An Algorithm to Resolve Dynamics in Minimal Outer Approximations

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Certificate

This is to certify that this dissertation entitled An Algorithm to Resolve Dynamics in Minimal Outer Approximations towards the partial fulfillment of the BS-MS dual degree program at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Arya Narnapatti at Rutgers University and Indian Institute of Science Education and Research under the supervision of Konstantin Mischaikow, Professor, Department of Mathematics , Rutgers University, during the academic year 2023-2024.

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This thesis is dedicated to Ace, a.k.a Aceu Kutty

Declaration

I hereby declare that the matter embodied in the report entitled An Algorithm to Resolve Dynamics in Minimal Outer Approximations are the results of the work carried out by me at the Department of Mathematics, Rutgers University and the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Konstantin Mischaikow and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

Arya Narnapatti , 20191020

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And lastly, but not leastly, Ace.

Abstract

Outer approximations present a way to conclude rigorous results about the dynamics of a continuous function $f : X \to X$ using combinatorial algorithms. In particular, information about the dynamics is captured by a lattice epimorphism ω from the lattice of forward invariant sets to the lattice of attractors associated with an outer approximation. Given a minimal outer approximation of a continuous function f, we explore the existence of a lift τ of ω. We show that this does not exist in general and introduce an algorithm *Resolve-OA* that aims to refine the minimal outer approximation to produce an outer approximation that preserves the information about the dynamics and for which a lift τ of ω exists. For simplicity, we focus on continuous functions from the unit cube $[0, 1]^d$ to itself. We introduce the notion of cubed complexes on the unit cube $[0, 1]^d$ and an operation of binary sub-division that allows us to refine the cubed complex. We present *Resolve-OA* in this context.

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Chapter 1

Introduction

Let X be a compact topological space. A continuous function $f : X \to X$ defines a discrete dynamical system on X, in which paths are given by repeatedly applying f, i.e., an initial point x_0 would in forward time follow the path $\{x_0, f(x_0), f^2(x_0), ...\}$. The broad goal is to understand and describe the dynamics of this system.

We are motivated by Conley's theory. Often in applications - for example, in population models used in ecology - the function f is not derived from first principles and may not completely capture the nonlinearities of the dynamics being modeled. In some cases, f could even be coming from black-box computation code [9]. The system could also be dependent on a parameter whose value is not known exactly. Conley called such systems 'rough equations' and noted that if such models were to be of use, they should be studied in 'rough terms' [2]. Studying solutions in terms of exact paths or even in terms of chain recurrence is not robust to small changes in the function or parameter and may not give an accurate description of the true dynamics. We instead look to make conclusions about the dynamics that are robust to perturbations.

A set $N \subseteq X$ is an attracting neighborhood for f if

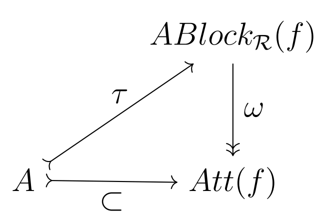
$$\omega(N) = \bigcap_{n \in \mathbb{Z}^+} \mathrm{cl}\left(\bigcup_{k \geq n} f^k(N)\right) \subseteq \mathrm{int}(N)$$

and a set $A \subseteq X$ is an attractor if there is an attracting neighbourhood N such that $A = \omega(N)$. A special subset of attracting neighborhoods are attracting blocks. $B \subseteq X$ is an attracting block if $f(\mathrm{cl}(B)) \subseteq \mathrm{int}(B)$. The set of attractors $Att(f)$, attracting neighborhoods $ANbhd(f)$, and attracting blocks $ABlock(f)$ each have a bounded distributive lattice structure [8, 7]. ω, in fact, defines a bounded lattice epimorphism from both $ANbhd(f)$ and $ABlock(f)$ to $Att(f)$.

We view attractors as the basis for a description of the dynamics that is robust [2]. Conley's fundamental theorem of dynamical systems tells us that the dynamics of a continuous function can always be decomposed into a chain recurrent part and a gradient-like part [11]. The attractors, or equivalently Morse sets, capture the chain-recurrent dynamics. Conley introduced Morse decompositions [2], a poset of invariant sets that captures the chain-recurrent dynamics, as well as some information about the gradient-like dynamics between the invariant sets in its partial order. Morse decompositions can be computed from the lattice of attractors. Attractors also capture the asymptotically observable dynamics of the system. Furthermore, there are at most only countably infinite attractors [8].

Attractors, however, are, in general, not directly computable. Attracting blocks are often readily computable, but can be uncountable in number [7]. Furthermore, attractors themselves are not robust to perturbations in f - simple bifurcation theory gives us numerous examples. Attracting blocks, however, are robust to small perturbations in f [4]. This makes the following theorem important.

Theorem 1.0.1. *[7](Theorem 1.2) For every finite sub-lattice $A \subseteq Att(f)$, there exists a bounded lattice monomorphism τ such that following diagram commutes:*



where $ABlock_{\mathcal{R}}(f)$ is the lattice of attracting blocks that are closed, regular sets.

This theorem guarantees that we can always view a finite resolution of the lattice of attractors through an index lattice of closed, regular attracting blocks.

In applications, when working with finite data or using numerical computation, one works not with the continuous function $f : X \to X$ but with a discretization of f [1]. Combinatorial dynamics presents a way to formalize this and derive rigorous results about the true dynamics of f [8]. X is discretized in general via a cell (usually CW) complex $\mathcal{X}$ such that $|\mathcal{X}| = X$, and f is approximated by a multi-valued map on the top-dimensional cells of $\mathcal{X}$ - notated by $\mathcal{F} : \mathcal{X}^{top} \rightrightarrows \mathcal{X}^{top}$. In particular, $\mathcal{F}$ is called an outer approximation of f if $f(|\mathcal{U}|) \subset \operatorname{int}(|\mathcal{F}(\mathcal{U})|)$ for all $\mathcal{U} \in \mathcal{X}^{top}$.

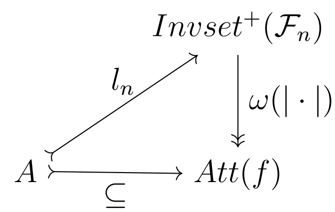
We define an attractor to be a subset of top-dimensional cells $\mathcal{A} \subseteq \mathcal{X}^{top}$ such that $\mathcal{F}(\mathcal{A}) = \mathcal{A}$ and forward invariant sets to be $\mathcal{N} \subseteq \mathcal{X}^{top}$ such that $\mathcal{F}(\mathcal{N}) \subseteq \mathcal{N}$. The set of attractors $Att(\mathcal{F})$ and forward-invariant sets $Invset^+(\mathcal{F})$ form bounded distributive lattices and $\omega : Invset^+(\mathcal{F}) \to Att(\mathcal{F})$ given by

$$\omega(\mathcal{N}) = \bigcap_{n\in\mathbb{N}} \left(\bigcup_{k\geq n} \mathcal{F}^k(\mathcal{N}) \right)$$

is a bounded lattice epimorphism - giving us a situation analogous to dynamics of a continuous f. In fact $|\mathcal{N}|$ for any forward invariant set $\mathcal{N} \in Invset^+(\mathcal{F})$ is an attracting block of f and we have the following commutative diagram [7]:

$$\begin{array}{ccc} Invset^+(\mathcal{F}) & \xrightarrow{\;|\cdot|\;} & ABlock_{\mathcal{R}}(f) \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ Att(\mathcal{F}) & \xrightarrow[\omega(|\cdot|)]{} & Att(f) \end{array}$$

In [8], the authors also define the notion of a convergent co-filtration of outer approximations $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ on a contracting co-filtration of discretizations $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ and show that every finite sub-lattice $A \subseteq Att(f)$ can be realized in $Invset^+(\mathcal{F}_n)$ for some $n_A \in \mathbb{N}$ i.e. for all $n \geq n_A$, the following diagram of bounded lattice homomorphism commutes (where l_n is called a lift of A):



$Att(\mathcal{F})$ also allows us to compute the Conley Index of invariant sets of the continuous function f being approximated [1]. The Conley Index is a topological index that provides us with information about the recurrent dynamics of the invariant set [10].

Since every attractor of $\mathcal{F}$ is a forward invariant set, $Att(\mathcal{F}) \subseteq Invset^+(\mathcal{F})$ as sets but $Att(\mathcal{F})$ is not a sub-lattice of $Invset^+(\mathcal{F})$ (see Figure 2.5). We are interested in a theorem analogous to Theorem 1.0.1 in the context of combinatorial dynamics. The lattice of attractors contains the required information to describe the global dynamics in terms of the recurrent and non-recurrent dynamics and, in general, can be much smaller than the lattice of forward invariant sets. However, unlike the lattice of forward invariant sets, it is not compatible with the topology of the cell complex $\mathcal{X}$, not allowing one to compute global information about the dynamics directly from it [6].

We want to find a lattice monomorphism $\tau : Att(\mathcal{F}) \to Invset^+(\mathcal{F})$ that acts as a lift of $\omega : Invset^+(\mathcal{F}) \to Att(\mathcal{F})$, i.e., $\omega \circ \tau$ is the identity of $Att(\mathcal{F})$. This allows us to embed $Att(\mathcal{F})$ into $Invset^+(\mathcal{F})$ while preserving the topological information.

The definition of an outer approximation is quite general; hence, it is easy to come up with examples of outer approximations $\mathcal{F}$ such that there is no lift of ω. Instead, we focus on the best outer approximation given a discretization $\mathcal{X}$, which we call the minimal outer approximation. We can ask the question if given a minimal outer approximation $\mathcal{F} : \mathcal{X}^{top} \rightrightarrows \mathcal{X}^{top}$ of a continuous $f : X \to X$, there always exists a lift of ω. However, it turns out this is not true (see Section 3.2).

A multi-valued map $\mathcal{F} : \mathcal{X}^{top} \rightrightarrows \mathcal{X}^{top}$ can also be viewed as a directed graph with vertex set $\mathcal{X}^{top}$ and edge set $\{(u, v) \mid v \in \mathcal{F}(u)\}$. We denote the condensation graph (graph of strongly connected components) of $\mathcal{F}$ by $SC(\mathcal{F})$. Being a directed acyclic graph, $SC(\mathcal{F})$ can be viewed as a poset. We call a strongly connected component $\mathcal{U} \in SC(\mathcal{F})$ a recurrent component or a Morse set if it has at least one edge. The Morse graph $M(\mathcal{F})$ is the sub-poset of recurrent components of $SC(\mathcal{F})$. Let $i : M(\mathcal{F}) \to SC(\mathcal{F})$ be the inclusion map.

Via Birkhoff's representation theorem, in [7], the authors show that the existence of a lift of ω is equivalent to the existence of a surjective poset morphism $\sigma : SC(\mathcal{F}) \to M(\mathcal{F})$ such that $\sigma \circ i$ is the identity on $M(\mathcal{F})$. Such a σ is called an order retraction of i. They introduce an algorithm that determines if an order retraction to i exists.

In this thesis, we introduce cubed complexes and show that they give a CW decomposition of the unit cube $[0,1]^d$ in $\mathbb{R}^d$. We also introduce a binary subdivision operation that allows us to refine a cubed complex. We present outer approximations in the context of cubed complexes and continuous functions $f : [0,1]^d \to [0,1]^d$ and define in this context, the notion of a refinement of an outer approximation. We provide a counter-example to the claim that there always exists a lift of $\omega : \mathit{Invset}^+(\mathcal{F}) \to \mathit{Att}(\mathcal{F})$ when $\mathcal{F}$ is a minimal outer approximation of some continuous f. We then present an algorithm *Resolve-OA* that takes as input a minimal outer approximation $\mathcal{F}$ of a continuous f on a cubed complex $\mathcal{C}$ and outputs a refinement $\mathcal{F}'$ of $\mathcal{F}$ which is an outer approximation of f on a refinement $\mathcal{C}'$ of $\mathcal{C}$. We conjecture that the output $\mathcal{F}'$ is such that $M(\mathcal{F}') \cong M(\mathcal{F})$ and $\mathcal{F}'$ has the property that an order retraction σ of $i : M(\mathcal{F}') \to SC(\mathcal{F}')$ exists. Hence equivalently, a lift τ of $\omega : \mathit{Invset}^+(\mathcal{F}') \to \mathit{Att}(\mathcal{F}')$ exists. If the conjecture is true, the output of *Resolve-OA* also allows the easy construction of σ and, hence, τ. It is important to note that the algorithm *Resolve-OA* assumes that for any cube c of a refinement of the input cubed complex $\mathcal{C}$, we can compute the value of the minimal outer approximation of f on c.

Chapter 2

Preliminaries

Notation

- $\mathbb{N} = \{0, 1, 2, 3...\}$.
- $\mathbb{Z}^+ = \{1, 2, 3...\}$.
- $[2^d] = \{a_1a_2...a_d \mid a_i \in \{0, 1\}\}$ - the set of binary strings of length d.
- $\mathbb{D} = \{\frac{p}{q} \in \mathbb{Q} \mid q = 2^n \text{ for an } n \in \mathbb{N}\}$ - the set of Dyadic rationals, i.e., numbers that have a terminating binary representation.

2.1 Posets and Lattices

2.1.1 Posets

Definition 2.1.1. A *partially ordered set* or a *poset* $(P, \leq_P)$ is a set P with a relation $\leq_P$ such that:

(i) (reflexivity) for all $p \in P$, $p \leq_P p$,
(ii) (anti-symmetry) for all $p, q \in P$, $p \leq_P q$ and $q \leq_P p \implies p = q$,
(iii) (transitivity) for all $p, q, r \in P$, $p \leq_P q$ and $q \leq_P r \implies p \leq_P r$.

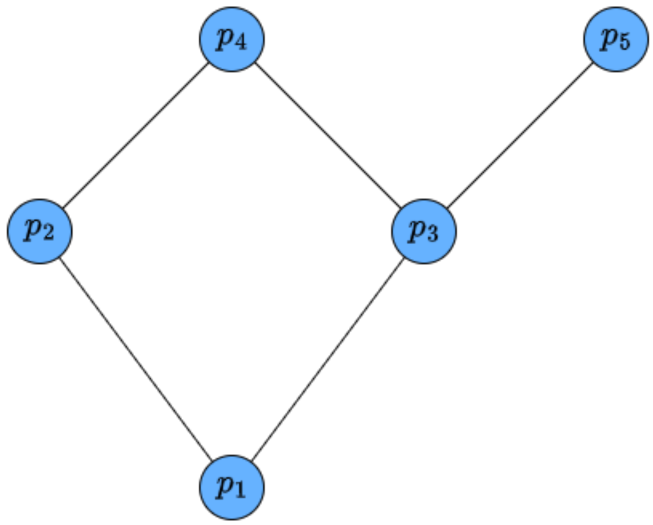


Figure 2.1: ***Example of a Hasse diagram.*** *A diagram of a poset P with partial order $p_1 \leq_P p_2$, $p_1 \leq_P p_3$, $p_1 \leq_P p_4$, $p_1 \leq_P p_5$, $p_2 \leq_P p_4$, $p_3 \leq_P p_4$, $p_3 \leq_P p_5$.*

We also use the notation:

(i) $p \geq_P q$ for: $q \leq_P p$.
(ii) $p <_P q$ or $q >_P p$ for: $p \leq_P q$ but $p \neq q$.

We represent posets using Hasse diagrams [3]. Figure 2.1 shows an example.

Definition 2.1.2. Given two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ an *order-preserving map* is a function $h : P \to Q$ such that for all $p_1, p_2 \in P$, $p_1 \leq_P p_2 \implies h(p_1) \leq_Q h(p_2)$.

Definition 2.1.3. An order-preserving map $h : P \to Q$ is:

(i) an *order injection* if h is injective,
(ii) an *order surjection* if h is surjective,
(iii) an *order embedding* if for all $p_1, p_2 \in P$, $p_1 \leq_P p_2 \iff h(p_1) \leq_Q h(p_2)$,
(iv) an *isomorphism* if it is a surjective order embedding. In that case, we say $P \cong Q$.

Remark 2.1.4. Every order embedding is an order injection.

Definition 2.1.5. A *sub-poset* of $(P, \leq_P)$ is given by a subset $I \subset P$ and the relation $\leq_I$ being just $\leq_P$ restricted to I.

Remark 2.1.6. For an order embedding $h : Q \to P$, $Q \cong h(Q)$, where $h(Q)$ is viewed as a sub-poset of P. Thus, for simplicity, we may denote the element $h(q) \in P$ by $p \in Q$ itself.

Definition 2.1.7. Consider a poset $(P, \leq_P)$:

(i) A *chain* of P is an $I \subseteq P$ that is totally ordered i.e., for any two $p, q \in I$, either $p \leq_P q$ or $q \leq_P p$.
(ii) An *anti-chain* of P is an $I \subseteq P$ such that for any two $p, q \in I$, $p \nleq_P q$ and $q \nleq_P p$.
(iii) A *down set* of P is an $I \subseteq P$ such that $p \in I$ and $q \leq_P p \implies q \in I$.
(iv) An *up set* of P is an $I \subseteq P$ such that $p \in I$ and $q \geq_P p \implies q \in I$.

Remark 2.1.8. I is a down set $\iff I^C = P \setminus I$ is an up set.

Let $O(P) = \{I \subseteq P \mid I \text{ is a down set}\}$ be the set of down sets of P.

Given a $p \in P$ we denote the *down set of* p by $\downarrow p = \{q \in P \mid q \leq p\}$ and the *up set of* p by $\uparrow p = \{q \in P \mid q \geq p\}$.

Given a $I \subseteq P$, a *maximal* element of I is a $p \in I$ such that $q \geq p$ and $q \in I$ implies $p = q$. Similarly, a *minimal* element of I is a $p \in I$ such that $q \leq p$ and $q \in I$ implies $p = q$.

For $p, q \in P$, we say p and q are *incomparable* if $p \nleq q$ and $q \nleq p$.

Remark 2.1.9. For a directed acyclic graph $G = (V, E)$, the relation $a \leq_G b$ if there exists a path from b to a in G, defines a poset structure on V.

2.1.2 Lattices

We introduce lattices as algebraic structures and show how they can viewed as posets. There is an equivalent way of introducing lattices as posets. For a more complete treatment, the interested reader may look at [3].

Definition 2.1.10. A *lattice* $(L, \vee, \wedge)$ is a set L along with two associative, commutative binary operations such that for all $a, b \in L$:

(i) $a \vee (a \wedge b) = a$,
(ii) $a \wedge (a \vee b) = a$.

$\vee$ is called the *join* operation and $\wedge$ is called the *meet* operation.

Remark 2.1.11. For any lattice $(L, \vee, \wedge)$, $\vee$ and $\wedge$ are idempotent, i.e. for all $a \in L$, $a \vee a = a$ and $a \wedge a = a$.

