Positivity Constraints in Flat-Space QFT

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled "Positivity Constraints in Flat-Space QFT" towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Kshitij Verma at Indian Institute of Science Education and Research under the supervision of Dr. Diptimoy Ghosh, Assistant Professor, Department of Physics, during the academic year 2023-2024.

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This thesis is dedicated to my teachers and my parents

Declaration

I hereby declare that the matter embodied in the report entitled "Positivity Constraints in Flat-Space QFT", are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Diptimoy Ghosh and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

Kshitij Verma

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Abstract

Low energy effective field theories (EFTs) find applications in several areas of physics. Using basic and well-accepted principles of quantum field theory—unitarity, causality, locality, etc., it is possible to constrain the space of unknown Wilson coefficients appearing in the EFTs. We try to study the positivity constraints on the Wilson coefficients in the Euler–Heisenberg theory in the presence of a light fermion by using unitarity of the S-matrix and analyticity of the scattering amplitudes. Naïvely, the procedure fails and the previously obtained constraints no longer hold true. However, this is just an artefact of the analysis and the constraints are not affected. Then we try to use the same techniques to find constraints in an effective field theory containing only gluons. We find that since the one-loop amplitude diverges in the forward limit, the previously obtained constrained no longer seem to be true. The problem this time is different from the one in the theory containing photons, and it is unclear if it can be resolved somehow.

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Chapter 1

Introduction

Often a full description of physics at high energy scales is unnecessary and cumbersome, and thus effective field theories (EFTs) become very convenient to calculate quantities in which we might be interested. For example, we do not need to know the internal structure of a proton if we are conducting a simple and not the most precise experiment measuring its e/m ratio by deflecting it under the influence of a magnetic field. And other times, it might also happen that we simply do not have understanding of the physics at higher energies. EFTs allow us to carry out calculations in that scenario as well, and might offer us some insight about the full UV theory. To construct an EFT Lagrangian, we start with a known Lagrangian and add all possible terms which obey the symmetries of the theory, and which also conform to our usual requirements such as being local, Lorentz invariant, and keeping the Hamiltonian Hermitian. Since these terms would have dimension greater than four, they are accompanied by appropriate negative powers of the cut-off Λ, above which we expect our EFT to fail. After this point, we would need more details about the UV theory to make accurate predictions. For more details, see [1, 2].

These Lagrangian terms would have some coefficients, usually called Wilson coefficients, in front of them. They cannot be deduced if we do not know about the UV theory however, and are thus arbitrary. However, it is in fact possible to put constraints on these parameters, and throw theories which violate them into the swampland, i.e., EFTs which cannot possibly come from a UV complete theory. These constraints are also helpful for experimental purposes. Obtaining these constraints is possible just from a few sacrosanct principles of physics—unitarity of the S-matrix, causality, etc. Earlier, these constraints on some coefficients were found in the form of some positivity relations [3], but now they have been extended to two-sided bounds on their ratios [4]. Also, at first, the amplitudes were calculated only up to tree level and only for scalar theories. Later on, work was done to see the effect loops have on these constraints, and separately extended to other theories—EFTs with fermions [5], Abelian gauge bosons [6, 7], and non-Abelian gauge bosons [8]. In the first part of the thesis, we will describe the procedure to obtain these constraints.

In the past, the Euler–Heisenberg Lagrangian [9] has been studied and the techniques listed above have found constraints on the coefficients in it. The amplitudes calculated to carry out the analysis were up to tree level in [6] and up to one-loop level in [7]. In the second part of the thesis, we will revisit this problem but with an added light fermion in the theory. This seems to spoil these constraints but this is just an artefact of the method used.

In the next part, we will take up an EFT which extends the non-Abelian gauge theory of the gluon. Some constraints on the coefficients in this theory were found in [8]. In the analysis concerning amplitudes, only tree-level amplitudes were used. In this thesis, we see that if we include loops, it appears that the amplitude analysis no longer works.

Chapter 2

Preliminaries

In this chapter, we shall review unitarity of S-matrix, analyticity of scattering amplitudes and crossing symmetry. Then we will see how using these properties, we can derive constraints on Wilson coefficients.

2.1 Unitarity of the S-matrix

In quantum field theory, the S-matrix connects the initial and final states of the Hilbert space. It is essentially the time evolution operator from $t = -\infty$ to $t = \infty$ when we consider the Heisenberg picture. Since the final state can be equal to the initial state, it is more useful to talk about the T-matrix, defined by the relation

$$S = 1 + iT. \tag{2.1}$$

When we consider scattering processes of particles, we take them to be definite momentum eigenstates. Thus, the matrix elements of this T-matrix can be used to define scattering amplitudes $\mathcal{M}$ by the equation:

$$\langle \text{final state}|T|\text{initial state}\rangle = (2\pi)^4\delta^4\left(\sum_i p_i\right)\mathcal{M}. \tag{2.2}$$

These scattering elements are tremendously important quantities because they allow us to calculate the differential cross sections, which are experimental observables, are in some sense the probability of the scattering happening in a differential solid angle element. We have the relation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto |\mathcal{M}|^2 \tag{2.3}$$

for general scattering processes. In particular, when we consider $2 \to 2$ scattering of identical particles, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\mathrm{CM}}^2}. \tag{2.4}$$

Since states have norm 1, and the S-matrix acting on a state outputs another state, which must also have norm 1, the S-matrix must be unitary:

$$\begin{aligned} 1 &= \langle f|f\rangle \\ &= \langle i|S^\dagger S|i\rangle\,, \end{aligned} \tag{2.5}$$

and hence,

$$S^\dagger S = 1. \tag{2.6}$$

In terms of the T-matrix, we have

$$(1 - iT^\dagger)(1 + iT) = 1 \tag{2.7}$$

$$T^\dagger T = i(T^\dagger - T) \tag{2.8}$$

To obtain the optical theorem, we sandwich these operators between an initial and a final state. Then using the definition of the scattering amplitude, we get

$$\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i) = i \sum_X \int d\Pi_X (2\pi)^4 \delta^4\left(p_i - p_X\right) \mathcal{M}(i \to X)\mathcal{M}^*(f \to X) \tag{2.9}$$

