Links, Quantum Groups and TQFTs

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by

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Certificate

This is to certify that this dissertation entitled Links, Quantum Groups and TQFTs towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Shruti Suresh Barapatre at Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rama Mishra, Professor, Department of Mathematics, during the academic year 2023-2024.

Rama Mishra
May 17, 2024

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This thesis is dedicated to my family.

Declaration

I hereby declare that the matter embodied in the report entitled Links, Quantum Groups and TQFTs are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rama Mishra and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

Shruti

15/05/2024

Shruti Suresh Barapatre

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Abstract

This thesis aims to provide a detailed understanding of quantum invariants of knots and links. We present them as the link invariants derived from a representation of some quantum group. The most celebrated invariant, the Jones polynomial, is shown to be obtained from the fundamental representation of the quantum group $U_h(\mathfrak{sl}_2(\mathbb{C}))$, making it an example of a quantum invariant. In this thesis, we focus on the computation of the Jones polynomial. We provide a closed-form expression for the Jones polynomial of the weaving links $W(3,m)$ and observe some patterns in the coefficients of the Jones polynomial of weaving links $W(4,m)$. We emphasise the role of quantum groups as a general machinery for generating link invariants. It is known that for every semi-simple lie algebra $\mathfrak{g}$, one can obtain a quantum group $U_h(\mathfrak{g})$ and define a new link invariant for each of its representations. We also discuss a general approach for constructing link invariants called 'Topological Quantum Field Theories (TQFTs)'. Quantum invariants are examples of 1-dimensional TQFTs. This thesis discusses the converse: every 1-dimensional TQFT arises from a representation of some quantum group. Looking at quantum invariants as 1-dimensional TQFTs allows us to generalise further and study invariants arising from n-dimensional TQFTs. Thus, one has a much broader definition of a quantum invariant, namely those arising from an n-dimensional TQFT.

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Introduction

Knot theory is the study of knots, which are smooth embeddings of S^1 in $\mathbb{R}^3$. More generally, links are smooth embeddings of finitely many copies of S^1 in $\mathbb{R}^3$. Thus, links are nothing but many knots that are linked together. The central problem in knot theory is classifying links up to ambient isotopy (Definition 1.0.2). Link invariants are important tools for distinguishing links. A link invariant is a map from the collection of all links to some co-domain, such that any two equivalent links have the same image. However, two non-equivalent links may also have the same image under a given invariant. So far, no complete invariant of links is known, which is useful. Thus, a need to create more and more invariants still exists.

One well-known polynomial invariant of links is the Jones polynomial. The Jones polynomial was originally defined by V. Jones (1984) [Jon85] via a representation of the braid groups in von Neumann algebras. Later, he showed that a polynomial invariant in two variables can be defined via Hecke algebra representations [Jon87]. The von Neumann algebras and Hecke algebras both provide solutions to the Yang-Baxter equation (which is the key for defining representations of braid groups). However, these algebras can not be extended further to define new link invariants. This is where quantum groups become quite useful.

Quantum groups were introduced by V. Drinfel'd [Dri88] and M. Jimbo (1985) [Jim85]. Quantum groups refer to 1 parameter deformations of the universal enveloping algebra (refer to the first chapter of [KS81]) of some complex lie algebra. They provide solutions to the Yang-Baxter equation. Thus, given a quantum group and its representation, we can define a link invariant. For instance, the Jones polynomial is obtained by the fundamental representation of the quantum group $U_h(\mathfrak{sl}_2(\mathbb{C}))$. For any integer $N \geq 2$, the quantum group $U_h(\mathfrak{sl}_N(\mathbb{C}))$ and its fundamental representation define a link invariant, which results in a countably infinite family of link invariants. Moreover, instead of $\mathfrak{sl}_N(\mathbb{C})$, we can take any semi-simple (See [Kyt11] for a definition) complex Lie algebra. Thus, quantum groups provide a machinery to generate link invariants. Link invariants arising in this fashion were termed as 'Quantum Invariants'.

The theory of quantum groups can be generalised further by regarding them as 1-dimensional Topological Quantum Field Theory (TQFT). TQFTs were defined by E. Witten in 1988 [Wit88]. It is known that 1-dimensional TQFTs give rise to link invariants. Later, it was found that even higher dimensional TQFTs can be used to define link invariants, providing a bigger domain for quantum invariants. Thus, a more general definition of a quantum invariant is the one arising from a TQFT.

Most of this thesis is expository in nature. We have provided a complete account for defining a quantum invariant. Using the definition of quantum trace, we computed the Jones polynomial of a doubly infinite family of links $W(n, m)$ known as *weaving links*. We have obtained some new results related to the coefficients of the Jones polynomial of $W(3, m)$ and $W(4, m)$ family. We have also given a perspective for Topological Quantum Field Theory and explained why they can be used to generalise quantum invariants.

Structure of the thesis

This thesis comprises of four chapters. In the first chapter, we cover the fundamentals of link theory. We introduce the theory of braid groups and link invariants. In the second chapter, we delve into the theory of Hopf algebra and quantum groups. We study the quantum group $U_h(\mathfrak{sl}_2(\mathbb{C}))$ and its representations. In the third chapter, we discuss the method of obtaining link invariants through quantum groups. Utilising this method, we compute the Jones polynomial of weaving links $W(3, m)$ and $W(4, m)$. We obtain a closed-form expression for the Jones polynomial of $W(3, m)$; for $W(4, m)$, we make some observations without proving them. Finally, in the fourth chapter, we discuss the theory of TQFTs. We investigate the relationship of 1-dimensional TQFTs with the representations of ribbon Hopf algebra, which are a specific kind of quantum groups.

Original Contribution

The computations presented in Section 3.4 (excluding the Subsection 3.4.1) and Section 3.5 are original. In particular

1. We proved that the Jones polynomial of weaving links $W(3,m)$ denoted by $V_{W(3,m)}(t)$ is given by the following:

$$V_{W(3,m)}(t) = \frac{(-1)^m}{t^m} C_m(-t) + t + t^{-1}.$$

   where $C_n(x)$ denotes the rank polynomial of the Lucas lattice of order n (Subsection 3.4.2).

2. We observed that the Jones polynomial of the weaving links $W(4,m)$ admits a normal distribution of coefficients (Subsection 3.5.1).

3. We computed and plotted the zeros of $V_{W(3,m)}(t)$ and $V_{W(4,m)}(t)$ for some values of m (Subsection 3.4.3 and 3.5.2).

4. We wrote SageMath and Mathematica codes (Appendix B) for the above calculations and observations. The SageMath code computes the matrices for generators of B_3. The Mathematica code computes the N-coloured Jones polynomial for weaving links $W(n,m)$ for $n, m \geq 1$.

Chapter 1

Links and their Invariants

In this chapter, we delve into the basics of knot theory. The results presented herein are drawn from The Knot Book by C. C. Adams [Ada94]. Intuitively speaking, a classical knot is a simple closed curve in $\mathbb{R}^3$ (or S^3). Formally,

Definition 1.0.1. *A classical knot is a smooth embedding of S^1 into $\mathbb{R}^3$ or S^3.*
A link is a smooth embedding of finitely many disjoint copies of S^1 in S^3.

Definition 1.0.2. *We say two links are ambient isotopic if an orientation-preserving homeomorphism of S^3 exists, which takes one link to another.*

The central problem in link theory is to classify all the links, i.e. given any two links determine whether they are isotopic or not. To do so, we define maps from the collection of all links to some co-domain (e.g. $\mathbb{N}$, Collection of groups, Polynomial rings, etc.). We call such a map a link invariant if given two isotopic links, their images under the map are equal. There are two main approaches for defining invariants of links. The first approach is the 'diagrammatic approach'. In this approach, we obtain a planar diagram corresponding to the link under consideration. Since a link is a subspace of $\mathbb{R}^3$, we can project it to a plane of $\mathbb{R}^3$.

Definition 1.0.3. *A projection of a link to a plane is said to be regular if the only singularities are transversal double points.*

Definition 1.0.4. *A regular projection together with the information of over/underpasses at every double point is called a link (or knot) diagram.*

Crossings are double points with information about the over/under pass in a link diagram. By only considering smooth embeddings, we eliminate the possibility of wild links. Hence, the minimal (taking minimum over all link diagrams) number of crossings for a link is always finite.
The following theorem by Reidemeister [Rei27] gives the relationship between isotopic links and their respective diagrams.

Theorem 1.0.1. *Two links L_1 and L_2 are ambient isotopic if and only if there exists a link diagram of L_1 that can be transformed into a link diagram of L_2 via a finite sequence of the following moves*† *(called the Reidemeister moves).*

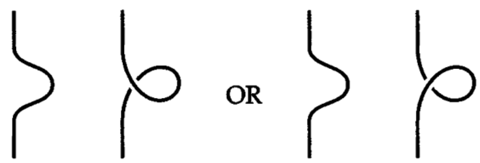


Figure 1.1: Type I Reidemeister move (Twist)

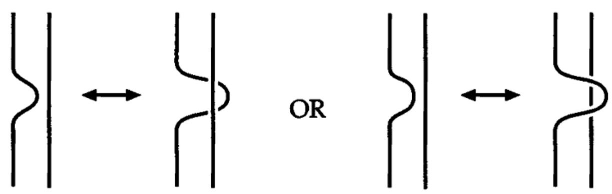


Figure 1.2: Type II Reidemeister move (Pinch)

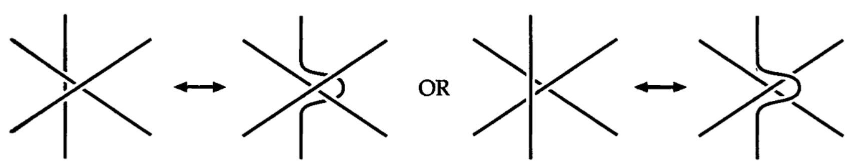


Figure 1.3: Type III Reidemeister move (Slide)

Given the above relationship, we can define link invariants by associating a quantity to each link diagram, such that the assignment is invariant under Reidemeister moves.

†Image Source: The Knot Book by C. C. Adams [Ada94]

1.1 Braids and Braid Groups

The second approach for defining link invariants is through braids and braid groups.

A braid is a set of n strands in $\mathbb{R}^3$, all of which are attached to a horizontal bar at the top and the bottom. Each strand intersects any horizontal plane between the two bars exactly once. Now, if we join the top bar with the bottom bar, we get what is called the closure of the braid (as shown in Figure 1.4), denoted by $\widehat{p}$ for a braid p. Closure of a braid is either a knot or a link. So, given a braid, we can obtain a link by taking its closure. The reverse direction is also true; given a link, we can always get a braid such that its closure is isotopic to the link we started with. This is known as the Alexander Theorem in link theory [Ale23]. A polynomial time algorithm exists to obtain a braid word for a given link. As we will see next, the set of braids on n strands forms a group, making braids valuable tools for studying links.

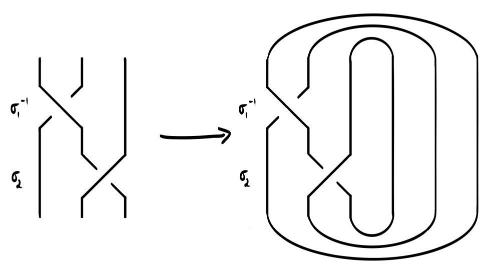


Figure 1.4: Closure of a Braid is a Link

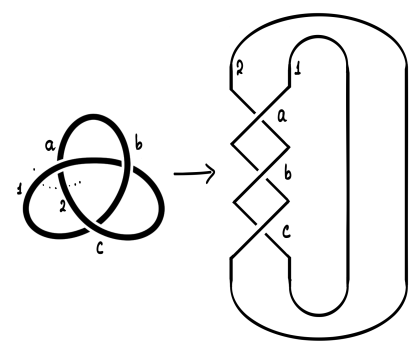


Figure 1.5: A braid for the trefoil knot

Definition 1.1.1. *The Artin braid group on n strands denoted by B_n is generated by the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$, modulo the following relations:*
i. Far commutativity, $\sigma_i \sigma_j = \sigma_j \sigma_i$ for all $|i - j| > 1$.
ii. Yang Baxter Relation,

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{1.1}$$

We can study braids using diagrams, as shown in Figure 1.6 and 1.7. The diagram for σ_i is denoted by the i^{th} strand passing under the $i + 1^{th}$ strand. And σ_i^{-1} is denoted by the i^{th} strand passing over the $i + 1^{th}$ strand.

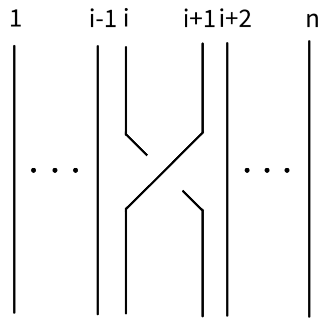


Figure 1.6: Diagram for the generator σ_i

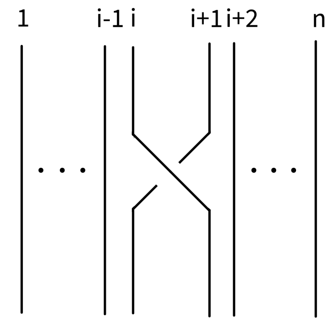


Figure 1.7: Diagram for the generator σ_i^{-1}

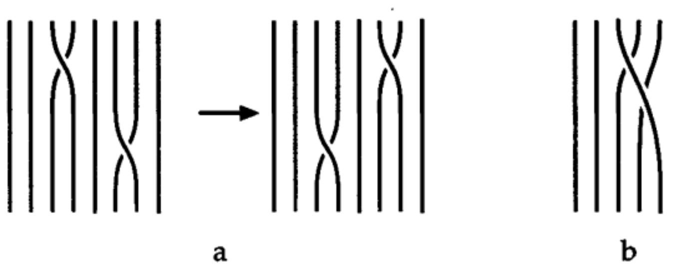


Figure 1.8: ‡(a) Far Commutativity for $|i - j| > 1$ (b) No such relation for $|i - j| = 1$

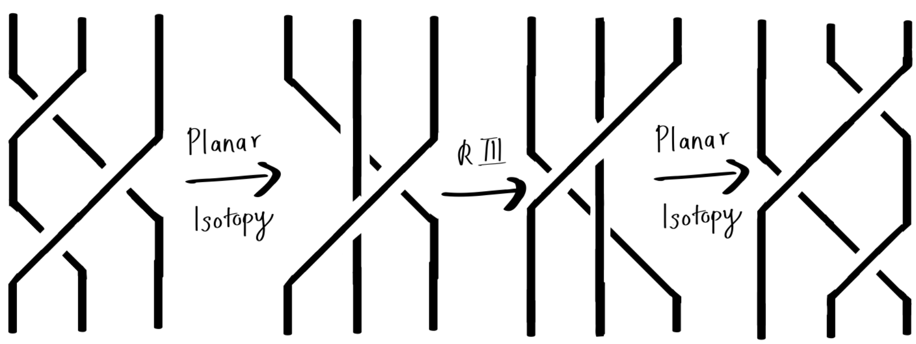


Figure 1.9: Yang Baxter Relation

Note that given a link L, there can be more than one braid whose closure is the link L,

‡Image Source: The Knot Book by C. C. Adams [Ada94]

i.e. the correspondence between the collection of braids and the collection of links is many to one. We have,

Definition 1.1.2. *Two braids are said to be Markov equivalent if their closures are isotopic links.*

The following theorem captures the Markov equivalence.

Theorem 1.1.1. *(Markov) Let $\alpha \in B_n$ and $\beta \in B_m$, $n \leq m$ be braids such that $\widehat{\alpha} = \widehat{\beta}$. Then α and β are related by a finite sequence of the following moves and their inverse operations:*
i. Right stabilization, $\alpha \mapsto \alpha\sigma_n \in B_{n+1}$, and
ii. Conjugation, $\beta \mapsto \gamma\beta\gamma^{-1}$ for some $\gamma \in B_m$.

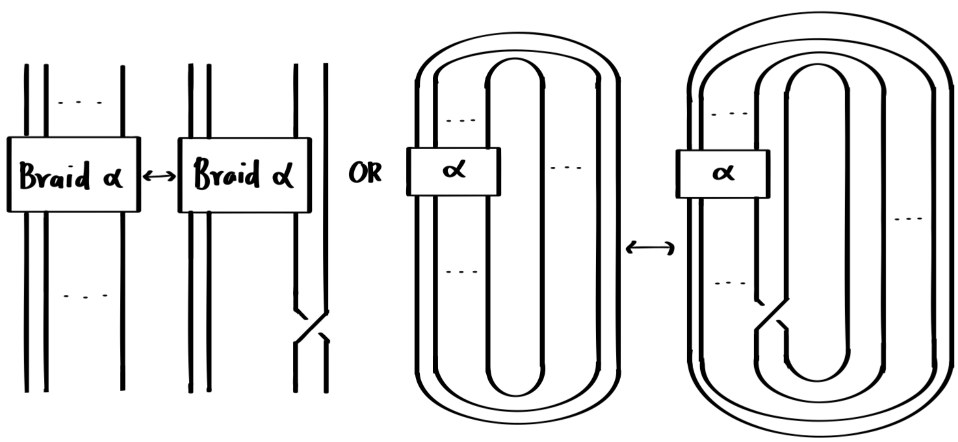


Figure 1.10: Right Stabilization

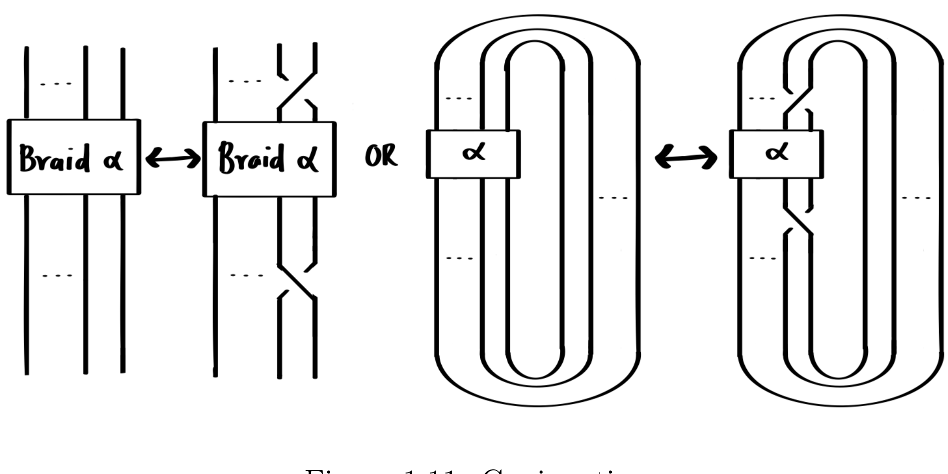


Figure 1.11: Conjugation

1.2 The Jones Polynomial

Our primary focus in this thesis will be on an invariant known as the 'Jones polynomial'. The Jones polynomial is an invariant of oriented links, discovered by Vaughan Jones [Jon85] in the year 1984 while he was working on von Neumann algebras. The Jones polynomial of an oriented link L, denoted by $V_L(t)$, is a Laurent polynomial in t if L has an odd number of components, and $t^{\frac{1}{2}}$ times a Laurent polynomial if L has even number of components.

Definition 1.2.1. *A finite dimensional von Neumann algebra is an algebra A_n, generated by an identity 1 and $n-1$ projections $e_1, \ldots, e_{n-1}$, modulo the following relations:*

1. $e_i^2 = e_i, e_i^* = e_i$,
2. $e_i e_{i\pm1} e_i = \frac{t}{(1+t)^2} e_i$ *where t is a complex number,*
3. $e_i e_j = e_j e_i$ *if* $|i-j| \geq 2$.

We can define a representation of B_n on A_n by sending the generating elements σ_i of B_n to $\sqrt{t}(te_i - (1-e_i))$. $V_L(t)$ is defined as a modification of some quantum trace function from A_n to $\mathbb{C}$. Precisely, the Jones polynomial is an assignment of a Laurent polynomial in $\sqrt{t}$ to oriented links such that the following is true:

1. The assignment is invariant under ambient isotopy,
2. $V_{\bigcirc}(t) = 1$,
3. For every triple of skein related link diagrams L_+, L_-, L_0 we have

$$\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0} \tag{1.2}$$

where three diagrams L_+, L_-, L_0 are said to be a skein triple if they are identical except for a region where they look like the three diagrams in Figure 1.12. The relation (1.2) is example of a skein relation, making the Jones polynomial a special case of a 'skein invariant'.

Let $\mathcal{L}$ denote the set of all ambient isotopy classes of oriented links and let F be a ring.