Lemma 2.1.12. *For all $a, b \in L$, $a \wedge b = a \iff a \vee b = b$.*

Proof. Assume $a \wedge b = a$, then $a \vee b = (a \wedge b) \vee b = b$. The other direction follows by switching the join and meet operations. □

Lemma 2.1.12 ensures that we can define a partial order $\leq_L$ on L:

$$a \leq_L b \iff a \wedge b = a \iff a \vee b = b.$$

Definition 2.1.13. A lattice L is *distributive* if for all $a, b, c \in L$:

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Definition 2.1.14. A lattice L is *bounded* if there exists $0, 1 \in L$ such that for all $a \in L$:

1. $a \vee 0 = a$,
2. $b \wedge 1 = a$.

Definition 2.1.15. Given two lattices $(L, \vee, \wedge)$ and $(K, \vee, \wedge)$, a *lattice homomorphism* is a function $h : L \to K$ such that for all $a, b \in L$:

(i) $h(a \vee b) = h(a) \vee h(a)$,
(ii) $h(a \wedge b) = h(a) \wedge h(a)$.

Definition 2.1.16. A lattice homomorphism $h : L \to K$ between two bounded lattices L, K is *bounded* if $h(0_L) = 0_K$ and $h(1_L) = 1_K$.

Definition 2.1.17. A lattice homomorphism $h : L \to K$ is:

(i) a *lattice monomorphism* if h is injective,
(ii) a *lattice epimorphism* if h is surjective,
(iii) a *lattice isomorphism* if h is injective and surjective.

Definition 2.1.18. $K \subset L$ is a *sub-lattice* of L if $a \in K$ and $b \in K$ implies $a \vee b, a \wedge b \in K$. If L is bounded, we also require that $0, 1 \in K$ for K to be a sub-lattice.

Definition 2.1.19. An $a \in L$ is a *join-irreducible* element if $a = b \vee c$ implies either $b = a$ or $c = a$.

We call $b \in L$ an *immediate predecessor* of $a \in L$ if $b <_L a$ and $b \leq_L b' <_L a \implies b = b'$.

Remark 2.1.20. $a \in L$ is a join-irreducible element if and only if a has a unique immediate predecessor.

Given a lattice L, we set $J(L) = \{a \in L \mid a \text{ is join-irreducible}\}$. $J(L)$ has a poset structure as a sub-poset of L.

We make some comments about finite lattices.

- For a finite lattice L, given a $I = \{a_1, ..., a_n\} \subseteq L$ we can define:

$$\bigvee I = a_1 \vee a_2 \vee ... \vee a_n,$$

$$\bigwedge I = a_1 \wedge a_2 \wedge ... \wedge a_n.$$

- We say I has a *maximum* element $max(I) = a$ if $a = \bigvee I$ and $a \in I$. Similarly, we say I has a *minimum* element $min(I) = a$ if $a = \bigwedge I$ and $a \in I$.

- Every finite lattice L is bounded with:

  (i) $0 = \bigwedge L = min(L)$,

  (ii) $1 = \bigvee L = max(L)$.

2.2 CW Complexes

Given two topological spaces X and Y and a continuous map $f : Y \supset A \to X$, by $X \cup_f Y$, we mean the space $X \sqcup Y/(x \sim y)$ if $f(y) = x$. We call this attaching Y to X via the gluing map f.

We call a topological space a k-*dimensional* cell or a k-*cell* if it is homeomorphic to $\mathcal{D}_k$ - the k-dimensional closed ball in $\mathbb{R}^k$.

We are only interested in CW complexes that are finite-dimensional and have a finite number of cells.

Definition 2.2.1. A topological space $\mathcal{X}$ is a *CW complex* if for a $n \in \mathbb{N}$ we have the following:

$$\emptyset = \mathcal{X}_{-1} \subseteq \mathcal{X}_0 \subseteq \mathcal{X}_1 \subseteq ... \subseteq \mathcal{X}_n = \mathcal{X}$$

where each $\mathcal{X}_k$ for $0 \leq k \leq n$ is obtained by attaching a finite number of k-cells to $\mathcal{X}_{k-1}$ via gluing maps in the sense: $\mathcal{X}_k = \mathcal{X}_{k-1} \cup_{\delta_1^k} e_1^k \cup_{\delta_2^k} e_2^k ... \cup_{\delta_l^k} e_l^k$ for k cells e_i^k and gluing maps $\delta_i^k : \delta e_i^k \rightarrow \mathcal{X}_k$.

The dimension of $\mathcal{X}$ as a CW complex is given by n.

2.3 Cubed Complexes

For our purpose, given a $d \in \mathbb{Z}^+$, a (binary) *cube* is a $c = (\mathbf{p}, \mathbf{b}, n)$ with $\mathbf{p} = (p_1, p_2, ..., p_d) \in \mathbb{D}^d$, $\mathbf{b} = (b_1, b_2, ...b_d) \in \{0,1\}^d$ and $n \in \mathbb{N}$ such that $\mathbf{b} = \{0, 0, ..., 0\}$ implies $n = 0$. Its geometric realization is given by the product of intervals $|c| = \prod_{j=1}^{d}[p_j, p_j + \frac{b_j}{2^k}] \subset \mathbb{R}^d$ - we also call this the product form of c.

- $\mathbf{p}$ identifies the position of the geometric realization of the cube in $\mathbb{R}^d$.
- The *dimension* of a cube is given by $dim(c) = \sum_{i=1}^{d} b_i = ||\mathbf{b}||_{L1}$, and we call cubes with dimension k k-cubes.
- n gives the size of the cube. We call $\frac{1}{2^n}$ the *side length* of the cube.
- We call d the *realization dimension* of c.

If $b_j = 1$, we call the interval $[p_j, p_j + \frac{b_j}{2^k}] = [p_j, p_j + \frac{1}{2^k}]$ *non-degenerate* and if $b_j = 0$, we call the interval $[p_j, p_j + \frac{b_j}{2^k}] = [p_j, p_j]$ *degenerate*. Given a non-degenerate interval $[p_j, p_j + \frac{1}{2^k}]$ we call the degenerate interval $[p_j, p_j]$ its *left-endpoint* and the degenerate interval $[p_j + \frac{1}{2^k}, p_j + \frac{1}{2^k}]$ its *right-endpoint*. Note that $dim(c)$ can also be interpreted as the number of non-degenerate intervals in its product form.

A cube c' is called a *face* of the cube c if the product form of c' can be obtained by replacing one or more non-degenerate intervals in the product form of c by one of their endpoints. Note that here the realization dimension of c and c' must be the same. If c' is a face of c

and c'' is a face of c', then c'' is also a face of c. Thus, the relation $c' \leq_C c$ if $c' = c$ or c' is a face of c is a partial order on the set of binary cubes. The *co-dimension* of a face c' of c is given by $codim(c', c) = dim(c) - dim(c')$ i.e., how many of the non-degenerate intervals in the product form of c must be replaced by an endpoint to obtain the product form of c'.

For example, for the cube $|c| = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$, the one dimensional faces are $[0, \frac{1}{2}] \times [0, 0]$, $[0, 0] \times [0, \frac{1}{2}]$, $[0, \frac{1}{2}] \times [\frac{1}{2}, \frac{1}{2}]$ and $[\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{2}]$; the zero dimensional faces are $[0, 0] \times [0, 0]$, $[0, 0] \times [\frac{1}{2}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{1}{2}] \times [0, 0]$, $[\frac{1}{2}, \frac{1}{2}] \times [\frac{1}{2}, \frac{1}{2}]$.

Remark 2.3.1. For any cube c, the boundary of its geometric realization $\partial|c|$ is equal to the union of the geometric realization of all its faces.

For $\mathbf{a} = (a_1, a_2, ..., a_d), \mathbf{b} = (b_1, b_2, ..., b_d) \in \{0, 1\}^d$, the relation $\mathbf{a} \leq_{\{0,1\}^d} \mathbf{b}$ if $a_i \leq b_i$ for all $1 \leq i \leq d$ defines a partial order in $\{0, 1\}^d$.

Remark 2.3.2. $c' = (\mathbf{p}', \mathbf{b}', n') \leq_C c = (\mathbf{p}, \mathbf{b}, n)$ if and only if there exist $\mathbf{x} \leq_{\{0,1\}^d} \mathbf{y} \leq_{\{0,1\}^d} \mathbf{b}$ such that:

(i) $n' = n$,
(ii) $\mathbf{p}' = \mathbf{p} + \frac{\mathbf{x}}{2^n}$,
(iii) $\mathbf{b}' = \mathbf{b} - \mathbf{y}$.

From this characterization we can also notice that the realization dimension of c and c' must be the same. The co-dimension is given by $codim(c', c) = ||\mathbf{y}||_{L1}$.

A *tree* $T = (V, E)$ is a connected, acyclic, undirected graph. A *rooted tree* (T, r) is a tree T with one of its nodes $r \in V$ designated as the root node. In a tree, there is a unique path from any node to any other. In particular, in a rooted tree, there is a unique path $\{r = v_0, v_1, ..., v_k = v\}$ from the root node r to any node $v \in V$. Here, v_{k-1} is a neighbor of v, and we call v_{k-1} the *parent* node of v. All other neighbors of v are called *children* of v. Note that u is a parent of v if and only if v is a child of u. For every edge $(u, v) \in E$ of a rooted tree - u is either the parent or a child of v. We set the following convention regarding edges (u, v) in rooted trees - u is the parent node of v. This also allows us to interpret a rooted tree as a directed acyclic graph and, hence, a poset.

Definition 2.3.3. Given a $n \in \mathbb{N}$, an *n-ary tree* is a rooted tree (T, r) such that every node has either n or 0 children.

Nodes with 0 children are called *leaves*, while the rest we call *branch nodes*. We denote the set of leaves by T_l and the set of branch nodes by T_b. Given a branch node v, we denote the set of its *children* by $ch(v)$. Given a non-root node v with parent u, we denote by $sib(v) = ch(u)$ the *siblings* of v.

Definition 2.3.4. Let $(\mathcal{T}^d, v_0)$ denote the complete, infinite 2^d-ary rooted tree which has a vertex set $V = \{v_i\}_{i \in \mathbb{N}} = \{v_0, v_1, v_2, ...\}$ and edge set $E_d = \{(v_i, v_{i2^d+j}) \mid i \in \mathbb{N}, 1 \leq j \leq 2^d\}$.

Each node v_i of $\mathcal{T}^d$ has 2^d children $ch(v_i) = \{v_{i2^d+1}, v_{i2^d+2}, ..., v_{i2^d+2^d} = v_{(i+1)2^d}\}$. Also, note that each node of $\mathcal{T}^d$ is a branch node - $\mathcal{T}^d_b = V$. In other words $\mathcal{T}^d$ has no leaf nodes - $\mathcal{T}^d_l = \emptyset$.

To each edge (v_i, v_{i2^d+j}) of $\mathcal{T}^d$ we can associate a binary string of length d - $a_1a_2..a_d \in [2^d]$ which is nothing but j written in binary. Figure 2.2 depicts $\mathcal{T}^2$ with the edges labeled by the associated binary strings.

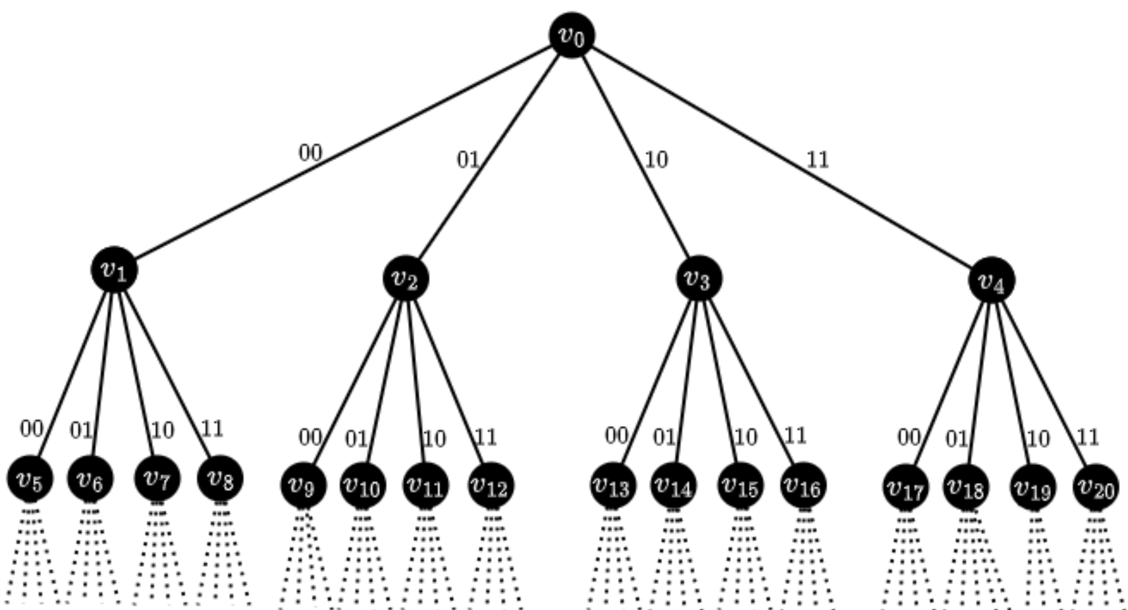


Figure 2.2: $\mathcal{T}^2$ - the complete, infinite 4-ary tree.

Note that for every $v_i \in V$, the labeling of the edges gives a bijection between the set of edges $\{(v_i, v) \mid v \in ch(v_i)\} = \{(v_i, v_{i2^d+j}) \mid 1 \leq j \leq 2^d\}$ and $[2^d]$. We can equivalently think of this as giving a bijection from $ch(v_i)$ to $[2^d]$ for each i.

We associate a cube to each node v_i of $\mathcal{T}^d$. To the root node v_0 we associate the d-dimensional unit cube $(\mathbf{0}, \mathbf{1}, 0)$ and say the geometric realization is $|v_0| = [0,1]^d$. For every other node v_i, there is a unique path from v_0 to v_i - say $\{v_0, v_{j_1}, v_{j_2}, ..., v_{j_k} = v_i\}$. This path is uniquely determined by the sequence of binary d-strings $\{a_{11}a_{12}...a_{1d},\ a_{21}a_{22}...a_{2d},\ ...,\ a_{k1}a_{k2}...a_{kd}\}$ where $a_{l1}a_{l2}...a_{ld}$ is the binary string associated to the edge $(v_{j_{l-1}}, v_{j_l})$. Define constants $z_j = \sum_{l=1}^{k} \frac{1}{2^l} a_{lj}$ and let $z = (z_1, z_2, ... z_d)$. We then label/associate each node v_i with a cube

$$v_i = (\mathbf{z}, \mathbf{1}, k)$$

with a geometric realization

$$|v_i| = \prod_{j=1}^{d} [z_j, z_j + \frac{1}{2^k}].$$

Proposition 2.3.5. *For any node v_i of $\mathcal{T}^d$, $|v_i| = |ch(v_i)|$.*

Proof. Let $x = (x_1, x_2, ..., x_d) \in |v_i| = \prod_{j=1}^{d}[z_j, z_j + \frac{1}{2^k}]$. For $1 \le j \le d$, if $x_j \le z_j + \frac{1}{2^{k+1}}$, set $a_j = 0$. Else, set $a_j = 1$. Then $x_j \in [z_j + \frac{a_j}{2^{k+1}}, c_j + \frac{a_j}{2^{k+1}} + \frac{1}{2^{k+1}}]$. But note that if $v_l \in ch(v_i)$ is such that the edge (v_i, v_l) is labeled by the binary string $a_1...a_d$, then $|v_l| = \prod_{j=1}^{d}[z_j + \frac{a_j}{2^{k+1}}, c_j + \frac{a_j}{2^{k+1}} + \frac{1}{2^{k+1}}]$ and $x \in |v_l|$. Hence $|v_i| \subseteq |ch(v_i)|$.

However for any $v_l \in ch(v_i)$, $|v_l| \subset |v_i|$ and hence $|v_i| \supseteq |ch(v_i)|$. □

Definition 2.3.6. A *cubed complex* on $[0,1]^d$ is given by a finite 2^d-ary rooted sub-tree $(T^{\mathcal{C}}, v_0)$ of $(\mathcal{T}^d, v_0)$, i.e, a sub-tree such that if v_i is a node of $T^{\mathcal{C}}$, then:

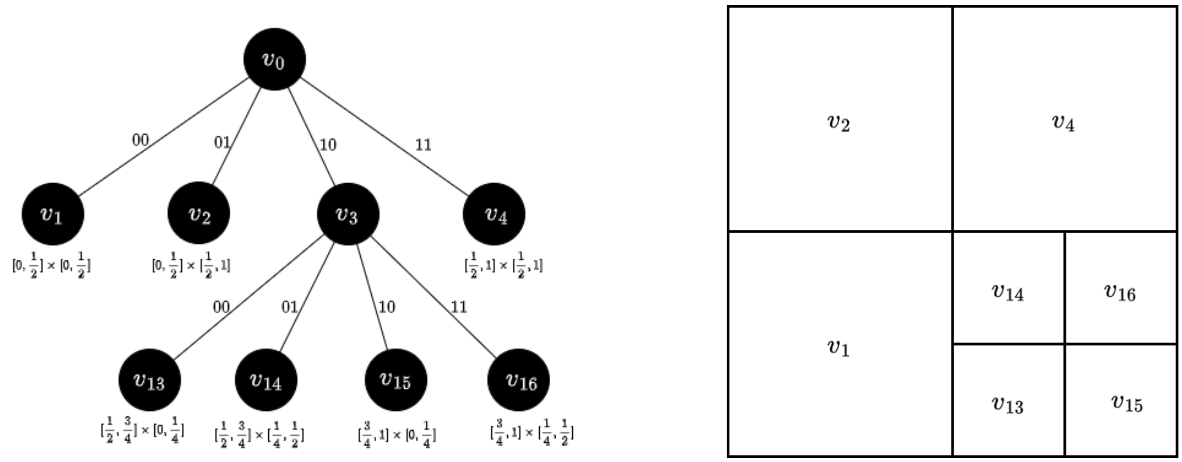
(i) the parent of v_i is node of $T^{\mathcal{C}}$,

(ii) for every $v_l \in sib(v_i)$, v_l is a node of $T^{\mathcal{C}}$.

Remark 2.3.7. Viewing $\mathcal{T}^d$ as a poset, condition (ii) of Definition 2.3.6 ensures that the vertices of any cubed complex are an up set of $\mathcal{T}^d$.

