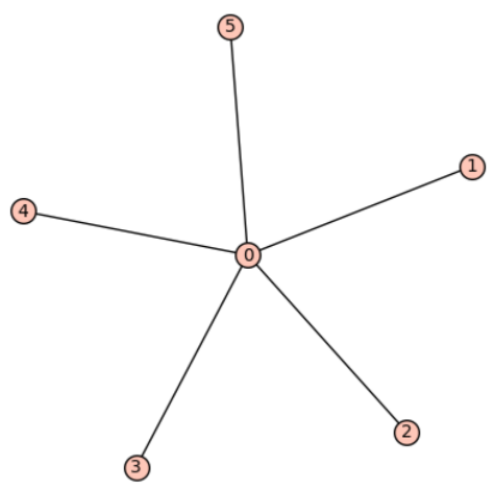


Figure 2.11: The star graph S_6

2.2 Homological algebra

In this section, we will introduce necessary topics from homological algebra that will be important in later chapters. Mainly, we will introduce the minimal free resolution and many invariants associated with it. Most of this material has been taken from [24] and [9]. All rings considered in this thesis will be commutative with an identity element.

Definition 2.40. Let R be a ring and A a monoid (a set with an associative binary operation "+" and an identity element). Then, R is said to have an A-grading if it can be decomposed into a direct sum of additive groups

$$R = \bigoplus_{a \in A} R_a$$

such that

$$R_m R_n \subset R_{m+n}$$

for all $m, n \in A$.

Definition 2.41. A non-zero element of R_n is called a **homogeneous** element of **degree** n. A **graded or homogeneous** ideal of R is defined to be an ideal that has a system of homogenous generators.

Remark 2.42. Since R is a direct sum of R_i, every $f \in R$ can be written uniquely as a direct sum of elements of R_i. Thus, f can uniquely be written as $f = \sum_{i \in A} f_i$, where $f_i \in R_i$, and all but finitely many f_i are 0. Each f_i is called the **homogeneous component** of f of degree i.

In this thesis, we will mainly be studying graded ideals. They have many equivalent characterisations.

Theorem 2.43. *If J is an ideal of the graded ring $R = \bigoplus_{a \in A}$, then the following are equivalent.*

- *J is a graded ideal.*
- *$J = \bigoplus_{i \in A} J_i$, where $J_i = R_i \cap J$.*
- *If $f \in J$, then every homogeneous component of f is in J.*

Proof. Refer to Chapter 1 in [24]. □

Example 2.44. The graded ring we will study in detail in later chapters is the multivariable polynomial ring. Let $S = k[x_1, \dots x_n]$ be the n variable polynomial ring, defined over the algebraically closed field k.

- Set $\deg(x_i) = 1$. Any monomial $x_i^{c_1} \cdots x_n^{c_n}$ has degree $c_1 + \cdots + c_n$. For $i \in \mathbb{N}$, let S_i be the $k-$vector space spanned by all monomials of degree i. For $i = 0$, we have $S_0 = k$. Then, we can see that $S = \bigoplus_{i \in \mathbb{N}} S_i$ is am $\mathbb{N}$-grading of the polynomial ring S.

- Set $\deg x_i = (0, \dots, 1 \dots, 0) \in \mathbb{N}^n$, (the i^{th} unit vector). Any monomial $x_i^{c_1} \cdots x_n^{c_n}$ has degree $(c_1, \dots, c_n)$. For $\mathbf{a} \in \mathbb{N}^n$, let $S_\mathbf{a}$ be the k-vector space spanned by all monomials of degree $\mathbf{a}$. For $\mathbf{a} = (0, \dots, 0)$, we have $S_\mathbf{a} = k$. Then, we can see that $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_\mathbf{a}$ is an $\mathbb{N}^n$-grading of the polynomial ring S. It is also known as a multigrading on S.

We will make use of the gradings in Example 2.44 several times in the subsequent chapters. It is also possible to generalise the definition of grading to modules.

Definition 2.45. Consider an A-graded ring R. A module M is said to be A-graded if it has a direct sum decomposition of additive groups $M = \bigoplus_{i \in A} M_i$, where M_i are such that $R_i M_j \subseteq M_{i+j}$, for all $i, j \in A$. All elements of M_i are called homogenous elements of degree i.

Definition 2.46. Let N and T be graded R-modules. We say that a homomorphism $\phi : N \to T$ has **degree** i, if $\deg(\phi(n)) = \deg n + i$ for all homogeneous elements $n \in N$. The space of all degree i homomorphisms of N and T is denoted by $\mathrm{Hom}_i(N, T)$. ϕ is said to be **graded or homogeneous** if $\phi \in \mathrm{Hom}_i(N, T)$ for some $i \in A$. If the map is a bijective homomorphism, then we call it a **graded isomorphism**.

Definition 2.47. Consider a graded R-module U. For $p \in A$, we denote by $U(-p)$ the graded R-module such that $U(-p)_i = U_{i-p}$ for all i. We say that $U(-p)$ is the **shifted** module of U by p degrees and p is called the **shift**.

There are many ways of constructing graded modules from a graded module U.

Theorem 2.48. *If $f : N \to T$ is a graded homomorphism, then* $\ker(f)$, $\mathrm{Im}(f)$ *and* $\mathrm{coker}(f)$ *are all graded.*

Proof. Refer to Proposition 2.9, [24]. $\square$

2.2.1 Resolutions

For this section, let A be a monoid and R be an A-graded ring.

Definition 2.49. A **complex** $(\mathbf{F}, \mathbf{d})$ over R is a sequence of R-modules and R-module homomorphisms

$$\mathbf{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow \dots$$

such that $d_{i-1}d_i = 0$ for all $i \in \mathbb{Z}$. The collection of maps $\mathbf{d} = \{d_i\}$ are called the **differentials** of $\mathbf{F}$. It is called a **left complex** if $F_i = 0$ for all $i < 0$. The complex $(\mathbf{F}, \mathbf{d})$ is said to be **graded** if F_i is a graded R-module and each d_i is a degree zero homomorphism for all i. If $\mathbf{F}$ is graded, we can write,

$$F_i = \bigoplus_{j \in A} F_{i,j}$$

Any element of $F_{i,j}$ is said to have **homological degree** i and **internal degree** j.

Remark 2.50. Since $d_{i-1}d_i = 0$, that implies that $\operatorname{Im}(d_i) \subseteq \ker(d_{i-1})$. Hence, in a complex, for all i, the image of d_i is contained in the kernel of d_{i-1}.

Definition 2.51. The **homology** of a complex is defined as

$$H_i(\mathbf{F}) = \frac{\ker(d_i)}{\operatorname{Im}(d_{i+1})} \text{ for all } i \in \mathbb{Z}.$$

The elements of $\ker(d_i)$ are called **cycles** and $\operatorname{Im}(d_{i+1})$ are called **boundaries**.

Definition 2.52. A complex is said to be **exact** if $H_i(\mathbf{F}) = 0$ for all i. A left complex is said to be **acyclic** if $H_i(\mathbf{F}) = 0$ for all $i > 0$.

Definition 2.53. If $(\mathbf{F}, \mathbf{d})$ and $(\mathbf{G}, \delta)$ are two complexes, then a homomorphism ψ of complexes is a set of homomorphisms $\{\psi_i\}$ where $\psi_i : F_i \to G_i$ is such that $\psi_{i-1}d_i = \delta_i\psi_i$. In other words, the following diagram commutes:

$$\begin{tikzcd}
F_i \arrow[r, "d_i"] \arrow[d, "\psi_i"'] & F_{i-1} \arrow[d, "\psi_{i-1}"] \\
G_i \arrow[r, "\delta_i"] & G_{i-1}
\end{tikzcd}$$

Definition 2.54. A **short exact sequence** of complexes is an exact complex of the form

$$0 \longrightarrow \mathbf{A} \xrightarrow{f} \mathbf{B} \xrightarrow{g} \mathbf{C} \longrightarrow 0$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are complexes and f and g are complex homomorphisms such that each

$$0 \longrightarrow A_i \xrightarrow{f_{i+1}} B_i \xrightarrow{g_{i+1}} C_i \longrightarrow 0$$

is a short exact sequence of R-modules.

Short exact sequences of complexes can tell us a lot about the homology of the individual complexes.

Theorem 2.55. *Given a short exact sequence of complexes,*

$$0 \longrightarrow \mathbf{A} \xrightarrow{f} \mathbf{B} \xrightarrow{g} \mathbf{C} \longrightarrow 0$$

there exists a connecting homomorphism $\delta_n : H_n(\mathbf{C}) \to H_{n-1}(\mathbf{A})$ for all n such that

$$\cdots \longrightarrow H_{n+1}(\mathbf{C}) \xrightarrow{\delta_{n+1}} H_n(\mathbf{A}) \longrightarrow H_n(\mathbf{B}) \longrightarrow H_n(\mathbf{C}) \xrightarrow{\delta_n} H_{n-1}(\mathbf{A}) \longrightarrow \dots$$

is an exact sequence.

Proof. Refer to Chapter 1, Section 13, [24]. □

The previous theorem is very fundamental and is used in proving many basic theorems in homological algebra.

Definition 2.56. A **free resolution** of a finitely generated R-module U is a complex

$$\mathbf{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} U \longrightarrow 0$$

such that:

- All F_i are finitely generated free R-modules, and
- $\mathbf{F}$ is an exact complex.

The free resolution is said to be graded if U is a graded module and $\mathbf{F}$ is a graded complex.

Theorem 2.57. *Every R-module U has a free resolution. In particular, if U is a finitely generated graded R-module, then it has a graded free resolution.*

Proof. Refer to Construction 4.2, [24]. □

Free resolutions are an interesting way of studying modules. We know that every finitely generated R-module can be written as a quotient of a free module, F/K, where F is free. By picking generators for K and considering the free module on this set, we can inductively construct a free resolution for the module. This is essentially the argument in Construction 4.2, [24]. This tells us that free resolutions give us some information on relations between generators of a module, relations between these relations and so on. Fixing particular types of generating sets can lead to different free resolutions.

Definition 2.58. Let $(R, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$ or an $\mathbb{N}$-graded k algebra, where $R_0 = k$. and $\mathfrak{m} = R_+$. A free resolution of the finitely generated graded R-module U

$$\mathbf{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} U \longrightarrow 0$$

is called **minimal** if $\operatorname{Im} d_{n+1} \subseteq \mathfrak{m}F_n$ for all $n \in \mathbb{N}$. It is called a graded minimal free resolution if $\mathbf{F}$ is also a graded free resolution.

Minimal-free resolutions turn out to have some very interesting properties.

Theorem 2.59. *Let $(R, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$ or an $\mathbb{N}$-graded k algebra, where $R_0 = k$ and the homogeneous maximal ideal $\mathfrak{m} = R_+$. Let U be a finitely generated graded R-module. Then the free resolution*

$$\mathbf{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} U \longrightarrow 0$$

is minimal if and only if for all n, F_n is constructed by taking the free module on a minimal set of generators (homogeneous in the case of a graded algebra R) for $\ker d_{n-1}$.

Proof. Refer to Theorem 4.7 in [24]. □

The above theorem tells us that finding the minimal free resolution is the same as picking a minimal homogeneous generating set for all $\ker d_{i-1}$.

Definition 2.60. Consider an A-graded ring R and let $p \in A$. A complex of the form:

$$0 \longrightarrow R(-p) \xrightarrow{1} R(-p) \longrightarrow 0$$

is called a **short trivial** complex. If $(\mathbf{F}, \mathbf{d})$ and $(\mathbf{G}, \delta)$ are complexes, then their **direct sum** is the complex $\mathbf{F} \oplus \mathbf{G}$ with modules $(\mathbf{F} \oplus \mathbf{G})_i = \mathbf{F}_i \oplus \mathbf{G}_i$ with differential $d'_i = d_i \oplus \delta_i$. A direct sum of short trivial complexes is called a **trivial** complex.

Now, we present the main theorem which captures the importance of this construction.

Theorem 2.61. *Let $(R, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$ or an $\mathbb{N}$-graded k algebra, where $R_0 = k$ and the homogeneous maximal ideal $\mathfrak{m} = R_+$. If $\mathbf{F}$ is a minimal free resolution of the graded R-module U, any free resolution for U is isomorphic to a direct sum of $\mathbf{F}$ with a trivial complex. In particular, the minimal free resolution of U is unique up to isomorphism.*

Proof. Refer to Chapter 9, [24]. □

Since the minimal free resolution is unique, we can define several associated invariants for a given R-module U.

2.2.2 Homological Invariants

Throughout this subsection, Let R be a $\mathbb{N}$-graded k algebra, where $R_0 = k$ and the homogeneous maximal ideal $\mathfrak{m} = R_+$. Let

$$\mathbf{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} U \longrightarrow 0$$

be the minimal graded free resolution of a graded finitely generated R-module U.

Definition 2.62. Let $\mathbf{F}$ be a minimal graded free resolution of a graded finitely generated R-module U. For $i \geq 1$, the submodule

$$\ker d_i = \operatorname{Im} d_{i+1}$$

of F_i is called the i^{th} **syzygy module** of U and is denoted by $\operatorname{Syz}_i(U)$.

Often it is difficult to obtain the exact description of the syzygy modules and the differentials in the minimal free resolution. Hence, the following invariants are widely studied.

Definition 2.63. The i^{th} **Betti number** of U over R is defined as:

$$\beta_i^R(U) := \operatorname{rank}(F_i).$$

Since the minimal free resolution is unique up to isomorphism, the Betti numbers are well-defined for any finitely generated graded R-module U. The main goal of this thesis is to study a certain property of the Betti numbers, for a class of ideals called binomial edge ideals.

Theorem 2.64.

$$\begin{aligned}\beta_i^R(U) &= \textit{number of minimal generators of } Syz_i^R(U)\\ &= \dim_k(\mathrm{Tor}_i^R(U,k))\end{aligned}$$

Proof. Follow Theorem 11.2 from [24]. □

By incorporating the grading, we can obtain the graded Betti numbers.

Definition 2.65. Let U be a finitely generated graded R-module. Then, $\beta_{i,j}(U)$ is defined as the total number of summands in the free module F_i in the minimal free resolution $\mathbf{F}$ of the form $R(-j)$.

From Theorem 2.64, we can see that

$$\beta_{i,j}(U) = \dim_k(\mathrm{Tor}_i^R(U,k)_j).$$

Remark 2.66. In light of the graded Betti numbers, $\beta_i(U)$ are called the **total Betti numbers** of U. It is easy to see from the definition that

$$\beta_i(U) = \sum_{j\in\mathbb{N}} \beta_{i,j}(U).$$

Theorem 2.67. *Let c be the minimal degree of a generator in a minimal system of homogeneous generators of U. Then,*

$$\beta_{i,j}(U) = 0$$

for all $j < i + c$. Hence, for any module, $\beta_{i,j}(U) = 0$ for all $j < i$.

Proof. This is proved in Proposition 12.3, [24]. □

This theorem is incorporated into the representation of the Betti numbers of a module into its **Betti table.** Here, the entry in the i^{th} row and j^{th} column, $\beta_{i,i+j}$ is the Betti number $\beta_{i,i+j}(U)$.

	β_0	β_1	β_2	$\ldots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\ldots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\ldots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\ldots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

In general, we also study several other invariants associated with the Betti numbers.

Definition 2.68. The **length** of a complex $\mathbf{G}$ is defined to be $\text{len}(\mathbf{G}) = \max\{i \mid G_i(U) \neq 0\}$. We say that $\mathbf{G}$ is a finite complex if its length is finite, otherwise, $\mathbf{G}$ is infinite. The **projective dimension** of a module U is an invariant of U defined as

$$\text{pd}(U) = \max\{i \mid \beta_i(U) \neq 0\}.$$

In other words, it is the length of the minimal free resolution.

The following theorem is a fundamental result which pioneered the study of homological invariants of ideals and modules.

Theorem 2.69. *(Hilbert's Syzygy Theorem) Let $S = k[x_1, \ldots x_n]$, where $x_1, x_2 \ldots, x_n$ are indeterminates. The minimal graded free resolution of a graded finitely generated S-module is finite and its length is at most n.*

Remark 2.70. The length of all graded finitely generated R- modules is finite, where R is a $\mathbb{N}$-graded k algebra, where $R_0 = k$ and the homogeneous maximal ideal $\mathfrak{m} = R_+$.

Another important invariant is the following.

Definition 2.71. The **Castelnuovo-Mumford regularity** of a graded finitely generated R-module U is defined as

$$\text{reg}(U) = \max\{j \mid \beta_{i,i+j}(U) \neq 0 \text{ for some } i\}.$$

Remark 2.72. Just like the projective dimension, the regularity is also finite. This can be seen as the number of summands in every F_i in the minimal free resolution $\mathbf{F}$ is finite. Hence, the number of distinct $j \in A$ such that $R(-j)$ is a summand is also finite.

The above results tell us that the Betti table for any graded finitely generated R-module has only finitely many non-zero entries. Hence, the table usually has a particular shape. The Betti numbers at the 'boundary' of the table are also widely studied.

Definition 2.73. A graded Betti number $\beta_{i,i+j}(U)$ of U is called **extremal**, if $\beta_{k,k+l}(U) = 0$ for all pairs $(k, l) \neq (i, j)$ with $k > i$ and $l > j$.

One important consequence of Theorem 2.61 is the following.

Corollary 2.74. *Let $\mathbf{F}$ be the minimal free resolution of a finitely generated graded R-module U and let $\mathbf{G}$ be another free resolution for U. Then,*

$$\operatorname{rank}(F_i)_j \leq \operatorname{rank}(G_i)_j.$$

for all homological degrees i and internal degrees j. In particular, $\beta_{i,j}(U) \leq \operatorname{rank}(G_i)$.

Proof. From Theorem 2.61, $\mathbf{G} = \mathbf{F} \oplus \mathbf{H}$, where $\mathbf{H}$ is a trivial complex. Thus, since all elements are free modules, $\operatorname{rank}(G_i)_j = \operatorname{rank}(F_i)_j + \operatorname{rank}(H_i)_j$. Thus, $\operatorname{rank}(F_i)_j \leq \operatorname{rank}(G_i)_j$. and hence $\beta_{i,j}(U) \leq \operatorname{rank}(G_i)$. □

This corollary allows us to get bounds on the Betti numbers, projective dimension and regularity of different modules, using the ranks of non-minimal free resolutions. We use this idea extensively in Chapter 5.

2.3 Monomial ideals

For general graded modules, studying the minimal free resolution is difficult. Only a little is known about the Betti numbers of general ideals and modules. A lot of work has been done on studying these invariants for different classes of ideals. One widely studied class are monomial ideals in polynomial rings. Throughout this section let k be a field, and $S = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables.

Definition 2.75. Consider any $(a_1, \ldots a_n) \in \mathbb{Z}^n$, where $a_i \geq 0$ for all i. Any product $x_1^{a_1} \cdots x_n^{a_n}$ is called a **monomial** in S.

A **monomial ideal** is an ideal in S generated by monomials.

We now give some basic properties of these ideals.

Theorem 2.76. *Let $I \subset S$ be an ideal. The following are equivalent:*

- *I is a monomial ideal.*
- *For all $f \in S$, $f \in I$ if and only if each monomial term of f is in I.*

Proof. Refer to Corollary 1.1.3 [10]. □

Theorem 2.77. *Consider the monomial ideal I. Let G be the generating set of I which is minimal with respect to divisibility. Then G is the unique minimal set of monomial generators.*

Proof. Refer to Proposition 1.1.6, [10]. $\square$

Theorem 2.78. *If I and J are monomial ideals, then*

- *$I + J$ is a monomial ideal,*
- *$I \cap J$ is a monomial ideal, and*
- *$I : J$ is a monomial ideal.*

Proof. Refer to Section 1.2, [10]. $\square$

All the above theorems illustrate why monomial ideals are easier to study. Given an ideal, there are some very useful monomial ideals associated with it.

Definition 2.79. A **monomial order** is a total order on $\mathrm{Mon}(S)$ (the set of monomials of S) where:

- $1 < u$ for all $u \in \mathrm{Mon}(S)$.
- if $u < v \in \mathrm{Mon}(S)$, then $uw < vw$ for all $w \in \mathrm{Mon}(S)$.

Example 2.80. Consider the polynomial ring $S = k[x_1, \dots, x_n]$. Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n) \in \mathbb{Z}^n$, with $a_i, b_i \geq 0$ for all i. The total order $<_{rev}$ of $\mathrm{Mon}(S)$ is defined by setting $x_1^{a_1} \dots x_n^{a_n} <_{rev} x_1^{b_1} \dots x_n^{b_n}$ if either one of the following holds.

- $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$
- $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the rightmost nonzero component of the vector $a - b$ is positive.

The total order $<_{rev}$ is a monomial order on S, called the **reverse lexicographic order** on S induced by the ordering $x_1 > x_2 > \dots > x_n$.

Definition 2.81. Fix a monomial order $<$ on $\mathrm{Mon}(S)$. Given $f = \sum_{u \in \mathrm{Mon}(S)} a_u u \in S$, the **initial monomial** of f, denoted by $\mathrm{in}_<(f)$ is the largest monomial with respect to $<$ such that $a_u \neq 0$.

Consider an ideal $I \subseteq S$. The **initial ideal** of I with respect to the monomial order $<$ is defined as

$$\mathrm{in}_<(I) = \langle \{\mathrm{in}_<(f) \mid 0 \neq f \in I\} \rangle.$$

Initial ideals are widely studied in a variety of contexts. They are intimately related to the theory of Gröbner basis, which is very important for computing generating sets for ideals in a polynomial ring. Their relation to Gröbner basis of an ideal also relates the homological invariants of I and $\mathrm{in}_<(I)$ as follows:

Theorem 2.82. *Fix a monomial order $<$. Consider a graded ideal $I \subset S$. Then*

$$\beta_{i,j}(I) \leq \beta_{i,j}(\text{in}_<(I)).$$

Proof. Refer to Theorem 22.9, [24]. □

2.3.1 Stanley-Reisner Correspondance

Certain types of monomial ideals have a nice combinatorial structure.

Definition 2.83. Consider a set P. A **simplicial complex** on P, denoted by Δ is a collection of subsets such that

- if $F \in \Delta$, then $F' \in \Delta$ for all $F' \subset F$,
- $\{i\} \in \Delta$ for all $i \in P$.

Each element of a simplicial complex Δ is called a **face**. A maximal face of Δ (with respect to inclusion) is called a **facet**.