§Image Source: The Knot Book by C. C. Adams [Ada94]

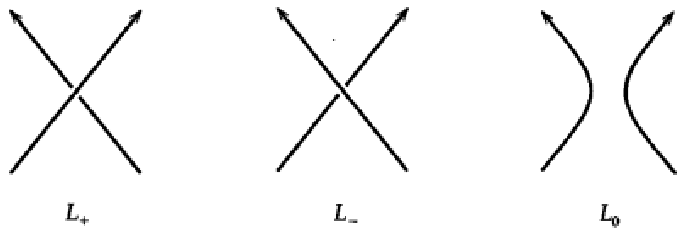


Figure 1.12: Skein Related Diagrams[§]

Definition 1.2.2. *A link invariant $P : \mathcal{L} \to F$ is a linear skein invariant if*

1. *$P(\bigcirc) = 1$, where $\bigcirc$ denotes the 1-component unknot,*
2. *There are 3 invertible elements $a_+, a_-, a_0 \in F$ such that for every skein triple L_+, L_-, L_0 we have*
$$a_+P(L_+) + a_-P(L_-) + a_0P(L_0) = 0.$$

By taking F to be the ring $\mathbb{Z}[l, l^{-1}, m, m^{-1}]$ and $a_+ = l, a_- = l^{-1}, a_0 = m$, we obtain the universal skein invariant. We have the following theorem as stated in Harpe et al. [dlHKW86]:

Theorem 1.2.1. *If $P : \mathcal{L} \to F$ is a skein invariant, then it is uniquely determined by the coefficients a_+, a_-, a_0 of the skein invariance relation.*

Kauffman defined a link invariant diagrammatically and proved that it satisfies the skein relation (1.2) and, hence, by the above theorem, must be the Jones polynomial. Kauffman's diagrammatic approach to the Jones polynomial uses the Kauffman bracket polynomial denoted by $<>$, whose domain is the collection of unoriented link diagrams.
The Kauffman bracket for any link L denoted by $< L >$, is a Laurent polynomial in the variable A with coefficients in $\mathbb{Z}$, which is defined by the following rules:

1. $< \bigcirc > = 1$,
2. $< \times > = A <)(> + B < \asymp >$
   $< \times > = A < \asymp > + B <)(>$,
3. $< L \cup \bigcirc >= C < L >$.

where $B = A^{-1}$ and $C = -A^2 - A^{-2}$.

The Jones polynomial is obtained by substituting $A = t^{-\frac{1}{4}}$ in the polynomial

$$X(L) = (-A^3)^{-\omega(L)} < L >$$

where $\omega(L)$ is the writhe of the corresponding link diagram L. Note that to define the writhe, the link has to be oriented. Thus, the domain of the polynomial X is the collection of all oriented links.

The Alexander polynomial, which is a well-known link invariant, was the only knot polynomial until the Jones polynomial was discovered. Unlike the Alexander polynomial, the Jones polynomial successfully distinguishes a lot of knots from their mirror images, for example, right-handed and left-handed trefoil. However, there are many non-isotopic links with the same Jones Polynomial, for example, the Conway knot and the Kinoshita-Terasaka knot, implying that the Jones polynomial is not a complete invariant. So, we need additional invariants that can distinguish links that previously known invariants cannot. One way to construct link invariants is by defining representations of braid groups on some algebra. Since the generating elements satisfy the Yang-Baxter equation (YBE), their images in the corresponding algebra must also satisfy the YBE. Therefore, finding link invariants comes down to finding algebras with elements that satisfy the YBE. One example of such an algebra is the von Neumann algebra, but it does not provide the machinery to generate more algebras. It was observed that representations of a (quasitriangular) quantum group naturally give rise to R-matrices, which are solutions to the quantum Yang-Baxter equation. The R matrix composed with the tensor flip map provides a solution to the YBE. Hence, the family of quantum groups is an excellent tool for defining link invariants. The construction of quantum groups depends on specific parameters, so changing the parameters can generate a new quantum group and, eventually, a new invariant. In the next chapter, we will discuss the theory of quantum groups.

Chapter 2

Quantum Groups

Quantum groups are defined as a one-parameter deformation of the universal enveloping algebra $U(\mathfrak{g})$ along with the Hopf algebra structure, where $\mathfrak{g}$ is a semisimple (See [Kyt11] for a definition) complex Lie algebra. Our focus will be on h-adic algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$, which is adequate to generate link invariants. In this chapter, we first develop the theory of Hopf algebra, then define quantum groups and study $U_h(\mathfrak{sl}_2(\mathbb{C}))$ in detail. The results in this chapter are well known. We will state only the results that are relevant to us and include the proof wherever required. The theory discussed in this chapter is taken from the book by Klimyk & Schmüdgen [KS81] and the lecture notes by Kalle Kytölä [Kyt11].

2.1 Hopf Algebra

2.1.1 Algebra

Definition 2.1.1. *An associative unital algebra is defined as a triple (A, μ, η) where*

1. *A is a vector space (or module) over $\mathbb{K}$ (For the rest of the thesis, we would assume $\mathbb{K} = \mathbb{C}$, unless stated otherwise).*

2. *$\mu : A \otimes A \to A$ is a bilinear map on A called the product or multiplication map such*

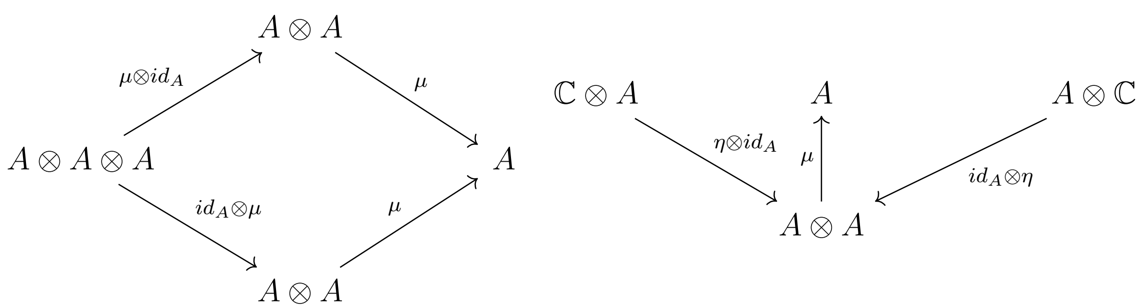
that it is associative, i.e.

$$\mu \circ (\mu \otimes id_A) = \mu \circ (id_A \otimes \mu).$$

3. $\eta : \mathbb{C} \to A$ *is a linear map called the unit map such that* $1 \in \mathbb{C}$ *is sent to the identity element of* A *i.e.*

$$\mu \circ (\eta \otimes id_A) = id_A = \mu \circ (id_A \otimes \eta).$$

The axioms of associativity and unitality can be depicted in the commutative diagrams shown below:



Opposite Algebra:

Let (A, μ, η) be an algebra. Define the tensor flip map as follows,

$$\tau_{A,A} : A \to A$$
$$\tau_{A,A}(a \otimes b) = b \otimes a.$$

Define the opposite product map by $\mu^{op} = \mu \circ \tau_{A,A}$.

Definition 2.1.2. *The opposite algebra of* A*, denoted by* A^{op} *is defined as the vector space* A *along with the algebra structure given by the maps* μ^{op} *and* η*.*

An algebra is called commutative if $\mu^{op} = \mu$. Defining the commutativity in terms of the opposite product map allows us to define co-commutativity similarly for co-algebras, as we will see in the Subsection 2.1.2.

Subalgebra and Ideal:

Let (A, μ, η) be an algebra. Subalgebras and ideals of A are defined in following ways:

Definition 2.1.3. *Let A' be a vector subspace of A. The subspace A' is said to be a subalgebra if the following is true:*

$$\begin{aligned} 1_A &\in A', \\ \mu(A' \otimes A') &\subset A'. \end{aligned}$$

Definition 2.1.4. *Let J be a vector subspace of A. The subspace J is called a left ideal of A if for every $a \in A$ and for every $k \in J$ the following holds:*

$$\mu(a, k) = ak \in J.$$

Additionally, the subspace J is called a right ideal of A if for every $a \in A$ and for every $k \in J$ the following holds:

$$\mu(k, a) = ka \in J.$$

Finally, J is called a two-sided ideal if it is both a left ideal and a right ideal.

For an ideal $J \subset A$, the quotient vector space A/J becomes an algebra by setting $\mu(a + J, b + J) = \mu(a, b) + J$.

Homomorphism of Algebras:

Definition 2.1.5. *A homomorphism between algebras (also known as an algebra map) (A_1, μ_1, η_1) and (A_2, μ_2, η_2) is a linear map $f : (A_1, \mu_1, \eta_1) \to (A_2, \mu_2, \eta_2)$ such that*

$$\begin{aligned} f(1_{A_1}) &= 1_{A_2}, \\ i.e.\ f \circ \eta_1 &= \eta_2, \\ f(\mu_1(a, b)) &= \mu_2(f(a), f(b)), \\ i.e.\ f \circ \mu_1 &= \mu_2 \circ (f \otimes f). \end{aligned}$$

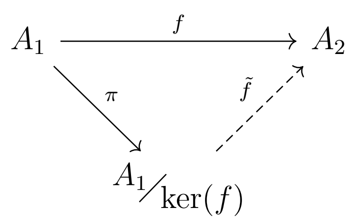
Two algebras $A_1 = (A_1, \mu_1, \eta_1)$ and $A_2 = (A_2, \mu_2, \eta_2)$ are said to be isomorphic (denoted by $A_1 \simeq A_2$) if there exist algebra homomorphisms $f : A_1 \to A_2$ and $f' : A_2 \to A_1$ such that

$f' \circ f = id_{A_1}$ and $f \circ f' = id_{A_2}$.
From now on, if the choice is obvious, we will not state the product and unit map. So, by A, we would mean a vector space along with some algebra structure.

Theorem 2.1.1. *(Isomorphism Theorem for Algebra) Let A_1, A_2 be two algebras. Let $f : A_1 \to A_2$ be an algebra homomorphism. Then*

1. $\mathrm{Im}(f) := f(A_1) \subset A_2$ *is a subalgebra,*
2. $\ker(f) := f^{-1}(\{0\}) \subset A_1$ *is an ideal,*
3. $A_1 / \ker(f) \simeq \mathrm{Im}(f)$.

More precisely, there exists an injective algebra homomorphism $\tilde{f} : A_1/\ker(f) \to A_2$ such that the following diagram commutes



where $\pi(a) = a + \ker(f)$.

Example 2.1.1. *The algebra of polynomials (with coefficients in $\mathbb{K}$) in indeterminate x is given by*

$$\begin{aligned} \mathbb{K}[x] &:= \{c_0 + c_1 x + ... + c_n x^n \,|\, n \in \mathbb{N},\ c_i \in \mathbb{K}\} \\ \mu(f(x), g(x)) &= fg(x) = f(x)g(x) \\ \eta(1) &= 1 \end{aligned}$$

Example 2.1.2. *Let G be a group. The group algebra of G is defined by*

$$\begin{aligned} \mathbb{K}[G] &:= \left\{ \sum_{g \in G} c_g e_g : c_g \in \mathbb{K} \text{ such that only finitely many } c_g \neq 0 \right\} \\ \mu(e_g, e_h) &= e_{g*h} \\ \eta(1) &= e_{id} \end{aligned}$$

*where * is the group operation and id is the group's identity element.*

2.1.2 Coalgebra

Broadly speaking, a coalgebra is defined by reversing the arrows' directions in the defining commutative diagrams of an algebra. Formally,

Definition 2.1.6. *A coassociative counital coalgebra is defined as a triple* (C, Δ, ϵ) *where*

1. C *is a vector space over* $\mathbb{C}$*.*
2. $\Delta : C \to C \otimes C$ *is a linear map called the coproduct or comultiplication map such that*

$$(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta.$$

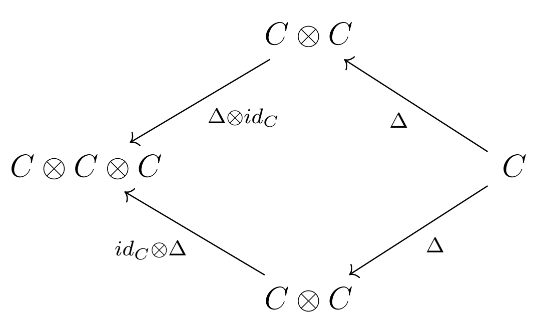
   The above property is called co-associativity.

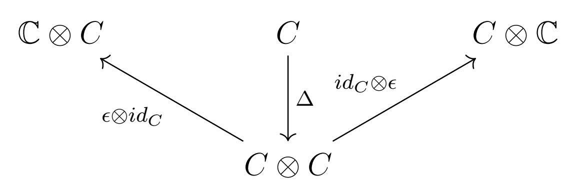
3. $\epsilon : C \to \mathbb{C}$ *is a linear map called the counit map such that*

$$(\epsilon \otimes id_C) \circ \Delta = id_C = (id_C \otimes \epsilon) \circ \Delta.$$

   The above property is called co-unitality.

The axioms of co-associativity and co-unitality can be depicted in the commutative diagrams shown below:





Co-opposite Co-Algebra:

Let (C, Δ, ϵ) be a coalgebra. Let $\tau_{C,C}$ be the tensor flip map defined earlier. Define the co-opposite product map by $\Delta^{cop} = \tau_{C,C} \circ \Delta$.

Definition 2.1.7. *The (co-)opposite coalgebra of C is defined as the vector space C along with the coalgebra structure given by the maps Δ^{cop} and ϵ.*

A coalgebra is called co-commutative if $\Delta^{cop} = \Delta$.

Subcoalgebra and Coideal:

Let $C = (C, \Delta, \epsilon)$ be a coalgebra. Subcoalgebras and coideals of C are defined in following ways:

Definition 2.1.8. *Let C' be a vector subspace of C. The subspace C' is said to be a subcoalgebra if the following is true:*

$$\Delta(C') \subset C' \otimes C'.$$

Definition 2.1.9. *Let J be a vector subspace of C. The subspace J is said to be a coideal if the following holds:*

$$\Delta(J) \subset J \otimes C + C \otimes J$$
$$\epsilon|_J = 0.$$

For $J \subset C$ a coideal, the quotient vector space C/J becomes a coalgebra by setting

$$\Delta_{C/J}(a + J) = \sum_{(a)} (a_{(1)} + J) \otimes (a_{(2)} + J)$$
$$\epsilon_{C/J}(a + J) = \epsilon(a)$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ (this notation is called the 'Sweedler Notation' and will be used throughout the thesis).

Homomorphism of Coalgebras:

Definition 2.1.10. *A homomorphism between coalgebras (also known as a coalgebra map)* $(C_1, \Delta_1, \epsilon_1)$ *and* $(C_2, \Delta_2, \epsilon_2)$ *is a linear map* $f : (C_1, \Delta_1, \epsilon_1) \to (C_2, \Delta_2, \epsilon_2)$ *such that*

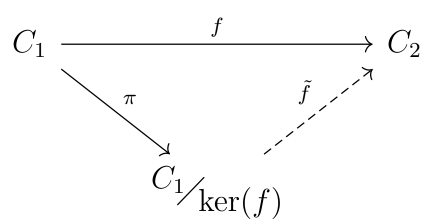
$$\Delta_2 \circ f = (f \otimes f) \circ \Delta_1,$$
$$\epsilon_2 \circ f = \epsilon_1.$$

Two coalgebras C_1, C_2 are said to be isomorphic (denoted by $C_1 \simeq C_2$) if there exist coalgebra homomorphisms $f : C_1 \to C_2$ and $f' : C_2 \to C_1$ such that $f' \circ f = id_{C_1}$ and $f \circ f' = id_{C_2}$.

Theorem 2.1.2. *(Isomorphism Theorem for coalgebra) Let* C_1, C_2 *be two coalgebras. Let* $f : C_1 \to C_2$ *be a coalgebra homomorphism. Then*

1. $\mathrm{Im}(f) := f(C_1) \subset C_2$ *is a sub coalgebra,*
2. $\ker(f) := f^{-1}(\{0\}) \subset C_1$ *is a coideal,*
3. $C_1 / \ker(f) \simeq \mathrm{Im}(f)$

More precisely, there exists an injective coalgebra homomorphism $\tilde{f} : C_1 / \ker(f) \to C_2$ *such that the following diagram commutes*



where $\pi(a) = a + \ker(f)$.

2.1.3 Bialgebra

A bialgebra is an algebra which has a compatible coalgebra structure. Formally,

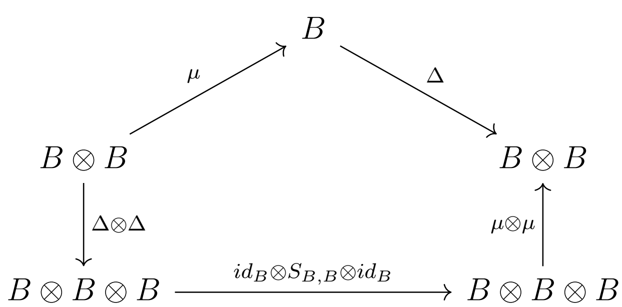
Definition 2.1.11. *A bialgebra is a quintuple* $(B, \mu, \Delta, \eta, \epsilon)$ *where* B *is a vector space and*

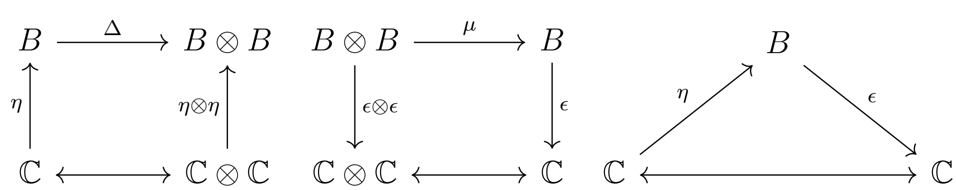
$$\begin{aligned} \mu &: B \otimes B \to B, \\ \Delta &: B \to B \otimes B, \\ \eta &: \mathbb{C} \to B, \\ \epsilon &: B \to \mathbb{C}, \end{aligned}$$

are linear maps such that (B, μ, η) *is an algebra and* (B, Δ, ϵ) *is a coalgebra. Furthermore, the following holds:*

$$\begin{aligned} &\Delta \circ \mu = (\mu \otimes \mu) \circ (id_B \otimes \tau_{B,B} \otimes id_B) \circ (\Delta \otimes \Delta), \\ &\Delta \circ \eta = \eta \otimes \eta, \\ &\epsilon \otimes \mu = \epsilon \otimes \epsilon, \\ &\epsilon \otimes \eta = id_{\mathbb{C}}. \end{aligned}$$

We can depict the above axioms as the following commutative diagrams





A linear map is said to be a homomorphism of bialgebras if it is both a homomorphism of algebras and a homomorphism of coalgebras.

2.1.4 Hopf Algebra

A Hopf algebra is a bialgebra with an additional antipode map. Formally,

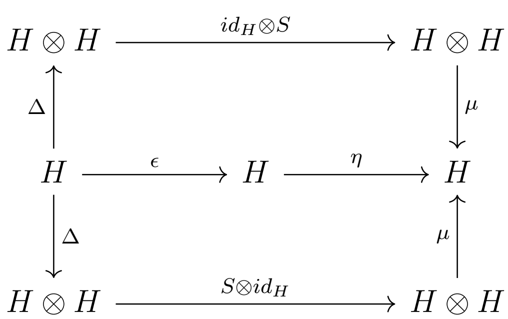
Definition 2.1.12. *A Hopf algebra is a sextuple $(H, \mu, \Delta, \eta, \epsilon, S)$ where H is a vector space and*

$$\begin{aligned} \mu &: H \otimes H \to H, \\ \Delta &: H \to H \otimes H, \\ \eta &: \mathbb{C} \to H, \\ \epsilon &: H \to \mathbb{C}, \\ S &: H \to H, \end{aligned}$$

are linear maps such that $(H, \mu, \Delta, \eta, \epsilon)$ is a bialgebra and the following holds:

$$\mu \circ (S \otimes id_H) \circ \Delta = \eta \circ \epsilon = \mu \circ (id_H \otimes S) \circ \Delta.$$

The map $S : H \to H$ is called the antipode map. The corresponding commutative diagram is given by:



Definition 2.1.13. *A linear map $f : (H_1, \mu_1, \Delta_1, \eta_1, \epsilon_1, S_1) \to (H_2, \mu_2, \Delta_2, \eta_2, \epsilon_2, S_2)$ is said to be a homomorphism of Hopf algebras if it is a homomorphism of bialgebras and $f \circ S_1 = S_2 \circ f$.*

Example 2.1.3. *The algebra of polynomials $H = \mathbb{C}[x]$ is endowed with a Hopf algebra* *structure with the following maps:*

$$\begin{aligned}
\mu(f(x), g(x)) &= f(x)g(x), \\
\eta(1) &= 1, \\
\Delta(x^n) &= \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}, \\
\epsilon(x^n) &= \delta_{n,0}, \\
S(x^n) &= (-1)^n x^n.
\end{aligned}$$

Example 2.1.4. *Let G be a finite group, define the Hopf algebra of complex-valued functions on G as follows:*

$$F(G) := \{f : G \to \mathbb{C}\} \tag{2.1}$$

with the following Hopf algebra structure:

$$\begin{aligned}
\mu(f \otimes g)(x) &= f(x)g(x), \\
\eta(1)(x) &= 1, \\
\Delta(f)(x \otimes y) &= f(x * y), \\
\epsilon(f) &= f(e), \\
S(f)(x) &= f(x^{-1}),
\end{aligned}$$

*where * is the group operation of G and e denotes the identity element of G. Note that since the product is defined as the point-wise multiplication in $\mathbb{C}$, it is commutative. So, given any group, we can associate a commutative Hopf algebra.*

Hopf Algebra structure on Dual Vector Space:

Given a Hopf algebra H, we can define a Hopf algebra structure on the dual vector space H^* by the following maps:

$$\begin{aligned}\langle \mu^*(a^* \otimes b^*), v\rangle &= \langle a^* \otimes b^*, \Delta(v)\rangle,\\ \langle \eta^*(1), v\rangle &= \epsilon(v),\\ \langle \Delta^*(a^*), v \otimes w\rangle &= \langle a^*, \mu(v \otimes w)\rangle,\\ \epsilon^*(a^*) &= \langle a^*, 1\rangle,\\ \langle S^*(a^*), v\rangle &= \langle a^*, S(v)\rangle,\end{aligned}$$

where $a^*, b^* \in H^*$ and $v, w \in H$.