Remark 2.3.8. Condition (i) of Definition 2.3.6 ensures that for any branch node of a cubed complex $T^{\mathcal{C}}$, all its children are nodes of $T^{\mathcal{C}}$. Further, by Proposition 2.3.5, for any node v_i of $T^{\mathcal{C}}$, $|v_i| = |ch(v_i)|$. From these it follows that $|T^{\mathcal{C}}_l| = |v_0| = [0,1]^d$.

Thus, the geometric realizations of the leaves of $T^{\mathcal{C}}$ give a tiling of the unit cube $[0,1]^d$. We show in Proposition 2.3.9 that they give a CW complex structure to $[0,1]^d$. Figure



(a) A cubed complex $T^{\mathcal{C}}$ of $[0,1]^2$. *(b) The geometric realization of $T^{\mathcal{C}}$.*

Figure 2.3: ***An example of a cubed complex $T^{\mathcal{C}}$ of the unit square $[0,1]^2$.*** *(a) depicts the rooted tree data structure with the product form of the leaves given. (b) shows the geometric realization of the leaves i.e., the top-dimensional cells.*

2.3a depicts an example of a cubed complex $T^{\mathcal{C}}$ with Figure 2.3b showing the geometric realization of the leaf nodes that give a tiling of $[0,1]^d$.

We call a leaf $c \in T_l^{\mathcal{C}}$ a top-dimensional cube of $T^{\mathcal{C}}$ and use the notation $\mathcal{C}^{top} = T_l^{\mathcal{C}}$ to denote the set of top-dimensional cubes.

Let $\tilde{\mathcal{C}} = \{c \mid c \leq_C c', c' \in \mathcal{C}^{top}\}$ be the set of top-dimensional cubes of $T^{\mathcal{C}}$ and all their faces. However, certain faces are repeated in this collection. For example, in Figure 2.3b, the face of the top-dimensional cube v_1 given by the cube $|c| = [\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{2}]$ overlaps with the faces of v_{13} and v_{14} given by $|c'| = [\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{4}]$ and $|c''| = [\frac{1}{2}, \frac{1}{2}] \times [\frac{1}{4}, \frac{1}{2}]$ respectively. In fact $|c| = |c'| \cup |c''|$. In such a case, we would like to consider c' and c'' as cubes of $T^{\mathcal{C}}$, but not c.

We set $\mathcal{C} = \{c \in \tilde{\mathcal{C}} \mid \nexists\, c' \in \tilde{\mathcal{C}}\ s.t\ dim(c') = dim(c)\ and\ |c'| \subseteq |c|\}$ - and call it the set of cubes of $T^{\mathcal{C}}$.

And $\mathcal{C}_k = \{c \in \mathcal{C} \mid dim(c) \leq k\}$ - the set of cubes of k-dimensional cubes of $T^{\mathcal{C}}$.

Given $\mathcal{C}$, one can construct $T^{\mathcal{C}}$ as the up set $\bigcup_{c \in \mathcal{C}^{top}} \uparrow c$ of $\mathcal{T}^d$. Hence, $\mathcal{C}$ is an equivalent description of the cubed complex $T^{\mathcal{C}}$.

Proposition 2.3.9. *Let $T^{\mathcal{C}}$ be a cubed complex and let $\mathcal{C}_k$ denote the k-dimensional cubes* *of $T^{\mathcal{C}}$. Then,*

$$\emptyset = |\mathcal{C}_{-1}| \subseteq |\mathcal{C}_0| \subseteq |\mathcal{C}_1| \subseteq ... \subseteq |\mathcal{C}_n| = |\mathcal{C}|$$

gives $[0,1]^d$ a CW complex structure.

Proof. First we note that from Remark 2.3.8 $|\mathcal{C}_n| = |\mathcal{C}^{top}| = |r| = [0,1]^d$.

We also note that any k-cube is homeomorphic to $\mathcal{D}_k$ and hence is a k-cell.

$|\mathcal{C}_0|$ is nothing but the union of all 0-dimensional faces (vertex points) of $\mathcal{C}^{top}$.

For $k \geq 1$, if $c \in \mathcal{C}_k$, then by Remark 2.3.1 $\partial|c| = \underset{c' <_C c}{\cup} |c'| \subseteq |\mathcal{C}_{k-1}|$. □

We define a partial order on $\mathcal{C}$, which is nothing but the face partial order coming from the CW complex structure on $\mathcal{C}$:

$$c' \leq_{\mathcal{C}} c \text{ if } \exists\, c'' \leq_C c \text{ in } \tilde{\mathcal{C}} \text{ s.t } dim(c'') = dim(c') \text{ and } |c'| \subseteq |c''|.$$

□

We also set the following notation.

1. Given a $v \in \mathcal{C} \setminus \mathcal{C}^{top}$, $top(v) = \{c \in \mathcal{C}^{top} \mid v <_{\mathcal{C}} c\}$.
2. Given a $v \in T^{\mathcal{C}}$, $z(v) = \{v' \in T^{\mathcal{C}}_l = \mathcal{C}^{top} \mid v' \leq_{T^{\mathcal{C}}} v\}$. We call a $v' \in z(v)$ a descendent of v and v an ancestor of such a v'.

2.4 Binary Sub-division

Definition 2.4.1. A cubed complex $T^{\mathcal{C}'}$ is a refinement of a cubed complex $T^{\mathcal{C}}$ if the set of nodes of $T^{\mathcal{C}}$ is a subset of the set of nodes of $T^{\mathcal{C}'}$.

Equivalently, a cubed complex $\mathcal{C}'$ is a refinement of $\mathcal{C}$ if the geometric realization of every $c' \in \mathcal{C}'$ is contained in the geometric realization of a $c \in \mathcal{C}$.

We call $c' \in \mathcal{C}'$ a *sub-cube* of a $c \in \mathcal{C}$ if $c' \leq c$ in $T^{\mathcal{C}'}$, i.e, $|c'| \subseteq |c|$.

Let c_0 be a fixed top-dimensional cube of a cubed complex $T^{\mathcal{C}} = (V, E)$ i.e. $c_0 \in T_l^{\mathcal{C}} = \mathcal{C}^{top}$. We describe a sub-division operation we call binary sub-division

$$T^{\mathcal{C}'} = BSD_{c_0}(T^{\mathcal{C}})$$

that outputs the refinement $T^{\mathcal{C}'} = (V', E')$ of $T^{\mathcal{C}}$ with $V' = V \cup ch(c_0)$.

Figure 2.4 shows an example of a binary sub-division applied on the cubed complex from Figure 2.3.

Proposition 2.4.2. *Any finite refinement of $T^{\mathcal{C}}$ can be obtained by a finite series of repeated binary subdivisions.*

Let $T^{\mathcal{C}'} = (V', E')$ be a refinement of $T^{\mathcal{C}} = (V, E)$. Consider the finite set of vertices $U = T_l' \setminus T_l$. These define a sub-poset of $\mathcal{T}_d$ such that the maximal elements are $T_l \subset V$. If we replace $T^{\mathcal{C}}$ by $BSD_c(T^{\mathcal{C}})$ for a $c \in T_l$ and U by $U \setminus \{c\}$, U continues to define a sub-poset of $\mathcal{T}_d$ such that the maximal elements are $T_l \subset V$. By continuing this process, picking a maximal element of U at each step, we will obtain $T^{\mathcal{C}'}$.

Let $X \subseteq [0, 1]^d$. The diameter of X is given by:

$$diam(A) = max(\{||x - y|| \mid x, y \in X\}).$$

Note that diameter of the geometric realization of a cube $c = (\mathbf{p}, \mathbf{b}, n)$ is $\frac{\sqrt{d}}{2^n}$. Hence as $n \to \inf$, $diam(|c|) \to 0$.

Let $X, Y \subseteq [0, 1]^d$. The distance between X and Y is given by:

$$D(X, Y) = min(\{||x - y|| \mid x \in X, y \in Y\}).$$

Remark 2.4.3. Let $T^{\mathcal{C}}$ be a cubical complex such that $max(\{n \mid c = (\mathbf{p}, \mathbf{b}, n) \in \mathcal{C}\}) = N$. Thus, the side length of the smallest cube in $\mathcal{C}$ is $\epsilon = \frac{1}{2^N}$. Then if $c_1, c_2 \in \mathcal{C}^{top}$ such that $d(|c_1|, |c_2|) < \epsilon$, then $|c_1| \cap |c_2| \neq \emptyset$.

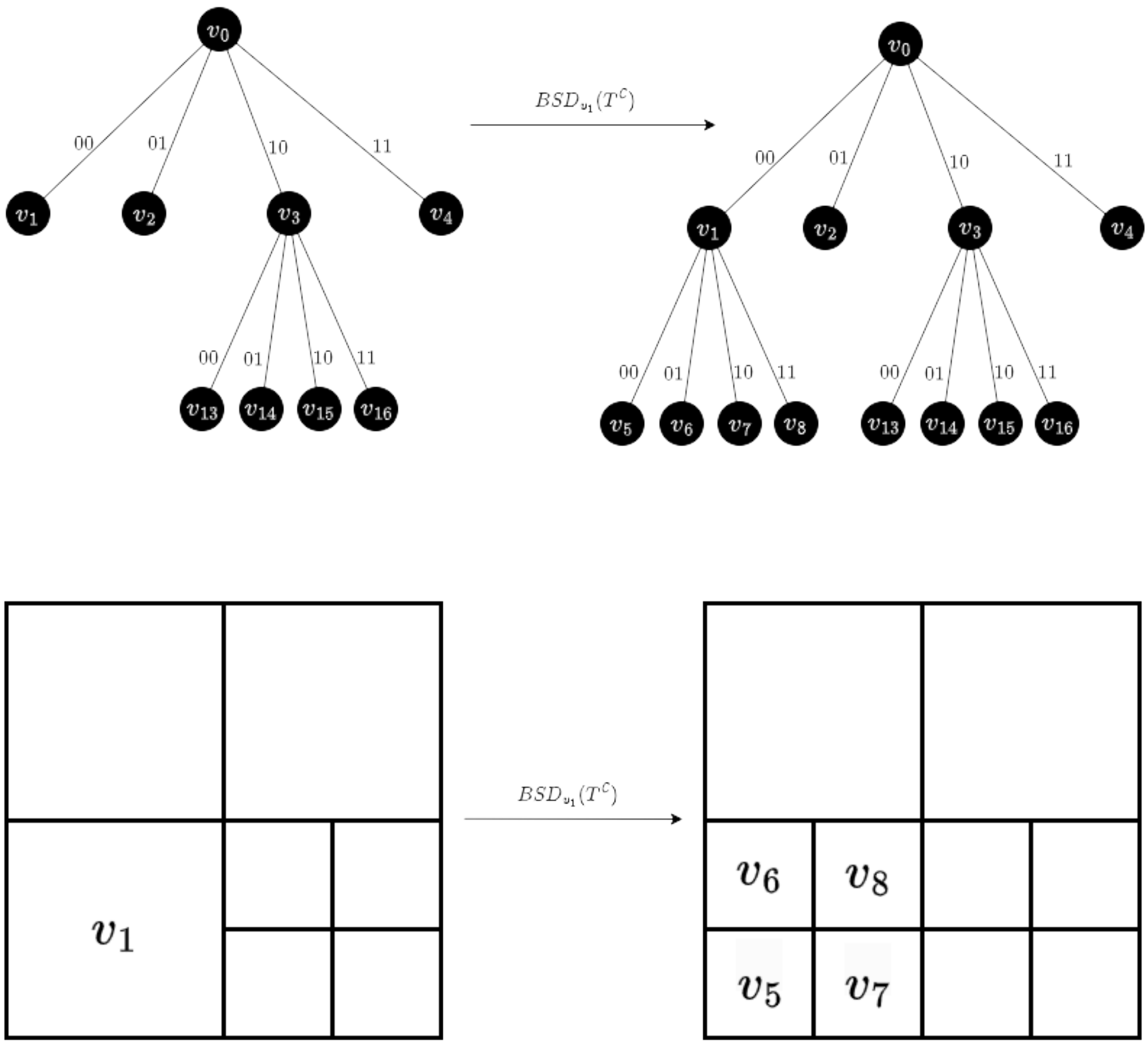


Figure 2.4: ***An example of Binary Sub-division*** *applied to the cubed complex from Figure 2.3. Above is the effect on $T^{\mathcal{C}}$ and below on the geometric realization $\mathcal{C}$.*

Let $T^{\mathcal{C}'}$ be the refinement of $T^{\mathcal{C}}$ in which every top-dimensional cube has side length ϵ. i.e., $T^{\mathcal{C}'}$ contains all vertices of the complete 2^d-ary tree up to level N. It is basically a grid of the unit cube.
For $T^{\mathcal{C}'}$ it is easy to see that $|c_1| \cap |c_2| = \emptyset$ implies that c_1 and c_2 are not neighbors and hence there must be another cube c 'in between' them. Hence, $d(|c_1|, |c_2|) > \epsilon$ Since every $c \in \mathcal{C}^{top}$ is a union of its descendants in $T^{\mathcal{C}'}$, $d(|c_1|, |c_2|) < \epsilon$ would imply there are descendants c_1' of c_1 and c_2' of c_2 such that $d(|c_1'|, |c_2'|) < \epsilon$ and hence $|c_1'| \cap |c_2'| \neq \emptyset \implies |c_1| \cap |c_2| \neq \emptyset$.

We also set some more notation. Let c be a node of $T^{\mathcal{C}}$. Then c is a top-dimensional cube for some $T^{\mathcal{C}'}$ such that $T^{\mathcal{C}}$ is a refinement of $T^{\mathcal{C}'}$. Let v be a face of c. Then, by $top^*(v)$, we denote the set of top-dimensional cubes in $T^{\mathcal{C}}$, which have a face of the same dimension as v contained in v. Or equivalently,

$$top^*(v) = \{c' \in \mathcal{C}^{top} \mid c' \in z(c'') \text{ for some } c'' \in top(v) \text{ and } |c'| \cap |v| \neq \emptyset\}.$$

2.5 Combinatorial Dynamics

Given a finite set A, a *combinatorial multi-valued map* on A is a function $\mathcal{F} : A \to 2^A$, which we denote by $\mathcal{F} : A \rightrightarrows A$. We extend this notation to subsets $U \subseteq A$ by defining $\mathcal{F}(U) = \bigcup_{u \in U} \mathcal{F}(u)$. This allows us to consider the dynamics of $\mathcal{F}$ on A via iteration.

We note $\mathcal{F} : A \rightrightarrows A$ can also be viewed as a directed graph $\mathcal{F} = (A, E)$ with vertex set A and edge set $E = \{(a, a') \mid a' \in \mathcal{F}(a)\}$.

We call $\mathcal{F} : A \rightrightarrows A$ *right-total* if $\mathcal{F}(u) \neq \emptyset$ for all $u \in A$.

Definition 2.5.1. A set $U \subseteq A$ is an *attractor* if $\mathcal{F}(U) = U$.

Definition 2.5.2. A set $U \subseteq A$ is *forward invariant set* if $\mathcal{F}(U) \subseteq U$.

We denote the set of attractors by $Att(\mathcal{F})$ and the set of forward invariant sets by $Invset^+(\mathcal{F})$. By definition, every attractor is a forward invariant set, and hence $Att(\mathcal{F})$ is a subset of $Invset^+(\mathcal{F})$.

Definition 2.5.3. Given any $U \subseteq A$, the *ω-limit set* of U is given by

$$\omega(U) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} \mathcal{F}^k(U).$$

Proposition 2.5.4. *We make some remarks about the ω-limit set. For every $U, V \subseteq A$:*

(i) There exists a n_* such that $\omega(U) = \bigcup_{k\geq n} \mathcal{F}^k(U)$ for all $n \geq n_*$.

(ii) For a right-total $\mathcal{F}$, $U \neq \emptyset \implies \omega(U) \neq \emptyset$.

(iii) $\omega(U)$ is an attractor for all $U \subseteq A$

(iv) For an attractor U, $\omega(U) = U$.

(v) For a forward invariant set U, $\omega(U)$ is the largest attractor contained in U.

(vi) $V \subseteq U \implies \omega(V) \subseteq \omega(U)$ and thus, $\omega(V \cap U) \subseteq \omega(V) \cap \omega(U)$.

(vii) $\omega(V \cup U) = \omega(U) \cup \omega(V)$ and thus, $\omega(U) = \bigcup_{u\in U} \omega(u)$.

Proof. (i). Let $A_n = \bigcup_{k\geq n} \mathcal{F}^k(U)$. Then $A_0 \supseteq A_1 \supseteq A_2$.... This is a nested sequence of finite sets, so it follows that there exists an n_* such that $A_{n_*} = A_n$ for all $n \geq n_*$. Thus $\omega(U) = \bigcap_{m\in\mathbb{N}} A_m = \bigcap_{m\geq n_*} A_m = A_n$ for any $n \geq n_*$.

The rest follow from (i). We refer the reader to [8] for more details. □

It is easy to see from the above proposition that:

1. $Invset^+(\mathcal{F})$ has a bounded distributive lattice structure with $\vee = \cup$, $\wedge = \cap$, $0 = \emptyset$ and $1 = A$.

2. $Att(\mathcal{F})$ has a bounded distributive lattice structure with $\vee = \cup$, $A \wedge A' = \omega(A \cap A')$, $0 = \emptyset$ and $1 = \omega(A)$.

Remark 2.5.5. For both $Att(\mathcal{F})$ and $Invset^+(\mathcal{F})$, the partial order is given by inclusion. This is as $A \leq_L B \iff A \vee B = A \cup B = B \iff A \subseteq B$ for both $L = Invset^+(\mathcal{F}), Att(\mathcal{F})$.

Though $Att(\mathcal{F})$ is a subset of $Invset^+(\mathcal{F})$, it is not a sub-lattice of $Invset^+(\mathcal{F})$ as the meet operations differ. Figure 2.5 shows an example of an $\mathcal{F} : A \rightrightarrows A$ for $A = \{a_1, a_2, a_3, a_4\}$. 2.5a shows $\mathcal{F}$ viewed as a directed graph, 2.5b shows $Invset^+(\mathcal{F})$ and 2.5c shows $Att(\mathcal{F})$. In $Invset^+(\mathcal{F})$, $\{a_1, a_2, a_3\} \wedge \{a_1, a_2, a_4\} = \{a_1, a_2, a_3\} \cap \{a_1, a_2, a_4\} = \{a_1, a_2\}$, however $\{a_1, a_2\} \notin Att(\mathcal{F})$.