If we set the initial the final state equal to the initial state, we get

$$\operatorname{Im} \mathcal{M}(i \to i) = 2E_{\mathrm{CM}} \left|\boldsymbol{p}_i\right| \sum_X \sigma(i \to X). \tag{2.10}$$

This clearly shows that the imaginary part of the amplitude must be a positive quantity, since everything on the right-hand side is positive. This is a non-perturbative result, but even though we usually calculate amplitudes perturbatively as asymptotic power series in coupling constants, which may not be unique, this equation holds at each order of the perturbation series.

For more details, consult [10, 11].

2.2 Analyticity of Scattering Amplitudes

It is generally taken that the scattering amplitudes are "real-boundary values of analytic functions", i.e. if we consider scattering amplitudes as functions of the complexified Mandelstam variables, then the physically relevant amplitudes for real values of the Mandelstam variables can be obtained at the boundaries of the domains where the amplitudes are analytic functions. This property can be derived by using the Wightman axiom—microscopic causality, i.e. spacelike separated fields either commute or anti-commute. Some other assumptions include the existence of a mass gap, and that the momentum transfer is small. A proper derivation is difficult, however more details can be found in the following references: [12, 13].

It is however possible to motivate this relation between causality and analyticity using an elementary toy model [14]. Suppose we have an input signal $f_{\text{in}}(t)$, and the output signal is given by

$$f_{\text{out}}(t) = \int_{\mathbb{R}} \mathrm{d}t' \, S(t - t') f_{\text{in}}(t'). \tag{2.11}$$

Here, the function S is analogous the the S-matrix. Considering the Fourier transforms of these functions, we get, using the convolution theorem,

$$\tilde{f}_{\text{out}}(\omega) = \tilde{f}_{\text{in}}(\omega)\tilde{S}(\omega). \tag{2.12}$$

Causality here means that only past events can influence future events. Hence $S(t) = 0$ for

$t < 0$. Now, in the equation

$$\tilde{S}(\omega) = \int_0^\infty e^{i\omega t} S(t), \tag{2.13}$$

ω was real, but we can take it to be a complex number in the upper-half plane, and the integral would still converge. Thus, it is reasonable to consider S-matrices which have $\tilde{S}(\omega)$ to be analytic in the upper-half plane. To get the values for the physically relevant real values of ω, we can take the limit:

$$\tilde{S}(\omega) = \lim_{\epsilon \to 0^+} \tilde{S}(\omega + i\epsilon). \tag{2.14}$$

Coming back to the case of quantum field theory, and restricting to the case of $2 \to 2$ scattering of particles with momenta p_i, where we take the initial particles to be incoming, and the final two particle to be outgoing, then we define the Mandelstam variables as

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2. \end{aligned} \tag{2.15}$$

We also have the relation

$$s + t + u = \sum_i m_i^2, \tag{2.16}$$

and hence only two Mandelstam variables are independent. For scattering of identical particles, analyticity for the scattering amplitude $\mathcal{M}(s,t)$ implies that it is analytic in s in the upper-half plane whenever $|t| < 4m^2$ [15]. When we take the forward scattering limit, $t \to 0$, then this implies

$$\mathcal{M}(s, t = 0) = \mathcal{M}^*(s^*, t = 0). \tag{2.17}$$

On the real axis, the amplitude can have poles at $s = m^2$ and $s = 3m^2$ due to propagators in tree-level Feynman diagrams, and branch cuts due to loops extending from $s = 0$ to $-\infty$, and from $s = 4m^2$ to ∞.

2.3 Crossing Symmetry

Crossing symmetry relates scattering amplitudes corresponding to two different processes. We shall derive this using the LSZ reduction formula, as also done in [7]. Consider $2 \to 2$ scattering of spin-1 massless particles having momenta p_i and polarizations ϵ_i. The polarization vectors can be written as

$$\epsilon^{\mu}_{\lambda}(p) = \frac{e^{i\lambda\phi}}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta\cos\phi - i\lambda\sin\phi \\ \cos\theta\sin\phi + i\lambda\cos\phi \\ -\sin\theta \end{pmatrix}, \tag{2.18}$$

where λ is the helicity and θ, ϕ correspond to the momentum vector in physical space

$$\boldsymbol{p} = |\boldsymbol{p}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta). \tag{2.19}$$

Then the LSZ reduction formula allows us to write the scattering amplitude in terms of correlation functions:

$$\begin{aligned} \mathcal{M}(p_1, p_2 \to p_3, p_4) = \int \prod_i \mathrm{d}^4 x_i \, e^{-i(p_1x_1 + p_2x_2 - p_3x_3 - p_4x_4)} \epsilon^{\mu_1}\epsilon^{\mu_2}\epsilon^{*\mu_3}\epsilon^{*\mu_4} \prod_i \partial_i^2 \\ \times \left\langle \Omega \middle| T\left\{\prod_i A_{\mu_i}(x_i)\right\} \middle| \Omega \right\rangle_{\text{connected}} . \end{aligned} \tag{2.20}$$

The order of the bosonic operators does not matter since they commute with each other. Now consider the process $p_1, -p_3 \to -p_2, p_4$. This is not a physical process since the temporal component of momenta cannot be negative. But we can analytically continue it along with the polarization vectors. If we choose

$$p_0, |\boldsymbol{p}| \to -p_0, -|\boldsymbol{p}|, \quad \theta, \phi \to \theta, \phi, \tag{2.21}$$

then we get

$$\epsilon^{\mu}_{\lambda}(p) = \epsilon^{*\mu}_{-\lambda}(-p). \tag{2.22}$$

Then the LSZ formula immediately gives us

$$\mathcal{M}(\lambda_1, p_1; \lambda_2, p_2 \to \lambda_3, p_3; \lambda_4, p_4) = \mathcal{M}(\lambda_1, p_1; -\lambda_3, -p_3 \to -\lambda_2, -p_2; \lambda_4, p_4). \tag{2.23}$$

More details along with derivations for particles with spins not equal to 1 can be found in [16].

