Unitarity via Quantum Off-Shell Recursions

A thesis submitted to the ***Physics Department*** *of*
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by

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Certificate

This is to certify that this thesis entitled ***"Unitarity via Quantum Off-Shell Recursions"*** towards partial fulfillment of the BS-MS dual degree program at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vatsal Garg at Asia Pacific Center for Theoretical Physics (APCTP), Pohang under the supervision of Prof. Kanghoon Lee, JRG Leader, APCTP, Pohang during the academic year 2023–2024.

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Declaration

I hereby declare that the matter embodied in the thesis entitled ***"Unitarity via Quantum Off-Shell Recursions"*** is the result of work carried out by me at Asia Pacific Center for Theoretical Physics, Pohang under the supervision of Prof. Kanghoon Lee and the same has not been submitted elsewhere for any other degree. Wherever others have contributed, every effort has been made to indicate this clearly with due reference to the literature and acknowledgement of collaborative research and discussions.

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(20191044)

This thesis is based on upcoming research work carried out by the author, his supervisor and collaborator.

This thesis is dedicated to all those who've ever called me by my name.

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Abstract

The challenge of accurately calculating differential cross-sections in QCD and the Standard Model has many applications, especially in analysing data from particle accelerators such as the LHC. The efficiency of traditional methods of cross-section calculation can be improved by bypassing some tedious intermediate steps and in order to do so we propose a new application of the quantum off-shell recursion framework in conjunction with the Largest Time Equation. We intend to use unitarity of the S-matrix to obtain higher point tree order cross-sections from lower point loop order amplitudes. We first review the quantum off-shell recursion framework with a pedagogical calculation in ϕ^4 theory. We then introduce a field-doubling prescription for ϕ^4 theory which when used in conjunction with the recursion framework allows us to exhibit unitarity explicitly. We finally obtain a proof-of-concept by deriving the tree order 6-point cross-section in the regular ϕ^4 theory from the 3-loop order 4-point amplitude in the field-doubled ϕ^4 theory.

Contents

— CHAPTER 1 —

Introduction

Scattering amplitudes lay at the heart of quantum field theories. Obtaining these amplitudes is necessary to study phenomena associated with particles. And to that end, Feynman diagrams have been employed in calculations for many decades. Julian Schwinger once said, "Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses."[18,19] As an idea the Feynman diagram is remarkably simple and easy to understand, and as a framework it is even more powerful.

But for problems involving gauge theories, the efficiency of the diagrammatic method comes under scrutiny. In calculations at higher precision (increasing loop order) or for a larger number of particles (increasing n), the count of Feynman diagrams relevant to the problem scales faster than a factorial function. There are far too many diagrams while their numerical evaluation is slow and computationally expensive.

n	2	3	4	5	6	7	8
No. of diagrams	4	25	220	2485	34300	559405	10525900

Table 1.1: *The number of tree Feynman diagrams contributing to the 'g g → ng' process for a pure non-Abelian gauge theory*[40].

Moreover, the building blocks of Feynman diagrams – *the Feynman rules* and *vertices* – are gauge-dependent for theories that have a gauge-redundancy in their actions and hence information gleaned from diagrams can vary based on the choice of gauge and field-redefinition.

There are modern techniques that aim to make calculations more efficient while exposing the mathematical structure of scattering amplitudes which wasn't as evident with Feynman diagrams. Using recursion relations is one such approach, of which there are two main classes: *on-shell* and *off-shell*. The famous BCFW (Britto–Cachazo–Feng–Witten) recursion relations[20,21] are an example of the former type, and in this setup one obtains tree amplitudes at higher points (more external particles) from tree amplitudes at lower points (as low as 3 external particles) by exploiting the analytic properties of on-shell amplitudes with a clever deformation[2]. An obvious advantage of using such a method when compared to Feynman diagrams is the fact that

one is working with gauge-invariant inputs (on-shell tree amplitudes). However, we are more interested in the BG (Berends–Giele) recursion relations[22,23] which depend on the recursive nature of interaction vertices of a theory. These recursion relations are *off-shell*, *i.e.* one or more external legs are taken off-shell which allows us to use objects called *currents*. These currents are then used to obtain amplitudes algebraically. Our interest in this type of recursion stems from the fact that on-shell recursion methods are not easily applicable to more general classes of theories and the efficiency of off-shell recursions tends to be higher as the number of external legs increases[28].

While the BG recursion provides an organised way of repackaging the Feynman diagrammatic expansion, there is a better way of constructing the off-shell recursions. The *perturbiner* method detailed by Rosly & Selivanov[24–26] generates the same recursions using classical equations of motion. The conventional perturbiner method which was limited to tree amplitudes has been adapted to incorporate Dyson–Schwinger equations allowing one to extend the recursion relations to loop orders. This novel framework – *quantum off-shell recursions*[1] – has a lot of advantages: it is efficient, it requires knowledge of just the action of the theory, it is systematic and hence can be very easily programmed on computers and most importantly, it is *highly* extensible. It has already been applied beyond the contexts of scattering amplitudes for regular quantum field theories, most recently in the realm of graviton scattering[5]. There is scope for applications in other areas too: Schwinger–Keldysh formalism, quantum field theory in curved background, and celestial holography to name a few.

§ 1.1 Goals and outline

We focus on applying the quantum off-shell recursion framework to the study of unitarity of the S-matrix. We shall use the Largest Time Equation[6] (a generalization of the famous optical theorem) to obtain higher point tree cross-section from lower point loop amplitudes. This runs against the conventional usage of the optical theorem, wherein one obtains loop amplitudes by stitching together tree amplitudes[27]. The quantum off-shell recursion framework is a perfect candidate for this investigation as the computation of loop amplitudes is no longer a drastic bottleneck. Establishing this result would mark significant progress in amplitude techniques as one could obtain QCD cross-sections skipping the laborious step of color decomposition.

The goals of this thesis are: to re-iterate the convenience and power of the quantum off-shell recursion framework; and to develop a computationally efficient method for calculating cross-sections in gauge theories using the Largest Time Equation.

The thesis is structured as follows:
In chapter 2 we discuss the background of this study, with a brief introduction to scattering amplitudes. We then take a look into unitarity methods and compare the approaches of generalized

unitarity and the Largest Time Equation. And finally we review color management in gauge theories.
In chapter 3 we will use the quantum off-shell recursion framework to obtain the 3-loop order 4-point amplitude in ϕ^4 theory.
In chapter 4 we introduce the field-doubling prescription for ϕ^4 which allows us to use the Largest Time Equation algebraically. We shall obtain the tree order 6-point cross-section in the regular theory from the 3-loop order 4-point amplitude in the field-doubled theory to demonstrate the utility of our method.
In chapter 5 we collect our results and summarise some insights gained from the calculations. We conclude by listing some of the future directions that can be taken from here.

We supply a few appendices at the end of the thesis.
In appendix A we begin the calculation for the pure Yang–Mills theory in the first-order formalism by deriving the Dyson–Schwinger equations for all fields involved.
In appendix B we list all of the non-standard notation used in this thesis for easy reference.

— CHAPTER 2 —

Background

In this chapter we shall acquaint ourselves with some important concepts related to the study of scattering amplitudes. Let us first try to understand what scattering amplitudes are.

§ 2.1 Scattering amplitudes

Scattering amplitudes are elementary building blocks of particle physics. They serve as a crucial link connecting theory and experiment. And to a large extent, the need to compute amplitudes was the driving force behind the historical development of quantum field theory. The study of amplitudes dates back to 1960s around the time when the S-matrix program was very much in vogue. There was a resurgence in 1980s around the time when Parke and Taylor published the monumental formula[29] responsible for massive simplification in calculations of QCD amplitudes. This kickstarted the modern field as we now know it, with countless developments made to push forward the state-of-the-art result by several loop orders.

Let us understand why amplitudes are so important. The key observable in collision experiments is the cross-section σ where classically,

$$\sigma = \frac{\text{no. of scattering events}}{\text{time} \times \text{projectile beam flux} \times \text{no. of target particles}}. \tag{2.1}$$

It is a measure of the probability of a given scattering process to take place in terms of energy/momenta of particles. However, a more useful quantity for physicists is the *differential cross-section* $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ which describes the angular dependence of the scattering event. This is actually the form of the differential cross-section for 2-to-2 scattering. For general n-particle scattering we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_n},$$

where $\vec{p}_1, \ldots, \vec{p}_n$ are the final 3-momenta of the particles involved. For now let us stick to the 2-to-2 scattering regime.

Analogous to the relationship between the probability density $|\psi|^2$ and the complex-valued

wavefunction (or the *probability amplitude*) ψ in quantum mechanics, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto |\mathcal{M}|^2,$$

where $\mathcal{M}$ is the so-called scattering amplitude. The time evolution operator that evolves momenta eigenstates from $t = -\infty$ to $t = \infty$, relevant to scattering is called the *scattering* or S-matrix. In some references, the scattering amplitude is defined as an element of the S-matrix

$$\mathcal{A} = \langle f|\mathcal{S}|i\rangle, \tag{2.2}$$

where $|i\rangle$ and $|f\rangle$ are the initial and final scattering states. There is a subtle difference between $\mathcal{A}$ and $\mathcal{M}$: the S-matrix itself splits into

$$\mathcal{S} = \mathbb{1} + i\mathcal{T}, \tag{2.3}$$

where $\mathbb{1}$ is the identity matrix (represents the trivial outcome where particles remain unscattered) and $\mathcal{T}$ is the *transfer matrix* which encodes all the interactions of a field theory and this is the part of the S-matrix that is indeed responsible for scattering events; we define $\mathcal{M}$ as follows,

$$\mathcal{T} = (2\pi)^4\delta^4(\Sigma k)\mathcal{M}, \tag{2.4}$$

where $\delta^4(\Sigma k)$ is enforcing 4-momentum conservation for external momenta explicitly, and the differential cross-section is proportional to the quantity

$$|\mathcal{M}|^2 \equiv |\langle f|\mathcal{M}|i\rangle|^2. \tag{2.5}$$

Therefore the quantity of interest is almost always $\mathcal{M}$ rather than $\mathcal{A}$, and hence hereafter we shall refer to $\mathcal{M}$ as the scattering amplitude unless explicitly mentioned.

The words ‘on-shell’ and ‘off-shell’ have appeared and will continue to do so frequently in this thesis. We shall now try to understand what they mean. Recall the relativistic dispersion relation,

$$E^2 = p^2c^2 + m_0^2c^4. \tag{2.6}$$

If we were to rearrange the expression, we get

$$E^2 - p^2c^2 = m_0^2c^4, \tag{2.7}$$

which now describes a hyperboloid in the energy-momentum space. This hyperboloid is also called the *mass shell*. Re-writing the above in natural units (and using the $-+++$ metric),

$$p^\mu p_\mu = -m_0^2. \tag{2.8}$$

A particle is said to be on-shell if its 4-momentum obeys the above equation *i.e.* its energy and momentum lies on the mass shell. In context of amplitudes, 'on-shell' implies that all incoming and outgoing particles are on-shell and momentum conservation is observed.

§ 2.1.1 *Calculating scattering amplitudes*

Now we shall take a look at how one goes about calculating amplitudes. For a very long time, the steps have largely remained the same.

Step 1. We read off the Feynman rules from the Lagrangian.

Step 2. We draw all Feynman diagrams up to a specific loop order[1].

Step 3. Perform regularization and renormalization as and when needed to treat the divergences

Step 4. Perform the integration and sum up all the terms.

However, due to advancements in the field we now have an updated pipeline[30] to go about our calculations more efficiently than before. For now let us continue to employ Feynman diagrams.

The first step is to use a computer program to generate Feynman diagrams. There exist programs for the simplification of the corresponding symbolic forms. The next step is to filter out the diagrams relevant for calculations. This makes calculations more efficient as not all diagrams are relevant for a given problem. The next step is classification of diagrams, even with a filter the number of Feynman diagrams can be daunting and one needs to simplify terms before integration begins. Depending on the approach: diagrammatic or algebraic, we either focus on identification of *graph topologies* or symbolic calculations to find and isolate arising tensor structures. There could be millions of Feynman diagrams but only a couple of hundred topologies. Categorisation into topologies saves a lot of computational effort and one usually goes further and identifies more fundamental structures called *minimal topology sets* on which symbolic substitutions are applied to obtain integral families. In the algebraic approach, after the identification of tensor structures we try to form integral families.

Integration can't be started just yet, there is more simplification to be done. IBP reduction is one of the most popular methods used to form *master integrals*. Each integral is a linear combination of simpler integrals known as master integrals, and solving just master integrals is enough. There exist many programs using which IBP reduction can be carried out. And finally we perform analytical or numerical integrations and arrive at the final expression.

We shall propose a pipeline based on the quantum off-shell recursion framework, that avoids the

[1] In the process of drawing Feynman diagrams, one can perform the diagrammatic expansion in terms of the number of loops. This expansion corresponds to the $\hbar$ expansion. The exponent of $\hbar$ counts the number of loops in a diagram and hence when we want to set the precision of the calculation, quantitatively we truncate the $\hbar$ expansion at some order.

use of Feynman diagrams entirely and works towards obtaining master integrals, so that at the end all we are left with is integration (which remains outside the scope of this thesis).

Useful references for this section are [2, 7, 31].

Next, we shall study unitarity methods used for calculating scattering amplitudes.

§ 2.2 Unitarity methods

Unitarity as an approach has been around for a very long time, but it didn't find its way into construction of amplitudes until much later[10]. The method of unitarity cuts is a powerful means of obtaining loop order amplitudes by exploiting the unitarity of the S-matrix. The key idea of this method is that one can construct loop amplitudes from on-shell tree amplitudes. Let us discuss the method of unitarity cuts at 1-loop order.

§ 2.2.1 *Unitarity cuts*

The method of unitarity cuts started as a framework for 1-loop order calculations. Let us first see how unitarity of the S-matrix comes into the picture. Enforcing unitarity gives,

$$\mathcal{S}^\dagger \mathcal{S} = \mathbb{1}, \tag{2.9a}$$

$$(\mathbb{1} - i\mathcal{T}^\dagger)(\mathbb{1} + i\mathcal{T}) = \mathbb{1}, \tag{2.9b}$$

$$\not{\mathbb{1}} + i\left(\mathcal{T} - \mathcal{T}^\dagger\right) + \mathcal{T}^\dagger\mathcal{T} = \not{\mathbb{1}}, \tag{2.9c}$$

$$-i\left(\mathcal{T} - \mathcal{T}^\dagger\right) = \mathcal{T}^\dagger\mathcal{T}. \tag{2.9d}$$

We have obtained the statement of the famous optical theorem in terms of the transfer matrix. At the level of matrix elements we can rewrite this as,

$$-i\langle \mathbf{f}|\mathcal{T} - \mathcal{T}^\dagger|\mathbf{i}\rangle = \sum_{\mathbf{n}} \langle \mathbf{f}|\mathcal{T}^\dagger|\mathbf{n}\rangle\langle \mathbf{n}|\mathcal{T}|\mathbf{i}\rangle. \tag{2.10}$$

Expanding the above expression perturbatively in coupling constant gives us at 1-loop order,

$$-i\left(\mathcal{M}^{(1)}_{\mathbf{i}\to\mathbf{f}} - \left(\mathcal{M}^{(1)}_{\mathbf{i}\to\mathbf{f}}\right)^*\right) = \sum\!\!\!\!\!\!\int_{\mathbf{n}} \mathcal{M}^{(0)}_{\mathbf{i}\to\mathbf{n}}\left(\mathcal{M}^{(0)}_{\mathbf{n}\to\mathbf{f}}\right)^*, \tag{2.11a}$$

$$2\,\mathrm{Im}\left(\mathcal{M}^{(1)}_{\mathbf{i}\to\mathbf{f}}\right) = \sum\!\!\!\!\!\!\int_{\mathbf{n}} \mathcal{M}^{(0)}_{\mathbf{i}\to\mathbf{n}}\left(\mathcal{M}^{(0)}_{\mathbf{n}\to\mathbf{f}}\right)^*, \tag{2.11b}$$

The imaginary part of the amplitude is related to the discontinuity across a branch cut singularity when one kinematic invariant (say, P^2) is positive while all others are negative and hence we have,

$$\text{Disc}_{P^2}\left(\mathcal{M}^{(1)}_{\mathbf{i}\to\mathbf{f}}\right) = \oint\!\!\!\!\!\!\sum_{\mathbf{n}} \mathcal{M}^{(0)}_{\mathbf{i}\to\mathbf{n}}\left(\mathcal{M}^{(0)}_{\mathbf{n}\to\mathbf{f}}\right)^{*}. \tag{2.11c}$$

We compute the discontinuity using Cutkosky's rules[11], and the act of taking the loop

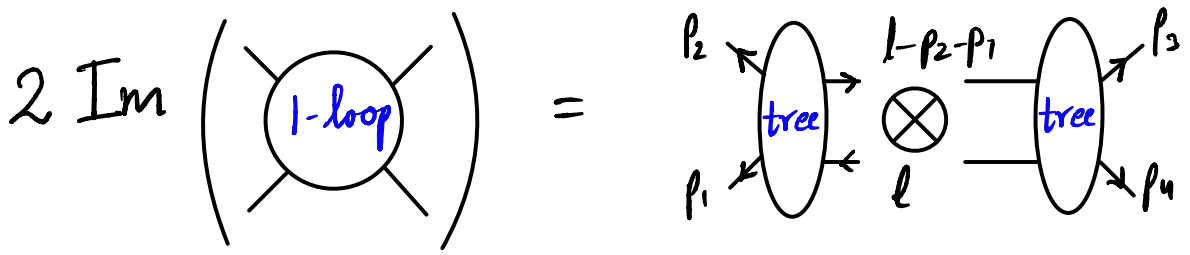

Figure 2.1: *Diagrammatic representation of the unitarity cut.* $(p_1 + p_2)^2$ *is the kinematic invariant.*

propagators on-shell and evaluating the Cutkosky's rules at momenta that *cross* the cut is also called the unitarity cut. The operation involves a summation over states that make it across the cut, and a phase space integral that picks out (positive energy) solutions for the on-shell condition. Here, the discontinuity is evaluated by a single continuous cut that factors the amplitude into two on-shell tree amplitudes. But in general, unitarity cuts can involve more than one continuous cut and take several propagators on-shell across other channels of momenta and factorize the discontinuity of the amplitude into more than two on-shell tree amplitudes. The method of reconstructing the full loop amplitude by systematic application of unitarity cuts at various momenta channels is called *generalized unitarity*[10].

Instead of the explicit loop order Feynman diagrams, one expands the 1-loop order amplitude in the basis of *master integrals* multiplied by rational functions of kinematic variables. The evaluation of these master integrals is easier than computing the loop integrals for each diagram and all that remains to determine the 1-loop amplitude is obtaining the coefficients. Let us define the set of basic 1-loop scalar integrals which make up the basis of master integrals,

$$\textbf{(tadpole)} \quad I_1(K_i) \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D}\frac{1}{l^2}, \tag{2.12a}$$

$$\textbf{(bubble)} \quad I_2\left(K_i, K_j\right) \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D}\frac{1}{l^2\left(l+K_j\right)^2}, \tag{2.12b}$$

$$\textbf{(triangle)} \quad I_3\left(K_i, K_j, K_k\right) \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D}\frac{1}{l^2(l-K_i)^2(l+K_k)^2}, \tag{2.12c}$$

$$\textbf{(box)} \quad I_4\left(K_i, K_j, K_k, K_l\right) \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D}\frac{1}{l^2(l-K_i)^2\left(l-K_i-K_j\right)^2(l+K_l)^2}, \tag{2.12d}$$

$$\vdots$$

where for I_n, n denotes the number of loop propagators and also the number of vertices in the diagram, and $K_i, K_j, \cdots$ are the sums of external momenta at each vertex.

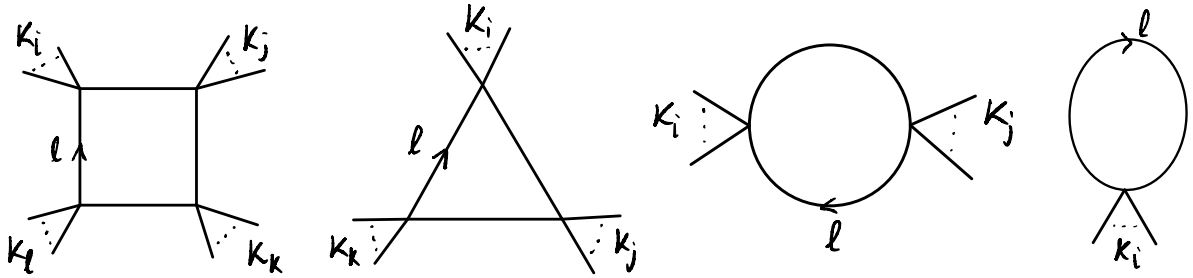

Figure 2.2: *Scalar 1-loop diagrams. K_i, K_j, K_k and K_l are all outgoing momenta.*

By integral reduction we have,

$$\mathcal{M}^{(1)} = \left(\sum_{\mathbf{K}=\{K_1,\ldots,K_n\}} c_1(\mathbf{K}) I_1(\mathbf{K}) + c_2(\mathbf{K}) I_2(\mathbf{K}) + c_3(\mathbf{K}) I_3(\mathbf{K}) + c_4(\mathbf{K}) I_4(\mathbf{K}) \right) + \mathcal{R}. \tag{2.13}$$

In $D = 4$, we only need these 4 types of 1-loop scalar integrals. c_1, c_2, c_3 and c_4 are the sets of rational coefficients that need to be determined, and $\mathcal{R}$ is a rational term that can't be directly computed from cuts in four dimensions. Since the coefficients are rational functions, the branch cuts are located only in the master integrals and we have,

$$\text{Disc}_{P^2} \mathcal{M}^{(1)} = \left(\sum_{i=1}^{4} \sum_{\mathbf{K}=\{K_1,\ldots,K_n\}} c_i(\mathbf{K}) \, \text{Disc} \, I_i(\mathbf{K}) \right). \tag{2.14}$$

We work with eq. (2.14) in the generalized unitarity method. It is an equality involving tree order amplitudes on both sides of the equation. And many terms on the RHS vanish because only a subset of the master integrals have a cut involving the given momentum P [2]. We solve the equation varying over all possible values of P. By applying multiple unitarity cuts, one obtains a set of linear equations that relate the coefficients $c_i(\mathbf{K})$ to the result of the cut on the LHS, each computed as a product of tree amplitudes. This effectively determines the 1-loop amplitude up to a rational term $\mathcal{R}$ which is determined independently and the method of doing so remains outside the scope of this discussion.

We shall now look at another attempt at perturbative unitarity in the form of the Largest Time Equation.

[2]The positivity of the kinematic invariant P^2 isolates the momentum channel P of interest where P is a sum of some of the external momenta.

§ 2.2.2 *Largest Time Equation*

The statement of optical theorem eq. (2.9d) holds true for the entire S-matrix *i.e.* in other words it is a non-perturbative statement. But when dealing with perturbation theory in QFT, it is not obvious why this statement would continue to hold true for truncated $\mathcal{T}$. Above we saw the usage of perturbative unitarity in the form of unitarity cuts at 1-loop order, but the Largest Time Equation is Veltman's attempt at proving perturbative unitarity at the level of individual Feynman diagrams.

A full proof of perturbative unitarity via the Largest Time Equation is quite involved. For the sake of illustrating the importance of this method, it is sufficient to include a sketch of the proof using scalar field theory. The arguments can be extended to fermions, gauge fields and spontaneously broken gauge theories. But they remain beyond the scope of this simple review.

Once again at the level of matrix elements we can rewrite the optical theorem as follows,

$$\langle \mathbf{f}|(i\mathcal{T})|\mathbf{i}\rangle + \langle \mathbf{f}|(i\mathcal{T})^\dagger|\mathbf{i}\rangle = -\sum_{\mathbf{n}} \langle \mathbf{f}|(i\mathcal{T})^\dagger|\mathbf{n}\rangle\langle \mathbf{n}|(i\mathcal{T})|\mathbf{i}\rangle. \tag{2.15}$$

But this time we focus on the imaginary part of the transfer matrix. The Largest Time Equation is the diagrammatic analogue of eq. (2.15). But usually Feynman diagrams do not have the notion of complex conjugation built in, to address that Veltman introduced it in the form of vertex colouring.

Recall that Feynman diagrams comprise vertices and multiple types of edges, but now to denote complex conjugation we introduce another type of vertex. One could define it to have a different colour from the usual type or say that it is circled as opposed to a regular uncircled vertex. Suppose that in the original scalar field theory (say, ϕ^3 theory) an uncircled vertex contributed a factor of $i\lambda$, the circled vertex now contributes a factor of $-i\lambda$. Given that we now have two different kinds of vertices, it is natural to have at least four different kinds of propagators.

The simplest kinds of propagators between two spacetime points x and y are given by,

$$\Delta^{\pm}(x-y) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_k} e^{\pm ik\cdot(x-y)}. \tag{2.16}$$

and notice that

$$\big(\Delta^{+}(x-y)\big)^* = \Delta^{+}(y-x) = \Delta^{-}(x-y). \tag{2.17}$$

We can rewrite the propagators with a clever integration trick. Recall $E_k = k^0$ and consider the

integral,

$$\int_{-\infty}^{\infty} \mathrm{d}k^0 \delta\Big((k^0)^2 - \vec{k}^2 - m^2\Big)\Theta(k^0) = \int_{-\infty}^{\infty} \mathrm{d}k^0 \left[\frac{\delta\left(k^0 - \sqrt{\vec{k}^2 + m^2}\right)}{\left|k^0 + \sqrt{\vec{k}^2 + m^2}\right|} + \frac{\delta\left(k^0 + \sqrt{\vec{k}^2 + m^2}\right)}{\left|k^0 - \sqrt{\vec{k}^2 + m^2}\right|}\right]\Theta(k^0). \tag{2.18a}$$

Integrating and picking up the positive solution $k^0 = \sqrt{\vec{k}^2 + m^2}$ gives,

$$\frac{1}{2k^0} = \frac{1}{2E_k} = \int_{-\infty}^{\infty} \mathrm{d}k^0 \delta\Big((k^0)^2 - \vec{k}^2 - m^2\Big)\Theta(k^0). \tag{2.18b}$$

Substituting in eq. (2.16) gives,

$$\Delta^+(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ik\cdot(x-y)} 2\pi\delta(k^2 + m^2)\Theta(k^0), \tag{2.18c}$$

$$\Delta^-(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} 2\pi\delta(k^2 + m^2)\Theta(k^0), \tag{2.18d}$$

replacing k with $-k$ gives,

$$\Delta^-(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ik\cdot(x-y)} 2\pi\delta(k^2 + m^2)\Theta(-k^0). \tag{2.18e}$$

In the above derivation we used the fact that $\delta(ab) = \frac{\delta(a)}{|b|} + \frac{\delta(b)}{|a|}$ and that Θ is the Heaviside step function. It shall be apparent soon why this form is more suitable. Using these, let us define the Feynman propagator D_F as,

$$D_F(x-y) = \Theta(x^0 - y^0)\Delta^+(x-y) + \Theta(y^0 - x^0)\Delta^-(x-y). \tag{2.19}$$

And we have the anti-Feynman propagator D_F^*,

$$D_F^*(x-y) = \Theta(x^0 - y^0)\Delta^-(x-y) + \Theta(y^0 - x^0)\Delta^+(x-y). \tag{2.20}$$

Let us assign these propagators as follows:

- $D_F(x-y)$ connects an *uncircled* vertex at x to an *uncircled* vertex at y.
- $D_F^*(x-y)$ connects a *circled* vertex at x to a *circled* vertex at y.
- $\Delta^+(x-y)$ connects a *circled* vertex at x to an *uncircled* vertex at y.
- $\Delta^-(x-y)$ connects an *uncircled* vertex at x to a *circled* vertex at y.