Proposition 2.5.6. *$\omega : Invset^+(\mathcal{F}) \rightrightarrows Att(\mathcal{F})$ is a bounded lattice epimorphsim.*

Proof. By Proposition 2.5.4(vii), for all $\mathcal{N}_1, \mathcal{N}_2 \in Invset^+(\mathcal{F})$:

$$\omega(\mathcal{N}_1 \vee \mathcal{N}_2) = \omega(\mathcal{N}_1 \cup \mathcal{N}_2) = \omega(\mathcal{N}_1) \cup \omega(\mathcal{N}_2) = \omega(\mathcal{N}_1) \vee \omega(\mathcal{N}_2)$$

.

In $Att(\mathcal{F})$:

$$\omega(\mathcal{N}_1) \wedge \omega(\mathcal{N}_2) = \omega(\omega(\mathcal{N}_1) \cap \omega(\mathcal{N}_2)) \subseteq \omega(\mathcal{N}_1 \cap \mathcal{N}_2)$$

$$= \omega(\omega(\mathcal{N}_1 \cap \mathcal{N}_2)) \subseteq \omega(\omega(\mathcal{N}_1) \cap \omega(\mathcal{N}_2)) = \omega(\mathcal{N}_1) \wedge \omega(\mathcal{N}_2)$$

$$\implies \omega(\mathcal{N}_1) \wedge \omega(\mathcal{N}_2) = \omega(\mathcal{N}_1 \cap \mathcal{N}_2) = \omega(\mathcal{N}_1 \wedge \mathcal{N}_2).$$

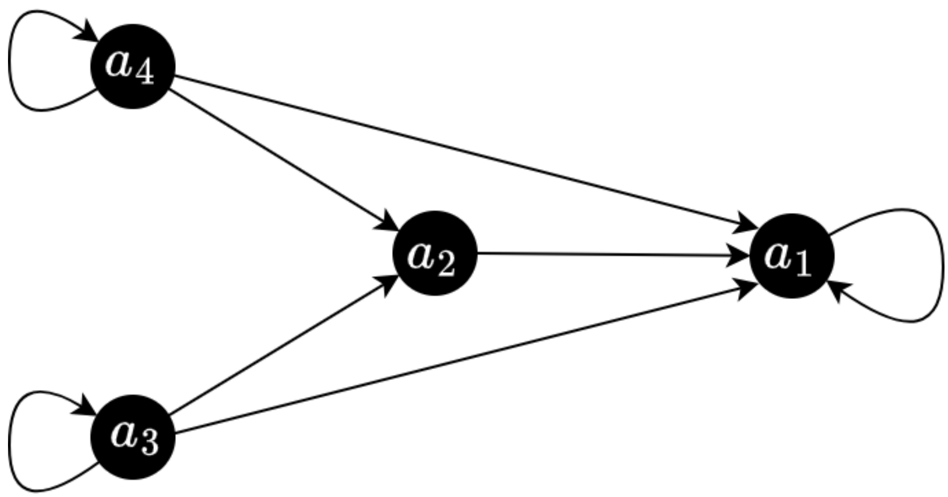
Thus, ω is lattice homomorphism. It is surjective as $\omega(\mathcal{A}) = \mathcal{A}$ for every attractor $\mathcal{A}$. Further, $\omega(\emptyset) = \emptyset$ and $\omega(A) = 1_{Att(\mathcal{F})}$. ☐

We are interested in studying dynamics on the unit cube $[0,1]^d$ and hence given a cubed complex $\mathcal{C}$ consider multi-valued maps on the set of top-dimensional cells $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$. We say $\mathcal{C}$ and $\mathcal{F}$ together define a combinatorial dynamical system on $[0,1]^d$.

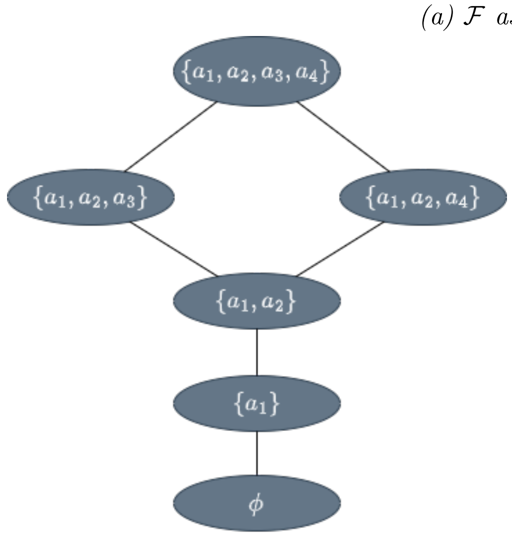
Definition 2.5.7. Let $\mathcal{C}$ be a cubed complex of $[0,1]^d$ and $f : [0,1]^d \to [0,1]^d$ be a continuous function. Then a multi-valued map $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ is an *outer approximation* of f if:

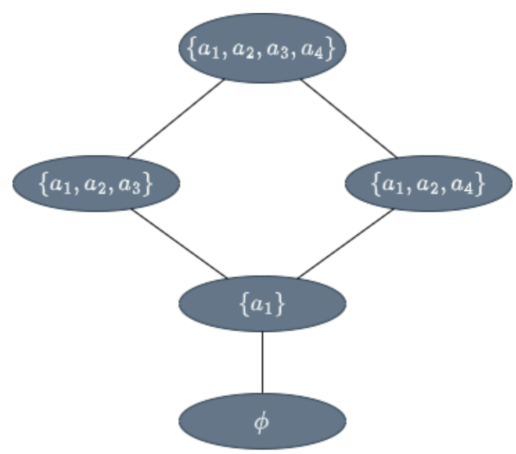
$$f(|c|) \subset \mathrm{Int}(|\mathcal{F}(c)|) \; \forall c \in \mathcal{C}^{top}.$$

We note that if $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ is an outer approximation, it must be right right-total as $f(|c|)$ must be non-empty for every $c \in \mathcal{C}^{top}$.



(a) $\mathcal{F}$ as a directed graph.





(b) $Invset^+(\mathcal{F})$ *(c) $Att(\mathcal{F})$*

Figure 2.5: ***An example of a combinatorial multi-valued map*** *$\mathcal{F} : A \to A$.*

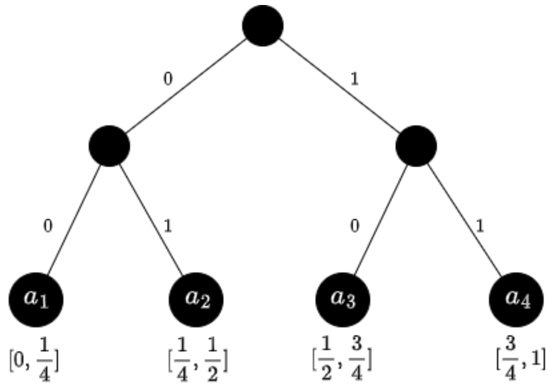


Figure 2.6: ***A cubed complex*** $T^{\mathcal{C}}$ *of the unit interval* $[0, 1]$.

The definition of an outer approximation is quite general; for example, consider the function $f : [0, 1] \to [0, 1]$ given by $f(x) = 0.2$ for all $x \in [0, 1]$. Consider the cubed complex $T^{\mathcal{C}}$ on $[0, 1]$ shown in Figure 2.6. Let $\mathcal{C}^{top} = \{a_1, a_2, a_3, a_4\} = A$ and then, $\mathcal{F}$ from Figure 2.5a is an outer approximation to f as $f(|a|) = \{0.2\} \subset \text{int}(|a_1|)$ and $a_1 \in \mathcal{F}(a)$ for all $a \in \mathcal{C}^{top}$. However, the dynamics of $\mathcal{F}$ seem to be more complicated than that of the constant function f.

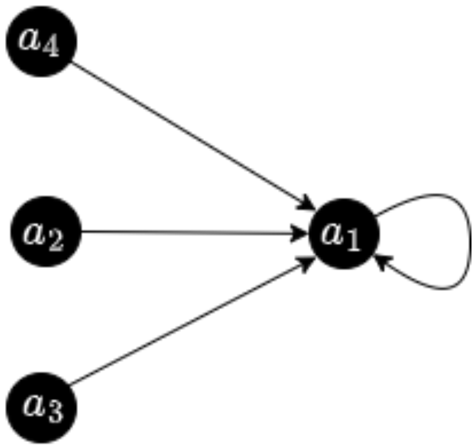
This is why we focus on minimal outer approximations.

Definition 2.5.8. The *minimal outer approximation* of a continuous $f : [0, 1]^d \to [0, 1]^d$ on a cubed complex $\mathcal{C}$ is given by $\mathcal{F}(c) = \{c' \in \mathcal{C} \mid |c'| \cap f(|c|) \neq \emptyset\}$ for all $c \in \mathcal{C}$.

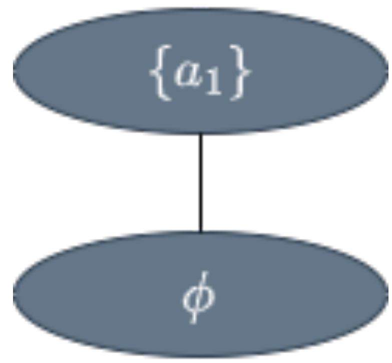
Given the cubed complex from Figure 2.6, we consider once again $f : [0, 1] \to [0, 1]$ given by $f(x) = 0.2$. $f(|a|) = \{0.2\} \subset \text{int}(|a_1|)$ for all $a \in \mathcal{C}^{top}$ and hence $\mathcal{F}'(a) = \{a_1\}$ for all $a \in \mathcal{C}^{top}$ is the minimal outer approximation of f, and is shown as a directed graph in Figure 2.7a. The lattice of attractors of $\mathcal{F}'$ - Figure 2.7b -is a lot simpler than that for $\mathcal{F}$ from Figure 2.5 and more accurately represents the true dynamics of f - which itself has only one attractor, a fixed point at 0.2. We see, though, that $Invset^+(\mathcal{F}')$ is a lot more complicated than $Invset^+(\mathcal{F})$.

However, we can embed $Att(\mathcal{F}')$ in $Invset^+(\mathcal{F}')$ through the bounded lattice monomorphism $\tau : Att(\mathcal{F}') \to Invset^+(\mathcal{F}')$ that takes $\emptyset$ to $\emptyset$ and $\{a_1\}$ to $\mathcal{C}^{top} = \{a_1, a_2, a_3, a_4\}$. Further, since $\omega(\emptyset) = \emptyset$ and $\omega(\mathcal{C}^{top}) = \{a_1\}$, $\omega \circ \tau = Id_{Att(\mathcal{F})}$ i.e., we say τ acts as a *lift* of ω.

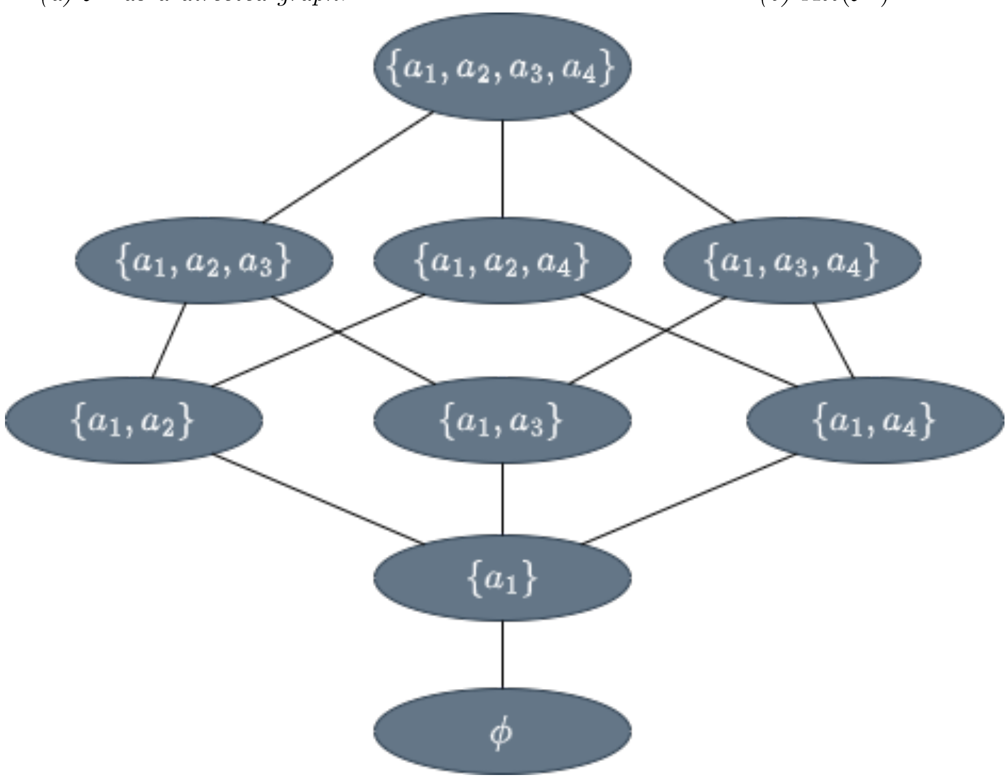
In general, we are interested in the question - given a minimal outer approximation $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ of some continuous $f : [0, 1] \to [0, 1]$, does there exists a lift τ of ω? In other



(a) $\mathcal{F}'$ as a directed graph.



(b) $Att(\mathcal{F}')$



(c) $Invset^+(\mathcal{F}')$

Figure 2.7: ***An example of a minimal outer approximation*** $\mathcal{F}'$.

words, does a bounded lattice monomorphism τ exist such that $\omega \circ \tau = Id_{Att(\mathcal{F})}$?

Given a directed graph $G = (V, E)$ a $S \subseteq V$ is *strongly connected* if for all $v, v' \in S$ such that $v \neq v'$ there are paths from $v \rightsquigarrow v'$ and from $v' \rightsquigarrow v$. A *strongly connected component* of G is a strongly connected $S \subseteq V$ that is maximal, i.e., if $S \subset S' \subseteq V$, then S' is not strongly connected. The *condensation graph* of G, also called the graph of strongly connected components of G, is the graph $SC(G)$ with the vertex set being the set of strongly connected components of G and edge set $\{(S, S') \mid \exists\, (v, v') \in E \text{ with } v \in S, v' \in S'\}$. $SC(G)$ is always a directed acyclic graph and hence can be viewed as a poset. For any $v \in V$, we denote by $SCC(v)$ the strongly connected component it belongs to in $SC(G)$.

Viewing $\mathcal{F}$ as a directed graph, we denote the associated condensation graph by $SC(\mathcal{F})$. We call a strongly connected component a *recurrent component* or a *Morse set* if it contains at least one edge. We denote by $M(\mathcal{F})$ the *Morse graph* - the sub-poset of Morse sets of $SC(\mathcal{F})$.

The recurrent dynamics of f is contained in the Morse sets for any outer approximation $\mathcal{F}$. Note that a non-recurrent component must be a singular top-dimensional cube c such that $c \notin \mathcal{F}(c)$. Further for any $c \in \mathcal{M} \in M(\mathcal{F})$, there exists a $c' \in M(\mathcal{F})$ such that $c' \in \mathcal{F}(c)$. This implies $\mathcal{M} \cap \mathcal{F}^k(c) \neq \emptyset$ for any $c \in \mathcal{M}$ and $k \in \mathbb{N}$.

Definition 2.5.9. Let $T^{\mathcal{C}'}$ be a refinement of $T^{\mathcal{C}}$. Then a $\mathcal{F}' : \mathcal{C}'^{top} \rightrightarrows \mathcal{C}'^{top}$ is a refinement of $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ if for every $c' \in \mathcal{C}'$, $c' \in z(c)$ for $c \in \mathcal{C} \implies \mathcal{F}'(c') \subseteq z(\mathcal{F}(c))$ i.e., $|\mathcal{F}'(c')| \subseteq |\mathcal{F}(c)|$.

2.6 Birkhoffs Representation Theorem

Birkhoff's representation Theorem allows us to translate between the category of finite distributive lattices and finite posets. In fact, it establishes a dual equivalence between the two categories. We present it here in a manner understandable to readers without a background in category theory.

We first note that for any poset P, the set of its down sets $O(P)$ has a bounded, distributive lattice structure with $\vee = \cup$, $\wedge = \cap$, $1 = P$ and $0 = \emptyset$.

Theorem 2.6.1. *[3] For any finite poset P, $P \cong J(O(P))$ and for any finite distributive* *lattice* L, $L \cong O(J(L))$.

Remark 2.6.2. Every joint irreducible element of $O(P)$ if given by $\downarrow p$ for some $p \in P$ and the isomorphism $P \cong J(O(P))$ is simply given by taking every $p \in P$ to $\downarrow p$.

Remark 2.6.3. For every $a \in L$, let $J_a(L) = \{b \in J(L) \mid b \leq_L a\}$. $J_a(L)$ is a down set of $J(L)$. The isomorphism $L \cong O(J(L))$ is given by taking every $a \in L$ to $J_a(L)$.

In fact every $a \in L$ can be written as $a = \bigvee J_a(L)$.

Theorem 2.6.4. *([7], Theorem 4.2),([3], Theorem 5.19), ([12], Theorem 10.4)(i) Let* L, K *be finite distributive lattices and* $h : K \to L$ *be a bounded lattice homomorphism. Then* $J(h) : J(L) \to J(K)$ *given by*

$$J(h)(a) = \bigwedge h^{-1}(\uparrow a) = min(h^{-1}(\uparrow a)) \text{ for all } a \in J(L)$$

is an order-preserving map such that h *is a bounded lattice monomorphism if and only if* $J(h)$ *is an order surjection and* h *is a bounded lattice epimorphism if and only if* $J(h)$ *is an order embedding. Further, for* $h : K \to L$ *and* $g : L \to M$*:*

$$J(g \circ h) = J(h) \circ J(g).$$

(ii) Let P, Q *be finite posets and* $h : Q \to P$ *be a order-preserving map. Then* $O(h) : O(P) \to O(Q)$ *given by*

$$O(h)(I) = h^{-1}(I) \text{ for all } I \in O(P)$$

is a bounded lattice homomorphism such that h *is an order surjection if and only if* $O(h)$ *is a bounded lattice monomorphism and* h *is an order embedding if and only if* $O(h)$ *is a bounded lattice epimorphism. Further, for* $h : Q \to P$ *and* $g : P \to R$*:*

$$O(g \circ h) = O(h) \circ O(g).$$

Proposition 2.6.5. *For any multi-valued map* $\mathcal{F} : A \rightrightarrows A$*:*

(i) $SC(\mathcal{F}) \cong J(Invset^+(\mathcal{F}))$,

(ii) $M(\mathcal{F}) \cong J(Att(\mathcal{F}))$.

Proof. We prove (i) and refer the reader to [7] for a complete proof of (ii).

We first show that $Invset^+(\mathcal{F}) \cong O(SC(\mathcal{F}))$; in fact, we show that they are the same lattice. Then, by Birkhoff's representation theorem, we can conclude that $SC(\mathcal{F}) \cong J(O(SC(\mathcal{F}) \cong J(Invset^+(\mathcal{F})$.

