Enumerating Subspaces Relative to Linear Operators

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by

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Certificate

This is to certify that this dissertation entitled Enumerating Subspaces Relative to Linear Operators towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Varun Shah at The Institute of Mathematical Sciences, Chennai (IMSc) under the supervision of Amritanshu Prasad, Professor, Department of Mathematics, during the academic year 2023-2024.

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Declaration

I hereby declare that the matter embodied in the report entitled Enumerating Subspaces Relative to Linear Operators are the results of the work carried out by me at the Department of Mathematics, The Institute of Mathematical Sciences, Chennai (IMSc), under the supervision of Amritanshu Prasad and the same has not been submitted elsewhere for any other degree. Wherever others have contributed, every effort is made to indicate this clearly with due reference to the literature and acknowledgement of collaborative research and discussions.

Varun Shah

This thesis is dedicated to my parents

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Abstract

This thesis serves three broad purposes. It delves into understanding the positivity phenomena observed in solutions to classical and contemporary enumeration problems in finite vector spaces, such as counting subspaces and flags of various types in a vector space. It offers a non-standard and elementary approach to grasping the geometry of Grassmannians and flag varieties. Finally, it serves as an exposition for the intriguing problem of comprehending the behavior of subspaces under repeated application of a linear map through the study of subspace profiles.

Contents

Introduction

It is a basic fact that the number of d-dimensional subspaces in the vector space $\mathbb{F}_q^n$ is given by the formula

$$(1)\qquad \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-d+1}-1)}{(q^d-1)(q^{d-1}-1)\cdots(q-1)}.$$

When we substitute in $q=1$, this formula surprisingly yields the binomial coefficient $\binom{n}{d}$, which is the number of subsets of $\{1,\ldots,n\}$ that have size d. While this may seem quite magical, a shift in perspective, due to Knuth [**Knu71**], clears it all up: viewing subspaces as $d\times n$ reduced row echelon form matrices, each subspace can be labelled by a subset of size d which records the pivot columns of the associated matrix. The number of matrices with a given pivot set is always a power of q; for example the number of 3×5 matrices with pivot set equal to $\{1,2,5\}$ is q^4.

$$\begin{pmatrix}1 & 0 & * & * & 0\\ 0 & 1 & * & * & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

So, the enumeration of these subspaces becomes a sum over these subsets, of powers of q, which naturally reduces to $\binom{n}{d}$ when $q=1$. Notable, this also shows that the formula 1, which is a priori only a rational function in q, simplifies into a polynomial in q with positive coefficients.

Butler's seminal work in the 1980s ventures into a related exploration, delving into the enumeration of subgroups of finite abelian p-groups $\mathbb{Z}_p^{\lambda_1}\times\cdots\times\mathbb{Z}_p^{\lambda_r}$, where λ is a partition of size n. When $\lambda=(1,\ldots,1)$, we obtain as a special case, the problem of enumerating linear subspaces in $\mathbb{Z}_p^n$, which can be solved using reduced row echelon form matrices as before. Birkhoff [**Bir35**], in the 30s, found a way to label subgroups of abelian p-groups using matrices akin to the reduced row echelon form matrices. Butler leveraged these *Birkhoff standard matrices* to prove that the number of subgroups is also given by a polynomial in p having positive integer coefficients which can be interpreted in terms of a combinatorial statistic on certain fillings of λ.

Meanwhile, nineteenth century geometers had introduced the *Grassmannian*, a geometric object whose points are parameterized by d-dimensional subspaces of complex vector spaces, to answer questions in enumerative geometry. The partitioning of the subspaces on the basis of pivot sets was already known to lend a geometrically meaningful decomposition of this space known as its *Schubert decomposition*. In that sense, Knuth's insight can be seen as an enumerative implication of the topological study of the collection of subspaces.

In the 1980s, contemporaries of Butler were investigating the geometry of the set of linear subspaces (as well as other linear algebraic objects) which are invariant under a nilpotent linear transformation, motivated by problems in geometric representation theory. This prompts questions on whether the geometric analysis of these spaces, similar to those of the Grassmannian, could offer a fresh perspective on Butler's enumerative and combinatorial results. Moreover, it raises the question of whether the positivity phenomenon observed in these counting problems may be stemming from the inherent geometric properties of these spaces, suggesting a deeper connection between geometry and enumerative combinatorics.

Structure of the thesis

This thesis primarily revolves around trying to understand the positivity phenomena. We see how this is pervasive in solutions to several classical and not-so-classical problems that have occupied enumerative combinatorialists for the last few decades. We argue that the positivity must be a consequence of the ability to organize the objects to be enumerated into geometric spaces that have a *nice* decomposition, such as an affine paving. In doing this, we accomplish our second aim, which is to develop an elementary, non-standard approach to the geometry of the Grassmannians and the flag varieties. Lastly, we aim to provide an introduction to the intriguing and challenging problem of understanding the behavior of subspaces under repeated application of a linear map.

The thesis is structured into three parts. The first part serves as a foundation, covering essential definitions and theorems from algebraic geometry and topology in Chapter 1, and introducing combinatorial objects such as partially ordered sets, permutations, and partitions, along with their diagrams and symmetric functions in Chapter 2. These objects play an important role in the theory we develop and will recur throughout the subsequent chapters.

The second part makes up the geometric content of this thesis. We begin by outlining the concept of paving an algebraic variety over graded posets in Chapter 3, enabling the computation of Betti numbers. This involves discussions on Poincaré duality and Borel-Moore homology. This framework is then applied to study the Grassmannian in Chapter 4, followed by the flag varieties in Chapter 5, and their fixed point varieties under a unipotent action in Chapter 6.

The final part focuses on enumeration. Chapter 7 investigates the lattice of subspaces of a vector space that are preserved under the action of any linear transformation, computing its invariant flag generating polynomial. The geometric analysis of part two allows us to count the number of flags in a vector space preserved by a nilpotent linear transformation, which leads to an explicit expression for the invariant flag generating polynomials for arbitrary operators. Lastly, Chapter 8 introduces the problem of enumerating subspaces by their profiles, as in [**BCRR92, Nie95, PR23a**]. We discuss the recent progress towards a solution as well as describing a new combinatorial proof of the case of diagonal linear transformations.

Original Contribution. The notion of an affine paving is well-known (e.g. [**Ful97**]), but we introduce a new perspective by defining affine pavings over graded posets in Definition 3.2. To the best of the author's knowledge, the discussion in Chapters 4 and 5 about the geometry of the Grassmannians and flag varieties is novel. While Chapter 6 is based on the work in [**Shi85**], we clarify several proofs and rectify some inaccuracies in the proofs provided therein, along with restating the definitions and theorems in modern terminology. As far as the author is aware, using the analysis of the geometry of the fixed point variety to deduce the enumerative and order-theoretic work of Butler is new. Finally, Chapter 8 is almost entirely original work.

Part 1

Preliminaries

CHAPTER 1

Some Geometry & Topology

1. Algebraic Geometry

All geometric objects we will study in this thesis are defined by algebraic equations. The basic idea of algebraic geometry is to understand the geometric and topological properties of these objects, such as their dimension, smoothness, and topology by using tools and techniques from algebra. In this section we will briefly discuss the requisite definitions and results from the subject, following conventions in [**Har77**] and [**SKKT00**].

1.1. Affine Varieties. Let $\mathbf{F}$ be any field. By $\mathbf{A}^n_{\mathbf{F}}$, or simply $\mathbf{A}^n$ (if $\mathbf{F}$ is understood), we shall mean the set of n-tuples of elements of $\mathbf{F}$. We call $\mathbf{A}^n_{\mathbf{F}}$ *affine n-space* over $\mathbf{F}$; its elements will be called *points*. In particular, $\mathbf{A}^1_{\mathbf{F}}$ is the affine line, $\mathbf{A}^2_{\mathbf{F}}$ the affine plane.

Definition 1.1. An *algebraic set* in $\mathbf{A}^n$ is the common zero set of a collection, $\{f_i\}_{i\in I}$, of polynomials in n variables defined over $\mathbf{F}$. We write

$$Y = Z(\{f_i\}_{i\in I}) \subseteq \mathbf{A}^n,$$

for this set of common zeros.

Remark 1.1. A priori an algebraic set Y may be defined as the zero set of a possibly infinite set of polynomials in $\mathbf{F}[x_1, \dots, x_n]$. However, Hilbert's Basis theorem (see [**AM69**, Theorem 7.5]) guarantees the existence of a finite subset whose zero set is also Y.

Example 1. When $\mathbf{F} = \mathbb{R}$ and $n = 2$, algebraic sets can be visualized as subsets of the Cartesian plane.

(1) The unit circle $\{(x, y) \in \mathbf{A}^2 \mid x^2 + y^2 = 1\}$ is an algebraic set because it is defined by the vanishing of the polynomial $x^2 + y^2 - 1 \in \mathbb{R}[x, y]$. More generally, any conic section is an algebraic set.
(2) The zero set of the polynomial $xy \in \mathbb{R}[x, y]$ is a pair of straight lines.
(3) The zero set of the polynomials x, y in $\mathbb{R}[x, y]$ is the singleton set $\{(0, 0)\}$ containing the origin. Similarly, any point in $\mathbf{A}^2$ is an algebraic set.

1.1.1. *Zariski Topology.* The affine space is the zero set of the zero polynomial, while the empty set is the zero set of the entire polynomial ring. Furthermore, it is easy to check that the the intersection of any family of algebraic sets as well as the union of two algebraic are algebraic sets. So, the collection of complements of algebraic sets form a topology, which we call the *Zariski topology on* $\mathbf{A}^n$.

Example 2. Choosing a basis for an n-dimensional F-vector space V identifies it with $\mathbf{A}_F^n$. Every vector in V is specified by a unique element of $\mathbf{A}_F^n$; these are its coordinates with respect to the basis. This allows us to define a ring of polynomial functions on V consisting of polynomials in the coordinates of the points of V. As a result, V is endowed with a Zariski topology.

1.1.2. *Irreducibility.* In Example 1, the pair of straight lines can be written as the union of two smaller algebraic sets: the x and y axes. However, the circle cannot be expressed as a union like this. We would like to capture this notion of indivisibility of algebraic sets.

Definition 1.2. A nonempty subset of a topological space X is *irreducible* if it cannot be expressed as the union of two proper subsets, each closed in the subspace topology.

Proposition 1.1. The closure of an irreducible subset of a topological space is also an irreducible subset.

PROOF. If $\overline{Y}$ can be written as the union $\overline{Y} = Y_1 \cup Y_2$ of two closed subsets, then Y can be expressed as a union $(Y_1 \cap Y) \cup (Y_2 \cap Y)$ of subsets, both closed in Y. Since Y is irreducible, it must be contained in one of the closed subsets, say $Y \subseteq Y_1$. This implies $\overline{Y} = Y_1$, so $\overline{Y}$ is irreducible. □

An *affine algebraic variety* is an irreducible algebraic subset of $\mathbf{A}^n$. Unless otherwise states, an affine variety will be equipped with the subspace topology. Any open subset of an affine variety is called a *quasi-affine variety.* For example, the affine space minus any point is a quasi-affine variety.

1.2. Projective Varieties. By $\mathbf{A}_{\mathbf{F}}^n$, or simply $\mathbf{A}^n$ (if $\mathbf{F}$ is understood), we shall mean the set of n-tuples of elements of $\mathbf{F}$. We call $\mathbf{A}_{\mathbf{F}}^n$ *affine n-space* over $\mathbf{F}$; its elements will be called *points*. In particular, $\mathbf{A}_{\mathbf{F}}^1$ is the affine line, $\mathbf{A}_{\mathbf{F}}^2$ the affine plane.

A *projective n-space* over $\mathbf{F}$, writeen $\mathbf{P}_{\mathbf{F}}^n$, or simply $\mathbf{P}^n$, is defined to be the set of equivalence classes of $(n+1)$-tuples $(a_0, a_1, \ldots, a_n)$ of elements of $\mathbf{F}$, not all zero, under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\varepsilon a_0, \ldots \varepsilon a_n)$ for all $\varepsilon \in \mathbf{F} - \{0\}$.

Elements of $\mathbf{P}^n$ are called *points* and if P is a point then any $(n+1)$-tuple $(a_0, \ldots, a_n)$ in the equivalence class P is called a set of *homogeneous coordinates* for P. We often write $P = [a_0, \ldots, a_n]$ using square brackets to indicate that $(a_0, \ldots, a_n)$ are homogeneous coordiates for P.

Unlike the affine case, a polynomial $f \in \mathbf{F}[x_0, \ldots, x_n]$ is not a well-defined function on $\mathbf{P}^n$. However, if f is a homogeneous polynomial of degree d, then $f(\varepsilon a_0, \ldots, \varepsilon a_n) = \varepsilon^d f(a_0, \ldots, a_n)$, ensuring that the property of f taking value zero or not is well-defined.

Definition 1.3. A projective *algebraic set* is the common zero set of a collection, $\{f_i\}_{i \in I}$, of homogeneous polynomials in $n+1$ variables defined over $\mathbf{F}$. We write

$$Y = Z(\{f_i\}_{i \in I}) \subseteq \mathbf{P}^n,$$

for this set of common zeros.

Arguing as we did in 1.1.1, we observe that the collection of complements of the algebraic sets form what is called the *Zariski topology on* $\mathbf{P}^n$. As a result, we can define a *Projective algebraic variety* to be an irreducible algebraic set in $\mathbf{P}^n$, together with the induced topology.

Example 3. Suppose V is a vector space, then the collection of lines in V is the associated projective space $\mathbf{P}V$. Just like before, choosing a basis of V identifies $\mathbf{P}V$ with $\mathbf{P}^{n-1}$. This allows us to speak of homogeneous polynomials on $\mathbf{P}V$ and also equip $\mathbf{P}V$ with a Zariski topology.

1.2.1. *Irreducible Components.* A topological space X is called *Noetherian* if it satisfies the *descending chain condition* for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \cdots$.

Example 4. Affine spaces $\mathbf{A}^n$ and Projective spaces $\mathbf{P}^n$ are Noetherian topological spaces. Since a subset of a Noetherian topological space is also Noetherian, this implies that all affine and projective algebraic sets are also Noetherian.

Proposition 1.2. In a Noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. These are called the *irreducible components* of Y.

Proof. This is [**Har77**, Proposition 1.5]. □

Since affine and projective spaces are Noetherian, Proposition 1.2 implies that every affine (resp. projective) algebraic set can be expressed uniquely as a finite union of affine (resp. projective) varieties.

1.3. Category of Quasi-Projective Varieties. We have defined affine and projective varieties. We now introduce *quasi-projective varieties*, a notion that encompasses both cases and the category of objects we will primarily be interested in.

Definition 1.4. An open subset of a projective variety is a *quasi-projective variety.*

In other words, quasi-projective varieties are locally closed subsets[1] of a projective space $\mathbf{P}^n$. The class of quasi-projective varieties includes all projective, affine and quasi-affine varieties. This is because any affine space $\mathbf{A}^n$ can be identified with the complement of the zero set of any homogeneous linear polynomial, such as a coordinate function x_0, on $\mathbf{P}^n$.

Remark 1.2. Heuristically, we can think of quasi-projective varieties as being defined by taking some polynomials equal to 0 and other polynomials to not be equal to 0.

[1] Locally closed sets are intersections of a closed set with an open set.

We will often simply say *variety* to mean quasi-projective variety. A *subvariety* of a variety is a subset which is itself a variety; it is easy to check that this is equivalent to the subset being irreducible and locally closed. Having defined the objects of the category, we can now define morphisms between them:

Definition 1.5. If $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$ are quasi-projective varieties, then a *morphism of varieties* is a map $F: X \to Y$ such that for each $P \in X$, there exist homogeneous polynomials $f_0, \ldots, f_m$ in $n+1$ variables such that F is given by $[f_0, \ldots, f_m]$ on some open set containing P. Often we will take the open set to be the one of the sets $\{x_i \neq 0\}$.

2. Singular Homology

Algebraic topology studies topological spaces by assigning them algebraic objects, such as groups and rings, which remain invariant under topological equivalences. In this thesis, we will compute topological invariants known as Betti numbers for various algebraic varieties that have traditionally interested combinatorialists. For this reason, we discuss the necessary background on singular homology, as outlined in standard references such as [**Hat02**].

2.1. Constructing Homology Groups.

Singular homology associates a sequence of abelian groups, known as the *singular homology groups*, to every topological space. The construction of these groups begins with the notion of a standard k-simplex, denoted Δ_k, which is the convex hull of the standard basis vectors, labeled $e_0, \ldots, e_k$, in $\mathbb{R}^{k+1}$.

Definition 2.1. A *singular k-simplex* of a topological space X is a continuous map from the standard k-simplex into X. The group $C_k(X)$ consists of formal linear combinations of singular k-simplices with complex coefficients, and its elements are referred to as *k-chains*.

For each $i = 0, \ldots, k$, there exists a map from the standard $(k-1)$-simplex to the face of the standard k-simplex not containing the standard basis vector e_i, which maintains the order of the vertices of the $(k-1)$-simplex. Composing this map with a singular k-simplex produces a singular $(k-1)$-simplex, σ^i, that is referred to as the *kth face of σ*. This also allows us to define the *boundary operator* $\partial = \partial_k : C_k(X) \to C_{k-1}(X)$ by linearly extending the action on simplices:

$$\partial_k(\sigma) = \sum_{i=0}^{k} (-1)^i \sigma^i,$$

to all chains.

The kernel of ∂_k forms the set of *k-cycles*, denoted by $Z_k(X)$, while the image of ∂_{k+1} is the set of *k-boundaries*, denoted by $B_i(X)$. An elementary calculation (c.f. [**Hat02**, Lemma 2.1, p. 105]) shows that the composition $\partial_{k-1}\partial_k = 0$, implying that for every k, $B_k(X) \subseteq Z_k(X)$.

This means we can represent the groups $C_k(X)$ and the maps ∂_k as a *chain complex*:

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} (0). \tag{2}$$

Definition 2.2. The *kth singular homology group* of X is the quotient

$$H_k(X) = Z_k(X)/B_k(X),$$

of the k-cycles by the k-boundaries.

The chain groups are complex vector spaces. The homology groups, which are quotients of subspaces of the chain groups, are therefore also vector spaces. In fact, the homology groups for every space we will encounter will be finite dimensional complex vector spaces. The *kth Betti number* of X, denoted by $b_k(X)$, is the dimension of $H_k(X)$.

Remark 2.1. Strictly speaking, we have defined singular homology with complex coefficients. The conventional approach, as developed in [**Hat02**], employs integer coefficients. In this approach, a k-chain is initially defined as an integer linear combination of singular k-simplices, leading to the homology groups being finitely generated abelian groups.

Utilizing complex coefficients often simplifies computations, while retaining the ability to capture the free part of the abelian group through the Betti numbers.

2.1.1. *Relative Homology.* If U is a subset of a topological space X, then denote by $C_k(A)$ the set of k-chains of X all of whose simplices take values in A. Then the boundary of a k-chain in $C_k(A)$ is a $(k-1)$-chain in $C_{k-1}(A)$. Therefore, the boundary operator descends to a map

$$\partial_k \colon C_k(X)/C_k(A) \to C_{k-1}(X)/C_{k-1}(A),$$

between the quotients, that we will call the groups of chains of X *relative to* A. We can define the relative *k-cycles* as the kernel of ∂_k.

Definition 2.3. The kth homology group of X *relative to* A is defined as the quotient

$$H_k(X, A) = \ker \partial_k / \mathrm{im} \partial_{k+1},$$

of the relative k-cycles by the relative k-boundaries. Note that when A is empty, the relative homology groups are the usual homology groups.

2.2. Functoriality. Consider pairs of topological spaces (X, A) and (Y, B), where $A \subset X$ and $B \subset Y$. We use the shorthand $f \colon (X, A) \to (Y, B)$ to mean that f is a continuous function from X to Y, which maps the subset A into B. Composing f with any chain in X gives a chain in Y, which induces a map between the chain groups of X and Y that commutes with the boundary operators. This map descends to a map between the homology groups, known as the *pushforward* of f, and denoted by $f_* \colon H_k(X, A) \to H_k(Y, B)$. The pushforward satisfies various desirable properties which can be easily checked:

(1) the pushforward of the identity map on a topological space is the identity map on its homology groups,
(2) the pushforward of compositions is the composition of pushforwards.

In other words, singular homology defines a functor from the category of topological spaces to the category of abelian groups.

2.3. Eilenberg-Steenrod axioms. Computing the homology groups of a topological space from first principles is usually impractical, so we give a list of fundamental properties of singular homology that for reasonably nice topological spaces completely characterize the singular homology groups. These are the *Eilenberg-Steenrod axioms*

(1) (Dimension axiom) The zeroth homology group of a space with a single point is isomorphic to $\mathbb{C}$, while all higher homology groups are trivial.

(2) (Homotopy axiom) If two continuous maps are homotopic to each other, then their pushforwards are equal. This implies, in particular, that two topological spaces with the same homotopy type have the same homology groups.

(3) (Additivity axiom) If a topological space X is the topological sum of subspaces X_a, then the homology groups of X are direct sums of the homology groups of the X_a.

(4) (Long exact sequence) Pushing forward the inclusions $(A, \varnothing) \to (X, \varnothing)$ and $(X, \varnothing) \to (X, A)$ gives a long exact sequence

$$\cdots \to H_k(A) \xrightarrow{i_*} H_k(X) \to H_k(X, A) \to H_{k-1}(A) \to \cdots$$

in homology.

(5) (Excision axiom) If U is a subset of A with closure contained in the interior of A, then the inclusion $(X - U, A - U) \to (X, A)$ induces an isomorphism

$$H_k(X - U, A - U) \xrightarrow{\sim} H_k(X, A),$$

between the homology groups.

Example 5. Euclidean space $\mathbb{R}^n$ is contractible, so the homotopy axiom implies that its homology groups coincide with the homology groups of a one point space. By the dimension axiom, $H_0(\mathbb{R}^n) = \mathbb{C}$, while $H_k(\mathbb{R}^n) = 0$ for all k larger than 0.

Example 6. Consider a convex neighborhood U of the origin in $\mathbb{R}^n$, and let $k > 1$. The convexity of U ensures that the closure of $\mathbb{R}^n - \overline{U}$ is contained in $\mathbb{R}^n - \{0\}$. Excising this set, we obtain $H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = H_k(U, U - \{0\})$.

As U is contractible, $H_k(U) = 0$. Consequently, by the long exact sequence in homology, the homology groups $H_k(U, U - \{0\})$ and $H_{k-1}(U - \{0\})$ are isomorphic. Further, the latter is isomorphic to $H_{k-1}(S^{n-1})$ by the homotopy axiom. In conclusion, $H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = H_{k-1}(S^{n-1})$.