2.4 Arc Analysis for Scalar Particles

In this section, we shall finally see how the properties described in the previous subsections can be used to derive constraints on Wilson coefficients. This procedure was first described in [3] but has been modified many times to improve it. Here, we present the version explained in [4]. Consider a $2 \to 2$ scattering process of identical scalar particles with mass m. Let

$$\mathcal{A}(s) = \mathcal{M}(s, t \to 0). \tag{2.24}$$

Then $\mathcal{A}$ has the following properties:

1. $\mathcal{A}$ is analytic in the upper-half plane. Thus, $\mathcal{A}(s) = \mathcal{A}^*(s^*)$. Apart from this, the amplitude may have poles on the real axis at $s = m^2, 3m^2$ and branch cuts on the negative real axis and from $s = 4m^2$ to ∞.
2. $\mathcal{A}$ has s-u crossing symmetry, and since $t = 0$ and $s + t + u = 4m^2$, we have $\mathcal{A}(s) = \mathcal{A}(u) = \mathcal{A}(4m^2 - s)$.
3. $\operatorname{Im}\mathcal{A} > 0$ because of the optical theorem.
4. As $|s| \to \infty$, we have $|\mathcal{A}(s)| \geq C s \log^2 s$ for some constant C. This is known as the Froissart–Martin bound [17, 18] and is true for gapped theories, i.e. when $m > 0$. It is however common to assume that it holds even when $m = 0$, as we shall do later.

Let us consider the integral

$$\oint \frac{\mathcal{A}(s')}{s'^{2n+1}} \, \mathrm{d}s' = 0, \tag{2.25}$$

for $n \geq 1$ on the contour shown below:

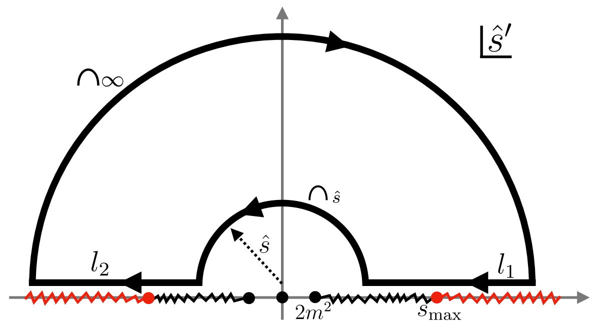

Figure 2.1: Contour (Credit[4]). The semicircles are centred at $\hat{s}' = s' = 2m^2$, and the smaller one has radius $\hat{s} = s$, while the larger semicircle is at infinity. The horizontal stretches of the contour are eps upwards of the real axis.

The integral is zero due to Cauchy's theorem because the amplitude is analytic in the upper-half plane. Using the Froissart–Martin bound,we can put the integral over the arc at infinity, $\cap_\infty$, to zero. Then, s-u symmetry and analyticity allow us to combine the integrals on the portions of the negative and positive real axes to get

$$\left(\int_{-\infty}^{-s} + \int_{s}^{\infty}\right) \frac{\mathcal{A}(s' + i\epsilon)}{(s' - 2m^2)^{2n+1}} \, \mathrm{d}s' = \int_{s}^{\infty} \frac{2i \, \mathrm{Im} \, \mathcal{A}(s')}{(s' - 2m^2)^{2n+1}} \, \mathrm{d}s' \, . \tag{2.26}$$

Thus, we finally get

$$a_n(s) := \frac{1}{\pi i} \int_{\cap_s} \frac{\mathcal{A}(s')}{(s' - 2m^2)^{2n+1}} \, \mathrm{d}s' = \frac{2}{\pi} \int_{s}^{\infty} \frac{\mathrm{Im} \, \mathcal{A}(s')}{(s' - 2m^2)^{2n+1}} \, \mathrm{d}s' > 0. \tag{2.27}$$

Note that the left-hand side is an IR-computable quantity, while the right-hand side requires us to know the full amplitude at arbitrarily high energies. It is not possible in general to find it, so we shall only use its positivity to get inequalities containing Wilson coefficients.

To exemplify the procedure, let us consider an EFT of a derivatively coupled scalar particle:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + \frac{c}{\Lambda^4} (\partial_\mu \phi)^4 + \ldots , \tag{2.28}$$

then the amplitude at the tree level would look as follows:

$$\mathcal{A} = c_0 + c_2(s - 2m^2)^2 + c_4(s - 2m^2)^4 + \dots, \tag{2.29}$$

where the coefficients c_i are related to the Lagrangian parameters—the Wilson coefficients. It is trivial to compute $a_n(s)$ from this IR expansion of the amplitude, and we get

$$a_n(s) = c_{2n} \geq 0. \tag{2.30}$$

These inequalities can then be converted into inequalities concerning the Wilson coefficients. However, since Lagrangians are not unique—due to integration by parts, and field redefinitions—the scattering amplitudes are better quantities to consider.

Many more bounds can be obtained and discussions pertaining to them can be found in [4, 19], but we don't use them here.

2.5 Arc Analysis for Gauge Bosons

The procedure outlined in the previous section works for scalar particles because then the amplitude possesses s-t-u symmetry, and in particular s-u symmetry. Some adaptations need to be made before applying it to scattering amplitudes of particles with non-zero spin. Here we will treat the case of spin-1 particles.