Consider any Feynman diagram comprising n vertices at spacetime points $x_1, \ldots, x_n$. Without loss of generality, we assume that some vertices are circled and the remaining are not. Now pick

one of these circled vertices and choose a reference frame where its spacetime point has the largest time component *i.e.* $x_m^0 > x_i^0$ for all $i \neq m$. Now take a copy of this diagram with just one modification, the vertex at x_m is now uncircled and the rest of the diagram is unchanged. This ensures that the integral represented by this diagram picks up an extra minus sign due to the *colour* of only a single vertex being changed. We shall now show that the sum of the integrals represented by these diagrams is 0.

The sum is trivially 0 if the vertex at x_m is an isolated point, but let us assume that it was connected to at least one other vertex at say, x_l. There are now two possibilities:

- If the vertex at x_l is uncircled, we effectively have the sum of $\Delta^+(x_m - x_l)$ (circled x_m to uncircled x_l) and $D_F(x_m - x_l)$ (uncircled x_m to uncircled x_l). Since x_m has the largest time, we see that $D_F(x_m - x_l) = \Delta^+(x_m - x_l)$. The relative minus sign between the integrals of the two diagrams ensures that the sum is $\Delta^+(x_m - x_l) - \Delta^+(x_m - x_l) = 0$.
- If the vertex at x_l is circled, we effectively have the sum of $D_F^*(x_m - x_l)$ (circled x_m to circled x_l) and $\Delta^+(x_m - x_l)$ (uncircled x_m to circled x_l). Since x_m has the largest time, we see that $D_F^*(x_m - x_l) = \Delta^-(x_m - x_l)$. The relative minus sign between the integrals of the two diagrams ensures that the sum is $\Delta^-(x_m - x_l) - \Delta^-(x_m - x_l) = 0$.

Now let us take $2^n - 1$ other copies of the original diagram such that we finally have all possible vertex configurations ranging from all vertices completely circled to all vertices completely uncircled with everything in between. We can form $2^{(n-1)}$ pairs of diagrams such that they only differ in the *colour* of the vertex at x_m. The pairwise sum of integrals represented by diagrams is 0 and hence the total sum of these 2^n integrals is 0. Let us denote the integral for the diagram comprised entirely of uncircled vertices with $I(x_1, \ldots, x_n)$ and the complex conjugation *i.e.* the integral for the diagram comprised entirely of circled vertices with $I^*(x_1, \ldots, x_n)$. We shall denote the integrals for all other kinds of diagrams (with mixed configuration of colour) with $\tilde{I}(x_1, \ldots, x_n)$.

We rewrite the sum of 2^n integrals as follows,

$$I(x_1, \ldots, x_n) + I^*(x_1, \ldots, x_n) = -\sum \tilde{I}(x_1, \ldots, x_n). \tag{2.21}$$

Equation (2.21) is the coveted Largest Time Equation. Veltman showed the analogy between this equation and eq. (2.15),

$$\begin{array}{ccccc} \langle \mathbf{f}|(i\mathcal{T})|\mathbf{i}\rangle & + & \langle \mathbf{f}|(i\mathcal{T})^\dagger|\mathbf{i}\rangle & = & -\sum_{\mathbf{n}} \langle \mathbf{f}|(i\mathcal{T})^\dagger|\mathbf{n}\rangle\langle \mathbf{n}|(i\mathcal{T})|\mathbf{i}\rangle \\ \wr & & \wr & & \wr \\ I(x_1, \ldots, x_n) & + & I^*(x_1, \ldots, x_n) & = & -\sum \tilde{I}(x_1, \ldots, x_n) \end{array} . \tag{2.22}$$

The analogy on the LHS is much more apparent than the analogy on the RHS of the two equations.

We shall heuristically establish the analogy on the RHS with the help of some explicit calculation.

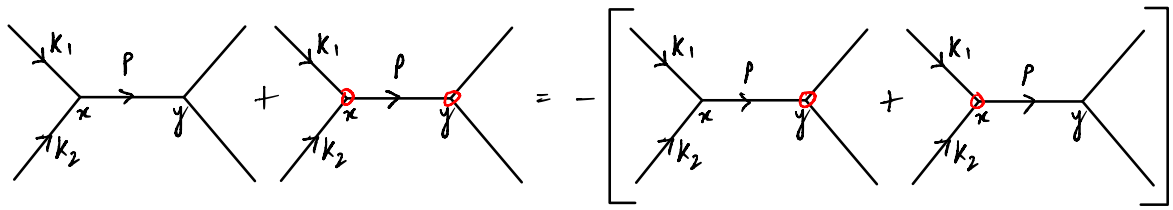

Figure 2.3: *The Largest Time Equation for 4-point tree diagram in ϕ^3 theory.*

On the RHS we have the following integrals,

$$-\lambda^2\left[\int \mathrm{d}^4x\,\mathrm{d}^4y\; e^{i(k_1+k_2)\cdot(x-y)}\Delta^-(x-y) + \int \mathrm{d}^4x\,\mathrm{d}^4y\; e^{i(k_1+k_2)\cdot(x-y)}\Delta^+(x-y)\right].$$

Substituting eq. (2.16) in the above expression and integrating over x^0 gives $\delta(k_1^0+k_2^0-p^0)$ for the Δ^- term and $\delta(k_1^0+k_2^0+p^0)$ for the Δ^+ term. Therefore the Δ^+ term vanishes and only the diagram with an uncircled vertex at x and a circled vertex at y survives. It is easy to see now that not all diagrams on the RHS of eq. (2.21) survive.

This is no coincidence, if one were to recall the alternate form for the Δ propagators:

$$\Delta^{\pm}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{ik\cdot(x-y)} 2\pi\delta(k^2+m^2)\Theta\Big(\pm k^0\Big), \tag{2.23}$$

one can adopt the interpretation that energy always flows from an uncircled vertex to a circled vertex. For Δ^- the flow of energy is along the direction of time whereas for Δ^+ the flow of energy is opposite to the direction of time. This is why the 2$^{\text{nd}}$ diagram on the RHS of fig. 2.3 was found to vanish. We can generalize this argument to more complicated diagrams and infer that only those diagrams which have a *physical*[3] flow of energy along the direction of time survive.

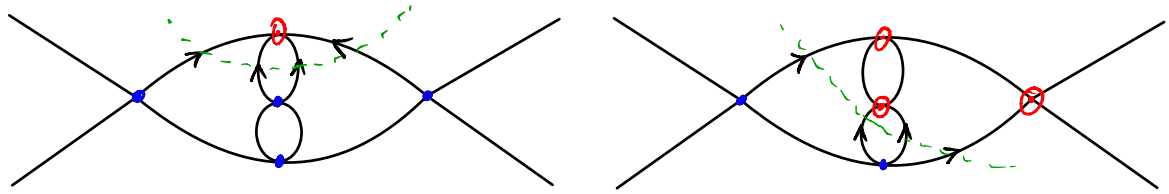

Figure 2.4: *Unphysical vs physical cuts.*

It is precisely only those diagrams which when cut across Δ propagators leave only uncircled vertices with external legs on one side and circled vertices with remaining external legs on the other. This kind of factorisation should remind us of what we saw in eq. (2.11). Cut lines are put on-shell by phase space integration accompanied by a summation over all states that cross the

[3] In accordance with the principle that energy can neither be created nor destroyed at a vertex.

cut. Hence we have heuristically shown that the RHS of the Largest Time Equation is analogous to the RHS of eq. (2.15), as side with uncircled vertices is a contribution to $i\mathcal{T}$ and the other side is a contribution to $(i\mathcal{T})^\dagger$.

If one can identify the $|\mathbf{i}\rangle$ state with the $|\mathbf{f}\rangle$ state, what we have on the RHS of eq. (2.15) is a contribution to some cross-section. This is how we shall use the Largest Time Equation in this thesis, we will compute the loop order amplitude and perform physical cuts and obtain contributions to the cross-section we desire. But finding cut diagrams is not a trivial task, as the complexity of Feynman diagrams increases naïve trial-and-error methods take exceedingly long times. We shall propose a means to incorporate the quantum off-shell recursion framework with the Largest Time Equation along with some algorithms to isolate cut diagrams easily.

If one were to compare the approaches of generalized unitarity and our method, we see that there are several differences. Firstly, generalized unitarity is a method of obtaining loop amplitudes whereas we intend to use the Largest Time Equation with *a priori* knowledge of the loop amplitude to obtain cross-sections. Secondly, generalized unitarity works best for some theories (in supersymmetric settings, the rational terms vanish) and extension to higher than 2-loop order is not easy[9]. But for our method, we have no such restrictions and in principle we can apply it to any theory and at any loop order.

Useful references for the subsection on generalized unitarity are [2, 9, 38]. Useful references for the subsection on the Largest Time Equation are [6, 13, 35, 36].

Next, we shall study color decomposition and cross-sections in gauge theories.

§ 2.3 Managing color

When dealing with gauge theoretic scattering amplitudes, disentangling color[4] and kinematical degrees of freedom proves to be an efficient choice of approach. This is because a factorization is possible that allows one to use partial or *color-ordered* or *color-stripped* amplitudes, which are independent of color and easier to construct. We just have to find a suitable basis spanning the color degrees of freedom of scattering amplitudes. For this section we shall deal with the gauge group SU(N) where N is the number of colors exclusively and limit our discussion to 1-loop order at most.

§ 2.3.1 *Color decomposition*

Gluon fields take the adjoint representation of the gauge group, and carry indices $a, b, \ldots$ that run from $1, \ldots, N^2 - 1$. Fermions like quarks take the fundamental (and anti-fundamental)

[4]The author has chosen to use the American spelling to distinguish between regular colour and QCD color.

representation and carry the fundamental (and anti-fundamental) indices $i, \ldots$ (and $\bar{i}, \ldots$) that run from $1, \ldots, N$. Let us choose the normalization of generators T^a such that,

$$\mathrm{Tr}\Big(T^a T^b\Big) = \delta^{ab}, \quad \Big[T^a, T^b\Big] = i\sqrt{2} f^{abc} T^c. \tag{2.24}$$

we also define $\tilde{f}^{abc} = \sqrt{2} f^{abc}$. We therefore have,

$$i\tilde{f}^{abc} = \mathrm{Tr}\Big(T^a \Big[T^b, T^c\Big]\Big). \tag{2.25}$$

The vertex structure in pure Yang–Mills theory tells us that the 3-gluon vertex is proportional to $\tilde{f}^{abc}$ and the 4-gluon vertex is proportional to $\tilde{f}^{abx}\tilde{f}^{xcd}$ + (perms. of b, c, d). The color factors of s-, t- and u-channel diagrams of the 4-gluon tree amplitude are

$$c_s \equiv \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}, \quad c_t \equiv \tilde{f}^{a_1 a_3 b}\tilde{f}^{b a_4 a_2}, \quad c_u \equiv \tilde{f}^{a_1 a_4 b}\tilde{f}^{b a_2 a_3}. \tag{2.26}$$

The Jacobi identity tells us that $c_s + c_t + c_u = 0$ and hence only two independent color structures exist for the 4-gluon tree amplitude. Upon explicit computation, we can conclude that

$$\mathcal{M}_4^{(0)} = g^2\Big(\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\tilde{\mathcal{M}}_4^{(0)}(1234) + (\text{perms. of } (234))\Big). \tag{2.27}$$

The amplitude on the LHS is called a *color-dressed* or a full amplitude, and the amplitude on the RHS is the color-stripped amplitude (denoted with a tilde). (1234) is a shorthand for helicity and momentum states on legs 1 to 4. In general we have,

$$\mathcal{M}_n^{(0)} = g^{n-2}\sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma_1}}T^{a_{\sigma_2}}\cdots T^{a_{\sigma_n}})\tilde{\mathcal{M}}_n^{(0)}(\sigma_1\sigma_2\cdots\sigma_n), \tag{2.28}$$

where $S_n/\mathbb{Z}_n$ is the set of all non-cyclic permutations in S_n, which is equivalent to S_{n-1}. Therefore for an n-point amplitude, we have a trace-basis of $(n-1)!$ elements. Things become more complicated when we look at 1-loop order, as more than one trace structure survives the simplification and color decomposed form contains leading and sub-leading order *primitive* color-stripped amplitudes,

$$\begin{aligned}\mathcal{M}_n^{(1)} = g^n \Bigg[& N\sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma_1}}T^{a_{\sigma_2}}\cdots T^{a_{\sigma_n}})\tilde{\mathcal{M}}_{n,1}^{(1)}(\sigma_1\sigma_2\cdots\sigma_n) \\ & + \sum_{i=2}^{\lfloor n/2\rfloor+1}\sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma_1}}T^{a_{\sigma_2}}\cdots T^{a_{\sigma_i}})\,\mathrm{Tr}(T^{a_{\sigma_i}}\cdots T^{a_{\sigma_n}})\tilde{\mathcal{M}}_{n,i}^{(1)}(\sigma_1\sigma_2\cdots\sigma_n)\Bigg].\end{aligned} \tag{2.29}$$

$\tilde{\mathcal{M}}^{(1)}_{n,i}$ are the primitive color-stripped amplitudes[5], and obtaining sub-leading order primitives from leading ones is non-trivial. They are related by a permutation sum,

$$\tilde{\mathcal{M}}^{(1)}_{n,i>1}(1,2,\ldots,c-1;c,c+1,\ldots,n) = (-1)^{c-1}\sum_{\sigma\in G}\tilde{\mathcal{M}}^{(1)}_{n,1}(\sigma_1,\sigma_2,\ldots,\sigma_n), \tag{2.30}$$

where if $\alpha_i \in \{\alpha\} \equiv \{c-1,c-2,\ldots,1\}$ and if $\beta_i \in \{\beta\} \equiv \{c,c+1,\ldots,n\}$ then G is the set of all permutations in S_n which preserve the cyclic ordering of α_i within $\{\alpha\}$ and β_i with $\{\beta\}$, while allowing for all possible relative orderings of the α_i with respect to β_i. One can think of G as a *riffle-shuffle* group[6] which allows cyclic permutations on the sub-stacks.

So far we discussed pure gluonic amplitudes, but things become even more complicated when we introduce quark-antiquark pairs. To exhibit the advantages of the color decomposition, it suffices to stop at the pure Yang–Mills theory. We shall list some properties of color-stripped amplitudes:

- **Cyclicity:**
$$\tilde{\mathcal{M}}_n(1,2,\ldots,n) = \tilde{\mathcal{M}}_n(2,\ldots,1,n) = \text{and so on}\ldots$$

- **Parity:**
$$\tilde{\mathcal{M}}_n(1,2,\ldots,n) = \tilde{\mathcal{M}}_n\big(\bar{1},\bar{2},\ldots,\bar{n}\big),$$
where $\bar{i}$ denotes helicity-inversion on the i^{th} leg

- **Reflection:**
$$\tilde{\mathcal{M}}_n(1,2,\ldots,n) = (-1)^n\tilde{\mathcal{M}}_n(n,n-1,\ldots,1),$$
this holds in presence of quark-antiquark pairs but only at tree order.

- **Photon decoupling:**
$$\sum_{\sigma\in\mathbb{Z}_{n-1}}\tilde{\mathcal{M}}^{(0)}_n(\sigma_1,\sigma_2,\ldots,\sigma_{n-1},n) = 0.$$
This identity follows from the fact that a pure gluonic amplitude with a single photon vanishes since $f^{0bc} = 0$. 0 is the color index of the $U(1)$ generator.

- **Gauge invariance**
Like their color-dressed counterparts, color-stripped amplitudes are gauge invariant.

These properties are helpful in reducing the number of independent color-stripped amplitudes significantly. We saw that the trace-basis has $(n-1)!$ elements, but Kleiss and Kuijf found that this basis is actually overcomplete[32]. A new basis was proposed[8] which was composed of structure constants instead of generators T^a. In this new basis we have the color decomposition

[5] $i = 1$ being the *leading* and the rest being the *sub-leading* order primitives.
[6] Credits to Varun Shah for the name.

for pure gluonic tree amplitude,

$$\mathcal{M}_n^{(0)} = (ig)^{n-2} \sum_{\sigma \in S_{n-2}} f^{a_1 a_{\sigma_2} x_1} f^{x_1 a_{\sigma_3} x_2} \cdots f^{x_{n-3} a_{\sigma_{n-1}} a_n} \tilde{\mathcal{M}}_n^{(0)}(1\sigma_2 \cdots \sigma_{n-1} n), \tag{2.31a}$$

$$\mathcal{M}_n^{(0)} = g^{n-2} \sum_{\sigma \in S_{n-2}} (F^{a_{\sigma_2}} \cdots F^{a_{\sigma_{n-1}}})_{a_1 a_n} \tilde{\mathcal{M}}_n^{(0)}(1\sigma_2 \cdots \sigma_{n-1} n), \tag{2.31b}$$

where $(F^a)_{bc} \equiv i f^{bac}$ is an SU(N) generator in the adjoint representation. For the pure gluonic 1-loop amplitude we have,

$$\mathcal{M}_n^{(1)} = g^n \sum_{\sigma \in S_{n-1}/\mathcal{R}} \mathrm{Tr}(F^{a_{\sigma_1}} \cdots F^{a_{\sigma_n}}) \tilde{\mathcal{M}}_{n,1}^{(1)}(\sigma_1, \ldots, \sigma_n), \tag{2.32}$$

where $\mathcal{R}$ is the reflection $\mathcal{R}(1, 2, \ldots, n) = (n, n-1, \ldots, 1)$. The utility of such a basis should be evident by now. It makes computation of amplitudes easier. But the drawbacks of color decomposition are apparent too, generally one is usually more interested in the evaluation of cross-section (dσ), which is proportional to the amplitude squared. Even up to just the next to leading order (NLO) contribution[7] we have to calculate $\mathcal{M}_n^{(0)} \times \left(\mathcal{M}_n^{(0),(1)}\right)^*$ summed over all color states. This immediately leads to a factorial complexity when doing the multiplication of color-dressed amplitude as we have to sum over the permutations of color-stripped amplitudes too[33].

The explicit summations with DDM basis are given below to indicate the non-triviality,

$$\textbf{(LO)} \quad \sum_{\text{colors}} \left|\mathcal{M}_n^{(0)}\right|^2 = \left(g^2\right)^{n-2} \sum_{i,j=1}^{(n-2)!} c_{ij} \tilde{\mathcal{M}}_i^{(0)} \left(\tilde{\mathcal{M}}_j^{(0)}\right)^*, \tag{2.33}$$

where the subscript on $\tilde{\mathcal{M}}$ now refers to the color-stripped amplitude $\tilde{\mathcal{M}}_n$ evaluated for the i^{th} permutation $\sigma_i \in S_{n-2}$. And c_{ij} matrix elements are defined as

$$c_{ij} \equiv \sum_{\text{colors}} (\sigma_i\{F^{a_2} \cdots F^{a_{n-1}}\})_{a_1 a_n} \left[\left(\sigma_j\{F^{a_2} \cdots F^{a_{n-1}}\}\right)_{a_1 a_n}\right]^*. \tag{2.34}$$

$$\textbf{(NLO)} \quad 2 \sum_{\text{colors}} \mathrm{Re}\left(\mathcal{M}_n^{(0)} \left(\mathcal{M}_n^{(1)}\right)\right) = 2\left(g^2\right)^{n-1} \mathrm{Re}\left[\sum_{i=1}^{(n-2)!} \sum_{j=1}^{(n-1)!/2} \tilde{c}_{ij} \tilde{\mathcal{M}}_i^{(0)} \left(\tilde{\mathcal{M}}_j^{(1)}\right)^*\right], \tag{2.35}$$

where the subscripts i, j retain similar meanings but for different permutation groups. And $\tilde{c}_{ij}$

[7] Cross-sections are expanded perturbatively too, with their perturbation parameter being the (square of) coupling constant. In our case, QCD cross-sections are expanded in the strong coupling constant $\alpha_S = g^2/4\pi$ as follows,

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{\mathrm{LO}} + \alpha_S \, \mathrm{d}\sigma^{\mathrm{NLO}} + \alpha_S^2 \, \mathrm{d}\sigma^{\mathrm{NNLO}} + \cdots$$

matrix elements are defined as

$$\tilde{c}_{ij} \equiv \sum_{\text{colors}} (\sigma_i \{F^{a_2} \cdots F^{a_{n-1}}\})_{a_1 a_n} \Big[\text{Tr}\big(F^{a_1} \sigma_j \{F^{a_2} \cdots F^{a_n}\}\big)_{a_1 a_n} \Big]^* . \tag{2.36}$$

Computation time can be saved if we obtain the color-dressed tree amplitude squared directly and all that remains is the summation over colors to obtain leading order contribution to the cross-section. One of the main goals of this thesis is to develop such a method using quantum off-shell recursion framework and the Largest Time Equation.

We shall now discuss how one deals with the color sum at the end of computations.

§ 2.3.2 *Color sum*

We can rewrite eq. (2.27) as,

$$\mathcal{M}_4^{(0)} = g^2 \sum_{\sigma \in S_3} \text{Tr}(T^{a_1} T^{a_{\sigma_2}} T^{a_{\sigma_3}} T^{a_{\sigma_4}}) \tilde{\mathcal{M}}_4^{(0)}(1 \sigma_2 \sigma_3 \sigma_4). \tag{2.37}$$

When computing the cross-section, we will have to deal with the product of traces summed over color indices

$$\text{Tr}(T^{a_1} T^{a_{\sigma_2}} T^{a_{\sigma_3}} T^{a_{\sigma_4}}) \, \text{Tr}(T^{a_1} T^{a_{\xi_2}} T^{a_{\xi_3}} T^{a_{\xi_4}}) \prod_{i,j} \delta^{a_{\sigma_i} a_{\xi_j}} .$$

We shall use the following identity,

$$(T^a)_{ij} (T^a)_{kl} = \delta^{il} \delta^{jk} - \frac{1}{N} \delta^{ij} \delta^{kl}. \tag{2.38}$$

However, there is a much simpler way to compute products of traces and for that we extend the gauge group SU(N) to U(N). This means including a *phase* generator T^0 which commutes with all generators of SU(N). We normalise it as

$$\text{Tr}(T^0 T^0) = 1. \tag{2.39}$$

And hence we take T^0 to be the diagonal matrix with entries $1/\sqrt{N}$. Equation (2.38) now simplifies as follows,

$$(T^a)_{ij} (T^a)_{kl} = \sum_{a=0}^{N^2-1} (T^a)_{ij} (T^a)_{kl}, \tag{2.40a}$$

$$= \sum_{a=1}^{N^2-1} (T^a)_{ij} (T^a)_{kl} + (T^0)_{ij} (T^0)_{kl}, \tag{2.40b}$$

$$= \delta^{il} \delta^{jk} - \frac{1}{N} \delta^{ij} \delta^{kl} + \frac{1}{N} \delta^{ij} \delta^{kl}, \tag{2.40c}$$

$$= \delta^{il}\delta^{jk}. \tag{2.40d}$$

We were able to extend our gauge group as we did because the U(1) gluon (also known as the photon) doesn't couple to SU(N) gluons. We shall now compute the different products of traces. First we calculate,

$$\begin{aligned}
\mathrm{Tr}(T^aT^bT^cT^d)\Big(\mathrm{Tr}(T^aT^bT^cT^d)\Big)^* &= \mathrm{Tr}(T^aT^bT^cT^d)\,\mathrm{Tr}((T^aT^bT^cT^d)^\dagger), && (2.41a)\\
&= \mathrm{Tr}(T^aT^bT^cT^d)\,\mathrm{Tr}(T^dT^cT^bT^a), && (2.41b)\\
&= (T^a)_{ij}(T^bT^cT^d)_{ji}(T^dT^cT^b)_{kl}(T^a)_{lk}, && (2.41c)\\
&= (T^a)_{ij}(T^a)_{lk}(T^bT^cT^d)_{ji}(T^dT^cT^b)_{kl}, && (2.41d)\\
&= \delta^{ik}\delta^{jl}(T^bT^cT^d)_{ji}(T^dT^cT^b)_{kl}, && (2.41e)\\
&= \mathrm{Tr}(T^bT^cT^dT^dT^cT^b), && (2.41f)\\
&= N\,\mathrm{Tr}(T^bT^cT^cT^b), && (2.41g)\\
&= N^2\,\mathrm{Tr}(T^bT^b), && (2.41h)\\
&= N^3\,\mathrm{Tr}(\mathbb{1}) = N^4. && (2.41i)
\end{aligned}$$

Next structure we calculate is,

$$\begin{aligned}
\mathrm{Tr}(T^aT^bT^cT^d)\Big(\mathrm{Tr}(T^aT^bT^dT^c)\Big)^* &= \mathrm{Tr}(T^aT^bT^cT^d)\,\mathrm{Tr}(T^cT^dT^bT^a), && (2.42a)\\
&= \mathrm{Tr}(T^bT^cT^dT^cT^dT^b), && (2.42b)\\
&= N\,\mathrm{Tr}(T^cT^dT^cT^d). && (2.42c)
\end{aligned}$$

Note that for any matrix we have,

$$(T^aAT^a)_{ij} = (T^a)_{ik}(A)_{kl}(T^a)_{lj} = \delta^{ij}\delta^{kl}(A)_{kl} = \mathrm{Tr}(A)\delta^{ij}. \tag{2.42d}$$

If $A = T^0$, then we get $\sqrt{N}\delta^{ij}$ but for all other generators we get 0. Using this we get,

$$\begin{aligned}
N\,\mathrm{Tr}(T^cT^dT^cT^d) &= N^{3/2}\,\mathrm{Tr}(T^0), && (2.42e)\\
&= N^2. && (2.42f)
\end{aligned}$$

All remaining trace products are listed as follows,

$$\begin{aligned}
\mathrm{Tr}(T^aT^bT^cT^d)\Big(\mathrm{Tr}(T^aT^cT^bT^d)\Big)^* &= N^2, && (2.43a)\\
\mathrm{Tr}(T^aT^bT^cT^d)\Big(\mathrm{Tr}(T^aT^cT^dT^b)\Big)^* &= N^2, && (2.43b)\\
\mathrm{Tr}(T^aT^bT^cT^d)\Big(\mathrm{Tr}(T^aT^dT^cT^b)\Big)^* &= N^2, && (2.43c)\\
\mathrm{Tr}(T^aT^bT^cT^d)\Big(\mathrm{Tr}(T^aT^dT^bT^c)\Big)^* &= N^2. && (2.43d)
\end{aligned}$$

Now on squaring eq. (2.27) and summing over all color indices, we will have terms proportional to $\left|\tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4)\right|^2$ and terms proportional to $\tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4)\left(\tilde{\mathcal{M}}_4^{(0)}(1\xi_2\xi_3\xi_4)\right)^*$ where $\sigma, \xi \in S_3$ and $\sigma \neq \xi$. As we saw earlier, the squares of amplitudes will be accompanied with a factor of N^4 coming from the color structure and the interference terms will appear with a factor of N^2. And we have,

$$\sum_{\text{colors}} \left|\mathcal{M}_4^{(0)}\right|^2 = g^4N^4 \sum_{\sigma\in S_3} \left|\tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4)\right|^2 + g^4N^2 \sum_{\substack{\sigma,\xi\in S_3\\ \sigma\neq\xi}} \tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4)\left(\tilde{\mathcal{M}}_4^{(0)}(1\xi_2\xi_3\xi_4)\right)^*. \tag{2.44}$$

We can simplify this further by using the photon decoupling identity for color-stripped amplitudes,

$$\sum_{\sigma\in S_3} \tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4) = 0. \tag{2.45}$$

Squaring and re-arranging the above equation gives,

$$\sum_{\sigma\in S_3} \left|\tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4)\right|^2 = -\sum_{\substack{\sigma,\xi\in S_3\\ \sigma\neq\xi}} \tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4)\left(\tilde{\mathcal{M}}_4^{(0)}(1\xi_2\xi_3\xi_4)\right)^*. \tag{2.46}$$

Inserting this into eq. (2.44) gives us,

$$\sum_{\text{colors}} \left|\mathcal{M}_4^{(0)}\right|^2 = g^4N^2\left(N^2-1\right) \sum_{\sigma\in S_3} \left|\tilde{\mathcal{M}}_4^{(0)}(1\sigma_2\sigma_3\sigma_4)\right|^2. \tag{2.47}$$

We were able to see that for 2-to-2 pure gluonic scattering, the cross-section is obtained from the sum of squares of color-stripped amplitudes. However, this neat structure may not be preserved when we introduce quarks into our theory.