Example 7. In [**Hat02**], it is shown that $H_k(S_n) = H_{k-1}(S_{n-1})$ for all k and n. So, we can deduce that $H_0(S^n) = H_n(S_n) = \mathbb{C}$, while all other homology groups vanish.

By the previous example, this also implies that $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \mathbb{C}$, while the other groups vanish.

CHAPTER 2

Combinatorial Preliminaries

The objective of this chapter is to familiarize the reader with certain important combinatorial objects, establish relevant notation and definitions, and compile essential results for subsequent chapters.

1. Partially ordered sets

A partially ordered set $\mathcal{P}$ (or poset, for short) is a set, together with a binary relation denoted $\leq$ (or $\leq_{\mathcal{P}}$ when there is a possibility of confusion), satisfying the following three axioms:

(1) For all $t \in \mathcal{P}$, $t \leq t$ (reflexivity).
(2) If $s \leq t$ and $t \leq s$, then $s = t$ (antisymmetry).
(3) If $s \leq t$ and $t \leq u$, then $s \leq u$ (transitivity).

We say that two elements s and t of P are *comparable* if one of $s \leq t$ or $t \leq s$ is true; otherwise s and t are *incomparable*.

Example 8. The set of natural numbers form a poset ordered as usual. For example, $2 \leq 3$ and $4 \leq 4$. In particular, for any finite n, the set $[n] = \{1, 2, \ldots, n\}$ forms a finite poset under this order.

Example 9. The set of divisors of any number n can also be ordered by divisibility. We define $i \leq j$ if i divides j. For example, if $n = 20$, we will have $2 \leq 4 \leq 20$.

Example 10. The set of all subsets of a given set can be *ordered by inclusion*, meaning $A \leq B$ if every element of A is also in B. In general, any collection of sets can be ordered by inclusion to form a poset: an example that will be particularly important is the poset of all linear subspaces of a given vector space.

1.1. Basic Properties of Posets.

Definition 1.1. A function $f\colon P \to Q$ from a poset P to a poset Q is *order-preserving* if $s \leq t$ in P implies that $f(s) \leq f(t)$. It is an *isomorphism* if it also has an order-preserving inverse. Two posets P, Q are said to be *isomorphic*, denoted $P \cong Q$ if there is an isomorphism from P to Q.

For example, if $P(S)$ denotes the poset of all subsets of a finite set S ordered by inclusion, then $P(S) \cong P(T)$ if and only if S and T are of the same size.

Definition 1.2. Analogously, a function $f\colon P \to Q$ is *order-reversing* if $s \le t$ in P implies that $f(t) \le f(s)$. An order-reversing map is an *anti-isomorphism* if it also has an order-reversing inverse. Two posets P, Q are said to be *anti-isomorphic* if there is an anti-isomorphism from P to Q.

Definition 1.3. A *subposet* of a poset P, is a subset Q of P and a partial ordering of Q inherited from the ordering of P. An *interval* $[s, t]$ in P is the subposet corresponding to the subset of all elements p such that $s \le p \le t$ in P.

Definition 1.4. A *chain* (or totally ordered set or linearly ordered set) is a poset in which any two elements are comparable. Thus the poset $[n]$ of Example 8 is a chain. A subset C of a poset P is called a chain if C is a chain when regarded as a subposet of P.

A poset Q is said to be a *refinement* of a poset P if there is an order preserving bijection from P to Q. Visually, if we think of P and Q as having the same ground set, then $\le_Q$ adds extra order relations to $\le_P$. A *linear extension* of a poset P is a chain which refines P.

1.1.1. *Hasse Diagrams.* If $s, t \in P$, then we say that t *covers* s if $s < t$ and no element $u \in P$ satisfies $s < u < t$. A finite poset P is completely determined by its cover relations. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $s < t$ then t is drawn above s.

Example 11. The Hasse diagram of the poset of divisors of 20 (rf. Example 9) is

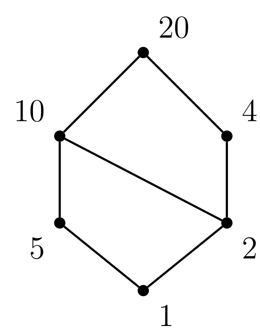


FIGURE 1. The poset of divisors of 20

1.1.2. *Graded Posets.* A *rank function* on a poset P is a function $\mathrm{rk}\colon P \to \underline{\mathbb{N}}$ such that $rk(s) = 0$ if s is a minimal element, and $\mathrm{rk}(t) = \mathrm{rk}(s) + 1$ if t covers s in P. If $\mathrm{rk}(s) = i$, we say that s has *rank* i. The Hasse diagram of a graded poset can be drawn so that for all i, the rank i elements all have the same vertical coordinate.

Definition 1.5. A poset P equipped with a rank function is called a *graded poset.* The *rank-generating* polynomial of a finite poset P is the polynomial

$$F(P, x) = \sum_{s \in P} x^{\mathrm{rk}(s)}.$$

Example 12. All examples of posets we have seen are graded. The poset of all subsets of a given set (see Figure 2) is graded, where the size of each subset is a rank function.

The poset of all divisors of a given number is graded by the rank function given by the number of prime factors, counted with multiplicity.

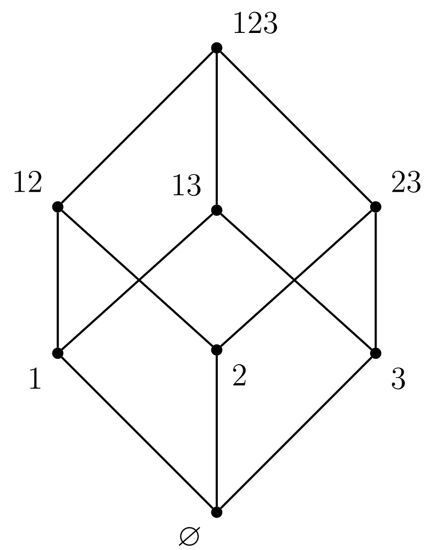


FIGURE 2. The poset of subsets of $\{1, 2, 3\}$

Example 13. The Tamari posets are a family of posets that cannot be graded.

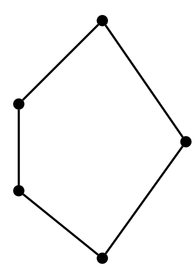


FIGURE 3. The smallest Tamari poset having 5 elements

1.1.3. *Lattices.* If s and t are elements of a poset P, then an *upper bound* is an element $u \in P$ such that $s \leq u$ and $t \leq u$. A *least upper bound* (or *join*) is an upper bound u satisfying $u \geq v$ for any other upper bound v. This least upper bound, when it exists, is unique and is denoted $s \vee t$ (read as *s join t*). Similarly, one defines the *greatest lower bound* (or *meet*) $s \wedge t$ (read as *s meet t*) when it exists.

A poset L is said to be a *lattice* if every pair of its elements has both a unique meet and a unique join.

Example 14. Every example of a poset we have encountered till now is a lattice. For instance, the poset of all subsets of a given set forms a lattice with meet being the intersection and join being the union.

The poset of divisors of a given natural number is a lattice where the meet of two numbers is their greatest common divisor, while join is their least common multiple.

The reader can check by inspection that the Tamari poset of five elements in Figure 3 is also a lattice.

Example 15. The following Hasse diagram defines a poset which is not a lattice, as can be easily seen.

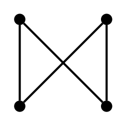


FIGURE 4. A poset which is not a lattice

1.2. Constructing new posets.

1.2.1. *Direct Products.* If P and Q are posets, then the *direct product* of P and Q is defined as the poset $P \times Q$ on the Cartesian product of the sets P and Q. In $P \times Q$, the order relation $\leq$ is defined such that $(s, t) \leq (s', t')$ if and only if $s \leq s'$ in P and $t \leq t'$ in Q.

It's worth noting that due to this definition, $P \times Q$ and $Q \times P$ are isomorphic, so we can carry out the product in any order.

If P and Q are graded with rank-generating functions $F(P, x)$ and $F(Q, x)$, then $P \times Q$ is a graded poset with rank function $\mathrm{rk}(s, t) = \mathrm{rk}_P(s) + \mathrm{rk}_Q(t)$ and $F(P \times Q, x) = F(P, x)F(Q, x)$.

1.2.2. *Dual of a Poset.* The *dual of a poset* is the poset P^* on the same set as P, but such that $s \leq t$ in P^* if and only if $t \leq s$ in P. If P and P^* are isomorphic, then P is called *self-dual*. The Hasse diagram of dual poset P^* is obtained by vertically reflecting (i.e. reflecting about the horizontal axis) the Hasse diagram of the poset P.

Remark 1.1. The identity function is an anti-isomorphism from P to P^*, so P is self-dual if and only if it is anti-isomorphic to itself.

Example 16. The rank generating polynomial of any self-dual poset exhibits an interesting property: if P is a self-dual graded poset with maximum rank n, then the number of elements of rank i is equal to the number of elements of rank $n - i$. In particular, the coefficients of $F(P, x)$ form a *palindromic* sequence.

1.3. Bruhat Order. The symmetric group $\mathfrak{S}_n$ denotes the set of all permutations $w \colon [n] \to [n]$. Any permutation w can be represented as a word $w = w_1 w_2 \cdots w_n$, with w_i being $w(i)$. For example, 123456 is the identity permutation in $\mathfrak{S}_6$.

We define a partial order $\leq$ on $\mathfrak{S}_n$, called the *Bruhat order*. If $v, w \in \mathfrak{S}_n$ are two permutations, then say $v \leq w$ if for all $i = 1, \ldots, n$, the subword $v_1 \cdots v_i$ when sorted is term-by-term dominated by the subword $w_1 \cdots w_i$ when sorted.

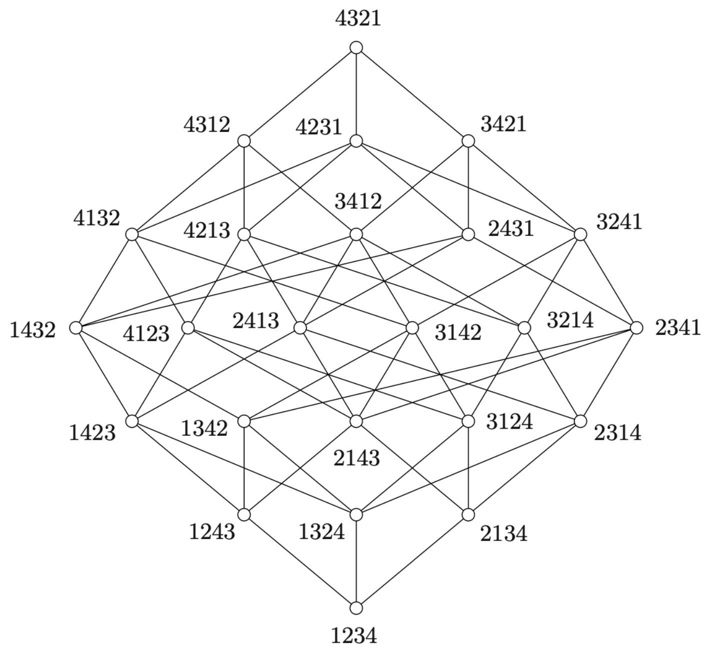


FIGURE 5. Bruhat Order on $\mathfrak{S}_4$ (taken from [**BB05**, p. 21])

Example 17. Consider the permutations 136245 and 561423 in $\mathfrak{S}_6$. Let us compare their subwords:

$$\operatorname{sort}(1) = 1 < 2 = \operatorname{sort}(2)$$
$$\operatorname{sort}(13) = 13 < 56 = \operatorname{sort}(56)$$
$$\operatorname{sort}(136) = 136 < 156 = \operatorname{sort}(561)$$
$$\operatorname{sort}(1362) = 1236 < 1456 = \operatorname{sort}(5614)$$
$$\operatorname{sort}(13624) = 12346 < 12456 = \operatorname{sort}(56142).$$

So, $136245 < 561423$ in Bruhat order. It is easily seen that the identity is the smallest element with respect to the bruhat order, while the permutation $w_0 = n, n-1 \cdots 1$ is the largest.

Let $w = w_1 \cdots w_n \in \mathfrak{S}_n$, then an *inversion* in w is a pair $i < j$ such that $w_i > w_j$. Define the *length* of w to be the number of inversions in w. A simple calculation shows that the length of the identity is zero, while the length of w_0 is $n(n-1)/2$.

Proposition 1.1. The symmetric group $\mathfrak{S}_n$ equipped with the Bruhat order is a graded poset with rank function given by the length of a permutation.

Observe that right multiplying a permutation w by w_0 has the effect of replacing any w_i in the corresponding word $w_1 \cdots w_n$ by $n - w_i + 1$. This means that $v \leq w$ implies that $w_0 v \geq w_0 w$, and multiplication by w_0 is an anti-isomorphism on $\mathfrak{S}_n$. This means that

Proposition 1.2. The Bruhat order is self-dual.

2. Diagrams and Tableaux

A *partition* $\lambda = (\lambda_1, \lambda_2 \ldots)$ is a weakly decreasing sequence of non-negative integers containing only finitely-many nonzero terms. We will not distinguish between two sequences which differ only by a string of zeroes at the end. For example, the sequences $(3, 2, 0)$, $(3, 2)$ and $(3, 2, 0, 0, 0)$ are all the same partition.

The nonzero λ_i are called the *parts* of λ, and the sum of the parts, denoted $|\lambda|$, is the size of the partition. If $|\lambda| = n$, we say that λ *is a partition of* n, and write $\lambda \vdash n$[1]

Sometimes it is convenient to describe a partition in *exponential notation*:

$$\lambda = 1^{m_1} 2^{m_2} \cdots$$

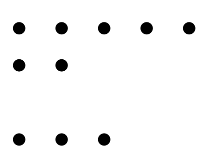
means that m_i many parts of λ equal i. For example, the partition $(4, 3, 3, 2, 1, 1)$ will be represented as $1^2 2 3^2 4$ in exponential notation.

A related notion to that of a partition is a *composition*. A sequence μ of positive integers is said to be a composition of n if $\sum_i \mu_i = n$. So, partitions of n can be viewed as special compositions where the entries in the sequence weakly decrease.

If the sequences are allowed to have zero entries then they are called *weak compositions*.

2.1. Diagrams.

A *diagram* of a weak composition μ may be formally defined as the set of points (i, j) in $\mathbb{Z}^2$ for all $j = 1, \ldots, \mu_i$. In drawing such diagrams, we shall adopt the convention, as with matrices, that the first coordinate i (the row index) increases as one goes downwards, and the second coordinate j (the column index) increases as one goes from left to right. For example, the diagram of the weak composition $(5, 2, 0, 3)$ is



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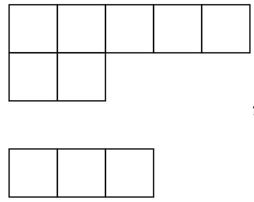
[1]To insert the symbol $\vdash$ in a document use `\vdash`.

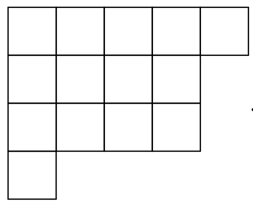
It is usually convenient to draw diagrams with squares instead of points, in which case the diagram is



,

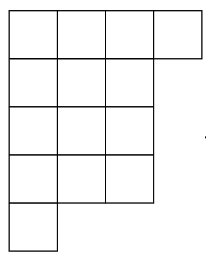
consisting of five squares in the first row, followed by two squares in the second row, none in the third row and three in the last row. We usually denote the diagram of a composition μ by the same symbol μ.

Example 18. The diagram of the partition $(5, 4, 4, 1)$ is



.

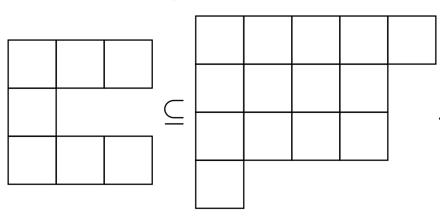
The *conjugate* of a partition λ is the partition λ' whose diagram consists of points (i, j) such that (j, i) belongs to the diagram of λ. So, the conjugate of the partition $(5, 4, 4, 1)$ is the partition $(4, 3, 3, 3, 1)$ with diagram



.

2.2. Partial Order on Diagrams. A partial order can be defined on the set of all weak compositions. We say $\mu \leq \mu'$ if, after padding both sequences with enough zeros, we have $\mu_i \leq \mu'_i$ for all $i \geq 1$. This relationship is visually intuitive when considering diagrams: $\mu \leq \mu'$ if and only if the diagram of μ is contained within the diagram of μ'. Thus, we refer to this order as the containment order, denoted by $\mu \subseteq \mu'$.

Example 19. Consider compositions $(3, 1, 3)$ and $(5, 4, 4, 1)$, then $(3, 1, 3) \subseteq (5, 4, 4, 1)$, as can be seen by the inclusion of the diagrams



.

By confining this ordering to the subset of all partitions, we derive a poset termed the *Young lattice*. This lattice is graded, whereby partitions are organized based on their size.

2.3. Tableaux. A *tableau*[2] on a partition λ is a function from the squares of the diagram λ to $\mathbb{N}$. A tableau can be graphically represented as a filling of the squares of λ by their respective images. For example,

(3)

3	2	8	8	8
1	1	9	1	
7	6	5	4	
6				

is a tableau on the partition $(5, 4, 4, 1)$. Suppose λ has size n, then the *weight* of a tableau on λ is a weak composition $\mu = (\mu_1, \ldots)$ such that for all i, μ_i is the number of entries of the tableau which are equal to i. The weight of the tableau (3), is equal to the composition $(3, 1, 1, 1, 1, 2, 1, 3, 1)$.

Let us now define some special kinds of tableaux that will recur in the thesis.

Definition 2.1. A tableau is said to be *weakly row increasing* (or simply, *row-weak*) if the entries in every row weakly increase on reading from left to right.

Example 20. The tableau in (3) is not weakly row increasing because the first row starts with a 3 followed by a 2. On the other hand, the tableau

1	1	2	2	3
4	5	6	7	
2	2	2	2	
6				

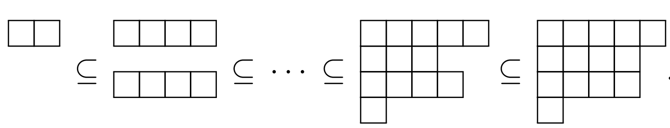
is weakly row increasing.

Definition 2.2. A tableau is said to be *weakly row and column increasing* (or simply *weak*) if the entries in every row weakly increase when read from left to right and the entries in every column weakly increase when read from top to bottom.

A tableau is said to be *semistandard* if it is weak and the entries strictly increase down each column. A semistandard tableaux of weight $(1, \ldots, 1)$ (n times) is said to be *standard*.

2.3.1. *Tableaux as chains.* Suppose we have a row-weak tableaux on a partition λ with weight equal to a composition μ. Then a handy trick is to view the tableaux as a chain in the poset of weak compositions. Of course, for all $i \geq 1$, the subset of squares of λ that have entries no greater than i is the diagram of some weak composition, and these diagrams form an increasing sequences of diagrams.

Example 21. Consider the tableau given in Example 20. Then we can think of it as the chain



[2]From the french word for table. Its plural form is tableaux.

If the tableau is a weak tableau, then we obtain a chain of Young diagrams. In fact, a standard tableaux is a maximal chain in the Young lattice.

2.3.2. *The column-sort.* We define an operation on the collection of tableaux, known as the *column-sort*, which simply sorts the rows of a tableau and makes them weakly increase when read from top to bottom. If we start with a row-weak tableau and then apply column-sort, obviously the resulting tableau will have weakly increasing columns. However, it's not immediately clear if the rows will also continue to remain weakly increasing.

Example 22. Let us apply column-sort to the tableau given in Example 20:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 4 & 5 & 6 & 7 \\ \cline{1-4} 2 & 2 & 2 & 2 \\ \cline{1-4} 6 \\ \cline{1-1} \end{array} \xrightarrow{\text{column}-\text{sort}} \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 4 & 5 & 6 & 7 \\ \cline{1-4} 6 \\ \cline{1-1} \end{array}.$$

Here, the resulting tableau has weakly increasing entries in each row and column.

Proposition 2.1. The operation column-sort maps row-weak tableaux into weak tableaux.

3. Symmetric Polynomials

The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbf{F}[x_1, \ldots, x_n]$ by permuting the variables $x_1, \ldots, x_n$. A polynomial $f(x_1, \ldots, x_n)$ is said to be a *symmetric polynomial* if $w \cdot f = f$ for all $w \in \mathfrak{S}_n$, i.e. if it is a fixed point for this action.

A basis of the space of symmetric polynomials is given by the *monomial symmetric polynomials* m_λ. For any partition λ, which can be written in exponential notation as $1^{m_1}2^{m_2}\cdots$, denote by $\mathbf{x}^\lambda = x_1^{\lambda_1}\cdots x_n^{\lambda_n}$. The polynomial m_λ is defined as

$$m_\lambda(x_1, \ldots, x_n) = \frac{1}{m_1!m_2!\cdots}\sum_{w\in\mathfrak{S}_n} w \cdot \mathbf{x}^\lambda.$$

Another basis of the space of symmetric polynomials is the collection of *complete homogeneous polynomials* indexed also by partitions. For any $r \geq 0$, we define the polynomials

$$h_r(x_1, \ldots, x_n) = \sum_{1\leq i_1\leq\cdots\leq i_r\leq n} x_{i_1}\cdots x_{i_r}.$$

In other words, h_r is the sum of every possible monomial of degree r. Then for a partition λ, define h_λ to be the product $h_{\lambda_1}h_{\lambda_2}\cdots$ of the polynomials corresponding to the parts of λ.

We can define a scalar product $\langle\cdot,\cdot\rangle$ on the space of symmetric functions with respect to which the m_λ and the h_λ are dual bases. This is known as the *Hall scalar product*. This much is all we will need for our purposes, but a complete treatment of the theory of symmetric functions can be found in [**Mac15, Ber09**].

Part 2

Geometry

CHAPTER 3

Paving a Variety by Affines

1. Poincaré Duality & Fundamental Classes

The Poincaré Duality theorem [**Hat02**, Theorem 3.30, p. 241] states that for all k, the kth homology groups and the $(n-k)$th cohomology groups of any n-dimensional oriented, compact, real manifold are isomorphic. In particular, if $\mathcal{N}$ is connected, the zeroth cohomology group being isomorphic $\mathbb{C}$ implies that the nth homology group is also isomorphic to $\mathbb{C}$. This means that there exists a generator, denoted $[\mathcal{N}]$, for the nth homology group called the *fundamental class* of $\mathcal{N}$.