Definition 2.84. The **dimension** of a face is defined as $\dim(F) = |F| - 1$. The dimension of the entire simplicial complex is defined as $\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}$.

Faces of dimension 0 are called **vertices** and faces of dimension one are called **edges**.

The following is an important example of a simplicial complex.

Example 2.85. (Clique complex) Consider a graph G with vertex set$V(G)$ and edge set $E(G)$. The **clique complex** of G is a simplicial complex $\Delta(G)$ where $F \subset V(G)$ is a face of $\Delta(G)$ if and only if the induced subgraph of G on F, G_F is a clique. Clearly, every set $\{v\} \in \Delta$, where $v \in V(G)$. If G_F is a clique and $F' \subset F$, then we know that $G_{F'}$ is also a clique. Hence $F' \in \Delta(G)$. Thus, we can see that $\Delta(G)$ is a simplicial complex.

Definition 2.86. Consider the polynomial ring $S = k[x_1, \ldots, x_n]$. For each $F \subset [n]$, we define

$$\mathbf{x}_F = \prod_{i \in F} x_i.$$

The **Stanley-Reisner ideal** of a simplicial complex Δ is defined as

$$I_\Delta = \langle \{\mathbf{x}_F \mid F \notin \Delta\} \rangle.$$

In other words, the Stanley-Reisner ideal is generated by the monomials associated with the minimal non-faces of Δ.

Remark 2.87. Note that I_Δ is a monomial ideal generated by square-free monomial generators. Such ideals are called square-free monomial ideals.

This correspondence between simplicial complexes and square-free monomial ideals goes both ways.

Definition 2.88. The **Stanley-Reisner complex** of a square-free monomial ideal I is the simplicial complex consisting of the monomials, not in I, that is

$$\Delta_I = \{F \subseteq [n] \mid \mathbf{x}_F \notin I\}.$$

Remark 2.89. It is important to note that the simplicial complexes here are not necessarily defined on $[n]$, but some subset of $[n]$. This is necessary due to the second condition in the definition of a simplicial complex.

Theorem 2.90. *Given a square-free monomial I and a simplicial complex Δ, the following are true:*

- $\Delta_{I_\Delta} = \Delta$.
- $I_{\Delta_I} = I$.

This is called the ***Stanley-Reisner Correspondence****.*

This correspondence is important in the study of monomial ideals. In several cases, it relates the homological properties of square-free monomial ideals to combinatorial properties of the corresponding Stanley-Reisner complex.

2.3.2 Simplicial Resolutions

In general, finding free resolutions for ideals is not an easy task. As discussed, it involves finding a generating set for the kernel of each differential map in the resolution. For monomial ideals, there are reliable methods of constructing free resolutions. In this section, we will present some important simplicial resolutions, including the Taylor and Lyubeznik resolution.

Simplicial resolutions involve casting various subsets of monomials as faces in a simplicial complex, with the differentials being the maps deleting vertices one at a time. All the material in this section has been taken from [20].

Construction 2.91. *Consider the polynomial ring $S = k[x_1, \dots x_n]$. Let the grading on S be the $\mathbb{N}^n$-grading described in Example 2.44. Let $M = \{m_1, \dots, m_r\}$ be a set of monomials generating a monomial ideal I in S. Consider a simplicial complex Δ on the set M. Fix an ordering on M, $m_{j_1} < m_{j_2} < \cdots < m_{j_r}$. We know each face in the simplicial complex is a subset of M. Consider a face $F \in \Delta$. For each F we associate the formal symbol $[F]$. We also give $[F]$ a multidegree (mdeg) as follows*

$$\deg_{\mathbb{N}^n}(F) = \deg_{\mathbb{N}^n}(\operatorname{lcm}\{m \mid m \in F\}).$$

Let H_s be the free S-module generated by the set $\{[F] \mid |[F]| = s\}$. We have a map $\phi_{s-1} : H_s \to H_{s-1}$ defined by:

$$[F] \longrightarrow \sum_{G \subset F, |G|=|F|-1} \epsilon_G^F \frac{\operatorname{lcm} F}{\operatorname{lcm} G}[G],$$

where ϵ_G^F is the map defined as:

$$\epsilon^F(G) = \begin{cases} 0 & |G| < |F| - 1 \\ 1 & G = F \setminus \{m_{j_k}\}, k \text{ is odd} \\ -1 & G = F \setminus \{m_{j_k}\}, k \text{ is even} \end{cases}$$

This construction thus gives us the sequence:

$$\mathbf{H}_{\boldsymbol{\Delta}} : 0 \longrightarrow H_r \xrightarrow{\phi_{r-1}} H_{r-1} \longrightarrow \cdots \longrightarrow H_1 \xrightarrow{\phi_0} H_0 \longrightarrow \frac{S}{I} \longrightarrow 0.$$

From the definition of ϕ_i, it can be checked that $\mathbf{H}_{\boldsymbol{\Delta}}$ is a complex, that is

$$\phi_{i-1} \circ \phi_i = 0 \text{ for all } 1 \leq i \leq r - 1.$$

In certain special cases, the complex from Construction 2.91 will be exact, and hence a resolution. We will describe two important cases where this happens.

Taylor Resolution

Let I be a monomial ideal with generating set M. In this case, the simplicial complex Δ on M is taken to be the simplex on M. In other words, every subset of M is a face of Δ. The faces of dimension i are precisely $\{G \subset M \mid |G| = i + 1\}$. Thus the complex described in

Construction 2.91 is:

$$\mathbf{H}_{\mathbf{\Delta}} : 0 \longrightarrow H_r \xrightarrow{\phi_{r-1}} H_{r-1} \longrightarrow \cdots \longrightarrow H_1 \xrightarrow{\phi_0} H_0 \longrightarrow \frac{S}{I} \longrightarrow 0$$

where $r = |M|$ and $H_i \neq 0$ for all $i \leq r$.

Theorem 2.92. *Let I be a monomial ideal I with generating set M. If Δ is the simplex on M, then the complex H_Δ is exact and hence, a resolution.*

Proof. Refer to Theorem 3.4, [20]. $\square$

Example 2.93. Let $S = k[x_1, x_2]$, with $m = \langle x_1, x_2 \rangle$. Let $I = \langle x_1x_2, x_1^3, x_2^2 \rangle$. Let the ordering be $x_1x_2 < x_1^3 < x_2^2$. Then the corresponding Taylor resolution $\mathbf{H}$ is:

$$0 \longrightarrow S[x_1x_2, x_1^3, x_2^2] \xrightarrow{\begin{pmatrix} 1 \\ -x_1^2 \\ x_2 \end{pmatrix}} \begin{matrix} S[x_1^3, x_2^2] \\ \oplus \\ S[x_1x_2, x_2^2] \\ \oplus \\ S[x_1x_2, x_1^3] \end{matrix} \xrightarrow{\begin{pmatrix} x_1^3 & -x_2^2 & 0 \\ x_1 & 0 & -x_2 \\ 0 & x_2 & -x_2^2 \end{pmatrix}} \begin{matrix} S[x_2^2] \\ \oplus \\ S[x_1^3] \\ \oplus \\ S[x_1x_2] \end{matrix} \xrightarrow{\begin{pmatrix} x_2 & x_1^3 & x_1x_2 \end{pmatrix}} S[\emptyset] \longrightarrow \frac{S}{I} \longrightarrow 0$$

Note that the Taylor Resolution is rarely minimal. In Example 2.93, we can see that $\mathrm{Im}(\begin{pmatrix} 1 \\ -x_1^2 \\ x_2 \end{pmatrix})$ is not contained in mH_2, as $1.[x_1x_2, x_1^3, x_2^2]$ is mapped to $\begin{pmatrix} 1 \\ -x_1^2 \\ x_2 \end{pmatrix}$ in H_2

Lyubeznik Resolution

Let I be a monomial ideal with generating set M. Fix an ordering on the monomials in M, say $m_1 < m_2 < \cdots < m_r$.

Definition 2.94. Given a monomial m, we define $\min(m) = \min_<\{m_i \in M \mid m_i \text{ divides } m\}$.

Given a set of monomials F,

$$\min(F) = \min_<\{m_i \in M \mid m_i \text{ divides } \operatorname{lcm}(F)\}.$$

Definition 2.95. A face F of a simplicial complex is said to be **rooted** if for all $G \subseteq F$, we have $\min(G) \in G$.

Consider the simplicial complex Δ on M, where the faces are given by $\{F \subseteq M \mid F \text{ is rooted}\}$. Hence, the set of all faces of dimension i is $\{F \subset M \mid F \text{ is rooted}, |F| = i+1\}$. This is called the **Lyubeznik simplicial complex** associated to I and $<$. Thus the associated chain complex is of the form:

$$\mathbf{H}_{\boldsymbol{\Delta}} : 0 \longrightarrow H_r \xrightarrow{\phi_{r-1}} H_{r-1} \longrightarrow \cdots \longrightarrow H_1 \xrightarrow{\phi_0} H_0 \longrightarrow \frac{S}{I} \longrightarrow 0.$$

Theorem 2.96. *Let I be a monomial ideal with generating set M. Fix an ordering $<$ on the monomials in M. Let Δ denote the Lyubeznik simplicial complex associated to M and $<$. Then, the associated chain complex is exact, and hence a resolution. This is called the* ***Lyubeznik*** *resolution.*

Proof. Refer to Theorem 6.5 in [20]. □

It can be seen from the definitions that the Lyubeznik resolution is a subcomplex of the Taylor resolution. Hence, it is 'closer' to the minimal free resolution. In general, both the Taylor and Lyubeznik resolutions are often distinct and far from minimal.

2.3.3 Mapping cone Construction

As discussed above, constructing a resolution for a module is often difficult. In this section, we introduce resolutions that can be constructed via resolutions of other modules. This idea is explored in the mapping cone construction.

Let I be a graded ideal of S and let the ring $R = \frac{S}{I}$. Let $\phi : (\mathbf{U}, \mathbf{d}) \to (\mathbf{U}', \mathbf{d}')$ be a map of complexes.

The **mapping cone** of ϕ is a complex $(\mathbf{W}, \delta)$ where:

$$W_i = U_{i-1} \oplus U'_i,$$

and the map $\delta_i : W_i \to W_{i-1}$ is given by

$$\begin{matrix} U'_i \\ \oplus \\ U_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} d'_i & \phi_{i-1} \\ 0 & d_{i-1} \end{pmatrix}} \begin{matrix} U'_{i-1} \\ \oplus \\ U_{i-2} \end{matrix}$$

Theorem 2.97. $(\mathbf{W}, \delta)$ *is a complex.*

Proof. Refer to Section 27, [24]. □

Theorem 2.98. *Assume that* $(\mathbf{U}, \mathbf{d})$ *and* $(\mathbf{U}', \mathbf{d}')$ *are free resolutions of the modules* V *and* V' *respectively and the complex map* ϕ *is induced from an injective homomorphism* $\psi : V \to V'$. *Consider a short exact sequence:*

$$0 \longrightarrow V \xrightarrow{\psi} V' \longrightarrow V'' \longrightarrow 0.$$

Then the mapping cone $(\mathbf{W}, \delta)$ *is a free resolution of* V''.

Proof. Refer to Section 27, [24]. □

Since Theorem 2.98 can be used for any exact sequence, it allows us to test this out on some well-known exact sequences.

Corollary 2.99. *Consider the graded ideals* $J, K \subset S$. *Let* $J + K = I$. *Then we have the following exact sequence.*

$$0 \longrightarrow J \cap K \longrightarrow J \oplus K \longrightarrow J + K \longrightarrow 0.$$

Consider any resolutions for J, K *and* $J \cap K$*:*

$$\mathbf{F} : \cdots F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} J \cap K \longrightarrow 0$$

$$\mathbf{G} : \cdots G_i \xrightarrow{d_i} G_{i-1} \xrightarrow{d_{i-1}} \cdots \longrightarrow G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} J \longrightarrow 0$$
$$\mathbf{H} : \cdots H_i \xrightarrow{d_i} H_{i-1} \xrightarrow{d_{i-1}} \cdots \longrightarrow H_1 \xrightarrow{d_1} H_0 \xrightarrow{d_0} K \longrightarrow 0$$

Then the complex:

$$\cdots \longrightarrow G_2 \oplus H_2 \oplus F_1 \longrightarrow G_1 \oplus H_1 \oplus F_0 \longrightarrow G_0 \oplus H_0 \longrightarrow I \longrightarrow 0$$

is a free resolution of I.

Proof. Since $\mathbf{G}$ and $\mathbf{H}$ are free resolutions of J and K, this implies that the complex $\mathbf{G} \oplus \mathbf{H}$ is a free resolution for $J \oplus K$.

Thus, from Theorem 2.98, we have that mapping cone $\mathbf{W}$ will be a free resolution for I. Hence the complex

$$\cdots \longrightarrow G_2 \oplus H_2 \oplus F_1 \longrightarrow G_1 \oplus H_1 \oplus F_0 \longrightarrow G_0 \oplus H_0 \longrightarrow I \longrightarrow 0$$

is a free resolution for I. $\square$

Remark 2.100. The above corollary can also give us information on the Betti numbers. If $\mathbf{L}$ denotes the minimal free resolution of I, then from Corollary 2.74, we have that

$$\beta_{i,j}(I) \leq \operatorname{rank}(G_i \oplus H_i \oplus F_{i-1})_j.$$

Now let $\mathbf{G}, \mathbf{H}$ and $\mathbf{F}$ be the minimal free resolutions of J, K and $J \cap K$ respectively. Since the minimal free resolution for $J \oplus K$ is $\mathbf{G} \oplus \mathbf{H}$, we have that

$$\operatorname{rank}(G_i \oplus H_i \oplus F_{i-1})_j = \operatorname{rank}(G_i)_j + \operatorname{rank}(H_i)_j + \operatorname{rank}(F_{i-1})_j = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

Hence,

$$\beta_{i,j}(I) \leq \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

This leads us to the idea of Betti splittings.

Betti splittings

Definition 2.101. Let I, J, and K be graded ideals with minimal generating sets $\mathfrak{G}(I)$, $\mathfrak{G}(J)$ and $\mathfrak{G}(K)$ such that $\mathfrak{G}(I)$ is the disjoint union of $\mathfrak{G}(J)$ and $\mathfrak{G}(K)$. Then $I = J + K$ is a **Betti splitting** if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i \in \mathbb{N} \text{ and (multi)degrees } j.$$

In other words, the mapping cone for the short exact sequence

$$0 \longrightarrow J \cap K \longrightarrow J \oplus K \longrightarrow J + K \longrightarrow 0,$$

is a minimal free resolution of $J + K = I$.

In this thesis, we extensively study Betti splittings for a class of ideals known as binomial edge ideals. The mapping cone will not be a minimal free resolution for most ideals. But under certain conditions, this free resolution becomes minimal.

Theorem 2.102. *Let I be a graded ideal in S, and suppose that J and K are graded ideals in S such that $\mathfrak{G}(I)$ is the disjoint union of $\mathfrak{G}(J)$ and $\mathfrak{G}(K)$. Suppose that for all i and all (multi)degrees j, $\beta_{i,j}(J \cap K) > 0$ implies that $\beta_{i,j}(J) = \beta_{i,j}(K) = 0$. Then*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \textit{ for all } i \textit{ and } j;$$

that is, $I = J + K$ is a Betti splitting.

Proof. This is from Theorem 2.3 [8]. Note that the proof in the reference presents the proof for monomial ideals, but the same proof works for graded ideals. We shall present it again for clarity.

Since $I = J + K$, we have the short exact sequence

$$0 \longrightarrow J \cap K \xrightarrow{\phi} J \oplus K \xrightarrow{\psi} J + K = I \longrightarrow 0.$$

This induces a long exact sequence in Tor, which restricts to a long exact sequence of vector spaces when taking the graded pieces,

$$\longrightarrow \mathrm{Tor}_i(k, J \cap K)_j \longrightarrow \mathrm{Tor}_i(k, J)_j \oplus \mathrm{Tor}_i(k, K)_j \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J \cap K)_j \longrightarrow$$

Fix some i and some (multi)degree j. First suppose $\beta_{i,j}(J \cap K) = 0$. By the hypothesis, if $\beta_{i-1,j}(J \cap K) \neq 0$, then $\beta_{i-1,j}(J) = 0$ and $\beta_{i-1,j}(K) = 0$. Hence this gives us the short exact sequence:

$$0 \longrightarrow \mathrm{Tor}_i(k, J)_j \oplus \mathrm{Tor}_i(k, K)_j \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J \cap K)_j \longrightarrow 0$$

Thus, if $\beta_{i,j}(J \cap K) = 0$, then $\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$ for all i and (multi)degrees j.

Instead, if we have that $\beta_{i-1,j}(J \cap K) = 0$, then we have the exact sequence,

$$0 \longrightarrow \mathrm{Tor}_i(k, J)_j \oplus \mathrm{Tor}_i(k, K)_j \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow 0,$$

which again gives us the desired formula.

Finally, assume that $\beta_{i,j}(J \cap K) \neq 0$. This tells us that $\beta_{i,j}(J) = 0$ and $\beta_{i,j}(K) = 0$. This gives us the exact sequence

$$0 \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J \cap K)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J)_j \oplus \mathrm{Tor}_{i-1}(k, K)_j \longrightarrow \cdots$$

If $\beta_{i-1,j}(J \cap K) = 0$, then that means that $\mathrm{Tor}_i(k, I)_j = \beta_{i,j}(I) = 0$ and hence, the formula holds. If $\beta_{i-1,j}(J \cap K) \neq 0$ then $\beta_{i-1,j}(J) = \beta_{i-1,j}(K) = 0$ which implies that $\beta_{i,j}(I) = \mathrm{Tor}_i(k, I)_j = \mathrm{Tor}_{i-1}(k, J \cap K)_j = \beta_{i-1,j}(J \cap K)$. Since $\beta_{i,j}(J) = 0$ and $\beta_{i,j}(K) = 0$, this agrees with the formula and hence proves the proposition. $\square$

The above theorem and variations to it will be very important throughout this thesis. It will help give important conditions on Betti splittings for certain binomial edge ideals. This theorem also gives nice conditions on Betti splittings for monomial ideals.

Definition 2.103. Let I be a monomial ideal in S. Let J be the ideal generated by all elements of $\mathfrak{G}(I)$ divisible by x_i, and let K be the ideal generated by all other elements of $\mathfrak{G}(I)$. We call $I = J + K$ an x_i-**partition** of I. If $I = J + K$ is also a Betti splitting, we call $I = J + K$ an x_i-**splitting**.

Theorem 2.104. *Let $I = J + K$ be an x_i-partition of the monomial ideal I in which all elements of J are divisible by x_i. If $\beta_{i,j}(J \cap K) > 0$ implies that $\beta_{i,j}(J) = 0$ for all i and multidegrees j, then $I = J + K$ is a Betti splitting. In particular, if the minimal graded free resolution of J is linear, then $I = J + K$ is a Betti splitting.*

Proof. Refer to Corollary 2.7, [8]. □

The above corollary can be applied to obtain conditions of the Betti splittings of some well-known graph ideals.

Definition 2.105. Consider a simple graph G, with $V(G) = [n]$. Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring. By associating each vertex to a variable in S, we can define the **edge ideal** of G as follows:

$$I(G) = \langle \{x_i x_j \mid \{i, j\} \in E(G)\} \rangle.$$

Definition 2.106. If i is a vertex of G that is not isolated and such that $G \setminus i$ is not a graph of isolated vertices, we call i a **splitting vertex** of G.

We can now apply Theorem 2.104,

Corollary 2.107. *Let G be a simple graph with edge ideal $I(G)$ and splitting vertex i. Let J be the ideal generated by all elements of $\mathfrak{G}(I)$ divisible by x_i, and K be generated by $\mathfrak{G}(I(G)) \setminus \mathfrak{G}(J)$. Then $I(G) = J + K$ is an x_i-splitting.*

Proof. J is just x_i multiplied by an ideal generated by variables, hence it has a linear resolution. Thus, from Theorem 2.104, the result follows. □

Thus, the above result tells us that splitting off a vertex from a graph induces a Betti splitting. One of the main goals of this thesis is to study this condition for another type of graph ideal that is, binomial edge ideals.

2.4 Binomial edge ideals

Just like edge ideals, binomial edge ideals are a type of ideals associated with finite simple graphs. They are defined as follows:

Definition 2.108. Consider a finite simple graph G, with $V(G) = [n]$. Let $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables. For $i, j \in [n]$, we denote $f_{ij} := x_i y_j - x_j y_i$. The **binomial edge ideal** J_G, is defined:

$$J_G := \langle \{f_{ij} \mid \{i, j\} \in E(G)\} \rangle.$$

Remark 2.109. We can see from the definition that J_G depends only on the edges of G. Hence, if G has an isolated vertex v, and $G' = G \setminus \{v\}$, then $J_G = J_{G'}$.

Binomial edge ideals are an interesting class of ideals. They were introduced in the early 2010s independently in [11] and [23] and have been shown to have some applications to conditional independence statements. Since they are ideals defined from a graph, the main way of studying the algebraic properties of these ideals is to relate them to the combinatorial properties of the corresponding graph.

2.4.1 Gröbner Basis

Here, we will recall the characterisation of the reduced Gröbner basis for the binomial edge ideal of any graph.

Definition 2.110. Let G be a simple graph on $\{0, \ldots, n\}$ and let i, j be two vertices of G with $i < j$. A path $\pi : i = i_0, i_1 \ldots, i_{r-1}, i_r = j$ is called **admissible** if:

1. $i_k \neq i_l$ for $k \neq l$,
2. for each $k = 1, \ldots, r-1$, one has either $i_k > j$ or $i_k < i$, and
3. For any proper subset $\{j_1, \ldots, j_s\}$ of $\{i_1, \ldots, i_r\}$, the sequence $i, j_1, \ldots, j_s, j$ is not a path.

Example 2.111. Consider the graph G with $V(G) = \{0, 1, 2, 3, 4, 5\}$ and $E(G) = \{\{0, 1\}, \{0, 2\}, \{0, 4\}, \{0, 5\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$, as in Figure 2.12. Consider the path $p = (2, 4, 3)$. We can see that all vertices are distinct and since there $\{2, 3\} \notin E(G)$, no subset of vertices forms a path between 2 and 3. Furthermore, $4 > 3 > 2$. Hence, we can see that p is an admissible path.