For any $a \in A$, $\mathcal{F}(a) \subseteq \downarrow_{SC(\mathcal{F})} SCC(a)$. Hence for any strongly connected component $S \in SC(\mathcal{F})$, $\mathcal{F}(S) \subseteq \downarrow_{SC(\mathcal{F})} S$. Thus if I is a down set of $SC(\mathcal{F})$ i.e. if $I \in O(SC(\mathcal{F}))$, $\mathcal{F}(I) \subseteq \bigcup\limits_{S \in I} \downarrow_{SC(\mathcal{F})} S = I$ and hence a forward invariant set.

Let $\mathcal{N} \in Invset^+(\mathcal{F})$ and let $a \in \mathcal{N}$. Let $a' \in \mathcal{N}$ such that $SCC(a') \leq SCC(a)$ i.e., there exists a path $\{a = a_0, a_1, ..., a_{k-1}, a_k = a'\}$ in $\mathcal{F}$. Then $a_1 \in \mathcal{N}$ as $a_1 \in \mathcal{F}(a)$. Similarly if we assume $a_{i-1} \in \mathcal{N}$, a_i must also be in $\mathcal{N}$. By induction $a_k = a' \in \mathcal{N}$. Thus if $a \in \mathcal{N}$, then $\downarrow_{SC(\mathcal{F})} SCC(a) \subseteq \mathcal{N}$ i.e., $\mathcal{N} \in O(SC(\mathcal{F}))$.

We have shown $O(SC(\mathcal{F}))$ and $Invset^+(\mathcal{F})$ are equivalent as sets. However, for both $\vee = \cup$ and $\wedge = \cap$, so they are equivalent as lattices.

By remark, the isomorphism $SC(\mathcal{F}) \cong J(Invset^+(\mathcal{F}))$ is in fact given by taking every $S \in SC(\mathcal{F})$ to $\downarrow_{SC(\mathcal{F})} S$.

For (ii) of the proposition, we do not provide a full proof but make the remark that every join irreducible attractor is given by $\downarrow_{SC(\mathcal{F})} \mathcal{M}$ for a $\mathcal{M} \in M(\mathcal{F})$ and the isomorphism $M(\mathcal{F}) \cong J(Att(\mathcal{F}))$ is given by taking every $\mathcal{M} \in M(\mathcal{F})$ to $\downarrow_{SC(\mathcal{F})\mathcal{M}}$. In other words, it is the isomorphism $SC(\mathcal{F}) \cong J(Invset^+(\mathcal{F}))$ restricted to $M(\mathcal{F})$.

□

Hence, corresponding to $\omega : Invset^+(\mathcal{F}) \to Att(\mathcal{F})$, we would have an order embedding $J(\omega) : J(Att(\mathcal{F})) \to J(Invset^+(\mathcal{F}))$ given by:

$$J(\omega)(\mathcal{A}) = min(\omega^{-1}(\uparrow \mathcal{A})).$$

However, for any forward invariant set $\mathcal{N}$, $\omega(\mathcal{N}) \subseteq \mathcal{N}$ i.e., $\omega(\mathcal{N}) \leq_{Invset^+(\mathcal{F})} \mathcal{N}$. Further, $\omega(\mathcal{A}) = \mathcal{A}$ for any attractor $\mathcal{A}$. Implying $min(\omega^{-1}(\uparrow \mathcal{A})) = \mathcal{A}$. i.e., $J(\omega)$ is nothing but the inclusion order embedding of $J(Att(\mathcal{F}))$ in $J(Invset^+(\mathcal{F}))$.

Similarly, corresponding to $Id_{Att(\mathcal{F})}$, $J(Id_{Att(\mathcal{F})}) = Id_{M(\mathcal{F})}$. Hence, by theorem , there exists a bounded lattice monomorphism $\tau : Att(\mathcal{F}) \to Invset^+(\mathcal{F})$ such that $\omega \circ \tau = Id_{Att(\mathcal{F})}$ if and only if there exists an order surjection $J(\tau) : J(Invset^+(\mathcal{F}) \to J(Att(\mathcal{F}))$ such that $J(\tau) \circ J(\omega) = J(\omega \circ \tau) = J(Id_{Att(\mathcal{F})}) = Id_{M(\mathcal{F})}$.

Further, since $SC(\mathcal{F}) \cong J(Invset^+(\mathcal{F}))$ and $M(\mathcal{F}) \cong J(Att(\mathcal{F}))$, the existence of such a lift τ is equivalent to the existence of an order surjection $\sigma : SC(\mathcal{F}) \to M(\mathcal{F})$ such that $\sigma \circ i = Id_{M(\mathcal{F})}$, where $i : M(\mathcal{F}) \to SC(\mathcal{F})$ is the inclusion map.

Chapter 3

Giving Some Context

3.1 The Conley Index

A pair of forward invariant sets $(\mathcal{N}_1, \mathcal{N}_0)$ is called a index pair if $\mathcal{N}_0 \subset \mathcal{N}_1$. For a field $\mathbb{F}$, under some weak conditions [1] $\mathcal{F}$ induces a map on relative homology:

$$\mathcal{F}_* : H_*(\downarrow_{\mathcal{C}} \mathcal{N}_1, \downarrow_{\mathcal{C}} \mathcal{N}_0; \mathbb{F}) \to H_*(\downarrow_{\mathcal{C}} \mathcal{N}_1, \downarrow_{\mathcal{C}} \mathcal{N}_0; \mathbb{F}).$$

The Conley index of the pair $(\mathcal{N}_1, \mathcal{N}_0)$ - notated by $Con_*(\mathcal{N}_1, \mathcal{N}_2; \mathbb{F})$ is given by the shift equivalence class of $\mathcal{F}_*$ (since $\mathbb{F}$ is a field, $\mathcal{F}_*$ is a linear transformation and the shift equivalence class is nothing but the rational canonical form of $\mathcal{F}_*$).

Let $\rho : M(\mathcal{F}) \to J(Att(\mathcal{F}))$ be the isomorphism discussed in Proposition 2.6.5. The Conley index of a Morse set $\mathcal{M} \in M(\mathcal{F})$ is given by:

$$Con_*(\mathcal{M}, \mathbb{F}) = Con_*(\rho(\mathcal{M}), \overleftarrow{\rho(\mathcal{M})}; \mathbb{F})$$

where $\overleftarrow{\rho(\mathcal{M})}$ is the unique immediate predecessor of $\rho(\mathcal{M})$.

Proposition 3.1.1. *For any index pair* $(\mathcal{N}_1, \mathcal{N}_0)$*:*

$$Con_*(\mathcal{N}_1, \mathcal{N}_2; \mathbb{F}) = Con_*(\omega(\mathcal{N}_1), \omega(\mathcal{N}_2); \mathbb{F}).$$

Thus, if τ is a lift of ω, the for any $\mathcal{M} \in M(\mathcal{F})$, the Conley index can be computed by:

$$Con_*(\mathcal{M}, \mathbb{F}) = Con_*(\tau(\rho(\mathcal{M})), \tau(\overleftarrow{\rho(\mathcal{M})}); \mathbb{F}).$$

The Conley Index is also defined for certain regions of phase space in the context of continuous maps (See [10] [5] for details), and we have that when $\mathcal{F}$ is an outer approximation of f, under some weak conditions:

$$Con_*(\mathcal{M}; \mathbb{F}) \sim Con_*(|\rho(\mathcal{M}) \setminus \overleftarrow{\rho(\mathcal{M})}|) \sim Con_*(|\tau(\rho(\mathcal{M})) \setminus \tau(\overleftarrow{\rho(\mathcal{M})})|).$$

$Con_*(\mathcal{M}; \mathbb{F})$ provides information about the invariant dynamics of f in the region $|\rho(\mathcal{M}) \setminus \overleftarrow{\rho(\mathcal{M})}|$ - which is in fact contained in $|\mathcal{M}|$.

There are theorems that say given the Conley index of a region and maybe some extra information, one can conclude the kind of invariant set contained in it [10]. For example they can guarantee the existence of a fixed point, a particular kind of fixed point, periodic orbits and even chaos. One fundamental but powerful result says that if the Conley Index of a region is non-trivial, then there must be a non-empty invariant set contained in it (however, the converse is not true, as we see in the next section).

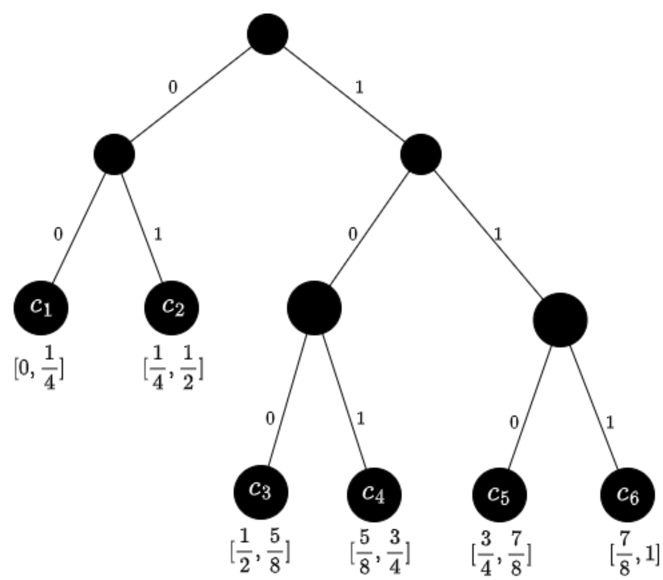


Figure 3.1: ***A cubed complex*** $T^{\mathcal{C}}$ *of the unit interval* $[0, 1]$.

3.2 A Counter-example

Consider the cubed complex $T^{\mathcal{C}}$ of the unit interval $[0,1]$ given by the rooted tree in Figure 3.1 and consider the continuous, in fact, piece-wise linear function $f:[0,1]\to[0,1]$ given as follows:

$$f(x)=\begin{cases}\frac{12}{5}x & \text{if } 0\le x\le\frac{1}{3}\\ -\frac{17}{11}x+\frac{217}{165} & \text{if } \frac{1}{3}\le x\le\frac{31}{60}\\ \frac{18}{13}x-\frac{31}{156} & \text{if } \frac{31}{60}\le x\le\frac{5}{8}\\ \frac{8}{17}x+\frac{19}{51} & \text{if } \frac{5}{8}\le x\le\frac{23}{30}\\ \frac{3}{2}x-\frac{5}{12} & \text{if } \frac{23}{30}\le x\le\frac{5}{6}\\ \frac{1}{4}x+\frac{5}{8} & \text{if } \frac{5}{6}\le x\le\frac{9}{10}\\ \frac{5}{2}x-\frac{7}{5} & \text{if } \frac{9}{10}\le x\le\frac{14}{15}\\ -9x+\frac{28}{3} & \text{if } \frac{14}{15}\le x\le 1\end{cases} \tag{3.1}$$

A graph of $y=f(x)$ is given in Figure 3.2 with the geometric realization of the cells of $\mathcal{C}^{top}$ shown on the x-axis.

Given $\mathcal{C}$ and f, we can compute the minimal outer approximation $\mathcal{F}$, which is shown in Figure 3.3, represented as a directed graph. Corresponding $SC(\mathcal{F})$, $M(\mathcal{F})$ are shown in Figure 3.5 and $Invset^{+}(\mathcal{F})$ and $Att(\mathcal{F})$ are shown in Figure 3.4.

We can see from Figure 3.4 that there is no bounded lattice monomorphism possible from $Att(\mathcal{F})\to Invset(\mathcal{F})$. In particular in $Att(\mathcal{F})$, $\{c_3,c_5,c_4\}$ is $\{c_1,c_2,c_3,c_5,c_4\}\wedge\{c_6,v_4,c_3,c_5,c_4\}$ as well as $\{c_3,c_4\}\vee\{c_5,c_4\}$; however, in $Invset^{+}(\mathcal{F})$ there is no element which acts as both.

Equivalently from Figure 3.5, we can see that there is no order surjection from $SC(\mathcal{F})$ to $M(\mathcal{F})$. In particular, consider $\{c_2\}$ - the only element of $SC(\mathcal{F})\setminus M(\mathcal{F})$. $\{c_2\}$ cannot be mapped to $\{c_1\}$ as $\{c_2\}\le_{SC(\mathcal{F})}\{c_6\}$ but $\{c_1\}\not\le_{SC(\mathcal{F})}\{c_6\}$. Similarly it cannot be mapped to $\{c_6\}$ as $\{c_6\}\not\le_{SC(\mathcal{F}}\{c_1\}$. $\{c_2\}$ cannot be mapped to $\{c_3\}$ either as $\{c_2\}\ge_{SC(\mathcal{F})}\{c_5\}$ but $\{c_3\}\not\le_{SC(\mathcal{F})}\{c_5\}$. Similarly it cannot be mapped to $\{c_5\}$ as $\{c_5\}\not\le_{SC(\mathcal{F}}\{c_3\}$.

In [7], the authors discuss a method to coarsen the lattice of attractors by excluding Morse sets with trivial Conley Index. In our counter-example, both the Morse sets $\{c_3\}$ and $\{c_5\}$

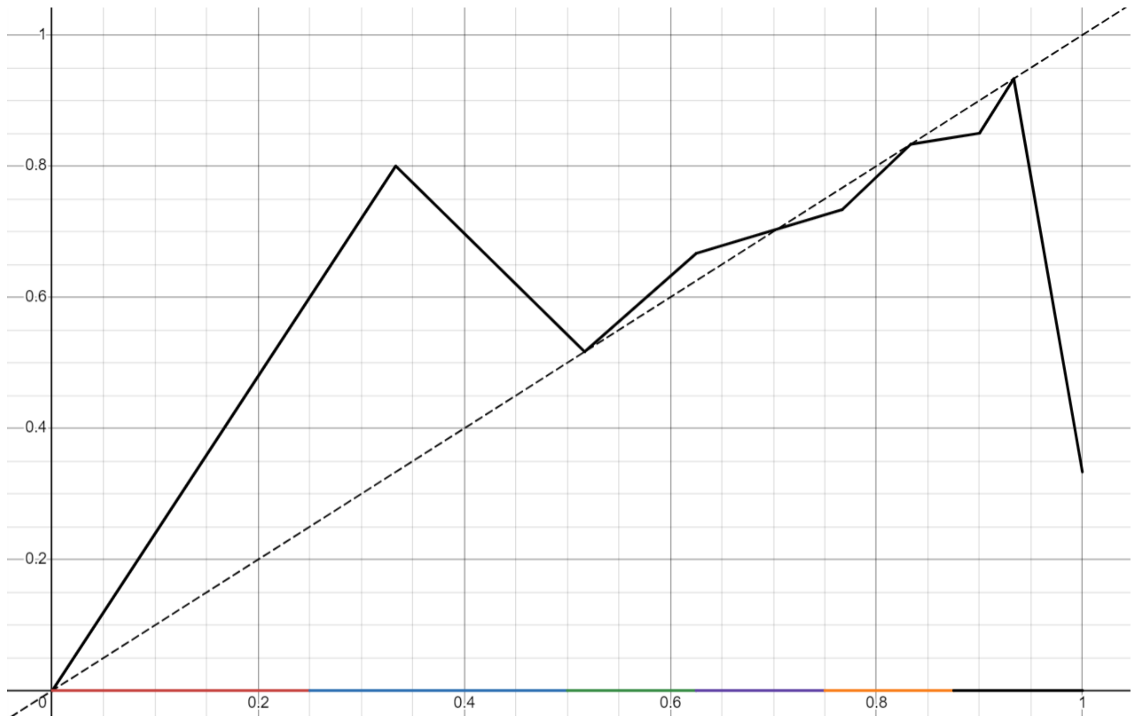


Figure 3.2: ***A graph of*** *$y - f(x)$ with the geometric realization of the top-dimensional cells $\mathcal{C}^{top}$ shown on the x-axis.*