The only thing we can try is: consider scattering amplitudes for different helicity in and out states. But even if we find an s-u symmetric amplitude, its imaginary part need not be positive. Thus, we consider linear combinations of different such amplitudes. Let $\mathcal{M}^{\lambda_1\lambda_2\lambda_3\lambda_4}$ denote the scattering amplitude of the $\gamma\gamma \to \gamma\gamma$ process, where the photons have helicities $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, respectively. Then, two such combinations are $\mathcal{M}^{++++} \pm \mathcal{M}^{++--} + \mathcal{M}^{+-+-}$. We can use the optical theorem directly to say that $\operatorname{Im}\mathcal{A}^{+-+-} \geq 0$. The same conclusion is true for $\mathcal{A}^{++++} \pm \mathcal{A}^{++--}$ but we need to choose the initial and final scattering states to be

$\frac{1}{\sqrt{2}}(|{+}{+}\rangle \pm |{-}{-}\rangle)$, respectively:

$$\frac{\langle{+}{+}| \pm \langle{-}{-}|}{\sqrt{2}} T \frac{|{+}{+}\rangle \pm |{-}{-}\rangle}{\sqrt{2}} = \frac{\mathcal{M}^{++++} \pm \mathcal{M}^{++--} \pm \mathcal{M}^{--++} + \mathcal{M}^{----}}{2} \\ = \mathcal{M}^{++++} \pm \mathcal{M}^{++--}, \tag{2.31}$$

and we assumed parity symmetry to get the second line ($\mathcal{M}^{\lambda_1\lambda_2\lambda_3\lambda_4} = \mathcal{M}^{-\lambda_1-\lambda_2-\lambda_3-\lambda_4}$). Hence, we conclude $\mathrm{Im}(\mathcal{A}^{++++} \pm \mathcal{A}^{++--}) \geq 0$.

Now, we can repeat the earlier analysis without any problem.

Chapter 3

QED

Let us restudy the Euler–Lagrangian theory but with an added light fermion. If we don't assume CP symmetry and consider the Lagrangian only up to $1/\Lambda^4$ order, then if we have four electromagnetic tensors with free indices, we can contract the indices using four metric tensors, or one Levi–Civita symbol and two metric tensors. We need not consider two Levi–Civita symbols because we can decompose that product into a linear combination of products of four metric tensors. Thus, we have four linearly independent 8-dimensional operators. One possible basis is

$$\begin{aligned} O_1 &= F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ O_2 &= F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu} \\ O_3 &= \widetilde{F}_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ O_4 &= \widetilde{F}_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}. \end{aligned} \tag{3.1}$$

The first two operators are CP-even, while the second two are CP-odd. Again for the time being, we shall assume that the theory only has CP-even operators. Thus, the Lagrangian is

$$\mathcal{L} = \overline{\psi}i\not{D}\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda^4}F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{c_2}{\Lambda^4}F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}. \tag{3.2}$$

An alternate way to write the same Lagrangian is using the operator $F_{\mu\nu}\widetilde{F}^{\mu\nu}F_{\alpha\beta}\widetilde{F}^{\alpha\beta}$ instead of O_2:

$$\mathcal{L} = \overline{\psi}i\not{D}\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1 + \frac{1}{2}c_2}{\Lambda^4}F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{\frac{1}{4}c_2}{\Lambda^4}F_{\mu\nu}\widetilde{F}^{\mu\nu}F_{\alpha\beta}\widetilde{F}^{\alpha\beta}. \tag{3.3}$$

Without the added fermion, previous work finds constraints on these coefficients— in [6] at tree-level and [7] at one-loop level.

3.1 Massless Fermion

For the moment, let us assume that the fermion is massless. We shall now calculate photon $2 \to 2$ scattering amplitudes for the helicity combinations mentioned before. For the light fermion, we only have one type of diagram—the box diagram. The calculation is not easy to do by hand, so we used the Mathematica packages—FeynRules[20], FeynArts[21], FeynCalc[22, 23, 24] and PackageX[25, 26] to assist with that.

Since these Mathematica packages only compute tensorial amplitudes, we shall also work in the center-of-momentum frame so that we can take their scalar product with the polarization vectors of the external photons ($\mathcal{M} = \mathcal{M}_{\alpha\beta\mu\nu}\epsilon_1^{\alpha}\epsilon_2^{\beta}\epsilon_3^{*\mu}\epsilon_4^{*\nu}$).

In this frame, we have

$$\begin{aligned} p_1 &= E(1, 0, 0, 1) \\ p_2 &= E(1, 0, 0, -1) \\ p_3 &= E(1, \sin\theta, 0, \cos\theta) \\ p_4 &= E(1, -\sin\theta, 0, -\cos\theta), \end{aligned} \tag{3.4}$$

and polarization vectors can be chosen to be

$$\begin{aligned}
\epsilon_1^\pm &= \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0) \\
\epsilon_2^\pm &= \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0) \\
\epsilon_3^\pm &= \frac{1}{\sqrt{2}}(0, \mp \cos\theta, -i, \pm \sin\theta) \\
\epsilon_4^\pm &= \frac{1}{\sqrt{2}}(0, \pm \cos\theta, -i, \mp \sin\theta),
\end{aligned} \tag{3.5}$$

where E is the energy of the photons, θ is the angle of deflection after the scattering, and in terms of Mandelstam variables, they are:

$$\begin{aligned}
E &= \sqrt{s/2} \\
\cos\theta &= 1 + \frac{2t}{s} = -1 - \frac{2u}{s} \\
\sin\theta &= 2\sqrt{\frac{tu}{s^2}}.
\end{aligned} \tag{3.6}$$