Grouplike Elements and Primitive elements:

Definition 2.1.14. *Let (C, Δ, ϵ) be a coalgebra. An element $a \in C$, $a \neq 0$ is said to be a grouplike element if $\Delta(a) = a \otimes a$.*

Definition 2.1.15. *Let $(B, \mu, \Delta, \eta, \epsilon)$ be a bialgebra. An element $x \in B$, $x \neq 0$ is said to be primitive if $\Delta(x) = x \otimes 1_B + 1_B \otimes x$.*

In the Example 2.1.3 if

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k} = x^n \otimes x^n$$

then $n = 0$, so $1 \otimes 1$ is the only grouplike element.

Remark 2.1.1. *If $x \in B$ is a primitive element, then its exponential, i.e. $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ (if the limit exists) is a grouplike element.*

Lemma 2.1.3. *The set of grouplike elements of a Hopf algebra forms a group with μ as the group operation.*

Proof. Let $(H, \mu, \Delta, \eta, \epsilon, S)$ be a Hopf algebra. Let G denote the set of grouplike elements of H. Since μ is associative by definition, it is enough to check the closure of G under μ, the

closure of G under taking inverses and the existence of the identity element in G.
i. Identity element belongs to G:

$$\begin{aligned}\Delta \circ \eta &= \eta \otimes \eta \\ \Delta \circ \eta(1) &= \eta(1) \otimes \eta(1) \\ \implies \Delta(1_H) &= 1_H \otimes 1_H \implies 1_H \in G\end{aligned}$$

ii. Closure: Let $a, b \in G$ then we have $\Delta(a) = a \otimes a$ and $\Delta(b) = b \otimes b$

$$\begin{aligned}\Delta \circ \mu &= (\mu \otimes \mu) \circ (id_H \otimes S_{H,H} \otimes id_H) \circ (\Delta \otimes \Delta) \\ \Delta \circ \mu(a \otimes b) &= (\mu \otimes \mu) \circ (id_H \otimes S_{H,H} \otimes id_H) \circ (\Delta \otimes \Delta)(a \otimes b) \\ \Delta \circ \mu(a \otimes b) &= (\mu \otimes \mu) \circ (id_H \otimes S_{H,H} \otimes id_H)(a \otimes a \otimes b \otimes b) \\ \Delta \circ \mu(a \otimes b) &= (\mu \otimes \mu)(a \otimes b \otimes a \otimes b) \\ \Delta \circ \mu(a \otimes b) &= (ab \otimes ab)\end{aligned}$$

Thus, $ab \in G$.
iii. Inverse: Let $a \in G$. We have by axioms of Hopf algebra

$$\begin{aligned}\mu \circ (S \otimes id_H) \circ \Delta = \eta \circ \epsilon = \mu \circ (id_H \otimes S) \circ \Delta \\ \mu \circ (S \otimes id_H) \circ \Delta(a) = \mu \circ (S \otimes id_H)(a \otimes a) = \mu(S(a) \otimes a)\end{aligned}$$

Similarly we have $\mu \circ (id_H \otimes S) \circ \Delta(a) = \mu(a \otimes S(a))$. Also we have $\eta \circ \epsilon(a) = \eta(1) = 1_H$. Combining we get

$$\mu(S(a) \otimes a) = 1_H = \mu(a \otimes S(a))$$

Thus, $S(a)$ is the inverse of a. The only thing left to show is that $S(a) \in G$. Let $\Delta(S(a)) = \sum_{(S(a))} S(a)_1 \otimes S(a)_2$. Then,

$$\begin{aligned}\Delta \circ \mu(1_H) &= \Delta \circ \mu(S(a) \otimes a) \\ &= (\mu \otimes \mu) \circ (id_H \otimes S_{H,H} \otimes id_H) \circ (\Delta \otimes \Delta)(S(a) \otimes a) \\ &= (\mu \otimes \mu) \circ (id_H \otimes S_{H,H} \otimes id_H)(\sum_{(S(a))} S(a)_1 \otimes S(a)_2 \otimes a \otimes a) \\ 1 \otimes 1 &= \mu \otimes \mu(\sum_{(S(a))} S(a)_1 \otimes a \otimes \sum_{(S(a))} S(a)_2 \otimes a) \\ 1 \otimes 1 &= \mu(\sum_{(S(a))} S(a)_1 \otimes a) \otimes \mu(\sum_{(S(a))} S(a)_2 \otimes a)\end{aligned}$$

This implies,

$$\begin{aligned}
&\mu(\sum_{(S(a))} S(a)_1 \otimes a) = 1, \\
&\mu(\sum_{(S(a))} S(a)_2 \otimes a) = 1 \\
&\implies \sum_{(S(a))} S(a)_1 = \sum_{(S(a))} S(a)_2 = S(a) \\
&\Delta(S(a)) = S(a) \otimes S(a) \implies S(a) \in G
\end{aligned}$$

G satisfies all the axioms of a group and thus is a group. ☐

Lemma 2.1.4. *The set of grouplike elements of the group algebra is the original group itself.*

Proof. Recall that the group algebra is given by

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} c_g e_g : c_g \in \mathbb{C} \text{ such that only finitely many } c_g \neq 0 \right\}$$

Let us denote the set of grouplike elements of $\mathbb{C}[G]$ by G'. The coproduct map on the group elements is given by $\Delta(e_g) = e_g \otimes e_g$; this implies $G \subset G'$. Let $a = \sum c_g e_g \in G'$, then we will show that $a \in G$. On the LHS, we will use $\Delta(a) = a \otimes a$, and on the RHS, we will extend Δ linearly.

$$\begin{aligned}
\Delta(a) = \Delta(\sum c_g e_g) &= \sum c_g \Delta(e_g) \\
a \otimes a &= \sum c_g (e_g \otimes e_g) \\
\sum (c_g e_g) \otimes \sum (c_g e_g) &= \sum c_g (e_g \otimes e_g) \\
\sum c_g c_h (e_g \otimes e_h) &= \sum c_g (e_g \otimes e_g) \\
\sum_{g \neq h} c_g c_h (e_g \otimes e_h) + \sum c_g^2 (e_g \otimes e_g) &= \sum c_g (e_g \otimes e_g)
\end{aligned}$$

Comparing coefficients on both the sides we get, $c_g c_h = 0$ for $g \neq h$ and $c_g^2 = 1$.
Claim: $c_g = 1$ for exactly one $g \in G$
Proof of the claim: Since $a \neq 0$ (by the definition of a grouplike element), we have $c_g = 1$ for some $g \in G$. Suppose there exists $h \in G$, $h \neq g$ such that $c_h = 1$ then $c_g c_h = 1$ but $c_g c_h = 0$ this leads to a contradiction. Thus, $c_g = 1$ for exactly one $g \in G$.

The above claim implies $a = e_g$ for some $g \in G$, thus $G' \subset G$ and $G = G'$. □

2.2 Representations of Associative Algebras

For this section, we refer to Chapter 4 of the book [GW09].

Definition 2.2.1. *Let A be an associative unital algebra over $\mathbb{C}$. A representation of A is a pair (ρ, V) where V is a vector space over $\mathbb{C}$ and $\rho : A \to \operatorname{End}(V)$ is an associative algebra homomorphism.*

Here $\operatorname{End}(V)$ is endowed with an associative unital algebra structure with matrix multiplication (i.e. composition of linear transformations) as the product map.
By the unitality axiom, we have $\rho(1_A) = I_V$, where $I_V : V \to V$ is the identity transformation.
Let $a \in A$ then $\rho(a) : V \to V$ and $\rho(a)(v) \in V$. Hence, the following map makes V an A-module.

$$A \times V \to V$$
$$(a, v) \mapsto \rho(a)(v)$$

Example 2.2.1. *The Trivial Representation of A over V is given by $\rho(a) = I_V$, $\forall a \in A$.*

Example 2.2.2. *Let $A = \mathbb{C}[x]$. Let V be a finite-dimensional vector space and let $T \in$* $\operatorname{End}(V)$*. Define a representation (ρ, V) of A by $\rho(f) = f(T)$ for $f \in \mathbb{C}[x]$.*

Definition 2.2.2. *(Invariant Subspace) Let $U \subset V$ be a linear subspace of a finite-dimensional vector space V. Let $\rho : A \to \operatorname{End}(V)$ be a representation. If $\rho(a)U \subset U$ for all $a \in A$, then we say U is an invariant subspace of (ρ, V).*

We can define a representation (ρ_U, U) by the restriction of $\rho(A)$ to U and a representation $(\rho_{V/U}, V/U)$ by quotient action of $\rho(A)$ to V/U.

Definition 2.2.3. *(Irreducible Representation) A representation is said to be irreducible if the only invariant subspaces of V are $\{0\}$ and V.*

Claim: Let $\ker(\rho) \coloneqq \{x \in A : \rho(x) = 0\}$. Then, $\ker(\rho)$ is a two-sided ideal of A.

Proof. Let $a \in A$, $k \in ker(\rho)$. Then $\rho(k) = 0$.

$$\rho(ak) = \rho(a)\rho(k) = \rho(a)0 = 0 \implies ak \in \ker(\rho)$$
$$\rho(ka) = \rho(k)\rho(a) = 0\rho(a) = 0 \implies ka \in \ker(\rho)$$

Thus, by the definition of a two-sided ideal, $\ker(\rho)$ is a two-sided ideal. □

Definition 2.2.4. *(Faithful Representation) A representation (ρ, V) is said to be faithful if* $\ker(\rho) = \{0\}$.

2.2.1 Direct Sum and Tensor of Representations

Let (ρ_1, V_1) and (ρ_2, V_2) be representations of A.

$$\rho_1 : A \to \mathrm{End}(V_1)$$
$$\rho_2 : A \to \mathrm{End}(V_2)$$

The two representations give a representation of $A \oplus A$ over the direct sum of vector spaces, i.e. the following:

$$(\rho_1 \oplus \rho_2)(g)(v_1 \oplus v_2) := \rho_1(g)(v_1) \oplus \rho_2(g)(v_2)$$

is a representation of $A \oplus A$ over $V_1 \oplus V_2$.

Similarly, we can define a representation of $A \otimes A$ over the tensor product of vector spaces, i.e. the following:

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) := \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$$

is a representation of $A \otimes A$ over $V_1 \otimes V_2$.

2.2.2 Intertwining Operator or Module Homomorphism

Definition 2.2.5. *Let (ρ, V) and (τ, W) be two representations of A. Let $T \in \mathrm{Hom}(V, W)$. T is called an intertwining operator or a module homomorphism if $T \circ \rho(a) = \tau(a) \circ T$, i.e. the following diagram commutes:*

$$
\begin{array}{ccc}
V & \xrightarrow{\rho(a)} & V \\
\Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T} \\
W & \xrightarrow{\tau(a)} & W
\end{array}
$$

Denote

$$\mathrm{Hom}_A(V, W) := \{T \in \mathrm{Hom}(V, W) : T \circ \rho(a) = \tau(a) \circ T \ \ \forall\, a \in A\}$$

Lemma 2.2.1. *Let* $T_1 \in \mathrm{Hom}_A(V, W)$ *and* $T_2 \in \mathrm{Hom}_A(W, X)$*. Then* $T_2 \circ T_1 \in \mathrm{Hom}_A(V, X)$*.*

Proof. We need to prove that $(T_2 \circ T_1) \circ \rho(a) = \tau'(a) \circ (T_2 \circ T_1)$ where

$$\begin{aligned} \rho(a) &: V \to V \\ \tau(a) &: W \to W \\ \tau'(a) &: X \to X \end{aligned}$$

We are given that $T_1 \circ \rho(a) = \tau(a) \circ T_1$ and $T_2 \circ \tau(a) = \tau'(a) \circ T_2$. Then

$$\begin{aligned} (T_2 \circ T_1) \circ \rho(a) &= T_2 \circ (T_1 \circ \rho(a)) \\ &= T_2 \circ \tau(a) \circ T_1 = \tau'(a) \circ (T_2 \circ T_1) \end{aligned}$$

This proves the result. □

Definition 2.2.6. *An intertwining operator* $T \in \mathrm{Hom}_A(V, W)$ *is called invertible if there exists* $T^{-1} \in \mathrm{Hom}_A(W, V)$ *such that*

$$\begin{aligned} T \circ T^{-1} &= I_W \in \mathrm{Hom}_A(W, W) \\ T^{-1} \circ T &= I_V \in \mathrm{Hom}_A(V, V) \end{aligned}$$

Definition 2.2.7. *(Equivalence of Representations) Two representations* (ρ, V) *and* (τ, W) *are said to be equivalent if there exists an invertible operator in* $\mathrm{Hom}_A(V, W)$ *and we write* $(\rho, V) \simeq (\tau, W)$*. One can check that this is an equivalence relation.*

Define $\mathrm{End}_A(V) := \mathrm{Hom}_A(V, V)$. Note that if $f, g \in \mathrm{End}_A(V)$ then $f \circ g \in \mathrm{End}_A(V)$ and

$I_V \in \mathrm{End}_A(V)$. Hence, $\mathrm{End}_A(V)$ is an associative unital algebra with composition as the product map.

Example 2.2.3. *Let G be a group and $A = \mathbb{C}[G]$ be the group algebra. Let (ρ, V) be a representation of A, then the map $g \mapsto \rho(e_g)$ is a group homomorphism from G to* $\mathrm{End}(V)$ *i.e. we have,*

$$\rho : G \to \mathrm{End}(V)$$
$$g \mapsto \rho(e_g)$$

Conversely, given a group homomorphism $\rho : G \to \mathrm{End}(V)$ we can define a representation of A as follows:

$$\tilde{\rho} : A \to \mathrm{End}(V)$$
$$\tilde{\rho}(f) = \sum_{g \in G} f(g)\rho(g)$$

where $f \in A = \mathbb{C}[G]$. Thus, we have a one-to-one correspondence between the group homomorphism from G to $\mathrm{End}(V)$ *and representations of A over V.*

2.2.3 Schur's Lemma

Lemma 2.2.2. *Let (ρ, V) and (τ, W) be irreducible representations of an associative algebra A. Let V, W have countable dimension over $\mathbb{C}$. Then*

$$\dim(\mathrm{Hom}_A(V, W)) = \begin{cases} 1 & \text{if } (\rho, V) \simeq (\tau, W) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first show that $\mathrm{Hom}_A(V, W)$ is not equal to $\{0\}$ if and only if (ρ, V) is isomorphic to (τ, W).

- Suppose $\mathrm{Hom}_A(V, W) = \{0\}$ then clearly (ρ, V) is not isomorphic to (τ, W).
- Conversely, suppose $\mathrm{Hom}_A(V, W) \neq \{0\}$ then there exists a non zero element T in $\mathrm{Hom}_A(V, W)$ non zero. Now, $\ker(T) \subset V$ and $\mathrm{Range}(T) \subset W$ are invariant subspaces of V and W respectively. Since (ρ, V) and (τ, W) are irreducible representations, we

have $\ker(T) = \{0\}$ or V and $\text{Range}(T) = \{0\}$ or W. $\ker(T)$ can not be equal to V, as that would imply $T = 0$, a contradiction.
Hence, $\ker(T) = \{0\}$, which implies $\text{Range}(T) = W$. So T is a bijection, hence an isomorphism from (ρ, V) to (τ, W). Therefore,

$$(\rho, V) \simeq (\tau, W).$$

Only thing left to show is that $\dim(\text{Hom}_A(V, W)) = 1$ when $(\rho, V) \simeq (\tau, W)$.
Let $(\rho, V) \simeq (\tau, W)$. If $S, T \in \text{Hom}_A(V, W)$ then $R = T^{-1}S \in \text{End}_A(V)$.
Assume for contradiction that R is not a multiple of the identity operator. Then for all $\lambda \in \mathbb{C}$ we have, $R - \lambda I \neq 0$, which implies $R - \lambda I$ is invertible.

- **Claim:** For any $0 \neq v \in V$ and distinct scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$, the vectors

$$(R - \lambda_1 I)^{-1}v, \ldots, (R - \lambda_m I)^{-1}v$$

are linearly independent.

- **Proof:** Suppose $\sum_i a_i(R - \lambda_i I)^{-1}v = 0$. Multiplying by $\prod_j (R - \lambda_j I)$ we get

$$\sum_i a_i \prod_{i \neq j} (R - \lambda_j)v = 0$$

Denote $\sum_i a_i \prod_{i \neq j}(x - \lambda_j)$ by $f(x)$. Then $f(R)v = 0$.
If $a_i \neq 0$ for some i then $f(x) \neq 0$ and it has a factorisation as follows:

$$f(x) = c(x - \mu_1)...(x - \mu_m)$$

with $c \neq 0$ and $\mu_i \in \mathbb{C}$.
We have,

$$f(R) = c(R - \mu_1 I)...(R - \mu_m I)$$

By assumption $R - \lambda_i I$ are invertible, which implies $f(R)$ is invertible. This contradicts the fact that $f(R)v = 0$. Thus, the vectors

$$(R - \lambda_1 I)^{-1}v, \ldots, (R - \lambda_m I)^{-1}v$$

are linearly independent.

We know that $\mathbb{C}$ does not have a countable dimension. The above result implies for every $\lambda_i \in \mathbb{C}$, we have a vector $(R - \lambda_i I)^{-1}v \in V$. So $\{(R - \lambda_i I)^{-1}v$ is an uncountably infinte collection of linearly independent vectors. This implies that V doesn't have a countable dimension. This leads to a contradiction to our assumption that V has a countable dimension. Thus, $R - \lambda I = 0$ for some λ this implies, $\text{Hom}_A(V, W) = \langle I \rangle$ (Generated as a complex vector space) and hence $\dim(\text{Hom}_A(V, W)) = 1$. □

2.2.4 Representations of a Hopf Algebra

Let H be a Hopf algebra. And ρ and ψ be two representations of H on a vector space V and W, respectively. Then, the coproduct map Δ allows us to define a representation of H on the vector space $V \otimes W$ as follows:

$$(\rho \otimes \psi)(a) := \rho \otimes \psi(\Delta(a))$$

Note that this representation is different from the one discussed in 2.2.1, as the former is a representation of H and the latter is a representation of $H \otimes H$.

Furthermore, the antipode map S allows us to define a representation of H on the dual vector space V^* by the following pairing:

$$\langle \rho^*(a)(v^*), v \rangle = \langle v^*, \rho(S(a))v \rangle$$

where $v^* \in H^*$ and $v \in H$.

2.3 Quasitriangular and Ribbon Hopf Algebra

The theory of quantum groups deals with non-commutative (and non co-commutative) Hopf algebra. While studying non-co-commutative Hopf algebras, we might demand that the non-co-commutativity is 'controlled' to some extent. Examples of such Hopf algebras are quasitriangular Hopf algebra. Many interesting properties arise when the quasitriangularity condition is added. More precisely, we have,

Definition 2.3.1. *(Quasitriangular Hopf Algebra) A quasitriangular Hopf algebra A is a* *pair (A, R) where A is a Hopf algebra and R in the completion $A\hat{\otimes}A$ satisfies*

$$\begin{aligned}(\Delta \otimes id)R &= R_{13}R_{23} \\ (id \otimes \Delta)R &= R_{13}R_{12} \\ \tau \circ \Delta(h) &= R(\Delta(h))R^{-1} \quad \forall h \in A\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}R &= \sum_i x_i \otimes y_i, \\ R_{12} &= \sum_i x_i \otimes y_i \otimes 1, \\ R_{13} &= \sum_i x_i \otimes 1 \otimes y_i, \\ R_{23} &= \sum_i 1 \otimes x_i \otimes y_i.\end{aligned}$$

The element R is called a universal R-matrix.

Note that the third equation of (2.2) says that even though A is not co-commutative, its opposite coproduct can be obtained by the coproduct map. The first and second equation of (2.2) implies

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.3}$$

This equation is called the quantum Yang-Baxter Equation.

Proposition 2.3.1. *Let H be a finite-dimensional Hopf algebra and let R' be the following map,*

$$\begin{aligned}R' : H^* &\to H \\ \phi &\mapsto \sum R^{(1)}\phi(R^{(2)})\end{aligned}$$

where $R = \sum R^{(1)} \otimes R^{(2)}$. Then, R satisfies the first equation of (2.2) *if and only if R' is a coalgebra map. And R satisfies the second equation of* (2.2) *if and only if R' is an antialgebra map.*

Proof. Note that $R'(\phi) = \mu \circ (id \otimes \phi)(R)$.

i. Let R satisfy the 1st equation of (2.2). Then,

$$\begin{aligned}
\Delta \circ R'(\phi) &= (id \otimes \mu) \circ (id \otimes id \otimes \phi) \circ (\Delta \otimes id)(R) \\
&= (id \otimes \mu) \circ (id \otimes id \otimes \phi) \circ R_{13}R_{23} && \text{By 1st equation of (2.2)} \\
&= (id \otimes \mu) \circ \sum R^{(1)} \otimes R^{(1)'} \otimes \phi(R^{(2)}R^{(2)'}) \\
&= (id \otimes \mu) \circ \sum R^{(1)} \otimes R^{(1)'} \otimes \Delta^* \circ \phi(R^{(2)} \otimes R^{(2)'}) \\
&= (R' \otimes R')(\Delta^* \circ \phi)
\end{aligned}$$

This implies R' is a coalgebra map.