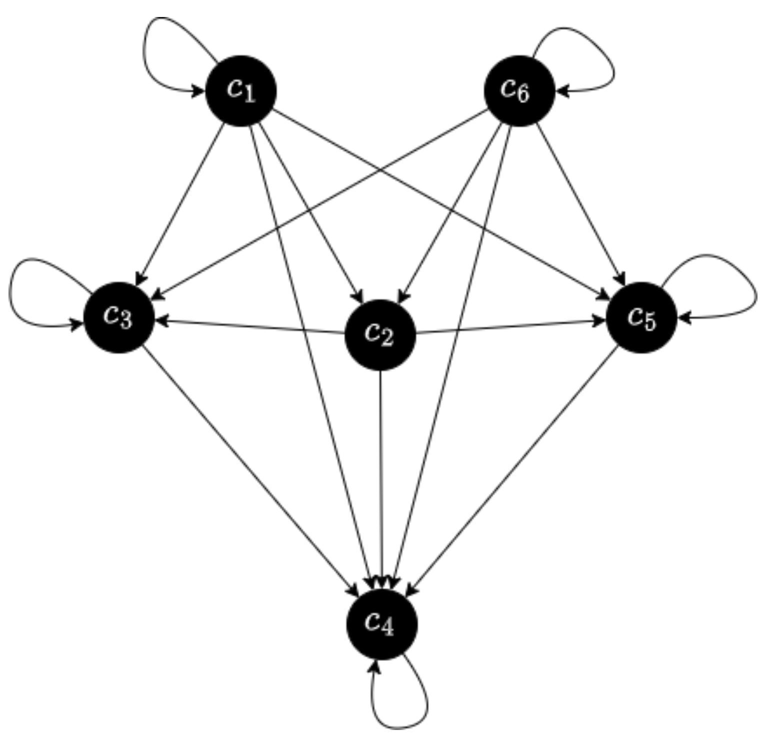
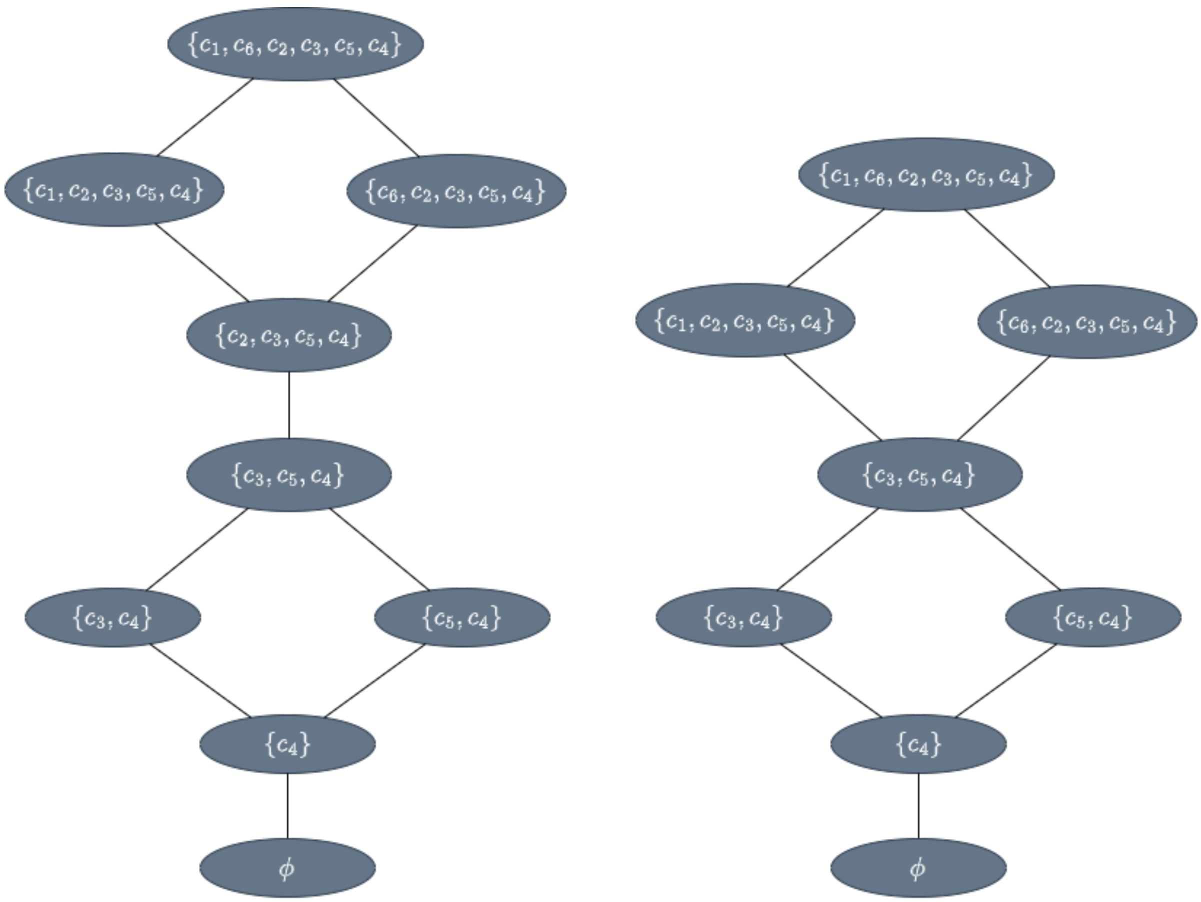
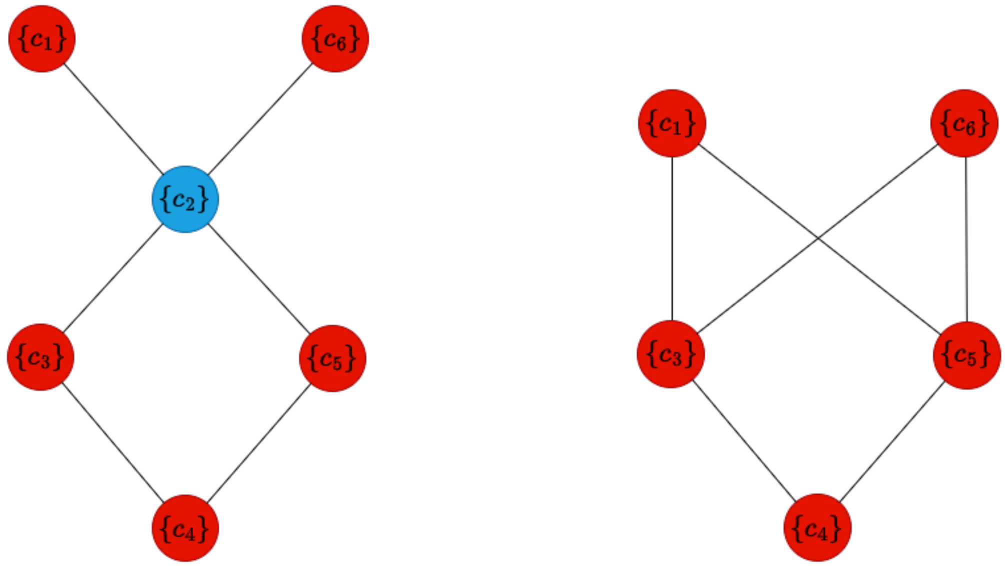


Figure 3.3: $\mathcal{F}$ as a directed graph.



(a) $Invset^+(\mathcal{F})$ *(b)* $Att(\mathcal{F})$

Figure 3.4: $Invset^+(\mathcal{F})$ *and* $Att(\mathcal{F})$

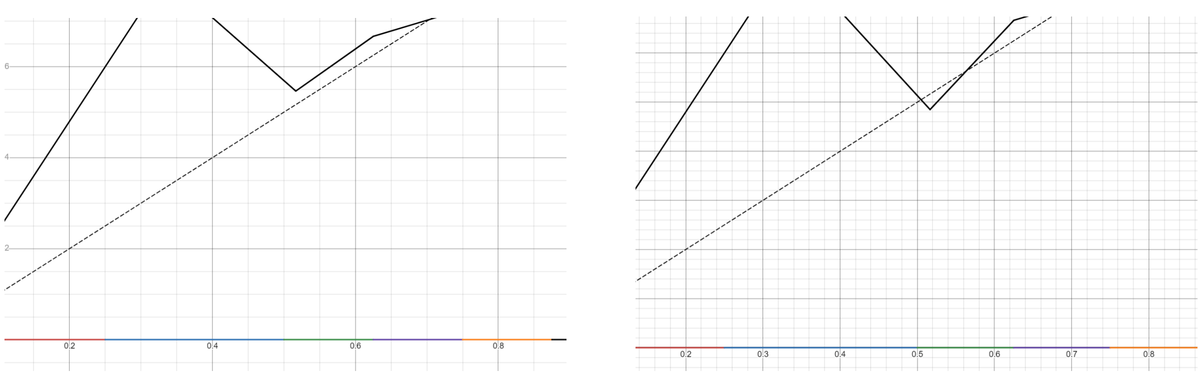


(a) $SC(\mathcal{F})$ *(b)* $M(\mathcal{F})$

Figure 3.5: $SC(\mathcal{F})$ *and* $M(\mathcal{F})$

have trivial Conley Index. However, we argue that both these regions contain interesting dynamics of the function f. In particular, both contain fixed points that seem to be bifurcation points.

For example, let us consider the fixed point in $|c_3|$. By perturbing f a small amount, as shown in Figure 3.6a, the fixed point disappears. However, if we perturb f a small amount, as shown in Figure 3.6b, we see interesting dynamics of an invariant set more complicated than a fixed point - which we may not want to discard in our description of the dynamics.



(a) *(b)*

Figure 3.6: f *perturbed in two ways.*

3.3 Some Useful Results

Let $i : Q \to P$ be an order embedding of posets. For simplicity, we denote an element $i(q) \in P$ for $q \in Q$ by q itself.

Definition 3.3.1. A $q \in Q$ is an *immediate predecessor in* Q or a Q*-predecessor* of a $p \in P \backslash Q$ if $q < p$, and $q \leq q' < p$ for a $q' \in Q$ implies $q = q'$.

Definition 3.3.2. A $q \in Q$ is an *immediate successor in* Q or a Q*-successor* of a $p \in P \setminus Q$ if $q > p$, and $q \geq q' > p$ for a $q' \in Q$ implies $q = q'$.

Given a $p \in P \setminus Q$, we denote by $pred_Q(p)$ the set of its Q-predecessors and by $succ_Q(p)$ the set of its Q-successors.

We note that for any $p \in P \setminus Q$, $pred_Q(p)$ and $succ_Q(p)$ are anti-chains of P.

Lemma 3.3.3. *([7], Lemma 2.3) Let σ be a retraction of the order embedding $i : Q \to P$ i.e. σ is a set map from P to Q such that $\sigma(q) = q$ for all $q \in i(Q)$. Then σ is an order retraction of i if and only if σ is order preserving on every chain contained in $P \setminus Q$ and for ever $p \in P \setminus Q$:*

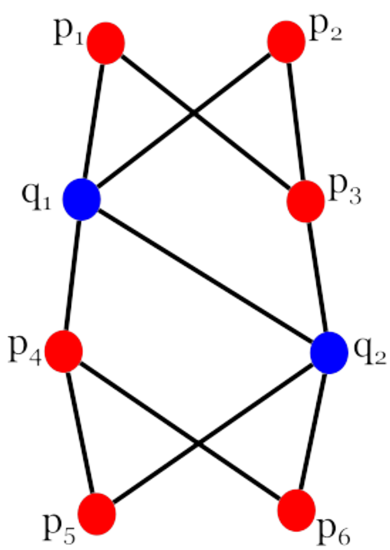
$$\sigma(p) \in \{q \in Q \mid pred_Q(p) \subseteq \downarrow q \text{ and } succ_Q(p) \subseteq \uparrow q\}.$$

Corollary 3.3.4. *If every $p \in P \setminus Q$ has a unique Q-predecessor, then the retraction σ of i that takes every $p \in P \setminus Q$ to its unique Q-predecessor is an order retraction.*

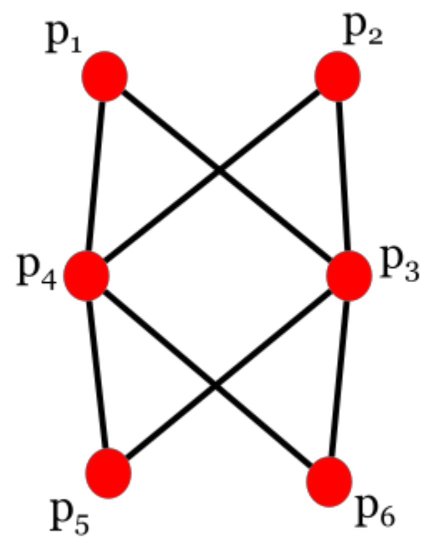
Similarly, if every $p \in P \setminus Q$ has a unique Q-successor, then the retraction σ of i that takes every $p \in P \setminus Q$ to its unique Q-successor is an order retraction.

However, it is not true that if every $p \in P \setminus Q$ has either a unique Q-predecessor or a unique Q-successor, an order retraction σ of i exists. An example of this is shown in Figure 3.7. Elements of $i(P)$ are in red and $Q \setminus i(p)$ are in blue. q_1 has a unique P-predecessor p_4 and q_2 has a unique P-successor p_3. However there is no order retraction of i. In particular, this is as $q_2 \leq_Q q_1$ but $p_4 \not\leq_Q p_3$.

Lemma 3.3.5. *Let $i : Q \to P$ be an order embedding. If there exists an up set Y of P such that every $p \in Y \setminus Q$ has a unique Q-successor and every $p \in Y^C \setminus Q$ has a unique Q-predecessor, then σ given by*



(a) ***Hasse diagram of*** *Q.*



(b) textbfHasse diagram of P,

Figure 3.7: ***An example of an order embedding*** *$i : P \to Q$.*

$$\sigma(p) = \begin{cases} p & \text{if } p \in Q \\ \text{The unique } Q\text{-predecessor of } p & \text{if } p \in Y^C \setminus Q \\ \text{The unique } Q\text{-successor of } p & \text{if } p \in Y \setminus Q \end{cases} \tag{3.2}$$

is an order retraction of i.

Proof. σ restricted to Y is an order retraction as every $p \in Y \setminus Q$ has a unique Q-successor. Similarly, σ restricted to Y^C is an order retraction as every $p \in Y^C \setminus Q$ has a unique Q-predecessor. Further if $p \in Y$ and $p' \in Y^C$ then $\sigma(p) \geq_P p >_P p' \geq_P \sigma(p')$.

□

Proposition 3.3.6. *If $\mathcal{U}$ is a minimal element of $SC(\mathcal{F})$, then $\mathcal{U} \in M(\mathcal{F})$.*

Proof: Let $\mathcal{U}$ be a minimal element of $SC(\mathcal{F})$ and assume $\mathcal{U} \notin M(\mathcal{F})$. Then $\mathcal{U} = \{c\}$ such that $c \notin \mathcal{F}(c)$. However, since $\mathcal{F}$ is an outer approximation, it is right total, i.e., $\mathcal{F}(c) \neq \emptyset$. This implies there must be another top-dimensional cube $c' \neq c$ such that $c' \in \mathcal{F}(c)$. However, that implies $SCC(c') <_{SC(\mathcal{F})} \mathcal{U}$ which is a contradiction as $\mathcal{U}$ is minimal. Thus $\mathcal{U}$ must be in $M(\mathcal{F})$.

Lemma 3.3.7. *Let $\mathcal{U}_1, \mathcal{U}_2 \in SC(\mathcal{F})$ such that they are incomparable and $|\mathcal{U}_1| \cap |\mathcal{U}_2| \neq \emptyset$. Then there exists at least one $\mathcal{U} \in SC(\mathcal{F})$ such that $\mathcal{U} <_{SC(\mathcal{F})} \mathcal{U}_1$ and $\mathcal{U} <_{SC(\mathcal{F})} \mathcal{U}_2$.*

Proof: Consider

$$V = f(|\mathcal{U}_1| \cap |\mathcal{U}_2|) \subseteq f(|\mathcal{U}_1|) \cap f(|\mathcal{U}_2|) \subseteq \text{int}(|\mathcal{F}(\mathcal{U}_1)|) \cap \text{int}(|\mathcal{F}(\mathcal{U}_2)|).$$

$V \cap |\mathcal{U}_1| = \emptyset$ as $\mathcal{U}_1 \not\leq \mathcal{U}_2$ and hence $\mathcal{U}_1 \notin \mathcal{F}(\mathcal{U}_2)$. Similarly, $V \cap |\mathcal{U}_2| = \emptyset$ as $\mathcal{U}_2 \not\leq \mathcal{U}_1$. However, V must intersect with at least one top-cell c and hence $SCC(c) = \mathcal{U}$ will be $<_{SC(\mathcal{F})} \mathcal{U}_1$ and $<_{SC(\mathcal{F})} \mathcal{U}_2$.

Proposition 3.3.8. *Let $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n \in SC(\mathcal{F})$ with $\mathcal{U} = \bigcup_{1 \leq i \leq n} \mathcal{U}_i$ such that $|\mathcal{U}|$ is connected and each $\mathcal{U}_i$ is either a minimal element of $M(\mathcal{F})$ or has only minimal elements of $M(\mathcal{F})$ as $M(\mathcal{F})$-predecessors. Then there exists a $\mathcal{M}$ that is a minimal element of $M(\mathcal{F})$ such that for every $1 \leq i \leq n$, $\mathcal{U}_i = \mathcal{M}$ or $\mathcal{M}$ is the unique $M(\mathcal{F})$-predecessor of $\mathcal{U}_i$.*

Proof: Let $\mathcal{M}$ be a minimal element of $M(\mathcal{F})$ and c be a neighboring cube of $\mathcal{M}$ (i.e $c \notin \mathcal{M}$ and $|\mathcal{M}| \cap |c| \neq \emptyset$). Then $SCC(c) >_{SC(\mathcal{F})} \mathcal{M}$ as $\mathcal{M}$ is minimal in $SC(\mathcal{F})$ and by Lemma 3.3.7 $\mathcal{M}$ and $SCC(c)$ cannot be incomparable.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two minimal elements of $M(\mathcal{F})$ such that $\mathcal{M}$ is an $M(\mathcal{F})$-Predecessor of some $\mathcal{U}_i$ and $\mathcal{M}'$ is an $M(\mathcal{F})$-Predecessor of some $\mathcal{U}_j$. From Proposition 2.5.4, there exist k and k' such that $\mathcal{M} \cap \mathcal{F}^l(\mathcal{U}) \neq \emptyset$ for all $l \geq k$ and $\mathcal{M}' \cap \mathcal{F}^l(\mathcal{U}) \neq \emptyset$ for all $l \geq k'$. Thus, if $K = max(k, k')$ both $\mathcal{M} \cap \mathcal{F}^K(\mathcal{U})$ and $\mathcal{M}' \cap \mathcal{F}^K(\mathcal{U})$ are non-empty. Also, since $\mathcal{M}$ and $\mathcal{M}'$ are $M(\mathcal{F})$-Predecessors of $\mathcal{U}_i$ and $\mathcal{U}_j$, any $\mathcal{U}'$ such that $\mathcal{U}_i \geq \mathcal{U}' > \mathcal{M}$ or $\mathcal{U}_j \geq \mathcal{U}' > \mathcal{M}'$ is a non-recurrent component. For such a $\mathcal{U}'$, if $\mathcal{U}' \in \mathcal{F}^l(\mathcal{U})$, then $\mathcal{U}' \notin \mathcal{F}^k(\mathcal{U})$ for all $k > l$. This implies none of the neighboring cubes of $\mathcal{M}$ or $\mathcal{M}'$ are in $\mathcal{F}^K(\mathcal{U})$. Which means $\mathcal{M}$ and $\mathcal{M}'$ must be in different connected components of $\mathcal{F}^K(\mathcal{U})$.

However, since $|\mathcal{U}|$ is connected, it implies that $f^n(|\mathcal{U}|)$ and hence $|\mathcal{F}^n(\mathcal{U})|$ is connected for every $n \in \mathbb{N}$. Which is a contradiction.

Chapter 4

Resolve-OA

We make use of the binary sub-division operation to refine a cubical complex $\mathcal{C}$. However, when a $c \in \mathcal{C}^{top}$ is sub-divided, we also must worry about how to alter $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ to define a multi-valued map on $\mathcal{C}'^{top}$ corresponding to $T^{\mathcal{C}'} = SubD_c(T^{\mathcal{C}})$. For this, we first define a *Subdivide* operation.

Definition 4.0.1. Let c_0 be a top-dimensional cube of $T^{\mathcal{C}}$, i.e., $c \in T^{\mathcal{C}}_l = \mathcal{C}^{top}$. We define a sub-division operation $Subdivide_{c_0}$ that takes as input the pair $(T^{\mathcal{C}}, \mathcal{F})$ where $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ is any multi-valued map on $T^{\mathcal{C}}$.

$$(T^{\mathcal{C}'}, \mathcal{F}') = Subdivide_{c_0}(T^{\mathcal{C}}, \mathcal{F})$$

.