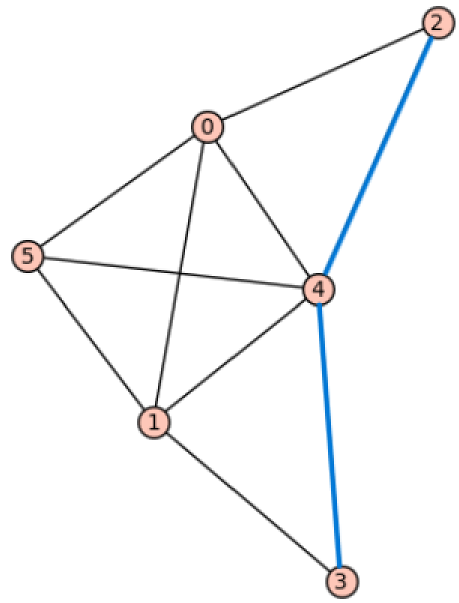


Figure 2.12: G with admissible path $(2, 4, 3)$.

Given an admissible path $\pi : i = i_0, i_1 \ldots, i_{r-1}, i_r = j$ from i to j where $i < j$, we associate the monomial:

$$u_\pi = (\prod_{i_k > j} x_{i_k})(\prod_{i_l < i} y_{i_l}).$$

Theorem 2.112. *Let $>$ be a monomial order with $x_0 > \cdots > x_n > y_0 > \cdots > y_n$. Let G be a simple graph and J_G denote the binomial edge ideal of G. Then the set of binomials:*

$$B = \bigcup_{i<j} \{u_\pi f_{i,j} : \pi \textit{ is an admissible path from } i \textit{ to } j\}$$

is a reduced Gröbner basis of J_G.

Proof. Refer to Theorem 2.1, [11]. □

This theorem can also tell us about the initial ideals of binomial edge ideals

Corollary 2.113. *Let $>$ be a monomial order with $x_0 > \cdots > x_n > y_0 > \cdots > y_n$. Let G be a simple graph and J_G denote the binomial edge ideal of G. Then, $\mathrm{in}_>(J_G)$ is a square-free monomial ideal.*

Proof. From Theorem 2.112, we can see that since

$$\bigcup_{i<j} \{u_\pi f_{i,j} : \pi \text{ is an admissible path from } i \text{ to } j\}$$

is a Gröbner basis, $\text{in}_>(J_G)$ is generated by

$$\bigcup_{i<j}\{u_\pi x_i y_j : \pi \text{ is an admissible path from } i \text{ to } j\}.$$

Hence, since all the vertices in any admissible path π are distinct, all the variables in $u_\pi x_i y_j$ are also distinct. Hence, $u_\pi x_i y_j$ is square-free for any admissible path π. Thus, $\text{in}_>(J_G)$ is a square-free monomial ideal. $\square$

The above theorem is a good illustration of how an algebraic property of J_G (the Gröbner basis) is related to a graph theoretic quantity (the admissible paths of G).

2.4.2 Minimal Primes

The characterisation of algebraic quantities using graph theoretic structures can be further seen while studying the primary decomposition of J_G.

Theorem 2.114. *J_G is a radical ideal.*

Proof. Refer to Corollary 2.2, [11] or Proposition 4.1, [23]. $\square$

Since J_G is radical, it can be written as an intersection of prime ideals.

Definition 2.115. Let G be a simple graph with $V(G) = [n]$. Consider $S \subseteq [n]$. Let $T = [n] \setminus S$, and let $G_1, \dots, G_{c(S)}$ be the connected components of $G[T]$. For each G_i we denote by the complete graph on the vertex set $V(G_i)$ as $\tilde{G}_i$. Then the ideal $P_S(G)$ is defined as:

$$P_S(G) = \langle\{\bigcup_{i\in S}\{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(s)}}\}\rangle.$$

Theorem 2.116. *$P_S(G)$ is a prime ideal.*

Proof. Each $J_{\tilde{G}_i}$ is the ideal of 2-minors of a generic $2 \times n_i$-matrix with $n_i = |V(G_i)|$. Thus, all $J_{\tilde{G}_i}$ as well as the ideal $\langle\bigcup_{i\in S}\{x_i, y_i\}\rangle$ are prime. Since all these prime ideals are in pairwise disjoint sets of variables, we can conclude that

$$P_S(G) = \langle\{\bigcup_{i\in S}\{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(s)}}\}\rangle = \langle\bigcup_{i\in S}\{x_i, y_i\}\rangle + \sum_{i=1}^{c(s)} J_{\tilde{G}_i}$$

is also prime. $\square$

The prime ideals $P_S(G)$ play an important role in characterising the primary decomposition of J_G.

Theorem 2.117. *Let G be a finite simple graph with $V(G) = [n]$. Then,*

$$J_G = \bigcap_{S \subseteq [n]} P_S(G).$$

Proof. Refer to theorem 3.2, [11] or Lemma 4.8, [23]. □

This leads us to a combinatorial characterisation of the minimal primes of J_G.

Theorem 2.118. *Let G be a connected finite simple graph with $V(G) = [n]$, and $S \subseteq [n]$. Let $T = [n] \setminus S$. Then $P_S(G)$ is a minimal prime ideal of J_G if and only if $S = \emptyset$ or for $S \neq \emptyset$, each $i \in S$ is a cut vertex of the graph $G[T \cup \{i\}]$.*

Proof. Refer to Corollary 3.9, [11]. □

This result tells us that the minimal primes for J_G are related to the cut points of an induced subgraph of G.

2.4.3 Regularity and Projective dimension

Homological invariants for binomial edge ideals have been widely studied. Specifically, there has been a lot of work on relating several interesting graph theoretic invariants of the graph G to homological invariants of J_G. Often, nice graph theoretic invariants can provide good bounds for otherwise hard-to-understand homological invariants. In this section, we will review well-known bounds on the Betti numbers, projective dimension and regularity of different types of binomial edge ideals.

Theorem 2.119. *Let G be a simple graph with $V(G) = [n]$,and let $S \subseteq [n]$. Then, for any $a \in \mathbb{N}^n$ with $a_j = 0$ for all $j \notin S$, we have*

$$\beta_{i,a}(J_G) = \beta_{i,a}(J_{G[S]}) \text{ for all } i \geq 0.$$

Proof. Refer to Lemma 2.1, [19]. □

Remark 2.120. The above theorem tells us that for any induced subgraph $G[S]$ on $S \subseteq [n]$, we have that

$$\beta_{i,j}(J_{G[S]}) \leq \beta_{i,j}(J_G) \text{ for all } i, j \in \mathbb{N}.$$

Furthermore, we have the following bound on the regularity for the graph of any binomial edge ideal.

Theorem 2.121. *Let G be a finite simple graph with $V(G) = [n]$. Let l be the length of the longest induced path of G. Then,*

$$l + 1 \leq \operatorname{reg}(J_G) \leq n,$$

where n is achieved if and only if G is a path graph.

Proof. Refer to Theorem 1.1, [19] and Theorem 7.36, [12]. □

There are also useful bounds on the projective dimension of the binomial edge ideal of any graph.

Theorem 2.122. *Let G be a connected graph with $V(G) = [n]$. Suppose that G is not the complete graph and that r is the vertex connectivity of G. Then,*

$$\operatorname{pd}(J_G) \geq n + r - 3.$$

Proof. Refer to Theorem 3.20, [1]. □

Theorem 2.123. *Let G be a connected graph on $[n]$. If f denotes the number of free vertices in G and $\operatorname{diam}(G)$ is the diameter of G, then*

$$\operatorname{pd}(J_G) \leq 2n - f - \operatorname{diam}(G)$$

Proof. Refer to Theorem 3 in [27]. □

By looking only at particular types of graphs, bounds for these homological invariants become stronger.

Theorem 2.124. *Let G be an indecomposable block graph on n vertices. Let f be the number of free vertices in G. Then, $\beta_{n-1,2n-f}(S/J_G)$ and $\beta_{n-1,2n-f}(S/\operatorname{in}<(J_G))$ are extremal Betti numbers of S/J_G and $S/\operatorname{in}(J_G)$, respectively. Moreover,*

$$\beta_{n-1,2n-f}(S/J_G) = \beta_{n-1,2n-f}(S/\operatorname{in}(J_G)) = f - 1$$

Proof. This lemma is proved in Theorem 6, [14]. □

Theorem 2.125. *Let G be a graph with $V(G) = [n]$. Let $c(G)$ denote the number of maximal cliques in G. Then,*

$$\operatorname{reg}(J_G) \leq c(G) + 1.$$

Proof. Refer to Theorem 3.5, [26]. □

Chapter 3

Betti Splittings of binomial edge ideals

In the previous sections, we have seen the technique of Betti splittings introduced for monomial ideals, with some applications to edge ideals. Our goal in this chapter is to explore similar kinds of splittings for binomial edge ideals. We first introduce a result by Saeedi Madani and Kiani in their paper [17], and rephrase it in the context of Betti splittings. We then extend this result and prove a more general version of the same. We also apply this result to obtain the 2nd Betti number of the binomial edge ideal of any tree.

Some of the results in this chapter are new. All results used from other sources will be mentioned.

3.1 Complete Betti splittings

In this section, we will describe some ways to break apart graphs, which translates to Betti splittings of the corresponding binomial edge ideals. For this, we will introduce some more graph theoretic terminology. Throughout, G will denote a finite simple graph, with vertices and edges of the graph G denoted by $V(G)$ and $E(G)$ respectively.

In the study of binomial edge ideals of graphs, free vertices in the graphs simplify the study of their homological properties. In this thesis, free vertices will appear multiple times, both in old and new results. We will now present some well-known results on the Betti numbers of the binomial edge ideals of some graphs, where free vertices play a major role.

Definition 3.1. A graph G is said to be **decomposable** if there exist two subgraphs G_1 and G_2 of G, and a decomposition $G = G_1 \cup G_2$ such that $V(G_1) \cap V(G_2) = \{v\}$, where v is a free vertex of G_1 and G_2.

Example 3.2. Consider the graph G with vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and edge set $E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \{1, 6\}, \{5, 6\}\}$. The graph G is decomposable, with $G_1 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ and $G_2 = \{\{1, 5\}, \{1, 6\}, \{5, 6\}\}$. This can be seen in Figure 3.1.

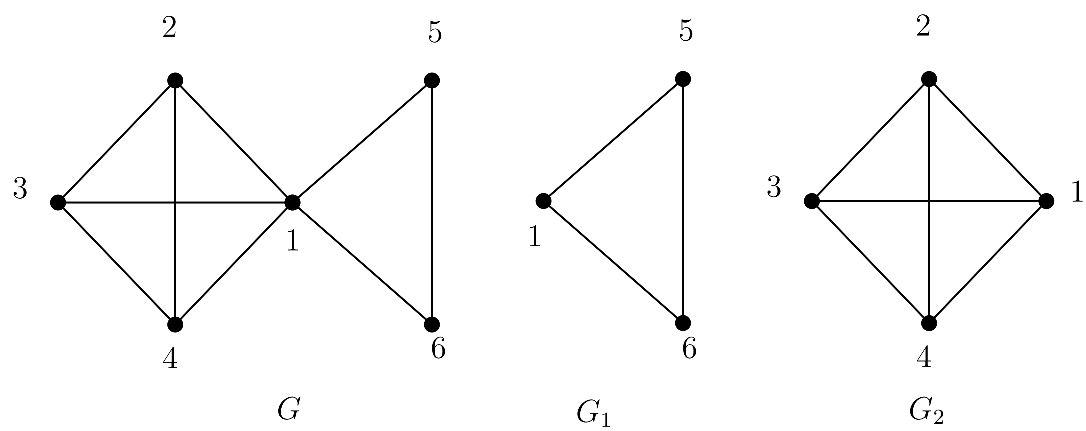


Figure 3.1:

For a module M, we can put all the graded Betti numbers together as a polynomial. It can be defined as follows:

Definition 3.3. The **Betti polynomial** is defined as the multivariable polynomial given by,

$$B_M(s, t) = \sum_{i,j} \beta_{i,j} s^i t^j$$

We have the following proposition concerning the Betti polynomial of the binomial edge ideal of decomposable graphs.

Theorem 3.4. *Let G be a decomposable graph, and let $G = G_1 \cup G_2$ be a decomposition of G. Then*

$$B_{S/J_G}(s, t) = B_{S/J_{G_1}}(s, t) B_{S/J_{G_2}}(s, t).$$

Proof. Refer to [14], Proposition 3. □

Remark 3.5. Note that for any ideal I the Betti numbers for I and S/I are closely related. It can be seen that $\beta_{i,j}(S/I) = \beta_{i-1,j+1}(I)$. The above proposition holds for S/I and must be rephrased for I.

This proposition gives us a way to obtain the Betti numbers of some graphs by breaking them down into smaller graphs. We shall now see an example of a Betti splitting in certain graphs.

Definition 3.6. Let G be a simple graph on the vertex set $V(G)$ and $e = \{i, j\} \notin E(G)$. Let $N_G(i)$ denote all the neighbours of the vertex i, i.e, $N_G(i) = \{v \in V(G) : \{i, v\} \in E(G)\}$. Then, we use G_e to denote the graph with,

$$V(G_e) = V(G) \text{ and } E(G_e) = E(G) \cup \{(k, l) : k, l \in N_G(i) \text{ or } k, l \in N_G(j)\}.$$

Similarly, if $v \in V(G)$, then we use G_v to denote the graph with.

$$V(G_v) = V(G) \text{ and } E(G_v) = E(G) \cup \{(k, l) : k, l \in N_G(v)\}.$$

Example 3.7. Consider the simple graph G with vertex set $V(G) = [7]$ and edge set $E(G) = \{(1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{5, 6\}, \{6, 7\}, \{7, 5\}\}$. We can see that $e = \{1, 7\}$ is not an edge in $E(G)$. Therefore, G_e is a simple graph with $V(G_e) = [7]$ and $E(G_e) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{2, 4\}, \{1, 3\}, \{5, 6\}, \{6, 7\}, \{7, 5\}\}$.

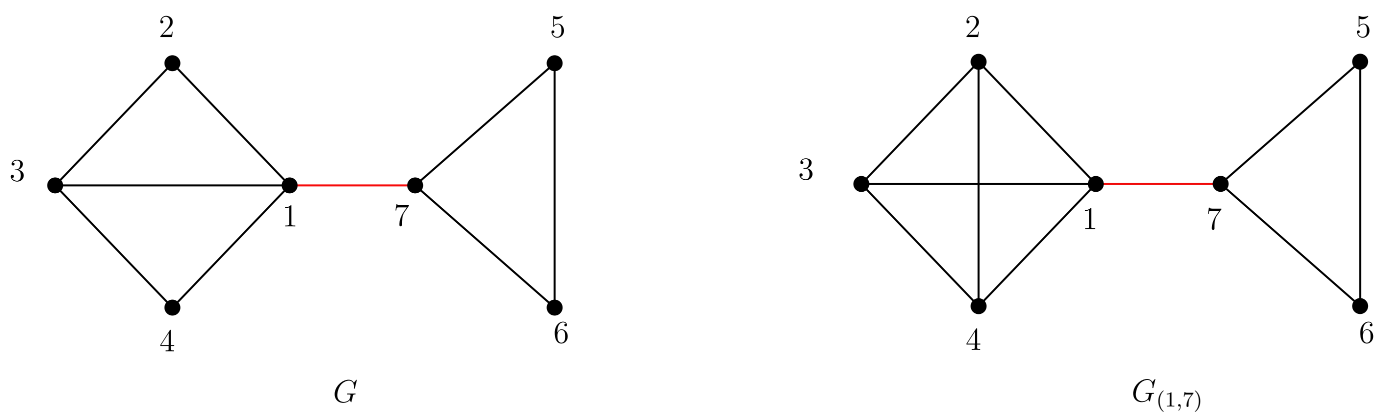


Figure 3.2:

Lemma 3.8. *Let G be a simple graph and $e = \{i, j\} \notin E(G)$ be a bridge in $G \cup e$. Let $f_e = x_i y_j - x_j y_i$. Then, $J_G : f_e = J_{G_e}$.*

Proof. Refer to [22], Theorem 3.4. $\square$

Definition 3.9. A **free cut edge** $e = \{u, v\}$ of a graph, is a cut edge, where both u and v are free vertices in $G \setminus e$.

Theorem 3.10. *Let G be a graph and let e be a free cut-edge of G. Then*

1. $\beta_{i,j}(J_G) = \beta_{i,j}(J_{G\setminus e}) + \beta_{i-1,j-2}(J_{(G\setminus e)})$.
2. $\operatorname{pd}(J_G) = \operatorname{pd}(J_{G\setminus e}) + 1$.
3. $\operatorname{reg}(J_G) = \operatorname{reg}(J_{G\setminus e}) + 1$.

Proof. The following proof is from [17] Proposition 3.9 We have $J_{(G\setminus e)_e} = J_{(G\setminus e)}$ since e is a free cut-edge of G. So, one may consider the short exact sequence

$$0 \longrightarrow \frac{S(-2)}{J_{(G\setminus e)} : f_e} \xrightarrow{\times f_e} \frac{S}{J_{(G\setminus e)}} \longrightarrow \frac{S}{J_G} \longrightarrow 0$$

By Lemma 3.8, we know that $[S/J_{(G\setminus e)}](-2) : f_e = [S/J_{(G\setminus e)_e}](-2)$. We have $J_{(G\setminus e)_e} = J_{G\setminus e}$ since e is a free cut edge of G. Let $\mathbf{E}$ be the minimal graded free resolution of $S/J_{G\setminus e}$. Now, consider the homomorphism of complexes $\phi : \mathbf{E}(-2) \to \mathbf{E}$, induced by multiplication by f_e. The mapping cone over the map ϕ resolves S/J_G. In addition, it is also minimal, because $\mathbf{E}$ is minimal and all the maps in the complex homomorphism ϕ are of positive degrees. $\square$

Lemma 3.11. *Let G be a simple graph. Consider an edge $e \in E(G)$. Then we have $J_{G\setminus e} : f_e \cong J_{(G\setminus e)} \cap \langle f_e \rangle$. Furthermore, if e is a cut edge, then $\beta_{r,j-2}(J_{(G\setminus e)_e}) = \beta_{r,j}(J_{(G\setminus e)} \cap \langle f_e \rangle)$*

Proof. From Lemma 3.8, we have $J_{(G\setminus e)_e} = J_{G\setminus e} : f_e$. By definition of quotient ideals, we have that, $J_{G\setminus e} : f_e \xrightarrow{\times f_e} J_{(G\setminus e)} \cap \langle f_e \rangle$ is an isomorphism of degree 2. Hence this means that:

$$\operatorname{Tor}_r^S(J_{G\setminus e} \cap \langle f_e \rangle, k)_j \cong \operatorname{Tor}_r^S(J_{G\setminus e} : f_e, k)_{j-2}.$$

This tells us that $\beta_{r,j-2}(J_{(G\setminus e)_e}) = \beta_{r,j}(J_{(G\setminus e)} \cap \langle f_e \rangle)$. $\square$

Remark 3.12. From the above lemma, we can see that the equation from Theorem 3.10, can be written as $\beta_{r,j}(J_G) = \beta_{r,j}(J_{G\setminus e}) + \beta_{r-1,j}(J_{(G\setminus e)} \cap \langle f_e \rangle)$. Since $\langle f_e \rangle$ is an ideal generated by one generator, we know that $\beta_{0,2}(\langle f_e \rangle) = 1$ and $\beta_{i,j}(\langle f_e \rangle) = 0$ for $i \neq 0$ and $j \neq 2$. This tells us that Theorem 5.15 (1), comes from a complete Betti splitting. In other words, $J_G = \langle f_e \rangle + J_{G\setminus e}$ is Betti splitting.

Hence, we showed that if G has a free cut-edge, then removing that edge leads to a Betti splitting. Naturally, it makes sense to wonder what would happen if the ends of the cut edge were not free. In the rest of this section, we will prove that if e is a cut edge with only one end free, then the removal of that edge will be a Betti splitting, thus extending the above result of Saeedi Madani and Kiani.

Theorem 3.13. *Let $e = \{u, v\} \in E(G)$, with $\deg v = 1$ (v is a pendent vertex). Then we have:*

1. *$J_G = J_{G\setminus e} + \langle f_e \rangle$ is a complete Betti Splitting.*

2. *$\beta_{r,j}(J_G) = \beta_{r,j}(J_{G\setminus e}) + \beta_{r-1,j-2}(J_{(G\setminus e)_e})$ for all $r \geq 1$ and $\beta_0(J_G) = \beta_{0,2}(J_G) = \beta_{0,2}(J_{G\setminus e}) + 1$*

Proof. 1. Consider $J_G = \langle f_e \rangle + J_{G\setminus e}$. Let the multigrading on J_G be given by the $\mathbb{N}^n$ grading. In other words, $\deg x_i = \deg y_i = i^{th}$ unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$. Therefore, all generators of $J_{G\setminus e} \cap \langle f_e \rangle$ are of the form $fx_v + gy_v$ and their multigraded Betti numbers occur within multidegrees $\mathbf{a}$, where its v^{th} component $\mathbf{a}_v$ is non-zero. Since $J_{G\setminus e}$ contains no generators having x_v or y_v, $\beta_{r,j}(J_{G\setminus e} \cap K) > 0$ implies that $\beta_{r,j}(K) = 0$ for all $r \in \mathbb{N}$ and $\mathbb{N}^n$ multidegrees j as defined above.

We have that $\beta_{0,2}(\langle f_e \rangle) = 1$ and $\beta_{i,j}(\langle f_e \rangle) = 0$ for $i \neq 0$ and $j \neq 2$ as $\langle f_e \rangle$ is a principal ideal. Since $J_{G\setminus e} \cap \langle f_e \rangle$ is generated by polynomials with degree 3 or more, this means that we have $\beta_{r,j}(J_{G\setminus e} \cap \langle f_e \rangle) > 0 \implies \beta_{r,j}(J) = 0$ for all $r \geq 0$ and degrees j. It is clear that since this is true for all degrees j, it holds for all multidegrees in $\mathbb{N}^n$ as well.

Therefore, from Theorem 2.102, this implies that (1) holds for all $\mathbb{N}^n$ multidegrees j. Since it is true for $\mathbb{N}^n$-multidegrees, we can combine them to obtain the same result with the degrees j in the standard grading. Hence we have:

$$\beta_{r,j}(J_G) = \beta_{r,j}(\langle f_e \rangle) + \beta_{r,j}(J_{G\setminus e}) + \beta_{r-1,j}(J_{G\setminus e} \cap \langle f_e \rangle) \text{ for all } r \in \mathbb{N}.$$

This shows that $J_G = \langle f_e \rangle + J_{G\setminus e}$ is a complete Betti splitting.