Conversely, if R' is a coalgebra map i.e. $\Delta \circ R'(\phi) = (R' \otimes R')(\Delta^* \circ \phi)$. Then we have

$$\begin{aligned}
(id \otimes id \otimes \phi) \circ (\Delta \otimes id)(R) &= \sum R^{(1)} \otimes R^{(1)'} \otimes \phi(R^{(2)}R^{(2)'}) \\
&= (id \otimes id \otimes \phi) \circ R_{13}R_{23} \\
\implies \Delta \otimes id(R) = R_{13}R_{23}
\end{aligned}$$

ii. Let R satisfy the 2nd equation of (2.2), then

$$\begin{aligned}
R'(\phi\psi) &= \sum R^{(1)}(\phi\psi)(R^{(2)}) \\
&= \mu \circ (\mu \otimes id) \circ (id \otimes \phi \otimes \psi) \circ (id \otimes \Delta)(R) \\
&= \mu \circ (\mu \otimes id) \circ (id \otimes \phi \otimes \psi)(R_{13}R_{12}) && \text{By 2nd equation of (2.2)} \\
&= \sum R^{(1)}R^{(1)'}\phi(R^{(2)'})\psi(R^{(2)}) \\
&= \sum R^{(1)}R^{(1)'}\langle \phi \otimes \psi, R^{(2)'} \otimes R^{(2)} \rangle \\
&= \sum R^{(1)}\psi(R^{(2)}) \sum R^{(1)'}\phi(R^{(2)'}) \\
&= R'(\psi)R'(\phi)
\end{aligned}$$

This implies R' is an antialgebra map.

Conversely, let R' be an antialgebra map i.e. $R'(\phi\psi) = R'(\psi)R'(\phi)$. So we have

$$R'(\phi \otimes \psi) = \mu \circ (\mu \otimes id) \circ (id \otimes \phi \otimes \psi) \circ (id \otimes \Delta)(R)$$

Since $R'(\phi\psi) = R'(\psi)R'(\phi)$, we have

$$\mu \circ (\mu \otimes id) \circ (id \otimes \phi \otimes \psi) \circ (id \otimes \Delta)(R) = \sum R^{(1)}\psi(R^{(2)}) \sum R^{(1)'}\phi(R^{(2)'})$$
$$= \mu \circ (\mu \otimes id) \circ (id \otimes \phi \otimes \psi) \circ R_{13}R_{12}$$
$$\implies id \otimes \Delta(R) = R_{13}R_{12}$$

□

The image of a universal R-matrix under any representation composed with the tensor flip map, i.e. $\tau \circ R$, satisfies the Yang-Baxter equation. Recall that the braid group generators also satisfy the YBE. Thus, given a representation of a quasitriangular Hopf algebra, we can always define a representation of braid groups. The method of generating link invariants through quantum groups is based on this relation.

A more useful class of Hopf algebra are ribbon Hopf algebras.

Definition 2.3.2. *[Saw95] A ribbon Hopf algebra is a tuple (H, R, G) where (H, R) is a quasitriangular Hopf algebra and G is a grouplike element satisfying*

$$G^{-1}uG^{-1} = S(u)$$
$$GaG^{-1} = S^2(a)\ \forall a \in H$$

where $u = \sum S(R^{(2)})R^{(1)}$ and $R = \sum R^{(1)} \otimes R^{(2)}$.

Given a representation of a ribbon Hopf algebra on a vector space V, we define a function from $\mathrm{End}(V)$ to the base field of the given Hopf algebra. We will see in the next chapter that this function is crucial for defining a link invariant.

Definition 2.3.3. *[Saw95] Let (H, R, G) be a ribbon Hopf algebra over $\mathbb{C}$ and let $\rho : H \to \mathrm{End}(V)$ be a representation of H. The quantum trace concerning V is the following function*

$$_qTr_V : \mathrm{End}(V) \to \mathbb{C}$$
$$_qTr_V(x) = Tr(\rho(G)x)$$

The quantity $_qTr_V(1)$ is called the quantum dimension of V. For convenience, we will call the element G the ribbon element.

2.4 The h-Adic Hopf Algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$

Recall the Example 2.1.4, given any group, the Hopf algebra of its complex-valued functions is commutative. Conversely, we have a non-trivial result that any commutative Hopf algebra arises as a function Hopf algebra of some group. What if we started with a non-commutative Hopf algebra? Then, we can say that it arises as a function Hopf algebra of some quantum group. The notion of quantum groups varies with usage; for us, the theory of quantum groups deals with non-commutative (and non-cocommutative) Hopf algebra. We focus on a class of quantum groups, namely deformations of the universal enveloping algebra of a Lie algebra. Here, we discuss the quantum group $U_h(\mathfrak{sl}_2(\mathbb{C}))$. Before defining $U_h(\mathfrak{sl}_2(\mathbb{C}))$, we need to define q-numbers and q-factorials.

2.4.1 q-Calculus

Let $q \in \mathbb{C}$ be a non-zero number.

Definition 2.4.1. *(q- Number) Let $a \in \mathbb{C}$. Define*

$$\begin{aligned} [a]_q \equiv [a] :&= \frac{q^a - q^{-a}}{q - q^{-1}} = \frac{e^{ah} - e^{-ah}}{e^h - e^{-h}} \\ &= \frac{\sinh ah}{\sinh h} \end{aligned}$$

where $q = e^h$. Clearly, $\lim_{q \to 1} [a]_q = a$.

Another useful expression is

$$[[a]]_q \equiv [[a]] := \frac{1 - q^a}{1 - q}.$$

The following relations can be directly derived from the properties of exponential:

For $m, n \in \mathbb{C}$ and $a, b, c \in \mathbb{C}$

$$\begin{aligned}
[m] &= q^{m-1} + q^{m-3} + \ldots + q^{-(m-1)} \\
[m+n] &= q^{n}[m] + q^{-m}[n] = q^{-n}[m] + q^{m}[n] \\
[m-n] &= q^{n}[m] - q^{m}[n] = q^{-n}[m] - q^{-m}[n] \\
0 &= [a][b-c] + [b][c-a] + [c][a-b] \\
[n] &= [2][n-1] - [n-2]
\end{aligned}$$

Definition 2.4.2. *(q- Factorial) q-factorials are defined in the following way:*

$$\begin{aligned}
[m]! &\coloneqq [1][2]\cdots[m], \\
[0]! &\coloneqq 1.
\end{aligned}$$

2.4.2 Definition and Properties of $U_h(\mathfrak{sl}_2(\mathbb{C}))$

Let h be an indeterminate. And let $\mathbb{C}[[h]]$ denote the ring of formal power series in h with complex coefficients. Then,

Definition 2.4.3. *$U_h(\mathfrak{sl}_2(\mathbb{C}))$ denotes the h-Adic algebra (i.e. an algebra over the ring $\mathbb{C}[[h]]$) with three generators E, F and H satisfying the following relations:*

$$\begin{aligned}
[H, E] &= 2E \\
[H, F] &= -2F \\
[E, F] &= \frac{e^{hH} - e^{-hH}}{e^{h} - e^{-h}} = \frac{\sinh hH}{\sinh h}
\end{aligned}$$

where $e^{hH} = 1 + hH + \dfrac{(hH)^2}{2!} + ... \in U_h(\mathfrak{sl}_2(\mathbb{C}))$.

And whose Hopf algebra structure is given by the following maps:

$$\begin{aligned}
\Delta(E) &= E \otimes e^{hH} + 1 \otimes E, \\
\Delta(F) &= F \otimes 1 + e^{-hH} \otimes F, \\
\Delta(H) &= H \otimes 1 + 1 \otimes H, \\
\epsilon(H) &= \epsilon(E) = \epsilon(F) = 0, \\
S(H) &= -H, \\
S(E) &= -Ee^{-hH}, \\
S(F) &= -e^{hH}F.
\end{aligned}$$

The h-adic Hopf algebra obtained in this way is called the h-adic quantum algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$. The set $\{E^l H^m F^n \mid l, m, n \in \mathbb{N}_0\}$ is a linear basis of $U_h(\mathfrak{sl}_2(\mathbb{C}))$. The quantum Casimir element of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is given by

$$C_h := EF + \frac{e^{h(H-1)} + e^{-h(H-1)}}{(e^h - e^{-h})^2} = FE + \frac{e^{h(H+1)} + e^{-h(H+1)}}{(e^h - e^{-h})^2}$$

It can be shown that $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is quasitriangular, and its universal R-matrix is given by

$$R = e^{h(H \otimes H)/2} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(1 - q^{-2})^n}{[n]_q!} E^n \otimes F^n$$

where $q = e^h$.
Let $K := e^{hH}$ be an element of H. Then we have,

$$\begin{aligned}
\Delta(K) &= \Delta(e^{hH}) = e^{h\Delta(H)} \\
&= e^{h(H \otimes I + I \otimes H)} = (e^{hH} \otimes I)(I \otimes e^{hH}) \\
&= e^{hH} \otimes e^{hH} = K \otimes K \\
\epsilon(K) &= e^{h\epsilon(H)} = e^{h*0} = e^0 = 1
\end{aligned} \tag{2.4}$$

This implies K is a grouplike element. One can check that K satisfies the axioms of a ribbon element, making $(U_h(\mathfrak{sl}_2(\mathbb{C})), R, K)$ a ribbon Hopf algebra.

2.4.3 Finite-dimensional representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$

Since $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is an algebra over the ring $\mathbb{C}[[h]]$, we consider representations of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on finite-dimensional linear spaces over the ring $\mathbb{C}[[h]]$. Given a finite-dimensional complex vector space V, the space $V[[h]]$ of formal power series in h with coefficients in V forms a linear space over $\mathbb{C}[[h]]$. So, a representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ will be an algebra homomorphism as follows:

$$\rho : U_h(\mathfrak{sl}_2(\mathbb{C})) \to \mathrm{End}(V[[h]]) = \mathrm{End}(V \otimes \mathbb{C}[[h]]) = \mathrm{End}(V) \otimes \mathrm{End}(\mathbb{C}[h]]).$$

Here $\mathrm{End}(V \otimes \mathbb{C}[[h]]) \cong \mathrm{End}(V) \otimes \mathrm{End}(\mathbb{C}[h]])$ because V is finite dimensional.

Let V be a finite-dimensional vector space over $\mathbb{C}$. Then by a finite-dimensional representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ we mean a finite-dimensional representation ρ on $V \otimes \mathbb{C}[[h]]$ such that $\rho(a)(v \otimes c) = \tilde{\rho}(v) \otimes c$ i.e. the representation doesn't transform elements of $\mathbb{C}[[h]]$ and hence can be defined entirely by its action on a basis of V. We will denote $\tilde{\rho}$ by ρ itself.

Here we discuss a class of finite-dimensional representations of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ denoted by T_l where $l \in \frac{1}{2}\mathbb{N}_0$, as described in [KS81].
Let $l \in \frac{1}{2}\mathbb{N}_0$ and let V_l denote the $(2l+1)$ dimensional complex vector space with a basis

$$\{e_i \mid i = -l, -l+1, ..., l-1, l\}.$$

Define T_l by its action on the generating elements E, F, H as follows:

$$\begin{aligned} T_l(E)e_m &= ([l-m][l+m+1])^{\frac{1}{2}} e_{m+1} \\ T_l(F)e_m &= ([l+m][l-m+1])^{\frac{1}{2}} e_{m-1} \\ T_l(H)e_m &= 2me_m \end{aligned}$$

Note that T_l is not irreducible, as subspaces $h^k V_l[[h]]$ are invariant subspaces. The representation is indecomposable, i.e. the underlying space $V_l[[h]]$ can not be decomposed as a direct sum of non-trivial invariant subspaces.

As discussed earlier, by representations of a quasitriangular Hopf algebra, we obtain a matrix solution to the YBE. The existence of such a matrix is the key to defining representations of braid groups. Furthermore, the ribbon Hopf algebra structure allows us to define the quantum trace function, which gives rise to a link invariant. The next chapter discusses this construction in detail, focusing on the Jones polynomial. By a quantum group, we would mean an h-adic Hopf algebra with a ribbon Hopf algebra structure.

Chapter 3

Quantum Groups and Quantum Invariants

Quantum groups, described in the previous chapter, are efficient tools for generating link invariants. Given a representation of a quantum group, we can define representations of braid groups. Modifying the quantum trace, we get a quantity which is invariant under Markov moves, thus defining a link invariant. We can keep the quantum group fixed and change its representation or change the quantum group itself. For every distinct pair of a quantum group and its representation, we get an invariant. The Jones polynomial is obtained by the fundamental representation of the quantum group $U_h(\mathfrak{sl}_2(\mathbb{C}))$. The N-coloured Jones polynomial is obtained by N-dimensional representation of the quantum group $U_h(\mathfrak{sl}_2(\mathbb{C}))$. By the fundamental representation of the quantum group $U_h(\mathfrak{sl}_n(\mathbb{C}))$, we obtain the HOMFLY-PT polynomial.

3.1 Quantum Invariants of Links

Quantum invariants of links broadly refer to a class of link invariants that arise by a representation of some quantum group. Let $(U_h(\mathfrak{g}), R, K)$ be a quantum group, where $\mathfrak{g}$ is a semisimple complex Lie algebra. And let (ρ, V) be a representation of $U_h(\mathfrak{g})$. Then for every $n \geq 1$, we obtain a representation of B_n over $V^{\otimes n}$ by sending the generators σ_i to $Id_{V^{\otimes i-1}} \otimes \tau \circ (\rho \otimes \rho)(R) \otimes Id_{V^{\otimes n-i-1}}$. We can then normalise the quantum trace function

${}_qTr_{V^{\otimes n}}$ to make it invariant under Markov moves. Thus, we can obtain a link invariant from a representation of a quantum group.

Remark 3.1.1. *1. When $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and the dimension of $V = 2$, we obtain the Jones polynomial.*

2. When $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and the dimension of $V = N$, we obtain the N-colored Jones polynomial.

3. Finally, when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and the dimension of $V = n$, we obtain the HOMFLY-PT polynomial.

In the upcoming sections, we will focus on the computation of the Jones polynomial through the fundamental representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$.

3.2 Finite dimensional representations of braid groups

We earlier defined a class of representations of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ denoted by T_l. Now, we can define representations of B_n as follows:

$$\tilde{T_l}^{\otimes n} : B_n \to GL(V[[h]]^{\otimes n})$$
$$\tilde{T_l}^{\otimes n} : \sigma_i \mapsto Id_{V[[h]]^{\otimes i-1}} \otimes \tau \circ (T_l \otimes T_l)(R) \otimes Id_{V[[h]]^{\otimes n-i-1}}$$

where τ is the tensor flip map.

Since the image of σ_i is invertible and satisfies the Yang-Baxter equation, $\tilde{T_l}^{\otimes n}$ is a group homomorphism. This directly follows from the defining properties of a universal R-matrix. From this point, we will denote $(T_l \otimes T_l)(R)$ by R. Given a braidword, its image will be a square matrix of order $(2l+1)^n$. Taking the quantum trace of this matrix gives a link invariant. The Jones polynomial is defined by fixing $l = \frac{1}{2}$.

3.3 The Jones Polynomial

Let $l = \frac{1}{2}$ then the dimension of V is $2 \times \frac{1}{2} + 1 = 2$ and the basis is $\{e_{-\frac{1}{2}}, e_{\frac{1}{2}}\}$. $T_{\frac{1}{2}}$ is defined as follows:

$$\begin{aligned} T_{\frac{1}{2}}(E)e_{-\frac{1}{2}} &= e_{\frac{1}{2}} & T_{\frac{1}{2}}(E)e_{\frac{1}{2}} &= 0 \\ T_{\frac{1}{2}}(F)e_{-\frac{1}{2}} &= 0 & T_{\frac{1}{2}}(F)e_{\frac{1}{2}} &= e_{-\frac{1}{2}} \\ T_{\frac{1}{2}}(H)e_{-\frac{1}{2}} &= -e_{-\frac{1}{2}} & T_{\frac{1}{2}}(H)e_{\frac{1}{2}} &= e_{\frac{1}{2}} \end{aligned}$$

Denote the images of E, F and H under $T_{\frac{1}{2}}^{\otimes n}$ by E_n, F_n and H_n respectively. Then,

$$E_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} F_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} H_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The R matrix is given by:

$$R = e^{h(H \otimes H)/2} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(1 - q^{-2})^n}{[n]_q!} E^n \otimes F^n$$

Note that $E_1^2 = F_1^2 = 0$. Thus, the image of R under the representation $T_{\frac{1}{2}} \otimes T_{\frac{1}{2}}$ is given by

$$R = e^{h(H \otimes H)/2}(1 + (q - q^{-1})(E \otimes F)).$$

We have,

$$E \otimes F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H \otimes H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I + (q - q^{-1})(E \otimes F) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, we get the matrix for R as below:

$$R = \begin{bmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{-1}{2}} & 0 & 0 \\ 0 & q^{\frac{-1}{2}}(q - q^{-1}) & q^{\frac{-1}{2}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{bmatrix}$$

$$q^{\frac{1}{2}} R = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix R doesn't satisfy the YBE; we must compose it with the tensor flip map to get the image of σ_1:

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \tau \circ q^{\frac{1}{2}} R = q^{\frac{1}{2}} \hat{R} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \\ &= \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \end{aligned}$$

For $n = 2$, we need to compute the images of E, F and H. Recall that the tensor product of representations ρ and γ is given by

$$(\rho \otimes \gamma)(a) = \rho \otimes \gamma(\Delta(a))$$

where $a \in U_h(\mathfrak{sl}_2(\mathbb{C}))$ and Δ is the coproduct map.
Using the definition, we get the following:

$$\begin{aligned} E_2 = T_{\frac{1}{2}} \otimes T_{\frac{1}{2}}(\Delta(E)) &= E_1 \otimes e^{hH_1} + I_2 \otimes E_1 \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & q & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & q & 1 & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
F_2 = T_{\frac{1}{2}} \otimes T_{\frac{1}{2}}(\Delta(F)) &= F_1 \otimes I_2 + e^{-hH_1} \otimes F_1 \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
H_2 = T_{\frac{1}{2}} \otimes T_{\frac{1}{2}}(\Delta(H)) &= H_1 \otimes I_2 + I_2 \otimes H_1 \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}
\end{aligned}
$$

The image of the ribbon element K is given by the linear operator which sends e_m to $q^{2m}e_m$. For a braid on n strands, its image under the representation will be a square matrix of order 2^n. Denote this matrix by M. Then, the image of M under the quantum trace function is given by

$$
{}_qTr_{[\frac{1}{2}]^{\otimes n}}(M) = Tr(M \times K^{\otimes n}) \tag{3.1}
$$

We can normalise the quantum trace to make it invariant under the Markov moves, which will be done in the upcoming sections.

It is difficult to compute quantum trace directly. The computation is made easier by decomposing the given matrix into block matrices, where the blocks are a scalar multiple of the identity matrix. To find the decomposition, we need to compute the span of the highest-weight vectors. The Highest weight vectors are eigenvectors of H_n, which also belong to the kernel of E_n. Here, we follow the computation method described in [MSS]. For $n = 2$ we have,

$$\ker(E_2) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ q \\ 0 \end{pmatrix} \right\}$$

Denote the first vector by w_0 and the second by w_1. Since $E_2 w_i = 0$ and $H_2 w_i = c_i w_i$, the span of w_i is defined by action of F_2 on w_i. We have,

$$F_2 w_0 = \begin{pmatrix} 0 \\ 1 \\ q^{-1} \\ 0 \end{pmatrix}, \; F_2^2 w_0 = \begin{pmatrix} q + q^{-1} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; F_2^3 w_0 = 0$$

$$F_2 w_1 = 0$$

We get the ordered basis $\{w_0, F_2 w_0, F_2^2 w_0, w_1\}$ and the base change matrix is given by:

$$Q = \begin{bmatrix} 0 & 0 & q + q^{-1} & 0 \\ 0 & 1 & 0 & -1 \\ 0 & q^{-1} & 0 & q \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We get the decomposition of $q^{\frac{1}{2}} \hat{R}$ as follows:

$$Q^{-1}(q^{\frac{1}{2}} \hat{R}) Q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -q^{-1} \end{bmatrix} = \begin{bmatrix} q Id_{[1]} & 0 \\ 0 & -q^{-1} Id_{[0]} \end{bmatrix}$$

Next, we have to normalise the quantum trace. The next two subsections justify the normalisation that we obtain in the end.

3.3.1 $n = 1$

For the trivial knot, the number of strands is equal to 1. Thus, its quantum trace is equal to

$$_qTr_{[\frac{1}{2}]}(Id_2) = Tr(K) = q^{-2\frac{1}{2}} + q^{2\frac{1}{2}} = q^{-1} + q$$

Since $V_{\bigcirc}(t) = 1$, we must normalise the quantum trace by the factor $q^{-1} + q$.