It outputs the pair $(T^{\mathcal{C}'}, \mathcal{F}')$ where $T^{\mathcal{C}'} = BSD_{c_0}(T^{\mathcal{C}})$ and $\mathcal{F}' : \mathcal{C}'^{top} \rightrightarrows \mathcal{C}'^{top}$ is given by:

$$\mathcal{F}'(c') = \begin{cases} \mathcal{F}(c') & \text{if } c' \notin ch(c_0) \text{ and } c_0 \notin \mathcal{F}(c') \\ (\mathcal{F}(c') \setminus \{c_0\}) \cup ch(c_0) & \text{if } c' \notin ch(c_0) \text{ and } c_0 \in \mathcal{F}(c') \\ \mathcal{F}(c_0) & \text{if } c' \in ch(c_0) \text{ and } c_0 \notin \mathcal{F}(c_0) \\ (\mathcal{F}(c_0) \setminus \{c_0\}) \cup ch(c_0) & \text{if } c' \in ch(c_0) \text{ and } c_0 \in \mathcal{F}(c_0) \end{cases} \tag{4.1}$$

Note that $\mathcal{C}'^{top} = (\mathcal{C}^{top} \setminus \{c\}) \cup ch(c)$. Hence, $\mathcal{F}'$ is a well defined multi-valued map on $\mathcal{C}'^{top}$.

If $c' \notin ch(c_0)$, then $\mathcal{F}'(c') = z(\mathcal{F}(c'))$ and if $c' \in ch(c_0)$, $\mathcal{F}'(c') = z(\mathcal{F}(c_0))$.

$\mathcal{F}'$ is a refinement of $\mathcal{F}$, however its dynamics are not very different. If $\{c_0\}$ is a non-recurrent component in $SC(\mathcal{F})$, then for each $c' \in ch(c_0)$, $\{c'\}$ would be a non-recurrent component in $SC(\mathcal{F}')$. Further $SC(\mathcal{F}) \setminus \{c_0\} = SC(\mathcal{F}') \setminus ch(c_0)$ and for any $S \in SC(\mathcal{F}) \setminus \{c_0\}$, $S \leq_{SC(\mathcal{F}')} (\geq_{SC(\mathcal{F}')})\{c'\} \iff S \leq_{SC(\mathcal{F})} (\geq_{SC(\mathcal{F})})\{c_0\}$ for all $c' \in ch(c_0)$. On the other hand, if $c_0 \in \mathcal{M}$ for some Morse set $\mathcal{M} \in M(\mathcal{F})$, then $\mathcal{M}' = (\mathcal{M} \setminus \{c\}) \cup ch(c)$ is a Morse set $\mathcal{M}' \in M(\mathcal{F}')$ such that $|\mathcal{M}| = |\mathcal{M}'|$.

Hence, the *Subdivide* operation is not enough to refine a minimal outer approximation $\mathcal{F}$ in a way to resolve the dynamics. To do so, we make the assumption that given our continuous function f of interest, for any cubed complex $T^{\mathcal{C}}$, and for any $c \in \mathcal{C}^{top}$, we can compute $\{c' \in \mathcal{C}^{top} \mid |c'| \cap f(|c|) \neq \emptyset\}$ - the value of the minimal outer approximation of f on c. We define another operation *ComputeMOA* that given a cubed complex $T^{\mathcal{C}}$, a $c_0 \in \mathcal{C}^{top}$ and our continuous function f of interest, takes as input an outer approximation $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$.

$$\mathcal{G} = ComputeMOA_{c_0}(\mathcal{F}; f)$$

.

Outputs a $\mathcal{G} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ given by

$$\mathcal{G}(c) = \begin{cases} \mathcal{F}(c) & \text{if } c \neq c_0 \\ \{c' \in \mathcal{C}^{top} \mid |c'| \cap f(|c|) \neq \emptyset\} & \text{if } c = c_0 \end{cases} \tag{4.2}$$

For a $p \in P$, we introduce the notation $\downarrow_1 p = \downarrow p \setminus \{p\} = \{q \in p \mid q <_P p\}$.

Definition 4.0.2. A $\mathcal{U} \in SC(\mathcal{F}) \setminus M(\mathcal{F})$ is bottom-resolved if every $\mathcal{U}' \in \downarrow_1 \mathcal{U} \setminus M(\mathcal{F})$ (i.e., $\mathcal{U}' < \mathcal{U}$ and $\mathcal{U}' \notin M(\mathcal{F})$) has a unique $M(\mathcal{F})$-Predecessor.

Let $BR(\mathcal{F})$ denote the set of bottom-resolved elements of $SC(\mathcal{F}) \setminus M(\mathcal{F})$.

Given a $\mathcal{U} \in BR(\mathcal{F}))$ we can consider the poset morphism $\sigma : \downarrow_1 \mathcal{U} \to M(\mathcal{F})$ given by:

$$\sigma(\mathcal{U}') = \begin{cases} \mathcal{U}' & \text{if } \mathcal{U}' \in M(\mathcal{F}) \\ \text{the unique } M(\mathcal{F})\text{-Predecessor of } \mathcal{U}' & \text{otherwise} \end{cases} \tag{4.3}$$

Definition 4.0.3. Let $\mathcal{U} \in BR(\mathcal{F})$ be bottom-resolved. A $v \in \mathcal{C} \setminus \mathcal{C}^{top}$ such that $top(v) \subseteq \mathcal{F}(\mathcal{U})$ is a problematic intersection for $\mathcal{U}$ if there does not exist a $\mathcal{M} \in M(\mathcal{F}) \cap \downarrow \mathcal{U}$ such that $\mathcal{M} \geq_{M(\mathcal{F})} \sigma(SCC(u))$ for all $u \in top(v)$.

Let $BR_0(\mathcal{F}) = \{\mathcal{U} \in BR(\mathcal{F}) \mid \mathcal{U} \text{ has no problematic intersections}\}$.

We first introduce the algorithm *Sort-Intersections*$(\mathcal{U}, Q)$, which takes as input a $\mathcal{U} \in BR(\mathcal{F})$ and a $Q \subseteq \mathcal{F}(\mathcal{U})$ and simply outputs the set of problematic intersections of $\mathcal{U}$ which lie in the interior of $|Q|$ i.e. satisfy $top(v) \subseteq Q$. The pseudo-code is described in **Algorithm 1**.

Algorithm 1 Sort-Intersections

Input: A cubed complex $T^{\mathcal{C}}$ of $[0,1]^d$; a multi-valued map $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$; a $\mathcal{U} \in BR(\mathcal{F})$; and a subset $Q \subseteq \mathcal{F}(\mathcal{U})$
set $A = \emptyset$
for $c \in \mathcal{C} \setminus \mathcal{C}^{top}$ such that $top(c) \subseteq Q$ **do**
  if c is a problematic intersection **then**
    Add c to A
  end if
end for
Output: A

We also introduce two division algorithms:

1. *Divide-1* takes as input a $\mathcal{U} \in BR_0(\mathcal{F})$. It involves sub-dividing $\mathcal{U}$ until each of its sub-cubes $\mathcal{U}'$ satisfies the condition that there exists a $\mathcal{M} \in M(\mathcal{F}) \cap \downarrow \mathcal{U}$ such that $\mathcal{M} \geq \sigma(SCC(v))$ for all $v \in \mathcal{F}(\mathcal{U}')$. i.e, it subdivides $\mathcal{U}$, computes the minimal outer approximation for all the children of $\mathcal{U}$, and checks which ones satisfy the above condition. Those that do not are further subdivided and this process is repeated until all the sub-cubes satisfy the condition. For each sub-cubes $\mathcal{U}'$, the $\mathcal{M}$ that ensures the above condition is satisfied if added to $\mathcal{F}'(\mathcal{U}')$ so that $\mathcal{M}$ is the unique $M(\mathcal{F}')$-predecessor of $\mathcal{U}'$. It outputs the sub-cubes of $\mathcal{U}$. The pseudo-code is described in **Algorithm 2**. In Proposition 4.0.5, we show it ends in finite steps.

2. *Divide-2* takes as input a set of non top-dimensional cells $V \subseteq (\mathcal{C} \setminus \mathcal{C}^{top})$. For each $v \in V$, it sub-divides the top cells that surround v, i.e., in $top^*(v)$, and computes the minimal outer

Algorithm 2 Divide-1

Input: A cubed complex $T^{\mathcal{C}}$ of $[0,1]^d$; a continuous function $f:[0,1]^d \to [0,1]^d$; an outer approximation $\mathcal{F}: \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ of f; and a $\mathcal{U} \in BR_0(\mathcal{F})$
Set $R = \mathcal{U} = \{c\}$, $S = \emptyset$, $T^{\mathcal{X}} = T^{\mathcal{C}}$ and $\mathcal{G} = \mathcal{F}$
while R is non-empty **do**
  Pick a c' in R
  $T^{\mathcal{X}}, \mathcal{G} = Subdivide_{c'}(T^{\mathcal{X}}, \mathcal{G})$
  for v in $ch(c')$ **do**
    $\mathcal{G} = ComputeMOA_v(\mathcal{G}; f)$
    if $\exists\, \mathcal{M} \in M(\mathcal{G}) \cap \downarrow \mathcal{U}$ such that $\mathcal{M} \geq M(\mathcal{G}) \cap \downarrow \{v\}$ **then**
      Set $\mathcal{G}(v) = \mathcal{G}(v) \cup \mathcal{M}$
      Add v to S
    else
      Add v to R
    end if
  end for
  Remove c' from R
end while
set $T^{\mathcal{C}'} = T^{\mathcal{X}}$ and $\mathcal{F}' = \mathcal{G}$
Output: $T^{\mathcal{C}'}, \mathcal{F}', S$

approximation of f for those sub-cells surrounding v. It checks which of these maps to $Y_v = \bigcap_{w \in top(v)} \downarrow w$ under $\mathcal{F}$. For those that do not, it sub-divides them further and repeats the process until all the top cells surrounding v map to Y under $\mathcal{F}$. It outputs B - the set of non-recurrent cubes that map to $z(Y_v)$ for each $v \in V$. For the rest of the sub-cubes, it sets $\mathcal{F}'$ to be $\mathcal{F}$ of its parent cell in $\mathcal{C}$. The pseudo-code is described in **Algorithm 3**. In Proposition 4.0.6 we show it ends in finite steps.

We then finally describe the algorithm $Divide - Meta$ which takes as input a $\mathcal{U} \in BR(\mathcal{F})$ along with $T^{\mathcal{C}}$ and $\mathcal{F}$, and outputs a refinement $T^{\mathcal{C}'}$ of $T^{\mathcal{C}}$ and $\mathcal{F}'$ of $\mathcal{F}$ such that all sub-cubes of $\mathcal{U}$ and all non-recurrent components in $\bigcup_{\mathcal{V} \in z(\mathcal{U})} \downarrow \mathcal{V}$ have a unique $M(\mathcal{F})$-Predecessor. It is shown visually in Figure 4.1, and the pseudo-code is described in **Algorithm 4**.

We are finally able to describe the algorithm *Resovle-OA*, described in **Algorithm 5**.

Conjecture 4.0.4. *Let $T^{\mathcal{C}}$ be a cubed complex of the unit cube $[0,1]^d$ and $\mathcal{F}: \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ be the minimal outer approximation of a continuous function $f:[0,1]^d \to [0,1]^d$ on $T^{\mathcal{C}}$. Resolve-OA($T^{\mathcal{C}}, f, \mathcal{F}$) returns a refinement $T^{\mathcal{C}'}$ of $T^{\mathcal{C}}$ and an outer approximation $\mathcal{F}': \mathcal{C}'^{top} \rightrightarrows \mathcal{C}'^{top}$ of f that is a refinement of $\mathcal{F}$ such that there exists an order retraction $\sigma : SC(\mathcal{F}') \to M(\mathcal{F}')$*

Algorithm 3 Divide-2

Input: A cubed complex $T^{\mathcal{C}}$ of $[0,1]^d$; a continuous $f : [0,1]^d \to [0,1]^d$; an outer approximation $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ of f; and a $V \subseteq (\mathcal{C} \setminus \mathcal{C}^{top})$

set $T^{\mathcal{X}} = T^{\mathcal{C}}$ and $\mathcal{G} = \mathcal{F}$

for $v \in V$ **do**

  set $Y = \bigcap_{\mathcal{V} \in top(v)} \downarrow SCC(\mathcal{V}) \subseteq SC(\mathcal{F})$

  set $X = top^*(v) \subseteq \mathcal{X}^{top}$

  $B = \emptyset$

  while X is non-empty **do**

    Pick an element x of X

    if $\mathcal{G}(x) \subseteq z(Y)$ **then**

      remove x from X

      if $\{x\} \in SC(\mathcal{G}) \setminus M(\mathcal{G})$ **then**

        add x to B

      end if

    else

      $T^{\mathcal{X}}, \mathcal{G} = Subdivide_x(T^{\mathcal{X}}, \mathcal{G})$

      for $c \in ch(x)$ **do**

        if $c \in top^*(v)$ **then**

          $\mathcal{G}' = ComputeMOA_c(\mathcal{G}; f)$

          if $\mathcal{G}'(c) \nsubseteq z(Y)$ **then**

            add c to X

          else

            set $\mathcal{G} = \mathcal{G}'$

            add c to B

          end if

        end if

      end for

      remove x from X

    end if

  end while

end for

set Z to be the union of the descendants of the cubes in B in $T^{\mathcal{X}}$

set $B' = c \in Z$ such that $c \in top^*(v)$ for some $v \in V$

for all $c \in Z \setminus B'$, set $\mathcal{G}(c)$ to be $z(\mathcal{F}(x))$ for the ancestor x of c in $T^{\mathcal{C}}$

set $T^{\mathcal{C}'} = T^{\mathcal{X}}$ and $\mathcal{F}' = \mathcal{G}$

Output: $T^{\mathcal{C}'}, \mathcal{F}', B'$

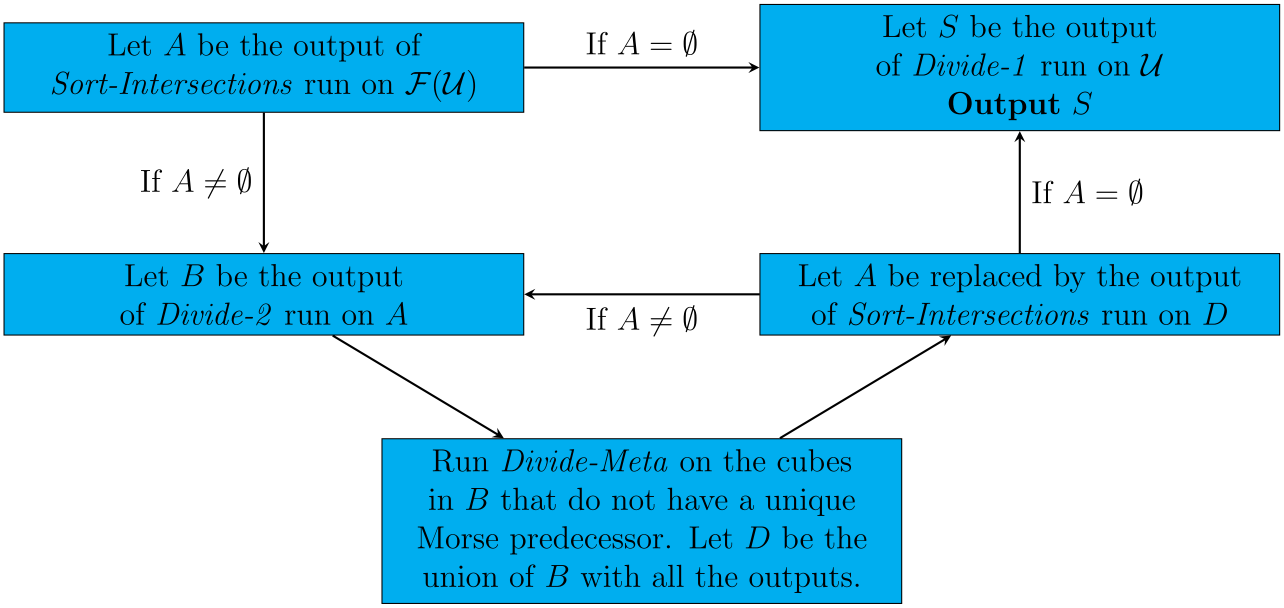


Figure 4.1: ***A diagram depicting*** *Divide-Meta.*

Algorithm 4 Divide-Meta

Input: A cubed complex $T^{\mathcal{C}}$ of $[0,1]^d$; a continuous $f : [0,1]^d \to [0,1]^d$; an outer approximation $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ of f; and a $\mathcal{U} \in BR(\mathcal{F})$;
set $k = 0$
set $T^{\mathcal{C}_0} = T^{\mathcal{C}}$ and $\mathcal{F}_0 = \mathcal{F}$
set $A = A_0 = \textit{Sort-Intersections}(T^{\mathcal{C}_0}, \mathcal{F}_0, \mathcal{U}, \mathcal{F}_0(\mathcal{U}))$
while A is non-empty **do**
    set $k = k + 1$
    set $(T^{\mathcal{X}} = T^{\mathcal{X}_k}, \mathcal{G} = \mathcal{G}_k, B = B_k) = \textit{Divide-2}(T^{\mathcal{C}_{k-1}}, f, \mathcal{F}_{k-1}, A_{k-1})$;
    set $C = C_k \subseteq B$ to be those elements of B that do not have a unique $M(\mathcal{F})$-predecessor
    set $B = B \setminus C$
    while C is non-empty **do**
        pick a minimal element c of $C \subset SC(\mathcal{G})$
        $T^{\mathcal{X}}, \mathcal{G}, R = \textit{Divide-Meta}(T^{\mathcal{X}}, f, \mathcal{G}, \{c\})$
        $B = B \cup R$
        remove c from C
    end while
    $T^{\mathcal{C}_k} = T^{\mathcal{X}}$, $\mathcal{F}_k = \mathcal{G}$ and $D_k = B$
    $A = A_k = \textit{Sort-Intersections}(T^{\mathcal{C}_k}, \mathcal{F}_k, \mathcal{U}, D_k)$
end while
$T^{\mathcal{C}'}, \mathcal{F}', S = \textit{Divide-1}(T^{\mathcal{C}_k}, \mathcal{F}_k, f, \mathcal{U})$
Output: $T^{\mathcal{C}'}, \mathcal{F}', S$

Algorithm 5 Resolve-OA

Input: A cubed complex $\mathcal{C}$ of $[0,1]^d$; a continuous $f : [0,1]^d \to [0,1]^d$; and a minimal outer approximation $\mathcal{F} : \mathcal{C}^{top} \rightrightarrows \mathcal{C}^{top}$ of f
Initialize $\mathcal{T} = SC(\mathcal{F}) \setminus M(\mathcal{F})$, $Y = \emptyset$
set $T^{\mathcal{X}} = T^{\mathcal{C}}$ and $\mathcal{G} = \mathcal{F}$
while $\mathcal{T}$ is non-empty **do**
  Pick a minimal element $\mathcal{U}$ of $\mathcal{T}$
  if $\mathcal{U}$ has a unique $M(\mathcal{F})$-predecessor **then**
    remove $\mathcal{U}$ from $\mathcal{T}$
  else if if every $\mathcal{U}' \in \uparrow_{\mathcal{T}} \mathcal{U}$ has a unique $M(\mathcal{F})$-successor **then**
    set $Y = Y \cup \uparrow_{\mathcal{T}} \mathcal{U}$
    set $\mathcal{T} = \mathcal{T} \setminus \uparrow_{\mathcal{T}} \mathcal{U}$
  else
    $T^{\mathcal{X}}, \mathcal{G}, S = \textit{Divide-meta}(T^{\mathcal{X}}, f, \mathcal{G}, \mathcal{U})$
    remove $\mathcal{U}$ from $\mathcal{T}$
  end if
end while
set $T^{\mathcal{C}} = T^{\mathcal{X}}$, $\mathcal{F}' = \mathcal{G}$ and $Y = z(Y)$
Output: $T^{\mathcal{C}'}, \mathcal{F}', Y$

of the inclusion map i.