2. By (1) and Lemma 3.11, we have that $\beta_{r,j}(J_G) = \beta_{r,j}(\langle f_e \rangle) + \beta_{r,j}(J_{G\setminus e}) + \beta_{r-1,j-2}((J_{G\setminus e)_e})$ for all $r \in \mathbb{N}$. Since $\langle f_e \rangle$ is an ideal generated by one generator, we know that $\beta_{0,2}(\langle f_e \rangle) = 1$ and $\beta_{i,j}(\langle f_e \rangle) = 0$ for $i \neq 0$ and $j \neq 2$. Hence, $\beta_{r,j}(J_G) = \beta_{r,j}(J_{G\setminus e}) + \beta_{r-1,j-2}(J_{(G\setminus v)_u})$ for all $r \geq 1$ and $\beta_0(J_G) = \beta_{0,2}(J_G) = \beta_{0,2}(J_{G\setminus e}) + 1$. □

In Theorem 3.13, we have proved that when there is a cut-edge e where one end is a pendant vertex, then removing e induces a complete Betti splitting. We can now use this to prove our desired result.

Corollary 3.14. *Consider a simple graph G. Let $e = \{u, v\} \in E(G)$, be a cut-edge where v is a free vertex in $G \setminus e$. Then we have:*

1. *$\beta_{r,j}(J_G) = \beta_{r,j}(J_{G\setminus e}) + \beta_{r-1,j-2}(J_{(G\setminus e)_e})$ for all $r \geq 1$,*

2. *$J_G = J_{G\setminus e} + \langle f_e \rangle$ is a complete Betti Splitting.*

Proof. Let G be connected with cut-edge $e = \{u, v\}$. Let G_1 and G_2 be the connected components of $G \setminus e$. Let $u \in V(G_1)$ and $v \in V(G_2)$. By definition, we know that v is a free vertex in G_2. Hence, we can see that G is a decomposable graph, with $G = (G_1 \cup \{e\}) \cup G_2$ (since pendant vertices are trivially free vertices and v is a pendant vertex of e). We shall prove the above splitting for the quotient S/J_G and then use Remark 3.11 to obtain the assertions. Recall that

$$\beta_{i,j}(\frac{S}{J_G}) = \sum_{i_1 \leq i, j_1 \leq j} \beta_{i_1,j_1}(\frac{S}{J_{G_1 \cup \{e\}}})\beta_{i-i_1,j-j_1}(\frac{S}{J_{G_2}}). \tag{3.1}$$

Since e is a cut-edge with a pendant vertex in $G_1 \cup \{e\}$, we can now apply Theorem 3.13. Thus,

$$\begin{aligned}\sum_{i_1 \leq i, j_1 \leq j} \beta_{i_1,j_1}(\frac{S}{J_{G_1 \cup \{e\}}})\beta_{i-i_1,j-j_1}(\frac{S}{J_{G_2}}) &= \sum_{2 \leq i_1 \leq i, j_1 \leq j} (\beta_{i_1,j_1}(\frac{S}{J_{G_1}}) + \beta_{i_1-1,j_1-2}(\frac{S}{J_{(G_1)_e}}))\beta_{i-i_1,j-j_1}(\frac{S}{J_{G_2}}) \\ &\quad + (\beta_{1,2}(\frac{S}{J_{G_1}}) + 1)\beta_{i-1,j-2}(\frac{S}{J_{G_2}}) + \beta_{i,j}(\frac{S}{J_{G_1 \cup \{e\}}}) + \beta_{i,j}(\frac{S}{J_{G_2}}).\end{aligned} \tag{3.2}$$

Now, by applying Theorem 3.13 to $\beta_{i,j}(\frac{S}{J_{G_1 \cup \{e\}}})$ and combining the equations we get

$$\begin{aligned} &= \sum_{1 \leq i_1 \leq i, j_1 \leq j} \beta_{i_1,j_1}(\frac{S}{J_{G_1}})\beta_{i-i_1,j-j_1}(\frac{S}{J_{G_2}}) + \beta_{i,j}(\frac{S}{J_{G_1}}) + \beta_{i,j}(\frac{S}{J_{G_2}}) \\ &\quad + \sum_{1 \leq i_1 \leq i, j_1 \leq j} \beta_{i_1-1,j_1-2}(\frac{S}{J_{(G_1)_e}})\beta_{i-i_1,j-j_1}(\frac{S}{J_{G_2}}) + \beta_{i-1,j-2}(\frac{S}{J_{(G_1)_e}}) + \beta_{i-1,j-2}(\frac{S}{J_{G_2}}) \\ &= \sum_{i_1 \leq i, j_1 \leq j} \beta_{i_1,j_1}(\frac{S}{J_{G_1}})\beta_{i-i_1,j-j_1}(\frac{S}{J_{G_2}}) + \sum_{i_1 \leq i-1, j_1 \leq j-2} \beta_{i_1,j_1}(\frac{S}{J_{(G_1)_e}})\beta_{i-1-i_1,j-2-j_1}(\frac{S}{J_{G_2}}). \end{aligned} \tag{3.3}$$

Since G_1 and G_2 are graphs on disjoint sets of vertices, J_{G_1} and J_{G_2} are ideals on disjoint sets of variables. Hence,

$$\sum_{i_1 \leq i, j_1 \leq j} \beta_{i_1,j_1}(\frac{S}{J_{G_1}})\beta_{i-i_1,j-j_1}(\frac{S}{J_{G_2}}) = \beta_{i,j}(\frac{S}{J_{G_1}+J_{G_2}}) = \beta_{i,j}(\frac{S}{J_{G_1\cup G_2}}) = \beta_{i,j}(\frac{S}{J_{(G\setminus e)}}), \tag{3.4}$$

Similarly, the same is true for $(G_1)_e$ and G_2. Note, that since v is already a free vertex of G_2, we have $(G \setminus e)_e = (G_1)_e \cup G_2$. Hence,

$$\begin{aligned}\sum_{i_1 \leq i-1, j_1 \leq j-2} \beta_{i_1,j_1}(\frac{S}{J_{(G_1)_e}})\beta_{i-1-i_1,j-2-j_1}(\frac{S}{J_{G_2}}) &= \beta_{i-1,j-2}(\frac{S}{J_{(G_1)_e}+J_{G_2}}) \\ &= \beta_{i-1,j-2}(\frac{S}{J_{(G_1)_e\cup G_2}}) \qquad (3.5) \\ &= \beta_{i-1,j-2}(\frac{S}{J_{(G\setminus e)_e}}). \qquad (3.6)\end{aligned}$$

Thus, combining Equation (3.5) with Equation (3.4) and Remark 3.5, we get:

$$\beta_{i,j}(J_G) = \beta_{i,j}(J_{G\setminus e}) + \beta_{i-1,j-2}(J_{(G\setminus e)_e}) \text{ for all } i \geq 1$$

Similar to Theorem 3.13, using Lemma 3.11, we can see that $J_G = J_{G\setminus e} + \langle f_e \rangle$ is a complete Betti splitting. $\square$

In general, having a cut edge $e = \{u, v\}$ where both u and v are not free will not be a complete Betti splitting. Even simple examples of this fail.

Example 3.15. Consider a simple graph G with $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{4, 5\}, \{5, 6\}\}$. Clearly $e = \{2, 5\}$ is a cut edge, where $\{2\}$ and $\{5\}$ are both not free. In this case, $J_G = J_{G\setminus e} + \langle f_e \rangle$ is not a Betti splitting.

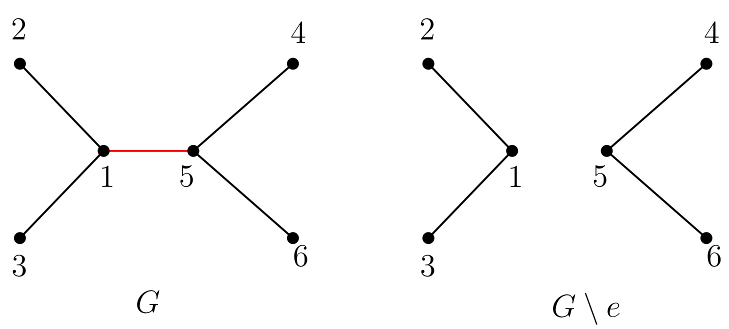


Figure 3.3: Non example of Example 3.15

3.2 Betti numbers of trees

In this section, we shall apply our results to study the Betti numbers of trees. We shall first describe the graded Betti numbers of the star graph. This will be followed by a result describing $\beta_2(J_T)$ and $\beta_{k,k+3}(T)$, for all trees T. We first start by surveying some important results on the linear strand of the Betti table of binomial edge ideals.

3.2.1 Linear strand

The linear strand of the Betti table for binomial edge ideals is well studied, [13]. The Betti numbers $\beta_{k,k+2}(J_G)$ are known for the binomial edge ideal for all graphs. Thus complete characterisation can be obtained through a study of the linear strand of determinantal facet ideals. These ideals are generated by certain minors of a matrix of indeterminates and are closely related to binomial edge ideals.

Definition 3.16. Consider an $m \times n$ matrix X and let S be an arbitrary set of maximal minors of X. The ideal generated by such a set S is called a **determinantal facet ideal** J_S.

Example 3.17. When $m = 1$, it is clear that X is just a row of indeterminates, $X_{1,j} = x_j$. We know the maximal minors here will be given by $M_{1,r} = x_r$. Hence, all the possible determinantal face ideals will be of the form $J_S = \langle x_{a_1}, x_{a_2}, \ldots, x_{a_k} \rangle$, where $S = \{M_{1,a_1}, \ldots, M_{1,a_k}\}$.

Example 3.18. When $m = 2$, the determinantal facet ideals will be generated by arbitrary sets of maximal minors of a $2 \times n$ matrix X of indeterminates. Denote the indeterminates by $X_{1,k} = x_k$ and $X_{2,l} = y_l$. Then, we can see that the maximal minor of the i^{th} and j^{th} column is $x_i y_j - x_j y_i$. Hence, given a set S of arbitrary maximal minors, we can see that the corresponding ideal will be $J_S = \langle \{x_i y_j - x_j y_i \mid M_{i,j} \in S\} \rangle$. In other words, it is a binomial edge ideal. Hence, binomial edge ideals turn out to be special cases of determinantal facet ideals.

In general, the linear strand of determinantal facet ideals has been classified. Note that if X is an $m \times n$ matrix of indeterminates, then the degree of all the maximal minors are the same and equal to $\min\{m, n\}$. Hence, if $m < n$, the linear strand will be the Betti numbers of the form $\beta_{i,i+m}(J_S)$.

Definition 3.19. Consider the set $[n] = \{1, \ldots, n\}$ and $m \leq n$. A collection of subsets C is called a m-**uniform clutter** if $|A| = m$ for all $A \in C$. The elements of C are called **circuits**.

We call $\Delta(C)$, the simplicial complex generated by C, generated as:

$$\Delta(C) = \langle \sigma \subset [n] \mid \text{Every subset of } \sigma \text{ of cardinality } m \text{ belongs to } C \rangle.$$

Remark 3.20. We can see that if $m < n$, then an m-clutter of $[n]$ determines a determinantal facet ideal. If C is an m-clutter, then the determinantal facet ideal J_C is generated by the maximal minors, whose columns are determined by the circuits of C. From the definition, it can also be seen that every determinantal facet ideal of an $m \times n$ matrix of indeterminates $(m < n)$ comes from an m-clutter of $[n]$.

Example 3.21. Consider $[4]$ and let the 3-clutter $C = \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}\}$. Thus, this gives us a 3×4 matrix of indeterminates X. Hence, the maximal minors are determined by the choice of three columns. Let $M_{a,b,c}$ be the maximal minor obtained from the a^{th}, b^{th} and c^{th} columns. Then, for the 3-clutter C, the maximal minors are of the form $S = \{M_{1,2,3}, M_{2,3,4}, M_{1,2,4}\}$.

Theorem 3.22. *Consider an m-clutter C. Let the corresponding determinantal facet ideal be denoted by J_C. Then the linear strand of its Betti table is given by:*

$$\beta_{i,i+m} = \binom{m+i-1}{m-1} f_{m+i-1}(\Delta(C)),$$

where $f_i(\Delta(C))$ denotes the number of faces of $\Delta(C)$ of dimension i.

Proof. This theorem is proved in [13]. □

Corollary 3.23. *Let G be a finite simple graph and J_G be the corresponding binomial edge ideal. Then, the linear strand of the Betti table of J_G is given by:*

$$\beta_{i,i+2}(J_G) = (i + 1)f_{i+1}(\Delta(G)),$$

where $\Delta(G)$ is the clique complex of the graph G and $f_{i+1}(\Delta(G))$ is the number of faces in $\Delta(G)$ of dimension $i + 1$.

Proof. From Example 3.18, we know that every binomial edge ideal is a determinantal facet ideal of a matrix of size $2 \times n$. Hence, substituting for m in Theorem 3.22, we get

$$\beta_{i,i+2}(J_G) = \binom{2+i-1}{2-1} f_{2+i-1}(\Delta(C)) = (i+1) f_{i+1}(\Delta(C)).$$

Now, the clutter C here corresponds to the edge set of the graph $E(G)$. Hence the faces of $\Delta(C)$ will be $\sigma \subset [n]$ such that all pairs of elements of σ are in C. In other words, it is a set of vertices σ such that there are edges between any two vertices of σ. Thus, σ must be a clique. Hence, all faces of $\Delta(C)$ are cliques of G. In other words, $\Delta(C) = \Delta(G)$. Thus,

$$\beta_{i,i+2}(J_G) = (i+1) f_{i+1}(\Delta(G)).$$

□

This previous result completely describes the linear strand of the Betti table for all binomial ideals. An interesting related question is about which binomial edge ideals have purely linear minimal free resolutions. These graphs have also been completely classified.

Theorem 3.24. *Let G be a finite simple graph and let J_G be its binomial edge ideal. Then the following are equivalent:*

1. *J_G has a linear resolution.*
2. *G is a complete graph.*

Proof. Refer to Theorem 2.1, [18]. □

Remark 3.25. Note that not all linear resolutions must be minimal, but if there exists a linear resolution of a module M, then its minimal free resolution must also be linear.

Thus, these theorems give us the total Betti numbers for complete graphs.

Corollary 3.26. *Let G be a complete graph on n vertices. Then*

$$\beta_i(J_G) = \beta_{i,i+2}(J_G) = (i+1)\binom{n}{i+2}.$$

Proof. Since G is the complete graph, $\Delta(G)$ is the simplicial complex of subsets of $[n]$. Hence, the number of elements of dimension $i+1$ is $\binom{n}{i+2}$. Thus from Corollary 3.23 and Theorem 3.24,

$$\beta_i(J_G) = \beta_{i,i+2}(J_G) = (i+1)\binom{n}{i+2}.$$

$\square$

Remark 3.27. When G is a complete graph, the binomial edge ideal is nothing but the ideal generated by all maximal minors of the $2 \times n$ matrix of indeterminates. Hence, it is a determinantal ideal. Thus, the same result can also be obtained from Theorem 3.24 and Proposition 2.2 (3) from [21].

Thus we have surveyed results characterising the linear strand and linear resolutions of binomial edge ideals.

3.2.2 Trees

The results on the linear strand of binomial edge ideals are useful to give some results on the Betti numbers of the binomial edge ideals of trees. We shall use them often in the many inductive proofs we discuss in this section.

Theorem 3.28. *Let S_n denote the star graph on n-vertices. Then we have:*

$$\beta_k(J_{S_n}) = \beta_{k,k+3}(J_{S_n}) = k\binom{n}{k+2} \qquad k \geq 1.$$

Proof. Let K_n denote the complete graph on n vertices. Consider the edge $e = \{0, i\}$. Since $S_n \setminus e \cong S_{n-1}$ $(S_n \setminus e)_e = K_{n-1}$, from Theorem 3.13, we have:

$$\beta_{k,j}(J_{S_n}) = \beta_{k,j}(J_{S_{n-1}}) + \beta_{k-1,j-2}(J_{K_{n-1}}) \quad \text{for all } k \geq 1.$$

We can now use induction to show the above assertion. For $n = 2$, we can see that S_2 is just an edge. We know that $\beta_{k,j}(J_{S_2}) = 0$ for all $k \geq 1$. Hence, we can see that it agrees with the above formula as $\binom{2}{r} = 0$ when $r > 2$. Now assume the formula holds for $n-1$. We must show that it holds for n.

From Corollary 3.26, we know that $\beta_{k,k+2}(K_n) = (k+1)\binom{n}{k+2}$ and $\beta_{k,j}(K_n) = 0$ if $j \neq k+2$.

Hence, using induction and Theorem 3.13, we can see that $\beta_{k,j}(J_{S_n}) = \beta_{k,j}(J_{S_{n-1}}) + \beta_{k-1,j-2}(J_{K_{n-1}}) = 0 + 0$, when $j \neq k+3$. This also tells us that:

$$\beta_{k,k+3}(J_{S_n}) = \beta_{k,k+3}(J_{S_{n-1}}) + \beta_{k-1,k+1}(J_{K_{n-1}}) = k\binom{n-1}{k+2} + k\binom{n-1}{k+1} = k\binom{n}{k+2}.$$

Thus, this verifies the above formula. $\square$

Remark 3.29. The above theorem helps characterise all the Betti numbers of S_n, since we know $\beta_0(J_{S_n}) = \beta_{0,2}(J_{S_n}) = n-1$.

The above theorem is a restatement of ([15], Proposition 3.8). It tells us the family of graphs S_n has regularity 3. We can also see that the regularity is achieved at $k=1$. Now, we shall try to use Theorem 3.13 to study the Betti numbers of general trees.

Lemma 3.30. *Let T be a tree with $v \in V(T)$ and let $S_v = \{u \in N_T(v) | \deg u > 1\}$. Then, there exists $a \in V(T)$ with $\deg a > 1$ such that:*

$$|S_a| \leq 1.$$

Proof. We can prove this via induction on $|V(T)|$. Let $|V(T)| = 2$. Then for all $v \in V(T)$, $|S_v| = 0$.

Now suppose it is true for all T such that $|V(T)| = k$. Consider a tree T' such that $|V(T')| = k+1$. Let $e = \{u, v\}$, where $\deg v = 1$. Hence, $T' \setminus e$ is a tree with $|V(T' \setminus e)| = k$. Hence, there exists $a \in V(T' \setminus e)$ such that $|S_a| \leq 1$.

- **Case 1:** $u \notin N_{T'}(a)$. In this case, the edge e doesn't contribute to the degrees of any vertex in $N_{T'}(a)$. If $u = a$, then only a degree 1 vertex is added to $N_G(a)$, hence $|S_a|$ also remains the same.

- **Case 2:** $u \in N_{T'}(a)$, $\deg u = 1$. Consider $N_{T'}(u) = \{a, v\}$ in T'. Since $\deg v = 1$, $|S_u| = 1$.

- **Case 3:** $u \in N_{T'}(a)$, $\deg u > 1$. Here, u is still the only vertex in $N_{T'}(a)$ whose degree is greater than one. Hence, $|S_a| = 1$.

Hence, the induction step has been shown in all possible cases. Therefore, the lemma holds. $\square$

Definition 3.31. A graph G is written in the form $T + K_m$, where T is a tree and K_m is a clique of size m, if G is such that $V(G) = V(T) \cup V(Q_m)$ and $E(G) = E(T) \cup E(Q_m)$, where $|V(T) \cap V(Q_m)| = 1$ and $E(T) \cap E(Q_m) = \emptyset$.

Example 3.32. Consider the graph G, with $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{6, 7\}\}$. Here, we can see that $G = T + K_3$, where T is the tree with $V(T) = \{1, 2, 3, 4, 5\}$ and $E(T) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{4, 5\}\}$ and K_3 is the clique of size 3, with $V(K_3) = \{4, 6, 7\}$ and $E(K_3) = \{\{4, 6\}, \{4, 7\}, \{6, 7\}\}$.

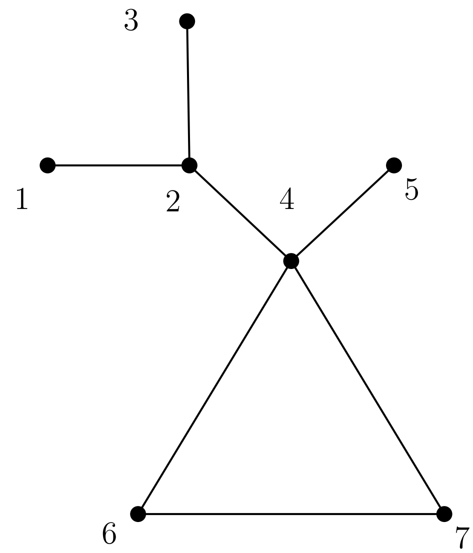


Figure 3.4: $G = T + K_3$

Using our previous results, we can obtain some information about the Betti numbers of any graph of the form $G = T + K_m$.

Lemma 3.33. *Consider a graph that can be expressed in the form $G = T + K_m$. If G and has n total vertices, then we have:*

$$\begin{aligned}\beta_1(J_G) &= \binom{n-1}{2} + 2\binom{m}{3} + \sum_{v_i \notin K_m} \binom{\deg v_i}{3} + \binom{\deg a - m + 1}{3} \\ &\quad + (n-m-1)\binom{m-1}{2} + (m-1)\binom{\deg a - m + 1}{2},\end{aligned}$$

where $\{a\} = V(T) \cap V(K_m)$.

Proof. We shall prove this lemma by induction on the number of vertices on the tree T. If $|V(T)| = 1$, this means that $E(T) = \emptyset$ and G is a complete graph. Hence, $n = m$. Therefore, we have the formula reduced to:

$$\beta_1(J_G) = \binom{n-1}{2} + 2\binom{n}{3} - \binom{n-1}{2} = 2\binom{n}{3}$$

Since this agrees with the formula for $\beta_1(K_n)$ from Corollary 3.26, the base case holds.

Consider a graph $G = T + K_m$. Now let us assume that the lemma is true for $|V(T)| = n - m$ (total number of vertices is $n-1$). We must show that it is true for $|V(T)| = n - m + 1$.