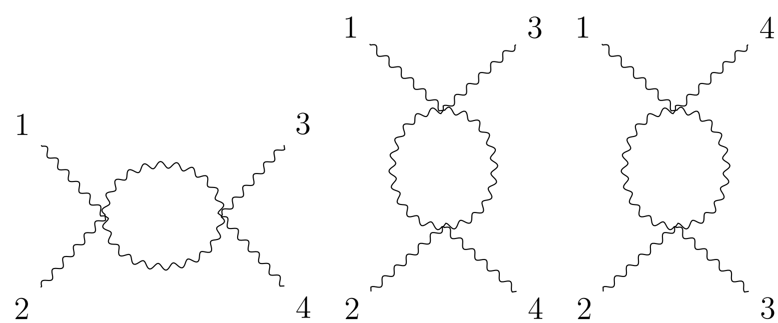

Figure 3.1: Bubble diagrams due to only the EFT operators

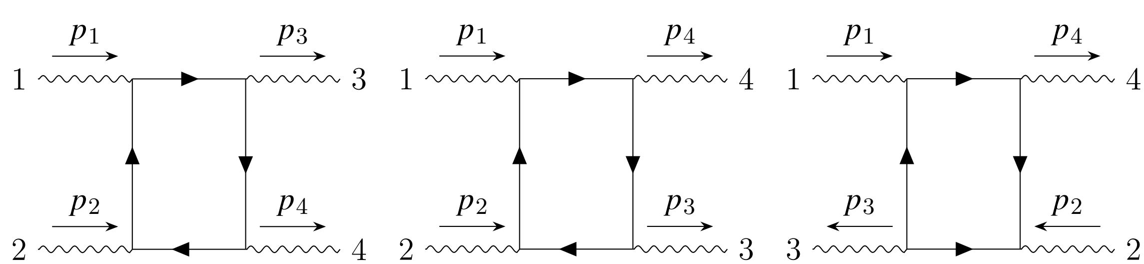

Figure 3.2: Box loop diagrams due to only the relevant dimension 4 operators

Up to e^4 order, the amplitudes come out to be

$$\begin{aligned}
\mathcal{M}^{++++} &= \frac{g_2}{\Lambda^4}s^2 - \frac{e^4}{4\pi^2 s^2}\left[(t^2+u^2)\left(\log\left(-\frac{1}{t}\right)-\log\left(-\frac{1}{u}\right)\right)^2 + 2s^2 + \pi^2(t^2+u^2)\right.\\
&\left. + 2(t^2-u^2)\left(\log\left(-\frac{1}{t}\right)-\log\left(-\frac{1}{u}\right)\right)\right],\\
\mathcal{M}^{++--} &= \frac{f_2}{\Lambda^4}(s^2+t^2+u^2) + \frac{e^4}{2\pi^2},\\
\mathcal{M}^{+-+-} &= \frac{g_2}{\Lambda^4}u^2 - \frac{e^4}{4\pi^2 u^2}\left[(t^2+s^2)\left(\log\left(-\frac{1}{t}\right)-\log\left(-\frac{1}{s}\right)\right)^2 + 2u^2 + \pi^2(t^2+s^2)\right.\\
&\left. + 2(t^2-s^2)\left(\log\left(-\frac{1}{t}\right)-\log\left(-\frac{1}{s}\right)\right)\right],
\end{aligned} \tag{3.7}$$

where g_2 and f_2 are related to the Lagrangian parameters c_1 and c_2 as

$$\begin{aligned}
g_2 &= 8c_1 + 6c_2,\\
f_2 &= 8c_1 + 2c_2.
\end{aligned} \tag{3.8}$$

They match the results obtained by Jikia et al [27]. Also note that no regularization has been done here. The amplitudes have no UV or IR divergences.

While the EFT contribution to the amplitude does not seem to have e^4, this factor is already absorbed inside c_i's, because these higher dimensional operators are also generated by square loop diagrams in the UV theory.

Now, using 2.27 and the amplitudes calculated in the above, we have

$$\frac{g_2 \pm f_2}{\Lambda^4} + \frac{e^4}{4\pi^2 s^2}\left(2\log\left(-\frac{s}{t}\right) - 1\right) \geq 0. \tag{3.9}$$

The second term diverges when we take t to zero, and the analysis requires us to take t to zero to make use of the optical theorem. Keeping in mind that $t = -s(1-\cos\theta)/2 \leq 0$, we note that this completely spoils the positivity bound if the fermion was not present as we can no longer say anything about any linear combination of f_2 and g_2.

3.2 Massive Fermion

The amplitudes obtained this time are presented below in terms of standard Passarino–Veltman functions [28] since the completely integrated amplitudes are very big:

$$\begin{aligned}\mathcal{M}^{++++} = {} & \frac{g_2}{\Lambda^4}s^2 + \frac{8\pi^2 e^4}{s^2}\Big[\left(t^2-u^2\right)\left(-\mathrm{B}_0(t)+\mathrm{B}_0(u)\right) \\ & + \left(4m^2 s - t^2 - u^2\right)\left(t\,\mathrm{C}_0(t) + u\,\mathrm{C}_0(u)\right) \\ & - m^2 s^2\left(2m^2 - s\right)\left(\mathrm{D}_0(s,t)+\mathrm{D}_0(s,u)\right) \\ & - \frac{1}{2}\left(4m^4 s^2 - 2m^2 s(t-u)^2 tu(t^2+u^2)\right)\mathrm{D}_0(t,u) - s^2\Big]\end{aligned} \tag{3.10}$$

$$\mathcal{M}^{++--} = \frac{f_2}{\Lambda^4}(s^2+t^2+u^2) + 8\pi^2 e^4\Big[-2m^4\Big(\mathrm{D}_0(s,t)+\mathrm{D}_0(s,u)+\mathrm{D}_0(t,u)\Big)+1\Big] \tag{3.11}$$