Furthermore, there exists a poset isomorphism $\rho : M(\mathcal{F}') \to M(\mathcal{F})$ such that $\rho(\mathcal{M})$ is the unique element of $M(\mathcal{F})$ such that $|\mathcal{M}| \subseteq |\rho(\mathcal{M})|$.

We begin working towards a partial proof with the following proposition.

Proposition 4.0.5. *The algorithm Divide $-$ 1 ends in finite steps.*

Proof. Let $N = max(\{n \mid c = (\mathbf{p}, \mathbf{b}, k) \in \mathcal{C}^{top}\})$. Then, the side length of the smallest top-dimensional cube in $\mathcal{C}$ is 2^{-N}. By Remark 2.4.3, if $c_1, c_2 \in \mathcal{C}^{top}$ such that $|c_1| \cap |c_2| = \emptyset$, then $D(|c_1|, |c_2|) \geq 2^{-N}$. Thus, if U is a set in $[0,1]^d$ with $diam(U) \leq 2^{-N}$ such that $U \cap |c_1| \neq \emptyset$ and $U \cap |c_2| \neq \emptyset$ for two cubes $c_1, c_2 \in \mathcal{C}^{top}$, then $|c_1| \cap |c_2| \neq \emptyset$.

Since f is continuous on a compact space $[0,1]^d$, it is uniformly continuous. i.e there exists ϵ such that $diam(V) < \epsilon$ implies that $diam(f(U)) < 2^{-N}$. Thus, if $\mathcal{U}' \in z(\mathcal{U})$ is a descendent of $\mathcal{U}$ in $T^{\mathcal{X}}$ with $diam(\mathcal{U}') < \epsilon$, then $diam(f(|\mathcal{U}'|)) < 2^{-N}$. Thus, $\bigcap_{\mathcal{V} \in \mathcal{G}(\mathcal{U}')} \mathcal{V} \neq \emptyset$ and since this cannot be a problematic intersection for $\mathcal{U}$, there exists $\mathcal{M} \in M(\mathcal{G}) \cap \downarrow \mathcal{U}$ such that $\mathcal{M} \geq M(\mathcal{G}) \cap \downarrow \mathcal{U}'$.

□

Proposition 4.0.6. *The algorithm Divide* -2 *ends in finite steps.*

Proof. Let $U = \text{int}(|\bigcap_{c \in top(v)} \downarrow SCC(c)|)$. $f(|v|) \subseteq U$ as $\mathcal{F}$ is an outer-approximation. Let $O = f^{-1}(U)$. O is an open set such that $|v| \subset O$.

Since $|v|$ and O^C are disjoint closed sets, there exists a finite distance ϵ between them. Thus, for any cube $c \in \mathcal{C}'^{top}$ with $diam(c) < \epsilon$ and $c \in top^*(v)$, we have that $f(|c|) \subseteq |U|$ and hence, $\mathcal{F}'(c) \subseteq z(\bigcap_{\mathcal{V} \in top(v)} \downarrow SCC(\mathcal{V}))$.

□

Conjecture 4.0.7. *Divide-Meta*$(T^{\mathcal{C}}, f, \mathcal{F}, \mathcal{U})$ *outputs a refinement* $T^{\mathcal{C}'}$ *of* $T^{\mathcal{C}}$ *and an outer approximation* $\mathcal{F}'$ *of* f *which is a refinement of* $\mathcal{F}$ *such that every non-recurrent component in* $\bigcup_{\mathcal{V} \in z(\mathcal{U})} \downarrow \mathcal{V}$ *has a unique* $M(\mathcal{F})$*-Predecessor.*

Furthermore, there exists a poset isomorphism $\rho : M(\mathcal{F}') \to M(\mathcal{F})$ *such that* $\rho(\mathcal{M})$ *is the unique element of* $M(\mathcal{F})$ *such that* $|\mathcal{M}| \subseteq |\rho(\mathcal{M})|$.

Notice that the algorithm *Divide-Meta* is recursive. Within *Divide-Meta*$(T^{\mathcal{C}}, f, \mathcal{F}, \mathcal{U})$, *Divide-Meta*$(T^{\mathcal{X}}, f, \mathcal{G}, \{c\})$ is called for a refinement $T^{\mathcal{X}}$ of $T^{\mathcal{C}}$, a refinement $\mathcal{G}$ of $\mathcal{F}$ that is an outer-approximation of f and a $\{c\} \in \downarrow_1 \mathcal{U}$ under $\mathcal{G}$. We make the following assumption which we call **assumption 1**: every time *Divide-Meta*$(T^{\mathcal{X}}, f, \mathcal{G}, \{c\})$ is called within *Divide-Meta*$(T^{\mathcal{C}}, f, \mathcal{F}, \mathcal{U})$, Conjecture 4.0.7 holds for *Divide-Meta*$(T^{\mathcal{X}}, f, \mathcal{G}, \{c\})$. Under this assumption, we can prove the conjecture holds true for *Divide-Meta*$(T^{\mathcal{C}}, f, \mathcal{F}, \mathcal{U})$. For the rest of the chapter, we work with this assumption.

Proposition 4.0.8. *Under assumption 1, for each* k*, there exists a poset isomorphism* $\rho_k : M(\mathcal{G}_k) \to M(\mathcal{F}_{k-1})$ *such that* $\rho(\mathcal{M})$ *is the unique element of* $M(\mathcal{F}_{k-1})$ *such that* $|\mathcal{M}| \subseteq |\rho(\mathcal{M})|$.

Proof. First note that if $\{c\} \in SC(\mathcal{F}_{k-1}) \setminus M(\mathcal{F}_{k-1})$ then for each $c' \in z(c)$, $\{c'\} \in SC(\mathcal{G}_k) \setminus M(\mathcal{G}_k)$ as $\mathcal{G}_k$ is a refinement of $\mathcal{F}_{k-1}$. Hence $|M(\mathcal{G}_k)| \subseteq |M(\mathcal{F}_{k-1})|$.

We also note that for every $c \in \mathcal{G}_k(\mathcal{U}) \setminus B_k$, $\mathcal{G}_k(c) = z(\mathcal{F}_{k-1}(c'))$ where c' is the ancestor of c in $T^{\mathcal{C}_{k-1}}$ and for any $c \in B_k$, $\{c\} \in SC(\mathcal{G}_k) \setminus M(\mathcal{G}_k)$.

Next we note that for each $c \in \mathcal{C}_{k-1}$, there exists a $c' \in z(c)$ such that $\mathcal{G}_k(c') = z(\mathcal{F}_{k-1}(c))$. Assume there is a c such that this is not the case. This means that there exists a $v \in A_{k-1}$ such that $c \in top(v)$ else c would never be subdivided and hence $z(c) = \{c\}$ with $\mathcal{G}_k(c) = z(\mathcal{F}_{k-1}(c))$. Then, when c is sub-divided, for those cubes $c' \in ch(c)$ such that $c' \in top^*(v)$, the minimal outer approximation would be computed. However, there exists at least one cube $c' \in ch(c)$ which is not in $top^*(v)$. This would be subdivided if there existed another $v' \in A_{k-1}$ such that $c' \in top^*(v')$. However, when c' is subdivided, there would exist a $c'' \in ch(c')$ such that $|c''| \subset \mathrm{int}(|c|)$. This would retain $\mathcal{G}_k(c'') = z(\mathcal{F}_{k-1}(c))$ and not be subdivided after as it is not in $top^*(v)$ for any $v \in A_{k-1}$. This is a contradiction.

Thus for every $\mathcal{M} \in M(\mathcal{F}_{k-1})$, $z(\mathcal{M}) \setminus B_k$ is a non-empty Morse set in $M(\mathcal{G}_k)$. Define $\rho'_k : M(\mathcal{F}_{k-1}) \to M(\mathcal{G}_k)$ by $\rho(\mathcal{M}) = z(\mathcal{M}) \setminus B_k$.

ρ is an isomorphism. For any path $\{c_0, c_1, ..., c_l\}$ in $\mathcal{F}_{k-1}$, for each $0 \leq i \leq l-1$, there exists a $c'_i \in z(c_i)$ such that $\mathcal{G}_k(c'_i) = z(\mathcal{F}_{k-1}(c_i)) \supseteq ch(c_{i+1})$. Hence there exists a path $\{c'_0, c'_1, ..., c'_l\}$ in $\mathcal{G}_k$. This means if $\mathcal{M}_1 \leq \mathcal{M}_2$ in $M(\mathcal{F}_{k-1})$, then $\rho'(\mathcal{M}_1) \leq \rho'(\mathcal{M}_2)$ in $M(\mathcal{G}_k)$. Further since $\mathcal{G}_k$ is a refinement of $\mathcal{F}_{k-1}$, $\rho'(\mathcal{M}_1) \leq \rho'(\mathcal{M}_2)$ in $M(\mathcal{G}_k)$ only if $\mathcal{M}_1 \leq \mathcal{M}_2$ in $M(\mathcal{F}_{k-1})$.

So $\rho_k = (\rho'_k)^{-1}$ is the isomorphism we desire. □

Proposition 4.0.9. *Under assumption 1, for each k, $\mathcal{U}$ is bottom-resolved under $\mathcal{F}_k$.*

Proof. We show this via induction. This is true for $\mathcal{F}_0$. Assume it is true for $\mathcal{F}_{k-1}$. By assumption 1, every cube in D_k has a unique $M(\mathcal{F})$-predecessor. It is left to show that for every $c \in \mathcal{F}_k(\mathcal{U}) \setminus D_k$, $SCC(c)$ is either a Morse set or has a unique $M(\mathcal{F}_k)$-predecessor.

We first show that for every $c \in \mathcal{G}_k(\mathcal{U}) \setminus B_k$, $SCC(c)$ is either a Morse set or has a unique $M(\mathcal{G}_k)$-predecessor. Then we note that again by the assumption, if $SCC(c) \leq \{c'\}$ for some $c' \in C_k$, every cube in $z(c)$ is either part of a Morse set or has a unique $M(\mathcal{F}_k)$-predecessor. Otherwise, c is not sub-divided and thus is either part of a Morse set or has a unique $M(\mathcal{F}_k)$-predecessor.

However, for every $c \in \mathcal{G}_k(\mathcal{U}) \setminus B_k$, $\mathcal{G}_k(c) = z(\mathcal{F}_{k-1}(c'))$ where c' is the ancestor of c in $T^{\mathcal{C}_{k-1}}$. This means $c \in \mathcal{M} \in M(\mathcal{G}_k)$ if $c' \in \rho(\mathcal{M}) \in M(\mathcal{F}_{k-1})$, Also $\mathcal{M}$ is the unique $M(\mathcal{G}_k)$-predecessor of c if $\rho_k(\mathcal{M})$ is the unique $M(\mathcal{F}_{k-1})$-predecessor of c'. □

Proposition 4.0.10. *Under assumption 1, For all k, if v is a problematic intersection of $\mathcal{U}$ under $\mathcal{F}_k$, then $top(v) \subseteq D_k$.*

Proof. Let v be a problematic intersection of $\mathcal{U}$ under $\mathcal{F}_k$. Assume $top(v) \subseteq ch(c)$ for some $c \in \mathcal{C}^{top}_{k-1}$. $SCC(c)$ is either a Morse set itself or has a unique $M(\mathcal{F}_{k-1})$ predecessor - say $\mathcal{M}$. Then $\mathcal{M} \geq \sigma(SCC(c'))$ for all $c' \in ch(c)$. Thus, such a v cannot be a problematic intersection. Thus, it must be that if v is a problematic intersection of $\mathcal{U}$ under $\mathcal{F}_k$, then v is contained in a problematic intersection v' of $\mathcal{U}$ under $\mathcal{F}_{k-1}$. In such a case, $top(v) \subseteq D_k$. □

Corollary 4.0.11. *A_k contains all the problematic intersections of $\mathcal{U}$ under $\mathcal{F}_k$.*

Proposition 4.0.12. *Under assumption 1, there exists a K such that $A_K = \emptyset$, i.e., the while loop ends after finite iterations.*

Proof. Given a Morse set $\mathcal{M} \in M(\mathcal{F})$, let $p(\mathcal{M})$ denote the length of the longest path in $M(\mathcal{F})$ from $\mathcal{M}$ to a minimal element of $M(\mathcal{F})$. If $\mathcal{M}' <_{M(\mathcal{F})} \mathcal{M}$, then $p(\mathcal{M}') < p(\mathcal{M})$.

For every k, let M_k denote the set $\{\mathcal{M} \in M(\mathcal{F}_k) \mid \exists\, c \in \mathcal{C}^{top}_k$ such that $c \in top(v)$ for some $v \in A_k$ and $\mathcal{M} = \sigma(SCC(c))\}$. Then let $p(A_k) = \max_{\mathcal{M} \in M_k} p(\mathcal{M})$.

We show that $p(A_{k+1}) < p(A_k)$. Let $c \in \mathcal{C}^{top}_k$ such that $c \in top(v)$ for some $v \in A_k$ and $p(\sigma(SCC(c))) = p(A_k) = x$. Then, c will be sub-divided and let $c' \in z(c)$ such that $c' \in top(v')$ for some $v' \in A_{k+1}$ such that $|v'| \subseteq |v|$. Assume $p(\sigma(SCC(c'))) = x$ as well. We know $M(\mathcal{F}_{k+1}) \cong M(\mathcal{F}_k)$ with isomorphism ρ_k. Since $\mathcal{F}_{k+1}(c') \subseteq z(\mathcal{F}_k(c))$, this must mean $\rho_k(\sigma(SCC(c'))) = \sigma(SCC(c))$. But that implies that $\sigma(SCC(c))$ (say $\mathcal{M}$) is in $\bigcap_{a \in top(v)} \downarrow SCC(a)$. Which in turn implies that $\mathcal{M} \leq \sigma(SCC(a))$ for all $a \in top(v)$. However, since $p(\mathcal{M}) = p(A_k)$, it cannot be that $\mathcal{M} < \sigma(SCC(a))$ for any $a \in top(v)$. Hence, v cannot be a problematic intersection. This is a contradiction.

Thus if $p(A_0) = K$, then $p(A_K) = 0$. Proposition 3.3.8 then implies that for each $v \in A_K$, there exists an $\mathcal{M} \in M(\mathcal{F}_K)$ such that $\mathcal{M}$ is the unique $M(\mathcal{F}_K)$-Predecessor for each $c \in top(v)$ and hence v is not a problematic intersection. Which is a contradiction. Hence $A_K = \emptyset$. □

Proposition 4.0.13. *If $(T^{\mathcal{C}}, \mathcal{F}', S) =$ Divide-1$(T^{\mathcal{C}}, f, \mathcal{F}, \mathcal{U})$ such that $\mathcal{F}(\mathcal{U}) = \{c \in \mathcal{C}^{top} \mid |c| \cap f(|\mathcal{U}| \neq \emptyset)\}$ Then $M(\mathcal{F}) = M(\mathcal{F}')$.*

Proof. Since only $\mathcal{U} = \{c\} \in SC(\mathcal{F}) \setminus M(\mathcal{F})$ and its sub-cells are sub-divided, the Morse sets themselves are unchanged. Assume $\mathcal{M}_1 >_{M(\mathcal{F})} \mathcal{M}_2$. If there exists a path from $\mathcal{M}_1$

to $\mathcal{M}_2$ in $SC(\mathcal{F})$ that doesn't contain $\mathcal{U}$ - that path remains in $\mathcal{F}'$ and $\mathcal{M}_1 >_{M(\mathcal{F}')} \mathcal{M}_2$. Further, for any path $\{\mathcal{M}_1, ..., \mathcal{V}, \mathcal{U}, \mathcal{V}', ..., \mathcal{M}_2\}$ in $SC(\mathcal{F})$, since $\mathcal{F}(\mathcal{U})$ is the minimal outer approximation, there will exist a descendent c' of the cube in $\mathcal{U}$ such that $\mathcal{F}'(c) \cap \mathcal{V} \neq \emptyset$ and hence $\{\mathcal{M}_1, ..., \mathcal{V}, \{c'\}, \mathcal{V}', ..., \mathcal{M}_2\}$ will be a path in $SC(\mathcal{F}')$. □

Every time *Divide-1* is run within *Divide-Meta*, the input $\mathcal{U}$ and $\mathcal{F}$ is such that the value of $\mathcal{F}$ on $\mathcal{U}$ is the minimal outer approximation of f.

Now we present a proof of Conjecture 4.0.4 under Assumption 1:

Proof. The output Y of *Resolve-OA* is a union of up sets and hence is itself an up set. Every non-recurrent element of Y has a unique $M(\mathcal{F}')$-successor. Y^C is a down set and every non-recurrent element of Y^C has a unique $M(\mathcal{F}')$-predecessor. Hence, by Lemma 3.3.5, we can define σ an order retraction of $i : M(\mathcal{F}) \to SC(\mathcal{F})$:

$$\sigma(\mathcal{U}) = \begin{cases} \mathcal{U} & \text{if } \mathcal{U} \in M(\mathcal{F}') \\ \text{The unique } M(\mathcal{F}')\text{-predecessor of } \mathcal{U} & \text{if } \mathcal{U} \in Y^C \setminus M(\mathcal{F}') \\ \text{The unique } M(\mathcal{F}')\text{-successor of } \mathcal{U} & \text{if } \mathcal{U} \in Y \setminus M(\mathcal{F}) \end{cases} \tag{4.4}$$

□

Remark 4.0.14. The output $\mathcal{F}'$ is not always a minimal outer approximation. Within *Divide-2*, the value of the multi-valued map for some of the sub-cubes is set to the value of the multi-valued map for its parent cube. This is what allows *Resolve-OA* to preserve the Morse graph.

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