Since $E(T) \neq \emptyset$, it follows from Lemma 3.30 that there exists $u \in V(T)$ such that $\deg u \neq 1$ and $|S_u| \leq 1$.

Case 1: $u \neq a$.

Consider $e = \{u, v\}$ with $\deg v = 1$. Inductively we know that:

$$\begin{aligned}
\beta_1(J_{G\setminus e}) &= \binom{n-2}{2} + 2\binom{m}{3} + \sum_{v_i \notin K_m, v_i \neq u} \binom{\deg v_i}{3} + \binom{\deg u - 1}{3} + \binom{\deg a - m + 1}{3} \\
&\quad + (n-m-2)\binom{m-1}{2} + (m-1)\binom{\deg a - m + 1}{2}.
\end{aligned}$$

From Theorem 3.13, we have $\beta_1(J_G) = \beta_1(J_{G\setminus e}) + \beta_0(J_{(G\setminus e)_e})$. Now, $(G \setminus e)_e$ is obtained by adding $\binom{\deg u - 1}{2}$ edges to $E(G \setminus e)$. Since T is a tree and $G = T + K_m$, we have $E(G) = n - m + \binom{m}{2}$. Hence, $G \setminus e$ has $n - m - 1 + \binom{m}{2} = n - 2 + \binom{m-1}{2}$ edges. This means that:

$$\beta_0(J_{(G\setminus e)_e}) = |E((G \setminus e)_e)| = n - 2 + \binom{m-1}{2} + \binom{\deg u - 1}{2}.$$

Therefore, substituting into $\beta_1(J_G) = \beta_1(J_{G\setminus e}) + \beta_0(J_{(G\setminus e)_e})$, and using the binomial identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ appropriately, we get:

$$\begin{aligned}
\beta_1(J_G) &= \binom{n-2}{2} + 2\binom{m}{3} + \sum_{v_i \notin K_m, v_i \neq u} \binom{\deg v_i}{3} + \binom{\deg u - 1}{3} + \binom{\deg a - m + 1}{3} \\
&\quad + (n-m-2)\binom{m-1}{2} + (m-1)\binom{\deg a - m + 1}{2} \\
&\quad + n - 2 + \binom{m-1}{2} + \binom{\deg u - 1}{2} \\
&= \binom{n-1}{2} + 2\binom{m}{3} + \sum_{v_1 \notin K_m} \binom{\deg v_i}{3} + \binom{\deg a - m + 1}{3} \\
&\quad + (n-m-1)\binom{m-1}{2} + (m-1)\binom{\deg a - m + 1}{2}.
\end{aligned}$$

Therefore, we obtain our desired formula.

Case 2: $u = a$.

Consider $e = \{a, v\}$ with $\deg v = 1$. Here, since $u = a$, we must modify the $\deg a$ in the inductive formula as well. Hence we have:

$$\begin{aligned}\beta_1(J_{G\setminus e}) &= \binom{n-2}{2} + 2\binom{m}{3} + \sum_{v_i \notin K_m} \binom{\deg v_i}{3} + \binom{\deg a - m}{3} \\ &\quad + (n-m-2)\binom{m-1}{2} + (m-1)\binom{\deg a - m}{2}\end{aligned}$$

Note that $|E(G \setminus e)_e|$ is obtained by adding edges between all vertices in $N_G(a)$. Hence, edges have to be added between all vertices of $N_G(a)$ in T. This amounts to a total of $\binom{\deg a-(m-1)-1}{2} = \binom{\deg a - m}{2}$. Edges must also be added between all vertices in K_m and vertices of $N_G(a)$ in T. This adds $(m-1)(\deg a - m)$ edges. Note that $\deg a - m = \binom{\deg a - m}{1}$. Hence, the total number of edges added is $\binom{\deg a - m}{2} + (m-1)\binom{\deg a - m}{1}$. Thus,

$$\beta_0(J_{(G\setminus e)_e}) = |E(G \setminus e)_e| = n - 2 + \binom{m-1}{2} + \binom{\deg a - m}{2} + (m-1)\binom{\deg a - m}{1}.$$

Using Theorem 3.13 and the identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ appropriately, we get:

$$\begin{aligned}\beta_1(J_G) &= \binom{n-2}{2} + 2\binom{m}{3} + \sum_{v_i \notin K_m} \binom{\deg v_i}{3} + \binom{\deg a - m}{3} \\ &\quad + (n-m-2)\binom{m-1}{2} + (m-1)\binom{\deg a - m}{2} \\ &\quad + n - 2 + \binom{m-1}{2} + \binom{\deg a - m}{2} + (m-1)\binom{\deg a - m}{1} \\ &= \binom{n-1}{2} + 2\binom{m}{3} + \sum_{v_1 \notin K_m} \binom{\deg v_i}{3} + \binom{\deg a - m + 1}{3} \\ &\quad + (n-m-1)\binom{m-1}{2} + (m-1)\binom{\deg a - m + 1}{2}.\end{aligned}$$

Thus, we get the desired formula. This completes the proof. □

Remark 3.34. The above formula can be used to obtain the first total Betti number of any tree. If $G = T$, it can be trivially written as $G = T + K_1$. Hence, $m = 1$. Let $T \cap K_1 = \{a\}$.

Therefore, from Lemma 3.33, we have:

$$\begin{aligned}\beta_1(J_T) &= \binom{n-1}{2} + 2\binom{1}{3} + \sum_{v_i \notin K_1}\binom{\deg v_i}{3} + \binom{\deg a - 1 + 1}{3} \\ &\quad +(n-1-1)\binom{1-1}{2} + (1-1)\binom{\deg a - m + 1}{2} \\ &= \binom{n-1}{2} + \sum_{v_i \notin K_1}\binom{\deg v_i}{3} + \binom{\deg a}{3} = \binom{n-1}{2} + \sum_{v_i}\binom{\deg v_i}{3}.\end{aligned}$$

This formula agrees with the corresponding formula for $\beta_1(J_T)$ obtained in Theorem 3.1, [15].

The above lemma will be very useful while calculating the second total Betti number of any tree.

Definition 3.35. Consider the graph P, with $V(P) = \{1,2,3,4,5,6\}$ and $E(P) = \{\{1,2\},\{2,3\},\{3,4\},\{2,5\},\{3,6\}\}$. Given a graph G, we define $P(G)$ to be the number of induced subgraphs of G isomorphic to P.

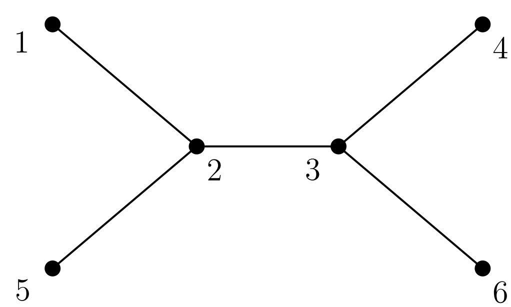


Figure 3.5: P

Theorem 3.36. *Let T be a tree. and J_T be its binomial edge ideal. Then:*

$$\beta_2(J_T) = \binom{n-1}{3} + 2\sum_{v_i}\binom{\deg v_i}{4} + \sum_{v_i}\binom{\deg v_i}{3}(1 + |E(T \setminus v_i)|) + P(T).$$

Proof. We can prove this using induction and the previous lemmas. For $n = 2$, we have that the tree is an edge. Since J_T a principal ideal, we have $\beta_2(J_T) = 0$, which agrees with the above formula. Assume the above formula is true for trees with $V(T) = n-1$. We must show that for any tree with $V(T) = n$, the formula holds.

Consider a tree T with $|V(T)| = n$. We know from Lemma 3.30 that there exists a vertex u such that $\deg u > 1$ and $|S_u| \leq 1$. Let $e = \{u, v\}$ be an edge such that v is a pendant vertex. Then, from Theorem 3.13, we have $\beta_2(J_T) = \beta_2(T \setminus e) + \beta_1(J_{(T\setminus e)_e})$. Let $T \setminus u$ denote the induced subgraph on $V(T) \setminus u$. Since T is a tree, by the choice of u, we can see that $(T \setminus e)_e = (T \setminus u) + K_{\deg u}$.

We have that $m = \deg u$ and the total number of vertices in $(T \setminus e)_e$ is $n - 1$. By definition, we have that $(T \setminus u) \cap K_{\deg u} = \{S_u\} = \{a\}$. Since in the construction $(T \setminus e)_e$ we add $m - 2$ edges to a, we can see that the degree of the vertex a goes from $\deg a$ to $\deg a + m - 2 = \deg a + \deg u - 2$. Thus we have that $\beta_1((T \setminus e)_e)$ is given by

$$\begin{aligned}
\beta_1((T \setminus e)_e) &= \binom{n-2}{2} + 2\binom{m}{3} + \sum_{v_1 \notin Q_m} \binom{\deg v_i}{3} + \binom{\deg a - m + 1}{3} \\
&\quad + (n - m - 2)\binom{m-1}{2} + (m-1)\binom{\deg a - m + 1}{2} \\
&= \binom{n-2}{2} + 2\binom{\deg u}{3} + \sum_{v_1 \notin Q_m} \binom{\deg v_i}{3} + \binom{\deg a - 1}{3} \\
&\quad + (n - \deg u - 2)\binom{\deg u - 1}{2} + (\deg u - 1)\binom{\deg a - 1}{2}
\end{aligned}$$

Note that $|(T \setminus e) \setminus u)| = |T \setminus u|$ and $|(T \setminus e) \setminus v_i)| = |(T \setminus v_i)| - 1$ for all $v_i \neq u$ in $T \setminus e$. Thus, combining the induction hypothesis with Theorem 3.13 we get

$$\begin{aligned}
\beta_2(J_T) &= \binom{n-2}{3} + \sum_{v_i \neq u} \binom{\deg v_i}{3} + \binom{\deg u - 1}{3} + 2\sum_{v_i \neq u} \binom{\deg v_i}{4} \\
&\quad + 2\binom{\deg u - 1}{4} + \sum_{v_i \neq u} \binom{\deg v_i}{3}(|E(T \setminus v_i)| - 1) + \binom{\deg u - 1}{3}(|E(T \setminus u)|) + P(T \setminus e) \\
&\quad + \binom{n-2}{2} + 2\binom{\deg u}{3} + \sum_{v_i \notin Q_m} \binom{\deg v_i}{3} + \binom{\deg a - 1}{3} \\
&\quad + (n - \deg u - 2)\binom{\deg u - 1}{2} + (\deg u - 1)\binom{\deg a - 1}{2}
\end{aligned}$$

Note that by the way we have chosen u all its neighbours except a will be degree one vertices in $T \setminus e$. Hence the term $\binom{\deg v_i}{3}$ is zero for all $v_i \in N_{T\setminus e}(u)$, where $v_i \neq a$. Hence, none of the v_i which contribute to the term $\sum_{v_i \neq u} \binom{\deg v_i}{3}$ in the above expression end up in $K_{\deg u}$ in $(T \setminus e)_e$. Using this observation on the term $\sum_{v_i \neq u} \binom{\deg v_i}{3}(|E(T \setminus v_i)| - 1)$ we can simplify the above expression. After using the identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ appropriately, we get:

$$\begin{aligned}
= \quad & \binom{n-1}{3} + \sum_{v_i \neq u} \binom{\deg v_i}{3} + \binom{\deg u - 1}{3} + 2\sum_{v_i \neq u} \binom{\deg v_i}{4} + 2\binom{\deg u - 1}{4} \\
& + \sum_{v_i \neq u,a} \binom{\deg v_i}{3}(|E(T) \setminus v_i)|) + \binom{\deg a}{3}(|E(T \setminus a)| - 1) + \binom{\deg u - 1}{3}(|E(T \setminus u)|) + P(T \setminus e) \\
& + 2\binom{\deg u}{3} + \binom{\deg a - 1}{3} + (n - \deg u - 2)\binom{\deg u - 1}{2} + (\deg u - 1)\binom{\deg a - 1}{2}
\end{aligned}$$

We can see that $E(T \setminus u)$ will have $n - \deg u - 1$ edges. The only elements of $P(T)$ which are not in $P(T \setminus e)$ are the induced subgraphs which contain the edge e. We also know the only adjacent vertex to u with a non-zero degree is a. Hence the total number will be $(\deg u - 2)\binom{\deg a - 1}{2}$. Therefore, combining all of these:

$$\begin{aligned}
= \quad & \binom{n-1}{3} + \sum_{v_i \neq u} \binom{\deg v_i}{3} + \binom{\deg u - 1}{3} + 2\sum_{v_i \neq u} \binom{\deg v_i}{4} + 2\binom{\deg u - 1}{4} \\
& + \sum_{v_i \neq u,a} \binom{\deg v_i}{3}(|E(T) \setminus v_i)|) + \binom{\deg a}{3}(|E(T \setminus a)| - 1) + \binom{\deg u - 1}{3}(|E(T \setminus u)|) \\
& + P(T) + 2\binom{\deg u}{3} + \binom{\deg a - 1}{3} + (|E(T \setminus u)| - 1)\binom{\deg u - 1}{2} + \binom{\deg a - 1}{2} \\
= \quad & \binom{n-1}{3} + \sum_{v_i \neq u} \binom{\deg v_i}{3} + \binom{\deg u - 1}{3} + 2\sum_{v_i \neq u} \binom{\deg v_i}{4} + 2\binom{\deg u - 1}{4} \\
& + \sum_{v_i \neq u,a} \binom{\deg v_i}{3}(|E(T) \setminus v_i)|) + \binom{\deg a}{3}(|E(T \setminus a)|) + \binom{\deg u}{3}(|E(T \setminus u)|) \\
& + P(T) + 2\binom{\deg u}{3} - \binom{\deg u - 1}{2} \\
= \quad & \binom{n-1}{3} + \sum_{v_i \neq u} \binom{\deg v_i}{3} + \binom{\deg u - 1}{3} + 2\sum_{v_i \neq u} \binom{\deg v_i}{4} + 2\binom{\deg u - 1}{4} \\
& + \sum_{v_i} \binom{\deg v_i}{3}(|E(T) \setminus v_i)|) + R(T) + 2\binom{\deg u}{3} - \binom{\deg u - 1}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{n-1}{3} + \sum_{v_i \neq u} \binom{\deg v_i}{3} + 2\binom{\deg u - 1}{3} + 2\sum_{v_i \neq u} \binom{\deg v_i}{4} + 2\binom{\deg u - 1}{4} \\
&\quad + \sum_{v_i} \binom{\deg v_i}{3} (|E(T) \setminus v_i)|) + P(T) + \binom{\deg u}{3} \\
&= \binom{n-1}{3} + 2\sum_{v_i \neq u} \binom{\deg v_i}{4} + \sum_{v_i} \binom{\deg v_i}{3} (1 + |E(T) \setminus v_i)|) + P(T).
\end{aligned}
$$

□

As seen in the previous section, the linear strand of binomial edge ideals is well-studied. But, similar characterisations do not exist for other strands. For a tree T, since all cliques have at most 2 vertices, the linear strand is such that $\beta_{k,k+2}(J_T) = 0$ for all $k \geq 1$. Hence, it becomes possible to use Theorem 3.13 to obtain the values of further strands.

Theorem 3.37. *Let T be a tree and J_T be its corresponding binomial edge ideal. Then,*

$$
\beta_{k,k+3}(J_T) = \sum_{v_j \in V(T)} (k-1) \binom{\deg v_j + 1}{k+1} \textit{ for all } k \geq 2.
$$

Proof. This can be proved using induction. Let $n = 2$. Then J_T is the binomial edge ideal of a single edge. Since this is a principal ideal, $\beta_{k,k+3}(J_T) = 0$ for all $k \geq 2$, which agrees with the formula. Suppose it is true for a T with $n-1$ vertices. Using Lemma 3.30, consider $e = \{u, v\}$ in T where u is such that $\deg u > 1$ and $|S_u| \leq 1$. Then, using Theorem 3.13, we get

$$
\beta_{k,k+3}(J_T) = \beta_{k,k+3}(J_{T \setminus e}) + \beta_{k-1,k+1}(J_{(T \setminus e)_e}).
$$

Hence, $\beta_{k,k+3}(J_T)$ depends on the linear strand of $(T \setminus e)_e$. We know the size of the clique in $(T \setminus e)_e$ is $\deg u$. Hence using Corollary 3.26 and the inductive hypothesis we get:

$$
\begin{aligned}
\beta_{k,k+3}(J_{G \setminus e}) &= \sum_{v_j \neq u} (k-1) \binom{\deg v_j + 1}{k+1} + (k-1) \binom{\deg u}{k+1}, \\
\beta_{k-1,k+1}(J_{(G \setminus e)_e}) &= (k-1) \binom{\deg u}{k}.
\end{aligned}
$$

Thus, substituting into Theorem 3.13 we get:

$$
\sum_{v_j \neq u} (k-1) \binom{\deg v_j + 1}{k+1}) + (k-1) \binom{\deg u}{k+1} + (k-1) \binom{\deg u}{k} = \sum_{v_j} (k-1) \binom{\deg v_j + 1}{k+1}
$$

$\square$

Chapter 4

Partial Betti Splittings

In the previous sections, we have seen examples of complete Betti splittings for monomial ideals and binomial edge ideals. We have also seen some conditions under which splitting an ideal leads to a Betti splitting. In the case of edge ideals, this condition translates to splitting off a vertex from the graph. In the case of binomial edge ideals, we shall show that splitting off a vertex is a partial Betti splitting. We shall also see how this partial splitting manifests for different types of graphs. From here on, Betti splittings are known as complete Betti splittings.

4.1 Conditions for partial splittings

While complete Betti splittings are rare, for many ideals, there are ways of decomposing generators such that some of the Betti numbers are still split. In the case of binomial edge ideals, defining the notion of a partial Betti splitting, where certain Betti numbers split, turns out to be useful.

Definition 4.1. Let I, J and K be graded ideals such that $\mathfrak{G}(I)$ is the disjoint union of $\mathfrak{G}(J)$ and $\mathfrak{G}(K)$. Then $I = J + K$ is an (r, s)**-Betti splitting** if:

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \quad \text{for all } (i, j) \text{ with } i \geq r \text{ or } j \geq i + s.$$

From the definition, we can see that a partial Betti splitting indicates that all Betti numbers beyond a certain row or column in the Betti table of the ideal are split. Such a notion can be handy, as it can give us information about important homological invariants such as the regularity and projective dimension.

Such a definition of partial Betti splittings allows us to slightly tweak conditions for complete Betti splittings to suit our needs. We will use a restatement of Theorem 2.102 for this. As we shall show, the proof remains unchanged except for minor details.

Theorem 4.2. *Let I, J and K be graded ideals such that $I = J + K$ and $\mathfrak{G}(I)$ is the disjoint union of $\mathfrak{G}(J)$ and $\mathfrak{G}(K)$. Suppose for a given i and (multi)degree j we have that:*

- $\beta_{i,j}(J \cap K) > 0$ *implies that* $\beta_{i,j}(J) = 0$ *and* $\beta_{i,j}(K) = 0$, *and*
- $\beta_{i-1,j}(J \cap K) > 0$ *implies that* $\beta_{i-1,j}(J) = 0$ *and* $\beta_{i-1,j}(K) = 0$.

Then we have:

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K). \tag{4.1}$$

Proof. Since $I = J + K$, we have the short exact sequence

$$0 \longrightarrow J \cap K \xrightarrow{\phi} J \oplus K \xrightarrow{\psi} J + K = I \longrightarrow 0$$

This induces a long exact sequence in Tor, which restricts to a long exact sequence of vector spaces when taking the graded pieces,

$$\longrightarrow \mathrm{Tor}_i(k, J \cap K)_j \longrightarrow \mathrm{Tor}_i(k, J)_j \oplus \mathrm{Tor}_i(k, K)_j \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J \cap K)_j \longrightarrow$$

Fix some i and some (multi)degree j. First suppose $\beta_{i,j}(J \cap K) = 0$. By the hypothesis, if $\beta_{i-1,j}(J \cap K) \neq 0$, that implies that $\beta_{i-1,j}(J) = 0$ and $\beta_{i-1,j}(K) = 0$. Hence this gives us the short exact sequence:

$$0 \longrightarrow \mathrm{Tor}_i(k, J)_j \oplus \mathrm{Tor}_i(k, K)_j \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J \cap K)_j \longrightarrow 0.$$

Since $\beta_{i,j}(J \cap K) = 0$, this gives us that $\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$.

Instead, if we have that $\beta_{i-1,j}(J \cap K) = 0$, then we have the exact sequence,

$$0 \longrightarrow \mathrm{Tor}_i(k, J)_j \oplus \mathrm{Tor}_i(k, K)_j \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow 0$$

which again gives us the desired formula.

Finally, assume that $\beta_{i,j}(J \cap K) \neq 0$. This tells us that $\beta_{i,j}(J) = 0$ and $\beta_{i,j}(K) = 0$. This gives us the exact sequence

$$0 \longrightarrow \mathrm{Tor}_i(k, I)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J \cap K)_j \longrightarrow \mathrm{Tor}_{i-1}(k, J)_j \oplus \mathrm{Tor}_{i-1}(k, K)_j \longrightarrow \cdots$$

If $\beta_{i-1,j}(J \cap K) = 0$, then that means that $\operatorname{Tor}_i(k, I)_j = \beta_{i-1,j}(I) = 0$ and hence, the formula holds. If $\beta_{i-1,j}(J \cap K) \neq 0$ then $\beta_{i-1,j}(J) = \beta_{i-1,j}(K) = 0$ which implies that $\beta_{i,j}(I) = \operatorname{Tor}_i(k, I)_j = \operatorname{Tor}_{i-1}(k, J \cap K)_j = \beta_{i-1,j}(J \cap K)$. Since $\beta_{i,j}(J) = 0$ and $\beta_{i,j}(K) = 0$, this agrees with the formula and hence proves the proposition. $\square$

We shall see that Theorem 4.2 is very useful to show partial splittings for some graded ideals.

4.2 Application to binomial edge ideals

The main goal of this section is to apply Theorem 4.2 to binomial edge ideals and obtain suitable partial splittings. We will show that splitting off a vertex from a graph corresponds to a partial splitting for its binomial edge ideal and we will also describe the (r, s) of the induced partial splitting.

Definition 4.3. Consider a graph G with $V(G) = [n]$ and its binomial edge ideal $I = J_G$. Let s be a vertex in $V(G)$. If J is the ideal generated by all elements in $\mathfrak{G}(I)$ of the form $fx_s + gy_s$ and K is the ideal generated by the rest of the elements of $\mathfrak{G}(I)$, we call $I = J + K$ an s**-partition**.