If $\mathcal{M}$ is an m-dimensional submanifold of $\mathcal{N}$, then the fundamental class $[\mathcal{M}]$ which generates the mth homology group of $\mathcal{M}$. The inclusion of $\mathcal{M}$ in $\mathcal{N}$ pushes the class $[\mathcal{M}]$ forward to the mth homology group of $\mathcal{N}$. Roughly speaking, this allows us to locate elements in the homology groups of $\mathcal{M}$ by identifying submanifolds of $\mathcal{N}$.

1.1. Borel-Moore Homology. If a manifold is not compact, a fundamental class need not exist. For example, $\mathbb{R}^n$ is an n-dimensional manifold. However, since it is contractible, its nth homology group is trivial. We can fix this by introducing a version of homology where Poincaré duality is ensured for every manifold.

Definition 1.1. Let X be a topological space that is embedded as a closed subspace of an m-dimensional oriented real manifold $\mathcal{M}$. The *Borel-Moore homology groups*, denoted $\overline{H}_i(X)$, are defined by the formula

$$\overline{H}_i(X) = H^{m-i}(\mathcal{M}, \mathcal{M} - X).$$

For an m-dimensional oriented manifold X, if we set $\mathcal{M} = X$, we obtain the equation $\overline{H}_i(X) = H^{m-i}(X)$ for Borel-Moore homology. In particular, any oriented (not necessarily compact) manifold has a fundamental class in Borel-Moore homology.

Proposition 1.1. When X is an m-dimensional compact, oriented manifold, the Borel-Moore homology coincides with the singular homology.

Proof. We have seen that the Borel-Moore homology groups $\overline{H}_i(X)$ are equal to the singular cohomology groups $H^{m-i}(X)$, but because X is compact, these are isomorphic to $H_i(X)$ by Poincaré duality. $\square$

2. Fundamental classes of algebraic varieties

Complex projective spaces and open sets of complex projective spaces taken together with the Euclidean topology are real manifolds. By Hilbert's basis theorem, a projective algebraic variety X in projective m-space is defined as the zero set of a finite set of homogeneous polynomials, say $f_1, \ldots, f_r$, on $\mathbb{C}\mathbf{P}^m$. In other words, X is the level set $F^{-1}(0)$ of 0 for the smooth function $F = (f_1, \ldots, f_r)\colon \mathbb{C}\mathbf{P}^m \to \mathbb{C}^r$. Similarly, every quasi-projective algebraic variety can be viewed as the level set for a smooth function on an open set in complex projective space.

According to the *constant rank level-set theorem* [**Tu11**, Theorem 11.2, p. 116], if $F = (f_1, \ldots, f_m)\colon \mathcal{N} \to \mathcal{M}$ is a smooth map of manifolds, and p is an element of $\mathcal{M}$ such that the *Jacobian matrix* $dF(p)$ of F

$$dF(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{pmatrix},$$

has constant rank in the level set $F^{-1}(p)$ in $\mathcal{N}$, then $F^{-1}(p)$ is a submanifold of $\mathcal{N}$. This motivates the following definition.

Definition 2.1. A quasi-projective variety X defined as the vanishing set of polynomials $f_1, \ldots, f_r$ in some open set of projective space is said to be *smooth* if Jacobian matrix of $F = (f_1, \ldots, f_r)$ has the same rank on all points of X.

It immediately follows that any smooth, quasi-projective variety is a smooth manifold, and when the variety is oriented, compact and connected, it has a fundamental class. The essential feature of Borel-Moore homology is that it extends the existence of a fundamental class to any algebraic set, without requiring smoothness, compactness or irreducibility.

Proposition 2.1. Let X be a projective algebraic set of dimension k. Then the $2k$th Borel-Moore homology group has a generator for each k-dimensional irreducible component of X, while the higher Borel-Moore homology groups vanish.

PROOF. Refer to [**Ful97**, Lemma 4, p. 219]. □

3. Affine Paving

The homology groups of various familiar topological spaces have been computed, for example, in [**Hat02**]. Let us describe them and make some important observations.

Example 23 (Spheres). A real n-sphere S^n is defined as the zero locus of the polynomial $x_0^2 + \cdots + x_n^2 - 1$ over $\mathbb{R}$. As we have have seen in Example 7, the zeroth and nth homology groups of S^n are isomorphic to $\mathbb{Z}$, while the remaining homology groups are trivial.

The sphere is the union of the singleton consisting the basis vector e_n and its complement. The complement is a subvariety which is isomorphic, as a variety, to an $\mathbb{R}^n$ via *stereographic projection*. Furthermore, the closure of the complement adds back the point $(1, 0, \ldots, 0)$, which can itself be viewed as $\mathbb{R}^0$. So, S^n can be decomposed into affine spaces, one of dimension n and one of dimension zero. Interestingly, for each non-trivial homology group, there is an affine space of the respective dimension.

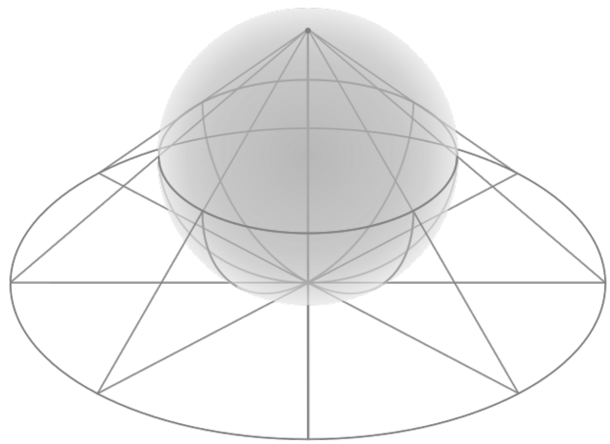


FIGURE 1. Stereographic Projection of S^2

Example 24 (Projective Space). The complex projective n-space $\mathbb{C}\mathbf{P}^n$ is a $2n$-dimensional real manifold. The odd numbered homology groups of $\mathbb{C}\mathbf{P}^n$ are all trivial, while the even numbered ones from 0 to $2n$ are isomorphic to $\mathbb{C}$.

The space $\mathbb{C}\mathbf{P}^n$ can also be decomposed into subvarieties, one for each $i = 0, 1, \ldots, n$, depending on the first nonzero homogeneous coordinate of any point. The ith subvariety will be isomorphic to an $(n-i)$-dimensional complex affine space, which is the same as $\mathbb{R}^{2(n-i)}$. As should be evident, the ranks of the homology groups are again reflected in the decomposition of the variety into affine spaces.

The observation in Examples 23 and 24 can be explained by means of a phenomenon known as an *affine paving* of a variety. The following version is due to Fulton [**Ful97**, Lemma 6, p. 222].

Definition 3.1. An affine paving of an algebraic variety X is a filtration $\varnothing = X_0 \subset \cdots \subset X_r = X$, satisfying, for all $i = 1, \ldots, r$, $X_i - X_{i-1}$ is isomorphic to an affine space.

Lemma 3.1. If a projective algebraic variety X has an affine paving, then for each i, the Betti number b_{2i} is equal to the number of affine spaces that are isomorphic to $\mathbb{C}^i$, while the odd Betti numbers are zero.

Before we dive into proving Lemma 3.1, let's see what happens to the long exact sequence in homology when working with Borel-Moore homology.

Lemma 3.2. If U is open in X, and Y is the complement of U in X, then there is a long exact seqeunce

$$\cdots \to \overline{H}_i(Y) \to \overline{H}_i(X) \to \overline{H}_i(U) \to \overline{H}_{i-1}(Y) \to \overline{H}_{i-1}(X) \to \overline{H}_{i-1}(U) \to \cdots, \tag{4}$$

in Borel-Moore homology.

PROOF. See [**Ful97**, Lemma 3, p. 219]. The idea of the proof is that if X can be embedded as a closed subspace of an orientable manifold $\mathcal{M}$, then $\mathcal{M} - Y$ is an orientable manifold containing U as a closed subspace. Then, this is the long exact sequence in cohomology applied to $\mathcal{M} - X \subset \mathcal{M} - Y \subset \mathcal{M}$. □

PROOF OF LEMMA 3.1. Let $X_0 \subset \cdots \subset X_r = X$ be the affine paving and let U_p be the difference $X_p - X_{p-1}$. We first argue by induction on p that the odd Betti numbers of X_p are zero. Assuming the result for $p-1$, X_{p-1} and U_p have trivial homology in odd dimensions. Plugging these into the long exact sequence (4), we deduce that X_p must also have odd homology trivial. As a result, its Betti numbers are also zero in odd dimensions.

Similarly, for each i, we argue by induction on p that $b_{2i}(X_p)$ is equal to the number of U_j isomorphic to $\mathbb{C}^i$, given $j \leq p$. The vanishing of odd homology for all involved spaces implies the existence of short exact sequences

$$0 \to \overline{H}_{2i}(X_{p-1}) \to \overline{H}_{2i}(X_p) \to \overline{H}_{2i}(U_p) \to 0,$$

which means that $\overline{H}_{2i}(X_p) = \overline{H}_{2i}(X_{p-1}) \oplus \overline{H}_{2i}(U_p)$. By induction, the assertion follows. □

Definition 3.2. Let $\mathcal{P}$ be a finite graded poset with rank function rk. A projective algebraic variety X is said to have an *affine paving over* $\mathcal{P}$ if there exist subvarieties U_a of X indexed by elements of $\mathcal{P}$ such that

(1) X is the disjoint union of the U_a,
(2) each U_a is isomorphic to the affine space $\mathbb{C}^{\mathrm{rk}(a)}$, and
(3) the closure $\overline{U_a}$ contains the subvarieties U_b whenever $b \leq a$ in $\mathcal{P}$.

Theorem 3.3. Let $\mathcal{P}$ be a finite graded poset and let X be a projective algebraic variety with an affine paving over $\mathcal{P}$. Then the odd Betti numbers of X are zero, while for each i, the Betti number $b_{2i}(X)$ is equal to the number of elements in $\mathcal{P}$ of rank i.

PROOF. Choose a linear extension $\mathcal{P}^*$ of $\mathcal{P}$ (e.g., by arbitrarily ordering elements of the same rank). If $\mathcal{P}$ has k elements, label its elements $a_1, \ldots, a_k$ so that $i \leq j$ implies that $a_i \leq a_j$ in $\mathcal{P}^*$. The subvarieties $X_i = \bigcup_{j \leq i} U_{a_j}$ define an affine paving of X, and so we can apply Lemma 3.1. □

CHAPTER 4

Grassmannians

Grassmannians are geometric spaces that parametrize all possible linear subspaces of a given dimension in a vector space. The study of these spaces can be traced back to nineteenth century enumerative geometry, which seeked answers to questions such as "How many lines intersect two given lines and a point in Euclidean space". These can be recast into understanding the intersections of certain subspaces of the Grassmanian, known as Schubert varieties. As we will discover in this chapter, Schubert varieties have a rich geometric structure that can be completely described by means of combinatorial objects known as partitions. This makes them amenable to studying various important geometric phenomena such as homology and smoothness etc.

1. The Plücker Map

Let us fix a field $\mathbf{F}$, and let $V = \mathbf{F}^n$ be an n-dimensional vector space with a standard basis $\{e_1, \ldots, e_n\}$. The *d-dimensional Grassmannian*, denoted by $\mathcal{G}_d(V)$, is the set of all d-dimensional linear subspaces of V. For $d = 1$, we obtain the set of lines in V, which is the projective space $\mathbf{P}V$ associated to V.

Let us denote by $C(d, n)$ the set of all d-subsets of $[n]$; we will often think of a set $\mathbf{s}$ in $C(d, n)$ as being an increasing sequence $s_1 < s_2 < \cdots < s_d$ of its elements. If W is a d-dimensional subspace of V, then the exterior power $\wedge^d W$ is a one-dimensional subspace of $\wedge^d V$. Consequently, we can view W as a point in $\mathbf{P} \wedge^d V$. The space $\wedge^d V$ has a basis labelled by elements of $C(d, n)$: each basis element $e_{\mathbf{s}} = e_{s_1} \wedge \cdots \wedge e_{s_d}$ is formed by taking the wedge product of d standard basis vectors chosen from the subset $\mathbf{s}$ specified by the index. Not distinguishing a vector from its coordinates with respect to this basis identifies $\mathbf{P} \wedge^d V$ with $\mathbf{P}^{\binom{n}{d}-1}$, which gives the *Plücker map*

$$\varphi\colon\ \mathcal{G}_d(V) \to \mathbf{P}^{\binom{n}{d}-1}.$$

Suppose that W is in $\mathcal{G}_d(V)$, then $\varphi(W)$ is the point $[w_1 \wedge \cdots \wedge w_d]$ in $\mathbf{P} \wedge^d V$ for any ordered basis $\{w_1, \ldots, w_d\}$ of W. Let us represent the basis as rows of a $d \times n$ matrix

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{d1} & w_{d2} & \cdots & \cdots & w_{dn} \end{pmatrix} \tag{5}$$

with entries defined by $w_i = \sum_j w_{ij} e_j$. Then the homogeneous coordinates of $\varphi(W)$ in $\mathbf{P}\wedge^d V$ are the $d \times d$ minors of the matrix (w_{ij}), which we denote

$$P_{\mathbf{s}} = \det(w_{is_j})_{1 \le i,j \le d}, \quad \mathbf{s} = s_1 < \cdots < s_d.$$

These are called the *Plücker coordinates* of W. In what follows we will use the notation $P_{s_1 \cdots s_d}$ even when the s_i do not form an increasing sequence.

Example 25. Suppose that U is the two-dimensional subspace of $\mathbb{C}^3$ spanned by the rows of the 2×3 matrix:

$$\begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \end{pmatrix}.$$

Let us compute the homogeneous coordinates of $\phi(U)$ in the projective space $\mathbf{P}^2$:

$$\begin{aligned} \phi(U) &= \left[\det\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \det\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}, \det\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}\right] \\ &= [2, 4, -6] = [1, 2, -3]. \end{aligned}$$

Proposition 1.1. The map ϕ is injective.

Proof. Suppose W is a d-dimensional subspace of V. Given $\varphi(W)$, we can recover W as the kernel of the map from V to $\wedge^{d+1} V$ which takes a vector v to its wedge product with some representative of $\varphi(W)$ in $\wedge^d V$. To see this, let us choose an ordered basis of V such that the first d basis vectors $w_1, \dots, w_d$ span W. Then the map sends a vector v to some scalar multiple of $v \wedge w_1 \wedge \cdots \wedge w_d$. If v is in the kernel, expressing it in terms of the basis shows that it must be spanned by $w_1, \dots, w_d$. □

Proposition 1.2. The image of ϕ is a closed subset of $\wedge^d V$.

We will need the following lemma to show this.

Lemma 1.3. An element $\eta \in \wedge^d V$ can be expressed as a wedge product of d vectors in V if and only if the kernel of map $\Phi_\eta \colon V \to \wedge^{d+1} V$, which sends v to $v \wedge \eta$ has dimension at least d.

Proof. If $\eta = w_1 \wedge \cdots \wedge w_d$, then the vectors $w_1, \dots, w_d$ are in the kernel of Φ_η. This means that the kernel has dimension at least d. To establish the reverse direction, let's assume that the kernel contains d linearly independent vectors, which we can conveniently take as the first d standard basis vectors without loss of generality. Expressing η with respect to this basis, first we wedge it with e_1. Since this must be zero, this allows us to deduce that the coefficients of the basis vectors $e_{\mathbf{s}}$ corresponding to subsets not containing 1 must be zero. Repeating this process with $e_2, \dots, e_d$, we arrive at the conclusion that the only basis vector with nonzero coefficients is $e_1 \wedge \cdots \wedge e_d$. In other words, $\eta = c e_1 \wedge \cdots \wedge e_d$ for some nonzero scalar c. □

Proof of Proposition 1.2. We will find a collection of homogeneous polynomials in the Plúcker coordinates such that the Grassmannian corresponds to the set of points where the polynomials vanish. We have a linear map

$$\Phi\colon \wedge^d V \to \mathrm{Hom}(V, \wedge^{d+1} V)$$

which sends $\eta \in \wedge^d V$ to the map Φ_η as defined in Lemma 1.3. Here, we interpret the space $\mathrm{Hom}(V, \wedge^{d+1} V)$ as the space of $n \times \binom{n}{d+1}$ matrices. By Lemma 1.3, the Grassmannian corresponds to the set of points whose image has rank at most $n-d$, or in other words, the set of points defined by the vanishing of all $(n-d+1) \times (n-d+1)$ minors of their image under Φ. These minors are all homogeneous polynomials in the Plücker coordinates, and as a result, $\mathcal{G}_d(V)$ is closed. □

The Plücker map is an embedding of the Grassmannian as a projective algebraic subset of $\mathbf{P} \wedge^d V$. We will show by the end of this chapter that it is indeed a projective variety.

Remark 1.1. The collection of homogeneous polynomials we found in the proof of Proposition 1.2 does not generate the full homogeneous vanishing ideal of the Grassmannian. An interested reader can read about the *Plücker relations*, a collection of quadratic polynomials which generate the ideal, in [**Man01**, Section 3.1.3].

2. Schubert Cells

The projective space $\mathbf{P} \wedge^d V$ can be divided into subsets $U_{\mathbf{s}}$, as $\mathbf{s}$ varies over all d-subsets of $[n]$, depending on what the index of the first nonzero homogeneous coordinate of a point is. A *Schubert cell* $\Omega_{\mathbf{s}}$ is defined as the intersection of the subset $U_{\mathbf{s}}$ with the Grassmannian. As a result, they are locally closed subsets of $\mathbf{P} \wedge^d V$ and the Grassmannian has a partition

$$\mathcal{G}_d(V) = \bigsqcup_{\mathbf{s} \in C(d,n)} \Omega_{\mathbf{s}},$$

into the Schubert cells.

2.0.1. *Canonical Bases.* We have defined Schubert cells as certain natural subsets of the Grassmannian viewed as a subset of projective space, sure. However, an obvious question arises: What subspaces do these points correspond to? Can we provide an equally natural description of these subspaces?

Definition 2.1. An ordered basis $\{w_1, \ldots, w_d\}$ of W is a canonical basis if (w_{ij}) is a matrix in reduced row echelon form, i.e. if there exists a sequence $1 \le c_1 < \cdots < c_d \le n$, such that (1) the entry $w_{ij} = 0$ whenever $j < c_i$, and (2) $w_{ic_j} = 1$ whenever $i = j$, and 0 otherwise.

Since every matrix corresponds to a unique matrix in reduced row echelon form, all subspaces have a unique canonical basis. Therefore, the indices $c_1, \ldots, c_d$ are also uniquely determined, and are called the set of *pivots* of the subspace or the basis. We will often also refer to the column number of any vector's first nonzero entry as its pivot.

Example 26. Suppose U is the two-dimensional subspace of $\mathbb{C}^4$ spanned by the rows of the 2×4 matrix

$$\begin{pmatrix} -1 & 2 & -1 & 3 \\ 2 & -4 & 2 & 1 \end{pmatrix}.$$

The point $\phi(U)$ has Plücker coordinates equal to $[0, 0, 1, 0, -2, 1]$, which implies that U is an element of the Schubert cell $\Omega_{\{1,4\}}$.

To find the canonical basis of U, we can put the above matrix into its reduced row echelon form:

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

This matrix has pivot set equal to $\{1, 4\}$, which is intriguingly also the index of the Schubert cell which contains U.

2.0.2. *Partial Order on $C(d, n)$.* The set of d-subsets of n can be made into a poset by defining $\mathbf{r} \leq \mathbf{s}$ if the sequence $\mathbf{s}$ is termwise larger than the sequence $\mathbf{r}$. In other words, if $\mathbf{r} = \{r_1 < \cdots < r_d\}$ and $\mathbf{s} = \{s_1 < \cdots < s_d\}$, then $r_i \leq s_i$ for all $i = 1, \ldots, d$.

Example 27. Suppose $\mathbf{r} = \{1, 3, 5\}$, $\mathbf{s} = \{2, 3, 6\}$ and $\mathbf{t} = \{1, 2, 6\}$. Then $\mathbf{r} \leq \mathbf{s}$ and $\mathbf{t} \leq \mathbf{s}$. However, $\mathbf{r}$ and $\mathbf{t}$ are not comparable.

Lemma 2.1. The lexicographic order is a refinement of $\leq$. In particular, if $\mathbf{r}$ precedes $\mathbf{s}$ lexicographically, then $\mathbf{r} \not\geq \mathbf{s}$.

Proof. If $\mathbf{r}$ precedes $\mathbf{s}$ lexicographically, then there is some i for which $r_i < s_i$, so it cannot be that $\mathbf{r} \geq \mathbf{s}$. □

Lemma 2.2. Suppose $W \in \mathcal{G}_d(V)$ has pivot set equal to $\mathbf{s}$, then the Plücker coordinate $P_{\mathbf{r}}$ is zero whenever $\mathbf{r} \not\geq \mathbf{s}$.

Proof. Once again, let us represent the canonical basis of W as the rows of a $d \times n$ matrix, as in (5). In each column numbered $1, \ldots, s_i - 1$, all entries except the the first $i - 1$ are zero, thus generating a subspace of at most $i - 1$-dimensions in $\mathbf{F}^i$. Now, suppose that $\mathbf{r} \not\geq \mathbf{s}$, then there is an i such that $r_i < s_i$. This implies that the first i columns of the submatrix with columns numbered $1, \ldots r_d$ must be linearly dependent. This ensures that the Plücker coordinate $P_{\mathbf{r}}$, being the determinant of the submatrix, must be zero. □

On the other hand, a similar argument using canonical bases shows that the coordinate indexed by the pivot set must be non-zero [1]. So we can deduce the following from Lemmas 2.1 and 2.2.

Proposition 2.3. The points of a Schubert cell $\Omega_{\mathbf{s}}$ correspond exactly to subspaces $W \in \mathcal{G}(d, V)$ with pivots equal to $\mathbf{s}$.

[1]The reader should verify these for Example 26.

2.1. Affine-ness of Cells. In preparation of investigating the geometry of the Schubert cells let us recast some of what we have discussed in more geometric terms. Define $V^d = V \times \cdots \times V$ (d times) to be the set of d-tuples of vectors in V. Viewing as $d \times n$ matrices with entries in k identifies V^d with the affine dn-space $\mathbf{A}^{d\times n}$. For every $\mathbf{s} \in C(d,n)$, the reduced row echelon form matrices with pivots $\mathbf{s}$ define a linear subvariety of V^d:

$$\mathrm{E}_{\mathbf{s}} = \{(x_{ij}) \in V^d : x_{ij} = 0 \text{ if } i \leq s_i \text{ and } x_{is_j} = \delta_{ij}\},$$

where δ_{ij} is 1 if $i = j$ and zero otherwise. Similarly, we can think of the wedge product as a map

$$\varpi = \varpi_d \colon V^d \to \wedge^d V$$

defined by $(v_1, \ldots, v_d) \mapsto v_1 \wedge \cdots \wedge v_d$. If $\mathbf{p}_d = \mathbf{p}\colon \wedge^d V - \{0\} \to \mathbf{P} \wedge^d V$ is the usual projection map, then the Plücker embedding of the Grassmannian can be thought of as the composition $\varpi\mathbf{p}$.