Useful references for this section are [2, 7, 8, 37].

Having gained some understanding of topics and developed motivation behind a key goal of this thesis, the next step is to perform a pedagogical calculation using the quantum off-shell recursion framework.

— CHAPTER 3 —

ϕ^4 *theory*

We shall now use the quantum off-shell recursion framework to derive the integrands for the 4-point amplitude of ϕ^4 theory up to 3-loop orders avoiding Feynman diagrams entirely! There are many steps involved in the calculation:

Step 1. Deriving the Dyson–Schwinger equation(s) of the theory

Step 2. Generating recursion relations using them by substitution of the perturbiner expansions

Step 3. Solving these recursion relations to obtain off-shell currents

Step 4. Performing the amputation of the off-shell leg and imposing the on-shell condition to obtain the amplitude integrands

Step 5. Reducing the full expression into a collection of sets called *topologies* (grouping terms based on graph isomorphisms)

Step 6. Finding the minimal spanning set of topologies and performing the integration

We shall focus on steps 1 to 5 in this thesis. Finding the minimal spanning set and performing the integration using the IBP reduction method remains to be a part of upcoming work.

§ 3.1 Deriving the Dyson–Schwinger equation

In the conventional tree order perturbiner method one relies on the classical equations of motion to construct the recursion relations, but as shown in [1] the Dyson–Schwinger equations are perfectly suitable to extend the formalism to loop orders. We shall use the representation of the Dyson–Schwinger equations used in [1]. The action used in our calculations is the classical action for ϕ^4 theory along with an external source $j(x) \equiv j_x$[1] with bare mass m and bare coupling λ for the fields,

$$S[\phi, j] = \int \mathrm{d}^4x \left[-\frac{1}{2}\partial^\mu \phi_x \partial_\mu \phi_x - \frac{1}{2}m^2\phi_x^2 - \frac{\lambda}{4!}\phi_x^4 + j_x\phi_x \right]. \tag{3.1}$$

[1] A similar convention shall be followed by fields where for any field Π, $\Pi(x) \equiv \Pi_x$

We denote integrations as follows,

$$\int_{x,y,\dots} \equiv \int \mathrm{d}^4x\, \mathrm{d}^4y \cdots, \quad \int_{p,q,\dots} \equiv \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\cdots, \tag{3.2}$$

and rewrite the action in terms of the kinetic operator $K_{xy} \equiv K(x,y)$ and the 4-point interaction vertex $V_{xyzw} \equiv V(x,y,z,w)$ as

$$S[\phi, j] = -\frac{1}{2}\int_{x,y} \phi_x K_{xy} \phi_y + \frac{\lambda}{4!}\int_{x,y,z,w} V_{xyzw}\phi_x\phi_y\phi_z\phi_w + \int_x j_x\phi_x, \tag{3.3}$$

where we have

$$K_{xy} = \left(-\Box_y + m^2\right)\delta^4(x-y), \tag{3.4a}$$

$$V_{xyzw} = -\delta^4(x-y)\delta^4(x-z)\delta^4(x-w). \tag{3.4b}$$

To derive the classical equation of motion for ϕ_x from eq. (3.3), we simply take a functional derivative with respect to ϕ_v,

$$\frac{\delta S}{\delta \phi_v} = -\int_y K_{vy}\phi_y - \frac{\lambda}{3!}\phi_v^3 + j_v, \tag{3.5a}$$

renaming $v \to x$,

$$\frac{\delta S}{\delta \phi_x} = -\int_y K_{xy}\phi_y - \frac{\lambda}{3!}\phi_x^3 + j_x, \tag{3.5b}$$

and because the action is extremal, LHS must vanish.

$$-\int_y K_{xy}\phi_y - \frac{\lambda}{3!}\phi_x^3 + j_x = 0. \tag{3.5c}$$

Equation (3.5c) is the required classical equation of motion. Now we shall define the propagators of this theory. The free propagator $D_{xy} \equiv D(x,y)$ is defined as the inverse of the kinetic operator,

$$\int_y K_{xy}D_{yz} = \delta^4(x-z), \tag{3.6a}$$

$$D_{xy} = \int_p \tilde{D}_p e^{ip\cdot(x-y)}. \tag{3.6b}$$

$\tilde{D}_p \equiv \tilde{D}(p)$ is the Fourier transform of the (position-space) free propagator and it is defined explicitly as[2],

$$\tilde{D}_p = \frac{1}{p^2 + m^2 - i\epsilon}. \tag{3.7}$$

The *dressed* propagator $\boldsymbol{D}_{xy}$ which includes all loop order corrections is related to the exact 2-point function as follows,

$$\boldsymbol{D}_{xy} = \int_p \tilde{\boldsymbol{D}}_p e^{ip\cdot(x-y)} = \frac{i}{\hbar}\langle 0|T\big(\phi_x \phi_y\big)|0\rangle, \tag{3.8}$$

where T is the time-ordering operator. And the momentum space dressed operator $\tilde{\boldsymbol{D}}_p$ with self-energy $\Pi\big(p^2\big)$ is,

$$\tilde{\boldsymbol{D}}_p = \frac{1}{p^2 + m^2 - \Pi\big(p^2\big)}, \tag{3.9}$$

where $\Pi\big(p^2\big)$ is given by all one-particle irreducible (1PI) diagrams in the 2-point function and it itself receives loop order corrections $\Pi = \Pi^{(1)} + \Pi^{(2)} + \cdots$

$$\frac{i}{\hbar}\Pi^{(1)}\Big(p^2\Big) = -\frac{\lambda}{2}\int_q \frac{1}{q^2 + m^2}, \tag{3.10a}$$

$$\begin{aligned}\left(\frac{i}{\hbar}\right)^2 \Pi^{(2)}\Big(p^2\Big) = \frac{\lambda^2}{4}\int_{q,r} \Bigg(&\frac{1}{\big(q^2 + m^2\big)\big(r^2 + m^2\big)^2} \\ &+ \frac{2}{3}\frac{1}{\big((q + r - p)^2 + m^2\big)\big(q^2 + m^2\big)\big(r^2 + m^2\big)}\Bigg).\end{aligned} \tag{3.10b}$$

Let us return to the derivation of the Dyson–Schwinger equation. From eq. (3.5c), we have to promote the scalar field ϕ to the operator $\hat{\phi}$ which is defined as,

$$\hat{\phi} \equiv \left(\varphi + \frac{\hbar}{i}\frac{\delta}{\delta j}\right). \tag{3.11}$$

φ is the *classical* field or in other words, the VEV (vacuum expectation value) of the scalar field ϕ in presence of a source.

$$\varphi_x \equiv \varphi_j(x) = \frac{\langle 0|\phi_x|0\rangle_j}{\langle 0\,|\,0\rangle_j}. \tag{3.12}$$

[2]The $i\epsilon$ factor is omitted from now on for convenience unless required.

It must be noted that φ is indeed the field of interest to which we apply the perturbiner expansion. Using the above ansatz on eq. (3.5c), we get[3]

$$-\int_y K_{xy}\left(\varphi_y + \frac{\hbar}{i}\frac{\delta}{\delta j_y}\right) - \frac{\lambda}{3!}\left(\varphi_x + \frac{\hbar}{i}\frac{\delta}{\delta j_x}\right)^3 + j_x = 0, \tag{3.13a}$$

$$-\int_y K_{xy}\varphi_y - \frac{\lambda}{3!}\varphi_x^3 - \frac{\lambda}{2}\frac{\hbar}{i}\varphi_x\frac{\delta\varphi_x}{\delta j_x} + \frac{\lambda}{3!}\hbar^2\frac{\delta^2\varphi_x}{\delta j_x\delta j_x} + j_x = 0, \tag{3.13b}$$

$$-\int_y K_{xy}\varphi_y - \frac{\lambda}{3!}\varphi_x^3 - \frac{\lambda}{2}\frac{\hbar}{i}\varphi_x\psi_{x,x} + \frac{\lambda}{3!}\hbar^2\psi_{x,x,x} + j_x = 0. \tag{3.13c}$$

Rewriting,

$$\int_y K_{xy}\varphi_y = j_x - \frac{\lambda}{3!}\varphi_x^3 - \frac{\lambda}{2}\frac{\hbar}{i}\varphi_x\psi_{x,x} + \frac{\lambda}{3!}\hbar^2\psi_{x,x,x}, \tag{3.13d}$$

$$\phi_x = \int_y D_{xy}\left(j_y - \frac{\lambda}{3!}\varphi_y^3 - \frac{\lambda}{2}\frac{\hbar}{i}\varphi_y\psi_{y,y} + \frac{\lambda}{3!}\hbar^2\psi_{y,y,y}\right). \tag{3.13e}$$

§ 3.1.1 *Descendant equations*

We now introduce *descendant* fields. Objects like $\frac{\delta^n\varphi}{\delta j\cdots\delta j}$ are crucial to our framework and can be treated as fields by themselves. Let us use the following notation,

(First descendant) $$\psi_{x,y} \equiv \frac{\delta\varphi_x}{\delta j_y}, \tag{3.14a}$$

(Second descendant) $$\psi_{x,y,z} \equiv \frac{\delta^2\varphi_x}{\delta j_y\delta j_z}, \tag{3.14b}$$

$$\vdots$$

moreover $\psi_{x,x} = \lim_{y\to x}\psi_{x,y}$ and $\psi_{x,x,x} = \lim_{\substack{y\to x\\ z\to x}}\psi_{x,y,z}$ and so on. These objects are the so-called descendant fields.

Equation (3.13e) is our coveted Dyson–Schwinger equation. Now observe that we have 3 undetermined fields but only 1 equation. We can take repeated functional derivatives with respect to j on the Dyson–Schwinger equation and generate equations with descendant fields as subjects. These equations shall be called *descendant equations*. Carefully taking a functional

[3] If we must be pedantic, one ought to note that in this kind of substitution $\left(\varphi_x + \frac{\hbar}{i}\frac{\delta}{\delta j_x}\right)^3$ should be treated carefully, this expression is actually

$$\lim_{\substack{y\to x\\ z\to x}}\left(\varphi_x + \frac{\hbar}{i}\frac{\delta}{\delta j_x}\right)\left(\varphi_y + \frac{\hbar}{i}\frac{\delta}{\delta j_y}\right)\left(\varphi_z + \frac{\hbar}{i}\frac{\delta}{\delta j_z}\right).$$

derivative once gives us,

$$\psi_{x,z} = D_{xz} - \frac{\lambda}{3!}\int_y D_{xy}\left(3\varphi_y^2\psi_{y,z} + 3\frac{\hbar}{i}\left(\varphi_y\psi_{y,y,z} + \psi_{y,y}\psi_{y,z}\right) - \hbar^2\psi_{y,y,y,z}\right). \tag{3.15}$$

However in the first descendant equation $\psi_{y,y,y,z}$ (the third descendant) appears for the first time. Now we have 2 equations and 4 undetermined fields. Taking a functional derivative again gives us,

$$\begin{aligned}\psi_{x,z,w} = -\frac{\lambda}{2!}\int_y D_{xy}\Bigg(&\varphi_y^2\psi_{y,z,w} + 2\varphi_y\psi_{y,z}\psi_{y,w} \\ &+ \frac{\hbar}{i}\varphi_y\psi_{y,y,z,w} + \frac{\hbar}{i}\psi_{y,w}\psi_{y,y,z} + \frac{\hbar}{i}\psi_{y,z}\psi_{y,y,w} + \frac{\hbar}{i}\psi_{y,y}\psi_{y,z,w}. \\ &-\frac{1}{3}\hbar^2\psi_{y,y,y,z,w}\Bigg)\end{aligned} \tag{3.16}$$

It is apparent now that taking additional functional derivatives will introduce newer descendants, always leaving us with fewer equations than variables. We perform an $\hbar$-expansion of the classical field and its descendants and truncate equations up to a particular order in $\hbar$. By doing so, we are fixing the highest loop order in calculation. We are allowed to perform a perturbative expansion as such, because the classical field is related to the generating function for connected diagrams $W[j]$[4] in the following manner,

$$\varphi_x \equiv \frac{\delta W[j]}{\delta j_x}. \tag{3.17}$$

The perturbative ($\hbar$) expansions for $\varphi_x, \psi_{x,y}, \psi_{x,y,z}, \ldots$ as listed as follows,

$$\varphi_x = \sum_{n=0}^{\infty}\left(\frac{\hbar}{i}\right)^n \varphi_x^{(n)}, \tag{3.18a}$$

$$\psi_{x,y} = \sum_{n=0}^{\infty}\left(\frac{\hbar}{i}\right)^n \psi_{x,y}^{(n)}, \tag{3.18b}$$

$$\psi_{x,y,z} = \sum_{n=0}^{\infty}\left(\frac{\hbar}{i}\right)^n \psi_{x,y,z}^{(n)}, \tag{3.18c}$$

$$\vdots$$

We take a simple example to show how the truncation works. Suppose we truncate the

[4]It is defined as,

$$\exp\left(\frac{i}{\hbar}W[j]\right) = Z[j] = \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}S[\phi, j]\right).$$

Dyson–Schwinger equation at 3-loop order, eq. (3.13e) is now of the form

$$\varphi_x = (\cdots) + \frac{\hbar}{i}(\cdots) + \left(\frac{\hbar}{i}\right)^2(\cdots) + \left(\frac{\hbar}{i}\right)^3(\cdots). \tag{3.19}$$

Let us see what fields contribute to the highest loop order, in our case since the highest loop order is 3 the fields contributing to it are $\varphi^{(3)}, \psi^{(2)}_{(\cdot,\cdot)}, \psi^{(1)}_{(\cdot,\cdot,\cdot)}$. This is because the first descendant $\psi_{(\cdot,\cdot)}$ comes attached with $\hbar$ and the second descendant $\psi_{(\cdot,\cdot,\cdot)}$ comes attached with $\hbar^2$. It seems enough to truncate the expansion of first descendant equation at 2-loop order.

$$\psi_{x,y} = (\cdots) + \frac{\hbar}{i}(\cdots) + \left(\frac{\hbar}{i}\right)^2(\cdots). \tag{3.20}$$

Again for the highest loop order, the contributing fields are $\psi^{(2)}_{(\cdot,\cdot)}, \psi^{(1)}_{(\cdot,\cdot,\cdot)}, \psi^{(0)}_{(\cdot,\cdot,\cdot,\cdot)}$. Because a tree order term has appeared, this ansatz is now complete. At the end, we need just 4 equations: the Dyson–Schwinger equation at 3-loop order, first descendant equation at 2-loop order, second descendant equation at 1-loop order and the third descendant equation at tree order. To summarise, we notice the pattern – if we truncate Dyson–Schwinger equation at l-loop order, then for the n^{th} descendant we need to truncate its equation at m-loop order such that $n + m = l$ and $n, m \geq 0$. Therefore, we end up needing just $l + 1$ equations. Because we are deriving the 3-loop order amplitude, we just need one more equation in addition to eqs. (3.13e), (3.15) and (3.16),

$$\begin{aligned}\psi_{x,z,w,v} = -\frac{\lambda}{2!}\int_y D_{xy}\Big(&\varphi_y^2\psi_{y,z,w,v} + 2\psi_{y,z}\psi_{y,w}\psi_{y,v} \\ &+ 2\varphi_y\psi_{y,z}\psi_{y,w,v} + 2\varphi_y\psi_{y,w}\psi_{y,z,v} + 2\varphi_y\psi_{y,v}\psi_{y,z,w}\Big)\end{aligned}. \tag{3.21}$$

We have all the equations we need, let us now derive the recursion relations using perturbiner expansions.

§ 3.2 Quantum perturbiner expansion

We first see how the quantum perturbiner expansion comes into being. A sketch of the derivation[1] shall be outlined. The generator $W[j]$ can be expanded with respect to j_x centered at $j_x = 0$,

$$W[j] = \sum_{n=2}^{\infty}\frac{1}{n!}\int_{x_1,x_2,\ldots,x_n}\frac{\delta^n W[j]}{\delta j_{x_1}\delta j_{x_2}\cdots\delta j_{x_n}}\bigg|_{j=0} j_{x_1}j_{x_2}\cdots j_{x_n}, \tag{3.22a}$$

$$= \frac{\hbar}{i}\sum_{n=2}^{\infty}\frac{1}{n!}\int_{x_1,x_2,\ldots,x_n} G_C(x_1,x_2,\ldots,x_n)\frac{ij_{x_1}}{\hbar}\frac{ij_{x_2}}{\hbar}\cdots\frac{ij_{x_n}}{\hbar}, \tag{3.22b}$$

where $G_C(x_1, \ldots, x_n)$ is the connected n-point correlation function by definition. Using the fact that $\varphi_x \equiv \frac{\delta W[j]}{\delta j_x}$, we can derive an expansion for the classical field,

$$\varphi_x = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{y_1, y_2, \ldots, y_n} G_C(x, y_1, y_2, \ldots, y_n) \frac{ij_{y_1}}{\hbar} \frac{ij_{y_2}}{\hbar} \cdots \frac{ij_{y_n}}{\hbar}. \tag{3.23}$$

The external source j_x is a parameter of the theory, we choose to define it in a manner so as to obtain scattering amplitudes from off-shell currents. Therefore we choose j_x to reproduce the LSZ (Lehmann–Symanzik–Zimmermann) reduction formula in eq. (3.22b).

$$j_x = \sum_{i'=1}^{N} \int_{y_{i'}} \boldsymbol{K}_{xy_{i'}} e^{-ik_{i'} \cdot y_{i'}} = \sum_{i'=1}^{N} \tilde{\boldsymbol{K}}(-k_{i'}) e^{-ik_{i'} \cdot x}, \tag{3.24}$$

where $k_{i'}$ are the on-shell external momenta for N particles and $\boldsymbol{K}_{xy}$ is the inverse of the dressed propagator $\boldsymbol{D}_{xy}$.

$$\int_y \boldsymbol{D}_{xy} \boldsymbol{K}_{yz} = \delta^4(x - z), \tag{3.25a}$$

$$\tilde{\boldsymbol{D}}_p \tilde{\boldsymbol{K}}_p = 1. \tag{3.25b}$$

This implies that

$$\tilde{\boldsymbol{K}}_p = p^2 + m^2 - \Pi\left(p^2\right). \tag{3.26}$$

We need to compute the 1PI 2-point function to explicitly define the source[5]. The source is also expanded in $\hbar$.

$$j_x = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n j_x^{(n)}, \tag{3.27}$$

and from eq. (3.10) we have at various loop orders,

$$j_x^{(0)} = \sum_{i'=1}^{N} \left(k_{i'}^2 + m^2\right) e^{-ik_{i'} \cdot x}, \tag{3.28a}$$

$$j_x^{(1)} = -\frac{i}{\hbar} \sum_{i'=1}^{N} \Pi^{(1)}\left(k_{i'}^2\right) e^{-ik_{i'} \cdot x}, \tag{3.28b}$$

$$j_x^{(2)} = -\left(\frac{i}{\hbar}\right)^2 \sum_{i'=1}^{N} \Pi^{(2)}\left(k_{i'}^2\right) e^{-ik_{i'} \cdot x}, \tag{3.28c}$$

$$\vdots$$

If we substitute the expansion of j_x into eq. (3.22b) at $n = N$ order specifically then we obtain the N-point scattering amplitude summed over all permutations of the external particles through

[5] As we shall see later, this requirement can be relaxed.

the LSZ reduction formula,

$$\sum_{\sigma\in S_N}\mathcal{M}_{\sigma(k_1,\ldots,k_N)} = \frac{\hbar}{i}(2\pi)^4\delta^4\big(k_{i_1\cdots i_N}\big)\sum_{i_1,\ldots,i_N}^{N}\tilde{G}_C\big(k_{i_1},k_{i_2},\ldots,k_{i_N}\big)\prod_{j=1}^{N}\frac{i}{\hbar}\tilde{\boldsymbol{K}}\Big(-k_{i_j}\Big), \tag{3.29}$$

where $k_{i_1\cdots i_N}\equiv k_{i_1}+\cdots+k_{i_N}$ and $\tilde{G}_C\big(k_{i_1},\ldots,k_{i_N}\big)$ is the connected N-point correlation function in momentum space. We now define the *quantum off-shell current* $\Phi_{i_1\cdots i_N}$ to be the amputated correlation function with one off-shell leg assigned the momentum $-k_{i_1\cdots i_N}$ (in accordance with momentum conservation).

$$\Phi_{i_1\cdots i_N}\equiv\tilde{G}_C\big(-k_{i_1\cdots i_N},k_{i_1},\ldots,k_{i_N}\big)\prod_{j=1}^{N}\frac{i}{\hbar}\tilde{\boldsymbol{K}}\Big(-k_{i_j}\Big), \tag{3.30}$$

which along with substitution of j_x expansion in eq. (3.23) leads to the *quantum* perturbiner expansion of the classical field φ_x

$$\begin{aligned}\varphi_x &= \sum_{i_1=1}^{N}\Phi_{i_1}e^{-ik_{i_1}\cdot x}+\sum_{1\le i_1<i_2}^{N}\Phi_{i_1i_2}e^{-ik_{i_1i_2}\cdot x}+\cdots+\sum_{1\le i_1<i_2<\cdots<i_n}\Phi_{i_1\cdots i_n}e^{-ik_{i_1\cdots i_n}\cdot x}+\cdots,\\ &= \sum_{\mathcal{P}}\Phi_{\mathcal{P}}e^{-ik_{\mathcal{P}}\cdot x}.\end{aligned} \tag{3.31}$$

where $\mathcal{P},\mathcal{Q},\mathcal{R},\ldots$ are *ordered* words that comprise letters $i,j,k,\ldots$ which are essentially multi-particle labels running from 1 to N. The length of a word is called its *rank* and denoted as $|\mathcal{P}|$. The action of the permutation group on the letters of a word leave the off-shell currents invariant *i.e.* for a rank-n word $\mathcal{P}$,

$$\Phi_{\sigma(\mathcal{P})}=\Phi_{\mathcal{P}},\quad\forall\,\sigma\in S_n. \tag{3.32}$$

The definition of these off-shell currents requires that words with repeated letters vanish, and this forces the maximum rank of a word to be the number of external particles N. We now define quantum perturbiner expansions for descendant fields too.

$$\psi_{x,y} = \int_p\Psi_{p|\emptyset}e^{ip\cdot(x-y)}+\sum_{\mathcal{P}}\int_p\Psi_{p|\mathcal{P}}e^{ip\cdot(x-y)}e^{-ik_{\mathcal{P}}\cdot x}, \tag{3.33a}$$

$$\psi_{x,y,z} = \sum_{\mathcal{P}}\int_{p,q}\Psi_{pq|\mathcal{P}}e^{ip\cdot(x-y)+iq\cdot(x-z)}e^{-ik_{\mathcal{P}}\cdot x}, \tag{3.33b}$$

$$\psi_{x,y,z,w} = \int_{p,q,r}\Psi_{pqr|\emptyset}e^{ip\cdot(x-y)+iq\cdot(x-z)+ir\cdot(x-w)}+\sum_{\mathcal{P}}\int_{p,q,r}\Psi_{pqr|\mathcal{P}}e^{ip\cdot(x-y)+iq\cdot(x-z)+ir\cdot(x-w)}e^{-ik_{\mathcal{P}}\cdot x}, \tag{3.33c}$$

$$\vdots$$

$\Psi_{p|\mathcal{P}}, \Psi_{pq|\mathcal{P}}, \ldots$ are the quantum off-shell currents associated with the descendant fields, and henceforth shall be referred to as the descendant currents for $\Phi_{\mathcal{P}}$. The momenta $p, q, \ldots$ are off-shell loop momenta.

Note that the first and third descendant currents have a zero-mode term. An off-shell current has two parameters, rank and order. $\Phi_{\mathcal{P}}$ is an order-1 quantity as it denotes an object with 1 off-shell leg, $\Psi_{p|\mathcal{P}}$ is an order-2 quantity as it denotes an object with 2 off-shell legs, and so on. The rank of a current denotes the number of on-shell legs instead. So this means that a zero-mode (rank-0) first descendant (order-2) current represents just 2 off-shell legs and no on-shell legs. Such an object is permitted by the theory (due to the presence of the propagator), but zero-mode for an order-3 current is not possible due to the theory missing a 3-point vertex and because ϕ^4 theory has no diagrams for odd number of external legs. Rank-0 order-4 objects are permitted by the theory due to the presence of a 4-point vertex, hence we see a zero-mode term for the third descendant current as well. Another thing worth noting is that the quantum off-shell current and its descendants have an $\hbar$-expansion because they are derived from the connected N-point correlation function in momentum space.