3.3.2 $n = 2$

The braid group B_2 is generated by σ_1. $\hat{\sigma_1}$ is the unknot.

$$\begin{aligned}
\sigma_1 &\mapsto \hat{R} \\
\sigma_1 &= q^{\frac{-1}{2}} \begin{bmatrix} qId_{[1]} & 0 \\ 0 & -q^{-1}Id_{[0]} \end{bmatrix} \\
_qTr(\sigma_1) &= q^{\frac{-1}{2}}(q_qTr(Id_{[1]}) - q^{-1}{}_qTr(Id_{[0]})) \\
&= q^{\frac{-1}{2}}(q(q^{-2} + 1 + q^2) - q^{-1}) \\
&= q^{\frac{-1}{2}}(q + q^3) = q^{\frac{3}{2}}(q^{-1} + q)
\end{aligned}$$

Normalising we get $q^{\frac{3}{2}}$, which is not equal to 1. Note that the writhe of this diagram (the diagram obtained by taking the closure of the braid σ_1) is 1. So the normalisation factor should be $q^{\frac{3}{2}\omega(L)}(q + q^{-1})$.

Let us verify if this normalisation factor works for σ_1^{-1}.

$$\begin{aligned}
\sigma_1^{-1} &= q^{\frac{1}{2}} \begin{bmatrix} q^{-1}Id_{[1]} & 0 \\ 0 & -qId_{[0]} \end{bmatrix} \\
_qTr(\sigma_1^{-1}) &= q^{\frac{1}{2}}(q^{-1}{}_qTr(Id_{[1]}) - q_qTr(Id_{[0]})) \\
&= q^{\frac{1}{2}}(q^{-1}(q^{-2} + 1 + q^2) - q) \\
&= q^{\frac{1}{2}}(q^{-3} + q^{-1}) = q^{\frac{-3}{2}}(q^{-1} + q)
\end{aligned}$$

The writhe for this diagram is -1. So, the normalising factor is $q^{\frac{-3}{2}}(q + q^{-1})$. Normalising the quantum trace, we get 1, the Jones polynomial for the unknot.

Now $\hat{\sigma_1}^2$ is the Hopf link. We have,

$$\begin{aligned}
\sigma_1^2 &= q^{-1}\begin{bmatrix} q^2 Id_{[1]} & 0 \\ 0 & q^{-2} Id_{[0]} \end{bmatrix} \\
{}_qTr(\sigma_1^2) &= q^{-1}(q^2\, {}_qTr(Id_{[1]}) + q^{-2}\, {}_qTr(Id_{[0]})) \\
&= q^{-1}(q^2(q^{-2}+1+q^2)+q^{-2}) \\
&= q^{-3}+q^{-1}+q+q^3
\end{aligned}$$

Normalising by $q^3(q+q^{-1})$ we get $q^{-5}+q^{-1}$. We know that the Jones polynomial of the oriented hopf link with linking number +1 is $-t^{\frac{1}{2}}-t^{\frac{5}{2}}$. So we need to multiply the normalised quantum trace with $(-1)^{1+\Lambda(L)}$, where $\Lambda(L)$ denote the number of components in the link L. Further, we need to substitute the modified quantum trace by $q=\frac{1}{\sqrt{t}}$. Note that this modification and substitution are consistent with the calculations of the base cases.
Our discussions so far have led us to the following result.

Let L be a link and let $\beta \in B_n$ be a braid such that $\hat{\beta}=L$. Denote the image of β under the representation $\tilde{T}_{[\frac{1}{2}]}^{\otimes n}$ by β itself. Then,

$$V_L(t=q^2) = (-1)^{1+\Lambda(L)}\frac{{}_qTr_{[\frac{1}{2}]^{\otimes n}}(\beta)}{q^{\frac{3}{2}\omega(L)}(q+q^{-1})} \tag{3.2}$$

Remark 3.3.1. *The expression on the right satisfies the skein relation* (1.2)*; hence, by Theorem 1.2.1, it must be equal to the Jones polynomial.*

3.4 Computation of Jones polynomial for $W(3,m)$

The weaving links $W(n,m)$ are a doubly infinite family of alternating links with the same planar projection as the Torus links $T(n,m)$. When n and m are co-prime, $W(n,m)$ comprises of weaving knots. These knots are examples of hyperbolic knots. X-S Lin conjectured that weaving knots have the maximum volume for a fixed crossing number. The volume-ish theorem [DL07] for alternating knots provides bounds on the volume in terms of the Jones polynomial. The Volume conjecture [Mur10] says that the volume of a knot's complement can be determined by its coloured Jones polynomial. Therefore, computing the Jones polynomial for Weaving knots (or links) can provide valuable results.

To begin computing, we can initiate by looking at the number of strands. When $n = 1$, we get the unknot, and for $n = 2$, $W(2, m)$ corresponds to the torus links, for which the closed form expression of the Jones polynomial is established. Consequently, we shall proceed with computing the Jones polynomial for $n = 3$, namely $W(3, m)$, utilising the formula (3.2). $W(3, m)$ is the closure of the braid $(\sigma_1\sigma_2^{-1})^m \in B_3$.

For $W(3, m)$ the number of strands is equal to 3. We obtain the following decomposed matrices for the generators σ_1 and σ_2 of B_3 with the help of SageMath (See the appendix for the code).

$$\sigma_1 = q^{\frac{-1}{2}} \begin{bmatrix} (q-q^{-1})Id_{[\frac{1}{2}]} & Id_{[\frac{1}{2}]} & 0 \\ Id_{[\frac{1}{2}]} & 0 & 0 \\ 0 & 0 & qId_{[\frac{3}{2}]} \end{bmatrix} = q^{\frac{-1}{2}} \begin{bmatrix} (q-q^{-1}) & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q \end{bmatrix}$$

$$\sigma_2 = q^{\frac{-1}{2}} \begin{bmatrix} q & 0 & 0 \\ -q^2 & -q^{-1} & 0 \\ 0 & 0 & q \end{bmatrix}$$

$$\sigma_2^{-1} = q^{\frac{1}{2}} \begin{bmatrix} q^{-1} & 0 & 0 \\ -q^2 & -q & 0 \\ 0 & 0 & q^{-1} \end{bmatrix}$$

For $m = 1$:

$$\sigma_1\sigma_2^{-1} = \begin{bmatrix} 1-q^{-2}-q^2 & -q & 0 \\ q^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} {}_qTr(\sigma_1\sigma_2^{-1}) &= (1-q^{-2}-q^2)\,{}_qTr(Id_{[\frac{1}{2}]}) +_q Tr(Id_{[\frac{3}{2}]}) \\ &= (1-q^{-2}-q^2)(q^{-1}+q) + (q^{-3}+q^{-1}+q+q^3) \\ &= q^{-1}+q \end{aligned}$$

Normalising we get $\frac{q^{-1}+q}{q^{-1}+q} = 1$. This is the correct answer since closure of $\sigma_1\sigma_2^{-1}$ is the unknot. Note that the writhe for $W(3, m)$ for this projection is 0, and the number of link components is odd, so the normalisation factor is $(q + q^{-1})$.

For $m = 2$, we get:

$$(\sigma_1\sigma_2^{-1})^2 = \begin{bmatrix} q^{-4} + 2(1-q^{-2}-q^2) + q^4 & q^{-1} - q + q^3 & 0 \\ q^{-1} - q - q^{-3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} {}_qTr((\sigma_1\sigma_2^{-1})^2) &= (q^{-4} + 2(1-q^{-2}-q^2) + q^4)_qTr(Id_{[\frac{1}{2}]}) -_q Tr(Id_{[\frac{1}{2}]}) +_q Tr(Id_{[\frac{3}{2}]}) \\ &= (q^{-4} + 2(1-q^{-2}-q^2) + q^4 - 1)(q^{-1}+q) + (q^{-3}+q^{-1}+q+q^3) \\ &= q^{-5} + q^5 \end{aligned}$$

Normalising, we get $\frac{q^{-5}+q^5}{q^{-1}+q} = q^{-4} - q^{-2} + 1 - q^2 + q^4$. Substituting $q^2 = \frac{1}{t}$, we get:

$$V_{FigureEight}(t) = t^{-2} - t^{-1} + 1 - t + t^2$$

Let us compute for an arbitrary m, let

$$(\sigma_1\sigma_2^{-1})^m := A(m) = \begin{bmatrix} M_{11}(m)Id_{[\frac{1}{2}]} & M_{12}(m)Id_{[\frac{1}{2}]} & 0 \\ M_{21}(m)Id_{[\frac{1}{2}]} & M_{22}(m)Id_{[\frac{1}{2}]} & 0 \\ 0 & 0 & Id_{[\frac{3}{2}]} \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$$

We have,

$$A(m+1) = \begin{bmatrix} M_{11}(m) & M_{12}(m) & 0 \\ M_{21}(m) & M_{22}(m) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-q^{-2}-q^2 & -q & 0 \\ q^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The above matrix multiplication gives the following recursions:

$$\begin{aligned} M_{11}(m+1) &= (1-q^{-2}-q^2)M_{11}(m) + q^{-1}M_{12}(m) \\ M_{12}(m+1) &= -qM_{11}(m) \\ M_{21}(m+1) &= (1-q^{-2}-q^2)M_{21}(m) + q^{-1}M_{22}(m) \\ M_{22}(m+1) &= -qM_{21}(m). \end{aligned}$$

Similarly, we have,

$$A(m+1) = \begin{bmatrix} 1-q^{-2}-q^2 & -q & 0 \\ q^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_{11}(m) & M_{12}(m) & 0 \\ M_{21}(m) & M_{22}(m) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the following recursions:

$$\begin{aligned}
M_{11}(m+1) &= (1-q^{-2}-q^2)M_{11}(m) - qM_{21}(m) \\
M_{12}(m+1) &= (1-q^{-2}-q^2)M_{12}(m) - qM_{22}(m) \\
M_{21}(m+1) &= q^{-1}M_{11}(m) \\
M_{22}(m+1) &= q^{-1}M_{12}(m).
\end{aligned}$$

Combining the two sets of recursions, we get $m \geq 1$

$$\begin{aligned}
M_{11}(m+1) &= q(1-q^{-2}-q^2)M_{21}(m+1) - qM_{21}(m) \\
M_{12}(m+1) &= -q^2M_{21}(m+1) \\
M_{22}(m+1) &= -qM_{21}(m)
\end{aligned}$$

and for $m \geq 2$

$$M_{21}(m+1) = (1-q^{-2}-q^2)M_{21}(m) - M_{21}(m-1)$$

with initial conditions: $M_{21}(1) = q^{-1}$ and $M_{21}(2) = -q^{-3} + q^{-1} - q$.

3.4.1 Digression: Posets and Rank polynomials

Definition 3.4.1. *[Sta97]Let P be a poset and let $I \subset P$. I is called an order ideal of P if it satisfies the following*

$$y \in I,\ \ z \leq y \implies z \in I.$$

Definition 3.4.2. *The lattice of order ideals of the poset P is defined as*

$$J(P) := \{I \subset P \,:\, I \text{ is an order ideal of } P\}.$$

We will refer to the lattice of order ideals as simply 'lattice'. Corresponding to a poset P, we can associate a polynomial called the rank polynomial, which we define as follows:

Definition 3.4.3. *The rank polynomial of the lattice of the poset P is defined as*

$$\begin{aligned} R_P(x) &= \sum_{I \in J(P)} x^{|I|} \\ &= \sum_{j \geq 0} W_P(j) x^j \end{aligned}$$

where $W_P(j) = |\{I \in J(P) \ : \ |I| = j\}|$ are called the j^{th} Whitney number of the lattice of the poset P.

Fibonacci Lattice:

Definition 3.4.4. *The fence poset of order n, denoted by Z_n is the poset $\{x_1, x_2, ..., x_n\}$ with the cover relations: $x_{2k-1} \lhd x_{2k} \rhd x_{2k+1}$, where $k \geq 1$.*

Since $|J(Z_n)| = f_n$ (n^{th} Fibonacci number), the lattice of the fence poset is called the Fibonacci lattice. Denote the rank polynomial of the Fibonacci lattice by $F_n(x)$ and denote the k^{th} Whitney number of the lattice of Z_n by $f_{n,k}$ then we have,

$$F_n(x) = \sum_{k \geq 0} f_{n,k} x^k.$$

Note that $f_{n,0} = 1$ since the empty set is the only order ideal with 0 elements. Moreover, $f_{n,n} = 1$, since the only order ideal with n elements is the poset itself.

Proposition 3.4.1. *[MZ02] The Whitney numbers $f_{n,k}$ satisfy the recurrence*

$$f_{n+4,k+2} = f_{n+2,k+2} + f_{n+2,k+1} + f_{n+2,k} - f_{n,k}$$

and the rank polynomial satisfies the recurrence

$$F_{n+4}(x) = (1 + x + x^2)F_{n+2}(x) - x^2 F_n(x).$$

Proof. Consider the fence poset Z_{n+4}.
Let $I \in J(Z_{n+4})$ be such that $|I| = k + 2$. The number of such I is $f_{n+4,k+2}$. The recurrence holds if $k + 2 \in \{0, 1\}$. Let $k + 2 \geq 2$. Let $x_1 \in Z_{n+4}$, then there are three possibilities:

1. $x_1, x_2 \in I$

2. $x_1 \in I$ $x_2 \notin I$

3. $x_1, x_2 \notin I$

- Case I: If $x_1, x_2 \in I$ then since $x_3 \leq x_2$, x_3 must belong to I. $I \setminus \{x_1, x_2\}$ is an order ideal of Z_{n+2} because x_1 and x_2 are not covered by elements of $Z_{n+4} \setminus x_1, x_2 \cong Z_{n+2}$. Moreover, $I \setminus \{x_1, x_2\} \subset Z_{n+2}$ is an order ideal of order k that contains x_3.
  Order ideals that don't contain x_3 can not contain x_4. So the number of order ideals of Z_{n+2} of order k, that don't contain $x_3 = f_{n,k}$. Thus,

$$|I \in J(Z_{n+4}) \text{ s.t } |I| = k+2,\ x_1, x_2 \in I| = f_{n+2,k} - f_{n,k}.$$

- Case II: If $x_1, \in I$ and $x_2 \notin I$ then $I \setminus \{x_1\}$ is an order ideal of $Z_{n+4} \setminus \{x_1, x_2\}$ of order $k+1$. Thus,

$$|I \in J(Z_{n+4}) \text{ s.t } |I| = k+2,\ x_1 \in I,\ \ x_2 \notin I| = f_{n+2,k+1}.$$

- Case III: If $x_1, x_2 \notin I$ then I is an order ideal of $Z_{n+4} \setminus \{x_1, x_2\}$ of order $k+2$. Thus,

$$|I \in J(Z_{n+4}) \text{ s.t } |I| = k+2,\ x_1, x_2 \notin I| = f_{n+2,k+2}.$$

Summing over all the cases we get,

$$f_{n+4,k+2} = f_{n+2,k+2} + f_{n+2,k+1} + f_{n+2,k} - f_{n,k}.$$

Multiplying by x^{k+2} and summing over $k+2 \geq 0$ we get

$$\sum_{k+2\geq 0} f_{n+4,k+2}x^{k+2} = \sum_{k+2\geq 0} f_{n+2,k+2}x^{k+2} + \sum_{k+2\geq 0} f_{n+2,k+1}x^{k+2} + \sum_{k+2\geq 0} f_{n+2,k}x^{k+2} - \sum_{k+2\geq 0} f_{n,k}x^{k+2}$$

$$F_{n+4}(x) = F_{n+2}(x) + xF_{n+2}(x) + x^2F_{n+2}(x) - x^2F_n(x)$$

$$\implies F_{n+4}(x) = (1 + x + x^2)F_{n+2}(x) - x^2F_n(x)$$

$\square$

Lucas Lattice:

Definition 3.4.5. *The crown poset of order n, denoted by C_n is the poset $\{x_1, x_2, ..., x_{2n}\}$ with the cover relations: $x_{2k-1} \lhd x_{2k} \rhd x_{2k+1},\ x_{2n} \rhd x_1$, where $k \geq 1$.*

Since $|J(C_n)| = L_{2n}$ ($2n^{th}$ Lucas number), the lattice of crown poset is called the Lucas lattice. Denote the rank polynomial of the Lucas lattice by $C_n(x)$ and denote the k^{th} Whitney number of C_n by $c_{n,k}$ then we have,

$$C_n(x) = \sum_{k\geq 0} c_{n,k}x^k.$$

Note that $c_{n,0} = 1$ since the empty set is the only order ideal with 0 elements. Furthermore, $c_{n,2n} = 1$, since the poset is the only order ideal with $2n$ elements.

Proposition 3.4.2. *[MZ02] The Whitney numbers $c_{n,k}$ satisfy the recurrence*

$$c_{n+2,k+2} = f_{2n+4,k+2} - f_{2n,k}$$

and the rank polynomial satisfies the recurrence

$$C_{n+2}(x) = F_{2n+4}(x) - x^2F_{2n}(x).$$

Refer to Murani & Salvi [MZ02] for the proof.

3.4.2 Closed form of Jones polynomial

We observed that the constant terms of the Jones polynomial of $W(3, m)$ were given by the m^{th} Whitney number of the Lucas lattice C_m [Inc]. More generally, we observed that the coefficients of $V_{W(3,m)}(t)$ were related to the Whitney numbers $c_{m,k}$. In this section, we state and prove the relation of the Jones polynomial with the rank polynomial of the Lucas lattice. Recall that

$$(\sigma_1\sigma_2^{-1})^m = \begin{bmatrix} M_{11}(m)Id_{[\frac{1}{2}]} & M_{12}(m)Id_{[\frac{1}{2}]} & 0 \\ M_{21}(m)Id_{[\frac{1}{2}]} & M_{22}(m)Id_{[\frac{1}{2}]} & 0 \\ 0 & 0 & Id_{[\frac{3}{2}]} \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$$

Proposition 3.4.3. *The entry $M_{21}(m)$ is given by,*

$$M_{21}(m) = \frac{(-1)^{m+1}}{q^{2m-1}} F_{2(m-1)}(-q^2)$$

where $F_n(x)$ is the rank polynomial of the Fibonacci lattice of order n.

Proof. Let $m = 1$, then the LHS $= q^{-1}$ and the RHS is

$$\frac{(-1)^2}{q^1} F_0(-q^2) = \frac{1}{q}.$$

Similarly, if $m = 2$ then the LHS $= -q^{-3} + q^{-1} - q$ and the RHS is

$$\begin{aligned}
\frac{(-1)^3}{q^3} F_2(-q^2) &= \frac{-1}{q^3}(-q^2 + q^4 - q^6) \\
&= -(-q^{-1} + q - q^3) \\
&= q^{-1} - q + q^3.
\end{aligned}$$

So, the claim is true for $m = 1, 2$. Assume that the claim is true for $m < p$ where $p \geq 2$. We have

$$\begin{aligned}
M_{21}(p) &= (1 - q^{-2} - q^2) M_{21}(p-1) - M_{21}(p-2) \\
&= (1 - q^{-2} - q^2) \frac{(-1)^p}{q^{2p-3}} F_{2(p-2)}(-q^2) - \frac{(-1)^{p-1}}{q^{2p-5}} F_{2(p-3)}(-q^2) \\
&= \frac{(-1)^{p+1}}{q^{2p-1}} (-q^2(1 - q^{-2} - q^2) F_{2p-4}(-q^2) - q^4 F_{2p-6}(-q^2)) \\
&= \frac{(-1)^{p+1}}{q^{2p-1}} ((-q^2 + 1 + q^4) F_{2p-4}(-q^2) - q^4 F_{2p-6}(-q^2)).
\end{aligned}$$

From the Proposition 3.4.1 we have

$$F_{2p-2}(-q^2) = (1 - q^2 + q^4) F_{2p-4}(-q^2) - q^4 F_{2p-6}(-q^2),$$

which gives

$$M_{21}(p) = \frac{(-1)^{p+1}}{q^{2p-1}} ((-q^2 + 1 + q^4) F_{2p-4}(-q^2) - q^4 F_{2p-6}(-q^2)) = \frac{(-1)^{p+1}}{q^{2p-1}} F_{2p-2}(-q^2).$$

Thus, by the principle of mathematical induction, we have

$$M_{21}(m) = \frac{(-1)^{m+1}}{q^{2m-1}} F_{2(m-1)}(-q^2)$$

for $m \geq 1$. □

Proposition 3.4.4. *The trace of the sub-matrix M^m is given by,*

$$Tr(M^m) = \frac{(-1)^m}{q^{2m}} C_m(-q^2)$$

where $C_n(x)$ denotes the rank polynomial of the Lucas lattice of order n.

Proof. Consider the trace of the matrix M^m,

$$Tr(M^m) = M_{11}(m) + M_{22}(m)$$

Let us verify the base case $m = 1$. The trace of M is $1 - q^{-2} - q^2$ and the RHS is:

$$\begin{aligned}\frac{(-1)}{q^2} C_1(-q^2) &= -q^{-2}(1 + (-q^2) + (-q^2)^2) \\ &= -q^{-2}(1 - q^2 + q^4) = -q^{-2} + 1 - q^2\end{aligned}$$

which is equal to the LHS. For $m > 1$ we substitute the expression of $M_{11}(m)$ and $M_{22}(m)$ in terms of $M_{21}(m)$. We get,

$$\begin{aligned}Tr(M^m) &= q(1 - q^2 - q^{-2})M_{21}(m) - qM_{21}(m-1) - qM_{21}(m-1) \\ &= q(1 - q^2 - q^{-2})\frac{(-1)^{m+1}}{q^{2m-1}} F_{2(m-1)}(-q^2) - 2q\frac{(-1)^m}{q^{2m-3}} F_{2(m-2)}(-q^2) \\ &= \frac{(-1)^m}{q^{2m}}(-q^2(1 - q^2 - q^{-2})F_{2(m-1)}(-q^2) - 2q^4 F_{2(m-2)}(-q^2)) \\ &= \frac{(-1)^m}{q^{2m}}((1 - q^2 + q^4)F_{2(m-1)}(-q^2) - q^4 F_{2(m-2)}(-q^2) - q^4 F_{2(m-2)}(-q^2)) \\ &= \frac{(-1)^m}{q^{2m}}(F_{2m}(-q^2) - q^4 F_{2(m-2)}(-q^2)) && \text{By Prop. 3.4.1} \\ &= \frac{(-1)^m}{q^{2m}} C_m(-q^2) && \text{By Prop. 3.4.2}\end{aligned}$$

$\therefore Tr(M^m) = \frac{(-1)^m}{q^{2m}} C_m(-q^2)$ for all $m \geq 1$.