Example 28. Suppose $n = 5$ and $d = 3$, then every element of V^d can be represented by a matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \end{pmatrix}.$$

Suppose we have a subset $S = \{1, 3, 4\} \in C(3, 5)$, then a generic element of E_S is a matrix of the form

$$\begin{pmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix},$$

where a $*$ can be replaced by any field element.

Proposition 2.3 is then saying that the morphism $\mathbf{p}\varpi$ is a bijection from the set of reduced row echelon matrices $\mathrm{E}_{\mathbf{s}}$ to the Schubert cell $\Omega_{\mathbf{s}}$. Turns out, we can say more.

Theorem 2.4. The map $\mathbf{p}\varpi\colon \mathrm{E}_{\mathbf{s}} \to \Omega_{\mathbf{s}}$ is an isomorphism of varieties.

warm-up 1. To prove Theorem 2.4, we just need to find a morphism inverse to $\mathbf{p}\varpi$. Let us first warm up with the case of $\mathbf{s} = \{1, \ldots, d\}$, where a generic element of $\mathrm{E}_{\mathbf{s}}$ is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & x_{1,d+1} & \cdots & x_{1,n} \\ 0 & 1 & \cdots & \cdots & 0 & x_{2,d+1} & \cdots & x_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 & x_{d,d+1} & \cdots & x_{d,n} \end{pmatrix}. \tag{6}$$

Then, we obtain among the Plücker coordinates, $P_{1,\ldots,i-1,j,i+1,\ldots,d} = x_{i,d+j}$. This gives rise to the required inverse, and so $\Omega_{1,\ldots,d}$ is isomorphic to $\mathrm{E}_{1,\ldots,d}$.

Proof of Theorem 2.4. We can easily permute the columns of every matrix in $\mathrm{E}_{\mathbf{s}}$ so that the the pivot columns are in positions $1, \ldots, d$. Every entry in a pivot column except the 1 is zero, so this operation transforms all the matrices into the form (6), while the Plücker coordinates might undergo shuffling and sign changes. However, these changes can be precisely described by the permutation applied. Consequently, we are still able to recover the matrix entries as Plücker coordinates, as discussed in warm-up 1. □

The sets $E_{\mathbf{s}}$ are affine spaces. A direct implication of Theorem 2.4 is that Schubert cells are isomorphic to affine spaces. Consequently, they are locally closed, irreducible (due to the irreducibility of affine spaces), subsets of the Grassmannian. In other words, they are subvarieties of the Grassmannian.

2.1.1. *Partitions.* The dimension of these affine spaces $E_{\mathbf{s}}$ can be computed by counting the free entries (represented by the $*$'s in Example 28) in the respective matrices. The number of $*$'s in the ith row is $\lambda_i = (n-d) - s_i + i$ and $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a partition.

The correspondence $\mathbf{s} \leftrightarrow \lambda$ establishes a bijection between the set of d-subsets of $[n]$ and the set of partitions whose Young diagrams fit within a $d \times (n-d)$ rectangle, the Young diagram of the partition d^{n-d}. As previously discussed, this set is the interval $[\varnothing, d^{n-d}]$ in the Young lattice, and the bijection is actually an anti-isomorphism of posets.

The upshot is that we can index the Schubert cells by partitions.

Theorem 2.5. For every partition $\lambda \subseteq d^{n-d}$, the Schubert cell $\Omega_\lambda \cong \mathbf{A}^{|\lambda|}$ is isomorphic to an affine space of dimension equal to the size of the partition.

3. Schubert Varieties

Schubert varieties $X_{\mathbf{s}}$ are defined as the closures of the Schubert cells $\Omega_{\mathbf{s}}$ in $\mathbf{P} \wedge^d V$. Given that the Schubert cells are quasi-projective varieties, this implies that Schubert varieties, as their name suggests, are projective algebraic varieties.

Proposition 3.1. For all $\mathbf{s} \in C(d,n)$, we have $X_{\mathbf{s}} = \bigsqcup_{\mathbf{r} \geq \mathbf{s}} \Omega_{\mathbf{r}}$.

We will show this by first locating for all $\mathbf{r} \geq \mathbf{s}$, the $\Omega_{\mathbf{r}}$ in $X_{\mathbf{s}}$. We put

$$\begin{aligned} D_{\mathbf{r}} &= \{(x_{ij}) \in V^d \mid x_{ij} = 0 \text{ if } j < r_i\} \text{ and} \\ E'_{\mathbf{r}} &= \{(x_{ij}) \in D_{\mathbf{r}} \mid x_{ir_i} \neq 0\}. \end{aligned}$$

Observe that any matrix in $E'_{\mathbf{r}}$ has pivot set equal to $\mathbf{r}$, so its image under $\mathbf{p}\varpi$ is also equal to the Schubert cell $\Omega_{\mathbf{r}}$. Moreover, the closure $\overline{E}'_{\mathbf{r}}$ in V^d is $D_{\mathbf{r}}$ and the set $D_{\mathbf{s}}$ contains the sets $E'_{\mathbf{r}}$ whenever $\mathbf{r} \geq \mathbf{s}$.

Lemma 3.2. Suppose $\mathbf{r} > \mathbf{s}$ are two d-subsets of $[n]$. Then we have the inclusion $\Omega_{\mathbf{r}} \subseteq X_{\mathbf{s}}$.

Proof. This follows simply from the observations made above. Because $\mathbf{r} \geq \mathbf{s}$, the set $E'_{\mathbf{r}}$ is contained in the set $D_{\mathbf{s}}$. Then we apply $\mathbf{p}\varpi$:

$$\begin{aligned} \Omega_{\mathbf{r}} &\subseteq \mathbf{p}(\varpi D_{\mathbf{s}} - \{0\}) \\ &= \mathbf{p}(\varpi \overline{E'_{\mathbf{s}}} - \{0\}). \end{aligned}$$

Recall that under a continuous function, the image of the closure of a set is contained in the closure of the image of that set. Because $\mathbf{p}\varpi$ is continuous, we get

$$\Omega_{\mathbf{r}} \subseteq \overline{\mathbf{p}\varpi \mathrm{E}'_{\mathbf{s}}} = \overline{\Omega_{\mathbf{s}}} = X_{\mathbf{s}},$$

as claimed. □

Next, we show that the Schubert variety X_S does not intersect the cells $\Omega_{\mathbf{r}}$ when $\mathbf{r} \not\geq \mathbf{s}$. Because the Schubert cells partition the Grassmannian, this would imply Proposition 3.1.

Lemma 3.3. Suppose $\mathbf{r}, \mathbf{s} \in C(d, n)$ and $\mathbf{r} \not\geq \mathbf{s}$. Then the intersection $\Omega_{\mathbf{r}} \cap X_{\mathbf{s}}$ is empty.

PROOF. Since $\mathbf{r} \not\geq \mathbf{s}$, the $\mathbf{s}$th Plücker coordinate of every point in the Schubert cell $\Omega_{\mathbf{s}}$ is zero. This means that the cell is contained in the closed subset $Z(P_{\mathbf{s}})$, defined by the vanishing of the $\mathbf{s}$th coordinate. Consequently, the Schubert variety $X_{\mathbf{s}}$, its closure, is also contained in $Z(P_{\mathbf{s}})$.

On the other hand, the $\mathbf{r}$th Plücker coordinate of any point in the Schubert cell $\Omega_{\mathbf{r}}$ cannot be zero, as shown in Lemma 2.1. Hence, the intersection $\Omega_B \cap X_S$ must be empty. □

Corollary 3.4. Suppose $\mathbf{r}, \mathbf{s} \in C(d, n)$. The following are equivalent:

(1) $\mathbf{r} \leq \mathbf{s}$,
(2) $X_{\mathbf{r}} \cap \Omega_{\mathbf{s}} \neq \varnothing$,
(3) $X_{\mathbf{r}} \supseteq X_{\mathbf{s}}$.

The subset $\{1, \ldots, d\}$ is the smallest element of $C(d, n)$, which means

$$X_{1,\ldots,d} = \bigsqcup_{\mathbf{s} \in C(d,n)} \Omega_{\mathbf{s}} = \mathcal{G}_d(V).$$

In particular, $\mathcal{G}_d(V)$ is a Schubert variety. We have finally shown that

Theorem 3.5. The Grassmannian is a projective variety.

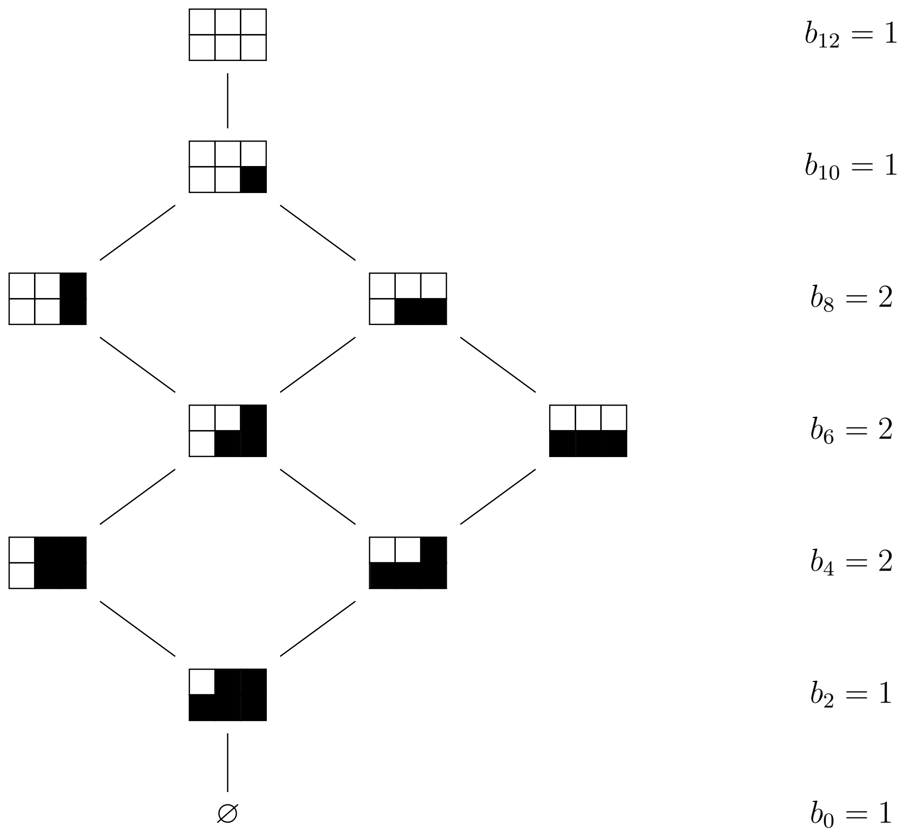
4. Betti numbers of the Grassmannian

The Schubert decomposition of the complex Grassmannian is a partition of the space into affine spaces indexed by partitions, which are elements of a graded poset $[\varnothing, d^{n-d}]$. Furthermore, Proposition 3.1 can be restated when we label the Schubert cells and Schubert varieties by partitions as follows. For all $\lambda \subseteq d^{n-d}$,

$$X_\lambda = \bigsqcup_{\mu \subseteq \lambda} \Omega_\mu. \tag{7}$$

These together imply that the Schubert decomposition is an affine paving over $[\varnothing, d^{n-d}]$. So, the Grassmannian has trivial Betti numbers in odd dimensions, while its $2i$th Betti number is equal to the number of partitions of size i contained in d^{n-d}.

Example 29. When $n = 5$ and $d = 2$, the Betti numbers of the Grassmannian are computed below:



Therefore, the Poincaré polynomial of the Grassmannian $\mathcal{G}_2(V)$ is $1 + t + 2t^2 + 2t^3 + 2t^4 + t^5 + t^6$.

CHAPTER 5

Flag Varieties

Instead of subspaces of fixed dimension in a vector space, we now study chains of linear subspaces, or flags. In studying the geometry of the collection of all flags we are led to their decomposition into cells indexed this time by permutations. Particularly, we will see how the combinatorics of permutations helps us in describing the enumerative and topological features of these spaces.

1. Complete Flag Varieties

A *flag of subspaces* in a vector space refers to a sequence of subspaces where each subspace is contained within the next one, starting from the zero subspace, going up to the entire space. The *type* of a flag denotes the sequence of codimensions of each subspace relative to the next. More precisely, a flag

$$W_\bullet = (\varnothing = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{m-1} \subseteq W_m = V)\,,$$

in V has type $\mu = (\mu_1, \ldots, \mu_m)$, where $\mu_i = \dim W_i - \dim W_{i-1}$. A *complete flag* is a flag of type $(1, \ldots, 1)$ (n times) and the *complete flag variety* is the set of all complete flags in V.

1.1. The Segre Embedding. Complete flags can be considered as elements of the product $\mathcal{G}_1(V) \times \cdots \times \mathcal{G}_{n-1}(V)$. Thus, $\mathcal{F}(V) \subseteq \mathbf{P}V \times \cdots \times \mathbf{P} \wedge^{n-1} V$. It's noteworthy that any product of projective spaces can be embedded into a larger projective space using the *Segre embedding.*

Let V_1, V_2 be vector spaces of dimensions k, l, respectively. The map $\mathbf{P}V_1 \times \mathbf{P}V_2 \to \mathbf{P}(V_2 \otimes V_2)$ defined by sending $([v_1], [v_2]) \to ([v_1 \otimes v_2])$ is well-defined and injective. This is known as the *Segre embedding.* In coordinates, the map sends an ordered pair $[a_0, \ldots, a_{k-1}] \times [b_0, \ldots, b_{l-1}]$ to $[\ldots, a_i b_j, \ldots]$ ordered lexicographically.

Lemma 1.1 ([**Har77**]). The image of the Segre embedding is an algebraic subset of $\mathbf{P}(V_1 \otimes V_2)$.

Let $N = \binom{n}{1} + \cdots + \binom{n}{n-1} - 1$. We can Segre embed the product of projective spaces

$$\psi\colon \mathbf{P}V \times \cdots \times \mathbf{P} \wedge^{n-1} V \to \mathbf{P}\left(V \otimes \cdots \otimes \wedge^{n-1} V\right) \cong \mathbf{P}^N,$$

and identify the complete flag variety $\mathcal{F}(V)$ with its image under ψ in $\mathbf{P}^N$.

Proposition 1.2. The image $\psi(\mathcal{F}(V))$ is an algebraic subset of $\mathbf{P}^N$.

PROOF. The proof is very similar to the proof of Proposition 1.2 and can be found in [**Gec03**, Theorem 3.3.11]. □

The map ψ is the *Plücker embedding* of the complete flag variety and the homogeneous coordinates of points of $\psi(\mathcal{F}(V))$ are their *Plücker coordinates*. In this section we will show that the image is irreducible and therefore has the structure of a projective variety. Henceforth, *points of the flag variety* will mean both the flags of subspaces and the corresponding points in projective space.

1.2. Schubert Decomposition of the Flag Variety. The Plücker coordinates on the flag variety are indexed by tuples of subsets of $[n]$ ordered lexicographically, so that the ith entry is a subset of size i. It easily follows from the description of the Segre embedding that the image of any point in the product of the Schubert cells $\Omega_{\mathbf{s}_i}$ is a point in the projective space $\mathbf{P}^N$ whose first nonzero homogeneous coordinate is indexed by the tuple $(\mathbf{s}_1, \dots, \mathbf{s}_n)$. Moreover, inclusion relations that are imposed among the subspaces of a flag manifest themselves at the level of the indexing subsets as well.

Proposition 1.3. Let $\mathbf{s}_1, \dots, \mathbf{s}_{n-1}$ be subsets of $[n]$ such that for all $i = 1, \dots, n-1$, the set $\mathbf{s}_i$ has size i. Then there is a flag $W_\bullet$ such that W_i has pivot set $\mathbf{s}_i$ if and only if the $\mathbf{s}_i$s form an *increasing sequence of subsets*. That means that the subsets satisfy

$$\varnothing \subset \mathbf{s}_1 \subset \cdots \subset \mathbf{s}_{n-1} \subset [n].$$

PROOF. Let us consider a flag $W_\bullet$, where for each $i = 1, \dots, n-1$, the subspace W_i has a pivot set equal to $\mathbf{s}_i$. The pivots are the column numbers containing the first nonzero entry of each canonical basis vector. Since every vector in the subspace is a linear combination of these basis vectors, their first nonzero coordinate must also appear in one of the columns indexed by $\mathbf{s}_i$. Furthermore, the inclusion $W_i \subset W_{i+1}$ implies that the subspace W_{i+1} also contains vectors with the first nonzero entry in positions indexed by the set $\mathbf{s}_i$. Consequently, $\mathbf{s}_i$ is contained in the set of pivots $\mathbf{s}_{i+1}$ of W_{i+1}.

Conversely, for any increasing sequence of subsets, we can construct a flag $W_\bullet$ such that each W_i is spanned by basis vectors corresponding to elements of the subset $\mathbf{s}_i$. Then W_i will have pivot set equal to $\mathbf{s}_i$. □

1.2.1. *Schubert Cells.* The natural action of the symmetric group $\mathfrak{S}_n$ on $[n]$ induces a free and transitive action on the collection of increasing sequences of subsets. As a result, there exists a bijective correspondence between $\mathfrak{S}_n$ and the set of increasing sequences.

Let $\mathbf{S}$ be the sequence

$$\varnothing \subseteq \{1\} \subseteq \{1,2\} \subseteq \cdots \subseteq \{1, \dots, n\},$$

where the ith element is the subset $\{1, \dots, i\}$. The bijection can be explicitly described as a permutation w being mapped to the sequence $w \cdot \mathbf{S}$, where the ith element is the set $\{w(1), \dots, w(i)\}$. Henceforth, we will not distinguish between a permutation w and the corresponding sequence $w\mathbf{S}$, often denoting the Plücker coordinates by permutations as P_w.

Example 30. If $n = 5$, then the permutation $32514 \in \mathfrak{S}_5$ maps the increasing sequence

$$\varnothing \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \{1,2,3,4\} \subseteq \{1,2,3,4,5\},$$

to the increasing sequence

$$\varnothing \subseteq \{3\} \subseteq \{2,3\} \subseteq \{2,3,5\} \subseteq \{1,2,3,5\} \subseteq \{1,2,3,4,5\}.$$

Definition 1.1. For each permutation $w \in \mathfrak{S}_n$, we define the *Schubert cell* Ω_w as the image under the Segre embedding of the set of complete flags whose set of pivots corresponds to w under the bijection described above.

This gives the *Schubert decomposition* of the complete flag variety

$$\mathcal{F}(V) = \bigsqcup_{w \in \mathfrak{S}_n} \Omega_w.$$

Note that the Schubert cell Ω_w is also the subset of points in $\mathbf{P}^N$ where the first nonzero Plücker coordinate is indexed by $w\mathbf{S}$. As a result, they are locally closed subsets of $\mathbf{P}^N$.

1.2.2. *The Dual Bruhat Order* $\mathfrak{S}_n$. If v and w are two permutations in S_n, then for each $i = 1, \ldots, n-1$, we can compare the subsets $\{w(1), \ldots, w(i)\}$ and $\{v(1), \ldots, v(i)\}$ in $C(i,n)$. This gives rise to a partial order on $\mathfrak{S}_n$.

Definition 1.2. If $v, w \in \mathfrak{S}_n$, then $v \leq w$ if for all $1 \leq i \leq n-1$, we have

$$\{w(1), \ldots, w(i)\} \leq \{v(1), \ldots, v(i)\} \text{ in } C(i,n).$$

Example 31. Let $n = 4$ and we have permutations $u = 1324, v = 4132$ and $w = 3412$ in $\mathfrak{S}_4$ written in one line notation. Then $v \leq u$ and $w \leq u$, but v and w are not comparable.

Remark 1.1. This partial order is dual to the Bruhat order we discussed in Section 2. In particular, it is graded.

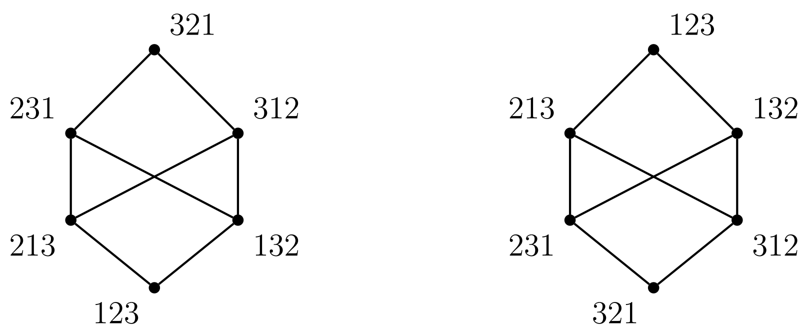


FIGURE 1. Bruhat and dual Bruhat orders on $\mathfrak{S}_3$

Proposition 1.4. If v and w are permutations in $\mathfrak{S}_n$ satisfying $v \not\leq w$, then the Plücker coordinate P_v of any flag in Ω_w is zero.

PROOF. Since $v \not\leq w$, there exists an i for which $\{v(1), \ldots, v(i)\} \not\geq \{w(1), \ldots, w(i)\}$ in $C(i,n)$. Lemma 2.2 implies that the homogeneous coordinate indexed by the subset $\{v(1), \ldots, v(i)\}$ of every subspace in the Schubert cell $\Omega_{w(1),\ldots,w(i)}$ is zero. This implies that the Plücker coordinate P_v of every point in Ω_w must also be zero. $\square$

1.3. Topology of the cells. A *basis for a complete flag* in V is an ordered basis of V such that for all $i = 1, \ldots, n-1$, the first i basis vectors constitute a basis for the ith subspace in the flag. We can represent a basis $\{w_1, \ldots, w_n\}$ of a flag $W_\bullet$ as rows of an $n \times n$ matrix

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{n1} & w_{n2} & \cdots & \cdots & w_{nn} \end{pmatrix},$$

with entries defined by $w_i = \sum_j w_{ij} e_j$.