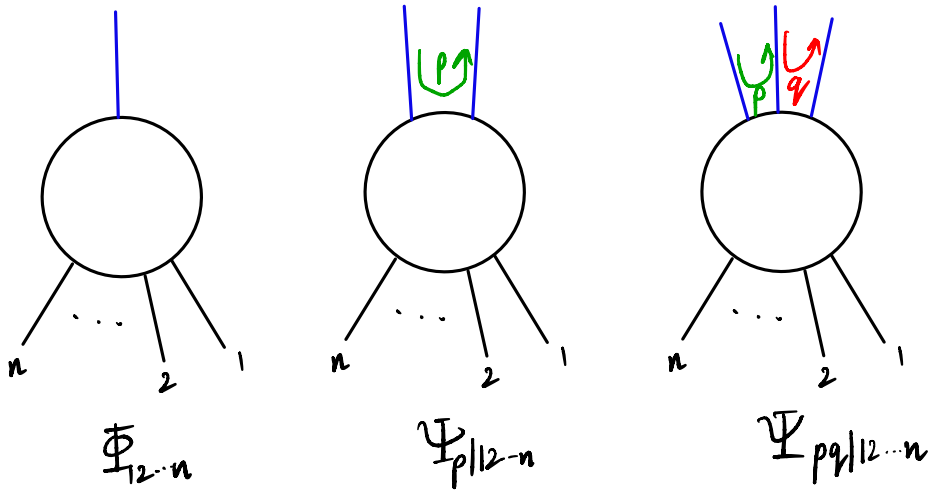

Figure 3.1: *A diagrammatic representation of off-shell currents. Blue lines are off-shell.*

§ 3.3 Deriving recursion relations

Now that we have the most important building blocks of our framework, let us derive recursion relations for the quantum off-shell current order by order.

§ 3.3.1 *Tree order*

Consider the tree order terms in eq. (3.13e),

$$\varphi_x^{(0)} = \int_y D_{xy}\left(j_y^{(0)} - \frac{\lambda}{3!}\left(\varphi_y^{(0)}\right)^3\right). \tag{3.34}$$

We substitute the perturbiner expansions on both sides and compare coefficients of $e^{-ik_{\mathcal{P}}\cdot x}$. Let us first check for words of rank > 1. This obviously excludes the source term as it contributes to just rank-1.

$$\sum_{\mathcal{P}} \Phi^{(0)}_{\mathcal{P}} e^{-ik_{\mathcal{P}}\cdot x} = -\frac{\lambda}{3!}\int_y D_{xy}\left(\sum_{\mathcal{Q}} \Phi^{(0)}_{\mathcal{Q}} e^{-ik_{\mathcal{Q}}\cdot y}\right)\left(\sum_{\mathcal{R}} \Phi^{(0)}_{\mathcal{R}} e^{-ik_{\mathcal{R}}\cdot y}\right)\left(\sum_{\mathcal{S}} \Phi^{(0)}_{\mathcal{S}} e^{-ik_{\mathcal{S}}\cdot y}\right), \tag{3.35a}$$

$$= -\frac{\lambda}{3!}\sum_{\mathcal{P}}\int_y\int_p \tilde{D}_p e^{ip\cdot(x-y)} \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \Phi^{(0)}_{\mathcal{Q}}\Phi^{(0)}_{\mathcal{R}}\Phi^{(0)}_{\mathcal{S}} e^{-ik_{\mathcal{P}}\cdot y}, \tag{3.35b}$$

$$= -\frac{\lambda}{3!}\sum_{\mathcal{P}}\int_y \frac{e^{iy\cdot(-p-k_{\mathcal{P}})}}{(2\pi)^4}\int \mathrm{d}^4p\, \tilde{D}_p e^{ip\cdot x} \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \Phi^{(0)}_{\mathcal{Q}}\Phi^{(0)}_{\mathcal{R}}\Phi^{(0)}_{\mathcal{S}}, \tag{3.35c}$$

$$= -\frac{\lambda}{3!}\sum_{\mathcal{P}}\int \delta^4(p+k_{\mathcal{P}})\,\mathrm{d}^4p\, \tilde{D}_p e^{ip\cdot x} \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \Phi^{(0)}_{\mathcal{Q}}\Phi^{(0)}_{\mathcal{R}}\Phi^{(0)}_{\mathcal{S}}, \tag{3.35d}$$

$$= -\frac{\lambda}{3!}\sum_{\mathcal{P}} \tilde{D}(-k_{\mathcal{P}}) \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \Phi^{(0)}_{\mathcal{Q}}\Phi^{(0)}_{\mathcal{R}}\Phi^{(0)}_{\mathcal{S}} e^{-ik_{\mathcal{P}}\cdot x}. \tag{3.35e}$$

Comparing coefficients on both sides,

$$\Phi^{(0)}_{\mathcal{P}} = -\frac{\lambda}{3!}\frac{1}{k^2_{\mathcal{P}}+m^2}\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \Phi^{(0)}_{\mathcal{Q}}\Phi^{(0)}_{\mathcal{R}}\Phi^{(0)}_{\mathcal{S}}. \tag{3.35f}$$

Abusing existing notation and using a new one we finally have,

$$\Phi^{(0)}_{\mathcal{P}} = -\frac{\lambda}{3!} D_{\mathcal{P}} \lceil \Phi\Phi\Phi \rfloor^{(0)}_{\mathcal{P}} \quad \forall\, |\mathcal{P}| > 1. \tag{3.35g}$$

There are several things to note here. Firstly, in eq. (3.35b), we combined the summations over three words on the RHS into two summations, one running over a global word $\mathcal{P}$ and a secondary summation that means to sum over all possible distributions of the letters of the ordered word $\mathcal{P}$ into non-empty ordered words $\mathcal{Q}, \mathcal{R}$ and $\mathcal{S}$. We then performed two integrations: one to obtain a δ-function in momentum space and the other to perform a δ-substitution. We finally compared coefficients on both sides for the common sum (over $\mathcal{P}$) and obtained the recursion relation eq. (3.35g). A new bracket $\lceil\cdot\rfloor^{(n)}_{\mathcal{P}}$ is defined as (in general for any current and any loop order),

$$\lceil \Phi\Phi\cdots\Psi_p\Psi_{qr}\cdots\rfloor^{(n)}_{\mathcal{P}} \equiv \sum_{\substack{a+b+c+d+\cdots=n\\ a,b,c,d,\ldots\geq 0}} \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}\cup\mathcal{T}\cdots} \Phi^{(a)}_{\mathcal{Q}}\Phi^{(b)}_{\mathcal{R}}\cdots\Psi^{(c)}_{p|\mathcal{S}}\Psi^{(d)}_{qr|\mathcal{T}}\cdots. \tag{3.36}$$

This bracket denotes the distributed summation of a word into non-empty ordered words and simultaneously the distributed summation of a particular loop order into non-negative loop orders, it shall greatly simplify the recursion relations. Moreover, the notation for propagators has been abused such that

$$D_{\mathcal{P}} \equiv \tilde{D}(-k_{\mathcal{P}}), \tag{3.37a}$$

$$D_{p|\mathcal{P}} \equiv \tilde{D}(p - k_{\mathcal{P}}), \tag{3.37b}$$

$$D_{pq|\mathcal{P}} \equiv \tilde{D}(p + q - k_{\mathcal{P}}), \tag{3.37c}$$

$$\vdots$$

Having derived the recursion relation for the tree order current $\Phi^{(0)}$, we shall find the initial condition on the lowest allowed rank for Φ. From the definition, it is obvious that rank-0 current $\Phi^{(0)}_{\emptyset}$ vanishes. Hence, we need to define the rank-1 tree order current and for that we use the expressions for source j_x.

$$\sum_{i'=1}^{N} \Phi^{(0)}_{i'} e^{-ik_{i'}\cdot x} = \int_y D_{xy} j^{(0)}_y, \tag{3.38a}$$

$$= \int_y \int_p \tilde{D}_p e^{ip(x-y)} \sum_{i'=1}^{N} \left(k^2_{i'} + m^2\right) e^{-ik_{i'}\cdot y}, \tag{3.38b}$$

performing the two integrations like earlier,

$$= \sum_{i'=1}^{N} \cancel{\tilde{D}(-k_{i'})\left(k^2_{i'} + m^2\right)} e^{-ik_{i'}\cdot x}. \tag{3.38c}$$

Comparing coefficients,

$$\Phi^{(0)}_{i'} = 1. \tag{3.38d}$$

Thus we see that rank-1 tree order currents are simply unity. Now let us move to the derivation of 1-loop order recursion relations.

§ 3.3.2 *1-loop order*

The equations we need are the 1-loop order terms in eq. (3.13e) and tree order terms in eq. (3.15),

$$\varphi^{(1)}_x = \int_y D_{xy} \left[j^{(1)}_y - \frac{\lambda}{2!} \left(\left(\varphi^{(0)}_y\right)^2 \varphi^{(1)}_y + \varphi^{(0)}_y \psi^{(0)}_{y,y} \right) \right], \tag{3.39a}$$

$$\psi^{(0)}_{x,z} = D_{xz} - \frac{\lambda}{2!} \int_y D_{xy} \left(\varphi^{(0)}_y\right)^2 \psi^{(0)}_{y,z}. \tag{3.39b}$$

We once again substitute the perturbiner expansions and obtain the recursion relations as earlier.

$$\Phi^{(1)}_{\mathcal{P}} = -\frac{\lambda}{2!} D_{\mathcal{P}} \left(\frac{1}{3} \lceil \Phi\Phi\Phi \rfloor^{(1)}_{\mathcal{P}} + \int_p \lceil \Phi\Psi_p \rfloor^{(0)}_{\mathcal{P}} \right) \quad \forall\, |\mathcal{P}| > 1, \tag{3.40a}$$

$$\Psi^{(0)}_{p|\mathcal{P}} = -\frac{\lambda}{2!} D_{p|\mathcal{P}} \lceil \Phi\Phi\Psi_p \rfloor^{(0)}_{\mathcal{P}} \quad \forall\, |\mathcal{P}| > 0. \tag{3.40b}$$

Once again, we have to set initial conditions for our recursion relations. Let us first check the zero-mode of the first descendant current.

$$\int_p \Psi^{(0)}_{p|\emptyset} e^{ip\cdot(x-z)} = D_{xz}, \tag{3.41a}$$

$$= \int_p \tilde{D}_p e^{ip\cdot(x-z)}. \tag{3.41b}$$

Comparing coefficients,

$$\Psi^{(0)}_{p|\emptyset} = \tilde{D}_p \equiv D_{p|\emptyset}. \tag{3.41c}$$

Now let us check the initial condition for 1-loop order off-shell current. The only rank-1 contributions are,

$$\sum_{i'=1}^{N} \Phi^{(1)}_{i'} e^{-ik_{i'}\cdot x} = \int_y D_{xy} \left[j^{(1)}_y - \frac{\lambda}{2!} \sum_{i'=1}^{N} \Phi^{(0)}_{i'} e^{-ik_{i'}\cdot y} \int_p \Psi^{(0)}_{p|\emptyset} \right], \tag{3.42a}$$

substituting from eqs. (3.10a) and (3.28b),

$$= \int_y D_{xy} \left(-\frac{i}{\hbar} \sum_{i'=1}^{N} \Pi^{(1)}\left(k^2_{i'}\right) e^{-ik_{i'}\cdot y} - \frac{\lambda}{2!} \sum_{i'=1}^{N} \int_p \tilde{D}_p e^{-ik_{i'}\cdot y} \right), \tag{3.42b}$$

$$= \int_y D_{xy} \left(\frac{\lambda}{2!} \sum_{i'=1}^{N} \int_q \frac{1}{q^2+m^2} e^{-ik_{i'}\cdot y} - \frac{\lambda}{2!} \sum_{i'=1}^{N} \int_p \tilde{D}_p e^{-ik_{i'}\cdot y} \right). \tag{3.42c}$$

Finally we have,

$$\Phi^{(1)}_{i'} = 0. \tag{3.42d}$$

It has been proposed[1] that all rank-1 loop-order off-shell currents vanish. While the proposition remains unproven, we shall assume it to be true and continue[6].

$$\Phi^{(n)}_{i'} = 0 \qquad \forall\, n \geq 1. \tag{3.43}$$

§ 3.3.3 *2- and 3-loop orders*

Now we shall list results for 2-loop and 3-loop orders directly. The equations relevant for 2-loop order recursion relations are 2-loop order terms from eq. (3.13e), 1-loop order terms from

[6] The verification of this assumption has been carried out up to 3-loop order currents.

eq. (3.15) and tree order terms from eq. (3.16).

$$\varphi_x^{(2)} = \int_y D_{xy}\left[j_y^{(2)} - \frac{\lambda}{2!}\left(\left(\varphi_y^{(0)}\right)^2\varphi_y^{(2)} + \left(\varphi_y^{(1)}\right)^2\varphi_y^{(0)} + \varphi_y^{(0)}\psi_{y,y}^{(1)} + \varphi_y^{(1)}\psi_{y,y}^{(0)} + \frac{1}{3}\psi_{y,y,y}^{(0)}\right)\right], \tag{3.44a}$$

$$\psi_{x,z}^{(1)} = -\frac{\lambda}{2!}\int_y D_{xy}\left[2\varphi_y^{(0)}\varphi_y^{(1)}\psi_{y,z}^{(0)} + \left(\varphi_y^{(0)}\right)^2\psi_{y,z}^{(1)} + \varphi_y^{(0)}\psi_{y,y,z}^{(0)} + \psi_{y,y}^{(0)}\psi_{y,z}^{(0)}\right], \tag{3.44b}$$

$$\psi_{x,z,w}^{(0)} = -\frac{\lambda}{2!}\int_y D_{xy}\left[\left(\varphi_y^{(0)}\right)^2\psi_{y,z,w}^{(0)} + 2\varphi_y^{(0)}\psi_{y,z}^{(0)}\psi_{y,w}^{(0)}\right]. \tag{3.44c}$$

We substitute the required perturbiner expansions and obtain the 2-loop order recursion relations,

$$\Phi_{\mathcal{P}}^{(2)} = -\frac{\lambda}{2!}D_{\mathcal{P}}\left(\frac{1}{3}\lceil\Phi\Phi\Phi\rfloor_{\mathcal{P}}^{(2)} + \int_p\left(\lceil\Phi\Psi_p\rfloor_{\mathcal{P}}^{(1)} + \int_q\frac{1}{3}\Psi_{pq|\mathcal{P}}^{(0)}\right)\right) \quad \forall\,|\mathcal{P}| > 1, \tag{3.45a}$$

$$\Psi_{p|\mathcal{P}}^{(1)} = -\frac{\lambda}{2!}D_{p|\mathcal{P}}\left(\lceil\Phi\Phi\Psi_p\rfloor_{\mathcal{P}}^{(1)} + \int_q\left(\lceil\Phi\Psi_{pq}\rfloor_{\mathcal{P}}^{(0)} + \lceil\Psi_p\Psi_q\rfloor_{\mathcal{P}}^{(0)}\right)\right) \quad \forall\,|\mathcal{P}| \geq 0, \tag{3.45b}$$

$$\Psi_{pq|\mathcal{P}}^{(0)} = -\frac{\lambda}{2!}D_{pq|\mathcal{P}}\left(\lceil\Phi\Phi\Psi_{pq}\rfloor_{\mathcal{P}}^{(0)} + 2\lceil\Phi\Psi_p\Psi_q\rfloor_{\mathcal{P}}^{(0)}\right) \quad \forall\,\mathcal{P} \geq 1. \tag{3.45c}$$

To derive the initial conditions for these recursion relations, it is enough to simply read off the recursion relations and substitute the initial conditions that we already know *e.g* $\Phi_i^{(0)}$, $\Psi_{p|\emptyset}^{(0)}$ etc. And the same continues for higher loop orders. This is why the knowledge of self-energy operator explicitly isn't needed. With the assumption that $\Phi_i^{(n)} = 0\ \forall\, n \geq 1$, we can construct and solve recursion relations easily. We truly don't need Feynman diagrams for this method! Now we finally state the 3-loop order recursion relations.

$$\Phi_{\mathcal{P}}^{(3)} = -\frac{\lambda}{2!}D_{\mathcal{P}}\left(\frac{1}{3}\lceil\Phi\Phi\Phi\rfloor_{\mathcal{P}}^{(3)} + \int_p\left(\lceil\Phi\Psi_p\rfloor_{\mathcal{P}}^{(2)} + \int_q\frac{1}{3}\Psi_{pq|\mathcal{P}}^{(1)}\right)\right) \quad \forall\,|\mathcal{P}| > 1, \tag{3.46a}$$

$$\Psi_{p|\mathcal{P}}^{(2)} = -\frac{\lambda}{2!}D_{p|\mathcal{P}}\left(\lceil\Phi\Phi\Psi_p\rfloor_{\mathcal{P}}^{(2)} + \int_q\left(\lceil\Phi\Psi_{pq}\rfloor_{\mathcal{P}}^{(1)} + \lceil\Psi_p\Psi_q\rfloor_{\mathcal{P}}^{(1)} + \frac{1}{3}\int_r\Psi_{pqr|\mathcal{P}}^{(0)}\right)\right) \quad \forall\,|\mathcal{P}| \geq 0, \tag{3.46b}$$

$$\Psi_{pq|\mathcal{P}}^{(1)} = -\frac{\lambda}{2!}D_{pq|\mathcal{P}}\left(\lceil\Phi\Phi\Psi_{pq}\rfloor_{\mathcal{P}}^{(1)} + 2\lceil\Phi\Psi_p\Psi_q\rfloor_{\mathcal{P}}^{(1)} + \int_r\left(\lceil\Phi\Psi_{pqr}\rfloor_{\mathcal{P}}^{(0)} + \lceil\Psi_p\Psi_{qr}\rfloor_{\mathcal{P}}^{(0)} + \lceil\Psi_q\Psi_{pr}\rfloor_{\mathcal{P}}^{(0)} + \lceil\Psi_r\Psi_{pq}\rfloor_{\mathcal{P}}^{(0)}\right)\right) \quad \forall\,|\mathcal{P}| \geq 1, \tag{3.46c}$$

$$\Psi_{pqr|\mathcal{P}}^{(0)} = -\frac{\lambda}{2!}D_{pqr|\mathcal{P}}\left(\lceil\Phi\Phi\Psi_{pqr}\rfloor_{\mathcal{P}}^{(0)} + 2\lceil\Psi_p\Psi_q\Psi_r\rfloor_{\mathcal{P}}^{(0)} + 2\lceil\Phi\Psi_p\Psi_{qr}\rfloor_{\mathcal{P}}^{(0)} + 2\lceil\Phi\Psi_q\Psi_{pr}\rfloor_{\mathcal{P}}^{(0)} + 2\lceil\Phi\Psi_r\Psi_{pq}\rfloor_{\mathcal{P}}^{(0)}\right) \quad \forall\,|\mathcal{P}| \geq 0. \tag{3.46d}$$

We have finally constructed all the recursion relations needed to compute the 4-point amplitude up to 3-loop orders. Note that these same recursions can be used for n-point amplitudes up to 3-loop orders, the derivation depends the truncation point of the $\hbar$-expansion but not the rank of the currents. We can also extrapolate the number of recursion relations needed for n-loop order

amplitudes, for n loops we would need

$$\frac{(n+1)(n+2)}{2} \text{ recursion relations.}$$

We now move on to discussing the implementation of this framework in Mathematica®, for solving the recursion relations and performing further simplifications.

§ 3.4 Solving and simplifying

The next step is to use a CAS to solve these recursion relations because at 3-loop order recursions are extremely tedious calculations to do by hand. We chose to use Mathematica® due to ease of access and logical syntax.

§ 3.4.1 *Implementation*

We shall discuss some common pitfalls awaiting someone doing this calculation. While setting up the code, it is important to design functions that take care of the distributed summation in eq. (3.35b). It is not easy to deal with the presence of zero-modes in the recursion relations either. The next step is to define the maximum rank of words used in the calculation. Here we are interested in the 4-point function, which is why we restrict the rank of the words to 3 (3 on-shell legs and 1 off-shell leg that will be brought on-shell after amputation). From the above recursion relations it should be apparent that they need to be solved in a particular order:

$$\Phi^{(0)} \to \Psi_p^{(0)} \to \Phi^{(1)} \to \Psi_{pq}^{(0)} \to \Psi_p^{(1)} \to \Phi^{(2)} \to \Psi_{pqr}^{(0)} \to \Psi_{pq}^{(1)} \to \Psi_p^{(2)} \to \Phi^{(3)}.$$

And to ensure that the code is efficient, it makes sense to create container variables for the results of the recursion relation so that we only have to solve each recursion only once. One should also notice that there are two different kinds of loop momenta at play in the recursions. For example, take a look at eq. (3.46a). On the LHS, we have a 3-loop order quantity which must mean that there are 3 loop momenta at play (say p, q and r). But on the RHS there are terms without explicit loop integrals and we only have 2 independent loop integrations (for p and q) but the third loop momenta is supposedly *hidden*. It is really important to take care of this distinction in the code lest something breaks during evaluations. Lastly, it is recommended to turn on parallelization to solve recursion relations as it improves the speed of calculations. But due to some technical details (parallelization overhead) benefits are best reaped for more complicated calculations (2- or 3-loop orders for example).

Once these things are dealt with, solving recursion relations should be straightforward and we proceed to obtain amplitudes from the off-shell currents.

§ 3.4.2 *Amputation*

We don't have the full amplitudes yet, to convert off-shell currents into amplitudes we use the following equation,

$$\mathcal{M}^{(n)}(k_1,\ldots,k_{N+1}) = \lim_{k^2_{1\cdots N}\to -m_p^2} \sum_{j=0}^{n} \tilde{\boldsymbol{K}}^{(j)}(k_{1\cdots N})\Phi^{(n-j)}_{1\cdots N}. \tag{3.47}$$

The above equation[7] means that we obtain the $N+1$-point n-loop order amplitude from rank-N current. The limit enforces the on-shell condition (eq. (2.8)) and we sum over various loop orders of the dressed kinetic operator (eq. (3.26)) and of the rank-N off-shell current. Momentum conservation is imposed by setting $k_{N+1} = -k_{1\cdots N}$. This also means that in the amplitude expression, the propagator $D_{1\cdots N}$ should vanish because this blows up when the on-shell condition is imposed. If it survives, then something was done incorrectly and hence it serves as a preliminary check for calculations.

However, using this equation turns out to be a rather inefficient method because we have to also calculate the higher loop order contributions to the kinetic operator which are non-trivial and then perform a multiplicative sum which increases the overall computational cost. A collaborator[8] suggested a much more efficient way of performing the amputation. This newer method of amputation is based on the physical meaning of *amputated* diagrams. To put it simply, amputated diagrams are those which don't have loops on external legs. A propagator of a theory contains all loop order corrections (in the form of self-energy operator) to the trivial tree order term which is just the identity. Thus, the suggested method of amputation was to take all propagators which carried the 'correct' momentum (after momentum conservation) for the off-shell leg $-k_{1\cdots N}$ (*i.e.* $D_{1\cdots N}$ in our case) and replace them with identity. We have essentially discarded all loop corrections on the external off-shell leg.

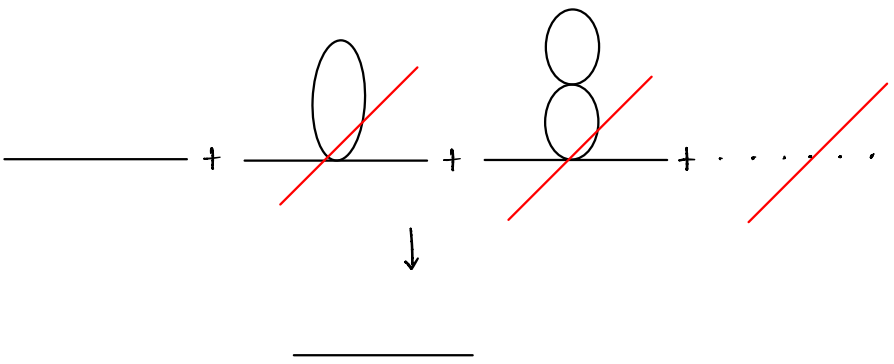

Figure 3.2: *Loop corrections being 'amputated'.*

[7] Also note that this equation differs in convention to the equation in [1], but the amplitudes obtained have the right sign convention.

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Algebraically,

$$D_{1\cdots N} \to \mathbb{1}, \quad (D_{1\cdots N})^n \to 0 \quad \forall\, n > 1. \tag{3.48}$$

This ansatz easily extends to theories with more than one kind of propagator, once we fix the external state of the off-shell leg we can identify the correct propagator to set to identity. This shall be demonstrated in the following chapter when we deal with the field-doubled theory.

§ 3.4.3 *Simplification of results*

Performing the said amputation gives us the amplitudes. Our results are as follows,

$$i\mathcal{M}_4^{(0)} = -\lambda, \tag{3.49a}$$

$$i\mathcal{M}_4^{(1)} = \frac{\lambda^2}{2!}\int_p D_{p|\emptyset}\big(D_{p|12} + D_{p|23} + D_{p|13}\big), \tag{3.49b}$$

$$i\mathcal{M}_4^{(2)} = -\frac{\lambda^3}{4!}\int_{p,q} D_{p|\emptyset} D_{q|\emptyset} \times (36 \text{ terms}\ldots), \tag{3.49c}$$

$$i\mathcal{M}_4^{(3)} = \frac{\lambda^4}{4!3!}\int_{p,q,r} D_{p|\emptyset} D_{q|\emptyset} D_{r|\emptyset} \times (813 \text{ terms}\ldots). \tag{3.49d}$$

These results agree with integrands obtained from Feynman diagrams. Let us see how these results can be further simplified.

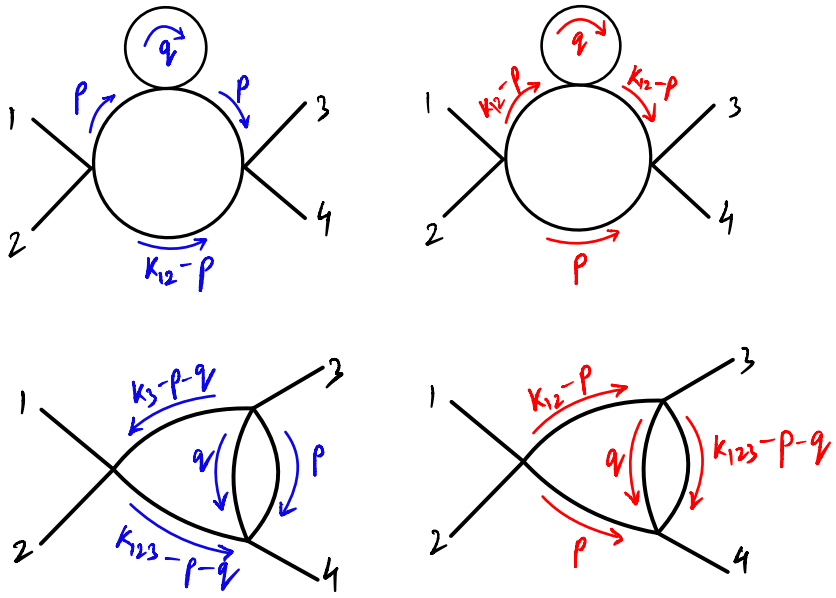

Figure 3.3: *Diagrams belonging to same topology, related to each other by loop momenta renaming*

If we consider two terms from the 2-loop amplitude:

$$\frac{\lambda^3}{3!} D_{p|\emptyset} D_{q|\emptyset} D_{pq|3} D_{pq|123}, \quad \frac{\lambda^3}{3!} D_{p|\emptyset} D_{q|\emptyset} D_{p|12} D_{pq|123},$$

and construct their Feynman diagrams (bottom row in fig. 3.3) and label the momenta properly, we notice that the two diagrams are related to each other by a simple renaming (or *shifts*) of loop momenta. These two diagrams belong to the same *topology*. We now discuss the methods to find these topologies and greatly simplify the amplitude expressions.