□

We have

$$ {}_qTr((\sigma_1\sigma_2^{-1})^m) = (q+q^{-1}) \times Tr(M^m) + (q^{-3}+q^{-1}+q+q^3). $$

Substituting the value of trace of M^m, we get,

$$ {}_qTr((\sigma_1\sigma_2^{-1})^m) = (q+q^{-1}) \times \frac{(-1)^m}{q^{2m}} C_m(-q^2) + (q^{-3}+q^{-1}+q+q^3). $$

Normalising and substituting $q^2 = \frac{1}{t}$ we obtain the Jones polynomial of $W(3,m)$

$$ V_{W(3,m)}(t) = (-1)^m t^m C_m(\frac{-1}{t}) + t + t^{-1}. $$

We have, $t^m C_m(\frac{-1}{t}) = \frac{1}{t^m} C_m(-t)$, thus we get the following formula:

$$ V_{W(3,m)}(t) = \frac{(-1)^m}{t^m} C_m(-t) + t + t^{-1} \tag{3.3} $$

The formula (3.3) is also obtained in [AC23] through a different method.

3.4.3 Zeros of $V_{W(3,m)}(t)$

For each $2 \leq m \leq 50$ we plotted the zeros of $V_{W(3,m)}(t)$. Here, we include some of the plots. We observed that the locus of zeros approaches a curve on the unit circle union, some part of the real line.

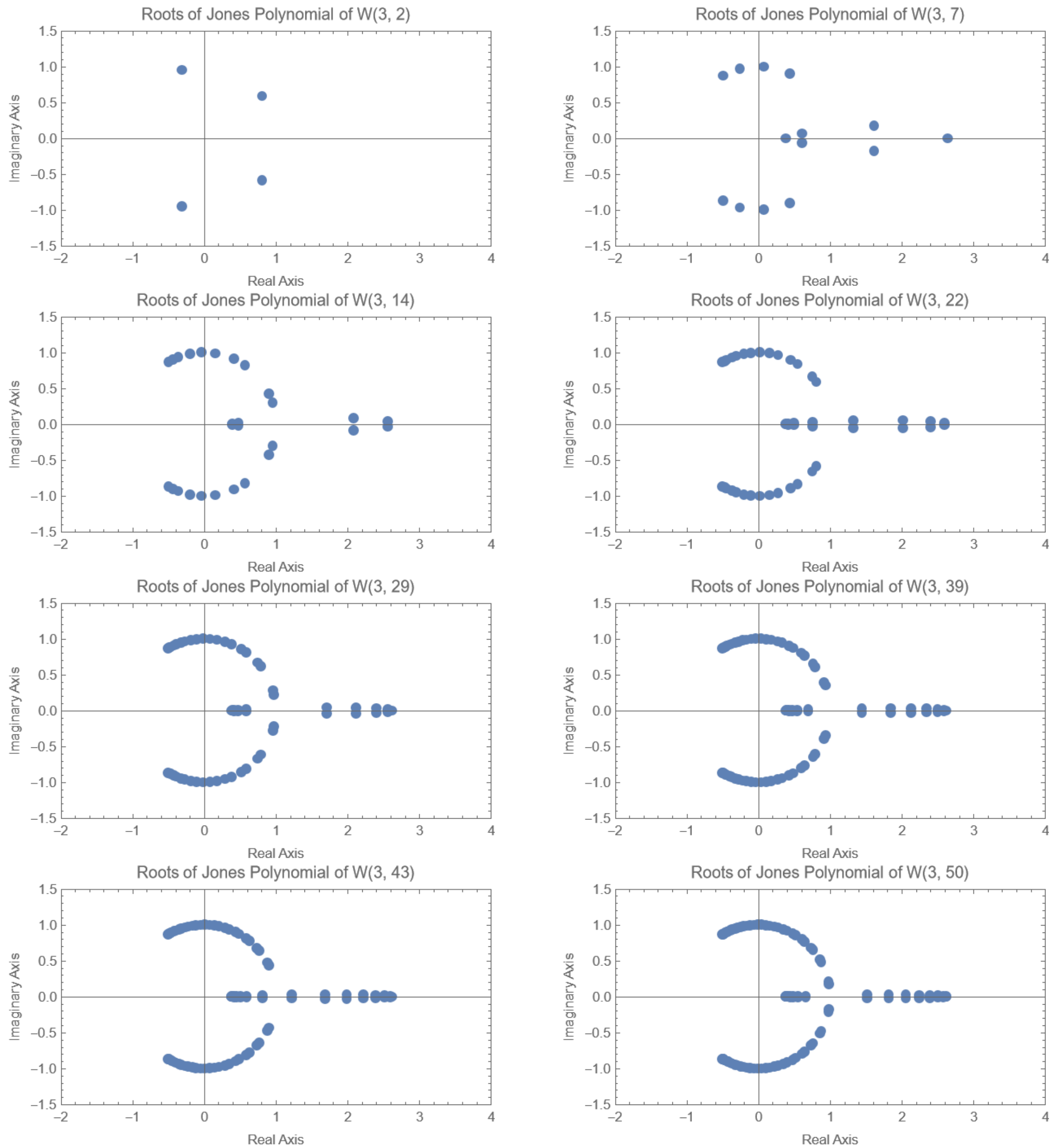


Figure 3.1: Distribution of zeros of $V_{W(3,m)}(t)$

3.5 Computation of Jones polynomial for $W(4,m)$

This section focuses on weaving links $W(4,m)$. Although we couldn't get a closed-form expression, we did observe some patterns in the coefficients of its Jones polynomial.

$W(4,m)$ is the closure of the braid $\sigma_1\sigma_2^{-1}\sigma_3 \in B_4$. The writhe of $W(4,m)$ with this projection is m. The matrices of the generating elements will be 16 by 16. The highest weight decomposition gives us six by six matrices. We obtain the following matrices with the help of a Mathematica code (included in the appendix).

$$\sigma_1 = \begin{bmatrix} \sqrt{q}Id_{[2]} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{q}Id_{[1]} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{q}Id_{[1]} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{q}Id_{[0]} & 0 & 0 \\ 0 & -\frac{1}{q^{7/2}} & -\frac{1}{q^{5/2}} & 0 & -\frac{1}{q^{3/2}}Id_{[1]} & 0 \\ 0 & 0 & 0 & -\frac{1}{q^{5/2}} & 0 & -\frac{1}{q^{3/2}}Id_{[0]} \end{bmatrix}$$

$$\sigma_2^{-1} = \begin{bmatrix} \frac{1}{\sqrt{q}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{q}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{q}} - q^{3/2} & 0 & \sqrt{q} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{q}} - q^{3/2} & 0 & \sqrt{q} \\ 0 & 0 & \sqrt{q} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{q} & 0 & 0 \end{bmatrix}$$

$$\sigma_3 = \begin{bmatrix} \sqrt{q} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{q}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{q}} & \frac{q^2-1}{q^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{q} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{q} & 0 \\ 0 & 0 & 0 & -\frac{1}{q^{5/2}} & 0 & -\frac{1}{q^{3/2}} \end{bmatrix}$$

$$\sigma_1\sigma_2^{-1}\sigma_3 = \begin{bmatrix} \sqrt{q}Id_{[2]} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0Id_{[1]} & \frac{1}{\sqrt{q}} & 0 & 0 & 0 \\ 0 & \frac{1-q^2}{\sqrt{q}} & -\frac{\left(q^2-1\right)^2}{q^{3/2}}Id_{[1]} & 0 & q^{3/2} & 0 \\ 0 & 0 & 0 & \frac{-q^4+q^2-1}{q^{3/2}}Id_{[0]} & 0 & -\frac{1}{\sqrt{q}} \\ 0 & -\frac{1}{q^{7/2}} & -\frac{1}{q^{5/2}} & 0 & -\frac{1}{q^{3/2}}Id_{[1]} & 0 \\ 0 & 0 & 0 & \frac{1-q^2}{q^{9/2}} & 0 & \frac{1}{q^{7/2}}Id_{[0]} \end{bmatrix}$$

In the Table A.2 we have included our computations of $V_{W(4,m)}(t)$ for $m \leq 10$. Note that [MR21] presents an algorithm for computing the Jones polynomial via Hecke Algebra. The authors also provide computations for some weaving knots; our calculations agree with theirs.

3.5.1 Some Observations on $V_{W(4,m)}(t)$

Denote the Jones polynomial of $W(4,m)$ by:

$$V_{W(4,m)}(t) = \sum_{-\min(m)}^{\max(m)} a_{m,i}t^i$$

Let $Sum(m) = \sum_{-\min(m)}^{\max(m)} |a_{m,i}|$. Then we obtain $\tilde{V}_{W(4,m)}(t)$ by modifying $V_{W(4,m)}(t)$ as follows:

$$\tilde{V}_{W(4,m)}(t) = \sum_{-\min(m)}^{\max(m)} \frac{|a_{m,i}|}{Sum(m)}t^i = \sum_{-\min(m)}^{\max(m)} \tilde{a}_{m,i}t^i$$

For a given m, we plotted i on the x axis and $\tilde{a}_{m,i}$ on the y axis. Note that due to the above modification, the sum of all the coefficients of $\tilde{V}_{W(4,m)}$ is equal to 1, so the mentioned graph can be thought of as a probability distribution and we can study its convergence as m increases.

Below are the graphs for some values of m. We observed that as m increases, the graph approaches a normal distribution.

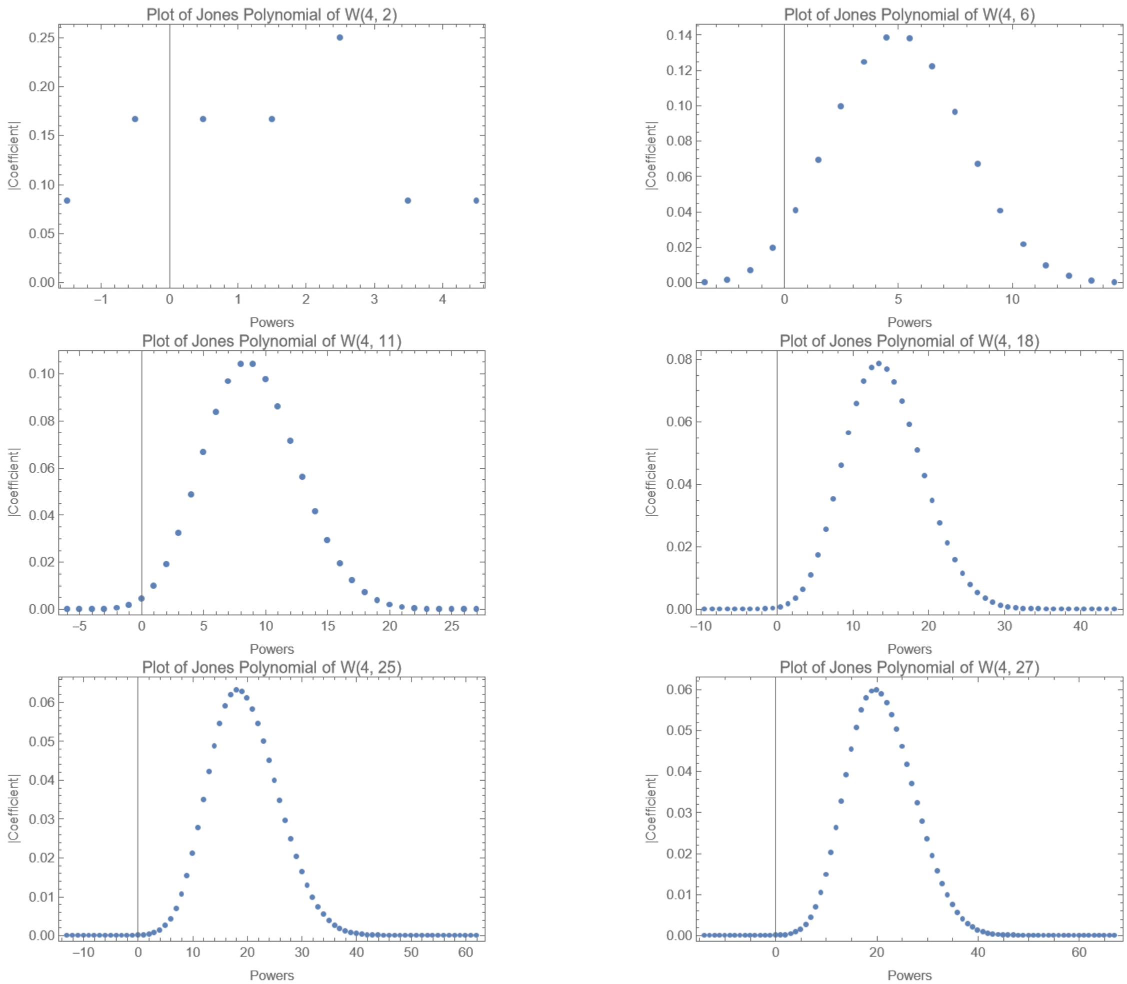


Figure 3.2: Distribution of coefficients of $V_{W(4,m)}(t)$

We also observed that the power of t, where the absolute value of the coefficient is highest, appears to increase linearly.

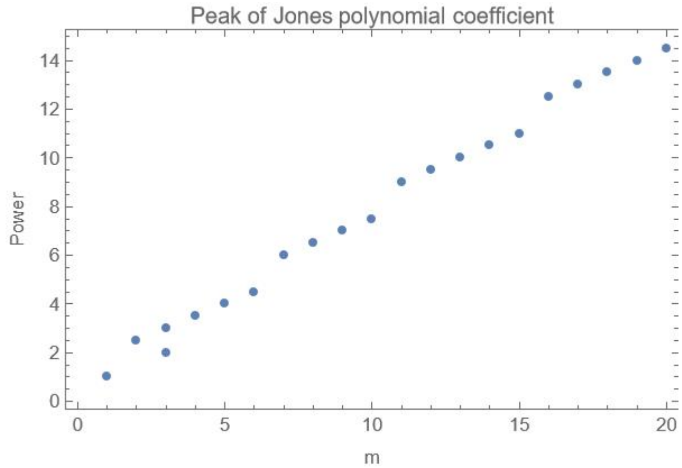


Figure 3.3: Graph of m vs power of peak

Remark 3.5.1. *The normal distribution property observed in weaving knots (and links) is not universal to all alternating knots. A specific class of alternating knots, known as Twist Knots, do not display a normal distribution of coefficients. Therefore, the normal distribution of coefficients seems to be a property unique to weaving knots (and links).*

3.5.2 Zeros of $V_{W(4,m)}(t)$

For each $2 \leq m \leq 50$ we plotted the zeros of $V_{W(4,m)}(t)$. Here, we include some of the plots.

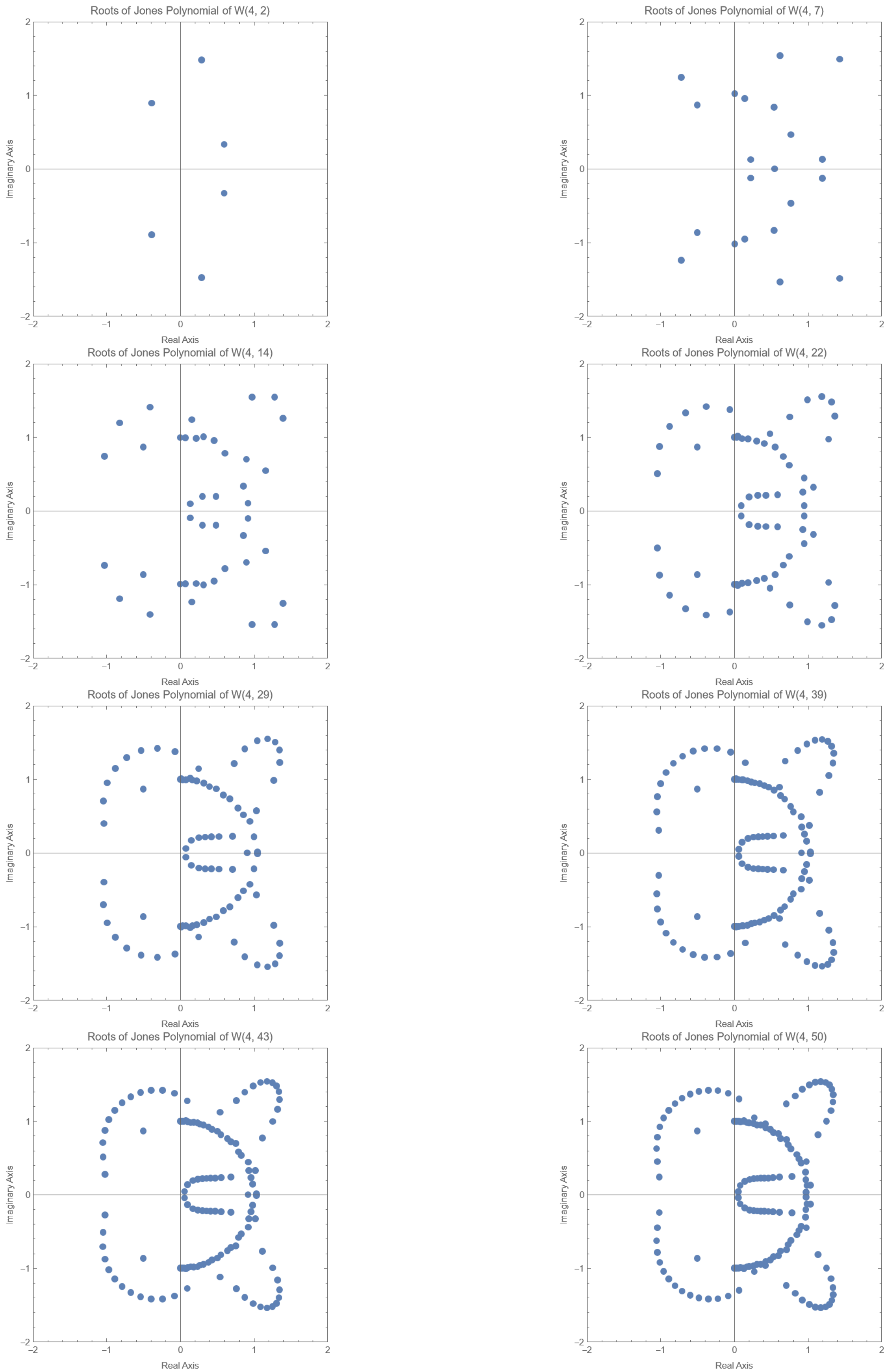


Figure 3.4: Distribution of zeros of $V_{W(4,m)}(t)$

3.6 The N-colored Jones Polynomial

To demonstrate the usefulness of the quantum groups, we compute the N-coloured Jones polynomial for the unknot.

The N-colored Jones polynomial is obtained by by N-dimensional representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$. Recall the representation T_l of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on $2l+1$ dimensional vector space, as defined earlier:

$$\begin{aligned} T_l(E)e_m &= ([l-m][l+m+1])^{\frac{1}{2}}e_{m+1}, \\ T_l(F)e_m &= ([l+m][l-m+1])^{\frac{1}{2}}e_{m-1}, \\ T_l(H)e_m &= 2me_m. \end{aligned}$$

For N-dimensional representation, $l = \frac{N-1}{2}$. The image of the ribbon element denoted by K is defined as

$$Ke_m = q^{2m}e_m$$

where $-l \leq m \leq l$.
Now, the image of the identity braid will be the identity matrix. Its quantum trace (before normalisation) will be the trace of the operator K.

$$Tr(K) = \sum_{m=-l}^{m=l} q^{2m} = \sum_{i=-N+1;step=2}^{i=N-1} q^i$$

Recall that m^{th} q-number is given by

$$[m] = q^{m-1} + q^{m-3} + \ldots + q^{-(m-1)}.$$

So $Tr(K) = [N]$, i.e. N-coloured Jones polynomial (unnormalised) of the unknot is $[N]$. Putting $N = 2, [2] = q + q^{-1}$ we retrieve the Jones polynomial.

Using quantum groups to compute the Jones polynomial involves only matrix multiplication, making it easier to code into a computer program and proving observed patterns.

The study of weaving links is fascinating, as they exhibit numerous intriguing properties. For a fixed value of n, we can explore the distribution of zeroes of $V_{W(n,m)}(t)$ and observe their convergence as m increases. Since for n odd, $V_{W(n,m)}(t)$ is symmetric, one can restrict their study to n odd. Moreover, one can work with different quantum invariants by taking a different quantum group and its representation, as mentioned for the N-coloured Jones polynomial. In the next chapter, we will see that the invariants from quantum groups are examples of 1-dimensional TQFTs. This perspective provides us with another way of generalising and obtaining new invariants.