It's important to note that row operations, such as scaling a row by a nonzero scalar or adding to a row a multiple of any row above it, do not alter the flag generated by the matrix. We begin with the first row and scale it appropriately to ensure that the first nonzero entry becomes one. Then, for every row below the first one, we add a suitable scalar multiple of the first row to eliminate all entries in the column corresponding to the leading one. This process is repeated for subsequent rows, resulting in a matrix where the first nonzero entry in each row is one, and every entry below a leading one in the same column is zero.

Indeed, for every $i = 1, \ldots, n-1$, the positions of the leading ones in the first i rows correspond to the pivots of the subspace W_i. Consequently, the permutation w corresponding to the Schubert cell containing the flag is encoded as the indices of the columns containing the leading ones in each row.

Definition 1.3. Any complete flag in the Schubert cell Ω_w has a unique *canonical basis* $\{x_1, \ldots, x_n\}$ which satisfies

$$\begin{aligned} x_{iw(i)} &= 1 \text{ for all } i = 1, \ldots, n-1, \text{ and} \\ x_{ij} &= 0 \text{ if } j < w(i) \text{ or } w^{-1}(j) < i. \end{aligned} \tag{8}$$

Example 32. Let us consider the flag in $\mathbb{C}^4$ generated by the rows of the 4×4 matrix

$$\begin{pmatrix} 0 & -2 & -4 & 0 \\ 2 & -1 & -4 & -6 \\ 0 & 4 & 8 & 2 \\ -1 & 0 & 4 & 1 \end{pmatrix}.$$

The row elimination procedure described above allows us to compute the canonical basis of the flag, obtained as the rows of the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The leading ones in each row (coloured red) form the permutation matrix of the permutation 2143, which means that the flag is an element of the Schubert cell Ω_{2143}.

Let us define $V^n = V \times \cdots \times V$ (n times) as the set of n-tuples of vectors in V, or equivalently, the affine space $\mathbf{A}^{n\times n}$ whose elements are $n \times n$ matrices with entries in $\mathbf{F}$. For $w \in \mathfrak{S}_n$, the collection of canonical bases of flags in the cell Ω_w can be viewed as the subset

$$\mathrm{E}_w = \{(x_{ij}) \in V^n \mid x_{iw(i)} = 1, x_{ij} = 0 \text{ if } j < w(i) \text{ or } w^{-1}(j) < i\}$$

of V^n defined by the linear conditions given in (8). This makes it a linear subvariety of V^n, thus it is isomorphic to an affine space.

Suppose a tuple of vectors in V^n forms a basis of a flag in V, then the computation of the Plücker coordinates of this flag can be described in terms of the various affine spaces as follows. We define a morphism between the two affine spaces

$$\varpi\colon V^n \to V \otimes \wedge^2 V \otimes \cdots \otimes \wedge^{n-1} V$$

by sending any tuple in V^n to a tensor where the ith component is computed by taking the wedge product of the first i vectors. We also have the usual projection map

$$\mathbf{p}\colon V \otimes \wedge^2 V \otimes \cdots \otimes \wedge^{n-1} V \to \mathbf{P}^N.$$

Then the Plücker embedding of the flag variety can be thought of as the composition $\mathbf{p}\varpi$.

Example 33. Suppose $n = 4$, then every element of V^n can be represented by a matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}.$$

Suppose we pick a permutation $2143 \in \mathfrak{S}_n$, then a generic element of E_w is a matrix of the form

$$\begin{pmatrix} 0 & 1 & * & * \\ 1 & 0 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where a $*$ can be replaced by any field element.

The discussion in the beginning of the section is then saying that the morphism $\varpi\mathbf{p}$ restricts to a bijection from the subvariety E_w to the Schubert cell Ω_w.

Theorem 1.5. In fact, the map $\mathbf{p}\varpi\colon \mathrm{E}_w \to \Omega_w$ is an isomorphism of varieties.

warm-up 2. To prove Theorem 1.5, we just need to find a morphism which is inverse to $\mathbf{p}\varpi$. Let us first warm up with the case of when w is the identity permutation, where an element of E_w is of the form

$$\begin{pmatrix} 1 & x_{12} & \cdots & \cdots & \cdots & x_{1n} \\ 0 & 1 & \cdots & \cdots & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 & x_{n-1,n} \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{9}$$

Then the Plücker coordinate indexed by the sequence of subsets

$$(\{1\}, \{1,2\}, \ldots, \{1,\ldots,i-1\}, \{1,\ldots,i-1,j\}, \{1,\ldots,i+1\}, \ldots, \{1,\ldots,n\}),$$

obtained by replacing the subset $\{1,\ldots,i\}$ in the sequence $\mathbf{S}$ with the subset $\{1,\ldots,i-1,j\}$, is equal to x_{ij}. This gives rise to the required inverse, and so when w is the identity permutation, the Schubert cell Ω_w is isomorphic to E_w.

Proof of Theorem 1.5. Applying the permutation w^{-1} to the columns of any matrix in E_w moves the leading ones into the diagonal entries. If the entry in position (i,j) in the resulting matrix is nonzero, then $(i, w(j))$ cannot satisfy the condition in (8), i.e. $i \geq j$ and $w(i) \geq w(j)$.

In particular, all the nonzero entries must be above the diagonal, and so this operation transforms all the matrices into the form (9), while the Plücker coordinates might undergo shuffling and sign changes. However, these changes can be precisely described by the permutation applied. Consequently, we are still able to recover the matrix entries as Plücker coordinates, as discussed in warm-up 1. □

Another implication of the proof is that the affine spaces $\mathrm{E}_{\mathbf{s}}$ are of dimension equal to the number of pairs $j > i$ satisfying $w(j) > w(i)$.

Definition 1.4 (depth of a permutation). A *non-inversion* in w is a pair $j > i$, such that $w(j) > w(i)$. The *depth*[1] of a permutation w, denoted $\mathrm{dep}(w)$, is defined as the number of non-inversions in w.

The depth of a permutation is the rank function for the dual Bruhat order.

According to Theorem 1.5, the Schubert cell Ω_w is isomorphic to an affine space of dimension $\mathrm{dep}(w)$. Consequently, they are locally closed, irreducible (due to the irreducibility of affine spaces), subsets of the complete flag variety. In other words, they are subvarieties of the complete flag variety.

Example 34. In Example 33, we saw that elements of E_{2143} are matrices of the form

$$\begin{pmatrix} 0 & 1 & * & * \\ 1 & 0 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We can apply the permutation inverse to 2143 (which is 2143 itself) to the columns to obtain

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[1] Recall that the length of a permutation is the number of inversions.

Note that the $*$s are now in the positions $(1,3),(1,4),(2,3),(2,4)$ which are also the non-inversions in 2143. So, the depth of 2143 is equal to 4.

2. Schubert Varieties

Schubert cells are locally-closed, irreducible subsets of $\mathbf{P}^N$, so their closures $X_w = \overline{\Omega_w}$ are projective varieties. These are the *Schubert varieties* of the complete flag variety. The following analogue of Proposition 3.1 can be proven.

Proposition 2.1. For all permutations $w \in \mathfrak{S}_n$ the Schubert variety X_w is equal to the disjoint union $\bigsqcup_{v \le w} \Omega_v$.

We will proceed in a manner identical to our proof of Proposition 3.1. Let us first locate the Schubert cells Ω_v for all permutations $v \le w$. Put

$$\begin{aligned} \mathrm{D}_w &= \{(x_{ij}) \in V^n \mid x_{ij} = 0 \text{ if } j < w(i)\} \text{ and} \\ \mathrm{E}'_w &= \{(x_{ij}) \in D_w \mid x_{iw(i)} \neq 0\}. \end{aligned}$$

Observe that the first i rows of any matrix in E'_w have pivot set $\{w(1),\dots,w(i)\}$, so its image under $\mathbf{p}\varpi$ is the Schubert cell Ω_w. Also, D_w is the closure $\overline{\mathrm{E}'_w}$ of E'_w in V^n and for all $v \le w$, it is easily verified that $\varpi(\mathrm{D}_w)$ contains $\varpi(\mathrm{E}'_v)$.

Lemma 2.2. Suppose $v, w \in \mathfrak{S}_n$ and $v \le w$. Then $\Omega_v \subseteq X_w$.

Proof. The lemma follows from a simple application of the observations above. Because $v \le w$, $\varpi(\mathrm{E}'_v) \subseteq \varpi(\mathrm{D}_w)$, and so applying $\mathbf{p}$:

$$\begin{aligned} \Omega_v &\subseteq \mathbf{p}(\varpi \mathrm{D}_w - \{0\}) \\ &= \mathbf{p}(\varpi \overline{\mathrm{E}'_w} - \{0\}). \end{aligned}$$

Recall that under a continuous function, the image of the closure of a set is contained in the closure of the image of the set. Because $\mathbf{p}\varpi$ is continuous

$$\Omega_v \subseteq \overline{\mathbf{p}\varpi \mathrm{E}'_w} = \overline{\Omega_w} = X_w,$$

as claimed. □

Lemma 2.3. Suppose $v, w \in \mathfrak{S}_n$ and $v \not\le w$. Then $\Omega_v \cap X_w = \varnothing$.

Proof. Because $v \not\le w$, it follows from 1.4 that any point in the Schubert cell Ω_w has the Plücker coordinate labelled by v equal to 0. Another way to say this is that the Schubert cell, Ω_w is contained in the zero set of the coordinate x_v. This is a closed set, so the closure $X_w \subseteq Z(x_v)$ as well. On the other hand, the vth coordinate of every point in the Schubert cell Ω_v is nonzero. So, the intersection $\Omega_v \cap X_w$ must be empty. □

The Schubert cells partition the complete flag variety, so Proposition 2.1 follows from Lemmas 2.2 and 2.3.

The identity permutation is the largest element under the dual Bruhat order on $\mathfrak{S}_n$. This means that the Schubert variety

$$X_{\mathrm{id}} = \bigsqcup_{v \leq \mathrm{id}} \Omega_v = \mathcal{F}(V)$$

is equal to the complete flag variety. So, we have finally shown that

Corollary 2.4. The complete flag variety is a projective algebraic variety.

3. Paving the complete flag variety

The Schubert decomposition of the complete flag variety (over $\mathbb{C}$) expresses it as a partition into affine spaces indexed by permutations, which are elements of a graded poset. This implies that the Schubert decomposition is an affine paving over the dual Bruhat order. So, the Grassmannian has trivial Betti numbers in odd dimensions, while its $2i$th Betti number is equal to the number of permutations in $\mathfrak{S}_n$ with i non-inversions. Because the Bruhat order is self-dual, this is also the number of permutations in $\mathfrak{S}_n$ with i inversions.

Example 35. The Bruhat order of $\mathfrak{S}_3$ is drawn in Figure 1. So, the Poincaré polynomial of the complete flag variety on $\mathbb{C}^3$ is $1 + 2t + 2t^2 + t^3$.

4. Partial Flag Varieties

The *partial flag variety* of type μ, denoted $\mathcal{F}_\mu(V)$, is the collection of all flags of type μ in V. When $\mu = (d, n-d)$, a flag of type μ represents a d-dimensional linear subspace of V, and when $\mu = (1, \ldots, 1)$ (repeated n times), the corresponding flag is a complete flag in V. Thus, the notion of partial flag varieties encompasses the Grassmannians and the complete flag variety as special cases. In this section, we will briefly see how our investigation of the goemetry of the Grassmannians and the complete flag variety can be carried out for general partial flag varities. This involved embedding them in a projective space, describing their Schubert decomposition, and studying the resulting Schubert cells and varieties.

4.1. Projective algebraic structure.

Let us fix a type $\mu = (\mu_1, \ldots, \mu_m)$ and let $d_i = \mu_1 + \cdots + \mu_i$ for all $i = 1, \ldots, m$. Then flags of type μ are elements of the product $\mathcal{G}(d_1, V) \times \cdots \times \mathcal{G}(d_{m-1}, V)$, which means that the flag variety $\mathcal{F}_\mu(V)$ can beviewed as a subset of the product $\mathbf{P}\wedge^{d_1} \times \cdots \times \mathbf{P}\wedge^{d_{m-1}}$ of projective spaces.

If $N = \binom{n}{d_1} + \cdots + \binom{n}{d_{m-1}} - 1$. Then the product of projective spaces can be Segre embedded into the projective space $\mathbf{P}^N$

$$\psi\colon \mathbf{P}\wedge^{d_1} \times \cdots \times \mathbf{P}\wedge^{d_{m-1}} \to \mathbf{P}\left(\wedge^{d_1} \otimes \cdots \otimes \wedge^{d_{m-1}}\right) \cong \mathbf{P}^N,$$

and we identify $\mathcal{F}_\mu(V)$ with its image under ψ in $\mathbf{P}^N$. As before, one can show that the image of a flag variety $\psi(\mathcal{F}_\mu(V))$ is a closed subset of $\mathbf{P}^N$.

4.1.1. *Plücker coordinates.* The homogeneous coordinates, which are also known as the Plücker coordinates, on the flag variety are indexed by tuples of subsets of $[n]$ ordered lexicographically, so that the ith subset is a subset of size d_i. The inclusion relations that are imposed among the subspaces of a flag manifest themselves at the level of the indexing subsets as well: the proof of Proposition 1.3 can be adapted to show that the coordinate indexed by a tuple $(\mathbf{s}_1, \ldots, \mathbf{s}_{m-1})$ of subsets, is zero, unless the $\mathbf{s}_i$ form an increasing sequence of subsets, i.e. they satisfy

$$\varnothing \subset \mathbf{s}_1 \subset \cdots \subset \mathbf{s}_{m-1} \subset [n].$$

4.2. Schubert Decomposition. The action of the symmetric group $\mathfrak{S}_n$ on $[n]$ leads to a transitive action on the collection of increasing sequences by application to each subset in the sequence, just as we saw in Section 1.2.1 for the complete flag variety. Let $\mathbf{S}$ be the increasing sequence

$$\{1, \ldots, d_1\} \subseteq \{1, \ldots, d_2\} \cdots \subseteq \{1, \ldots, d_{m-1}\},$$

then its stabilizer is the subgroup of permutations that permute the first μ_1 elements, the next μ_2 elements, and so on. Particularly, this subgroup is isomorphic to $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_m}$, which is often termed as *Young subgroups*.

As a result, the collection of increasing sequences is in a bijective correspondence with the space of cosets $\mathfrak{S}_n/\mathfrak{S}_\mu$ of the Young subgroup, and a coset $w\mathfrak{S}_\mu$ corresponds to the sequence $w\mathbf{S}$. Every coset must contain a unique element of minimal depth; we denote the set of minimal-depth coset representatives by $\mathfrak{S}^\mu$. The minimal depth coset representatives w are characterized by the property that for all $i = 1, \ldots, m$, we have

$$w(j) > w(k) \text{ for all } j < k \text{ in } \{d_{i-1} + 1, \ldots d_i\}.$$

Definition 4.1. Let $w \in \mathfrak{S}^\mu$ be a minimal depth coset representative. Then a *Schubert cell* Ω_w is the image under the Segre embedding ψ of the collection of flags of type μ whose set of pivots corresponds to the coset of w under the bijection described above.

This gives the *Schubert decomposition* of the partial flag variety

$$\mathcal{F}_\mu(V) = \bigsqcup_{w \in \mathfrak{S}^\mu} \Omega_w.$$

Note that the Schubert cell Ω_w is also the subset of points in $\mathbf{P}^N$ where the first nonzero homogeneous coordinate is the one indexed by the sequence $w\mathbf{S}$. Consequently, the Schubert cells are locally closed subsets of $\mathbf{P}^N$.

Theorem 4.1. Let $w \in \mathfrak{S}^\mu$, then the Schubert cell Ω_w is isomorphic to the affine space $\mathbf{A}^{\mathrm{dep}(w)}$.

PROOF. The proof of Theorem 1.5 can be adapted to prove this. □

A direct implication of the theorem is that Schubert cells are locally closed, irreducible (due to the irreducibility of affine spaces), subsets of the flag variety. In other words, they are quasi-projective subvarieties. So, for any $w \in \mathfrak{S}^\mu$ we can define the *Schubert variety*, denoted X_w, to be the closure of the Schubert cell Ω_w.

The various Schubert varieties for the flag variety of type μ satisfy inclusion relations according to the subposet $\mathfrak{S}^\mu$ of the dual Bruhat order. For all $w \in \mathfrak{S}^\mu$ the Schubert variety $X_w = \bigsqcup_{v \leq w} \Omega_v$, where the disjoint union is over $v \in \mathfrak{S}^\mu$. Let w be the permutation of minimum depth in the set $\mathfrak{S}^\mu$. Then the corresponding Schubert variety, X_w, contains all the Schubert cells, and we have

$$X_w = \bigsqcup_{v \leq w} \Omega_v = \mathcal{F}_\mu(V).$$

This finally shows that

Corollary 4.2. The partial flag varieties are projective algebraic varieties.

CHAPTER 6

Fixed Point Varieties

The fixed points of a linear transformation T on a vector space V are the solutions to the homogeneous system $Tv - v = 0$ of linear equations, so they form an algebraic subset of the associated projective space. Moreover, any linear mapping on V induces linear transformations on the various exterior powers of V and their tensor products. We conclude that the sets of points fixed by such transformations in the Grassmannian and flag varieties are algebraic sets as well. In this chapter, we follow the work of Shimomura [**Shi85**], and study the geometry of this space of fixed points, when T is a unipotent linear transformation.

1. Invariant Subspaces

Proposition 1.1. Suppose we have two similar linear transformations on a vector space V. Then there is an automorphism of the projective space $\mathbf{P}V$ which maps the fixed point set of one to the other.

Proof. Suppose the linear automorphism X conjugates one of the transformations to the other. Application of the linear map X induces an isomorphism of the projective space $\mathbf{P}V$, sending the set of fixed points of one transformation to the set of fixed points of the other. □

If T is a unipotent transformation of V, then $T - I$ is a nilpotent transformation. The transformations T and $T - I$ have the same invariant subspaces. The similarity classes of unipotent and nilpotent linear transformations are defined by their Jordan canonical forms, which correspond to partitions indicating the sizes of the blocks.

Example 36. Consider the following unipotent matrix along with its corresponding nilpotent matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Both their Jordan blocks are of size 2, 2 and 1, and so correspond to the partition $(2, 2, 1)$.

Definition 1.1. The *shape* of a unipotent(or nilpotent) linear transformation is the partition obtained from the sizes of the Jordan blocks.

Fix a partition λ and label the squares of the diagram of λ from 1 to n, going down each column from top to bottom, starting with the rightmost column and then moving left, as in Example 37.

Example 37. The squares of the partition $(4, 3, 2)$ will be labelled

c_7	c_4	c_2	c_1
c_8	c_5	c_3	
c_9	c_6		

.

Let us define a linear transformation $\mathfrak{N}$ on V by specifying its action on the standard basis vectors as follows.

(1) If the square labelled c_i is in the first column of λ, then we send the standard basis e_i to zero.
(2) If the square immediately to the left of the square labelled c_i is labelled c_j, then map e_i to e_j.

It is evident that this transformation is a nilpotent linear transformation on V with shape equal to λ. Certainly, by adding the identity map to this transformation, we obtain a unipotent transformation on V also of shape λ.

Example 38. Labelling the diagram as in Example 37, for instance, the corresponding operator will send e_2 to e_4, which will be mapped to e_7, and then to 0.

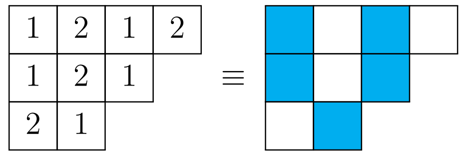
Remark 1.1. The choice of how we label the squares of the diagram might appear arbitrary at first glance, but it actually possesses a notable property. Let us consider the map $\mathfrak{N}$ as a function defined on the ordered set $\{1, \ldots, n, \infty\}$, where i is viewed as the basis vector e_i and ∞ represents the zero vector in V. Then $\mathfrak{N}$ preserves order: $i \leq j$ implies that $\mathfrak{N}i \leq \mathfrak{N}j$.

The reader should check, for instance, that if we were to label the squares row by row instead of column by column, this property would not always hold.

Definition 1.2. We denote by $\mathrm{Fix}(\mathcal{G}_d(V))$ the subset of the Grassmannian $\mathcal{G}_d(V)$ whose points correspond to the subspaces that are invariant under the nilpotent linear transformation $\mathfrak{N}$ described earlier.

The Schubert cells of the Grassmannian are indexed by d-subsets of $[n]$; it will be helpful to think of these as tableaux of weight $(d, n-d)$ on λ. To illustrate, given a subset $\mathbf{s}$ in $C(d, n)$, we place $1s$ in the squares c_i corresponding to elements of $\mathbf{s}$, and place 2s in the remaining squares. Frequently, we will colour in the squares to represent the 1s, while the uncoloured squares will be understood to contain 2s.

Example 39. The tableau of shape $(4, 3, 2)$ associated to the subset $\{2, 3, 6, 8\} \subset [9]$ is



Definition 1.3. If τ is a tableau of shape λ and weight $(d, n-d)$, then we denote by $\mathrm{Fix}(\Omega_\tau)$ the set of fixed points contained in the Schubert cell corresponding to $\mathbf{t}$.

Clearly $\mathrm{Fix}(\Omega_\tau) = \Omega_\tau \cap \mathrm{Fix}(\mathcal{G}_d(V))$, so it is a locally closed subset of $\mathbf{P} \wedge^d V$.

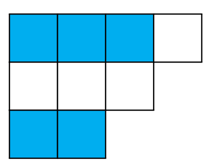
Proposition 1.2. The fixed point set $\mathrm{Fix}(\Omega_\tau)$ is nonempty if and only if the tableau τ is *row-weak*.

Proof. If the tableau τ is row-weak, then any entry to the left of a 1 must also be a 1. Consequently, the subspace in the Schubert cell Ω_τ, spanned by the standard basis vectors e_i corresponding to elements of τ, is $\mathfrak{N}$-invariant.

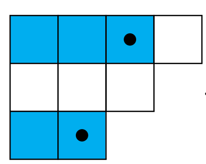
Conversely, consider a subspace in Ω_τ. If the entry in the c_ith square of λ is a 1, then there exists of a vector in the subspace with its leading nonzero entry in position i. According to Remark 1.1, applying $\mathfrak{N}$ will yield a vector whose pivot is positioned as indicated by the square to the left of c_i. If the subspace were $\mathfrak{N}$-invariant, then this shows that any entry to the left of a 1 must also be a 1, proving that τ must be row-weak. □

Definition 1.4. Let τ be a row-weak tableau of weight $(d, n-d)$. The label of the rightmost cell in its row which contains a 1 (if one exists) is referred to as an *initial number* for τ.