The fully evaluated forms are not very complicated for $t = 0$ and are as follows,

$$\begin{aligned}\mathcal{A}^{++++} = {} & \frac{g_2}{\Lambda^4}s^2 + \frac{e^4}{4\pi^2 s^2}\Bigg[2\sqrt{s\left(s-4m^2\right)}\left(s-2m^2\right)\log\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right) \\ & + 4\left(s-m^2\right)\sqrt{s\left(4m^2+s\right)}\log\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right) \\ & + \left(4m^4+2m^2 s - s^2\right)\log^2\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right) \\ & + \left(4m^4-2m^2 s\right)\log^2\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right) - 6s^2\Bigg]\end{aligned} \tag{3.12}$$

$$\begin{aligned}\mathcal{A}^{++--} = {} & \frac{2f_2}{\Lambda^4}s^2 + \frac{e^4}{2\pi^2 s^2}\Bigg[-2m^2\sqrt{s\left(s-4m^2\right)}\log\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right) \\ & - 2m^2\sqrt{s\left(4m^2+s\right)}\log\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right) \\ & + 2m^4\log^2\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right) \\ & + 2m^4\log^2\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right) + s^2\Bigg]\end{aligned} \tag{3.13}$$

$$\mathcal{A}^{+-+-}(s) = \mathcal{A}^{++--}(-s) \tag{3.14}$$

As a consistency check, we can compare this expression with the literature when $s, t \ll m$ as that is tantamount to integrating out the fermion. By evaluating the path integral for a constant A^μ field, we get the operators [29]

$$\frac{e^4}{5760\pi^2 m^4}\left(4(F^2)^2 + 7(F\tilde{F})^2\right), \tag{3.15}$$

which gives

$$\begin{aligned} \mathcal{M}^{++++} &= \frac{11e^4}{720\pi^2 m^4}s^2, \\ \mathcal{M}^{++--} &= -\frac{e^4}{240\pi^2 m^4}(s^2 + t^2 + u^2), \end{aligned} \tag{3.16}$$

which match our answer for the low energy limit when we ask Mathematica to calculate the Taylor series.

Let

$$I_\pm\left(\frac{s}{m^2}\right) := \frac{m^4}{e^4}\frac{1}{2\pi i}\int_{\cap_s} \frac{\mathcal{A}^{++++} \pm \mathcal{A}^{++--} + \mathcal{A}^{+-+-}}{s'^3}\,\mathrm{d}s', \tag{3.17}$$

where these $\mathcal{A}_i$ contain the contribution from the relevant operators only. Then, the bounds we obtain are:

$$\frac{g_2 \pm f_2}{\Lambda^4} + \frac{e^4}{m^4}I_\pm\left(\frac{s}{m^2}\right) \geq 0,$$

or equivalently,

$$\frac{g_2 \pm f_2}{e^4} \geq -\frac{\Lambda^4}{m^4}I_\pm\left(\frac{s}{m^2}\right). \tag{3.18}$$

Some values of the dimensionless function I are given below:

s/m^2	$I_+(s/m^2)$	$I_-(s/m^2)$
0.01	1.13×10^{-3}	1.97×10^{-3}
0.1	1.13×10^{-3}	1.97×10^{-3}
1	1.13×10^{-3}	1.97×10^{-3}
10	5.01×10^{-4}	5.81×10^{-4}
100	1.06×10^{-5}	1.07×10^{-5}
1000	1.63×10^{-7}	1.63×10^{-7}
10^6	3.37×10^{-13}	3.37×10^{-13}

Table 3.1: Values of $I_\pm$ for different values of s/m^2

As an example, let us assume that the fermion is the electron whose mass is about 0.5 MeV. This theory is an EFT which is valid till $\Lambda = \sqrt{s} = 100$ MeV, which is approximately the mass of the muon. Thus, if we take s only till $0.1\Lambda^2$, we get the bound

$$\frac{g_2 \pm f_2}{e^4} \geq -\left(\frac{100}{0.5}\right)^4 I_\pm\left(0.1\cdot\left(\frac{100}{0.5}\right)^2\right) = -19.75, \tag{3.19}$$

and if we stretch the limits of the EFT till $s = \Lambda^2$, we get

$$\frac{g_2 \pm f_2}{e^4} \geq -0.26 \tag{3.20}$$

For the general case, we can state the bounds in terms of two independent dimensionless quantities s/Λ^2 and m^2/Λ^2:

s/Λ^2 \ m^2/Λ^2	10^{-6}	10^{-4}	10^{-2}	1
0.01	2206.90	1062.38	11.26	0.001
0.1	27.90	16.27	5.01	0.001
0.5	1.28	0.81	0.36	0.001
1	0.34	0.22	0.11	0.001

Table 3.2: Negative of the lower bound for $(g_2+f_2)/e^4$ for different values of s/Λ^2 and m^2/Λ^2

m^2/Λ^2 \ s/Λ^2	10^{-6}	10^{-4}	10^{-2}	1
0.01	2206.97	1069.52	19.70	0.002
0.1	27.90	16.28	5.81	0.002
0.5	1.28	0.81	0.37	0.002
1	0.34	0.22	0.11	0.002

Table 3.3: Negative of the lower bound for $(g_2 - f_2)/e^4$ for different values of s/Λ^2 and m^2/Λ^2

Now suppose that we experimentally measure the EFT coefficients and want to find out whether we can say something about the cutoff Λ. Note that by experiments, we get to know the values c_i/Λ^4, but not c_i themselves. We can then shift around the terms in the inequality to get

$$I_\pm\left(\frac{s}{\Lambda^2}\cdot\frac{\Lambda^2}{m^2}\right) \geq -\frac{g_2 \pm f_2}{\Lambda^4}\frac{m^4}{e^4}. \tag{3.21}$$

If $g_2 \pm f_2$ are positive, then this inequality just says that Λ^4 is greater than a negative number, which does not say much. If however, at least one of $g_2 \pm f_2$ is negative, then this does give us something. Without any knowledge of Λ, we can safely take the parameter s/Λ^2 to be equal to 0.1, and we can stretch it to 1. As $I_\pm$ are decreasing functions with respect to Λ^2/m^2, as we continue to increase this parameter, there will come a time after which the inequality will no longer be satisfied. This would give us the upper bound for Λ^2/m^2, and thus Λ.