Chapter 4

TQFTs and Quantum Invariants

An n dimensional TQFT is a functor from the n dimensional cobordism category to the category of finite dimensional vector spaces. In this chapter, we discuss a significant result which says that 1-dimensional TQFTs are in one-to-one correspondence with finite-dimensional representations of quasitriangular quantum groups. Hence, the invariants discussed in the previous chapter are examples of 1-dimensional TQFTs. Bringing TQFT into the picture enables us to generalise quantum invariants further. One can expect to obtain link invariants for any arbitrary n dimensional TQFT. For example, Khovanov homology is a 2-dimensional TQFT. Hence, TQFTs are quite useful for defining link invariants.

4.1 Axiomatic definition of TQFT

In [Ati88] M. Atiyah gave the following axiomatic definition of TQFTs:
A topological quantum field theory (TQFT) Z in dimension $d+1$ over a ground ring F is the following data:

- To each oriented closed smooth d dimensional manifold Σ, an associated finitely generated F-module $Z(\Sigma)$.

- To each oriented smooth $d+1$ dimensional manifold M (with boundary ∂M), an associated element $Z(M) \in Z(\partial M)$.

From now onwards, by a d dimensional manifold, we would mean an oriented closed smooth d dimensional manifold, and by a $(d+1)$ dimensional manifold, we would mean an oriented smooth $(d+1)$ dimensional manifold. The above data must satisfy the following axioms:

1. Z is functorial: By functoriality of Z we mean that if

   $$f : \Sigma \to \Sigma'$$

   is an orientation-preserving diffeomorphism, then Z induces an isomorphism of vector spaces

   $$Z(f) : Z(\Sigma) \to Z(\Sigma').$$

   And $Z(gf) = Z(g)Z(f)$ for any morphism

   $$g : \Sigma' \to \Sigma".$$

   Lastly, if f extends to an orientation preserving diffeomorphism from M to M', where $\partial(M) = \Sigma$, $\partial(M') = \Sigma'$, then $Z(f)$ maps $Z(M)$ to $Z(M')$.

2. Z is involutory:

   $$Z(\Sigma^*) = Z(\Sigma)^*$$

   where Σ^* is Σ with opposite orientation and $Z(\Sigma)^*$ is the dual of $Z(\Sigma^*)$.

3. Z is multiplicative:
   a. For any disjoint union of d dimensional manifold $\Sigma \cup \Sigma'$ we have,

   $$Z(\Sigma \cup \Sigma') = Z(\Sigma) \otimes Z(\Sigma')$$

   b. Let $\partial(M_1) = \Sigma_1 \cup \Sigma_3$ and $\partial(M_2) = \Sigma_3^* \cup \Sigma_2$. And let M denote the manifold obtained by glueing M_1 and M_2 along Σ_3.

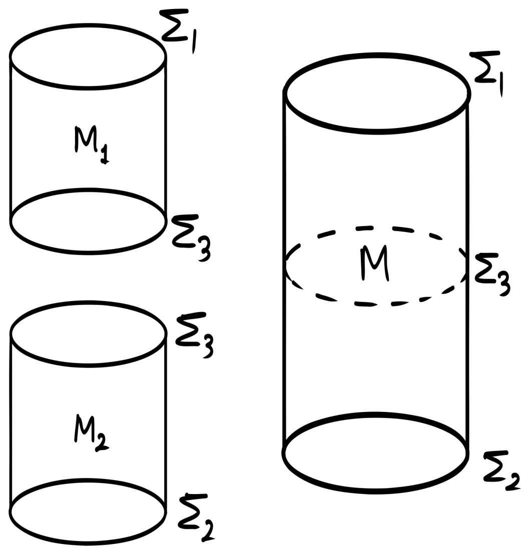


Figure 4.1: Two manifolds glued along their common boundary

Then,

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle$$

where $\langle\rangle$ denote the natural pairing

$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2).$$

To understand this, note that $Z(M_1) \in Z(\Sigma_1) \otimes Z(\Sigma_3)$ and $Z(M_2) \in Z(\Sigma_3)^* \otimes Z(\Sigma_2)$. Since $\partial(M) = \Sigma_1 \cup \Sigma_2, Z(M) \in Z(\Sigma_1) \otimes Z(\Sigma_2)$. Thus, $Z(M)$ is the image of $Z(M_1) \otimes Z(M_2)$ under the natural pairing map.

Let M be a $(d+1)$ dimensional manifold with $\partial(M) = \Sigma_1 \cup \Sigma_0^*$. Then

$$\begin{aligned} Z(M) \in Z(\partial(M)) &= Z(\Sigma_0^*) \otimes Z(\Sigma_1) \\ &= Hom(Z(\Sigma_0), Z(\Sigma_1)). \end{aligned}$$

This implies that any cobordism M between Σ_0 and Σ_1 can be seen as inducing a linear transformation as follows:

$$Z(M) : Z(\Sigma_0) \to Z(\Sigma_1).$$

4.2 Non triviality axioms

In addition to the axioms and properties defined above, we impose some non-triviality axioms. Note that *3a* implies that for the empty d dimensional manifold ϕ, $Z(\phi)$ is idempotent. This implies $Z(\phi) = F$ or ϕ. We rule out the possibility that $Z(\phi) = \phi$ and impose:

$$Z(\phi) = F$$

for the empty d dimensional manifold, where F is the base field.

Observe that, *3b* implies that for the empty $d + 1$ dimensional manifold $Z(\phi) \in F$ and $Z(\phi)$ is idempotent. Thus, it can take values 0 or 1. We impose the following:

$$Z(\phi) = 1$$

for the empty $(d + 1)$ dimensional manifold.

4.3 TQFT as a functor

Definition 4.3.1. *The d+1 dimensional cobordism category denoted by $Bord_{d+1}$ is the category whose objects are d-dimensional closed, oriented smooth manifolds and whose morphisms are d+1 dimensional cobordisms.*

By d+1 dimensional cobordism, we mean a d+1 dimensional compact, oriented smooth manifold M whose boundary $\partial(M)$ is a disjoint union of d dimensional closed, oriented smooth manifolds.

Notice that the domain of Z is exactly $Bord_{d+1}$, and the codomain is the category of finite dimensional vector spaces. So, a $d+1$ dimensional TQFT can be defined as a monoidal, dual functor from the $d+1$ dimensional Cobordism category to the finite-dimensional vector spaces category. We have the following:

- Z being functorial is equivalent to Z being a functor,

- Z being involutory is equivalent to Z being a dual functor,
- Z being multiplicative is equivalent to Z being a monoidal functor.

4.4 Category of Oriented Tangles

This section discusses the category of Oriented Tangles as described in [Tur89].

Definition 4.4.1. *A tangle is a finite family of disjoint, oriented circles and segments which are properly embedded in $\mathbb{R}^2 \times [0,1]$. Precisely, a (k,l) tangle is defined as an oriented one-dimensional smooth, compact submanifold L of $\mathbb{R}^2 \times [0,1]$ such that*

$$\partial(L) \coloneqq L \cap (\mathbb{R}^2 \times \{0,1\}) = \{(i,0,0)|i = 1,2,...,k\} \cup \{(j,0,1)|j = 1,2,...,l\}$$

For instance, a (0,0) tangle is an oriented link of circles.

Define:

$$s(L) = (\epsilon_1, ...\epsilon_k)$$
$$t(L) = (\mu_1, ...\mu_l)$$

where ϵ_i(respectively μ_j) is 1 or -1 depending on whether the tangent vector to L at the point $(i,0,0)$ (respectively $(j,0,1)$) is directed upwards or downwards (we need to fix one convention and work with it).

Definition 4.4.2. *Two tangles L and L' are said to be isotopic if there exists an isotopy of $\mathbb{R}^2 \times [0,1]$ which takes L to L' and which is identity on $\mathbb{R}^2 \times \{0,1\}$.*

Definition 4.4.3. *The category* Tang *of oriented tangles is defined as the category where the objects are finite sequences of ± 1. And the morphisms from ϵ to μ are oriented tangles L such that $s(L) = \epsilon$ and $t(L) = \mu$.*

Given two tangles L and L' such that $t(L) = s(L')$, we define the composition $L' \circ L$ as the tangle obtained by translating L' by the vector $(0,0,1)$, glueing it to L along their common ends and contracting it twice along the vertical axis.

The juxtaposition of tangles next to each other makes the category Tang a monoidal category. Moreover, it is a strictly monoidal category since, $(L \otimes L') \otimes L" = L \otimes (L' \otimes L")$ for all $L, L', L" \in$ Morph(Tang). The generating objects (generated by a sequence of $\circ$ and $\otimes$) are (1) and (-1). And Morph(Tang) is generated by the following eight tangles.

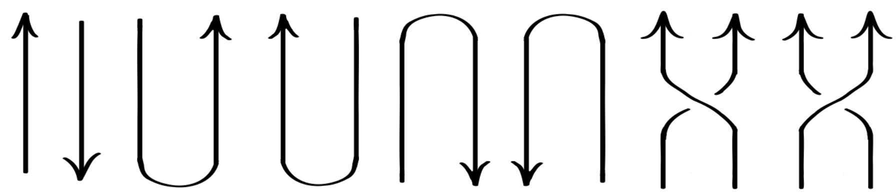


Figure 4.2: Generating tangles for the category of oriented tangles

The category Tang is equipped with a dual structure given by $(1)^* = (-1)$ and $L^* = L$ with opposite orientation.

4.5 1-dimensional TQFT from quantum groups

Notice that the 1-dimensional cobordism category is exactly the category of oriented tangles. So, to define a 1-dimensional TQFT, we only need to define the images for the generating objects and morphisms. We will see that given a representation of a ribbon hopf algebra, we can naturally define a 1-dimensional TQFT.

Let (H, R, G) be a ribbon hopf algebra. Let $\rho : H \to End(V)$ be a representation of H. Recall that $\rho^* : H \to End(V^*)$ given by the following

$$\langle \rho^*(a)v^*, v\rangle = \langle v^*, \rho(S(a))v\rangle,$$

defines a representation of H on V^*. Denote,

$$\begin{aligned} \rho_+ &:= \rho, \\ \rho_- &:= \rho^*, \\ R_{ij} &:= \rho_i \otimes \rho_j(R), \\ G_i &:= \rho_i(G). \end{aligned}$$

We define a 1-dimensional TQFT as described by Sawin [Saw95].

$$\begin{aligned}
F((1)) =& V \quad F((-1)) = V^* \\
F(\uparrow) :& V \to V = id_V \\
F(\downarrow) :& V^* \to V^* = id_{V^*} \\
F(\cup^{\uparrow}) :& \mathbb{C} \to V^* \otimes V \\
& c \mapsto c \sum_{\alpha} v_\alpha^* \otimes G_1^{-1}(v_\alpha) \\
F({}^{\uparrow}\cup) :& \mathbb{C} \to V \otimes V^* \\
& c \mapsto c \sum_{\alpha} v_\alpha \otimes v_\alpha^* \\
F(\cap_{\downarrow}) :& V \otimes V^* \to \mathbb{C} \\
& x \otimes y^* \mapsto y^*(G_1(x)) \\
F({}_{\downarrow}\cap) :& V^* \otimes V \to \mathbb{C} \\
& y^* \otimes x \mapsto y^*(x) \\
F(\times) :& V \otimes V \to V \otimes V \\
& x \otimes y \mapsto \tau \circ R_{11}(x \otimes y) \\
F(\times) :& V \otimes V \to V \otimes V \\
& x \otimes y \mapsto \tau \circ R_{11}^{-1}(x \otimes y)
\end{aligned}$$

We can check that the map described above satisfies the axioms of a 1-dimensional TQFT. We state an important result, without proving as follows [Saw95]:

Proposition 4.5.1. *Let f be a tangle, and F be a 1-dimensional TQFT constructed from a ribbon hopf algebra. Then, the image of the closure of f, under F, equals the quantum trace of f.*

The above proposition justifies choosing the quantum trace function to define link invariants.

4.6 Every 1-dimensional TQFT arises from some quantum group

In the previous section, we defined a 1-dimensional TQFT from a representation of a ribbon hopf algebra. The converse is also true; every 1-dimensional TQFT arises from a representation of some ribbon Hopf algebra [Saw95]. We will discuss a weaker version of the converse; specifically, every 1-dimensional TQFT arises from a representation of some quasitriangular hopf algebra.

Let F be a 1-dimensional TQFT and let $B = F($$)$. Then B satisfies the YBE, and $R = \tau \circ B$ satisfies the QYBE.

Let $n = \dim(F(1))$, then $R \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. We can also see R as an element of $M_n(M_n(\mathbb{C}))$. Thus we will use the notation R^{ik}_{jl} for $R \in M_n(M_n(\mathbb{C}))$.
We will build a quasitriangular hopf algebra whose fundamental representation will give rise to F. We will discuss the construction of two hopf algebras as described in [Maj90].

4.6.1 Bialgebras $A(R)$ and $U(R)$

Definition 4.6.1. *$A(R)$ is defined as a non-commutative bialgebra generated by one and n^2 generators u^i_j and modulo some relations.*

$$A(R) := \frac{1, \{u^i_j\}^n_{i,j=1}}{R(u \otimes Id_n)(Id_n \otimes u) - (Id_n \otimes u)(u \otimes Id_n)R}$$

where u is a $n \times n$ matrix with its $(i,j)^{th}$ entry given by u^i_j. The following equations give the associated coproduct and counit maps on generators.

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j$$
$$\epsilon(u^i_j) = \delta^i_j$$

The image of a non-generating element under these maps is obtained by extending the maps as algebra homomorphisms.

One must check if the above maps are well defined and compatible with the algebra structure (i.e. $A(R)$ is a bialgebra).

Definition 4.6.2. *$U(R)$ is defined as a non-cocommutative bialgebra generated by one and $2n^2$ generators and modulo some relations.*

$$U(R) := \frac{1, \{L_j^{\pm i}\}_{i,j=1}^n}{\begin{array}{c}(L^\pm \otimes Id_n)(Id_n \otimes L^\pm)R - R(Id_n \otimes L^\pm)(L^\pm \otimes Id_n) \\ (L^- \otimes Id_n)(Id_n \otimes L^+)R - R(Id_n \otimes L^+)(L^- \otimes Id_n)\end{array}}$$

where L^+, L^- are $n \times n$ matrix with $[L^+_{ij}] = L_j^{+i}$ and $[L^-_{ij}] = L_j^{-i}$. The following equations give the associated coproduct and counit maps by extending them as algebra maps.

$$\Delta(L_j^{\pm i}) = \sum_k L_k^{\pm i} \otimes L_j^{\pm k}$$
$$\epsilon(L_j^{\pm i}) = \delta_j^i$$

One must check if the above maps are well defined and compatible with the algebra structure (i.e. $U(R)$ is a bialgebra).

4.6.2 Hopf algebras $\hat{A}(R)$ and $\hat{U}(R)$

We define a pairing $\langle , \rangle$ between $A(R)$ and $U(R)$ i.e. a map from $A(R) \times U(R)$ to $\mathbb{C}$, as follows:

$$\langle u, L^+ \rangle = R^+ := R,$$
$$\langle u, L^- \rangle = R^- := \tau \circ R^{-1}.$$

In terms of generators, we get:

$$\langle u_j^i, L_l^{+k} \rangle = (R^+)^{ik}_{jl}.$$
$$\langle u_j^i, L_l^{-k} \rangle = (R^-)^{ik}_{jl}.$$

Note that if $R^+ = R^-$ then $\langle u, L^+ - L^- \rangle = R^+ - R^- = 0$ even if $L^+ \neq L^-$. So, this pairing is not non-degenerate. We want a non-degenerate pairing so that $A(R)$ is dual of

$U(R)$. Let

$$K_1 = \{\phi \in A(R) \,|\, \langle \phi, h \rangle = 0 \ \forall h \in U(R)\}$$
$$K_2 = \{h \in A(R) \,|\, \langle \phi, h \rangle = 0 \ \forall \phi \in A(R)\}$$

Define

$$\hat{A}(R) := A(R)/K_1$$
$$\hat{U}(R) := U(R)/K_2$$

Then $\langle , \rangle$ defines a non-degenerate pairing between $\hat{A}(R)$ and $\hat{U}(R)$. Since both bialgebras are finite dimensional we have $\hat{A}(R)^* = \hat{U}(R)$.
Now we define a Hopf algebra structure on $\hat{A}(R)$ and $\hat{U}(R)$ by giving an antipode map as follows:

$$S' : \hat{A}(R) \to \hat{A}(R)$$
$$u^i_j \mapsto f_{ij}$$

where we see the elements of codomain as maps from $\hat{U}(R)$ to $\mathbb{C}$ and f_{ij} is defined by its image on the generators, given by

$$f_{ij}(L^{+k}_l) = (R^{-1})^{ik}_{jl},$$
$$f_{ij}(L^{-k}_l) = \tau(R)^{ik}_{jl}.$$

Similarly, we define

$$S : \hat{U}(R) \to \hat{U}(R),$$
$$L^{+i}_j \mapsto g^+_{ij},$$
$$L^{-i}_j \mapsto g^-_{ij}.$$

where

$$g^+_{ij}(u^k_l) = (R^{-1})^{ki}_{lj},$$
$$g^-_{ij}(u^k_l) = \tau(R)^{ki}_{lj}.$$

We get antipode maps from S and S' by extending them as an antialgebra map.

4.6.3 $\hat{U}(R)$ is quasitriangular

Recall the Proposition 2.3.1, to define a quasitriangular structure on a Hopf algebra H, it is enough to give a map from H^* to H, which is both an antialgebra map and a coalgebra map. We define a quasitriangular structure on $\hat{U}(R)$ by giving such a map.
Define,

$$\begin{aligned} R' : \hat{A}(R) &\to \hat{U}(R) \\ u^i_j &\mapsto L^{+i}_j \end{aligned}$$

and extend as an antialgebra and coalgebra map.

Now, we define a representation of $\hat{U}(R)$ on a vector space V of dimension n, called the fundamental representation.

Definition 4.6.3. *The fundamental representation of $\hat{U}(R)$ on a vector space V of dimension n is given by the following map:*

$$\begin{aligned} \rho : \hat{U}(R) &\to End(V) \\ L^{+i}_j &\mapsto [R^{ki}_{lj}]^n_{k,l=1} \\ L^{-i}_j &\mapsto [\tau(R^{-1})^{ki}_{lj}]^n_{k,l=1} \end{aligned}$$

and extending it as an algebra map.

Viewing R' as an element of $\hat{U}(R)\hat{\otimes}\hat{U}(R)$, we can show that its image under the fundamental representation is R. Thus, given a 1-dimensional TQFT F, we constructed a quasitriangular hopf algebra such that its fundamental representation defines F.

In this chapter, we discussed an important connection of 1-dimensional TQFT with quantum invariants (i.e. the invariants coming from a representation of a quantum group). The equivalence of 2-dimensional TQFTs with Frobenius Algebra is stated [Abr96]. Algebraic structures on three and 4-dimensional TQFTs are discussed in [CY94]. Thus, besides providing a general machinery for defining link invariants, the algebraic structures on TQFTs might offer a rigorous way to compute the defined invariants, making TQFTs an important tool.

Conclusion and Future Directions

This thesis aimed to highlight the importance of quantum groups and TQFTs in generating link invariants. The following is a summary of the main ideas and results presented in the thesis:

1. We discussed how link invariants can be constructed using quantum groups.
2. We wrote a SageMath and Mathematica code for computing the N-coloured Jones polynomial for the weaving links family. These codes encode the method of obtaining link invariants from quantum groups.
3. We focused on computing the Jones polynomial for weaving links $W(3, m)$ and $W(4, m)$ and made some observations. In particular, we obtained a closed-form expression of the Jones polynomial for $W(3, m)$.
4. We explored the theory of TQFTs and their correspondence with representations of quantum groups.

In the future, the following areas of study can be pursued:

1. We can study the distribution of zeros of the Jones polynomial for weaving links. Since the Jones polynomial of the weaving links $W(n, m)$ is symmetric for n odd, we expect to observe exciting patterns in their Jones polynomial.
2. Along with the Jones polynomial, we can compute other quantum invariants such as the coloured Jones or the HOMFLY-PT polynomial. These computations can be efficiently coded into a computer program.

3. It is known that for alternating links, the Jones polynomial and signature determine the ranks of Khovanov Homology. Thus, we can also attempt to compute higher dimensional TQFTs, such as the Khovonov Homology and Heegaard Floer Homology.

4. The existence of algebraic structures on TQFTs up to 4 dimensions has been proven, and we can try to incorporate these structures to compute link invariants.

5. We can study how quantum invariants relate to geometric invariants. For instance, the volume conjecture relates the hyperbolic volume with the coloured Jones polynomial. Being able to compute the Colored Jones polynomial will enable us to check the Volume conjecture.

Quantum invariants span a vast subject with diverse algebras and superalgebras, yielding intriguing invariants. Hence, quantum invariants are a promising field of study.

Appendix A

Jones polynomial for some weaving links

Here we include the computation of the Jones polynomial for weaving links $W(3,m)$, $W(4,m)$ and $W(5,m)$ for $1 \leq m \leq 10$. These computations were done using the Mathematica code which is included in Appendix B.