Let us assume that we are working with 2-loop order terms in the amplitude. We first generate a list of all possible shifts based on linear combinations of external momenta $k_1, k_2, \cdots, k_n$ and loop momenta p, q. Shifts were chosen to be of the form,

$$l \to a k_{\mathcal{P}} + b l_{\mathcal{P}}, \tag{3.50}$$

where $l \in \{p, q\}$ and $a, b \in \{1, -1\}$. And we have

$$l_\emptyset = 0, \quad l_1 = p, \quad l_2 = q, \quad l_{12} = p + q. \tag{3.51}$$

We then develop a criterion to filter out shifts that are incompatible[9] with each other. For example:

$$(p \to k_1, q \to k_2), \qquad \text{(Incompatible)}$$
$$(p \to k_{12} - p, q \to q). \qquad \text{(Compatible)}$$

The remaining shifts are implemented simultaneously and we obtain new forms for propagators in the form of a tuple. For example after implementing shifts,

$$D_{p|\emptyset} D_{q|\emptyset} D_{pq|3} D_{pq|123} \to \left\{ D_{p|12} D_{q|3} D_{pq|12} D_{pq|\emptyset}, D_{p|\emptyset} D_{q|\emptyset} D_{p|12} D_{pq|123}, \cdots \right\}. \tag{3.52}$$

We implement a weighted-sum procedure, where a preference is set for propagators and this allows us to isolate a single element from the tuple and thus guaranteeing[10] a canonical form for each topology. An example of weights is shown:

$$\begin{aligned} D_{p|\emptyset} &\to 10^{14}, \\ D_{q|\emptyset} &\to 10^{13}, \\ &\vdots \\ D_{12|pq} &\to 10^{-2}, \\ D_{123|pq} &\to 10^{-3}. \end{aligned}$$

In this scheme, we are preferring propagators $D_{p|\emptyset}, D_{q|\emptyset}$ over $D_{12|pq}, D_{123|pq}$, and hence we

[9] Here incompatibility refers to fact that on shifting loop momenta the resultant momenta configurations no longer have any physical interpretation.

[10] Weights need to be chosen in proper manner to ensure a noticeable difference in sums.

shall see that the canonical forms of topologies will more likely have the former propagators over the latter. It is apparent from the description of the method that this is not much better than a brute-force attempt and while it works successfully up to 2-loop orders, the ansatz proves to be terribly inefficient for 3-loop order terms. This apparent failure led us to discover several references[3,4,14] on topology identification. An algorithm[3,4] (referred to as *Pak algorithm* in some texts) seemed promising but its implementation wasn't compatible with our approach of quantum off-shell currents.

We believe that the field-doubling prescription is a good candidate for implementing loop momenta shifts. The structure of the amplitudes of such a theory would be richer given that there are more than one kinds of propagators and vertices. In the next chapter, we shall discuss the field-doubling prescription to use the Largest Time Equation with the quantum off-shell recursion framework.

This chapter is heavily based on [1] and served as both a review and an extension of the reference.

CHAPTER 4

Field-doubled ϕ^4 theory

In this chapter we focus on the field-doubling prescription. We shall calculate the 3-loop order 4-point amplitude in this theory and from it obtain the tree order 6-point cross-section via unitarity. The steps of the calculation broadly remain the same as before but we first elaborate upon the field-doubling prescription and emphasize changes from the regular ϕ^4 theory. After which we shall derive the Dyson–Schwinger equations and proceed from there.

§ 4.1 Field-doubling

Recall eq. (2.11a). We can generalize it to higher loop orders and obtain

$$-i\left(\mathcal{M}^{(a)}_{\mathbf{i}\to\mathbf{f}} - \left(\mathcal{M}^{(a)}_{\mathbf{i}\to\mathbf{f}}\right)^*\right) = \sum\!\!\!\!\!\!\int_{\mathbf{n}} \mathcal{M}^{(b)}_{\mathbf{i}\to\mathbf{n}}\left(\mathcal{M}^{(c)}_{\mathbf{n}\to\mathbf{f}}\right)^*, \tag{4.1}$$

where the loop orders satisfy $a > b$ and $a > c$. We aim to obtain the RHS of eq. (4.1) from the LHS, and for that we construct a *field-doubled* action as follows,

$$S_{f.d}[\phi, j] = S[\phi, j] - S^*[\phi, j]. \tag{4.2}$$

This action should generate amplitudes proportional to the LHS of eq. (4.1). To keep track of the complex-conjugated action, it is useful to define indices + and −. Therefore, we now work with a pair of scalar fields $\phi^\pm$ and their associated external sources $j^\pm$, we denote them collectively with capital indices $A, B, \ldots \in \{+, -\}$. The action is defined as

$$S\left[\phi^A, j^A\right] = S\left[\phi^+, j^+\right] - S[\phi^-, j^-], \tag{4.3a}$$

$$= -\frac{1}{2}\int_{x,y} \phi^A_x K^{AB}_{xy} \phi^B_y - \frac{\lambda^A}{4!}\int_x V^{ABCDE}\phi^B_x\phi^C_x\phi^D_x\phi^E_x + \int_x \eta^{AB} j^A_x \phi^B_x, \tag{4.3b}$$

where we picked the form similar to eq. (3.3) and

$$\eta^{AB} \equiv \begin{pmatrix} \eta^{++} & \eta^{+-} \\ \eta^{-+} & \eta^{--} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.4}$$

and the non-vanishing components of V^{ABCDE} are

$$V^{+++++} = 1, \quad V^{-----} = -1. \tag{4.5}$$

The kinetic operator K^{AB}_{xy} is defined as,

$$K^{AB}_{xy} \equiv \begin{pmatrix} K^{++}_{xy} & 0 \\ 0 & K^{--}_{xy} \end{pmatrix} = \begin{pmatrix} \left(-\Box_y + m^2\right)\delta^4(x-y) & 0 \\ 0 & -\left(-\Box_y + m^2\right)\delta^4(x-y) \end{pmatrix}. \tag{4.6}$$

We now define the propagators of the theory D^{AB}_{xy} by the equation

$$\int_y K^{AB}_{xy} D^{BC}_{yz} = \delta^{AC}\, \delta^4(x-z). \tag{4.7}$$

And we obtain,

$$D^{++}_{xy} = \int_p \frac{e^{ip\cdot(x-y)}}{p^2 + m^2 - i\epsilon}, \tag{4.8a}$$

$$D^{+-}_{xy} = \int_p e^{ip\cdot(x-y)} 2\pi\delta(p^2+m^2)\Theta(p^0), \tag{4.8b}$$

$$D^{-+}_{xy} = -\int_p e^{ip\cdot(x-y)} 2\pi\delta(p^2+m^2)\Theta(-p^0), \tag{4.8c}$$

$$D^{--}_{xy} = -\int_p \frac{e^{ip\cdot(x-y)}}{p^2 + m^2 + i\epsilon}. \tag{4.8d}$$

Note that we had the freedom to define D^{+-}_{xy} and D^{-+}_{xy} propagators as we saw fit, and we chose the definition that implemented Cutkosky's rules[11] so as to implement cuts algebraically. The momentum space propagators are then given by,

$$\tilde{D}^{++}_{p} = \frac{1}{p^2 + m^2 - i\epsilon}, \tag{4.9a}$$

$$\tilde{D}^{+-}_{p} = 2\pi\delta(p^2+m^2)\Theta(p^0), \tag{4.9b}$$

$$\tilde{D}^{+-}_{p} = -2\pi\delta(p^2+m^2)\Theta(-p^0), \tag{4.9c}$$

$$\tilde{D}^{--}_{p} = -\frac{1}{p^2 + m^2 + i\epsilon}. \tag{4.9d}$$

We proceed to derive the Dyson–Schwinger equations of the theory.

§ 4.2 Dyson–Schwinger equations and descendants

Like earlier, we take a functional derivative with respect to ϕ_z^F in eq. (4.3b) and rename indices to obtain the classical equations of motion,

$$-\int_y K_{xy}^{AB}\phi_y^B - \frac{\lambda^E}{3!}V^{ABCDE}\phi_x^B\phi_x^C\phi_x^D + \eta^{BA}j_x^B = 0, \tag{4.10a}$$

$$-\int_y K_{xy}^{AB}\phi_y^B - \frac{\lambda^E}{3!}V^{ABCDE}\phi_x^B\phi_x^C\phi_x^D + \eta^{AB}j_x^B = 0, \tag{4.10b}$$

where we exploited the symmetry of η. Now when we promote the fields to operators, care must be taken to symmetrize terms that were symmetric earlier *i.e.* after $\phi \longrightarrow \hat{\phi} = \left(\varphi + \frac{\hbar}{i}\frac{\delta}{\delta j}\right)$ we get,

$$\int_y K_{xy}^{AB}\hat{\phi}_y^B = -\frac{\lambda^E}{3!}V^{ABCDE}\frac{1}{3!}\left(\hat{\phi}_x^B\hat{\phi}_x^C\hat{\phi}_x^D + \text{perms.}\right) + \eta^{AB}j_x^B, \tag{4.11a}$$

$$\int_y K_{xy}^{AB}\varphi_y^B = -\frac{\lambda^E}{3!}V^{ABCDE}\frac{1}{3!}\left(\left(\varphi_x^B + \frac{\hbar}{i}\frac{\delta}{\delta j_x^B}\right)\left(\varphi_x^C\varphi_x^D + \frac{\hbar}{i}\frac{\delta\varphi_x^D}{\delta j_x^C}\right) + \text{perms.}\right) + \eta^{AB}j_x^B, \tag{4.11b}$$

after careful expansion, we get

$$\int_y K_{xy}^{AB}\varphi_y^B = \eta^{AB}j_x^B - \frac{\lambda^E}{3!}V^{ABCDE}\left(\varphi_x^B\varphi_x^C\varphi_x^D + 3\frac{\hbar}{i}\varphi_x^{(B}\psi_{x,x}^{CD)} - \hbar^2\psi_{x,x,x}^{(BCD)}\right), \tag{4.11c}$$

$$\int_{x,y} D_{zx}^{FA}K_{xy}^{AB}\varphi_y^B = \int_x D_{zx}^{FA}\left[\eta^{AB}j_x^B - \frac{\lambda^E}{3!}V^{ABCDE}\left(\varphi_x^B\varphi_x^C\varphi_x^D + 3\frac{\hbar}{i}\varphi_x^{(B}\psi_{x,x}^{CD)} - \hbar^2\psi_{x,x,x}^{(BCD)}\right)\right], \tag{4.11d}$$

$$\varphi_z^F = \int_x D_{zx}^{FA}\left[\eta^{AB}j_x^B - \frac{\lambda^E}{3!}V^{ABCDE}\left(\varphi_x^B\varphi_x^C\varphi_x^D + 3\frac{\hbar}{i}\varphi_x^{(B}\psi_{x,x}^{CD)} - \hbar^2\psi_{x,x,x}^{(BCD)}\right)\right]. \tag{4.11e}$$

Performing appropriate renaming,

$$\varphi_x^A = \int_y D_{xy}^{AB}\left[\eta^{BC}j_y^C - \frac{\lambda^F}{3!}V^{BCDEF}\left(\varphi_y^C\varphi_y^D\varphi_y^E + 3\frac{\hbar}{i}\varphi_y^{(C}\psi_{y,y}^{DE)} - \hbar^2\psi_{y,y,y}^{(CDE)}\right)\right]. \tag{4.11f}$$

where we used the usual notation for descendant fields. Now we explicitly write eq. (4.11f) after selecting the non-vanishing components of V,

$$\begin{aligned}\varphi_x^A = &\int_y D_{xy}^{A+}\left(j_y^+ - \frac{\lambda^+}{3!}\left(\varphi_y^+\right)^3 - \frac{\lambda^+}{2!}\frac{\hbar}{i}\varphi_y^+\psi_{y,y}^{++} + \frac{\lambda^+}{3!}\hbar^2\psi_{y,y,y}^{+++}\right)\\ &- \int_y D_{xy}^{A-}\left(j_y^- - \frac{\lambda^-}{3!}\left(\varphi_y^-\right)^3 - \frac{\lambda^-}{2!}\frac{\hbar}{i}\varphi_y^-\psi_{y,y}^{--} + \frac{\lambda^-}{3!}\hbar^2\psi_{y,y,y}^{---}\right).\end{aligned} \tag{4.12}$$

We have derived the Dyson–Schwinger equations for this theory. Let us now derive the descendant equations by repeatedly taking functional derivatives with respect to the sources j^A,

$$\begin{aligned}\psi^{AA'}_{x,z} &= D^{A+}_{xz}\eta^{+A'} + D^{A-}_{xz}\eta^{-A'} \\ &\quad - \frac{\lambda^+}{3!}\int_y D^{A+}_{xy}\left(3\left(\varphi^+_y\right)^2\psi^{+A'}_{y,z} + 3\frac{\hbar}{i}\varphi^+_y\psi^{++A'}_{y,y,z} + 3\frac{\hbar}{i}\psi^{+A'}_{y,z}\psi^{++}_{y,y} - \hbar^2\psi^{+++A'}_{y,y,y,z}\right) \\ &\quad + \frac{\lambda^-}{3!}\int_y D^{A-}_{xy}\left(3\left(\varphi^-_y\right)^2\psi^{-A'}_{y,z} + 3\frac{\hbar}{i}\varphi^-_y\psi^{--A'}_{y,y,z} + 3\frac{\hbar}{i}\psi^{-A'}_{y,z}\psi^{--}_{y,y} - \hbar^2\psi^{---A'}_{y,y,y,z}\right)\end{aligned} \tag{4.13}$$

Above are the 4 first descendant equations,

$$\begin{aligned}\psi^{AA'A''}_{x,z,w} &= -\frac{\lambda^+}{2!}\int_y D^{A+}_{xy}\left(\left(\varphi^+_y\right)^2\psi^{+A'A''}_{y,z,w} + 2\varphi^+_y\psi^{+A'}_{y,z}\psi^{+A''}_{y,w}\right. \\ &\qquad \left. + \frac{\hbar}{i}\varphi^+_y\psi^{++A'A''}_{y,y,z,w} + \frac{\hbar}{i}\psi^{+A''}_{y,w}\psi^{++A'}_{y,y,z} + \frac{\hbar}{i}\psi^{+A'A''}_{y,z,w}\psi^{++}_{y,y} + \frac{\hbar}{i}\psi^{+A'}_{y,z}\psi^{++A''}_{y,y,w}\right) \\ &\quad + \frac{\lambda^-}{2!}\int_y D^{A-}_{xy}\left(\left(\varphi^-_y\right)^2\psi^{-A'A''}_{y,z,w} + 2\varphi^-_y\psi^{-A'}_{y,z}\psi^{-A''}_{y,w}\right. \\ &\qquad \left. + \frac{\hbar}{i}\varphi^-_y\psi^{--A'A''}_{y,y,z,w} + \frac{\hbar}{i}\psi^{-A''}_{y,w}\psi^{--A'}_{y,y,z} + \frac{\hbar}{i}\psi^{-A'A''}_{y,z,w}\psi^{--}_{y,y} + \frac{\hbar}{i}\psi^{-A'}_{y,z}\psi^{--A''}_{y,y,w}\right)\end{aligned} \tag{4.14}$$

Above are the 8 second descendant equations, and notice that we truncated them at 1-loop order. This is in accordance with the truncation rule mentioned at the end of section 3.1.1.

$$\begin{aligned}\psi^{AA'A''A'''}_{x,z,w,v} &= -\frac{\lambda^+}{2!}\int_y D^{A+}_{xy}\left(\left(\varphi^+_y\right)^2\psi^{+A'A''A'''}_{y,z,w,v} + 2\psi^{+A'}_{y,z}\psi^{+A''}_{y,w}\psi^{+A'''}_{y,v}\right. \\ &\qquad \left. + 2\varphi^+_y\psi^{+A'}_{y,z}\psi^{+A''A'''}_{y,w,v} + 2\varphi^+_y\psi^{+A''}_{y,w}\psi^{+A'A'''}_{y,z,v} + 2\varphi^+_y\psi^{+A'''}_{y,v}\psi^{+A'A''}_{y,z,w}\right) \\ &\quad + \frac{\lambda^-}{2!}\int_y D^{A-}_{xy}\left(\left(\varphi^-_y\right)^2\psi^{-A'A''A'''}_{y,z,w,v} + 2\psi^{-A'}_{y,z}\psi^{-A''}_{y,w}\psi^{-A'''}_{y,v}\right. \\ &\qquad \left. + 2\varphi^-_y\psi^{-A'}_{y,z}\psi^{-A''A'''}_{y,w,v} + 2\varphi^-_y\psi^{-A''}_{y,w}\psi^{-A'A'''}_{y,z,v} + 2\varphi^-_y\psi^{-A'''}_{y,v}\psi^{-A'A''}_{y,z,w}\right)\end{aligned} \tag{4.15}$$

Above are the 16 third descendant equations truncated at tree order. We now have all the equations. Note that all these fields have their own $\hbar$-expansions. We shall now substitute the perturbiner expansions and obtain the recursion relations order by order.

§ 4.3 Obtaining the recursion relations

First let us list the perturbiner expansions for these new fields. The form remains the same as earlier, but the currents now gain indices.

$$\varphi^A_x = \sum_{\mathcal{P}} \Phi^A_{\mathcal{P}} e^{-ik_{\mathcal{P}}\cdot x}, \tag{4.16a}$$

$$\psi^{AB}_{x,y} = \int_p \Psi^{AB}_{p|\emptyset} e^{ip\cdot(x-y)} + \sum_{\mathcal{P}} \int_p \Psi^{AB}_{p|\mathcal{P}} e^{ip\cdot(x-y)} e^{-ik_{\mathcal{P}}\cdot x}, \tag{4.16b}$$

$$\psi^{ABC}_{x,y,z} = \sum_{\mathcal{P}} \int_{p,q} \Psi^{ABC}_{pq|\mathcal{P}} e^{ip\cdot(x-y)+iq\cdot(x-z)} e^{-ik_{\mathcal{P}}\cdot x}, \tag{4.16c}$$

$$\psi^{ABCD}_{x,y,z,w} = \int_{p,q,r} \Psi^{ABCD}_{pqr|\emptyset} e^{ip\cdot(x-y)+iq\cdot(x-z)+ir\cdot(x-w)} + \sum_{\mathcal{P}} \int_{p,q,r} \Psi^{ABCD}_{pqr|\mathcal{P}} e^{ip\cdot(x-y)+iq\cdot(x-z)+ir\cdot(x-w)} e^{-ik_{\mathcal{P}}\cdot x}. \tag{4.16d}$$

We shall take a step towards simplifying the cluttered notation. While listing $\hbar$-expansions of the Dyson–Schwinger equations or of the descendant equations, we adopt the following convention,

$$\varphi^{A,(0)}_x \to \varphi^A_0, \quad \psi^{AA',(0)}_{x,z} \to \psi^{AA'}_0, \quad \psi^{\pm A',(0)}_{y,z} \to \psi^{\pm A'}_0, \quad j^{\pm,(0)}_y \to j^{\pm}_0, \quad \cdots, \tag{4.17}$$

where we have dropped the position space indices completely in favour of loop order. One can easily recover those indices by examining $A, A', A'', \ldots$ or $\pm$ indices. If one observes the Dyson–Schwinger equations (or its descendants), one realises that the signature indices and position space indices are in some sense *linked*. A comes with x, A' comes with z, and so on; but $\pm$ always come with y[1]. We can also ignore the position space integral as its presence too can be inferred while recovering the position space indices. Also note that propagators are defined according to the following convention,

$$D^{AB}_{\mathcal{P}} \equiv \tilde{D}^{AB}(-k_{\mathcal{P}}), \tag{4.18a}$$

$$D^{AB}_{p|\mathcal{P}} \equiv \tilde{D}^{AB}(p - k_{\mathcal{P}}), \tag{4.18b}$$

$$D^{AB}_{pq|\mathcal{P}} \equiv \tilde{D}^{AB}(p + q - k_{\mathcal{P}}), \tag{4.18c}$$

$$D^{AB}_{p|\emptyset} \equiv \tilde{D}^{AB}(p), \tag{4.18d}$$

$$\vdots$$

§ 4.3.1 *Tree order*

Collecting the tree order terms in eq. (4.12),

$$\varphi^A_0 = \left[D^{A+}\left(j^+_0 - \frac{\lambda^+}{3!}\left(\varphi^+_0\right)^3 \right) - D^{A-}\left(j^-_0 - \frac{\lambda^-}{3!}\left(\varphi^-_0\right)^3 \right) \right]. \tag{4.19}$$

Substituting the perturbiner expansions gives us,

$$\Phi^{A,0}_{\mathcal{P}} = -\frac{\lambda^+}{3!} D^{A+}_{\mathcal{P}} \lceil \Phi^+\Phi^+\Phi^+ \rfloor^{(0)}_{\mathcal{P}} + \frac{\lambda^-}{3!} D^{A-}_{\mathcal{P}} \lceil \Phi^-\Phi^-\Phi^- \rfloor^{(0)}_{\mathcal{P}} \quad \forall\, |\mathcal{P}| > 1. \tag{4.20}$$

[1] With a single exception of $D^{A+}_{xz}\eta^{+A'}$ term in the first descendant equations.

To define the initial conditions we must first define the external sources as follows,

$$j_x^A = \sum_{i'=1}^{N} \eta^{AB} \tilde{\boldsymbol{K}}^{BC}(-k_{i'}) \varepsilon_{i'}^C e^{-ik_{i'}\cdot x}. \tag{4.21}$$

The definition looks a lot more complex than it was for regular ϕ^4 theory. But $\tilde{\boldsymbol{K}}^{AB}$ is the dressed kinetic operator for this theory and like we saw earlier, we don't need to know the explicit forms of the loop order corrections, and η was added to make our (sign) conventions compatible with regular ϕ^4 theory. The ε_i^A term is simply the polarization of the i^{th} leg. For theories with more than one kinds of fields, polarization *vectors* tend to make an appearance as they allow us to set the choice of fields at different external legs. In our problem, we shall isolate the u-channel 2-to-2 scattering and hence fix the polarizations as follows,

$$\varepsilon_1^+ = 1, \quad \varepsilon_2^+ = 0, \quad \varepsilon_3^+ = 0, \quad \varepsilon_1^- = 0, \quad \varepsilon_2^- = 1, \quad \varepsilon_3^- = 1, \tag{4.22}$$

which means that the field on the 1$^{\text{st}}$ leg is ϕ^+, and the fields on the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ legs are ϕ^-[2]. We now derive the initial conditions,

$$\sum_{i'=1}^{N} \Phi_{i'}^{A,0} e^{-ik_{i'}\cdot x} = \int_y D_{xy}^{A+} j_0^+ - D_{xy}^{A-} j_0^-, \tag{4.23a}$$

$$= \int_{y,p} e^{ip\cdot(x-y)} \Big(\tilde{D}_p^{A+} j_0^+ - \tilde{D}_p^{A-} j_0^- \Big), \tag{4.23b}$$

$$= \sum_{i'=1}^{N} \int_{y,p} e^{ip\cdot(x-y)} e^{-ik_{i'}y} \Big(\tilde{D}_p^{A+} \eta^{++} \tilde{\boldsymbol{K}}^{+B,0}(-k_{i'}) \varepsilon_{i'}^B - \tilde{D}_p^{A-} \eta^{--} \tilde{\boldsymbol{K}}^{-B,0}(-k_{i'}) \varepsilon_{i'}^B \Big), \tag{4.23c}$$

performing the position space integration,

$$= (2\pi)^4 \sum_{i'=1}^{N} \int_p \delta^4(p + k_{i'}) e^{ip\cdot x} \Big(\tilde{D}_p^{A+} \tilde{K}^{+B}(-k_{i'}) \varepsilon_{i'}^B + \tilde{D}_p^{A-} \tilde{K}^{-B}(-k_{i'}) \varepsilon_{i'}^B \Big), \tag{4.23d}$$

from the structure of the kinetic operator we have,

$$= (2\pi)^4 \sum_{i'=1}^{N} \int_p \delta^4(p + k_{i'}) e^{ip\cdot x} \Big(\tilde{D}_p^{A+} \tilde{K}^{++}(-k_{i'}) \varepsilon_{i'}^+ + \tilde{D}_p^{A-} \tilde{K}^{--}(-k_{i'}) \varepsilon_{i'}^- \Big), \tag{4.23e}$$

[2]The off-shell leg has ϕ^+ field and hence $\varepsilon_4^+ = 1$, but in our framework this fact comes into play later and will be brought up while discussing amputation.

performing the momentum space integration gives,

$$= \sum_{i'=1}^{N} e^{-ik_{i'}\cdot x}\Big(\tilde{D}^{A+}(-k_{i'})\tilde{K}^{++}(-k_{i'})\varepsilon_{i'}^{+} + \tilde{D}^{A-}(-k_{i'})\tilde{K}^{--}(-k_{i'})\varepsilon_{i'}^{-}\Big), \tag{4.23f}$$

$$= \sum_{i'=1}^{N}\Big(\delta^{A+}\varepsilon_{i'}^{+} + \delta^{A-}\varepsilon_{i'}^{-}\Big)e^{-ik_{i'}\cdot x}, \tag{4.23g}$$

$$= \sum_{i'=1}^{N}\varepsilon_{i'}^{A}e^{-ik_{i'}\cdot x}. \tag{4.23h}$$

Comparing coefficients on both sides,

$$\Phi_{i'}^{A,0} = \varepsilon_{i'}^{A}. \tag{4.23i}$$

The derivation is rather straightforward with just one thing that we must pay attention to, in eq. (4.23d) we made the substitution $\tilde{\boldsymbol{K}}^{0} \to \tilde{K}$. This is justified because the tree order dressed propagator is simply the *undressed* propagator[3] and likewise, the tree order kinetic operator must be equal to the undressed kinetic operator.

§ 4.3.2 *1-loop order*

Collecting 1-loop order terms in eq. (4.12) and tree order terms in eq. (4.13),

$$\begin{aligned}\varphi_1^{A} = \Bigg[&D^{A+}\Big(j_1^{+} - \frac{\lambda^{+}}{2!}\Big(\big(\varphi_0^{+}\big)^2\varphi_1^{+} + \varphi_0^{+}\psi_0^{++}\Big)\Big)\\ &-D^{A-}\Big(j_1^{-} - \frac{\lambda^{-}}{2!}\Big(\big(\varphi_0^{-}\big)^2\varphi_1^{-} + \varphi_0^{-}\psi_0^{--}\Big)\Big)\Bigg],\end{aligned} \tag{4.24a}$$

$$\psi_0^{AA'} = D^{AB}\eta^{BA'} - \frac{\lambda^{+}}{2!}D^{A+}\big(\varphi_0^{+}\big)^2\psi_0^{+A'} + \frac{\lambda^{-}}{2!}D^{A-}\big(\varphi_0^{-}\big)^2\psi_0^{-A'}. \tag{4.24b}$$

On substituting the perturbiner expansions we get,

$$\begin{aligned}\Phi_{\mathcal{P}}^{A,1} = &-\frac{\lambda^{+}}{2!}D_{\mathcal{P}}^{A+}\Big(\frac{1}{3}\big\lceil\Phi^{+}\Phi^{+}\Phi^{+}\big\rfloor_{\mathcal{P}}^{(1)} + \int_p\big\lceil\Phi^{+}\Psi_p^{++}\big\rfloor_{\mathcal{P}}^{(0)}\Big)\\ &+\frac{\lambda^{-}}{2!}D_{\mathcal{P}}^{A-}\Big(\frac{1}{3}\big\lceil\Phi^{-}\Phi^{-}\Phi^{-}\big\rfloor_{\mathcal{P}}^{(1)} + \int_p\big\lceil\Phi^{-}\Psi_p^{--}\big\rfloor_{\mathcal{P}}^{(0)}\Big)\end{aligned} \qquad \forall\,|\mathcal{P}| > 1, \tag{4.25a}$$

$$\Psi_{p|\mathcal{P}}^{AA',0} = -\frac{\lambda^{+}}{2!}D_{p|\mathcal{P}}^{A+}\big\lceil\Phi^{+}\Phi^{+}\Psi_p^{+A'}\big\rfloor_{\mathcal{P}}^{(0)} + \frac{\lambda^{-}}{2!}D_{p|\mathcal{P}}^{A-}\big\lceil\Phi^{-}\Phi^{-}\Psi_p^{-A'}\big\rfloor_{\mathcal{P}}^{(0)} \quad \forall\,|\mathcal{P}| \geq 1. \tag{4.25b}$$

Like before we make the assumption that,

$$\Phi_i^{A,n} = 0 \quad \forall\, n \geq 1. \tag{4.26}$$

[3]The 'dress' in *dressed* refers to loop order corrections that the *bare* or *undressed* propagator receives.