For demonstration purposes, suppose we find the right-hand side to be -10^{-12}, then for $s/\Lambda^2 = 0.1$, we get $\Lambda/m \leq 2400$ approximately; and for $s/\Lambda^2 = 1$, we get $\Lambda/m \leq 750$. A more realistic number would be -10^{-15}, which can be obtained by supposing $m = 1$ GeV, $\Lambda = 1$ TeV and $(g_2 \pm f_2)/e^4 = -10^{-3}$. This gives $\Lambda/m \leq 14000$ for $s/\Lambda^2 = 0.1$, and $\Lambda/m \leq 4500$ for $s/\Lambda^2 = 1$.

These bounds are not particularly useful because we could have also obtained it if we had assumed that the UV theory is weakly coupled, i.e. $|c_i/e^4| \sim 1$. This then gives us $\Lambda/m \sim 1000$. By contrast, $\Lambda/m = 2000$ would imply $(g_2 \pm f_2)/e^4 = 16$, which would come from a strongly coupled UV theory.

3.3 Scalar QED

Suppose that we have a light charged scalar instead of a fermion. Then, the Lagrangian is

$$\mathcal{L} = \left|D_\mu \phi\right|^2 - m^2|\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda^4}F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{c_2}{\Lambda^4}F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}. \tag{3.22}$$

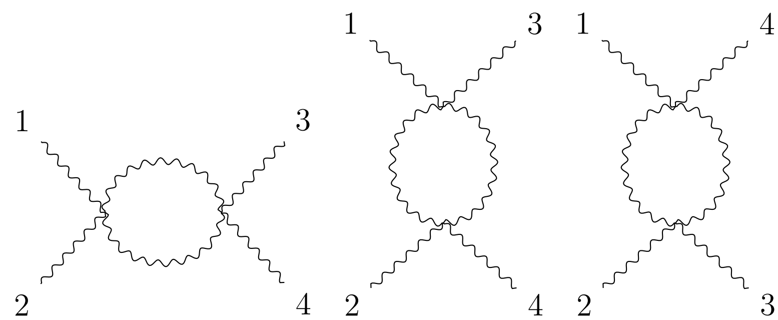

Figure 3.3: Bubble diagrams due to only the EFT operators

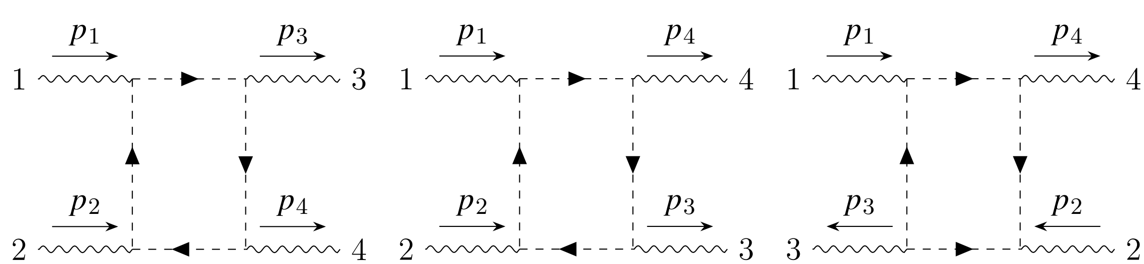

Figure 3.4: Box loop diagrams due to only the relevant dimension 4 operators

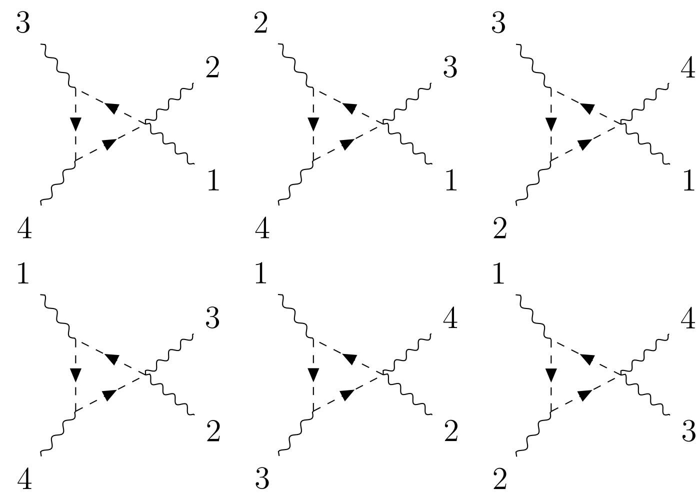

Figure 3.5: Triangle loop diagrams also coming only from the relevant dimension 4 operators

This time, we have three types of diagrams involving the light particle—the bubble diagrams coming from the EFT, the triangle diagrams and the box diagram with the light scalar running inside the loops. The amplitudes obtained are

$$\begin{aligned}\mathcal{M}^{++++} = \frac{4\pi^2}{s^2}\Big[&s(t-u)(\mathrm{B}_0(u)-\mathrm{B}_0(t)) - 2\left(2m^2 s - tu\right)\left(u\,\mathrm{C}_0(u) + t\,\mathrm{C}_0(t)\right)\\ &+ \left(2m^4s^2 + 4m^2stu + t^2u^2\right)\mathrm{D}_0(u,t) + 2m^4s^2(\mathrm{D}_0(s,u)+\mathrm{D}_0(s,t)) + s^2\Big]\end{aligned} \tag{3.23}$$

$$\mathcal{M}^{++--} = 4\pi^2 e^4\Big[2m^4(\mathrm{D}_0(s,u)+\mathrm{D}_0(s,t)+\mathrm{D}_0(u,t)) - 1\Big], \tag{3.24}$$

and again $\mathcal{M}^{+-+-}(s,t,u) = \mathcal{M}^{++++}(u,t,s)$.