Remark 4.4. If G is the graph of the binomial edge ideal $I = J_G$ and $I = J + K$ is an s-partition as in Definition 4.3, then we can see that J is the binomial edge ideal of the graph $G_1 = \{\{s, k\} \mid k \in N_G(i)\}$ and K is the binomial edge ideal of the graph $G_2 = G \setminus \{s\}$.

Example 4.5. Consider the graph G with $V(G) = [5]$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{4, 5\}\}$. Fix $1 \in [5]$. Then, G_1 is the graph with $V(G_1) = \{1, 2, 4\}$ and $E(G_1) = \{\{1, 2\}, \{1, 4\}\}$ and G_2 is the graph with $V(G_2) = \{2, 3, 4, 5\}$ and $E(G) = \{\{2, 3\}, \{3, 4\}, \{4, 5\}\}$, then $J_G = J_{G_1} + J_{G_2}$ is a 1-partition. The graphs G, G_1 and G_2 are given in Figure 4.1.

Example 4.6. Let G be a graph with an edge $e = \{u, v\}$ such that v is a pendant vertex. Since v has degree one, f_e is the only generator which is of the form $fx_v + gy_v$. Hence $J_G = J_{G \setminus e} + \langle f_e \rangle$ is a v-partition.

Since every Betti splitting $I = J + K$ involves the intersection $J \cap K$, the following lemma is useful.

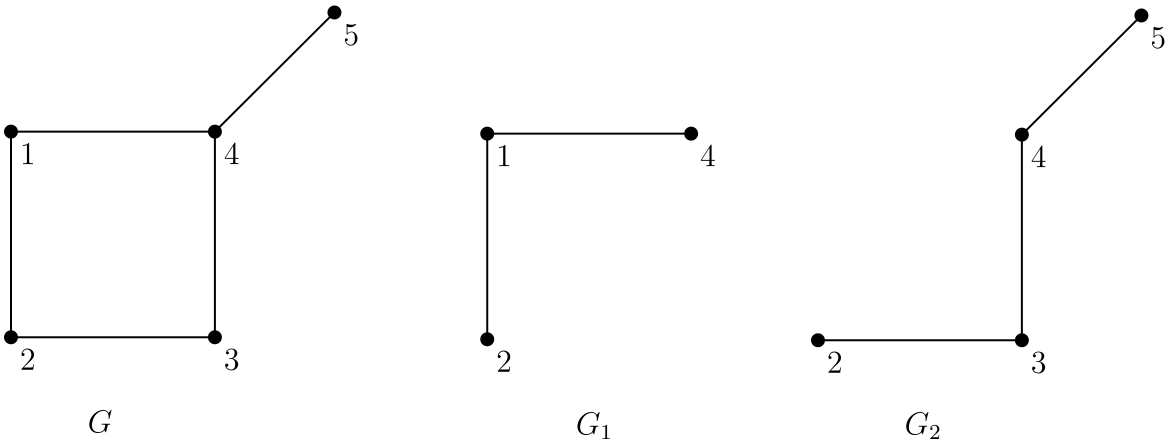


Figure 4.1: A 1-partition of J_G

Lemma 4.7. *Consider the graph G on $[n]$ and let J_G be its binomial edge ideal. Let $J_G = J_{G_1} + J_{G_2}$ be an s-partition of I, where G_1 and G_2 are as described in Remark 4.4. Denote the minimal degree 3 generators of $J_{G_1} \cap J_{G_2}$ by $\mathfrak{G}(J_{G_1} \cap J_{G_2})_3$. Then:*

$$\mathfrak{G}(J_{G_1} \cap J_{G_2})_3 = \{x_s f_{a,b}, y_s f_{a,b} \mid a, b \in N_G(s) \text{ and } \{a,b\} \in E(G)\}.$$

In other words, $(J_{G_1} \cap J_{G_2})_3 = (x_s H + y_s H)_3$, where H is the binomial edge ideal of the induced graph on $N_G(s)$.

Proof. Let the vertices of $N_G(s)$ be denoted by $\{v_1, \ldots, v_k\}$. Since all generators of $J_{G_1} \cap J_{G_2}$ have degree ≥ 3, it is clear that the minimum generators of degree 3 form a basis for the vector space $(J_{G_1} \cap J_{G_2})_3$. Hence, we need to prove that the proposed set is a k-basis for $(J_{G_1} \cap J_{G_2})_3$. Let $B(V)$ denote the basis of the vector space V. Then,

$$B((J_{G_1})_3) = \{x_i f_{a,s}, y_i f_{a,s} \mid a \in \{v_1, \ldots, v_k\} \text{ and } i \in \{1, \ldots, n\}\},$$

$$B((J_{G_2})_3) = \{x_i f_{a,b}, y_i f_{a,b} \mid f_{a,b} \in \mathfrak{G}(J_{G_2}) \text{ and } i \in \{0, \ldots, n\}\}.$$

It is easily seen that the above sets generate $(J_{G_1})_3$ and $(J_{G_2})_3$ respectively. Linear independence is inferred by considering the $\mathbb{N}^n$ multigrading where $\deg x_i = \deg y_i = (0, \ldots, 1, \ldots, 0)$. Since the only elements in both $B((J_{G_1})_3)$ and $B((J_{G_2})_3)$ with the same multidegree; $x_i f_{a,b}$ and $y_i f_{a,b}$ are linearly independent, this means that any linear combination of elements from $B((J_{G_1})_3)$ or $B((J_{G_2})_3)$ will be zero if and only if all coefficients of the elements in the linear combination are zero. This tells us that the above sets must be a k-basis of $(J_{G_1})_3$ and $(J_{G_2})_3$ respectively.

Consider $P = \sum_{e\in B((J_{G_1})_3)} c_e e \in J_{G_1} \cap J_{G_2}$, where c_e are constants. Let us look at $x_i f_{a,s}$ where $i \notin \{v_1, \ldots, v_k\}$. We can rewrite $x_i f_{a,s}$ as:

$$x_i f_{a,s} = x_i(x_a y_s - x_s y_a) = y_s(x_i x_a) - x_s(x_i y_a).$$

It is also clear, since $i \notin \{v_1, \ldots, v_k\}$ that the term $y_s x_i x_a$ in P, when written as an element in $(J_{G_1})_3$, only comes from the basis element $x_i f_{a,s}$. Since P is in $(J_{G_2})_3$ as well, we can also write

$$P = R + y_s(cx_i x_a + L) = Q + y_s(\sum_{f_{a,b}\in\mathfrak{G}(K)} c'_e f_{a,b}), \tag{4.2}$$

where no terms of R and Q are divisible by y_s and L does not have any monomial terms divisible by $x_i x_a$. Clearly the above equations implies that $cx_i x_a + L = \sum_{f_{a,b}\in\mathfrak{G}(K)} c'_e f_{a,b}$. Now by introducing a grading where $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$ for all i, we can see that $x_i x_a$ is of degree $(2, 0)$ but the degree of each term $f_{a,b}$ in $\mathfrak{G}(K)$, is $(1, 1)$. Hence, for Equation (4.2) to hold, $c = 0$. The same argument can be made for $y_i f_{a,s}$ where $i \notin \{v_1, \ldots, v_k\}$.

Now consider the case where $i \in \{v_1, \ldots, v_k\}$. Here, we can see that the term $y_s x_i x_a$ when written as an element of $(J_{G_1})_3$ only comes from the basis elements $x_i f_{a,s}$ and $x_a f_{i,s}$. As before, to make sure there are no elements of degree $(2, 0)$, the coefficients of $x_i f_{a,s}$ and $x_a f_{i,s}$ must cancel. It is also clear that $cx_i f_{a,s} - cx_a f_{i,s} = cx_s(x_a y_i - x_i y_a) = cx_s f_{a,i}$. The same argument can be applied to $y_s y_i x_a$ where $i \in \{1, \ldots, k\}$. Hence from this and the above equation:

$$P = \sum_{f_{a,s}\in\mathfrak{G}(J_{G_1}), i\in[n]} c_{i,a} x_i f_{a,s} + c'_{i,a} y_i f_{a,s} = \sum_{a,i\in N_G(s)} c_{i,a} x_s f_{a,i} + c'_{i,a} y_s f_{a,i}.$$

Written in terms of the basis of $(J_{G_2})_3$, we can see that

$$P = \sum_{a,i\in N_G(s)} c_{i,a} x_s f_{a,i} + c'_{i,a} y_s f_{a,i} = x_s(\sum_{f_{a,b}\in\mathfrak{G}(K)} d_{a,b} f_{a,b}) + y_s(\sum_{f_{a,b}\in\mathfrak{G}(K)} d'_{a,b} f_{a,b}),$$

where $d_{a,b}, d'_{a,b}$ are all arbitrary constants. Equating coefficients of y_s gives us:

$$\sum_{a,i\in N_G(s)} c'_{a,i} f_{a,i} = \sum_{f_{a,b}\in\mathfrak{G}(J_{G_2})} d'_{a,b} f_{a,b}.$$

Since $\{f_{i,j} | \{i, j\} \in V(G)\}$ is a linearly independent set, this implies that $c'_{a,i} = 0$ for all $a, i \in N_G(s)$ where $\{a, i\} \notin E(G)$. The same argument can be made for the coefficients of

x_s. Thus:

$$P = \sum_{a,i \in N_G(s), f_{a,i} \in \mathfrak{G}(K)} c_{a,i} x_s f_{a,i} + c'_{a,i} y_s f_{a,i}.$$

The proposed set spans $(J_{G_1} \cap J_{G_2})_3$. Since $f_{a,i} \in \mathfrak{G}(K)$, it is clear that $x_s f_{a,i}$, $y_s f_{a,i} \in J_{G_2}$. We also have $x_s(x_a y_i - x_i y_a) = x_a(x_s y_i - x_i y_s) - x_i(x_s y_a - x_a y_s) \in J_{G_1}$ and $y_s(x_a y_i - x_i y_a) = y_a(x_s y_i - x_i y_s) - y_i(x_s y_a - x_a y_s) \in J_{G_1}$, which means that this set is in $(J \cap K)_3$. To establish linear independence, we consider the $\mathbb{N}^n$-multigrading $\deg x_i = \deg y_i = (0, \ldots, 1, \ldots, 0)$ on the set $\{x_s f_{a,b}, y_s f_{a,b} \mid a, b \in N_G(s) \text{ and } \{a, b\} \in E(G)\}$. As before the only elements with the same multidegree are $x_s f_{a,b}$ and $y_s f_{a,b}$. Since these elements are linearly independent, this means that any linear combination of elements from the proposed set will be zero if and only if all coefficients of the elements in the linear combination are zero. Hence $\{x_s f_{a,b}, y_s f_{a,b} \mid a, b \in N_G(s) \text{ and } \{a, b\} \in E(G)\}$ is a k-basis of $\mathfrak{G}(J_{G_1} \cap J_{G_2})_3$ and the proposition follows. □

This result is very interesting, as it shows that the degree three generators of $J_{G_1} \cap J_{G_2}$ can be written in terms of the generators of a binomial edge ideal. In particular, there will be degree three generators of $J_{G_1} \cap J_{G_2}$, only when there is a triangle in G containing s. When $J_{G_1} \cap J_{G_2}$ has some degree three generators, then the linear strand will be of the form $\beta_{k,k+3}(J_{G_1} \cap J_{G_2})$. We will use some further Betti splittings to characterise the linear strand in this case.

Theorem 4.8. *Consider a graph G and let G' be the induced subgraph on $N_G(s)$. Now consider the s-partition $J_G = J_{G_1} + J_{G_2}$. Then, we have:*

$$\beta_{k,k+3}(J_{G_1} \cap J_{G_2}) = 2\beta_{k,k+2}(J_{G'}) + \beta_{k-1,k+1}(J_{G'}) \quad \textit{for all } k \geq 0.$$

Proof. From Lemma 4.7, we have that the minimal degree 3 generators for $J_{G_1} \cap J_{G_2}$ are

$$\{x_s f_{a,b}, y_s f_{a,b} \mid a, b \in N_G(s) \text{ and } \{a, b\} \in E(G)\}.$$

Since, $J_{G_1} \cap J_{G_2}$ is generated in degree 3 or higher, this tells us that there are no minimal generators of smaller degrees. Hence, if I is the ideal generated by $\{x_s f_{a,b}, y_s f_{a,b} \mid a, b \in N_G(s)$ and $\{a, b\} \in E(G)\}$, then $\beta_{k,k+3}(J_{G_1} \cap J_{G_2}) = \beta_{k,k+3}(I)$.

We now consider the partition $I = I_x + I_y$, where $\mathfrak{G}(I_x) = \{x_s f_{a,b} \mid \{a, b\} \in E(G')\}$ and $\mathfrak{G}(I_y) = \{y_s f_{a,b} \mid \{a, b\} \in E(G')\}$.

Claim.
$$I_x \cap I_y = \langle \{x_s y_s f_{a,b} \mid \{a,b\} \in E(G')\}\rangle.$$

Proof. It is clear that each $x_s y_s f_{a,b} \in I_x \cap I_y$. For the other inclusion, consider $g \in I_x \cap I_y$. Since g is in both I_x and I_y, we can write it as:

$$g = x_s(\sum k_{a,b} f_{a,b}) = y_s(\sum k'_{a,b} f_{a,b}),$$

where $k_{a,b}$ and $k'_{a,b}$ are non-zero polynomials. Since, none of $\{f_{a,b}\}$ are divisible by x_s or y_s, we know that some terms of $k_{a,b}$ are divisible by y_s, for all $(a,b) \in G'$. Denote all the $k_{a,b}$ which are divisible by y_s with $\bar{k_{a,b}}$. Hence, we can write:

$$g = x_s(\sum \bar{k}_{a,b} f_{a,b} + L) = y_s(\sum k'_{a,b} f_{a,b})$$

where $\bar{k}_{a,b}$ are non-zero polynomials divisible by y_s, and no term of L is divisible by y_s. Since g must be divisible by y_s, we have that $y_s \mid L$. But since no element of L is divisible by y_s, this implies that $L = 0$.

Hence, we can write $g = x_s(\sum \bar{k}_{a,b} f_{a,b})$. If $\bar{k}_{a,b} = y_s h_{a,b}$, then $g = x_s y_s(\sum h_{a,b} f_{a,b}) \in \langle \{x_s y_s f_{a,b} \mid \{a,b\} \in E(G')\}\rangle$. □

Now we have that $G(I_x) = \{x_0 f_{a,b} \mid \{a,b\} \in E(G')\}, G(I_y) = \{y_0 f_{a,b} \mid \{a,b\} \in E(G')\}$ and $G(I_x \cap I_y) = \{x_0 y_0 f_{a,b} \mid \{a,b\} \in E(G')\}$. It is clear that $J_{G'} \xrightarrow{\times x_0} I_x$, $J_{G'} \xrightarrow{\times y_0} I_y$ and $J_{G'} \xrightarrow{\times x_0 y_0} I_x \cap I_y$ are all isomorphisms of degree 1, 1 and 2 respectively. Now, consider the $\mathbb{N}^n$ multigrading on I_x, I_y and $I_x \cap I_y$. Let $\deg x_0 = \deg y_0 = (1, \ldots, 0)$. The isomorphisms of the ideals give us:

$$\mathrm{Tor}_i^S(I_x, k)_{(1,j)} \cong \mathrm{Tor}_i^S(I_y, k)_{(1,j)} \cong \mathrm{Tor}_i^S(J_{G'}, k)_j \text{ and } \mathrm{Tor}_i^S(I_x \cap I_y, k)_{(2,j)} \cong \mathrm{Tor}_i^S(J_{G'}, k)_j$$

where j is some multigraded degree. By combining all the multigraded Betti numbers, we can see that $\beta_{i,j}(I_x) = \beta_{i,j}(I_y) = \beta_{i,j-1}(J_{G'})$ and $\beta_{i,j}(I_x \cap I_y) = \beta_{i,j-2}(J_{G'})$. It is also clear that all Betti numbers of $I_x \cap I_y$ occur only on multidegrees $(2,j)$ while all Betti numbers of I_x and I_y occur only at $(1,j)$. Hence, by using Theorem 2.102 and combining all multidegrees, we have $\beta_{i,j}(I) = \beta_{i,j}(I_x) + \beta_{i,j}(I_y) + \beta_{i-1,j}(I_x \cap I_y)$. Therefore,

$$\beta_{k,k+3}(J_{G_1} \cap J_{G_2}) = \beta_{k,k+3}(I) = \beta_{k,k+2}(J_{G'}) + \beta_{k,k+2}(J_{G'}) + \beta_{k-1,k+1}(J_{G'})$$

□

From this theorem, we can see that the linear strand of $J_{G_1} \cap J_{G_2}$ is intimately related to the linear strand $J_{G'}$. Hence, we can use this and Theorem 4.2 to get a partial splitting for the binomial edge ideal of any graph G.

Theorem 4.9. *Let J_G be the binomial edge ideal of a graph G and let $J_G = J_{G_1} + J_{G_2}$ be an s-partition of G, as defined above. Let $c(s)$ be the size of the largest clique that s is a part of. Then,*

$$\beta_{i,j}(J_G) = \beta_{i,j}(J_{G_1}) + \beta_{i,j}(J_{G_2}) + \beta_{i-1,j}(J_{G_1} \cap J_{G_2}) \quad \text{for all } (i,j) \text{ with } i \geq c(s) \text{ or } j \geq i+4. \tag{4.3}$$

Or in other words, $J_G = J_{G_1} + J_{G_2}$ is a $(c(s), 4)$-Betti Splitting.

Proof. From the previous theorem, we know that

$$\beta_{k,k+3}(J_{G_1} \cap J_{G_2}) = \beta_{k,k+2}(J_{G'}) + \beta_{k,k+2}(J_{G'}) + \beta_{k-1,k+1}(J_{G'}).$$

From Corollary 3.23 we have that $\beta_{k,k+2}(J_{G'}) = (k+1)f_{k+1}(\Delta(G))$, where $f_i(\Delta(G))$ is the number of faces of $\Delta(G)$ of dimension i. We know that the largest clique in G' is of size $c(s)-1$. Hence, $\beta_{k,k+2}(J_{G'}) = 0$ for all $k \geq c(s)-2$. Therefore, this means that $\beta_{k,k+3}(J_{G_1} \cap J_{G_2}) = 0$ for all $k \geq c(s)-1$.

Consider the multigrading on $J_G = J_{G_1} + J_{G_2}$ to be given by the $\mathbb{N}^n$ grading, in other words, $\deg x_i = \deg y_i = i^{th}$ unit vector $(0,\ldots,0,1,0,\ldots,0)$. Therefore, all generators of $J_{G_1} \cap J_{G_2}$ are of the form $fx_s + gy_s$ and their multigraded Betti numbers occur within multidegrees $\mathbf{a}$ such that its s^{th} component, $\mathbf{a_s}$ is non-zero. Since J_{G_2} contains no generators of the form $fx_s + gy_s$, $\beta_{i,j}(J_{G_1} \cap J_{G_2}) > 0$ implies that $\beta_{i,j}(J_{G_2}) = 0$ for all $i \in \mathbb{N}$ and multidegrees j as defined above.

From Theorem 3.28, since G_1 is a star graph,

$$\beta_i(J_{G_1}) = \beta_{i,i+3}(J_{G_1}) = i\binom{n}{i+2} \quad i \geq 1.$$

Hence, we can see that for all multidegrees $j = (j_1, \ldots, j_n)$ with $\sum_k j_k \geq i+4$, we have:

1. $\beta_{i,j}(J_{G_1} \cap J_{G_2}) > 0$ implies that $\beta_{i,j}(J_{G_1}) = 0$, and

2. $\beta_{i-1,j}(J_{G_1} \cap J_{G_2}) > 0$ implies that $\beta_{i-1,j}(J_{G_1}) = 0$.

Since the minimal degree of the generators of $J_{G_1} \cap J_{G_2}$ is 3, and $\beta_{k,k+3}(J_{G_1} \cap J_{G_2}) = 0$ for all $k \geq c(s)-1$, we also have that $\beta_{i,j}(J_{G_1} \cap J_{G_2}) > 0$ implies that $\beta_{i,j}(J_{G_1}) = 0$ for all $i \geq c(s)-1$ and multidegrees j.

Therefore, from Theorem 4.2, we have

$$\beta_{i,j}(J_G) = \beta_{i,j}(J_{G_1}) + \beta_{i,j}(J_{G_2}) + \beta_{i-1,j}(J_{G_1} \cap J_{G_2}),$$

for all i and multidegrees j with $i \geq c(s)$ or $\sum_{k=1}^{n} j_k \geq i+4$. Thus, the result holds for $\mathbb{N}^n$ multidegrees j. Since it is true for all N^n multidegrees, we can combine them to obtain the same result in the standard grading. □

This result can give us nice splittings for some big classes of graphs.

Corollary 4.10. *Let I be the binomial edge ideal of a triangle-free graph T and let $I = J+K$ be an s-partition of T, as defined above. Then,*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i \geq 1 \text{ and multidegrees } j. \tag{4.4}$$

or in other words, $I = J+K$ is a (1,4)-Betti Splitting.

Proof. This follows directly from the Theorem 2.8, as in a triangle-free graph G, G' will have no edges. Hence, $c(s) = 1$. □

Remark 4.11. In general a $(1, \_)$-Betti splitting is just a complete Betti splitting. Hence, Corollary 4.10 says that splitting off a vertex is a complete Betti splitting for binomial edge ideals of triangle-free graphs. Notice, that in Theorem 3.13, splitting off the edge $e = \{u, v\}$ is equivalent to splitting off the pendant vertex v. Hence, the complete Betti splitting seen there turns out to be a special case of Corollary 4.10.

The above theorem and corollary can tell us a lot about the Betti numbers for several families of graphs. One notable example is the family of bipartite graphs, which is triangle-free. One difficulty that occurs while using Betti splittings, is that information about $J_{G_1} \cap J_{G_2}$ is necessary. For example, if we had a characterisation of the minimal generators of $J_{G_1} \cap J_{G_2}$ for a triangle-free graph G, using Corollary 4.10, a general formula for $\beta_1(G)$ can be given. In that direction, we present the following conjecture:

Conjecture 4.12. *Let T be a triangle-free graph. If m denotes the number of edges in T, and $C^*(T)$ is the set of induced cycles of T then:*

$$\beta_1(T) = \binom{m}{2} + \sum_{v_i} \binom{\deg v_i}{3} + \sum_{C_i \in C^*(T)} |C_i| - 1$$

In Lemma 4.7, we characterize the degree 3 generators for $J_{G_1} \cap J_{G_2}$. A similar procedure can be applied to calculate the generators of degree 4 and above. Unfortunately, the calculation becomes extremely tedious and characterizing the minimal generators for any degree n is a tough task without a modification to the methodology.