And it is not too hard to find out the initial conditions for $\Psi^{AA',0}_{p|\emptyset}$,

$$\int_p \Psi^{AA',0}_{p|\emptyset} e^{ip\cdot(x-z)} = D^{AB}_{xz}\eta^{BA'}, \tag{4.27a}$$

$$= \int_p \tilde{D}^{AB}_{p} e^{ip\cdot(x-z)}\eta^{BA'}. \tag{4.27b}$$

Comparing both sides,

$$\Psi^{AA',0}_{p|\emptyset} = D^{AB}_{p|\emptyset}\eta^{BA'}. \tag{4.27c}$$

§ 4.3.3 *2- and 3-loop orders*

It is now possible to use the initial conditions from earlier to derive the higher order initial conditions directly from the recursion relations. Collecting 2-loop order terms in eq. (4.12), 1-loop order terms in eq. (4.13) and tree order terms in eq. (4.14),

$$\begin{aligned}\varphi^A_2 = \Bigg[& D^{A+}\left(j^+_2 - \frac{\lambda^+}{2!}\left(\left(\varphi^+_0\right)^2\varphi^+_2 + \left(\varphi^+_1\right)^2\varphi^+_0 + \varphi^+_0\psi^{++}_1 + \varphi^+_1\psi^{++}_0 + \frac{1}{3}\psi^{+++}_0\right)\right) \\ & - D^{A-}\left(j^-_2 - \frac{\lambda^-}{2!}\left(\left(\varphi^-_0\right)^2\varphi^-_2 + \left(\varphi^-_1\right)^2\varphi^-_0 + \varphi^-_0\psi^{--}_1 + \varphi^-_1\psi^{--}_0 + \frac{1}{3}\psi^{---}_0\right)\right)\Bigg],\end{aligned} \tag{4.28a}$$

$$\begin{aligned}\psi^{AA'}_1 = & -\frac{\lambda^+}{2!}D^{A+}\left(2\varphi^+_0\varphi^+_1\psi^{+A'}_0 + \left(\varphi^+_0\right)^2\psi^{+A'}_1 + \phi^+_0\psi^{++A'}_0 + \psi^{++}_0\psi^{+A'}_0\right) \\ & + \frac{\lambda^-}{2!}D^{A-}\left(2\varphi^-_0\varphi^-_1\psi^{-A'}_0 + \left(\varphi^-_0\right)^2\psi^{-A'}_1 + \phi^-_0\psi^{--A'}_0 + \psi^{--}_0\psi^{-A'}_0\right),\end{aligned} \tag{4.28b}$$

$$\psi^{AA'A''}_0 = -\frac{\lambda^+}{2!}D^{A+}\left(\left(\varphi^+_0\right)^2\psi^{+A'A''}_0 + 2\varphi^+_0\psi^{+A'}_0\psi^{+A''}_0\right) + \frac{\lambda^-}{2!}D^{A-}\left(\left(\varphi^-_0\right)^2\psi^{-A'A''}_0 + 2\varphi^-_0\psi^{-A'}_0\psi^{-A''}_0\right). \tag{4.28c}$$

Substituting perturbiner equations to obtain 2-loop order recursion relations,

$$\begin{aligned}\Phi^{A,2}_{\mathcal{P}} = & -\frac{\lambda^+}{2!}D^{A+}_{\mathcal{P}}\left(\frac{1}{3}\lceil\Phi^+\Phi^+\Phi^+\rfloor^{(2)}_{\mathcal{P}} + \int_p\left(\lceil\Phi^+\Psi^{++}_p\rfloor^{(1)}_{\mathcal{P}} + \int_q\frac{1}{3}\Psi^{+++,0}_{pq|\mathcal{P}}\right)\right) \\ & + \frac{\lambda^-}{2!}D^{A-}_{\mathcal{P}}\left(\frac{1}{3}\lceil\Phi^-\Phi^-\Phi^-\rfloor^{(2)}_{\mathcal{P}} + \int_p\left(\lceil\Phi^-\Psi^{--}_p\rfloor^{(1)}_{\mathcal{P}} + \int_q\frac{1}{3}\Psi^{---,0}_{pq|\mathcal{P}}\right)\right)\end{aligned} \qquad \forall\,|\mathcal{P}| > 1, \tag{4.29a}$$

$$\begin{aligned}\Psi^{AA',1}_{p|\mathcal{P}} = & -\frac{\lambda^+}{2!}D^{A+}_{p|\mathcal{P}}\left(\lceil\Phi^+\Phi^+\Psi^{+A'}_p\rfloor^{(1)}_{\mathcal{P}} + \int_q\left(\lceil\Phi^+\Psi^{++A'}_{pq}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{++}_q\Psi^{+A'}_p\rfloor^{(0)}_{\mathcal{P}}\right)\right) \\ & + \frac{\lambda^-}{2!}D^{A-}_{p|\mathcal{P}}\left(\lceil\Phi^-\Phi^-\Psi^{-A'}_p\rfloor^{(1)}_{\mathcal{P}} + \int_q\left(\lceil\Phi^-\Psi^{--A'}_{pq}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{--}_q\Psi^{-A'}_p\rfloor^{(0)}_{\mathcal{P}}\right)\right)\end{aligned} \qquad \forall\,|\mathcal{P}| \geq 0, \tag{4.29b}$$

$$\Psi^{AA'A'',0}_{pq|\mathcal{P}} = -\frac{\lambda^+}{2!}D^{A+}_{pq|\mathcal{P}}\left(\lceil\Phi^+\Phi^+\Psi^{+A'A''}_{pq}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^+\Psi^{+A'}_{p}\Psi^{+A''}_{q}\rfloor^{(0)}_{\mathcal{P}}\right) + \frac{\lambda^-}{2!}D^{A-}_{pq|\mathcal{P}}\left(\lceil\Phi^-\Phi^-\Psi^{-A'A''}_{pq}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^-\Psi^{-A'}_{p}\Psi^{-A''}_{q}\rfloor^{(0)}_{\mathcal{P}}\right) \quad \forall\,|\mathcal{P}| \geq 1. \tag{4.29c}$$

Collecting 3-loop order terms in eq. (4.12), 2-loop order terms in eq. (4.13), 1-loop order terms in eq. (4.14) and tree order terms in eq. (4.15),

$$\begin{aligned}\varphi^A_3 = \Big[& D^{A+}\Big(j^+_3 - \frac{\lambda^+}{3!}\Big((\varphi^+_1)^3 + 3!\varphi^+_0\varphi^+_1\varphi^+_2 + 3(\varphi^+_0)^2\varphi^+_3 \\ & +3\varphi^+_2\psi^{++}_0 + 3\varphi^+_1\psi^{++}_1 + 3\varphi^+_0\psi^{++}_2 + \psi^{+++}_1\Big)\Big) \\ & - D^{A-}\Big(j^-_3 - \frac{\lambda^-}{3!}\Big((\varphi^-_1)^3 + 3!\varphi^-_0\varphi^-_1\varphi^-_2 + 3(\varphi^-_0)^2\varphi^-_3 \\ & +3\varphi^-_2\psi^{--}_0 + 3\varphi^-_1\psi^{--}_1 + 3\varphi^-_0\psi^{--}_2 + \psi^{---}_1\Big)\Big)\Big]\end{aligned}, \tag{4.30a}$$

$$\begin{aligned}\psi^{AA'}_2 = & -\frac{\lambda^+}{2!}D^{A+}\Big((\varphi^+_0)^2\psi^{+A'}_2 + 2\varphi^+_0\varphi^+_1\psi^{+A'}_1 + 2\varphi^+_0\varphi^+_2\psi^{+A'}_0 + (\varphi^+_1)^2\psi^{+A'}_0 \\ & +\varphi^+_1\psi^{++A'}_0 + \varphi^+_0\psi^{++A'}_1 + \psi^{++}_0\psi^{+A'}_1 + \psi^{++}_1\psi^{+A'}_0 + \frac{1}{3}\psi^{+++A'}_0\Big) \\ & + \frac{\lambda^-}{2!}D^{A-}\Big((\varphi^-_0)^2\psi^{-A'}_2 + 2\varphi^-_0\varphi^-_1\psi^{-A'}_1 + 2\varphi^-_0\varphi^-_2\psi^{-A'}_0 + (\varphi^-_1)^2\psi^{-A'}_0 \\ & +\varphi^-_1\psi^{--A'}_0 + \varphi^-_0\psi^{--A'}_1 + \psi^{--}_0\psi^{-A'}_1 + \psi^{--}_1\psi^{-A'}_0 + \frac{1}{3}\psi^{---A'}_0\Big)\end{aligned}, \tag{4.30b}$$

$$\begin{aligned}\psi^{AA'A''}_1 = & -\frac{\lambda^+}{2!}D^{A+}\Big((\varphi^+_0)^2\psi^{+A'A''}_1 + 2\varphi^+_0\varphi^+_1\psi^{+A'A''}_0 + 2\varphi^+_1\psi^{+A'}_0\psi^{+A''}_0 + 2\varphi^+_0\psi^{+A'}_1\psi^{+A''}_0 \\ & +2\varphi^+_0\psi^{+A'}_0\psi^{+A''}_1 + \varphi^+_0\psi^{++A'A''}_0 + \psi^{+A'}_0\psi^{++A''}_0 + \psi^{+A''}_0\psi^{++A'}_0 + \psi^{++}_0\psi^{+A'A''}_0\Big) \\ & + \frac{\lambda^-}{2!}D^{A-}\Big((\varphi^-_0)^2\psi^{-A'A''}_1 + 2\varphi^-_0\varphi^-_1\psi^{-A'A''}_0 + 2\varphi^-_1\psi^{-A'}_0\psi^{-A''}_0 + 2\varphi^-_0\psi^{-A'}_1\psi^{-A''}_0 \\ & +2\varphi^-_0\psi^{-A'}_0\psi^{-A''}_1 + \varphi^-_0\psi^{--A'A''}_0 + \psi^{-A'}_0\psi^{--A''}_0 + \psi^{-A''}_0\psi^{--A'}_0 + \psi^{--}_0\psi^{-A'A''}_0\Big)\end{aligned}, \tag{4.30c}$$

$$\begin{aligned}\psi^{AA'A''A'''}_0 = & -\frac{\lambda^+}{2!}D^{A+}\Big((\varphi^+_0)^2\psi^{+A'A''A'''}_0 + 2\psi^{+A'}_0\psi^{+A''}_0\psi^{+A'''}_0 \\ & +2\varphi^+_0\psi^{+A'}_0\psi^{+A''A'''}_0 + 2\varphi^+_0\psi^{+A''}_0\psi^{+A'A'''}_0 + 2\varphi^+_0\psi^{+A'''}_0\psi^{+A'A''}_0\Big) \\ & + \frac{\lambda^-}{2!}D^{A-}\Big((\varphi^-_0)^2\psi^{-A'A''A'''}_0 + 2\psi^{-A'}_0\psi^{-A''}_0\psi^{-A'''}_0 \\ & +2\varphi^-_0\psi^{-A'}_0\psi^{-A''A'''}_0 + 2\varphi^-_0\psi^{-A''}_0\psi^{-A'A'''}_0 + 2\varphi^-_0\psi^{-A'''}_0\psi^{-A'A''}_0\Big)\end{aligned}. \tag{4.30d}$$

Substituting perturbiner expansions one last time,

$$\Phi^{A,3}_{\mathcal{P}} = -\frac{\lambda^+}{2!}D^{A+}_{\mathcal{P}}\left(\frac{1}{3}\lceil\Phi^+\Phi^+\Phi^+\rfloor^{(3)}_{\mathcal{P}} + \int_p\left(\lceil\Phi^+\Psi^{++}_p\rfloor^{(2)}_{\mathcal{P}} + \int_q\frac{1}{3}\Psi^{+++,1}_{pq|\mathcal{P}}\right)\right) + \frac{\lambda^-}{2!}D^{A-}_{\mathcal{P}}\left(\frac{1}{3}\lceil\Phi^-\Phi^-\Phi^-\rfloor^{(3)}_{\mathcal{P}} + \int_p\left(\lceil\Phi^-\Psi^{--}_p\rfloor^{(2)}_{\mathcal{P}} + \int_q\frac{1}{3}\Psi^{---,1}_{pq|\mathcal{P}}\right)\right) \qquad \forall\,|\mathcal{P}| > 1, \quad (4.31a)$$

$$\Psi^{AA',2}_{p|\mathcal{P}} = -\frac{\lambda^+}{2!}D^{A+}_{p|\mathcal{P}}\left(\lceil\Phi^+\Phi^+\Psi^{+A'}_p\rfloor^{(2)}_{\mathcal{P}} + \int_q\left(\lceil\Phi^+\Psi^{++A'}_{pq}\rfloor^{(1)}_{\mathcal{P}} + \lceil\Psi^{++}_q\Psi^{+A'}_p\rfloor^{(1)}_{\mathcal{P}} + \int_r\frac{1}{3}\Psi^{+++A',0}_{pqr|\mathcal{P}}\right)\right) + \frac{\lambda^-}{2!}D^{A-}_{p|\mathcal{P}}\left(\lceil\Phi^-\Phi^-\Psi^{-A'}_p\rfloor^{(2)}_{\mathcal{P}} + \int_q\left(\lceil\Phi^-\Psi^{--A'}_{pq}\rfloor^{(1)}_{\mathcal{P}} + \lceil\Psi^{--}_q\Psi^{-A'}_p\rfloor^{(1)}_{\mathcal{P}} + \int_r\frac{1}{3}\Psi^{---A',0}_{pqr|\mathcal{P}}\right)\right) \qquad \forall\,|\mathcal{P}| \geq 0, \quad (4.31b)$$

$$\Psi^{AA'A'',1}_{pq|\mathcal{P}} = -\frac{\lambda^+}{2!}D^{A+}_{pq|\mathcal{P}}\left(\lceil\Phi^+\Phi^+\Psi^{+A'A''}_{pq}\rfloor^{(1)}_{\mathcal{P}} + 2\lceil\Phi^+\Psi^{+A'}_p\Psi^{+A''}_q\rfloor^{(1)}_{\mathcal{P}} + \int_r\left(\lceil\Phi^+\Psi^{++A'A''}_{pqr}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{+A'}_p\Psi^{++A''}_{qr}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{+A''}_q\Psi^{++A'}_{pr}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{++}_r\Psi^{+A'A''}_{pq}\rfloor^{(0)}_{\mathcal{P}}\right)\right) + \frac{\lambda^-}{2!}D^{A-}_{pq|\mathcal{P}}\left(\lceil\Phi^-\Phi^-\Psi^{-A'A''}_{pq}\rfloor^{(1)}_{\mathcal{P}} + 2\lceil\Phi^-\Psi^{-A'}_p\Psi^{-A''}_q\rfloor^{(1)}_{\mathcal{P}} + \int_r\left(\lceil\Phi^-\Psi^{--A'A''}_{pqr}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{-A'}_p\Psi^{--A''}_{qr}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{-A''}_q\Psi^{--A'}_{pr}\rfloor^{(0)}_{\mathcal{P}} + \lceil\Psi^{--}_r\Psi^{-A'A''}_{pq}\rfloor^{(0)}_{\mathcal{P}}\right)\right) \qquad \forall\,|\mathcal{P}| \geq 1, \quad (4.31c)$$

$$\Psi^{AA'A''A''',0}_{pqr|\mathcal{P}} = -\frac{\lambda^+}{2!}D^{A+}_{pqr|\mathcal{P}}\left(\lceil\Phi^+\Phi^+\Psi^{+A'A''A'''}_{pqr}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Psi^{+A'}_p\Psi^{+A''}_q\Psi^{+A'''}_r\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^+\Psi^{+A'}_p\Psi^{+A''A'''}_{qr}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^+\Psi^{+A''}_q\Psi^{+A'A'''}_{pr}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^+\Psi^{+A'''}_r\Psi^{+A'A''}_{pq}\rfloor^{(0)}_{\mathcal{P}}\right) + \frac{\lambda^-}{2!}D^{A-}_{pqr|\mathcal{P}}\left(\lceil\Phi^-\Phi^-\Psi^{-A'A''A'''}_{pqr}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Psi^{-A'}_p\Psi^{-A''}_q\Psi^{-A'''}_r\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^-\Psi^{-A'}_p\Psi^{-A''A'''}_{qr}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^-\Psi^{-A''}_q\Psi^{-A'A'''}_{pr}\rfloor^{(0)}_{\mathcal{P}} + 2\lceil\Phi^-\Psi^{-A'''}_r\Psi^{-A'A''}_{pq}\rfloor^{(0)}_{\mathcal{P}}\right) \qquad \forall\,|\mathcal{P}| \geq 0. \quad (4.31d)$$

Having produced all the recursion relations up to 3-loop orders, we can proceed to solving them. The next step would be to obtain the results, simplify them and demonstrate unitarity.

§ 4.4 Interpretation of results

We work our way to obtaining the cross-section from amplitudes. First, we perform the amputation.

§ 4.4.1 *Amputation*

With this theory, amputation is not as straightforward because we now have 4 propagators to potentially set to identity and 2 off-shell currents to potentially amputate. Picking the correct propagator to set to identity depends on the *signature* of the off-shell current and picking the signature of the current to amputate in turn depends on the desired polarization of the off-shell leg. The meaning of $\Phi^{A}_{1\cdots N}$ is that we have N on-shell legs $(1,\ldots,N)$ each of whose polarizations have been set and the polarization of the off-shell leg is given by A. In our example, we are interested in the u-channel of 2-to-2 scattering and hence wish to set the polarization of the 4^{th} leg as that of the 1^{st} leg. Therefore we amputate the Φ^{+}_{123} currents. And we set D^{++}_{123} to identity (line joining λ^{+} vertex and + external state.). With the following replacements

$$D^{++}_{123} \to 1, \quad D^{+-}_{123} \to 0, \quad D^{-+}_{123} \to 0, \quad D^{--}_{123} \to 0, \quad (D^{++}_{123})^{l} \to 0 \ (\forall\, l > 1), \tag{4.32}$$

amputation is complete and we shall obtain our amplitudes. Next we shall filter out terms that don't contribute to physical cuts that we desire.

§ 4.4.2 *Filtering physical cuts*

We calculated the 4-point amplitude up to 3-loop orders because we aim to obtain the 6-point tree order cross-section. This can understood from power counting of the coupling constant (in the regular theory),

$$\mathcal{M}_4 = \lambda\bar{\mathcal{M}}^{(0)}_4 + \lambda^2\bar{\mathcal{M}}^{(1)}_4 + \lambda^3\bar{\mathcal{M}}^{(2)}_4 + \lambda^4\bar{\mathcal{M}}^{(3)}_4 + \mathcal{O}(\lambda^5), \tag{4.33a}$$
$$\mathcal{M}_6 = \lambda^2\bar{\mathcal{M}}^{(0)}_6 + \lambda^3\bar{\mathcal{M}}^{(1)}_6 + \lambda^4\bar{\mathcal{M}}^{(2)}_6 + +\mathcal{O}(\lambda^5), \tag{4.33b}$$
$$\mathcal{M}_8 = \lambda^3\bar{\mathcal{M}}^{(0)}_8 + \lambda^4\bar{\mathcal{M}}^{(1)}_8 + \mathcal{O}(\lambda^5), \tag{4.33c}$$
$$\vdots$$

Above we expanded the amplitude in terms of the coupling constant and $\bar{\mathcal{M}}$ represents the part of the amplitude stripped of the coupling constant. It should be apparent from above that

$$\mathcal{M}^{(3)}_4 \overset{\text{cuts}}{\cong} \mathcal{M}^{(0)}_6\left(\mathcal{M}^{(0)}_6\right)^{*} + \mathcal{M}^{(1)}_4\left(\mathcal{M}^{(1)}_4\right)^{*} + \mathcal{M}^{(0)}_4\left(\mathcal{M}^{(2)}_4\right)^{*} + \mathcal{M}^{(2)}_4\left(\mathcal{M}^{(0)}_4\right)^{*}. \tag{4.34}$$

The absence of $\mathcal{M}_4^{(0)}\left(\mathcal{M}_8^{(0)}\right)^*$ and $\mathcal{M}_4^{(0)}\left(\mathcal{M}_6^{(1)}\right)^*$ terms is a consequence of the fact that unitarity cuts are feasible only if the intermediate states are the same on both sides of the cut. We desire to only isolate terms on the LHS of eq. (4.34) that lead to the $\mathcal{M}_6^{(0)}\left(\mathcal{M}_6^{(0)}\right)^*$ (henceforth referred to as the tree-tree) structure. After isolating such terms we could perform the cutting procedure and

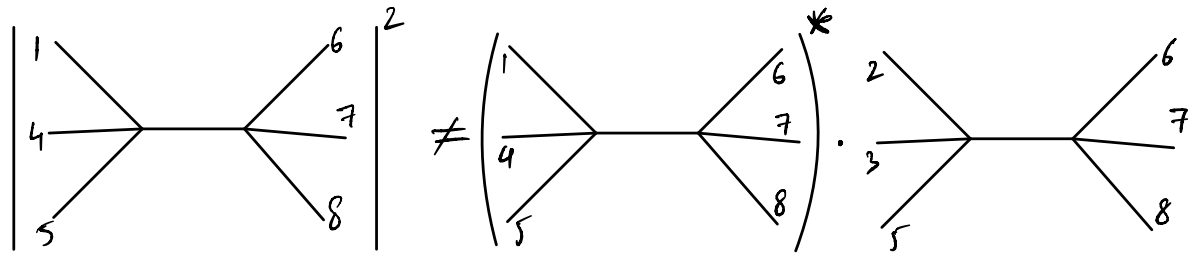

Figure 4.1: *The term cross-section refers to LHS above, whereas what we obtain is the RHS. By identification one may relate the two, but there is a subtle difference.*

obtain the cross-section. Let us illustrate this with the example of the 1-loop order amplitude. Similar to eq. (4.34), we have at 1-loop,

$$\mathcal{M}_4^{(1)} \overset{\text{cuts}}{\cong} \mathcal{M}_4^{(0)}\left(\mathcal{M}_4^{(0)}\right)^*. \tag{4.35}$$

The result we obtain for the 1-loop order field-doubled amplitude is,

$$i\mathcal{M}_{4,u}^{(0)} = \int_p \frac{1}{2}\Delta+^p_{23}\Delta-^p\lambda^+\lambda^-, \tag{4.36}$$

where in our code, we used the following convention,

$$\begin{pmatrix} D^{++}_{pqr|\mathcal{P}} & D^{+-}_{pqr|\mathcal{P}} \\ D^{-+}_{pqr|\mathcal{P}} & D^{--}_{pqr|\mathcal{P}} \end{pmatrix} \equiv \begin{pmatrix} D+^{pqr}_{\mathcal{P}} & -\Delta+^{pqr}_{\mathcal{P}} \\ \Delta-^{pqr}_{\mathcal{P}} & -D-^{pqr}_{\mathcal{P}} \end{pmatrix}. \tag{4.37}$$

And $D+, D-, \Delta+$ and $\Delta-$ propagators behave like the ones described in section 2.2.2. The Δ propagators denote the physical cut and the result should be a product of terms with $D+, \lambda^+$ on one side and $D-, \lambda^-$ on the other (representing the $(i\mathcal{T})^\dagger$ and $(i\mathcal{T})$ sides of the optical theorem respectively). The cutting procedure is as follows,

$$\int_p \frac{1}{2}\Delta+^p_{23}\Delta-^p\lambda^+\lambda^- = \frac{1}{2}\not{\sum}\lambda^+\lambda^-, \tag{4.38a}$$

where the loop integral has been replaced with a phase space integral over two physical states after substitution of the expression for the Δ propagators. The role of this phase space integral is to pick out as many on-shell states as the number of propagators cut. We introduce a summation over these on-shell states to *distribute* them across lines being cut. In our case this translates to two external states labelled by say, 5 and 6, being each assigned the loop momentum p after carrying out the phase space integral,

$$= \frac{1}{2}\Big[\left(\lambda^+\lambda^-\right)_{p\to 5} + \left(\lambda^+\lambda^-\right)_{p\to 6}\Big], \tag{4.38b}$$

$$= \lambda^+\lambda^-, \tag{4.38c}$$

$$= \Big[\Big(i\mathcal{M}^{(0)}(1^+4^+5^+6^+)\Big) \times \Big(-i\mathcal{M}^{(0)}(2^-3^-5^-6^-)\Big)\Big]. \tag{4.38d}$$

We just showed the algebraic method of performing the cut. There was no need to draw a

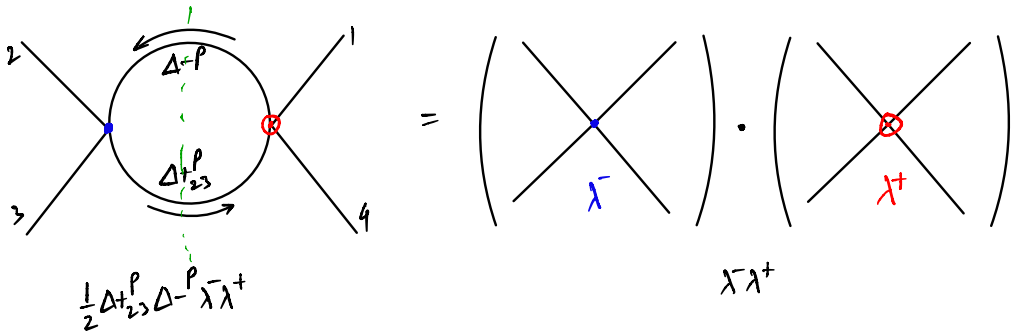

Figure 4.2: *The diagrammatic representation of eq.* (4.38)

Feynman diagram and find the right cut diagram. However, the case of 1-loop amplitude is very simple. We didn't have to filter any terms. But in the case of the 3-loop amplitude we must apply a filter. First using power counting we shall define a canonical form for terms that have the physical cut. But this constaint is not enough to completely discard unphysical cuts. Then we define a filter using the structure of loop momenta in these terms. Together these filters are sufficient to isolate the necessary terms.