Example 40. The tableau on the partition $(4, 3, 2)$ in Example 39 is not row-weak. However,



is row-weak. The set of initial numbers (marked with a $\bullet$) of this tableau is $\{2, 6\}$:



1.1. The val **statistic.** In Section 2.1, we introduced linear subvarieties E_τ (indexed by d-subsets $\mathbf{s}$ there) of V^d, which consist of canonical bases in V with pivot set τ. Let $\tau_1 < \cdots < \tau_d$ be the elements of τ. Then, we define a function ρ on E_τ taking values in V^d, which replaces the jth vector of the input with $\mathfrak{N}$ applied to the ith vector whenever the square labeled τ_j is immediately to the left of the square labeled τ_i in λ. The function ρ is linear, so the image $\rho(\mathrm{E}_\tau)$ is also a linear subvariety of V^d.

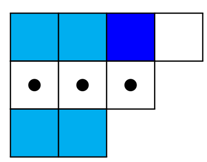
Theorem 1.3. The restriction of the function $\mathbf{p}\varpi$ (as defined in 2.1) to $\rho(\mathrm{E}_\tau) \subset V^d$ is an isomorphism onto the fixed point set $\mathrm{Fix}(\Omega_\tau)$.

Proof. Let $v_1, \ldots, v_d$ be linearly independent vectors in V. The map $\mathbf{p}\varpi$ assigns to the tuple $(v_1, \ldots, v_d)$ the point in the Grassmannian corresponding to the subspace they span. It can be easily verified that our construction of ρ ensures that $\mathbf{p}\varpi$ maps $\rho(\mathrm{E}_\tau)$ into $\mathrm{Fix}(\Omega_\tau)$.

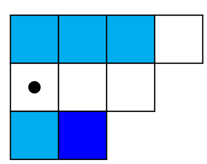
Therefore, all we need to do is provide an inverse morphism. The map $\mathbf{p}\varpi$ is an isomorphism between the sets $\mathrm{E}_\tau \to \Omega_\tau$, so the composition $\rho(\mathbf{p}\varpi)^{-1}$ defines a map from $\mathrm{Fix}(\Omega_\tau)$ to $\rho(\mathrm{E}_\tau)$. This is the required inverse. □

We have shown that the fixed point set $\mathrm{Fix}(\Omega_\tau)$ of any Schubert cell is isomorphic to an affine space. So, they are subvarieties of the Grassmannian. Any element of $\rho(\mathrm{E}_\tau)$ only depends on the choice of the vectors where the pivot positions are equal to an initial number of τ. So, the dimension of $\mathrm{Fix}(\Omega_\tau)$ can be computed by counting, for each initial number, the number of 2s in λ which are in a cell with a larger label.

Example 41. The tableau in 40 has two initial numbers. Let us colour the first one in blue and indicate the squares it counts with bullets:



Similarly, for the second initial number:



So, the dimension of the affine space is $3 + 1 = 4$.

We can carry out a more refined counting. Suppose the square labelled j in λ contains a 2. We denote by $\mathrm{val}(c_j)$ the number of initial numbers in τ that are smaller than j. We set $\mathrm{val}(c) = 0$ whenever c is a square with entry 1. It is evident that the dimension of $\mathrm{Fix}(\Omega_\tau)$ is equal to the sum of the values of all the squares in λ; we shall call this the *value* of τ.

Furthermore, if the entry in c_j is a 2, then any initial number $i < j$ is located either

(1) In a row above j and in the same column, or in a column to the right of j.
(2) In a row below j and in a column to the right of j.

Hence, $\mathrm{val}(j)$ is simply the number of entries in the same column above j and the number of entries in the column immediately to the right below j which are equal to 1.

Example 42. Things are often easier to parse using examples. Let us again consider the tableau from Example 40, drawn with the labels:

7	4	2	1
8	5	3	
9	6		

Then, the pair $(8, 2)$ can instead be counted as the pair $(8, 7)$. Similarly, the pair $(8, 6)$ is counted by itself.

Remark 1.2. When τ is a weakly row and column increasing tableau (i.e. a weak tableau), then the entry below a 2 cannot be a 1. This implies that the value of a square with entry 2 is simply the number of 1s in the same column above it.

Theorem 1.4. If τ is a row-weak tableau on λ, then the subvariety $\mathrm{Fix}(\Omega_\tau)$ of Ω_τ is isomorphic to the affine space $\mathbf{A}^{\mathrm{val}(\tau)}$.

1.2. Inclusion Relations. The inclusion of Schubert varieties in the Grassmannian was determined by the ordering of the d-subsets of $[n]$. For example, when $n = 9$ and $d = 5$, the Schubert variety X_{24679} is contained in the Schubert variety X_{12479} because $\{1, 2, 4, 7, 9\} \leq \{2, 4, 7, 6, 9\}$ in $C(d, n)$.

However, these two correspond to the following tableaux on λ:

7	4	2	1
8	5	3	
9	6		

and

7	4	2	1
8	5	3	
9	6		

Given that the values of both tableaux are equal to 4, we cannot generally expect the inclusion of the closures of the fixed point subsets of the Schubert cells. However, we can identify certain special inclusion relations.

1.2.1. *The column-sort operation.* Recall the *column-sort* operation, denoted $\mathcal{S}$, on tableaux which sorts each column to make the entries in each column weakly increase from top to bottom. According to Proposition 2.1, column-sorting a row-weak tableau results in a weak-tableau. To that end, for a weak tableau τ, let $\mathcal{S}^{-1}(\tau)$ be the set of all row-weak tableau which give τ upon column-sorting.

Proposition 1.5. For all $\gamma \in \mathcal{S}^{-1}(\tau)$, we have $\mathrm{Fix}(\Omega_\gamma) \subseteq \overline{\mathrm{Fix}(\Omega_\tau)}$.

Proof. See [**Shi85**, Theorem 2.5]. □

1.3. Irreducible Components. If W is in $\mathrm{Fix}(\mathcal{G}_d(V))$, then the restriction of the nilpotent transformation $\mathfrak{N}$ to W is also nilpotent, and has shape contained in λ. Let us identify a weak tableau τ with the Young diagram defined by the entries which equal 1, and define Y_τ to be the collection of subspaces in $\mathrm{Fix}(\mathcal{G}_d(V))$ such that the shape of $\mathfrak{N}$ restricted to W is τ.

Clearly, any subspace in Y_τ must be an element of the cell $\mathrm{Fix}(\Omega_{\tau'})$ for some row-weak τ' satisfying $\mathcal{S}(\tau') = \tau$. That is,

$$Y_\tau = \bigcup_{\tau' \in \mathcal{S}^{-1}(\tau)} \mathrm{Fix}(\Omega_{\tau'}).$$

Lemma 1.6. If $\tau < \gamma$ are weak tableaux of weight $(d, n-d)$ on λ, then for all row-weak $\tau' \in \mathcal{S}^{-1}(\tau)$, we have $\tau' \not\geq \gamma$.

Proof. If $\mathbf{r} \leq \mathbf{s}$ in $C(d,n)$, then for all $k \in 1, \ldots, n$, the set $\mathbf{r} \cap \{1, \ldots, k\}$ is larger in size than the set $\mathbf{s} \cap \{1, \ldots, k\}$. This is because if $s_i \leq k$ then $r_i \leq k$ as well.

Applying this observation to τ and γ for $k = \lambda_1 + \cdots + \lambda_i$, we deduce that the number of 1s of τ in the first i columns of λ from the left must be smaller than the number of 1s in γ. Since the column-sort operation does not change the number of 1s in any column of τ', τ' cannot be larger than γ. □

Proposition 1.7. If τ is a weak tableau, then Y_τ is an irreducible, locally closed subset of $\mathrm{Fix}(\mathcal{G}_d(V))$.

Proof. If $\mathrm{Fix}(X_\tau)$ is the fixed point subset of the Schubert variety X_τ, then using Lemma 1.6, we can express Y_τ as the difference

$$Y_\tau = \mathrm{Fix}(X_\tau) - \bigcup_{\substack{\gamma \text{ weak} \\ \tau < \gamma}} \mathrm{Fix}(X_\gamma)$$

of two closed sets, which implies that Y_τ is locally closed.

That it is irreducible follows because $\tau > \tau'$ for any $\tau' \in \mathcal{S}^{-1}(\tau)$, so Y_τ is equal to the closure of the irreducible set $\mathrm{Fix}(\Omega_\tau)$ in Y_τ. □

The fixed point variety is the union of the Y_τ, and therefore, also of their closures. According to Proposition 1.2, this provides us with the decomposition of $\mathrm{Fix}(\mathcal{G}_d(V))$ into its irreducible components:

$$\mathrm{Fix}(\mathcal{G}_d(V)) = \bigcup_{\tau \text{ weak}} \overline{Y}_\tau.$$

2. Fixed Points of Flag Varieties

Let $\mu = (\mu_1, \ldots, \mu_m)$ be a composition of n, and for all $i = 1, \ldots, m$, let $d_i = \mu_1 + \cdots + \mu_i$. The set of flags of type μ in V such that each constituent subspace is preserved by $\mathfrak{N}$ corresponds to the fixed point set of the flag variety $\mathcal{F}_\mu(V)$ under the associated unipotent action. We denote this space by $\mathrm{Fix}(\mathcal{F}_\mu(V))$.

The Schubert cells of $\mathcal{F}_\mu(V)$ are indexed by increasing sequences of subsets of $[n]$ such that the size of the ith subset is d_i. Similar to the approach used for the Grassmannians, we can represent such an increasing sequence as a tableau of weight μ on λ. Given an increasing sequence $\tau = \tau_1, \ldots, \tau_{m-1}$, we assign a 1 to the squares with labels in τ_1, 2 to the squares with labels in τ_2 but not in τ_1, and so on.

Example 43. Let $n = 9$ and $\mu = (3, 1, 5)$. The tableau on $\lambda = (4, 3, 2)$ corresponding to the increasing sequence

$$\{3, 5, 9\} \subseteq \{1, 3, 5, 9\} \subseteq \{1, \ldots, 9\},$$

is

1	3	3	2
3	1	1	
3	3		

.

Definition 2.1. If τ is a tableau of weight μ on λ, then we denote by $\mathrm{Fix}(\Omega_\tau)$ the collection of fixed points in the Schubert variety Ω_τ of the flag variety. We have

$$\mathrm{Fix}(\mathcal{F}_\mu(V)) = \bigsqcup_\tau \mathrm{Fix}(\Omega_\tau).$$

If a flag $W_\bullet$ is in $\mathrm{Fix}(\Omega_\tau)$, then by definition, its constituents W_i are elements of $\mathrm{Fix}(\Omega_{\tau_i})$ for all $i = 1, \ldots, m-1$. As a result, the weight $(d_i, n - d_i)$ tableau on λ corresponding to τ_i are row-weak. It is easily verified that this is equivalent to τ itself being row-weak. So, we have shown one direction of the following lemma.

Lemma 2.1. The fixed point subset $\mathrm{Fix}(\Omega_\tau)$ of Ω_τ is nonempty if and only if τ is a row-weak tableau.

Proof. We have shown earlier that τ being row-weak is a necessary condition. Conversely, suppose that τ is a row-weak tableau. For all $i = 1, \ldots, m-1$, let W_i be the d_i-dimensional $\mathfrak{N}$-invariant subspace of V that is spanned by the basis vectors e_j corresponding to the squares of λ with entries at most i. Then $W_\bullet$ is an element of the Schubert cell Ω_τ. Therefore, the intersection $\mathrm{Fix}(\Omega_\tau)$ is nonempty. □

2.1. The value of a tableau.

Definition 2.2. Let τ be a row-weak tableau of weight μ on λ. Define its value $\mathrm{val}(\tau)$ recursively as follows:

(1) If μ has only one part, put $\mathrm{val}(\tau) = 0$.
(2) If $\mu = (\mu_1, \mu_2)$ has two parts, val is defined as in Theorem 1.4.
(3) If $\mu = (\mu_1, \ldots, \mu_{m-1}, \mu_m)$ and $m > 2$, put $\mu' = (\mu_1, \ldots, \mu_{m-1})$. Let τ' be the μ'-tableau obtained by deleting the squares with figure m from τ and rearranging the rows into a partition. Then τ' is a weak μ'-tableau of shape λ', where λ' is a Young diagram with $n - \mu_m$ cells. Finally, define

$$\mathrm{val}(\tau) = \mathrm{val}(\tau') + \mathrm{val}(\tau_{(m-1)}).$$

Theorem 2.2. The fixed point set $\mathrm{Fix}(\Omega_\tau)$ is isomorphic to the affine space $\mathbf{A}^{\mathrm{val}(\tau)}$.

Proof. This follows from Proposition 1.3. □

Remark 2.1. According to Theorem 2.2, the fixed point set of the Schubert varieties are subvarieties of $\mathrm{Fix}(\mathcal{F}_\mu(V))$. It has been shown that this decomposition of the fixed point set $\mathrm{Fix}(\mathcal{F}_\mu(V))$ of the flag variety into affine spaces gives rise to an affine paving of the space (See [**BO11**]).

2.2. Irreducible Components. Let τ be a weight μ weak tableau. Then, analogous to the what we did for the Grassmannian, we can put $\mathcal{S}^{-1}(\tau)$ be the set of all row-weak tableau which give τ upon column-sorting. Then we have the following analogue of Proposition 1.5.

Proposition 2.3. For all $\gamma \in \mathcal{S}^{-1}(\tau)$, the fixed point set of the Schubert variety $\mathrm{Fix}(\Omega_\gamma)$ is contained in the closure $\overline{\mathrm{Fix}(\Omega_\tau)}$ of the fixed point set of the Schubert variety $\mathrm{Fix}(\Omega_\tau)$.

If τ is a weak tableau, then the diagrams τ_i are Young diagrams of certain partitions. We can put

$$Y_\tau = \left\{ W_\bullet \in \mathrm{Fix}(F_\mu(V)) \,\middle|\, \begin{matrix} \mathfrak{N}|_{W_i} \text{ is a nilpotent map} \\ \text{of shape } \tau_i \end{matrix} \right\}.$$

Theorem 2.4. Let τ be a weight μ weak tableau of shape λ. Then the subset Y_τ is an irreducible, locally closed subvariety of $\mathrm{Fix}(F_\mu(V))$.

Proof. By the definition of Ω_β^λ, we have that Y_τ can be written as the union

$$Y_\tau = \bigcup_{\tau' \in \mathcal{S}^{-1}(\tau)} \mathrm{Fix}(\Omega_{\tau'})$$

of fixed point subsets of the Schubert varieties indexed by tableaux τ' which give τ upon column-sorting. If we denote the fixed point set of a Schubert variety X_τ by $\mathrm{Fix}(X_\tau)$, then we can express Y_τ as the difference

$$Y_\tau = \mathrm{Fix}(X_\tau) - \bigcup_{\substack{\gamma \text{ weak} \\ \gamma > \tau}} \mathrm{Fix}(X_\gamma).$$

Hence Y_α is locally closed. Furthermore, it is irreducible because it is the closure of the irreducible set $\mathrm{Fix}(\Omega_\tau)$ in Y_τ. □

So, the fixed point variety is the union of the Y_τ, and therefore, also of their closures. According to Proposition 1.2 and Theorem 2.4, this provides us with the decomposition of $\mathrm{Fix}(\mathcal{F}_\mu(V))$ into its irreducible components:

$$\mathrm{Fix}(\mathcal{F}_\mu(V)) = \bigcup_{\tau \text{ weak}} \overline{Y}_\tau.$$

2.3. Components of maximum dimension. While we've managed to identify the irreducible components in $\mathrm{Fix}(\mathcal{F}_\mu(V))$, there's a lingering issue. We haven't yet excluded the possibility of inclusions between the various $\overline{Y}_\tau$. For instance, in the cases of the Grassmannians and the flag varieties, all Schubert varieties were contained within a single Schubert variety.

Understanding these inclusions between the various closures and describing the tableaux that genuinely index irreducible components is generally challenging. However, we can say something: if $\overline{Y}_\tau$ has maximum dimension among all the other sets, then it cannot be included in some larger irreducible component. Consequently, $\overline{Y}_\tau$ must itself be an irreducible component; now we provide an elegant description of the tableaux that achieve maximum dimension.

Suppose τ is a weak, weight μ tableau of shape λ. Then for all $i = 1, \ldots, m-1$, the d_i-tableau α_i corresponds to a Young diagram of size d_i. It follows from Theorem 1.4 that $\mathrm{val}(\alpha)$ can be calculated by counting for each entry i in λ the number of entries smaller than i in the same column. Because the tableau is weak, these entries have to be above the i, so the contribution from the rth column of la is at most

$$(\lambda'_r - 1) + (\lambda'_r - 2) + \cdots + 1 + 0 = \binom{\lambda'_r}{2}.$$

For the equality to hold, every entry in τ should count all entries above it, meaning the entries should be strictly increasing down every column.

Proposition 2.5. The value $\mathrm{val}(\alpha)$, of a weak tableau τ on a partition λ is at most $\sum_i \binom{\lambda'_i}{2}$. The equality holds if and only if τ is a semistandard Young tableaux.

Corollary 2.6. The sets $\overline{Y}_\tau$, where τ is semistandard, constitute the irreducible components of $\mathrm{Fix}(\mathcal{F}_\mu(V))$ with maximum dimension. This, combined with Remark 2.1, implies that the top Betti number of this space equals the Kostka number $K_{\mu,\lambda}$. This number represents the count of semistandard weight μ tableaux on the diagram of λ.

Part 3

Enumeration

CHAPTER 7

Lattice of invariant subspaces

An invariant subspace of a linear mapping $T\colon V \to V$ is a subspace that is preserved by T. The collection of all invariant subspaces ordered by inclusion forms a poset that we shall denote by $\mathcal{L}(T)$. The intersection of two invariant subspaces is invariant and is the largest subspace contained in both of them, while the sum of two invariant subspaces is the smallest invariant subspace containing both of them. So, the poset $\mathcal{L}(T)$ is equipped with meet and join operations. In other words,

Proposition 0.1. The poset $\mathcal{L}(T)$ is a lattice.

Proposition 0.2. The lattice of invariant subspaces of two similar linear transformations are isomorphic.

Proof. If two linear transformations are similar, then there is a linear automorphism X which conjugates one to the other. Applying X results in an isomorphism between the two lattices. □

Lemma 0.3. Let $T\colon V \to V$ be a linear transformation. Then the lattice $\mathcal{L}(T)$ is self-dual.

Proof. Consider a linear transformation $T\colon V \to V$. The annihilator of a subspace W in V is defined as the set of vectors in V whose dot product with every vector in W is zero. It is easy to check that when W is invariant under T, its annihilator is a subspace which is invariant under the transpose of T. The function that maps a subspace to its annihilator is an order-reversing bijection.

Moreover, any linear transformation is similar to its transpose. This implies that the lattice of subspaces invariant under T is isomorphic to the lattice of subspaces invariant under the transpose. These together imply that the lattice $\mathcal{L}(T)$ is self-dual. □

Proposition 0.4. Every interval in $\mathcal{L}(T)$ is self-dual.

Proof. If W and W' are T-invariant subspaces of V such that W' contains W, then T naturally gives rise to linear maps on both W' and the quotient W'/W. Quotienting out W leads to an inclusion preserving bijection between the subspaces of W' containing W and the subspaces of W'/W. This is the fourth isomorphism theorem for vector spaces.

In fact, the above correspondence is a bijection between the T-invariant subspaces of the two vector spaces. So, the interval $[W, W']$ in the lattice $\mathcal{L}(T)$ is isomorphic, as a poset, to the lattice of T-invariant subspaces of W'/W. It follows from Lemma 0.3 that this is self-dual. $\square$

Example 44. Let us compute the invariant subspaces for a diagonalisable operator that has all its eigenvalues distinct. Each eigenvalue corresponds to a one-dimensional eigenspace, which is obviously invariant. Furthermore, the subspace generated by any subset of the eigenvectors is also invariant.

In other words, for each subset of $[n]$, the subspace generated by the basis vectors corresponding to elements of the subset is invariant. It requires a straightforward calculation to check that these are all the invariant subspaces. This shows, in particular, that the lattice of invariant subspaces is isomorphic to the lattice of subsets of $[n]$, which is often known as the *Boolean lattice*.

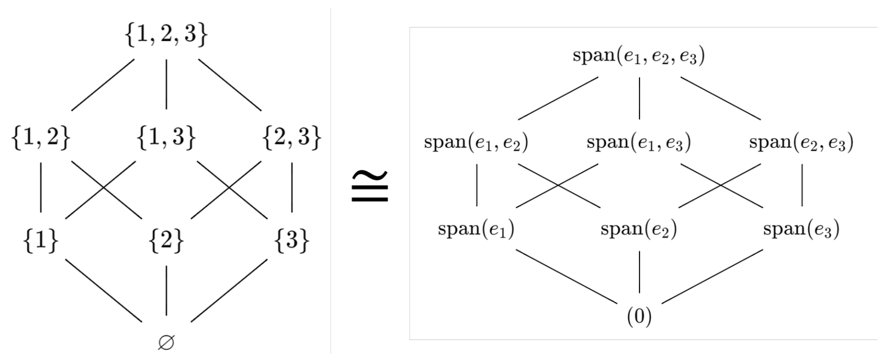


FIGURE 1. Lattice of invariant subspaces for a diagonal linear map on $\mathbf{F}^3$.

1. Invariant flag generating polynomial

Let us revisit the concept of a *flag of subspaces* in a vector space, which is a sequence of subspaces where each subspace is contained within the next one, starting from the zero subspace and ending at the entire space. If $T\colon V \to V$ is a linear map, then a flag is said to be *T-invariant* if all of its constituent subspaces remain invariant under T.

The *type* of a flag is the sequence of codimensions of each subspace relative to the next. More precisely, a flag

$$W_\bullet = (\varnothing = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{m-1} \subseteq W_m = V)\,,$$

in V has type $\mu = (\mu_1, \ldots, \mu_m)$, where $\mu_i = \dim W_i - \dim W_{i-1}$. There exists an inclusion-reversing bijection on the interval $[W_{i-1}, W_{i+1}]$ because it is self-dual. The subspace W_i is an element in the interval and will be mapped to a subspace W_i' that has codimension equal to $\dim W_{i+1} - \dim W_i$ relative to W_{i-1}. The flag obtained by replacing W_i with W_i' in $W_\bullet$ is a flag of type equal to the μ but with μ_i and μ_{i+1} swapped. This leads to a bijection between the collection of flags of types differing by a swap of two adjacent terms. What's more, any permutation can be written as a product of such swaps, which gives us

Theorem 1.1. Let μ and μ' be two sequences with entries differing only up to a permutation. Then there is a bijective correspondence between the collection of flags in V of type μ and of type μ'.