The above ideas can also give us some nice results regarding the projective dimension and regularity.

Corollary 4.13. *Consider a graph G and let $J_G = J_{G_2} + J_{G_2}$ be an s- partition. Then:*

1. *If* $\mathrm{pd}(J_G) \geq c(s)$*, then:*

$$\mathrm{pd}(J_G) = \max\{\mathrm{pd}(J_{G_1}), \mathrm{pd}(J_{G_2}), \mathrm{pd}(J_{G_1} \cap J_{G_2}) + 1\}$$

2. *If* $\mathrm{reg}(J_G) \geq 4$*, then:*

$$\mathrm{reg}(J_G) = \max\{\mathrm{reg}(J_{G_2}), \mathrm{reg}(J_{G_1} \cap J_{G_2}) - 1\}$$

Proof. Given that $\mathrm{pd}(J_G) \geq c(s)$, we know that there is a partial splitting for all $\beta_{i,j}(J_G)$, for all $i \geq c(s)$. Hence, $\mathrm{pd}(J_G) = \max\{\mathrm{pd}(J_{G_1}), \mathrm{pd}(J_{G_2}), \mathrm{pd}(J_{G_1} \cap J_{G_2}) + 1\}$.

Similarly, if $\mathrm{reg}(J_G) \geq 4$, we know that there is a partial splitting for all $\beta_{i,j}(J_G)$, for all $i \geq c(s)$. Hence, $\mathrm{reg}(J_G) = \max\{\mathrm{reg}(J_{G_1}), \mathrm{reg}(J_{G_2}), \mathrm{reg}(J_{G_1} \cap J_{G_2}) - 1\}$. Since $\mathrm{reg}(J_{G_1}) = 3$, we have $\mathrm{reg}(J_G) = \max\{\mathrm{reg}(J_{G_2}), \mathrm{reg}(J_{G_1} \cap J_{G_2}) - 1\}$. □

Thus finding an (r, s)-splitting, can make the problem of finding the projective dimension and regularity of J_G a little simpler.

4.3 Partial splittings of initial ideals

In the previous section, we gave conditions to obtain a partial Betti splitting for binomial edge ideals, via $J_G = J_{G_1} + J_{G_2}$ from Theorem 4.9. Note that even though $J_G = J_{G_1} + J_{G_2}$, this equality does not hold at the level of initial ideals. In other words, there are graphs G with vertices $s \in V(G)$ such that $\mathrm{in}(J_G) \neq \mathrm{in}(J_{G_1}) + \mathrm{in}(J_{G_1})$. In this section, we consider graphs G and vertices s such that $\mathrm{in}(J_G) = \mathrm{in}(J_{G_1}) + \mathrm{in}(J_{G_1})$. In this case, we prove that this induces a partial Betti splitting on the initial ideals and characterises the corresponding (r, s).

The following lemma proves to be useful for our purposes.

Lemma 4.14. *Let G be a finite simple graph and let $J_G = J_{G_1} + J_{G_2}$ be an s-partition of G. If $\operatorname{in}(J_G) = \operatorname{in}(J_{G_1}) + \operatorname{in}(J_{G_2})$, then*

$$\operatorname{in}(J_{G_1} \cap J_{G_2}) = \operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2}). \tag{4.5}$$

Proof. This result follows directly from Lemma 1.3, [2]. □

The above lemma is rather surprising, as $\operatorname{in}(J_G)$ does not appear in Equation (4.5). But, since it allows us to write $\operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2})$ as $\operatorname{in}(J_{G_1} \cap J_{G_2})$, it is useful to obtain a partial splitting.

Lemma 4.15. *Let $J_G = J_{G_1} + J_{G_2}$ be an s-partition, with G_1 and G_2 as in Remark 4.4. Then we have that $\operatorname{reg}(\operatorname{in}(J_{G_1})) = 3$.*

Proof. This follows directly from Corollary 3.3, [28], as $\operatorname{reg}(J_{G_1}) = 3$. □

The above lemma tells us that the regularity for the initial ideal of any star graph is at most three. Using this, we can obtain a partial Betti splitting for $\operatorname{in}(J_G)$.

Lemma 4.16. *Consider an s-partition, $J_G = J_{G_1} + J_{G_2}$. Then, the degree three generators for the initial ideal of $\operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2})$ is given by:*

$$\mathfrak{G}(\operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2}))_3 = \{x_s x_a y_b, y_s x_a y_b \mid a < b \in N_G(s), \{a, b\} \in E(G)\}.$$

Proof. From Lemma 4.14, we know that the degree three generators of $\operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2})$ are the same as that of $\operatorname{in}(J_{G_1} \cap J_{G_2})$ when $J_G = J_{G_1} + J_{G_2}$ is an s-partiton. From Lemma 4.7 we have,

$$\mathfrak{G}(J_{G_1} \cap J_{G_2})_3 = \{x_s f_{a,b}, y_s f_{a,b} \mid a, b \in N_G(s), \{a, b\} \in E(G)\}. \tag{4.6}$$

We know all the minimal generators of $J_{G_1} \cap J_{G_2}$ of degree three form a basis of $(J_{G_1} \cap J_{G_2})_3$. Consider any degree three polynomial g in $J_{G_1} \cap J_{G_2}$. It can be written as:

$$g = \sum_{e_i \in \mathfrak{G}(J_{G_1} \cap J_{G_2})} k_i e_i.$$

where k_i are in the field K. It follows from Equation (4.6) that distinct e_i's have distinct monomial terms. Note that every generator in Equation (4.6) is made of distinct monomials. where k_i are in the field K. It follows from Equation (4.6) that distinct $e_i' s$ have distinct monomial terms. Hence, this means that the coefficient of any monomial in g is the same as the coefficient of some $e_i \in \mathfrak{G}(J_{G_1} \cap J_{G_2})$. Hence, the leading term of g is the same as the leading term of some e_i.

We know the leading terms of all e_i are of the form $\{x_s x_a y_b, y_s x_a y_b \mid a < b \in N_G(s), \{a,b\} \in E(G)\}$. Thus from the above argument, we have

$$\mathfrak{G}(\text{in}(J_{G_1}) \cap \text{in}(J_{G_2}))_3 = \mathfrak{G}(\text{in}(J_{G_1} \cap J_{G_2}))_3 = \{x_s x_a y_b, y_s x_a y_b \mid a, b \in N_G(s), \{a,b\} \in E(G)\}.$$

□

Our goal is to characterise the strand $\beta_{k,k+3}(\text{in}(J_{G_1}) \cap \text{in}(J_{G_2}))$. To do this, we use the help of edge ideals. The following lemmas are useful:

Lemma 4.17. *Let G be a finite simple graph, with edge ideal $I(G)$. Then:*

$$\beta_{i,j}(I(G)) = \sum_{S \subseteq V(G), |S|=j} \#\,\text{comp}(G[S]^c) - 1 \textit{ for all } i \geq 0$$

where $G[S]^c$ is the complement of the induced subgraph of G on S.

Proof. This is proved in Proposition 2.1, [25]. □

Now consider the set $W := \{x_a y_b | \; a < b \in N_G(s), \{a,b\} \in E(G)\}$. We can see that this is the generating set of the edge ideal of some bipartite graph $\tilde{G}_s$ as follows:

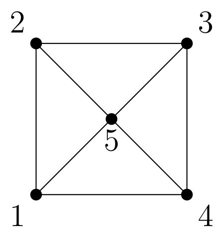
Definition 4.18. The graph $\tilde{G}_s$ is obtained from G as the graph with $V(\tilde{G}_s) = N_G(s) \sqcup N_G(s)$. In other words, $V(\tilde{G}_s) = \{i_A, i_B \mid i \in N_G(s)\}$. The edges are given by $E(\tilde{G}_s) = \{\{i_A, j_B\} \mid i < j, \{i,j\} \in E(G)\}$.

From the definition, we can see that the two independent sets of $\tilde{G}_s$ are $\{i_A \in V(\tilde{G}_s) \mid i \in N_G(s)\}$ and $\{i_B \in V(\tilde{G}_s) \mid i \in N_G(s)\}$ and that $\tilde{G}_s$ is a bipartite graph. We can also see that the edge ideal of $\tilde{G}_s$ is generated by W. An example of G and $\tilde{G}_s$ is given in Example 4.19.

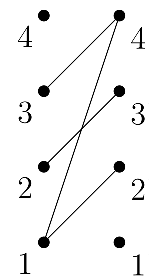
Example 4.19. Consider a graph G with $V(G) = \{1,2,3,4,5\}$ and $E(G) = \{\{1,2\},\{2,3\}, \{3,4\},\{4,1\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$. The corresponding bipartite graph $\tilde{G}_5$ is given by $V(\tilde{G}_5) = \{1,2,3,4,1',2',3',4'\}$ and $E(\tilde{G}_5) = \{1,2'\},\{2,3'\},\{3,4'\},\{1,4'\}\}$.

Theorem 4.20. *Consider a graph G and its s- partition $J_G = J_{G_1} + J_{G_2}$. Let $\tilde{G}_s$ be its corresponding bipartite graph as defined above. Let the edge ideal of $\tilde{G}_s$ be denoted by $I(\tilde{G}_s)$. Then, we have:*

$$\beta_{k,k+3}(\text{in}(J_{G_1}) \cap \text{in}(J_{G_2})) = 2\beta_{k,k+2}(I(\tilde{G}_s)) + \beta_{k-1,k+1}(I(\tilde{G}_s)).$$



G



$\tilde{G}_5$

Figure 4.2: G and $\tilde{G_5}$, in Example 4.19

Proof. From the discussion above,

$$\mathfrak{G}(\text{in}(J_{G_1}) \cap \text{in}(J_{G_2}))_3 = \{x_s x_a y_b, y_s x_a y_b \mid a < b \in N_G(s), \{a,b\} \in E(G)\}.$$

We also know that there are no minimal generators of a smaller degree. Hence, if I is the ideal generated by $\mathfrak{G}(\text{in}(J_{G_1}) \cap \text{in}(J_{G_2}))_3$, then $\beta_{k,k+3}(\text{in}(J_{G_1}) \cap \text{in}(J_{G_2})) = \beta_{k,k+3}(I)$.

Now consider the partition $I = I_x + I_y$, where $\mathfrak{G}(I_x) = \{x_s x_a y_b \mid a < b \in N_G(s), \{a,b\} \in E(G)\}$ and $\mathfrak{G}(I_y) = \{y_s x_a y_b \mid a < b \in N_G(s), \{a,b\} \in E(G)\}$.

Claim.

$$I_x \cap I_y = \langle \{x_s y_s x_a y_b \mid a < b, \{a,b\} \in E(G)\} \rangle.$$

Proof. Since the intersection of two monomial ideals is generated by the least common multiple of their generators, we have that

$$\mathfrak{G}(I_x \cap I_y) = \{\text{lcm}(x_s x_a y_b, y_s x_c y_d) \mid a,b,c,d \in N_G(s), \{a,b\}, \{c,d\} \in E(G)\}.$$

Case 1: $a \neq c$ and $b \neq d$. $\text{lcm}(x_s x_a y_b, y_s x_c y_d) = x_s y_s x_a x_c y_b y_d$.
Case 2: $a \neq c$ but $b = d$ $\text{lcm}(x_s x_a y_b, y_s x_c y_d) = x_s y_s x_a y_b y_d$.
Case 3: $a = c$ but $b \neq d$, $\text{lcm}(x_s x_a y_b, y_s x_c y_d) = x_s y_s x_a y_b y_d$.
Case 4: $a = c$ and $b = d$, then $\text{lcm}(x_s x_a y_b, y_s x_c y_d) = x_s y_s x_a y_b$.

Hence, we can see that $x_s y_s x_a y_b \mid \text{lcm}(x_s x_a y_b, y_s x_c y_d)$ for all $\{a,b\}$ and $\{c,d\} \in E(G)$. Since $\text{lcm}(x_s x_a y_b, y_s x_a, y_b) = x_s y_s x_a y_b \in \mathfrak{G}(I_x \cap I_y)$ for all $\{a,b\}$, we know that $\{x_s y_s x_a y_b \mid a < b, \{a,b\} \in E(G)\} \subseteq I_x \cap I_y$. Thus,

$$I_x \cap I_y = \langle \{x_s y_s x_a y_b \mid a < b, \{a,b\} \in E(G)\} \rangle.$$

This proves the claim. $\square$

Now we have that $\mathfrak{G}(I_x) = \{x_s x_a y_b \mid a < b, \{a,b\} \in E(G)\}, \mathfrak{G}(I_y) = \{y_s x_a y_b \mid a < b, \{a,b\} \in E(G)\}$ and $\mathfrak{G}(I_x \cap I_y) = \{x_s y_s x_a y_b \mid a < b, \{a,b\} \in E(G)\}$. It is clear that $I(G'_s) \xrightarrow{\times x_s} I_x$, $I(G'_s) \xrightarrow{\times y_s} I_y$ and $I(G'_s) \xrightarrow{\times x_s y_s} I_x \cap I_y$ are all isomorphisms of degree 1, 1 and 2 respectively. Now, consider the $\mathbb{N}^n$ multigrading on S, $\deg x_s = \deg y_s = (0, \ldots, 1, \ldots, 0)$. The isomorphisms of the ideals give us:

$$\operatorname{Tor}_i^S(I_x, k)_{(1,j)} \cong \operatorname{Tor}_i^S(I_y, k)_{(1,j)} \cong \operatorname{Tor}_i^S(I(\tilde{G}_s), k)_j \text{ and } \operatorname{Tor}_i^S(I_x \cap I_y, k)_{(2,j)} \cong \operatorname{Tor}_i^S(I(\tilde{G}_s), k)_j$$

where j is some multigraded degree. By combining all the multigraded Betti numbers, we can see that $\beta_{i,j}(I_x) = \beta_{i,j}(I_y) = \beta_{i,j-1}(J_{G'})$ and $\beta_{i,j}(I_x \cap I_y) = \beta_{i,j-2}(I(\tilde{G}_s))$. It is also clear that all Betti numbers of $I_x \cap I_y$ occur only on multidegrees $(2, j)$ while all Betti numbers of I_x and I_y occur only at $(1, j)$. As before, by using Theorem 2.102 and combining all multidegrees, we have $\beta_{i,j}(I) = \beta_{i,j}(I_x) + \beta_{i,j}(I_y) + \beta_{i-1,j}(I_x \cap I_y)$. Therefore,

$$\begin{aligned}
\beta_{k,k+3}(\operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2})) &= \beta_{k,k+3}(I) \\
&= \beta_{k,k+2}(I(\tilde{G}_s)) + \beta_{k,k+2}(I(\tilde{G}_s)) + \beta_{k-1,k+1}(I(\tilde{G}_s)) \\
&= 2\beta_{k,k+2}(I(\tilde{G}_s)) + \beta_{k-1,k+1}(I(\tilde{G}_s).
\end{aligned}$$

$\square$

Now, we are ready to prove the main result of this section.

Theorem 4.21. *Consider a graph G and its s-partition $J_G = J_{G_1} + J_{G_2}$ such that $\operatorname{in}(J_G) = \operatorname{in}(J_{G_1}) + \operatorname{in}(J_{G_2})$. Let $\tilde{G}_s$ denote the corresponding bipartite graph, as in Definition 4.18. If $K_{m,n}$ is the largest induced complete bipartite subgraph of $\tilde{G}_s$. Then we have:*

$$\beta_{i,j}(\operatorname{in}(J_G)) = \beta_{i,j}(\operatorname{in}(J_{G_1})) + \beta_{i,j}(\operatorname{in}(J_{G_2})) + \beta_{i-1,j}(\operatorname{in}(J_{G_1} \cap J_{G_2})) \tag{4.7}$$

for all (i, j) with $i \geq c'(s)$ or $j \geq i + 4$, where $c'(s) = m + n$. In other words, $\operatorname{in}(J_G) = \operatorname{in}(J_{G_1}) + \operatorname{in}(J_{G_2})$ is a $(c'(s), 4)$-Betti Splitting.

Proof. From the previous theorem, we know that $\beta_{k,k+3}(\operatorname{in}(J_{G_1} \cap J_{G_2})) = \beta_{k,k+2}(I(G'_s)) + \beta_{k,k+2}(I(\tilde{G}_s)) + \beta_{k-1,k+1}(I(G'_s))$. From Lemma 4.17 we have

$$\beta_{k,k+2}(I(\tilde{G}_s)) = \sum_{P \subseteq V(G), |P| = k+2} \#\operatorname{comp}(\tilde{G}_s[P]^c) - 1$$

where $\#\operatorname{comp}(\tilde{G}_s[P]^c)$ is the number of connected components of the complement of $\tilde{G}_s[P]$, the induced subgraph of $\tilde{G}_s$ the vertices in P.

Now, consider $k = c'(s) - 1$. Hence, for all P, with $|P| \geq c'(s) + 1$, we can see that P must contain vertices from the independent sets of $\tilde{G}_s$, A and B. Let P_A and P_B be the corresponding sets of vertices from A and B respectively. Since the largest complete bipartite subgraph of $\tilde{G}_s$ has $c'(s)$ vertices, we know that $\tilde{G}_s[P]^c$ must have at least one edge from P_A to P_B. Since P_A and P_B are both subsets of independent sets in $\tilde{G}_s$, G_{P_A} and G_{P_B} are both complete graphs. This implies that $\#\operatorname{comp}(\tilde{G}_s^c) - 1 = 0$. Therefore, this means that $\beta_{k,k+3}(J_{G_1} \cap J_{G_2}) = 0$ for all $k \geq c(s) - 1$.

Consider the ideals $\operatorname{in}(J_G) = \operatorname{in}(J_{G_1}) + \operatorname{in}(J_{G_2})$, with multigrading $\deg x_i = \deg y_i = i^{th}$ unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$. Therefore, since all generators of $\operatorname{in}(J_{G_1})$ are divisible by x_s or y_s, the generators of $\operatorname{in}(J_{G_1} \cap J_{G_2})$, are also divisible by x_s or y_s and their multigraded Betti numbers occur within only multidegrees j, where j_s is non-zero. Since $\operatorname{in}(J_{G_2})$ contains no generators divisible by x_s or y_s, $\beta_{i,j}(\operatorname{in}(J_{G_1} \cap J_{G_2})) > 0$ implies that $\beta_{i,j}(\operatorname{in}(J_{G_2})) = 0$ for all $i \in \mathbb{N}$ and multidegrees j as defined above.

From Lemma 4.15, the regularity of $\operatorname{in}(J_{G_1})$ is 3. Hence, we can see that for all multidegrees $j = (j_1, \ldots, j_n)$ with $\sum_k j_k \geq i + 4$, we have:

1. $\beta_{i,j}(\operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2})) > 0$ implies that $\beta_{i,j}(J_{G_1}) = 0$, and
2. $\beta_{i-1,j}(\operatorname{in}(J_{G_1}) \cap \operatorname{in}(J_{G_2})) > 0$ implies that $\beta_{i-1,j}(\operatorname{in}(J_{G_1})) = 0$.

Since the minimal degree of the generators of $\operatorname{in}(J_{G_1} \cap J_{G_2})$ is 3, and $\beta_{k,k+3}(\operatorname{in}(J_{G_1} \cap J_{G_2})) = 0$ for all $k \geq c'(s) - 1$, we also have that $\beta_{i,j}(\operatorname{in}(J_{G_1} \cap J_{G_2})) > 0$ implies that $\beta_{i,j}(\operatorname{in}(J_{G_1})) = 0$ for all $i \geq c'(s) - 1$ and multidegrees j.

Therefore, from Theorem 4.2, we have

$$\beta_{i,j}(\operatorname{in}(J_G) = \beta_{i,j}(\operatorname{in}(J_{G_1}) + \beta_{i,j}(\operatorname{in}(J_{G_2}) + \beta_{i-1,j}(\operatorname{in}(J_{G_1} \cap J_{G_2})),$$

for all i and multidegrees j with $i \geq c'(s)$ or $\sum_{k=1}^{n} j_k \geq i + 4$. Thus, the result holds for $\mathbb{N}^n$ multidegrees j. Since it is true for $\mathbb{N}^n$ multidegrees, we can combine them to obtain the same result in the standard grading. $\square$

Chapter 5

Bounds on Homological Invariants

In the previous sections, we studied various complete and partial Betti splittings for binomial edge ideals. In this chapter, we explore a different topic of bounds for various homological invariants of binomial edge ideals. In particular, we shall give a bound on the maximum possible total degree of any Betti number of the binomial edge ideal of any graph. Using this result, we shall partially recover many known results on bounds for the regularity and projective dimension of the binomial edge ideals of different types of graphs.

Theorem 5.1. *Let G be a simple graph on n vertices and let f be the number of free vertices in G. Then*

$$\max\{j \mid \beta_{i,j}(J_G) \neq 0\} \leq 2n - f.$$

Proof. Let $>$ be the monomial order with $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Consider $\text{in}(J_G)$ with its reduced Grobner basis as the generating set $\mathfrak{G}(\text{in}(J_G))$. From Theorem 2.112, we know that $\mathfrak{G}(\text{in}(J_G)) = \{u_\pi x_i y_j \mid \text{where } \pi \text{ is an admissible path with endpoints } i < j\}$.

Now, consider the Taylor resolution on the set of monomials $\mathfrak{G}(\text{in}(J_G))$, as defined in Section 2.3.2, with the $\mathbb{N}^{2n}$ multidegree, defined as $\deg x_i = i^{th}$ unit vector $= (0, \ldots, 1, \ldots, 0)$ and $\deg y_i = (n+i)^{th}$ unit vector. Hence, we have that T_i is the free modules generated with basis $\{[F] \mid F \subseteq \mathfrak{G}(\text{in}(J_G)), |F| = i\}$ and map $\phi_i : T_i \to T_{i-1}$ such that:

$$\phi_i(F) = \sum_{G \subset F, |G| = |F|-1} \epsilon_G^F \frac{\text{lcm}(F)}{\text{lcm}(G)}[G].$$

with ϵ_G^F as defined in Construction 2.91.