First let us discuss how one can isolate exactly those terms which contribute to tree-tree structure. We shall make use of the well-known fact that for any connected tree with n vertices, there are $n-1$ edges. Now let us focus on one side of the cut, *i.e.* the side with $D+$ propagators and λ^+ vertices. From this theorem we have the constraint that post-cutting, we must be left with $(D+)^{n-1}(\lambda^+)^n$ structure. Likewise for the other side we must have $(D-)^{n-1}(\lambda^-)^n$. And using the fact that for a theory with ϕ^n-like interaction we have the following relations

$$L = P - V + 1, \tag{4.39a}$$

$$nV = N + 2P, \tag{4.39b}$$

eliminating V,

$$L = P - \frac{(N+2P)}{n} + 1, \tag{4.39c}$$

$$(L-1) + \frac{N}{n} = P\left(\frac{n-2}{n}\right), \tag{4.39d}$$

$$P = \frac{n(L-1)+N}{n-2}. \tag{4.39e}$$

where L is the number of loops, V is the number of vertices, P is the number of internal lines and N is the number of external lines. And finally recalling eq. (4.34), we use the correspondence between $(2n+1)$-loop 4-point amplitude and the $2(n+2)$-point tree order cross-section. Putting all of these mathematical facts together gives us a canonical form for diagrams that contribute to cuts,

$$f\left(\lambda^+\lambda^-\right)^{(n+1)/2} \underbrace{(D+\cdots D+)}_{\frac{n-1}{2}\text{ terms}} \underbrace{(\Delta\cdots\Delta)}_{n+1\text{ terms}} \underbrace{(D-\cdots D-)}_{\frac{n-1}{2}\text{ terms}}$$

where f is the symmetry factor attached to the term and n is the loop order of the field-doubled theory. This formula is valid for all theories with a 4-point interaction vertex with 4 external legs (2-to-2 scattering). However, not all terms of this form give us 'physical' cuts. Some terms have *unphysical* cuts which don't contribute to the RHS of eq. (2.15). An additional filter is needed.

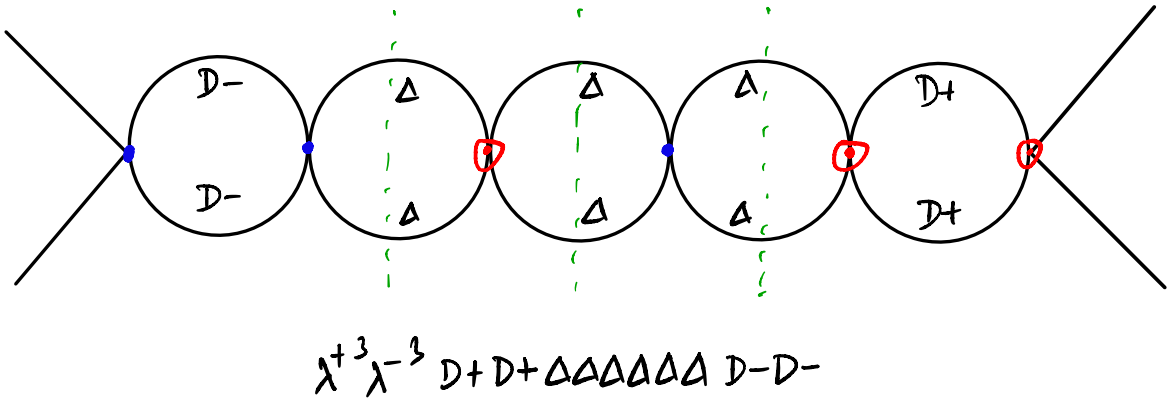

Figure 4.3: *This diagram has the canonical form, but we see that it allows for unphysical cuts.*

We propose a filter based on the momenta structure of Δ propagators. By this we mean that we shall examine the momenta carried by these propagators. Let us take the example of 3-loop order terms, then we have the following momenta

$$\underbrace{p, q, r}_{\text{loop momenta}} \quad , \quad \underbrace{k_1, k_2, k_3}_{\text{external momenta}}$$

then using the loop momenta as a basis, we check the linear independence of the momenta of the 4 Δ propagators. If the smallest linearly independent set is smaller than the basis, we discard the

term. For example,

$$\frac{1}{8}(\lambda^+)^2(\lambda^-)^2 D{+}^{p}\Delta{+}^{r}\Delta{+}^{r}_{23}\Delta{-}^{r}\Delta{-}^{r}D{-}^{q} \text{ is discarded,}$$
$$\text{but } \frac{1}{12}(\lambda^+)^2(\lambda^-)^2 D{+}^{r}\Delta{+}^{pq}_{3}\Delta{+}^{qr}_{123}\Delta{-}^{q}\Delta{-}^{pqr}_{13}D{-}^{p} \text{ is kept.}$$

In the former term, there is only r but in the latter term we have[4] $p+q$, $q+r$, $-q$ and $-p-q-r$ which can be row-reduced to the basis. The filtering criterion is that for the canonical n-loop order terms, the momenta structure for $(n+1)$ Δ propagators must of rank-n in the sense explained above. With these two filters in place, we successfully isolate the diagrams that reproduce the tree-tree structure. Next we shall discuss the process of loop momenta shifting.

§ 4.4.3 *Loop momenta shifting*

Unlike in regular ϕ^4 theory, we have a lot more granular information encoded in terms comprising the amplitude. It is possible for us to fix a canonical form for the Δ propagators that acts as a constraint, while leaving $D+$ and $D-$ propagators *free*. We pick the form,

$$\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}$$

and rename (as discussed in section 3.4.3) the loop momenta p, q, and r in each term to achieve this form[5]. The terms now essentially only differ in the momenta configurations of $D+$ and $D-$ propagators. This results in a very good constraint (at least for 3-loop order) and classifies all the topologies up to permutations (in p, q and r). We go one step further and fix a preferred permutation to get rid of that redundancy as well. This finally leaves us with the following terms of interest,

$$\begin{aligned}\frac{i\mathcal{M}'^{(3)}_4}{(\lambda^-\lambda^+)^2} = &\frac{1}{6}D{+}^{p}_{23}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{p}_{23} + \frac{1}{2}D{+}^{p}_{23}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{q}_{23}\\
&+ D{+}^{pq}_{123}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{pr}_{2} + \frac{1}{2}D{+}^{pq}_{123}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{p}_{23}\\
&+ \frac{1}{2}D{+}^{pq}_{1}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{p}_{23} + \frac{1}{2}D{+}^{p}_{23}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{pq}_{2}\,.\\
&+ \frac{1}{2}D{+}^{p}_{23}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{pq}_{3} + \frac{1}{4}D{+}^{pq}_{1}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{pq}_{2}\\
&+ \frac{1}{4}D{+}^{pq}_{1}\Delta{+}^{pqr}_{23}\Delta{-}^{p}\Delta{-}^{q}\Delta{-}^{r}D{-}^{pq}_{3}\end{aligned} \tag{4.40}$$

[4] $\Delta{+}(p) = \Delta{-}(-p)$.

[5] There are multiple ways of renaming momenta of Δ propagators so that we end up with the canonical form. This is because the canonical form is invariant under permutations of p, q and r. But it suffices to pick any one of such renamings.

$\mathcal{M}'^{(3)}_4$ denotes the filtered physical terms from $\mathcal{M}^{(3)}_4$. Further simplification may be possible but we haven't explored that direction yet. We shall now try and obtain the cross-section from above.

§ 4.4.4 *From amplitude to cross-section*

In order to obtain the tree-tree form, we need to take care of a few things. Recall that the polarizations set for the problem are,

$$1^+, \quad 4^+, \quad 2^-, \quad 3^-.$$

And because now we have four lines to cut, we sum over four physical states 5, 6, 7 and 8. The resultant terms should look like,

$$\left[\left(i\mathcal{M}^{(0)}(1^+4^+5^+6^+7^+8^+)\right) \times \left(-i\mathcal{M}^{(0)}(2^-3^-5^-6^-7^-8^-)\right)\right].$$

An identification must be made so as to ensure that labels appearing on $D+, \Delta+$ propagators are 1 and 4 only, we make the following replacements in propagator indices

$$123 \to 4, \quad 23 \to 14. \tag{4.41}$$

Next we have to set the Δ propagators to unity (representing the completion of the phase space integral) and assign the on-shell loop momenta p, q and r to all permutations of physical states 5, 6, 7 and 8. But with one important caveat, for the sake of future comparison we replace labels containing 8 with its complement *i.e.* for example in the term,

$$D+_{148}D-_{238} \xrightarrow{\text{by momentum conservation}} D+_{567}D-_{567}. \tag{4.42}$$

This is done because while comparing results with 6-point tree order amplitudes, we shall use the off-shell recursion framework to generate the amplitude and treat the 8^{th} leg as the off-shell leg which is never explicitly named in recursion relations. After massaging of terms, we get

$$\begin{aligned} i\mathcal{M}'^{(3)}_4 = \left(\lambda^-\lambda^+\right)^2 &\left(D+_{145} + D+_{146} + D+_{147} + D+_{156} + D+_{157}\right. \\ &\left. + D+_{167} + D+_{456} + D+_{457} + D+_{467} + D+_{567}\right) \times \\ &\left(D-_{235} + D-_{236} + D-_{237} + D-_{256} + D-_{257}\right. \\ &\left. + D-_{267} + D-_{356} + D-_{357} + D-_{367} + D-_{567}\right) \end{aligned} \tag{4.43}$$

Next we shall derive the 6-point tree order amplitudes and check our result.

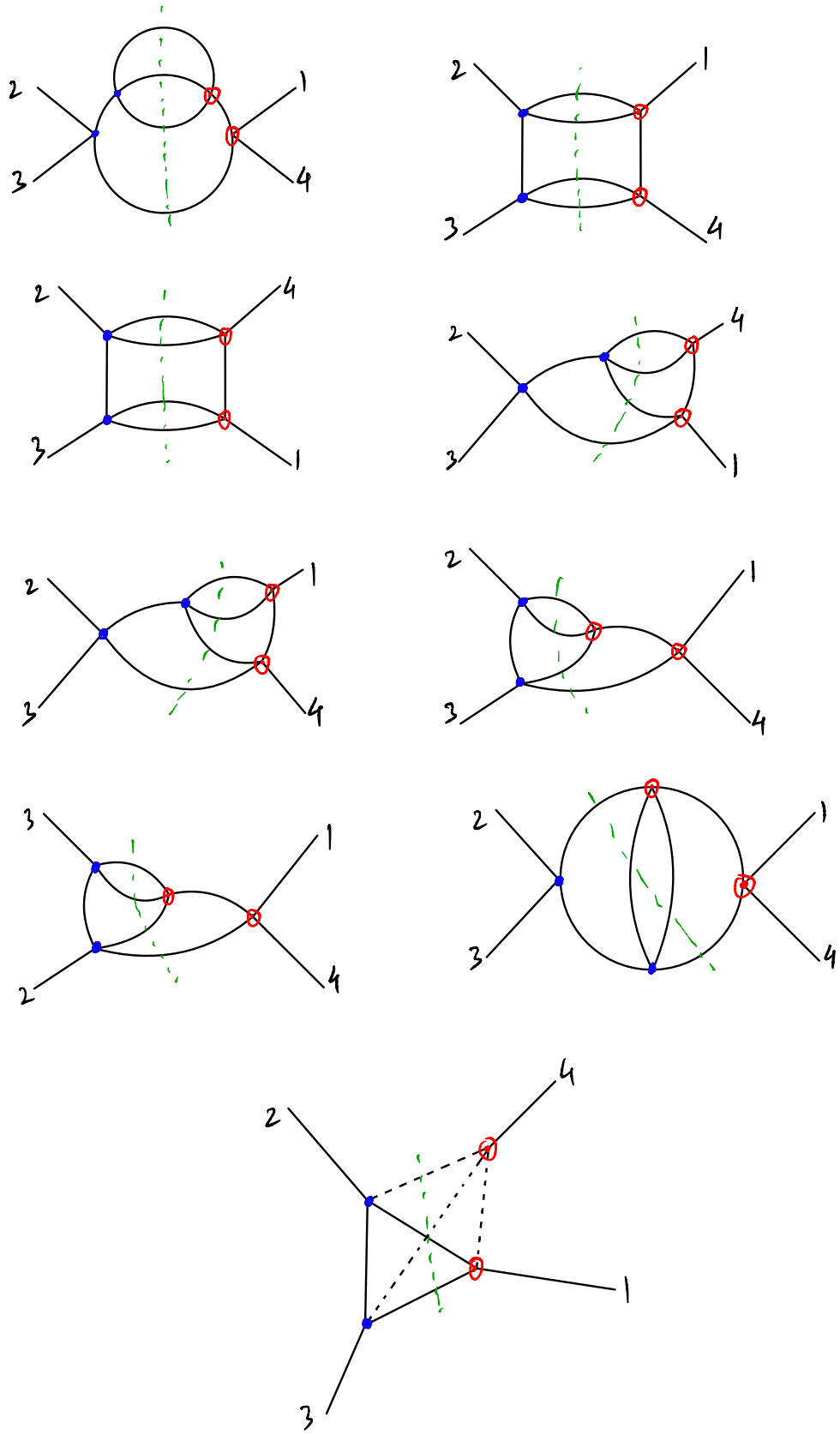

Figure 4.4: *The diagrammatic representation of eq.* (4.40)*. These are the only diagrams that contribute to physical cuts at 3-loop orders.*

§ 4.4.5 *6-point tree order*

To derive the tree level result, we must use the correct particle labels and set the word accordingly. We choose two words 14567 and 23567 for our recursion relations. We can use eq. (3.35g) with $D \to D+$ and $\Phi \to \Phi^+$. Solving these recursion relations gives us Φ^+_{14567} and Φ^+_{23567}. We perform the amputation by setting $D+_{14567}$ and $D+_{23567}$ to unity respectively. We now have $i\mathcal{M}^{(0)}_{1^+4^+5^+6^+7^+8^+}$ and $i\mathcal{M}^{(0)}_{2^+3^+5^+6^+7^+8^+}$. To flip the signature on the second term we perform complex conjugation by the following replacements,

$$\lambda^+ \to -\lambda^-, \quad D+ \to D-. \tag{4.44}$$

Explicitly we have,

$$\begin{aligned} i\mathcal{M}^{(0)}_{1^+4^+5^+6^+7^+8^+} &= \left(\lambda^+\right)^2 \left(D+_{145} + D+_{146} + D+_{147} + D+_{156} + D+_{157}\right. \\ &\quad \left. + D+_{167} + D+_{456} + D+_{457} + D+_{467} + D+_{567}\right), \end{aligned} \tag{4.45a}$$

$$\begin{aligned} -i\mathcal{M}^{(0)}_{2^-3^-5^-6^-7^-8^-} &= \left(\lambda^-\right)^2 \left(D-_{235} + D-_{236} + D-_{237} + D-_{256} + D-_{257}\right. \\ &\quad \left. + D-_{267} + D-_{356} + D-_{357} + D-_{367} + D-_{567}\right), \end{aligned} \tag{4.45b}$$

And finally we confirm that

$$\boxed{i\mathcal{M}'^{(3)}_4 = \left[\left(i\mathcal{M}^{(0)}(1^+4^+5^+6^+7^+8^+)\right) \times \left(-i\mathcal{M}^{(0)}(2^-3^-5^-6^-7^-8^-)\right)\right]} \tag{4.46}$$

Thus via unitarity, we were able to obtain a higher point cross-section from a lower point loop amplitude! This entire calculation serves as a proof-of-concept for the utility of our framework. It should be apparent now how one can use perturbative unitarity in this manner for their calculations. In future work, we shall demonstrate this for colored theories.

CHAPTER 5

Discussion

We conclude this thesis with a discussion on the results, conclusions drawn from them and some considerations for future work.

§ 5.1 Results

We outlined two main goals for this thesis in section 1.1 and we shall lay out the results in that context.

We have exhibited the utility of the framework through calculations. We showed how one starts out with just the action and proceeds to obtain the expressions for amplitudes. The procedure is very straightforward and completely algebraic in nature, we don't even have to construct the loop order corrections for the self-energy operator. Implementation is also relatively simple. We were able to obtain the 3-loop 4-point amplitude and perform loop momenta shifting up to 2-loop orders. We were able to adapt our framework to the field-doubling prescription easily.

We have schematically shown that perturbative unitarity is applicable for obtaining cross-sections using our framework. Starting out with the action for the field-doubled theory we were able to obtain cross-sections at tree order for the regular theory.

We were able to perform loop momenta shifting up to 3-loop orders in this theory. The procedure for isolating physical cut diagrams was also laid out systematically and it remains to be seen whether it can be extended to higher loop orders. We verifiably obtained the 6-point cross-section at tree order and this serves as an important proof-of-concept for our method. In future work, we shall extend it to colored theories. Our algorithms can be generalized for more complicated theories as no special assumptions were made.

We were able to demonstrate the Largest Time Equation algebraically.

§ 5.2 Conclusions

We began with a discussion on the background for this thesis so that the readers could grasp key concepts needed for calculations. We presented a short review of two different approaches to unitarity: generalized unitarity and the Largest Time Equation. We contrasted the two approaches and showcased how we were going to implement the latter for our method. We also discussed how one tackles color in gauge theories and what it means to calculate cross-sections. We presented a case for our motivations and elaborated on the goals we worked towards. The quantum off-shell recursion framework was a candidate for our work and we showed with a pedagogical calculation how it was used in action. We then took a step at exhibiting unitarity with the field-doubling prescription and successfully obtained the tree order 6-point cross-section from the 3-loop 4-point amplitude.

We observed the advantages of the framework and understand why it can potentially serve as a great alternative to Feynman diagram based approaches to amplitudes and beyond. We also observed how the field-doubling prescription is relatively simple and results in recursion relations with exactly double the number of terms from the regular theory. We realise that calculations grow to become more tedious and complicated but the added benefits of being able to isolate specific channels of scattering at will and performing loop momenta shifting effortlessly make the trade-off worth it. Although we shall indeed work on automating the process of field doubling given a regular theory.

We derived the principles for selection of relevant physical cut diagrams from graph-theoretic notions and momenta constraints, we remain curious to see whether there exist other ways to constrain terms. We are investigating the possibility of obtaining only these terms directly from modified recursion relations. If we are successful, the efficiency of our method would greatly increase.

We were able to perform loop momenta shifting in the field-doubled theory and we plan to adapt those algorithms for use in regular theories. We aim to publish a Mathematica® package based on our code that would prepare amplitudes for integration.

We also plan on working towards performing integration in tandem with solving the recursion relations. Currently we solve the recursion relations and obtain unevaluated integrals at the end, which means we have to perform multiple loop integrals at once. But if we were to solve the recursion relations up to 1-loop order and then perform the integration and then solve the recursion relations up to 2-loop order and then perform the integration for the new loop momentum and so on, we might increase the overall efficiency of computation as we are tackling relatively simple integrals at each stage rather than a single monstrous one at the end.

Now to talk about colored theories, we see a lot of work ahead of us still. We need to derive the

recursion relations up to 1-loop order in all helicity configurations. A CAS implementation of the pure Yang–Mills theory still remains tricky, the complexity of the theory requires ingenious solutions. We shall simplify the Dyson–Schwinger equations by adopting the spinor-helicity formalism which will lead to cancellations of terms. New notation for descendant fields and currents is in the pipeline. We shall then use the color-dressed perturbiner expansions[1,34] and solve for the 1-loop order 4-point amplitude. And then proceed with field-doubling to obtain cross-sections.

We feel that this framework's potential hasn't still been fully tapped into. With the submission of supervisor's group's latest work[5], we shall move towards applying the quantum off-shell recursion framework to perturbative methods in General Relativity. We aim to perform gravitational wave calculations for spinning binary black hole systems[12,15] using our framework and reproduce state-of-the-art results. Adapting the worldline[16] and biadjoint scalar[17] theories for use with the framework are also possibilities worth investigating.

— APPENDIX A

Pure Yang–Mills theory

So far we have dealt with scalar theories, we now introduce a non-Abelian gauge theory. In this appendix, we shall derive the Dyson–Schwinger equations for the pure Yang–Mills theory in the first-order formalism.

§ A.1 First-order formalism

The action for pure Yang–Mills theory with the Feynman–'t Hooft gauge condition $\partial_\mu A^{a\mu} = 0$ and with the associated Faddeev–Popov ghosts is given by,

$$\begin{aligned} S[A, c, \overline{c}, j, \eta, \overline{\eta}] = \int_x \lim_{y\to x} -\frac{1}{4} F_x^{a\mu\nu} F_y^{a\mu\nu} - \frac{1}{2}\partial_x^\mu A_x^{a\mu} \partial_y^\nu A_y^{a\nu} - \partial_x^\mu \overline{c}_x^a \mathcal{D}_y^{ab\mu} c_y^b \\ + \int_x A_x^{a\mu} j_x^{a\mu} + \overline{c}_x^a \eta_x^a + \overline{\eta}_x^a c_x^a \end{aligned}, \tag{A.1}$$

where $j, \overline{\eta}, \eta$ are the sources for $A, c, \overline{c}$ fields respectively. ∂_x here means that the derivative acts on a quantity located at x, this notation is important because we can easily deal with quantities like $\partial_x^\mu Y_x^a Z_y^{a\mu}$ where the partial derivative is contracted with a quantity it doesn't act on. We work with the following conventions,

$$F_x^{a\mu\nu} = \partial_x^\mu A_x^{a\nu} - \partial_x^\nu A_x^{a\mu} + \lim_{y\to x} g f^{abc} A_x^{b\mu} A_y^{c\nu} + J_x^{a\mu\nu}, \tag{A.2a}$$

$$\mathcal{D}_x^{ab\mu} = \delta^{ab}\partial_x^\mu - g f^{abc} A_x^{c\mu}. \tag{A.2b}$$

Where J is the external source of the field strength tensor. Expanding the covariant derivative $\mathcal{D}$ in eq. (A.1),

$$\begin{aligned} S_{\text{YM}} = \int_x \lim_{y\to x} -\frac{1}{4} F_x^{a\mu\nu} F_y^{a\mu\nu} - \frac{1}{2}\partial_x^\mu A_x^{a\mu} \partial_y^\nu A_y^{a\nu} - \partial_x^\mu \overline{c}_x^a \partial_y^\mu c_y^a \\ + \int_x \lim_{\substack{y\to x \\ z\to x}} g f^{abc} A_x^{a\mu} \partial_y^\mu \overline{c}_y^b c_z^c + \int_x A_x^{a\mu} j_x^{a\mu} + \overline{c}_x^a \eta_x^a + \overline{\eta}_x^a c_x^a \end{aligned}. \tag{A.3}$$

Usually one expands the field strength tensor and obtains the vertices and propagators of the theory. But using a subtle trick, we can avoid using the 4-point vertices usually found in this theory and greatly simplify calculations. This 'trick' is also referred to as the first-order formalism where the field strength tensor too is treated like an independent field[1]. Notice,

$$-\frac{1}{4}F_x^{a\mu\nu}F_y^{a\mu\nu} = \frac{1}{4}F_x^{a\mu\nu}F_y^{a\mu\nu} - \frac{1}{2}F_x^{a\mu\nu}F_y^{a\mu\nu}, \tag{A.4a}$$

$$= \frac{1}{4}F_x^{a\mu\nu}F_y^{a\mu\nu} - \frac{1}{2}F_x^{a\mu\nu}\left(\partial_y^\mu A_y^{a\nu} - \partial_y^\nu A_y^{a\mu} + \lim_{z\to y} g f^{abc} A_y^{b\mu} A_z^{c\nu} + J_y^{a\mu\nu}\right), \tag{A.4b}$$

$$\begin{aligned} = \frac{1}{4}F_x^{a\mu\nu}F_y^{a\mu\nu} - \frac{1}{2}F_x^{a\mu\nu}\partial_y^\mu A_y^{a\nu} + \frac{1}{2}F_x^{a\mu\nu}\partial_y^\nu A_y^{a\mu} \\ - \lim_{z\to y}\frac{1}{2} g f^{abc} F_x^{a\mu\nu} A_y^{b\mu} A_z^{c\nu} - \frac{1}{2}F_x^{a\mu\nu} J_y^{a\mu\nu} \end{aligned}. \tag{A.4c}$$

Substituting the above in eq. (A.3) and performing integration by parts to simplify a few terms gives us,

$$\begin{aligned} S_{\text{YM}} = \int_x \lim_{y\to x} \frac{1}{4}F_x^{a\mu\nu}F_y^{a\mu\nu} - \frac{1}{2}F_x^{a\mu\nu}\partial_y^\mu A_y^{a\nu} + \frac{1}{2}F_x^{a\mu\nu}\partial_y^\nu A_y^{a\mu} + \frac{1}{2}A_x^{a\mu}\partial_y^\mu\partial_y^\nu A_y^{a\nu} - \int_{x,y}\overline{c}_x^a K_{xy}^{ab} c_y^a \\ + \int_x \lim_{\substack{y\to x\\ z\to x}} g f^{abc}\left(A_x^{a\mu}\partial_y^\mu \overline{c}_y^b c_z^c - \frac{1}{2}F_x^{a\mu\nu}A_y^{b\mu}A_z^{c\nu}\right) + \int_x A_x^{a\mu} j_x^{a\mu} + \overline{c}_x^a \eta_x^a + \overline{\eta}_x^a c_x^a - \frac{1}{2}F_x^{a\mu\nu}J_x^{a\mu\nu}. \end{aligned} \tag{A.5}$$

§ A.1.1 *Field-redefinition*

One peculiar feature of the action in first-order formalism is that there is no propagator for $A_x^{a\mu}$ field. Without defining a propagator we won't be able to derive anything meaningful for the gluon fields. We can solve this problem by a simple field-redefinition of the field strength tensor,

$$F'^{a\mu\nu}_x = \lim_{y\to x} g f^{abc} A_x^{b\mu} A_y^{c\nu} + J_x^{a\mu\nu}, \tag{A.6}$$

which implies the substitution,

$$F_x^{a\mu\nu} \to F'^{a\mu\nu}_x + \partial_x^\mu A_x^{a\nu} - \partial_x^\nu A_x^{a\mu}. \tag{A.7}$$

[1]This explains why we defined its external source.