An obvious thing to try, as combinatorialists, is to count the number of invariant flags of a given type. However, this is only meaningful when we take the field $\mathbf{F} = \mathbf{F}_q$ to be the finite field with q elements, whence there are always only finitely many flags. Let $\alpha_T(\mu; q)$ denote the number of T-invariant flags, then it is an immediate consequence of Theorem 1.1 that if two sequences μ, μ' differ by a permutation, then $\alpha_T(\mu; q) = \alpha_T(\mu'; q)$.

A useful way to record the $\alpha_T(\mu; q)$ is via the *generating polynomial* for T-invariant flags:

$$F_T(x_1, \ldots, x_m; q) = \sum_{\mu} \alpha_T(\mu; q) x_1^{\mu_1} \cdots x_m^{\mu_m}$$

Corollary 1.2. The polynomial F_T is a symmetric polynomial that can be expanded as

$$F_T = \sum_{\mu \vdash n} \alpha_T(\mu; q) m_\mu$$

with respect to the monomial symmetric functions.

2. Factorising the lattice

In Example 44, we saw how every invariant subspace of a diagonalizable operator with distinct eigenvalues is in a sense built up from certain special invariant subspaces: the eigenspaces. This decomposition enabled us to explicitly describe and enumerate the points and flags in the lattice of invariant subspaces. However, not all linear transformations are diagonalizable. Nevertheless, the theory of modules over a principal ideal domain provides a method for decomposing them into simpler pieces.

2.1. Primary decomposition of linear maps. The primary decomposition of a linear transformation T on V expresses the vector space V as a direct sum of smaller invariant subspaces, on the basis of their behaviour with respect to repeated application of T.

Definition 2.1. Let $f(t)$ be any irreducible monic polynomial over $\mathbf{F}$. Given a linear transformation T on V, put

$$V_f = \{v \in V \mid f^r(T)v = 0 \text{ for some } r \in \mathbb{N}\}.$$

Then V_f is T-invariant and so we can define the f-primary part of T to be the restriction of T to V_f.

Theorem 2.1. Let $T\colon V \to V$ be a linear transformation whose minimal polynomial factorises into powers $f_1^{a_1}, \ldots, f_r^{a_r}$, of irreducible polynomials. Then V decomposes into non-zero subspaces

$$V = V_{f_1} \oplus \cdots \oplus V_{f_r},$$

and the minimal polynomial of the restriction of T to each subspace V_{f_i}, denoted T_{f_i}, is $f_i^{a_i}$.

PROOF. The first assertion follows upon viewing V along with the T action as an $F[t]$-module and applying [**Jac85**, Theorem 3.11]. The second can then be easily checked. $\square$

Example 45. Let us describe the primary decomposition of diagonalisable linear transformations. An operator with eigenvalues $\nu_1, \ldots, \nu_r$ has minimal polynomial $(t-\nu_1)\cdots(t-\nu_r)$, and the polynomial $t-\nu_r$ annihilates the eigenspace with eigenvalue ν_r. Consequently, we can express the linear transformation as a direct sum of maps acting on the eigenspaces through scalar multiplication by the corresponding eigenvalue.

More broadly, the primary decomposition of any linear transformation over an algebraically closed field, such as $\mathbb{C}$, aligns perfectly with the transformation's Jordan canonical form. The vector space decomposes into the *generalized eigenspaces*, and the linear map restricted to each of these subspaces is represented by the Jordan blocks corresponding to the respective eigenvalues.

Definition 2.2. Let $f(t)$ be an irreducible monic polynomial. A linear transformation T is called *f-primary* if $T = T_f$. By Theorem 2.1, this is the same as the minimal polynomial of T having a unique irreducible factor.

2.1.1. *Structure of primary transformations.* We have succeeded in simplifying the study of a linear transformation to understanding several nicely-behaved linear transformations acting on a smaller space via the primary decomposition. As we will now see, primary transformations can themselves be written as a direct sum of certain indecomposable transformations – the cyclic transformations.

Definition 2.3. Suppose T is a linear transformation on V. A subspace W of V is said to be a *cyclic subspace* if there exists a vector v in W such that the sequence $v, Tv, T^2v, \ldots$ generates V. If V is itself a cyclic subspace, then T is called a *cyclic transformation.*

Example 46. Suppose $e_1, \ldots e_n$ are the standard basis vectors of V. The *shift operator* maps for all $i = 1, \ldots, n-1$, the ith standard basis vector to the $i+1$st standard basis vector, and maps e_n to 0. Then the shift operator is a cyclic transformation because e_1 is a cyclic vector. In general, any vector with a nonzero component in the direction of e_1 will be cyclic.

Any diagonalizable linear map with distinct eigenvalues is also cyclic. Without loss of generality, let us assume that the standard basis vectors $e_1, \ldots, e_n$ are eigenvectors with eigenvalues $\nu_1, \ldots, \nu_n$, respectively. For any $(c_1, \ldots, c_n) \in V$, there is a polynomial f such that $f(\nu_i) = c_i$ for all $i = 1, \ldots, n$. The polynomial f evaluated on the linear map sends the vector $(1, \ldots, 1)$ to $(c_1, \ldots, c_n)$. This means that $(1, \ldots, 1)$ must be a cyclic vector. Similarly, one can show that a vector is cyclic, provided none of its coordinates are zero.

Cyclic transformations are especially interesting because of the following lemma.

Lemma 2.2. Let g be any polynomial over the field $\mathbf{F}$. Upto similarity, there exists a unique cyclic linear transformation with minimal polynomial g.

Proof. If T is a cyclic linear transformation with minimal polynomial g, where $g = a_0 + a_1 t + \cdots + a_d t^d$ is a polynomial of degree d, then the collection $\{v, Tv, \ldots, T^{d-1}v\}$ must form a basis of V. With respect to this basis, the map T can be represented by the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix},$$

which is known as the *companion matrix of* g. Consequently, every cyclic linear transformation with minimal polynomial g can be represented by the companion matrix, implying that they are all similar. □

Theorem 2.3. Let f be an irreducible monic polynomial over $\mathbf{F}$. if $T\colon V \to V$ is an f-primary linear transformation, then there exist T-cyclic subspaces $W_1, \ldots, W_r$ of V satisfying

$$V = W_1 \oplus \cdots \oplus W_r.$$

Proof. See [**Jac85**, Section 3.8]. □

In the notation of Theorem 2.3, the cyclic subspaces $W_1, \ldots, W_r$ are all f-primary. So there is, after possibly reordering the subspaces, a partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_r > 0$, such that the restriction of T to W_i has minimal polynomial f^{λ_i}. We can deduce using Lemma 2.2 that there is a bijective correspondence between partitions and similarity classes of f-primary transformations. We will denote by $T_{f,\lambda}$ a f-primary linear transformation corresponding to λ and say that λ is the *shape* of $T_{f,\lambda}$.

Applying Theorems 2.1 and 2.3, any linear map T on an n-dimensional vector space V uniquely specifies irreducible polynomials $f_1, \ldots, f_r$, and partitions $\lambda^1, \ldots, \lambda^r$, such that T can be written as the direct sum:

$$T = T_{f_1,\lambda^1} \oplus \cdots \oplus T_{f_r,\lambda^r}. \tag{10}$$

This information can be organized into the *Jordan datum of* T, which is a function from the set of all irreducible monic polynomials over $\mathbf{F}$ taking values in the set of all partitions, which sends the polynomials f_i to the partitions λ^i and the rest to the empty partition. Conversely, any such function Θ leads to an operator T_Θ, provided the function satisfies $\sum_f \deg(f)|\Theta(f)| = n$, which is forced by the dimensions of the spaces on left and right hand side in (10) being equal.

Corollary 2.4. The Jordan datum gives rise to a bijective correspondence between the set of all partition-valued functions satisfying the above property and the set of similarity classes of linear transformations on an n-dimensional vector space V.

2.2. Application to Lattices. We will now see how the algebraic machinery of primary decomposition can be useful in better understanding the lattices of invariant subspaces.

Theorem 2.5. If T is a linear map with primary decomposition $T = T_{f_1} \oplus \cdots \oplus T_{f_r}$, then the lattice $\mathcal{L}(T)$ factorises into

$$\mathcal{L}(T) \cong \mathcal{L}\left(T_{f_1}\right) \times \cdots \times \mathcal{L}\left(T_{f_r}\right),$$

the product of the lattices corresponding to the primary parts of T.

Proof. Suppose W is a T-invariant subspace of V. For each $i = 1, \ldots, r$, the subspace W_{f_i}, defined as in Definition 2.1, is equal to the intersection $W \cap V_{f_i}$. The intersection is a T_{f_i}-invariant subspace of V_{f_i}; this leads to an order-preserving map from the lattice $\mathcal{L}(T)$ to the product of the lattices corresponding to the primary parts of T.

It is evident that the map is surjective. Moreover, according to Theorem 2.1, W decomposes as the direct sum $W = W_{f_1} \oplus \cdots \oplus W_{f_r}$, indicating that the W_{f_i} determine W and estabishing that the map is injective. □

In terms of the invariant flag generating polynomial, Theorem 2.5 states that

Corollary 2.6. If T is a linear map on $\mathbf{F}_q^n$ with primary decomposition $T = T_{f_1} \oplus \cdots \oplus T_{f_r}$, then the invariant flag generating polynomial F_T factorises as

$$F_T(x_1, \ldots, x_m; q) = \prod_{i=1}^{r} F_{T_{f_i}}(x_1, \ldots, x_m; q). \tag{11}$$

3. The case of Nilpotent Operators

In this section, we will use the geometric machinery developed in part two to enumerate flags of various types which are invariant under a nilpotent operator of a given shape. Let us start with a warm-up and take the zero map as our nilpotent operator.

warm-up 3 (Invariant subspaces). All subspaces are invariant with respect to the zero map, so counting d-dimensional invariant subspaces is the same as counting the points on some Grassmannian. The Schubert decomposition of the Grassmannian $\mathcal{G}_d(V)$ in terms of partitions is

$$\mathcal{G}_d(V) = \bigsqcup_{\lambda \subseteq d^{n-d}} \Omega_\lambda.$$

Taking sizes on both sides we get

$$\begin{aligned} |\,\mathcal{G}_d(V)| &= \sum_{\lambda \subseteq d \times (n-d)} \mathbf{A}^{|\lambda|} \\ &= \sum_{\lambda \subseteq d \times (n-d)} q^{|\lambda|} \end{aligned}$$

For all $i = 0, \ldots, d(n-d)$, the number of partitions $\lambda \subseteq d \times (n-d)$ of size i is equal to the $2i$th Betti number of the Grassmannian. This means that

Theorem 3.1. The number of d-dimensional subspaces of V is equal to the polynomial

$$\sum_{i=0}^{d(n-d)} b_{2i}(\mathcal{G}_d(V))q^i,$$

which is also the Poincaré polynomial of the Grassmannian evalauted at q. In particular, the count is a polynomial in q with positive integer coefficients.

warm-up 4 (Invariant flags). Let $\mu = (\mu_1, \ldots, \mu_m)$ be a composition of n. Then to count flags of type μ in V which are invariant under the zero map is also equivalent to counting the points on the flag variety $\mathcal{F}_\mu(V)$. This space has Schubert decomposition:

$$\mathcal{F}_\mu(V) = \bigsqcup_{w \in \mathfrak{S}^\mu} \Omega_w.$$

Let us take sizes on both sides:

$$\begin{aligned} |\mathcal{F}_\mu(V)| &= \sum_{w \in \mathfrak{S}^\mu} \mathbf{A}^{\ell(w)} \\ &= \sum_{w \in \mathfrak{S}^\mu} q^{\ell(w)}. \end{aligned}$$

The cells form an affine paving of the flag variety over $\mathfrak{S}^\mu$, so for all $i \geq 0$, the number of $w \in \mathfrak{S}^\mu$ with $\ell(w) = i$ equals the $2i$th Betti number of the flag variety.

Theorem 3.2. The number of flags of type μ in V is equal to

$$\sum_{i \geq 0} b_{2i}(\mathcal{F}_\mu(V))q^i,$$

i.e. the Poincaré polynomial of the Grassmannian evaluated at q. In particular, the count is a polynomial in q having positive integer coefficients.

Corollary 3.3. The number of complete flags in V is equal to

$$\sum_{i=0}^{n(n-1)/2} b_{2i}(\mathcal{F}(V))q^i.$$

Let λ be a partition, then the number of flags in V of type μ that are invariant under a nilpotent operator of shape λ can similarly be calculated used the decomposition of the space $\mathrm{Fix}(\mathcal{F}_\mu(V))$ into cells:

Theorem 3.4. The number of flags of type μ in V invariant under a nilpotent operator of shape λ is equal to the sum over row-weak tableau

$$\sum_{\tau \text{ row-weak}} q^{\mathrm{val}(\tau)} = \sum_{i \geq 0} b_{2i}(\mathrm{Fix}(\mathcal{F}_\mu(V)))q^i, .$$

As a result, this is also a polynomial in q with positive integer coefficients.

4. Reduction to Nilpotence

We have shown that the number of flags of a given type invariant under a nilpotent operator of shape λ is the generating polynomial for the val statistic on certain fillings of the Young diagram of λ. We will now see how this allows us to enumerate flags invariant under any operator.

Lemma 4.1 (Jordan-Chevalley decomposition of primary operators). Let $f(t)$ be an irreducible monic polynomial over the field $\mathbf{F}$ satisfying $f'(t) \neq 0$, and $T\colon V \to V$ be an f-primary linear transformation. Then there are linear maps S, Q, both polynomials in T, such that $T = S + Q$, $f(S) = 0$ and Q is nilpotent.

Proof. This is a consequence of the more general Jordan-Chevalley decomposition of endomorphisms of V, as in [**Baj11**, p. 14]. A more elementary proof carried out in the case when $\mathbf{F}$ is a subfield of complex numbers can be found in [**HK61**, Theorem 8, p. 217] (if the reader can get their hands on the first edition.) The argument can be made to work in the more general situation as long as $f'(t)$ is not zero. Taylor's formula for polynomials only holds when working over fields of characteristic zero, but the proof in [**HK61**] requires only an equation of the form $f(a+b) \equiv f(a) + f'(a)b \bmod b^2$, which is always valid. □

If T is as in 4.1, then S and Q are called the *semisimple and nilpotent parts* of T, respectively. The matrix S is invertible, so the algebra of polynomials in S over $\mathbf{F}$ forms a field, denoted by $\mathbf{K}$ and the linear action of S on V turns V into a $\mathbf{K}$-vector space. Additionally, T, S and Q commute among themselves because the transformations S and Q are polynomials in T; as a result, T and Q are $\mathbf{K}$-linear maps.

We denote by subscripts, such as $\mathcal{L}_{\mathbf{K}}(T)$ or $\mathcal{L}_{\mathbf{F}}(T)$, the field over which we are considering the subspaces invariant under T. The following theorem, due to Brickman and Fillmore [**BF67**, Theorem 6], establishes a relation between the lattice of T-invariant subspaces with the lattice of subspaces invariant under its nilpotent part Q.

Theorem 4.2. In the notation as above, the lattices $\mathcal{L}_{\mathbf{F}}(T)$ and $\mathcal{L}_{\mathbf{K}}(Q)$ are isomorphic.

Proof. The linear map S is a polynomial in T, so $W \in \mathcal{L}_{\mathbf{F}}(T)$ implies that $W \in \mathcal{L}_{\mathbf{F}}(S)$. Thus W is $\mathbf{K}$-linear, and so the lattices $\mathcal{L}_{\mathbf{F}}(T) = \mathcal{L}_{\mathbf{K}}(T)$ are identical. Finally, over $\mathbf{K}$, the linear map S acts as a scalar and so every subspace is S-invariant. So, if W is preserved by T, then it must also be preserved by $T-S$, which is equal to Q. Therefore, $\mathcal{L}_{\mathbf{K}}(T) = \mathcal{L}_{\mathbf{K}}(Q)$. □

Remark 4.1. In Theorem 4.2, is the shape of the nilpotent transformation Q equal to the shape of the f-primary T we started with? Suppose Q, when viewed as an $\mathbf{K}$-linear transformation, has shape λ. Then V decomposes as a direct sum $V = V_1 \oplus \cdots \oplus V_r$ of $\mathbf{K}$-linear subspaces cyclic with respect to Q such that for all $i = 1, \ldots, r$, the application $Q^{\lambda_i} V_i$ is trivial.

On the other hand, the $\mathbf{K}$-linear subspaces V_i are a fortiori also $\mathbf{F}$-linear. Furthermore, the subspaces being Q-cyclic over $\mathbf{K}$ implies that they are T-cyclic over $\mathbf{F}$. This is because

S and Q are polynomials in T, allowing the action of T to emulate any action induced by Q and S. Finally, it is not hard to check that modulo Q, the polynomial f evaluated on T is equal to f evaluated on S, which is zero. This means that for all $i = 1, \ldots, r$, we get $f(T)^{\lambda_i} V_i = Q^{\lambda_i} V_i = 0$, and so the shapes agree.

Definition 4.1. If T is a nilpotent linear transformation of shape λ, then denote by F_λ the invariant flag generating function, F_T, of T. In Section 3, we have shown that the coefficients of the monomials of F_λ are always polynomials in q.

Corollary 4.3. If T is a linear map on $\mathbf{F}_q^n$ with Jordan datum Θ, then the invariant flag generating polynomial F_T is

$$F_T(x_1, \ldots, x_m; q) = \prod_f F_{\Theta(f)}(x_1^{\deg f}, \ldots, x_m^{\deg f}; q^{\deg f}).$$

Proof. This follows from (11), Theorem 4.2 and Remark 4.1. □

Corollary 4.4. For every linear map, the number of invariant flags of a given type is a polynomial in q with positive integer coefficients.

4.1. Similarity Class Types. Suppose we are working over a finite field $\mathbf{F} = \mathbf{F}_q$ with q elements. Then Theorem 4.2 says something very interesting: the lattice of invariant subspaces for an f-primary transformation is isomorphic to the lattice of invariant subspaces of a nilpotent transformation, viewed as a linear map over a field extension of degree equal to the degree of f, having the same shape as the primary transformation. But there is a unique such field extension and all nilpotent transformations of the same shape are conjugate, which means that the lattice of an f-primary transformation depends on f only through its degree. This motivates the following definition.

Definition 4.2. Jordan data Θ_1 and Θ_2 are said to be *of the same type* if there is a degree preserving relabelling γ of the set of all monic irreducible polynomials over $\mathbf{F}$, such that $\Theta_1 = \Theta_2 \circ \gamma$. As a result, the type of Θ only records the degrees of the polynomials (and not the polynomials themselves) for which Θ takes a certain value λ.

Two linear transformations are said to be of the same *similarity class type* if their Jordan data are of the same type. This leads to a partition of the collection of all linear transformations, much coarser than the similarity classes. However, Theorem 4.2 implies that

Theorem 4.5. The lattice of invariant subspaces of a linear transformation only depends on its similarity class type. In other words, if T, T' are of the same type, then the lattices $\mathcal{L}(T)$ and $\mathcal{L}(T')$ are isomorphic.

As a result, the invariant flag generating polynomial of a linear map is also only a function of its similarity class type.

CHAPTER 8

Enumerating Subspaces by Profile

So far, we have extensively examined subspaces of a vector space that are preserved under the action of an operator. In this final chapter of the thesis, we will introduce the notion of the profile of a subspace and delve into recent progress made toward enumerating subspaces based on their profiles. This topic is particularly intriguing because various aspects of the problem have connections to the theory of symmetric functions, orthogonal polynomials, classical combinatorial objects, free probability theory, and more.

1. Profiles

Let V be a n-dimensional vector space over an arbitrary field $\mathbf{F}$.

Definition 1.1. Let Δ be a linear mapping on V and $\mu = (\mu_1, \ldots, \mu_k)$ be a sequence of positive integers satisfying $\mu_1 + \cdots + \mu_k \leq n$. A subspace $W \subseteq V$ has Δ-profile μ if

$$\dim(W + \Delta W + \cdots \Delta^{i-1} W) = \mu_1 + \cdots \mu_i \text{ for each } 1 \leq i \leq k$$

and

$$W + \Delta W + \cdots \Delta^k W = W + \Delta W + \cdots \Delta^{k-1} W.$$

Remark 1.1. Suppose the subspace $W \subseteq V$ has Δ-profile $\mu = (\mu_1, \ldots, \mu_k)$ and $\mu_1 + \cdots + \mu_k < n$. The subspace $V' = W + \Delta W + \cdots$ is then easily seen to be Δ-invariant and under the restriction of Δ to V', W has profile equal to μ with $\mu_1 + \cdots + \mu_k = \dim V'$. As a result, we will assume that the entries of a profile add up to n.

1.1. Properties of Profiles.

The sequences that occur as profiles of subspaces have the additional property of being integer partitions.

Lemma 1.1. Suppose W has Δ-profile $\mu = (\mu_1, \ldots, \mu_k)$, then $\mu_1 \geq \mu_2 \cdots \geq \mu_k$.

Proof. Put $U_0 = (0)$ and $U_j = W + \cdots \Delta^{j-1} W$ for all $j = 1, \ldots, k$. Then for each $j = 1, \ldots, k$, the subspace U_j is Δ-invariant and contains U_{j-1}. Moreover, Δ maps the subspace U_{j-1} into the subspace U_j. As a result, Δ descends to a map $U_j/U_{j-1} \to U_{j+1}/U_j$. Since $U_{j+1} \subseteq U_j + \Delta(U_j)$, this map is a surjection. Therefore, $u_j = \dim(U_j/U_{j-1}) \geq \dim(U_{j+1}/U_j) = u_{j+1}$. □

Example 47. Profiles of subspaces generalize many classical ways to describe the behaviour of a subspace under an operator. For instance, if $W \subseteq V$ is an m-dimensional subspace invariant under Δ, then it has profile equal to (m). On the other hand, if $v \in V$

is a cyclic vector for Δ, then the one-dimensional subspace generated by v has Δ-profile $(1, 1, \ldots, 1)$ (n times).

Proposition 1.2. If Δ, Δ' are similar linear transformations on V, then the set of subspaces that have Δ-profile μ is in bijection with the set of subspaces that have Δ'-profile μ.

PROOF. Since the linear maps Δ, Δ' are similar, there is an invertible linear map X such that $\Delta' X = X\Delta$. If $W \subseteq V$ has Δ-profile equal to μ, then for each $i = 1, \ldots, k$, we have

$$\begin{aligned}\mu_1 + \cdots \mu_i &= W + \Delta W + \cdots \Delta^{i-1} W \\ &= XW + X\Delta W + \cdots X\Delta^{i-1} W \\ &= XW + \Delta' XW + \cdots (\Delta')^{i-1} XW.\end{aligned}$$

So, the map $W \to XW$ is the required bijection. □

2. Enumerating Subspaces of a Given Profile

When $F = \mathbf{F}_q$, there are only finitely many subspaces of W, so we denote by $\sigma(\mu, \Delta)$ the number of subspaces V with Δ-profile μ.