From the Construction 2.91, we can see that the final term in the Taylor resolution is the free module generated by one element, $S[\mathfrak{G}(\text{in}(J_G))]$. Hence, its multidegree will be

$\deg_{\mathbb{N}^{2n}}(\mathrm{lcm}(\mathfrak{G}(\mathrm{in}(J_G))))$. Since $\mathrm{in}(J_G)$ is a square-free ideal in $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, we know that the maximum possible $\mathbb{N}^{2n}$ multidegree will be $(1, \ldots, 1, \ldots, 1)$. Since this is a refinement of the $\mathbb{N}^n$ multidegree defined by $\deg x_i = \deg y_i = (0, \ldots, 1, \ldots, 0)$, this implies that the maximum possible $\mathbb{N}^n$ multidegree will be $(2, \ldots, 2)$.

Fix a free vertex v in G. We shall now consider the Lyubeznik resolution as defined in Section 2.3.2, on the set of monomials $\mathfrak{G}(\mathrm{in}(J_G))$. To study this resolution, we need a total ordering on $\mathfrak{G}(\mathrm{in}(J_G))$. Represent the elements of $\mathfrak{G}(\mathrm{in}(J_G))$ with $\{m_1, \ldots, m_k\}$. Consider any total ordering which satisfies the following property:

If $x_v \mid m_i$ or $y_v \mid m_i$, and $x_v \nmid m_j$ and $y_v \nmid m_j$, then $m_j < m_i$.

In simpler terms, we want a total order where any monomial containing x_v or y_v is greater than a monomial that does not contain either of them. An example of such an ordering is the lexicographic ordering, starting with the v^{th} unit vector.

Since every face F in the Lyubeznik simplicial complex is rooted (refer to Section 2.3.2), we know that for all $E \subset F$, $\min(E) \in E$. In other words, the smallest monomial according to the total ordering in $\mathfrak{G}(\mathrm{in}(J_G))$ which divides $\mathrm{lcm}(E)$ is in E for all $E \subset F$.

Claim. *If F is a subset of $\mathfrak{G}(\mathrm{in}(\mathrm{J_G}))$, with $m_i, m_j \in F$ and $i \neq j$ such that $x_v \mid m_i$ and $y_v \mid m_j$, then F is not rooted.*

Proof. Consider $E = \{m_i, m_j\} \subset F$ such that $x_v \mid m_i$ and $y_v \mid m_j$. Since m_i and m_j are in $\mathfrak{G}(\mathrm{in}(J_G))$, they are of the form $u_{\pi_1} x_i y_j$ and $u_{\pi_2} x_k y_l$, where $\pi_1 : i = l_0, l_1, \ldots, l_s = j$ and $\pi_2 : k = k_0, \ldots, k_r = l$ are admissible paths with endpoints $i < j$ and $k < l$ respectively. Since $x_v \mid m_i$ and $y_v \mid m_j$, this implies that v must be a vertex in both π_1 and π_2.

Since v is free, all its neighbours have edges between them. By the definition of admissible, we know that no subset of vertices from π_1 or π_2 form a path. If u_k, v, u_{k+1} were the neighbours of v in the path π_1, then since v is free, there would be an edge between u_k and u_{k+1}, which would imply that π_1 is not admissible. The same argument can be made for π_2 as well. Hence, the only possibility of such an m_i and m_j is if v is an endpoint of π_1 and π_2. Therefore, we assume $m_i = u_{\pi_1} x_v y_j$ and $m_j = u_{\pi_2} x_k y_v$.

We know all vertices in π_2 are such that $k_i < k$ or $k_i > v$ and all vertices in π_1 are such that $l_i < v$ or $l_i > j$. Let l_q be the first vertex in π_1 such that $k < l_q < v$. Note that there is an edge from k_{r-1} to l_1 since v is free. Thus we have that $\pi' : k, \ldots, k_{r-1}, l_1, \ldots, l_q$ is a path in G. All vertices in π' are clearly either $< k$ or $> l_q$. Hence, π' is a walk on G which satisfies Property 2 of being an admissible path. Hence, this implies that we can take a subset of vertices $\{j_1, \ldots, j_t\}$ such that $\pi'' : k, j_1 \ldots, j_t, l_q$ is an admissible path.

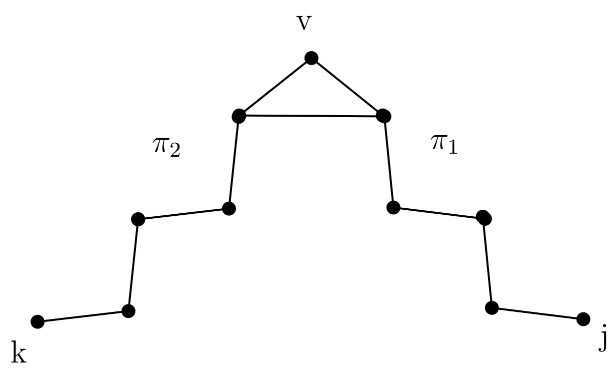


Figure 5.1: π_1 and π_2

Consider the monomial $u_{\pi''}x_k y_{l_q}$. Note that for any vertex $k_i \in \pi_1$, $k_i < k$ or $k_i > v$, and hence from the definition of l_q, $k_i > l_q$. Hence the monomial x_{k_i} or y_{k_i} associated to k_i in $u_{\pi''}$ is the same as in u_{π_1}. Similarly, all $l_1, \ldots, l_{j-1}$ are such that $l_i > j > i_q$ or $l_i < v$, which from the definition of l_q, implies that $l_i < k$. Hence the monomial x_{l_i} or y_{l_i} associated to l_i in $u_{\pi''}$ is the same as in u_{π_2}. It is also clear that $x_k \mid u_{\pi_2}x_k y_v, u_{\pi''}x_k y_{l_q}$ and $y_{l_q} \mid u_{\pi_1}x_k y_v, u_{\pi''}x_k y_{l_q}$. Hence, this implies that $u_{\pi''}x_k y_{l_q}$ divides $\mathrm{lcm}(m_1, m_2)$. Thus, from the total ordering, since π'' doesn't contain v, it is less than both $u_{\pi_1}x_v y_j$ and $u_{\pi_2}x_k y_v$, which implies that $\min(G) \notin G$.

In case there exists no such i_q, consider $\pi' : k, \ldots, k_{r-1}, i_1, \ldots, j$. Since none of the vertices in π_1 are in the interval (k, v), we know that all vertices in π_1 are either less than v or greater than j. Hence, π' satisfies Property 2 in the definition of an admissible path. Hence there exists a subset of vertices $\{j_1, \ldots, j_t\}$ such that $\pi'' : k, j_1 \ldots, j_t, j$ is an admissible path. As before, it can also be seen that $u_{\pi'}x_k y_j$ will divide $\mathrm{lcm}(u_{\pi_1}x_v y_j, u_{\pi_2}x_k y_v)$, from the definition of $u_{\pi'}$.

Thus, in either case, we will have $\min(G) \notin G$. Hence, F cannot be rooted. $\square$

From the above claim, no face in the Lyubeznik simplicial complex will contain monomials having both x_v and y_v for the free vertex v. Therefore, looking at the $\mathbb{N}^n$ multidegree, it is clear that $\mathrm{lcm}(F)$ when F is rooted has $\mathrm{mdeg}(\mathrm{lcm}(F))_v \leq 1$.

Since we chose any arbitrary free vertex in G, this can be used to give free resolutions with this property for any free vertex in G. Therefore, since the minimal free resolution $\mathbb{F}$ of $\mathrm{in}(J_G)$ has the property that $\mathrm{rank}(\mathbb{F}_i)_j \leq \mathrm{rank}(\mathbb{F}'_i)_j$, where $\mathbb{F}'$ is any free resolution of $\mathrm{in}(J_G)$, this means that for all $\mathbb{N}^n$ multidegrees $\mathbf{a}$ such that $\beta_{i,\mathbf{a}}(\mathrm{in(J_G)}) \neq 0$, we have that $\mathbf{a}_v \leq 1$, for all free vertices v in G. Therefore, from the Taylor resolution and the above arguement, we have that

$$\max\{j \mid \beta_{i,j}(\mathrm{in}(J_G)) \neq 0\} \leq \sum_{i\in[n], i \text{ is not free}} 2 + \sum_{i \text{ is free}} 1 = 2n - f.$$

Since $\beta_{i,j}(J_G) \leq \beta_{i,j}(\mathrm{in}(J_G))$, the maximum possible total degree j of J_G is less than that of $\mathrm{in}(J_G)$. Hence,

$$\max\{j \mid \beta_{i,j}(J_G) \neq 0\} \leq \max\{j \mid \beta_{i,j}(\mathrm{in}(J_G)) \neq 0\} \leq 2n - f.$$

□

Remark 5.2. The following theorem can be restated in a form more representing the regularity as follows:

$$\max\{i + r \mid \beta_{i,i+r}(J_G) \neq 0\} \leq 2n - f.$$

In different cases, this can lead to some nice bounds, as with the corollary below.

Corollary 5.3. *If J_G is such that it has only one extremal Betti number, then we have*

$$\mathrm{pd}(J_G) + \mathrm{reg}(J_G) \leq 2n - f.$$

Proof. In the case that J_G has only a single extremal Betti number, that means that the regularity is achieved at the projective dimension as well. In other words, $\beta_{\mathrm{pd},\mathrm{pd}+\mathrm{reg}}(J_G) \neq 0$. Therefore, from Theorem 5.1, we have that $\mathrm{pd}(J_G) + \mathrm{reg}(J_G) \leq 2n - f$. □

Using further bounds on the projective dimension and regularity respectively, we can use the above to obtain bounds on the other. One example is below

Corollary 5.4. *If G is a connected graph which is r-vertex connected (refer to Definition 2.15) and J_G has exactly one extremal Betti number, then*

$$\mathrm{reg}(J_G) \leq n - f - r + 3.$$

Proof. From Corollary 5.3 and Theorem 2.122 we have

$$n + r - 3 + \mathrm{reg}(J_G) \leq 2n - f.$$

Hence, $\mathrm{reg}(J_G) \leq n - f - r + 3$. □

This corollary can be used to partially obtain some well-known bounds on the regularity of block graphs.

Corollary 5.5. *If G is an indecomposable block graph which is J_G has exactly one extremal Betti number, then*

$$\operatorname{reg}(J_G) \leq n - f + 2.$$

Proof. If G is an indecomposable block graph and $G \neq K_n$, then we know that it must be $1-$vertex connected. Hence, from Corollary 5.4, if J_G has a single extremal Betti number, then $\operatorname{reg}(J_G) \leq n - f + 2$. □

Remark 5.6. The above corollary was proved in Theorem 8, [14] where they show that for any indecomposable block graph G, the ideal J_G has a single Betti number if and only if $\operatorname{reg}(J_G) = n - f + 2$.

We can also obtain other bounds on the regularity of block graphs.

Corollary 5.7. *Let G be an indecomposable block graph. Then,*

$$\operatorname{pd}(J_G) + \operatorname{reg}(J_G) \geq 2n - f.$$

Proof. From Theorem 2.124, we know that $\beta_{n-2,2n-f}(J_G) \neq 0$. Hence, from Theorem 5.1, this implies that

$$\max\{j \mid \beta_{i,j}(J_G) \neq 0\} = 2n - f.$$

Hence, $\operatorname{pd}(J_G) + \operatorname{reg}(J_G) \geq 2n - f$. □

Corollary 5.8. *Let G be an indecomposable block graph. Then,*

$$\operatorname{reg}(J_G) \geq \operatorname{diam}(G)$$

Proof. From Corollary 5.7 and Theorem 2.123,

$$\operatorname{reg}(J_G) + 2n - \operatorname{diam}(G) - f \geq 2n - f.$$

Thus, $\operatorname{reg}(J_G) \geq \operatorname{diam}(G)$ □

Remark 5.9. The above lower bound for the regularity in block graphs is weaker than the bound given in Theorem 8, [14]. But this bound will apply to other types of graphs which satisfy the condition that $\operatorname{pd}(J_G) + \operatorname{reg}(J_G) \geq 2n - f$.

Chapter 6

Future Directions/Conjectures

There are many directions in which one can proceed while further studying the Betti numbers of binomial edge ideals. We shall now list a few ideas and conjectures we had during this project.

1. One of the main results of this thesis is applying Theorem 3.13 to obtain the second Betti number of any tree. On studying the formula of the Betti numbers, we can see some interesting patterns. For example, while checking the second Betti number of trees, we can see that it depends on the term $P(T)$, which is the number of a particular type of induced subgraph present in T. Furthermore, we can see that the term $2\sum_{v_i}\binom{\deg v_i}{4}$ is similar to a term from the second Betti number of the induced star graph on each vertex. Similarly, from Theorem 3.1, [15], we know that the formula for the first Betti number of any tree is given by:

   Theorem 6.1. *Let G be a tree with $V(G) = [n]$. Then,*

   $$\beta_1(J_G) = \beta_2(S/J_G) = \beta_{2,4}(S/J_G) = \binom{n-1}{2} + \sum_{v \in V(G)} \binom{\deg v}{3}.$$

   Again, here we can see that the term $\binom{\deg v}{3}$ for each v is nothing but the first Betti number of the induced star graph centred at each vertex.

   Hence, it might be reasonable to think that higher Betti numbers depend upon certain induced subgraphs in the tree T. Furthermore, since we can see that $P(T)$ has a larger diameter than any induced star graph, it might be true that the higher Betti numbers depend upon the induced subgraphs of larger diameter. Going forward, understanding what types of induced subgraphs show up in higher Betti numbers and in what form, might be an interesting endeavour.

2. Another important result we have proved in this thesis, is Theorem 4.9, which tells us that splitting off the induced graph on a vertex v is a $(c(v), 4)$-Betti splitting. This idea of a partial Betti splitting has been first introduced in this thesis. Another way of framing this result is to say that the mapping cone of I obtained from the exact sequence

$$0 \longrightarrow J \cap K \xrightarrow{\phi} J \oplus K \xrightarrow{\psi} J + K = I \longrightarrow 0,$$

agrees with the minimal free resolution of I for all F_i with $i \geq c(v)$. Hence, this same property can be investigated for other exact sequences as well. One line of inquiry could be to study other types of ideals and try to understand if there could be a partial Betti splitting in those cases as well. In particular, the class of monomial ideals could be a possible source of interesting results. An example of this is the partial Betti splitting proved in Theorem 4.21 for the initial ideals of certain binomial edge ideals.

One can also study partial splittings for other exact sequences of binomial edge ideals. One such important sequence comes from Lemma 4.8, [23]. This exact sequence has been used several times in different contexts and slightly different forms (See also Theorem 1.1, [6] and Theorem 1, [27].) Let i be some vertex in $V(G)$, that is not free. We have $J_G = Q_1 \cap Q_2$ where $Q_1 = J_{G\setminus v} + \langle x_v, y_v\rangle$ and $Q_2 = J_{G_v}$. Clearly, $Q_1 + Q_2 = J_{G_v\setminus v} + \langle x_v, y_v\rangle$ Consider the exact sequence

$$0 \longrightarrow S/J_G \longrightarrow S/Q_1 \oplus S/Q_2 \longrightarrow S/(Q_1 + Q_2) \longrightarrow 0. \tag{6.1}$$

One can study if the mapping cone of this exact sequence agrees with the minimal free resolution of J_G and from what i this happens. This idea could be helpful for inductive arguments and can be used to tackle some conjectures on homological invariants such as the regularity of block graphs.

3. The technique of using partial Betti splittings may also apply to many other well-known problems. One such problem is on the extremal Betti numbers of binomial edge ideals. In the past, Herzog has conjectured the following (Introduction, [14]):

Conjecture 6.2. *If the initial ideal of a graded ideal $I \subset S$ is a square-free monomial ideal, then the extremal Betti numbers of I and $\mathrm{in}_>(I)$ coincide in their positions and values.*

This has been proved for toric rings by Strumfels in [29]. Since binomial edge ideals also have square-free initial ideals, the conjecture should still apply. Partial Betti splittings could be a useful tool to study the case where the extremal Betti numbers occur after the point where the partial splittings begin.

Another important problem is the subadditivity problem for binomial edge ideals.

Definition 6.3. Consider a graph G and its binomial edge ideal J_G. We define

$$t_i(J_G) = \sup\{j \mid \beta_{i,j}(J_G) \neq 0\}.$$

In other words, t_i is the i^{th} maximal graded shift of the minimal free resolution of J_G.

The subadditivity problem is defined as follows:

Problem 6.4. *Is it true that for any binomial edge ideal that*

$$t_a(S/J_G) + t_b(S/J_G) \geq t_{a+b}(S/J_G) \text{ for all } a, b \geq 1.$$

For what kind of graphs can this property hold?

As with the previous conjecture, having a partial Betti splitting can give some insight into this question for $\beta_{i,j}(J_G)$, with $i \geq c(v)$, or $i + j \geq 4$.

4. Finally, the last important result we have proved in this thesis is Theorem 5.1. In particular, we have shown that $\max\{j \mid \beta_{i,j} > 0\} \leq 2n - f$. As we have seen, this bound is achieved in the case of block graphs. An interesting question is to investigate whether this bound is obtained for other types of graphs as well. To that end, we make the following conjecture:

Conjecture 6.5. *Let G be a chordal graph, with binomial edge ideal J_G. Then we have*

$$\max\{j \mid \beta_{i,j}(J_G) \neq 0\} = 2n - f,$$

where f is the number of free vertices in G.

We expect the following method can be proved using an inductive argument, following a slightly modified version of the exact sequence, 6.1, but the details are non-trivial and need to be worked out.

Bibliography

[1] Arindam Banerjee and Luis Núñez-Betancourt, *Graph connectivity and binomial edge ideals*, Proceedings of the American Mathematical Society **145** (2016), no. 2, 487–499.

[2] Aldo Conca, *Gorenstein Ladder Determinantal Rings*, Journal of the London Mathematical Society **54** (1996), no. 3, 453–474.

[3] Reinhard Diestel, *Graph theory, 5th edition*, Graduate texts in mathematics, 2017.

[4] G. A. Dirac, *On rigid circuit graphs*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **25** (1961), 71–76.

[5] Shalom Eliahou and Michel Kervaire, *Minimal resolutions of some monomial ideals*, Journal of Algebra **129** (1990), no. 1, 1–25.

[6] Viviana Ene, Jürgen Herzog, and Takayuki Hibi, *Cohen-Macaulay binomial edge ideals*, Nagoya Mathematical Journal **204** (2011), 57–68.

[7] Giuseppe Favacchio, Johannes Hofscheier, Graham Keiper, and Adam Van Tuyl, *Splittings of toric ideals*, Journal of Algebra **574** (2021), 409–433.

[8] Christopher Francisco, Tai Hà, and Adam Van Tuyl, *Splittings of monomial ideals*, Proceedings of the American Mathematical Society **137** (2008), 3271–3282.

[9] Eloisa Grifo, *Lecture notes on homological algebra*, Fall 2023.

[10] Jürgen Herzog and Takayuki Hibi, *Monomial Ideals*, Springer London, London, 2011.

[11] Jürgen Herzog, Takayuki Hibi, Freyja Hreinsdóttir, Thomas Kahle, and Johannes Rauh, *Binomial edge ideals and conditional independence statements*, Advances in Applied Mathematics **45** (2010), 317–333.

[12] Jürgen Herzog, Takayuki Hibi, and Hidefumi Ohsugi, *Binomial Ideals*, Graduate Texts in Mathematics, vol. 279, Springer International Publishing, Cham, 2018.

[13] Jürgen Herzog, Dariush Kaini, and Sara Saeedi Madani, *The linear strand of determinantal facet ideals*, Michigan Mathematical Journal **66** (2017), no. 1, 107–123.

[14] Jürgen Herzog and Giancarlo Rinaldo, *On the Extremal Betti Numbers of Binomial Edge Ideals of Block Graphs*, The Electronic Journal of Combinatorics (2018), P1.63.

[15] A. V. Jayanthan, Arvind Kumar, and Rajib Sarkar, *Almost complete intersection binomial edge ideals and their Rees algebras*, Journal of Pure and Applied Algebra **225** (2021), no. 6, 106628.

[16] A. V. Jayanthan, N. Narayanan, and B. V. Raghavendra Rao, *Regularity of binomial edge ideals of certain block graphs*, Proceedings - Mathematical Sciences **129** (2019), no. 3, 36.

[17] Dariush Kiani and Sara Saeedi Madani, *The regularity of binomial edge ideals of graphs*, (2013), arXiv:1310.6126.

[18] Dariush Kiani and Sara Saeedi Madani, *Binomial Edge Ideals of Graphs*, The Electronic Journal of Combinatorics (2012), P44–P44.

[19] Kazunori Matsuda and Satoshi Murai, *Regularity bounds for binomial edge ideals*, Journal of Commutative Algebra **5** (2013), no. 1, 141–149.

[20] Jeff Mermin, *Three Simplicial Resolutions*, De Gruyter, April 2012, pp. 127–142.

[21] Rosa M. Miró-Roig, *Betti numbers of determinantal ideals*, Journal of Algebra **318** (2007), no. 2, 653–668.

[22] Fatemeh Mohammadi and Leila Sharifan, *Hilbert Function of Binomial Edge Ideals*, Communications in Algebra **42** (2014), no. 2, 688–703.

[23] Masahiro Ohtani, *Graphs and Ideals generated by some 2-minors*, November 2009.

[24] Irena Peeva, *Graded Syzygies*, Springer London, 2011.

[25] Mike Roth and Adam Van Tuyl, *On the Linear Strand of an Edge Ideal*, Communications in Algebra **35** (2007), no. 3, 821–832.

[26] Mohammad Rouzbahani Malayeri, Sara Saeedi Madani, and Dariush Kiani, *A proof for a conjecture on the regularity of binomial edge ideals*, Journal of Combinatorial Theory, Series A **180** (2021), 105432.

[27] Mohammad Rouzbahani Malayeri, Sara. Saeedi Madani, and Dariush. Kiani, *On the depth of binomial edge ideals of graphs*, Journal of Algebraic Combinatorics **55** (2022), no. 3, 827–846.

[28] Sara Saeedi Madani and Dariush Kiani, *Binomial edge ideals of regularity 3*, Journal of Algebra **515** (2018), 157–172.

[29] Bernd Sturmfels, *Polynomial equations and convex polytopes*, The American Mathematical Monthly **105** (1998), no. 10, 907–922.