Performing the substitution in eq. (A.5) and dropping the prime from the new field strength tensor gives,

$$\begin{aligned} S_{\mathrm{YM}} = &\int_x A_x^{a\mu} j_x^{a\mu} + \overline{c}_x^a \eta_x^a + \overline{\eta}_x^a c_x^a - \frac{1}{2} F_x^{a\mu\nu} J_x^{a\mu\nu} - \int_{x,y} \overline{c}_x^a K_{xy}^{ab} c_y^a \\ &+ \int_x \lim_{\substack{y\to x \\ z\to x}} g f^{abc} \left(A_x^{a\mu} \partial_y^\mu \overline{c}_y^b c_z^c - \frac{1}{2} F_x^{a\mu\nu} A_y^{b\mu} A_z^{c\nu} - \partial_x^\mu A_x^{a\nu} A_y^{b\mu} A_z^{c\nu} \right) \\ &+ \int_x \lim_{y\to x} \frac{1}{4} F_x^{a\mu\nu} F_y^{a\mu\nu} - \frac{1}{2} \partial_x^\mu A_x^{a\nu} J_y^{a\mu\nu} + \frac{1}{2} \partial_x^\nu A_x^{a\mu} J_y^{a\mu\nu} + \frac{1}{2} A_x^{a\mu} \partial_y^\nu \partial_y^\nu A_y^{a\mu} \end{aligned} , \tag{A.8}$$

where we rewrite the $A_x^{a\mu} \partial_y^\nu \partial_y^\nu A_y^{a\mu}$ term as $-A_x^{a\mu} K_{xy}^{ab\mu\nu} A_y^{b\nu}$ and finally obtain,

$$\begin{aligned} S_{\mathrm{YM}} = &\int_x A_x^{a\mu} j_x^{a\mu} + \overline{c}_x^a \eta_x^a + \overline{\eta}_x^a c_x^a - \frac{1}{2} F_x^{a\mu\nu} J_x^{a\mu\nu} + \int_{x,y} -\overline{c}_x^a K_{xy}^{ab} c_y^a - \frac{1}{2} A_x^{a\mu} K_{xy}^{ab\mu\nu} A_y^{b\nu} \\ &+ \int_x \lim_{\substack{y\to x \\ z\to x}} g f^{abc} \left(A_x^{a\mu} \partial_y^\mu \overline{c}_y^b c_z^c - \frac{1}{2} F_x^{a\mu\nu} A_y^{b\mu} A_z^{c\nu} - \partial_x^\mu A_x^{a\nu} A_y^{b\mu} A_z^{c\nu} \right) \\ &+ \int_x \lim_{y\to x} \frac{1}{4} F_x^{a\mu\nu} F_y^{a\mu\nu} - \frac{1}{2} \partial_x^\mu A_x^{a\nu} J_y^{a\mu\nu} + \frac{1}{2} \partial_x^\nu A_x^{a\mu} J_y^{a\mu\nu} \end{aligned} . \tag{A.9}$$

The main changes after the field-redefinition are the vanishing of $F\partial A$ terms, the conversion of $A\partial\partial A$ to $A\Box A$ and the appearance of the ∂AAA vertex. Having derived the final action, we can move to derivations of the classical equations of motion for all fields.

§ A.2 Dyson–Schwinger equations

Deriving the Dyson–Schwinger equations for this theory isn't as straightforward as earlier. We have different kinds of fields, each with their own external source. Promotion of the bare fields to operators will prove to be a challenging step.

§ A.2.1 *Faddeev–Popov ghosts*

The classical equations of motion for the anti-ghost field $\overline{c}$ are given by $\frac{\delta^L S}{\delta \overline{c}} = 0$ and for eq. (A.9) we have,

$$\int_y K_{xy}^{ab} c_y^b = \eta_x^a + \lim_{y\to x} g f^{abc} \left(A_x^{b\mu} \partial_y^\mu c_y^c + \partial_x^\mu A_x^{b\mu} c_y^c \right), \tag{A.10}$$

where we performed an integration by parts on the $A\partial\overline{c}c$ term in the action and exploited the anti-symmetry of f^{abc}. We then promote the ghost fields to operators $\hat{c} \equiv \left(C + \frac{\hbar}{i} \frac{\delta^L}{\delta \overline{\eta}} \right)$ and gluon

fields to operators $\hat{A} \equiv \left(\mathcal{A} + \frac{\hbar}{i}\frac{\delta}{\delta j}\right)$ and symmetrize,

$$\int_y K^{ab}_{xy} C^b_y = \eta^a_x + \lim_{y\to x} g f^{abc} \left[\mathcal{A}^{b\mu}_x \partial^\mu_y C^c_y + \partial^\mu_x \mathcal{A}^{b\mu}_x C^c_y + \frac{1}{2}\frac{\hbar}{i}\left(\partial^\mu_y\left(\psi^{b\mu,1c}_{x,y} + \psi^{1c,b\mu}_{y,x}\right) + \partial^\mu_x\left(\psi^{b\mu,1c}_{x,y} + \psi^{1c,b\mu}_{y,x}\right)\right)\right], \tag{A.11}$$

where the descendants are now defined as,

$$\psi^{\cdot,1c}_{\cdot,y} \equiv \frac{\delta^L \cdot}{\delta \overline{\eta}^c_y}, \quad \psi^{\cdot,2c}_{\cdot,y} \equiv \frac{\delta^R \cdot}{\delta \eta^c_y}, \quad \psi^{\cdot,b\mu}_{\cdot,y} \equiv \frac{\delta \cdot}{\delta j^{b\mu}_y}, \quad \psi^{\cdot,b\mu\nu}_{\cdot,y} \equiv \frac{\delta \cdot}{\delta J^{b\mu\nu}_y}. \tag{A.12}$$

Now the order of indices matters and we must be careful while symmetrizing. The final form of Dyson–Schwinger equations for ghost fields is,

$$C^a_x = \int_y D^{ab}_{xy}\eta^b_y + \int_y D^{ab}_{xy} \lim_{z\to y} g f^{bcd}\left[\mathcal{A}^{c\mu}_y \partial^\mu_z C^d_z + \partial^\mu_y \mathcal{A}^{c\mu}_y C^d_z + \frac{1}{2}\frac{\hbar}{i}\left(\partial^\mu_z\left(\psi^{c\mu,1d}_{y,z} + \psi^{1d,c\mu}_{z,y}\right) + \partial^\mu_y\left(\psi^{c\mu,1c}_{y,z} + \psi^{1d,c\mu}_{z,y}\right)\right)\right]. \tag{A.13}$$

The classical equations of motion for the ghost field c are given by $\frac{\delta^R S}{\delta c} = 0$ and for eq. (A.9) we have,

$$\int_y K^{ab}_{xy}\overline{c}^b_y = \overline{\eta}^a_x + \lim_{y\to x} g f^{abc} A^{b\mu}_x \partial^\mu_y \overline{c}^c_y, \tag{A.14}$$

from here we can derive the Dyson–Schwinger equations for anti-ghost fields by promotion to $\hat{\overline{c}} \equiv \left(\overline{C} + \frac{\hbar}{i}\frac{\delta^R}{\delta\eta}\right)$,

$$\overline{C}^a_x = \int_y D^{ab}_{xy}\overline{\eta}^b_y + \int_y D^{ab}_{xy} \lim_{z\to y} g f^{bcd}\left[\mathcal{A}^{c\mu}_y \partial_z \overline{C}^d_z + \frac{1}{2}\frac{\hbar}{i}\partial^\mu_z\left(\psi^{c\mu,2d}_{y,z} + \psi^{2d,c\mu}_{z,y}\right)\right]. \tag{A.15}$$

§ A.2.2 *Field strength tensor*

The classical equations of motion for the field strength tensor are given by taking a functional derivative on eq. (A.9) with respect to F and setting it to 0,

$$F^{a\mu\nu}_x = \lim_{y\to x} g f^{abc} A^{b\mu}_x A^{c\nu}_y + J^{a\mu\nu}_x. \tag{A.16}$$

This is simply the definition of the new field strength tensor, it shouldn't surprise us. We can promote the fields to operators $\hat{F} \equiv \left(\mathcal{F} + \frac{\hbar}{i}\frac{\delta}{\delta J}\right)$ and we get,

$$\mathcal{F}^{a\mu\nu}_x = J^{a\mu\nu}_x + \lim_{y\to x} g f^{abc}\left(\mathcal{A}^{b\mu}_x \mathcal{A}^{c\nu}_y + \frac{1}{2}\frac{\hbar}{i}\left(\psi^{b\mu,c\nu}_{x,y} + \psi^{c\nu,b\mu}_{y,x}\right)\right). \tag{A.17}$$

An interesting thing to note is that there is no propagator defined for this field.

§ A.2.3 *Gluons*

The classical equations of motion for gluon fields are given by $\frac{\delta S}{\delta A} = 0$, and we have,

$$\int_y K^{ab\mu\nu}_{xy} A^{b\nu}_y = j^{a\mu}_x - \partial^\nu_x J^{a\mu\nu}_x + \lim_{y\to x} g f^{abc} \left[\partial^\mu_x \overline{c}^b_x c^c_y + F^{b\mu\nu}_x A^{c\nu}_y + \partial^\mu_x A^{b\nu}_x A^{c\nu}_y - \partial^\nu_x A^{b\nu}_x A^{c\mu}_y\right]. \tag{A.18}$$

We promote all fields to operators as shown earlier, only now we have to be careful with the $\partial\overline{c}c$ term because it needs to be anti-symmetrized owing to the Grassmannian nature of derivatives.

$$\begin{aligned}\mathcal{A}^{a\mu}_x = \int_y D^{ab\mu\nu}_{xy}\left(j^{b\nu}_y - \partial^\lambda_y J^{b\nu\lambda}_y\right) + \int_y D^{ab\mu\nu}_{xy} \lim_{z\to y} g f^{bcd} \Big[&\partial^\nu_y \overline{C}^c_y C^d_z + \mathcal{F}^{c\nu\lambda}_y \mathcal{A}^{d\lambda}_z \\ &+ \partial^\nu_y A^{c\lambda}_y A^{d\lambda}_z - \partial^\lambda_y A^{c\lambda}_y A^{d\nu}_z + \frac{1}{2}\frac{\hbar}{i}\Big(\left(\psi^{c\nu\lambda,d\lambda}_{yz} + \psi^{d\lambda,c\nu\lambda}_{zy}\right) + \partial^\nu_y\left(\psi^{1d,2c}_{zy} - \psi^{2c,1d}_{yz}\right). \\ &+ \partial^\nu_y\left(\psi^{c\lambda,d\lambda}_{yz} + \psi^{d\lambda,c\lambda}_{zy}\right) - \partial^\lambda_y\left(\psi^{c\lambda,d\nu}_{yz} + \psi^{d\nu,c\lambda}_{zy}\right)\Big)\Big]\end{aligned} \tag{A.19}$$

We have finally derived all the Dyson–Schwinger equations for this theory. To proceed, one has to simply take functional derivatives with respect to the external sources. However, to list them here explicitly would require 16 equations. We are working on a new notational system that would allow us to treat these equations more efficiently and would make implementation easier. We leave the next steps in the procedure for future work.

— APPENDIX B —

Notation

In this appendix, we shall list all the non-standard notation used (and abused) in this thesis. We shall also list some standard notation which has been abused leading to potential confusion.

§ B.1 Chapter 2

- σ represents both the cross-section and later in the chapter it is used to represent a member of the permutation group.
- $\mathcal{A}$ and $\mathcal{M}$ are both used to represent amplitudes in literature, in this thesis we highlight a subtle difference between the two: $\mathcal{M}$ represents just the interacting part of the theory whereas $\mathcal{A}$ represents the full S-matrix element.
  - While defining $\mathcal{M}$, we yet again abused notation to simultaneously represent the matrix and its elements. But in literature, once the final and initial states of the scattering event have been fixed it is common to drop any notation associated with them. In this thesis $\mathcal{M}$ has almost exclusively been used to denote the matrix element rather than the matrix.
- $\not\int\!\!\!\sum$ refers to the act of performing the phase space integral and taking the combinatorial sum over resultant on-shell states.
- $\tilde{\mathcal{M}}$ represents the color-stripped amplitude.
- $\mathcal{M}_n^{(0),(1)}$ means $\mathcal{M}_n^{(0)}$ or $\mathcal{M}_n^{(1)}$.

§ B.2 Chapter 3

- Integrals are represented as,

$$\int_{x,y,\ldots} \equiv \int \mathrm{d}^4x\, \mathrm{d}^4y \cdots, \quad \int_{p,q,\ldots} \equiv \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{d}^4q}{(2\pi)^4} \cdots .$$

- For any field $\Pi_x \equiv \Pi(x)$.
- D_{xy} is the position space propagator, $\tilde{D}_p$ is the momentum space propagator, $\boldsymbol{D}_{xy}$ is the dressed propagator and $\tilde{\boldsymbol{D}}_p$ is the dressed propagator in momentum space.
  - K_{xy}, $\tilde{K}_p$, $\boldsymbol{K}_{xy}$ and $\tilde{\boldsymbol{K}}_p$ are related in a similar manner.
- T is the time-ordering operator, not to be confused with T^a which are the generators of $\mathrm{SU}(N)$ in the fundamental representation.
- ϕ is the scalar field, φ is the classical field or the VEV of the scalar field, $\hat{\phi}$ is the field operator and Φ is the off-shell current.
- Descendant fields are defined as,

$$\psi_{x,y} \equiv \frac{\delta \varphi_x}{\delta j_y}, \quad \psi_{x,y,z} \equiv \frac{\delta^2 \varphi_x}{\delta j_y \delta j_z}, \quad \cdots .$$

  The order of the descendant can be determined by the number of terms in subscripts on ψ. First descendant has 2 terms, second descendant has 3 terms and so on.
  - Ψ is the associated current and the order of the descendant current is determined by the number of loop momenta it has in subscripts. First descendant has 1 loop momenta, second descendant has 2 loop momenta and so forth.
- $\sum_{\sigma \in S_N} \mathcal{M}_{\sigma(k_1,\ldots,k_N)}$ represents summation of amplitudes over all permutations of its external legs.
- $k_{i_1 \cdots i_N} = k_{i_1} + \cdots + k_{i_N}$ where i_j are particle labels.
- We introduce notation for the distributed summation over words and loop momenta,

$$\left\lceil \Phi\Phi \cdots \Psi_p \Psi_{qr} \cdots \right\rfloor_{\mathcal{P}}^{(n)} \equiv \sum_{\substack{a+b+c+d+\cdots=n \\ a,b,c,d,\ldots \geq 0}} \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}\cup\mathcal{T}\cdots} \Phi_{\mathcal{Q}}^{(a)} \Phi_{\mathcal{R}}^{(b)} \cdots \Psi_{p|\mathcal{S}}^{(c)} \Psi_{qr|\mathcal{T}}^{(d)} \cdots .$$

- A shorthand is used to represent propagators in recursion relations,

$$D_{\mathcal{P}} \equiv \tilde{D}(-k_{\mathcal{P}}), \quad D_{p|\mathcal{P}} \equiv \tilde{D}(p - k_{\mathcal{P}}), \quad D_{pq|\mathcal{P}} \equiv \tilde{D}(p + q - k_{\mathcal{P}}), \quad D_{p|\emptyset} \equiv \tilde{D}(p), \ \cdots .$$

- In eq. (3.47), m_p refers to the physical mass

§ B.3 Chapter 4

Notation largely remains unchanged from regular ϕ^4 theory.

- p^0 is the time component of 4-momentum p.
- We introduce a shorthand for writing out the Dyson–Schwinger equations and their descendants,

$$\varphi_x^{A,(0)} \to \varphi_0^A, \quad \psi_{x,z}^{AA',(0)} \to \psi_0^{AA'}, \quad \psi_{y,z}^{\pm A',(0)} \to \psi_0^{\pm A'}, \quad j_y^{\pm,(0)} \to j_0^{\pm}, \quad \cdots .$$

- For currents $\Phi_{\mathcal{P}}^{A,0}$, $\Psi_{p|\mathcal{P}}^{AB,0}$, $\Psi_{pq|\mathcal{P}}^{ABC,0}$, $\cdots$ the number in subscripts is the loop order.
- ε is the polarization, should not be confused with ϵ in Feynman's $i\epsilon$ prescription.
- Yet another representation for propagators is defined,

$$\begin{pmatrix} D^{++}_{pqr|\mathcal{P}} & D^{+-}_{pqr|\mathcal{P}} \\ D^{-+}_{pqr|\mathcal{P}} & D^{--}_{pqr|\mathcal{P}} \end{pmatrix} \equiv \begin{pmatrix} D{+}^{pqr}_{\mathcal{P}} & -\Delta{+}^{pqr}_{\mathcal{P}} \\ \Delta{-}^{pqr}_{\mathcal{P}} & -D{-}^{pqr}_{\mathcal{P}} \end{pmatrix}.$$

- $\mathcal{M}'^{(3)}_4$ represents all physical cut diagrams from the 3-loop order 4-point amplitude.

§ B.4 Appendix A

- $\mathcal{D}$ is the covariant derivative for gauge theories.
- $\dfrac{\delta^L}{\delta\cdot}$ and $\dfrac{\delta^R}{\delta\cdot}$ are left-handed and right-handed Grassmannian derivatives.
- Classical fields or the VEVs are defined as,

$$\mathcal{A} \equiv \mathrm{VEV}(A), \quad \mathcal{F} \equiv \mathrm{VEV}(F), \quad C \equiv \mathrm{VEV}(c), \quad \overline{C} \equiv \mathrm{VEV}(\overline{c}).$$

- Descendant fields are defined as

$$\psi_{\cdot,y}^{\cdot,1c} \equiv \frac{\delta^L \cdot}{\delta\overline{\eta}_y^c}, \quad \psi_{\cdot,y}^{\cdot,2c} \equiv \frac{\delta^R \cdot}{\delta\eta_y^c}, \quad \psi_{\cdot,y}^{\cdot,b\mu} \equiv \frac{\delta\cdot}{\delta j_y^{b\mu}}, \quad \psi_{\cdot,y}^{\cdot,b\mu\nu} \equiv \frac{\delta\cdot}{\delta J_y^{b\mu\nu}}.$$

Bibliography

[1] Kanghoon Lee. "Quantum off-shell recursion relation". In: *JHEP* 05 (2022), p. 051. DOI: 10.1007/JHEP05(2022)051. arXiv: 2202.08133 [hep-th].

[2] Henriette Elvang and Yu-tin Huang. *Scattering Amplitudes in Gauge Theory and Gravity*. Cambridge University Press, Apr. 2015. ISBN: 978-1-316-19142-2, 978-1-107-06925-1.

[3] Jens Simon Hoff. "Methods for multiloop calculations and Higgs boson production at the LHC". PhD thesis. 2015. DOI: 10.5445/IR/1000047447.

[4] Alexey Pak. "The Toolbox of modern multi-loop calculations: novel analytic and semi-analytic techniques". In: *J. Phys. Conf. Ser.* 368 (2012). Ed. by Liliana Teodorescu et al., p. 012049. DOI: 10.1088/1742-6596/368/1/012049. arXiv: 1111.0868 [hep-ph].

[5] Kyoungho Cho, Kwangeon Kim, and Kanghoon Lee. "Binary Black Holes and Quantum Off-Shell Recursion". In: (Nov. 2023). arXiv: 2311.01284 [hep-th].

[6] M. J. G. Veltman. "Unitarity and causality in a renormalizable field theory with unstable particles". In: *Physica* 29 (1963), pp. 186–207. DOI: 10.1016/S0031-8914(63)80277-3.

[7] Simon Badger et al. "Scattering Amplitudes in Quantum Field Theory". In: *Lect. Notes Phys.* 1021 (2024), pp. DOI: 10.1007/978-3-031-46987-9. arXiv: 2306.05976 [hep-th].

[8] Vittorio Del Duca, Lance J. Dixon, and Fabio Maltoni. "New color decompositions for gauge amplitudes at tree and loop level". In: *Nucl. Phys. B* 571 (2000), pp. 51–70. DOI: 10.1016/S0550-3213(99)00809-3. arXiv: hep-ph/9910563.

[9] Hjalte Axel Frellesvig. "Generalized Unitarity Cuts and Integrand Reduction at Higher Loop Orders". PhD thesis. Copenhagen U., 2014.

[10] Zvi Bern et al. "Fusing gauge theory tree amplitudes into loop amplitudes". In: *Nucl. Phys. B* 435 (1995), pp. 59–101. DOI: 10.1016/0550-3213(94)00488-Z. arXiv: hep-ph/9409265.

[11] R. E. Cutkosky. "Singularities and discontinuities of Feynman amplitudes". In: *J. Math. Phys.* 1 (1960), pp. 429–433. DOI: 10.1063/1.1703676.

[12] Zvi Bern et al. “Spinning black hole binary dynamics, scattering amplitudes, and effective field theory”. In: *Phys. Rev. D* 104.6 (2021), p. 065014. DOI: 10.1103/PhysRevD.104.065014. arXiv: 2005.03071 [hep-th].

[13] Horaţiu Năstase. *Introduction to Quantum Field Theory*. Cambridge University Press, 2020. ISBN: 978-1-108-49399-4, 978-1-108-66563-6, 978-1-108-62499-2. DOI: 10.1017/9781108624992.

[14] Vladyslav Shtabovenko. “FeynCalc goes multiloop”. In: *J. Phys. Conf. Ser.* 2438.1 (2023), p. 012140. DOI: 10.1088/1742-6596/2438/1/012140. arXiv: 2112.14132 [hep-ph].

[15] Zvi Bern et al. “Binary Dynamics through the Fifth Power of Spin at O(G2)”. In: *Phys. Rev. Lett.* 130.20 (2023), p. 201402. DOI: 10.1103/PhysRevLett.130.201402. arXiv: 2203.06202 [hep-th].

[16] Zvi Bern et al. “Quantum Field Theory, Worldline Theory, and Spin Magnitude Change in Orbital Evolution”. In: (Aug. 2023). arXiv: 2308.14176 [hep-th].

[17] Stephen G. Naculich. “Color-factor symmetry of the amplitudes of Yang-Mills and biadjoint scalar theory using perturbiner methods”. In: *JHEP* 06 (2023), p. 084. DOI: 10.1007/JHEP06(2023)084. arXiv: 2304.01287 [hep-th].

[18] Julian S. Schwinger. “Quantum Electrodynamics - An Individual View”. In: *J. Phys. Colloq.* 43.C8 (1982), pp. 409–423. DOI: 10.1051/jphyscol:1982826.

[19] Julian S. Schwinger. “Renormalization Theory of Quantum Electrodynamics: An Individual View”. In: *1st International Symposium on the History of Particle Physics*. 1980.

[20] Ruth Britto et al. “Direct proof of tree-level recursion relation in Yang-Mills theory”. In: *Phys. Rev. Lett.* 94 (2005), p. 181602. DOI: 10.1103/PhysRevLett.94.181602. arXiv: hep-th/0501052.

[21] Ruth Britto, Freddy Cachazo, and Bo Feng. “New recursion relations for tree amplitudes of gluons”. In: *Nucl. Phys. B* 715 (2005), pp. 499–522. DOI: 10.1016/j.nuclphysb.2005.02.030. arXiv: hep-th/0412308.

[22] Frits A. Berends and W. T. Giele. “Recursive Calculations for Processes with n Gluons”. In: *Nucl. Phys. B* 306 (1988), pp. 759–808. DOI: 10.1016/0550-3213(88)90442-7.

[23] Frits A. Berends, W. T. Giele, and H. Kuijf. “Exact and Approximate Expressions for Multi - Gluon Scattering”. In: *Nucl. Phys. B* 333 (1990), pp. 120–159. DOI: 10.1016/0550-3213(90)90225-3.

[24] A. A. Rosly and K. G. Selivanov. “On amplitudes in selfdual sector of Yang-Mills theory”. In: *Phys. Lett. B* 399 (1997), pp. 135–140. DOI: 10.1016/S0370-2693(97)00268-2. arXiv: hep-th/9611101.

[25] A. A. Rosly and K. G. Selivanov. "Gravitational SD perturbiner". In: (Oct. 1997). arXiv: hep-th/9710196.

[26] K. G. Selivanov. "SD perturbiner in Yang-Mills + gravity". In: *Phys. Lett. B* 420 (1998), pp. 274–278. DOI: 10.1016/S0370-2693(97)01514-1. arXiv: hep-th/9710197.

[27] Zvi Bern and Yu-tin Huang. "Basics of Generalized Unitarity". In: *J. Phys. A* 44 (2011), p. 454003. DOI: 10.1088/1751-8113/44/45/454003. arXiv: 1103.1869 [hep-th].

[28] Tanju Gleisberg et al. "How to calculate colourful cross sections efficiently". In: (Aug. 2008). arXiv: 0808.3672 [hep-ph].

[29] Stephen J. Parke and T. R. Taylor. "Perturbative QCD Utilizing Extended Supersymmetry". In: *Phys. Lett. B* 157 (1985). [Erratum: Phys.Lett.B 174, 465 (1986)], p. 81. DOI: 10.1016/0370-2693(85)91216-X.

[30] Marvin Gerlach, Florian Herren, and Martin Lang. "tapir: A tool for topologies, amplitudes, partial fraction decomposition and input for reductions". In: *Comput. Phys. Commun.* 282 (2023), p. 108544. DOI: 10.1016/j.cpc.2022.108544. arXiv: 2201.05618 [hep-ph].

[31] Johannes M. Henn and Jan C. Plefka. *Scattering Amplitudes in Gauge Theories*. Vol. 883. Berlin: Springer, 2014. ISBN: 978-3-642-54021-9. DOI: 10.1007/978-3-642-54022-6.

[32] Ronald Kleiss and Hans Kuijf. "Multi - Gluon Cross-sections and Five Jet Production at Hadron Colliders". In: *Nucl. Phys. B* 312 (1989), pp. 616–644. DOI: 10.1016/0550-3213(89)90574-9.

[33] Walter Giele, Zoltan Kunszt, and Jan Winter. "Efficient Color-Dressed Calculation of Virtual Corrections". In: *Nucl. Phys. B* 840 (2010), pp. 214–270. DOI: 10.1016/j.nuclphysb.2010.07.007. arXiv: 0911.1962 [hep-ph].

[34] Sebastian Mizera and Barbara Skrzypek. "Perturbiner Methods for Effective Field Theories and the Double Copy". In: *JHEP* 10 (2018), p. 018. DOI: 10.1007/JHEP10(2018)018. arXiv: 1809.02096 [hep-th].

[35] Gerard 't Hooft and M. J. G. Veltman. "DIAGRAMMAR". In: *NATO Sci. Ser. B* 4 (1974), pp. 177–322. DOI: 10.1007/978-1-4684-2826-1_5.

[36] M. J. G. Veltman. *Diagrammatica: The Path to Feynman rules*. Vol. 4. Cambridge University Press, May 2012. ISBN: 978-1-139-24339-1, 978-0-521-45692-0.

[37] Kirill Melnikov. *Lecture notes for QFT II*. https://www.ttp.kit.edu/~melnikov/TTP2/notes/notes18.pdf. 2017.

[38] Ruth Britto. "Loop Amplitudes in Gauge Theories: Modern Analytic Approaches". In: *J. Phys. A* 44 (2011), p. 454006. DOI: 10.1088/1751-8113/44/45/454006. arXiv: 1012.4493 [hep-th].

[39] François Gelis. *Quantum Field Theory*. Cambridge University Press, July 2019. ISBN: 978-1-108-48090-1, 978-1-108-57590-4.

[40] Michelangelo L. Mangano and Stephen J. Parke. "Multiparton amplitudes in gauge theories". In: *Phys. Rept.* 200 (1991), pp. 301–367. DOI: 10.1016/0370-1573(91)90091-Y. arXiv: hep-th/0509223.

[41] G. 't Hooft. "The Conceptual Basis of Quantum Field Theory". In: *Philosophy of physics*. Ed. by Jeremy Butterfield and John Earman. 2007, pp. 661–729. DOI: 10.1016/B978-044451560-5/50010-5.