We can readily deduce the enumerative analogue of Proposition 1.2.

Proposition 2.1. Let μ be a partition. If Δ and Δ' are similar linear maps on V, then $\sigma(\mu, \Delta) = \sigma(\mu, \Delta')$.

In [**BCRR92**], Bender, Coley, Robbins and Rumsey propose the following combinatorial problem.

Problem 1. Given μ and Δ compute $\sigma(\mu, \Delta)$.

Using the Möbius inversion on the lattice of subspaces of V, they show that the $\sigma(\mu, \Delta)$ for various values of μ satisfy a large system of linear equations. They solve these equations in two cases to obtain compact formulae for the $\sigma(\mu, \Delta)$. When Δ is a simple operator (i.e. its characteristic polynomial is irreducible), we have

$$\sigma(\mu, \Delta) = \frac{q^n - 1}{q^{\mu_1} - 1} \prod_{j \geq 2} q^{\mu_j(\mu_j - 1)} \begin{bmatrix} \mu_{j-1} \\ \mu_j \end{bmatrix},$$

and when Δ is a regular nilpotent operator (equivalantly, nilpotent with one-dimensional null space), then

$$\sigma(\mu, \Delta) = \prod_{j \geq 2} q^{\mu_j^2} \begin{bmatrix} \mu_{j-1} \\ \mu_j \end{bmatrix}.$$

2.1. The Chen-Tseng Recursion.

Definition 2.1. Let r be a positive integer and suppose $\Delta \in M_n(\mathbf{F}_q)$. Given sequences $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\nu = (\nu_1, \dots, \nu_r)$ of nonnegative integers, denote by $\phi(\lambda, \nu; \Delta)$ the number of flags $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_r$ of subspaces of $\mathbf{F}_n^q$ such that

$$\begin{aligned} \dim W_i &= \lambda_i && \text{for } 1 \le i \le r \\ \dim(W_i \cap \Delta^{-1} W_i) &= \nu_i && \text{for } 1 \le i \le r \\ \Delta W_i &\subseteq W_{i+1} && \text{for } 1 \le i \le r-1. \end{aligned}$$

If $\lambda = \nu$, any subspace W_i satisfying the first two conditions must be Δ-invariant. So, $\phi(\lambda, \lambda; \Delta)$ is equal to the number of Δ-invariant flags $W_1 \subseteq \cdots \subseteq W_r$ of type λ.

Lemma 2.2. Given a partition $\mu = (\mu_1, \dots, \mu_k)$, let $m_i = \mu_1 + \cdots + \mu_i$ for $1 \le i \le k$. We have

$$\sigma(\mu, \Delta) = \phi(\lambda, \nu; \Delta)$$

where $\lambda = (m_k, \dots, m_1)$ and $\nu = (m_k, m_{k-1} - \mu_k, \dots, m_1 - \mu_2)$.

Proof. Subspaces U with Δ-profile μ are in a one-to-one correspondence with flags $U_1 \subseteq \cdots \subseteq U_k$ where $U_i = U + \Delta U + \cdots + \Delta^{i-1} U$ for $1 \le i \le k$. It can be easily checked that these satisfy the above conditions. Conversely, suppose a flag $W_\bullet$ satisfies the conditions. We have

$$\begin{aligned} \lambda_i - \nu_i &= \dim W_i - \dim(W_i \cap \Delta^{-1} W_i) \\ &= \dim\left(\frac{W_i + \Delta^{-1} W_i}{\Delta^{-1} W_i}\right) \\ &= \dim\left(\frac{\Delta W_i + W_i}{W_i}\right). \end{aligned}$$

This means that $\dim(\Delta W_i + W_i) = m_{i+1}$, which along with the third condition forces $W_{i+1} = W_i + \Delta W_i$. ∎

Chen and Tseng [**CT13**, Lemma 2.7] proved that the $\phi(\lambda, \nu; \Delta)$ (and particularly the $\sigma(\mu, \Delta)$ by Lemma 2.2) satisfy a recursion in which the base cases are of the form $\phi(\eta, \eta; \Delta)$. Solving the recursion in many cases has led to alternate proofs of the two cases answered in [**BCRR92**] and some more (See [**GR12, CT13, AR22b, AR22a**]).

For $\Delta \in M_n(\mathbf{F}_q)$, the recursion involves coefficients that are products of q-binomial coefficients that depend only on λ and ν. These observations, together with Lema 2.2, imply the following result.

Theorem 2.3. There are polynomials $g_{\mu\lambda}(t) \in \mathbb{Z}[t]$ for every partition μ and every partition λ of n such that

$$\sigma(\mu, \Delta) = \sum_{\lambda \vdash n} g_{\mu\lambda}(q) \alpha_\Delta(\lambda; q),$$

for every prime power q and every $\Delta \in M_n(\mathbf{F}_q)$.

Recall that the invariant flag generating polynomial $F_\Delta(x_1, \ldots, x_k; q)$ of Δ is defined as $\sum_{\lambda \vdash n} \alpha_\Delta(\lambda; q) m_\lambda$. Putting $G_\mu(x_1, \ldots, x_k; q) = \sum_{\lambda \vdash n} g_{\mu\lambda} h_\lambda$, where h_λ is the complete homogeneous polynomial associated to the partition λ, we can restate Theorem 2.3 as:

Theorem 2.4. There is a symmetric polynomial $G_\mu(x_1, \ldots, x_k; q)$ in $\mathbb{Z}[x_1, \ldots, x_k; t]$ such that

$$\sigma(\mu, \Delta) = \langle G_\mu, F_\Delta \rangle,$$

for every prime power q and every $\Delta \in M_n(\mathbf{F}_q)$.

So, we have reduced computing $\sigma(\mu, \Delta)$ and solving Problem 1 to computing the symmetric polynomials G_μ. An expression for these in terms of classically studied polynomials called the *q-Whittaker* polynomials was found in [**Ram23**].

2.2. Similarity Class Types. An interesting implication of Theorem 2.3 is that the quantity $\sigma(\mu, \Delta)$ only depends on the operator Δ via the lattice of invariant subspaces of Δ. In Chapter 2, we proved that the lattice of invariant subspaces of an operator only depended on its similarity class type. So $\alpha_\Delta(\mu; q)$ and as a result, $\sigma(\mu, \Delta)$ also only depend on the similarity class type. We have the following strengthening of Proposition 2.1

Corollary 2.5. Let μ be a partition. If Δ and $\Delta' \in \mathrm{End}(V)$ have the same similarity class type, then $\sigma(\mu, \Delta) = \sigma(\mu, \Delta')$.

Problem 2. We conjecture that we can similarly strengthen Proposition 1.2: Let V be a finite dimensional vector space defined over an arbitrary field. Let μ be a partition, Δ and Δ' operators on V with the same similarity class type. There exists an explicit bijection between the collection of subspaces of V that have Δ-profile μ and those that have Δ'-profile μ.

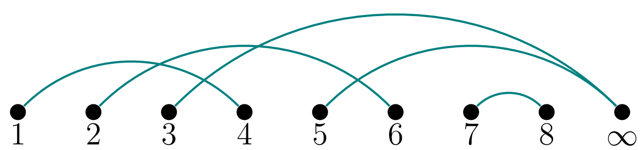
3. Diagonal Operators

The problem of enumerating subspaces by profiles also interacts with several well-studied combinatorial objects; this connection first manifests itself when studying the quantities $\sigma(\mu, \Delta)$ for diagonalizable operators Δ. Prasad and Ram [**PR23a**] compute $\sigma(\mu, \Delta)$ when the operator has distinct eigenvalues, in terms of a combinatorial statistic on certain discrete objects known as set partitions. In this section, we will provide an alternate proof of the result of Prasad and Ram, as well as generalize the theorem to all diagonalizable operators. Let us first assume that Δ has distinct eigenvalues.

3.1. Two-Part Profiles. Consider a d-dimensional subspace W of V such that the subspace along with its image under Δ span all of V; that is $W + \Delta W = V$. In this situation, the profile of W looks pretty simple: just two parts, $(d, n-d)$.

3.1.1. *Chord Diagrams.* A *chord diagram* is a visual representation of an involution w in $\mathfrak{S}_n$. We arrange n nodes labelled $1, \ldots, n$ along the X-axis. and to their right, a node labelled ∞. A circular arc lying above the X-axis is used to connect the elements of each transposition of w, while each fixed point of w is connected to the node ∞.

The chord diagram of the involution $(1,4)(2,6)(7,8)$ in the set $\mathfrak{S}_8$ is shown below:



We will refer to the left end of each arc as an *opening node*, and the right end as a *closing node*. In the above example, the opening nodes are $1, 2, 3, 5, 7$, while the closing nodes are $4, 6$ and ∞. A *crossing* is a pair of arcs $[(i,j),(k,\ell)]$ such that $i < k < j < \ell$. The chord diagram above has four crossings, namely $[(1,4),(2,6)]$, $[(1,4),(3,\infty)]$, $[(2,6),(3,\infty)]$, and $[(2,6),(5,\infty)]$. Let $v(w)$ denote the number of crossings of the chord diagram of an involution w.

3.1.2. *Subspaces and Chord Diagrams.* To every subspace W of V with Δ-profile equal to $(d, n-d)$, we can associate a chord diagram on $n+1$ nodes with d opening nodes. Let $w_1, \ldots, w_d$ be the echelon basis of W and $c_1 < \cdots < c_d$ the corresponding pivots. The chord diagram has $c_1, \ldots, c_d$ as its opening nodes; for all $i = 1, \ldots, d$, if the set

$$\text{pivots}(W + \Delta(w_1 + \cdots + w_i)) - \text{pivots}(W + \Delta(w_1 + \cdots + w_{i-1}))$$

is nonempty an arc connecting the ith opening node to the unique element, and if the set is empty, then to ∞.

Proposition 3.1. The number of subspaces of V with Δ-profile $(d, n-d)$, that correspond to a chord diagram w is equal to

$$(q-1)^{n-d} q^{\binom{d}{2}} \cdot q^{v(w)}.$$

Corollary 3.2. The number of subspaces of $V$with Δ-profile $(d, n-d)$ is given by the sum

$$(q-1)^{n-d} q^{\binom{d}{2}} \cdot \sum_{w} q^{v(w)}.$$

3.2. Set Partitions. A *set partition* $A = \{A_1, \ldots, A_m\}$ of $[n]$ is a decomposition

$$[n] = A_1 \cup \cdots \cup A_m,$$

of $[n]$ into pairwise disjoint non-empty subsets $A_1, \ldots, A_m$ of $[n]$. The subsets $A_1, \ldots, A_m$ are called the *blocks* of A. The *shape* of the set partition A is the list of cardinalities of $A_1, A_2, \ldots, A_m$ sorted in weakly decreasing order. So the shape of A is a partition of size n.

Example 48. The partition $A = \{1,2,7\} \cup \{3,6\} \cup \{4,5\}$ of the set $\{1, \ldots, 7\}$ will often be written as $127|36|45$. The shape of the partition is $(3,2,2)$.

We shall denote by $\Pi(n)$ the set of all partitions of $[n]$, and when $\lambda \vdash n$ is a partition of n, by $\Pi(\lambda)$ the set of partitions of $\{1, 2, \ldots, n\}$ of shape λ.

3.2.1. *The Interlacing Statistic.* We will denote by $\underline{\mathbb{N}}$ the set $\mathbb{N}$ along with an additional element ∞, that is taken to be larger than every element of $\mathbb{N}$.

Definition 3.1 (The arcs of a set). Suppose $A \subset \mathbb{N}$ has elements $a_1, \ldots, a_l$ written in increasing order. Its jth arc is the pair (a_j, a_{j+1}) for $j = 1, \ldots, l-1$, and its lth arc is (a_l, ∞).

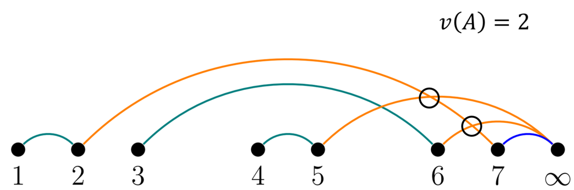


FIGURE 1. interlacing number of the partition 127|36|45.

Definition 3.2 (Interlacing Number). Let $n \in \mathbb{N}$. Let $A = A_1|\cdots|A_m \in \Pi_n$ with $|A_i| = l_i$. The *interlacing number*, denoted $v(A)$, of A is the number of crossings (as in Section 3.1.1) between the jth arcs of the sets $A_1, \ldots, A_m$ for all $j = 1, \ldots, \min_i l_i$.

Example 49. The interlacing number of the partition $A = 127|36|45$ is 2. The arcs, together with the two crossings that contribute to $v(A)$, are shown in Figure 3.2.1.

Table 1 shows the arcs and the number of interlacings for some more set partitions of the set $\{1, \ldots, 7\}$. The first, second, and third arcs are shown in different colours. Only crossing arcs of the same colour contribute to the interlacing number.

3.3. The General Case. A consequence of the discussion of the two-part profiles case is a new proof of the following theorem (See [**PR23a**, Theorem 4.7]) of Prasad and Ram:

Theorem 3.3. If $\Delta \in M_n(\mathbf{F}_q)$ is diagonalizable with distinct eigenvalues then

$$\sigma(\mu, \Delta) = q^{\sum_{j\geq 2} \binom{\mu_j}{2}} (q-1)^{\sum_{j \geq 2} \mu_j} b_{\mu'}(q),$$

where μ' is the partition conjugate to μ and

$$b_\lambda(q) = \sum_{A \in \Pi(\lambda)} q^{v(A)}.$$

This approach simply extends to understanding the case when Δ is an arbitrary diagonalizable matrix.

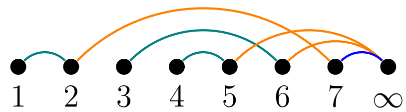
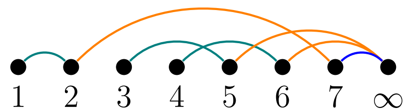
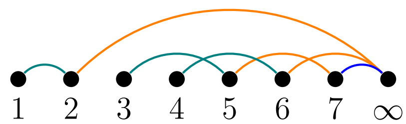
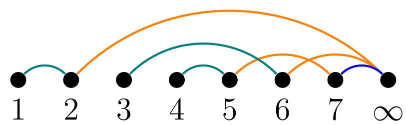
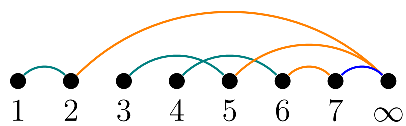
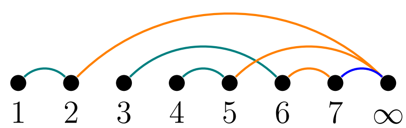
A	Arcs of A	$v(A)$
127|36|45		2
127|35|46		3
12|357|46		2
12|36|457		1
12|35|467		1
12|367|45		0

TABLE 1. Interlacing numbers of some set partitions of $\{1, \ldots, 7\}$

Definition 3.3. A partition A of $[n]$ is said to be a *refinement* of a partition B if every block of A is contained in some block of B. Refinement is a partial order on $\Pi(n)$ and we write $A \leq B$ to mean A *refines* B.

It is easily checked that any two partitions have a meet and a join, so refinement makes $\Pi(n)$ into a lattice. The smallest element of this lattice is the discrete partition, whose blocks are singletons.

Definition 3.4. Let $B \in \Pi(n)$. A partition $A \in \Pi(n)$ is *transverse* to B if each block of A has exactly one element from every block of B. In other words, A is transverse to B if their meet, $A \wedge B$, is the discrete partition.

Example 50. The partitions 127|36|45 and 13|246|57 are transverse, while 127|36|45 and 13|247|56 are not.

Denote by $\Pi(\lambda, B)$ the collection of partitions of $[n]$ of shape λ which are transverse to B.

Let Δ be a diagonalizable matrix with eigenvalues $d_1, \ldots, d_k$. If the d_i-eigenspace is of dimension ν_i then we may assume without loss of generality that $\Delta = \bigoplus_i d_i I_{\nu_i}$. Define A_Δ to be the set partition such that i and j are in the same block whenever the i-th and the j-th diagonal entries of Δ are equal.

Theorem 3.4. If $\Delta \in M_n(\mathbf{F}_q)$ is as above then

$$\sigma(\mu, \Delta) = q^{\sum_{j\geq 2} \binom{\mu_j}{2}} (q-1)^{\sum_{j\geq 2} \mu_j} b^{\Delta}_{\mu'}(q)$$

where μ' is the partition conjugate to μ and

$$b^{\Delta}_{\lambda}(q) = \sum_{A \in \Pi(\lambda, A_\Delta)} q^{v(A)}.$$

We note that this specializes the main theorem of [**PR23a**] because when Δ has distinct eigenvalues, A_Δ is the discrete partition and so every partition is transverse to A_Δ.

Remark 3.1. We have seen, in this thesis, how we can geometrically interpret why the solutions to various counting problems in finite vector spaces end up being polynomials with positive integer coefficients. While counting subspaces by profile leads to polynomials, the coefficients can be negative integers as well. A natural direction of research is to interpret this phenomenon geometrically, in particular the positive polynomials $b_\lambda(q)$ that show up in the solution of the Δ diagonalizable case.

Bibliography

[Agg23] Divya Aggarwal. Enumeration of splitting subsets of endofunctions on finite sets, 2023.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[AR22a] Divya Aggarwal and Samrith Ram. Polynomial matrices, splitting subspaces and Krylov subspaces over finite fields. *Finite Fields Appl.*, 83:Paper No. 102081, 16, 2022.

[AR22b] Divya Aggarwal and Samrith Ram. Splitting subspaces of linear operators over finite fields. *Finite Fields Appl.*, 78:Paper No. 101982, 17, 2022.

[Baj11] Beata Bajorska. On jordan-chevalley decomposition. *Silesian J. Pure Appl. Math.*, 1:7–27, 2011.

[BB05] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[BCRR92] Edward A. Bender, Raymond Coley, David P. Robbins, and Howard Rumsey, Jr. Enumeration of subspaces by dimension sequence. *J. Combin. Theory Ser. A*, 59(1):1–11, 1992.

[Ber09] François Bergeron. *Algebraic combinatorics and coinvariant spaces*. CMS Treatises in Mathematics. Canadian Mathematical Society, Ottawa, ON; A K Peters, Ltd., Wellesley, MA, 2009.

[BF67] L. Brickman and P. A. Fillmore. The invariant subspace lattice of a linear transformation. *Canadian J. Math.*, 19:810–822, 1967.

[Bir35] Garrett Birkhoff. Subgroups of abelian groups. *Proceedings of the London Mathematical Society*, s2-38(1):385–401, 1935.

[BO11] Jonathan Brundan and Victor Ostrik. Cohomology of Spaltenstein varieties. *Transform. Groups*, 16(3):619–648, 2011.

[But94] Lynne M. Butler. Subgroup lattices and symmetric functions. *Mem. Amer. Math. Soc.*, 112(539):vi+160, 1994.

[CT13] Eric Chen and Dennis Tseng. The splitting subspace conjecture. *Finite Fields Appl.*, 24:15–28, 2013.

[Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[Gec03] Meinolf Geck. *An introduction to algebraic geometry and algebraic groups*, volume 10 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2003.

[GR12] Sudhir R. Ghorpade and Samrith Ram. Enumeration of splitting subspaces over finite fields. In *Arithmetic, geometry, cryptography and coding theory*, volume 574 of *Contemp. Math.*, pages 49–58. Amer. Math. Soc., Providence, RI, 2012.

[Har77] Robin Hartshorne. *Algebraic geometry*, volume No. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HK61] Kenneth Hoffman and Ray Kunze. *Linear algebra*. Prentice-Hall Mathematics Series. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1961.

[HS79] R. Hotta and N. Shimomura. The fixed-point subvarieties of unipotent transformations on generalized flag varieties and the Green functions. Combinatorial and cohomological treatments cent. *Math. Ann.*, 241(3):193–208, 1979.

[Jac85] Nathan Jacobson. *Basic algebra. I*. W. H. Freeman and Company, New York, second edition, 1985.

[Knu71] Donald E. Knuth. Subspaces, subsets, and partitions. *J. Combinatorial Theory Ser. A*, 10:178–180, 1971.

[KT21] Steven N. Karp and Hugh Thomas. q-Whittaker functions, finite fields, and Jordan forms. *Sém. Lothar. Combin.*, 85B:Art. 57, 12, 2021.

[LB18] V. Lakshmibai and Justin Brown. *Flag varieties*, volume 53 of *Texts and Readings in Mathematics*. Hindustan Book Agency, Delhi, second edition, 2018. An interplay of geometry, combinatorics, and representation theory.

[Mac15] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley.

[Man01] Laurent Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci*, volume 6 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés, 3. [Specialized Courses].

[Nie95] Harald Niederreiter. The multiple-recursive matrix method for pseudorandom number generation. *Finite Fields Appl.*, 1(1):3–30, 1995.

[PR23a] Amritanshu Prasad and Samrith Ram. Set partitions, tableaux, and subspace profiles under regular split semisimple matrices, 2023.

[PR23b] Amritanshu Prasad and Samrith Ram. Splitting subspaces and a finite field interpretation of the Touchard-Riordan formula. *European J. Combin.*, 110:Paper No. 103705, 11, 2023.

[PR24] Amritanshu Prasad and Samrith Ram. Enumeration of anti-invariant subspaces and Touchard's formula for the entries of the q-Hermite Catalan matrix. *Adv. in Appl. Math.*, 154:Paper No. 102654, 18, 2024.

[Pra07] Amritanshu Prasad. Representations of $gl_2(f_q)$ and $sl_2(f_q)$, and some remarks about $gl_n(f_q)$, 2007.

[Ram23] Samrith Ram. Subspace profiles over finite fields and q-whittaker expansions of symmetric functions, 2023.

[Shi85] Naohisa Shimomura. The fixed point subvarieties of unipotent transformations on the flag varieties. *J. Math. Soc. Japan*, 37(3):537–556, 1985.

[SKKT00] Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen, and William Traves. *An invitation to algebraic geometry*. Universitext. Springer-Verlag, New York, 2000.

[Spa76] N. Spaltenstein. The fixed point set of a unipotent transformation on the flag manifold. *Indag. Math.*, 38(5):452–456, 1976. Nederl. Akad. Wetensch. Proc. Ser. A **79**.

[Sta12] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[Tu11] Loring W. Tu. *An introduction to manifolds*. Universitext. Springer, New York, second edition, 2011.

[Zha07] Yufei Zhao. On the bruhat order of the symmetric group and its shellability. 2007.