$W(3,1)$	1
$W(3,2)$	$t^2 + \frac{1}{t^2} - t - \frac{1}{t} + 1$
$W(3,3)$	$-t^3 - \frac{1}{t^3} + 3t^2 + \frac{3}{t^2} - 2t - \frac{2}{t} + 4$
$W(3,4)$	$t^4 + \frac{1}{t^4} - 4t^3 - \frac{4}{t^3} + 6t^2 + \frac{6}{t^2} - 7t - \frac{7}{t} + 9$
$W(3,5)$	$-t^5 - \frac{1}{t^5} + 5t^4 + \frac{5}{t^4} - 10t^3 - \frac{10}{t^3} + 15t^2 + \frac{15}{t^2} - 19t - \frac{19}{t} + 21$
$W(3,6)$	$t^6 + \frac{1}{t^6} - 6t^5 - \frac{6}{t^5} + 15t^4 + \frac{15}{t^4} - 26t^3 - \frac{26}{t^3} + 39t^2 + \frac{39}{t^2} - 47t - \frac{47}{t} + 52$
$W(3,7)$	$-t^7 - \frac{1}{t^7} + 7t^6 + \frac{7}{t^6} - 21t^5 - \frac{21}{t^5} + 42t^4 + \frac{42}{t^4} - 70t^3 - \frac{70}{t^3} + 98t^2 + \frac{98}{t^2} - 118t - \frac{118}{t} + 127$
$W(3,8)$	$t^8 + \frac{1}{t^8} - 8t^7 - \frac{8}{t^7} + 28t^6 + \frac{28}{t^6} - 64t^5 - \frac{64}{t^5} + 118t^4 + \frac{118}{t^4} - 184t^3 - \frac{184}{t^3} + 248t^2 + \frac{248}{t^2} - 295t - \frac{295}{t} + 313$
$W(3,9)$	$-t^9 - \frac{1}{t^9} + 9t^8 + \frac{9}{t^8} - 36t^7 - \frac{36}{t^7} + 93t^6 + \frac{93}{t^6} - 189t^5 - \frac{189}{t^5} + 324t^4 + \frac{324}{t^4} - 480t^3 - \frac{480}{t^3} + 630t^2 + \frac{630}{t^2} - 737t - \frac{737}{t} + 778$
$W(3,10)$	$t^{10} + \frac{1}{t^{10}} - 10t^9 - \frac{10}{t^9} + 45t^8 + \frac{45}{t^8} - 130t^7 - \frac{130}{t^7} + 290t^6 + \frac{290}{t^6} - 542t^5 - \frac{542}{t^5} + 875t^4 + \frac{875}{t^4} - 1250t^3 - \frac{1250}{t^3} + 1600t^2 + \frac{1600}{t^2} - 1849t - \frac{1849}{t} + 1941$

Table A.1: Jones polynomial of the weaving knots $W(3,m)$ for $m \leq 10$

$W(4,1)$	1
$W(4,2)$	$-t^{9/2} + t^{7/2} - 3t^{5/2} + 2t^{3/2} - \frac{1}{t^{3/2}} - 2\sqrt{t} + \frac{2}{\sqrt{t}}$
$W(4,3)$	$t^7 - 4t^6 + 8t^5 - 11t^4 + 13t^3 - 13t^2 - \frac{1}{t^2} + 11t + \frac{5}{t} - 8$
$W(4,4)$	$-t^{19/2} + 5t^{17/2} - 15t^{15/2} + 29t^{13/2} - 46t^{11/2} + 56t^{9/2} - 63t^{7/2} + 58t^{5/2} - 50t^{3/2} + \frac{7}{t^{3/2}} - \frac{1}{t^{5/2}} + 33\sqrt{t} - \frac{20}{\sqrt{t}}$
$W(4,5)$	$t^{12} - 6t^{11} + 21t^{10} - 51t^9 + 99t^8 - 162t^7 + 220t^6 - 260t^5 + 271t^4 - 248t^3 - \frac{1}{t^3} + 202t^2 + \frac{9}{t^2} - 139t - \frac{35}{t} + 80$
$W(4,6)$	$-t^{29/2} + 7t^{27/2} - 28t^{25/2} + 78t^{23/2} - 174t^{21/2} + 328t^{19/2} - 543t^{17/2} + 781t^{15/2} - 990t^{13/2} + 1116t^{11/2} - 1122t^{9/2} + 1009t^{7/2} - 807t^{5/2} + 562t^{3/2} - \frac{54}{t^{3/2}} + \frac{11}{t^{5/2}} - \frac{1}{t^{7/2}} - 331\sqrt{t} + \frac{157}{\sqrt{t}}$
$W(4,7)$	$t^{17} - 8t^{16} + 36t^{15} - 113t^{14} + 281t^{13} - 589t^{12} + 1084t^{11} - 1787t^{10} + 2646t^9 - 3528t^8 + 4242t^7 - 4606t^6 + 4523t^5 - 4009t^4 - \frac{1}{t^4} + 3187t^3 + \frac{13}{t^3} - 2239t^2 - \frac{77}{t^2} + 1358t + \frac{273}{t} - 686$
$W(4,8)$	$-t^{39/2} + 9t^{37/2} - 45t^{35/2} + 157t^{33/2} - 431t^{31/2} + 991t^{29/2} - 1991t^{27/2} + 3581t^{25/2} - 5862t^{23/2} + 8788t^{21/2} - 12088t^{19/2} + 15240t^{17/2} - 17590t^{15/2} + 18566t^{13/2} - 17899t^{11/2} + 15706t^{9/2} - 12466t^{7/2} + 8841t^{5/2} - 5506t^{3/2} + \frac{436}{t^{3/2}} - \frac{104}{t^{5/2}} + \frac{15}{t^{7/2}} - \frac{1}{t^{9/2}} + 2930\sqrt{t} - \frac{1284}{\sqrt{t}}$
$W(4,9)$	$t^{22} - 10t^{21} + 55t^{20} - 211t^{19} + 634t^{18} - 1588t^{17} + 3457t^{16} - 6715t^{15} + 11852t^{14} - 19250t^{13} + 28986t^{12} - 40590t^{11} + 52836t^{10} - 63810t^9 + 71322t^8 - 73602t^7 + 69937t^6 - 60940t^5 - \frac{1}{t^5} + 48389t^4 + \frac{17}{t^4} - 34667t^3 - \frac{135}{t^3} + 22092t^2 + \frac{654}{t^2} - 12273t - \frac{2220}{t} + 5781$
$W(4,10)$	$-t^{49/2} + 11t^{47/2} - 66t^{45/2} + 276t^{43/2} - 901t^{41/2} + 2443t^{39/2} - 5733t^{37/2} + 11963t^{35/2} - 22623t^{33/2} + 39301t^{31/2} - 63366t^{29/2} + 95494t^{27/2} - 135064t^{25/2} + 179474t^{23/2} - 223790t^{21/2} + 261196t^{19/2} - 284492t^{17/2} + 288262t^{15/2} - 270773t^{13/2} + 234758t^{11/2} - 186726t^{9/2} + 135101t^{7/2} - 87882t^{5/2} + 50592t^{3/2} - \frac{3605}{t^{3/2}} + \frac{935}{t^{5/2}} - \frac{170}{t^{7/2}} + \frac{19}{t^{9/2}} - \frac{1}{t^{11/2}} - 25238\sqrt{t} + \frac{10604}{\sqrt{t}}$

Table A.2: Jones polynomial of the weaving knots $W(4,m)$ for $m \leq 10$

$W(5,1)$	1
$W(5,2)$	$t^4+\frac{1}{t^4}-2t^3-\frac{2}{t^3}+4t^2+\frac{4}{t^2}-5t-\frac{5}{t}+5$
$W(5,3)$	$t^6+\frac{1}{t^6}-6t^5-\frac{6}{t^5}+16t^4+\frac{16}{t^4}-30t^3-\frac{30}{t^3}+44t^2+\frac{44}{t^2}-54t-\frac{54}{t}+59$
$W(5,4)$	$t^8+\frac{1}{t^8}-8t^7-\frac{8}{t^7}+32t^6+\frac{32}{t^6}-86t^5-\frac{86}{t^5}+177t^4+\frac{177}{t^4}-292t^3-\frac{292}{t^3}+407t^2+\frac{407}{t^2}-491t-\frac{491}{t}+521$
$W(5,5)$	$t^{10}+\frac{1}{t^{10}}-10t^9-\frac{10}{t^9}+50t^8+\frac{50}{t^8}-170t^7-\frac{170}{t^7}+443t^6+\frac{443}{t^6}-943t^5-\frac{943}{t^5}+1683t^4+\frac{1683}{t^4}-2570t^3-\frac{2570}{t^3}+3431t^2+\frac{3431}{t^2}-4047t-\frac{4047}{t}+4280$
$W(5,6)$	$t^{12}+\frac{1}{t^{12}}-12t^{11}-\frac{12}{t^{11}}+72t^{10}+\frac{72}{t^{10}}-292t^9-\frac{292}{t^9}+906t^8+\frac{906}{t^8}-2296t^7-\frac{2296}{t^7}+4935t^6+\frac{4935}{t^6}-9175t^5-\frac{9175}{t^5}+14934t^4+\frac{14934}{t^4}-21518t^3-\frac{21518}{t^3}+27709t^2+\frac{27709}{t^2}-32138t-\frac{32138}{t}+33749$
$W(5,7)$	$t^{14}+\frac{1}{t^{14}}-14t^{13}-\frac{14}{t^{13}}+98t^{12}+\frac{98}{t^{12}}-462t^{11}-\frac{462}{t^{11}}+1659t^{10}+\frac{1659}{t^{10}}-4851t^9-\frac{4851}{t^9}+12024t^8+\frac{12024}{t^8}-25910t^7-\frac{25910}{t^7}+49299t^6+\frac{49299}{t^6}-83636t^5-\frac{83636}{t^5}+127435t^4+\frac{127435}{t^4}-175469t^3-\frac{175469}{t^3}+219521t^2+\frac{219521}{t^2}-250618t-\frac{250618}{t}+261847$
$W(5,8)$	$t^{16}+\frac{1}{t^{16}}-16t^{15}-\frac{16}{t^{15}}+128t^{14}+\frac{128}{t^{14}}-688t^{13}-\frac{688}{t^{13}}+2808t^{12}+\frac{2808}{t^{12}}-9304t^{11}-\frac{9304}{t^{11}}+26080t^{10}+\frac{26080}{t^{10}}-63534t^9-\frac{63534}{t^9}+136981t^8+\frac{136981}{t^8}-264626t^7-\frac{264626}{t^7}+461928t^6+\frac{461928}{t^6}-732946t^5-\frac{732946}{t^5}+1061982t^4+\frac{1061982}{t^4}-1410522t^3-\frac{1410522}{t^3}+1722973t^2+\frac{1722973}{t^2}-1940581t-\frac{1940581}{t}+2018673$
$W(5,9)$	$t^{18}+\frac{1}{t^{18}}-18t^{17}-\frac{18}{t^{17}}+162t^{16}+\frac{162}{t^{16}}-978t^{15}-\frac{978}{t^{15}}+4473t^{14}+\frac{4473}{t^{14}}-16569t^{13}-\frac{16569}{t^{13}}+51822t^{12}+\frac{51822}{t^{12}}-140706t^{11}-\frac{140706}{t^{11}}+338110t^{10}+\frac{338110}{t^{10}}-728967t^9-\frac{728967}{t^9}+1424185t^8+\frac{1424185}{t^8}-2539557t^7-\frac{2539557}{t^7}+4154934t^6+\frac{4154934}{t^6}-6261732t^5-\frac{6261732}{t^5}+8719593t^4+\frac{8719593}{t^4}-11248065t^3-\frac{11248065}{t^3}+13469497t^2+\frac{13469497}{t^2}-14997438t-\frac{14997438}{t}+15542507$
$W(5,10)$	$t^{20}+\frac{1}{t^{20}}-20t^{19}-\frac{20}{t^{19}}+200t^{18}+\frac{200}{t^{18}}-1340t^{17}-\frac{1340}{t^{17}}+6790t^{16}+\frac{6790}{t^{16}}-27814t^{15}-\frac{27814}{t^{15}}+96040t^{14}+\frac{96040}{t^{14}}-287540t^{13}-\frac{287540}{t^{13}}+761465t^{12}+\frac{761465}{t^{12}}-1809618t^{11}-\frac{1809618}{t^{11}}+3900924t^{10}+\frac{3900924}{t^{10}}-7689503t^9-\frac{7689503}{t^9}+13945635t^8+\frac{13945635}{t^8}-23378065t^7-\frac{23378065}{t^7}+36354014t^6+\frac{36354014}{t^6}-52585709t^5-\frac{52585709}{t^5}+70910204t^4+\frac{70910204}{t^4}-89299515t^3-\frac{89299515}{t^3}+105174071t^2+\frac{105174071}{t^2}-115971368t-\frac{115971368}{t}+119802312$

Table A.3: Jones polynomial of the weaving knots $W(5,m)$ for $m \leq 10$

Appendix B

SageMath and Mathematica Code

- We provide a SageMath program to compute the matrix for generators of the braid group on three strands under the fundamental representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ (the representation involved in computing the Jones polynomial).

```
#Prerequisites
q = SR.var('q')
h = SR.var('h')
E_1=matrix(SR,[[0,0],[1,0]])
F_1=matrix(SR,[[0,1],[0,0]])
H_1=matrix(SR,[[-1,0],[0,1]])
K_1 = Matrix([[q^-1, 0],[0,q]])
R=matrix(SR,[[q,0,0,0],[0,q-q^-1,1,0],[0,1,0,0],[0,0,0,q]])

#Recursions for E, F and H
cacheH = {} # Dict to store cached results
cacheE = {}
cacheF = {}

def H(n):
  if n in cacheH:
    return cacheH[n]
    elif n==1:
        result = H_1
    else:
        result= H(n-1).tensor_product(identity_matrix(2)) +
```

```
identity_matrix(2**(n-1)).tensor_product(H_1)
    cacheH[n] = result.subs(h==log(q)).simplify_full()
    return result.subs(h==log(q)).simplify_full()

def E(n):
    if n in cacheE:
        return cacheE[n]
    elif n==1:
        result = E_1
    else:
        result = E(n-1).tensor_product(exp(h*H_1)) + identity_matrix(2**(n
-1)).tensor_product(E_1)
    cacheE[n] = result.subs(h==log(q)).simplify_full()
    return result.subs(h==log(q)).simplify_full()

def F(n):
    if n in cacheF:
        return cacheF[n]
    elif n==1:
        result = F_1
    else:
        result = F(n-1).tensor_product(identity_matrix(2)) + exp(-1*h*H(n
-1)).tensor_product(F_1)
    cacheF[n] = result.subs(h==log(q)).simplify_full()
    return result.subs(h==log(q)).simplify_full()

def K(n):
    if n==1:
        return K_1
    else:
        return K(n-1).tensor_product(K_1)

#Eigenvectors of E
def P(n):
    return E(n).eigenvectors_right()

def w(j,n):
    return P(n)[0][1][j]

def Span(i,n): # Span of highest weight vectors
    Q=[]
```

```
    f=F(n)
    for j in range(n+1):
        if f**j*w(i,n) !=zero_vector(2**n):
            Q.append(F(n)^j*w(i,n))
        else:
            break
    return Q

def Q(n): # Base Change Matrix
    Q =[]
    for i in range(len(P(n)[0][1])):
        Q +=Span(i,n)
    return ((matrix(Q).subs(h==log(q))).simplify_full()).transpose()

def Qi(n):
    return Q(n).inverse()

def A(j,n):
    return identity_matrix(power(2,j-1)).tensor_product(R.tensor_product(
        identity_matrix(power(2,n-j-1))))

def Sig(j,n):
    B = A(j,n)
    return (Qi(n)*B*Q(n)).simplify_full()

def pos(j,n):
    if j==0:
        return 0
    else:
        r =0
        for k in range(j):
            r+= len(Span(k,n))
        return r

def d(j,n): #Columns and Rows to be deleted
    if j==0:
        r = [x for x in range(len(Span(j,n)))]
        r.remove(0)
        return r
    else:
        r = [x for x in range(len(Span(j,n)))]
```

```
        r.remove(0)
        return [x + pos(j,n) for x in r]

def Sigma(i,n):#Decomposed matrices for generators
    dcol = []
    for j in range(len(P(n)[0][1])):
        dcol += d(j,n)
    C = Sig(i,n).delete_columns(dcol,check=True)
    R = C.delete_rows(dcol,check=True)
    return R

print(Sigma(1,3))

print(Sigma(2,3))
```

The output of the above code is the following matrices:

```
[(q^2 - 1)/q           1           0]
[          1           0           0]
[          0           0           q]
[   q    0    0]
[-q^2 -1/q    0]
[   0    0    q]
```

- We also provide a Mathematica code to compute N-colored Jones polynomial for weaving knots (and links) $W(n, m)$.

```
qnum[m_] := Sum[q^(-m + 1 + i), {i, 0, 2*m - 1, 2}] (*q-number*)

qfact[m_] :=
 Module[{prod}, prod = 1; Do[prod = prod*qnum[i], {i, 1, m}];
  Expand[prod]](*q-facrtorial*)

(*Matrix of H under representation of dimension N*)
H1[N_] := Module[{l, matrix}, l = (N - 1)/2;
  matrix =
   Table[If[j == i, Times[2, (-l + i - 1)], 0], {i, 1, N}, {j, 1,
     N}];
  matrix]

(*Matrix of E under representation of dimension N*)
E1[N_] := Module[{l, matrix},
  l = (N - 1)/2;
  matrix =
   Table[If[j == i - 1, Power[Times[qnum[2*l - j + 1], qnum[j]], 0.5],
      0], {i, 1, N}, {j, 1, N}];
  matrix]

(*Matrix of F under representation of dimension N*)
F1[N_] := Module[{l, matrix},
  l = (N - 1)/2;
  matrix =
   Table[If[j == i + 1,
     Power[Times[qnum[2*l - j + 2], qnum[j - 1]], 0.5], 0], {i, 1,
     N}, {j, 1, N}];
  matrix]

coord[i_, N_] := List[Floor[(i - 1)/N], Mod[(i - 1), N]]

(*The Tensor Flip Matrix*)
Tensorflip[N_] := Module[{flip, ls}, flip = IdentityMatrix[N^2];
  ls = Range[N^2];
  Do[If[i != j && coord[i, N][[2]] == coord[j, N][[1]] &&
     coord[i, N][[1]] == coord[j, N][[2]],
    flip[[All, {i, j}]] = flip[[All, {j, i}]];
```

```
    ls = DeleteCases[ls, j]; ls = DeleteCases[ls, i];], {i, ls}, {j,
    ls}];
  flip]

(*Matrix of R under representation of dimension N*)
R[N_] := Module[{prod, sum, Rwithoutflip},
  prod = MatrixExp[(h/2)*KroneckerProduct[H1[N], H1[N]]];
  sum = IdentityMatrix[N^2];
  Do[sum +=
    Divide[Times[q^((i^2 + i)/2), Power[1 - q^(-2), i]], qfact[i]]*
     KroneckerProduct[MatrixPower[E1[N], i],
      MatrixPower[F1[N], i]], {i, 1, N - 1}];
  Rwithoutflip = Simplify[prod . sum] /. h -> Log[q];
  Tensorflip[N] . Rwithoutflip]

(*Matrix of K under representation of dimension N*)
K1[N_] := Module[{l, matrix},
  l = (N - 1)/2;
  matrix =
   Table[If [j == i, Power[q, Times[2, -l + i - 1]], 0], {i, 1,
     N}, {j, 1, N}];
  matrix]

(*Matrix of K under representation of dimension N tensored n times. n \
here denote the number of strands*)
K[N_, n_] := If[n == 1, K1[N], KroneckerProduct[K[N, n - 1], K1[N]]]

(*Matrix of generator sigma i as an element of B_n, under N-colored \
representation*)
A[N_, n_, i_] :=
 KroneckerProduct[IdentityMatrix[N^(i - 1)],
  KroneckerProduct[R[N], IdentityMatrix[N^(n - i - 1)]]]

(*Matrix of weaving knot W(n,m) under N-colored representation*)
W[N_, n_, m_] :=
 Module[{B}, B = IdentityMatrix[N^n];
  Do[If[Mod[i, 2] == 1, B = B . A[N, n, i],
    B = B . Inverse[A[N, n, i]]], {i, 1, n - 1}];
  Simplify[MatrixPower[B, m]]]

Writhe[n_, m_] := If[Mod[n, 2] == 1, 0, m]
```

```
(*Normalised quantum trace*)
Qtrace[N_, n_, m_] :=
 Rationalize[
  Cancel[Rationalize[
    Simplify[
     Rationalize[
      Cancel[Tr[W[N, n, m] . K[N, n]]/(q^(1.5*Writhe[n, m]) qnum[N])],
       0]], 0]]]

(*N-colored Jones polynomial of W(n,m)*)
J[N_, n_, m_] :=
 Rationalize[
  Simplify[Qtrace[N, n, m]*(-1)^(1 + GCD[n, m]) /. q -> t^-0.5], 0]

J[2, 3, 3]
```

The output of the above code is the Jones polynomial of $W(3,3)$, which is presented below.

```
4 - 1/t^3 + 3/t^2 - 2/t - 2 t + 3 t^2 - t^3
```

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