And for the $t \to 0$ limit,

$$\begin{aligned}\mathcal{A}^{++++} &= -2m^2\left(m^2+s\right)\log^2\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right)\\&+2m^2\sqrt{s\left(s-4m^2\right)}\log\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right)\\&-\left(s-2m^2\right)\sqrt{s\left(4m^2+s\right)}\log\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right)\\&-2m^4\log^2\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right)+3s^2,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\mathcal{A}^{++--} &= 2m^2\sqrt{s\left(s-4m^2\right)}\log\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right)\\&+2m^2\sqrt{s\left(4m^2+s\right)}\log\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right)\\&-2m^4\log^2\left(\frac{\sqrt{s\left(s-4m^2\right)}+2m^2-s}{2m^2}\right)\\&-2m^4\log^2\left(\frac{\sqrt{s\left(4m^2+s\right)}+2m^2+s}{2m^2}\right)-s^2,\end{aligned} \tag{3.26}$$

and $\mathcal{A}^{+-+-}(s) = \mathcal{A}^{++++}(-s)$ as before. We can again compare it to the known results in the low energy limit [29], where the generated operators are

$$\frac{e^4}{23040\pi^2 m^4}\left(7(F^2)^2 + (F\tilde{F})^2\right) \tag{3.27}$$

to give the amplitude

$$\begin{aligned}\mathcal{M}^{++++} &= \frac{e^4}{360\pi^2 m^4}s^2,\\ \mathcal{M}^{++--} &= \frac{e^4}{480\pi^2 m^4}(s^2+t^2+u^2).\end{aligned} \tag{3.28}$$

The end result is the same—total positivity is lost, but it is almost recovered when we take $s/\Lambda^2 \approx 1$ and $m/\Lambda \approx 1$.

Chapter 4

Yang–Mills

Now we wish to repeat the same analysis for SU(3) gauge theory but for now let us consider only the theory with gluons and ghosts. For the moment, let us only consider the dimension 4 operators. Thus the Lagrangian is,

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \partial_\mu \overline{c}^a \partial^\mu c^a + g f^{abc} \partial_\mu \overline{c}^a A^{b\mu} c^c \tag{4.1}$$

We will again calculate the 2 to 2 scattering amplitudes. This time, there are more diagrams because the gluon interacts with itself and because there are ghosts.

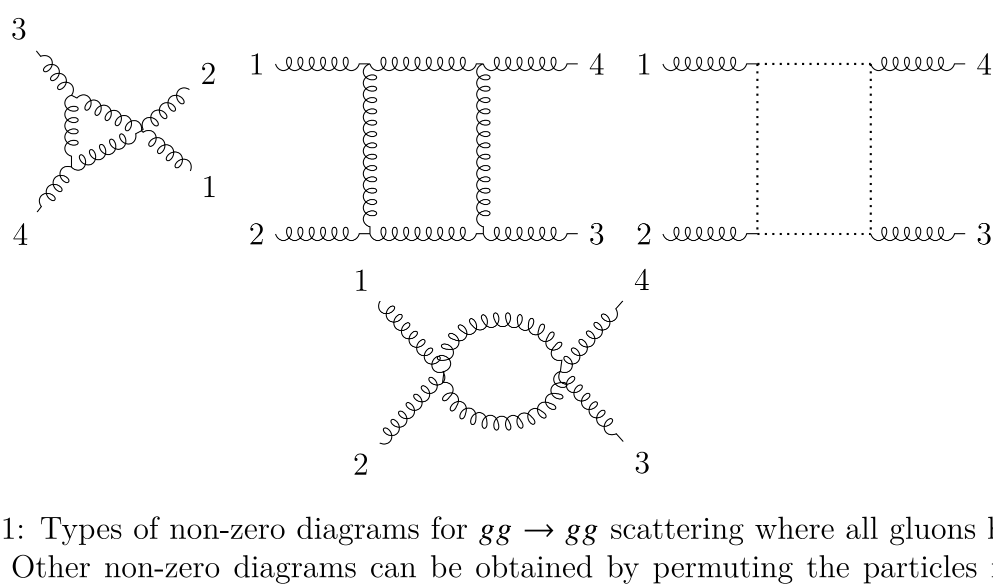

Figure 4.1: Types of non-zero diagrams for $gg \to gg$ scattering where all gluons have color index 1. Other non-zero diagrams can be obtained by permuting the particles inside the shown diagrams. The rest of the diagrams are zero due to SU(3) color structure constants.

4.1 Calculation of Scattering Amplitude $gg \to gg$

The amplitude in consideration is the state with two gluons of color index 1 and positive helicities going to the same state. The individual diagrams have UV and/or IR divergences, but the final amplitude only has an IR divergence, which is regulated using dimensional regularization.

$$
\begin{aligned}
\mathcal{M}(s,t,u) = \frac{9g^4}{16\pi^2 s^2 tu}\Bigg(& 2s^4 \log\left(\mu^2\right) \log\left(-\frac{1}{u}\right) - 2s^4 \log\left(\mu^2\right) \log\left(-\frac{1}{s}\right) + 2s^4 \log\left(-\frac{1}{t}\right) \log\left(-\frac{1}{u}\right) \\
& - 2s^4 \log\left(-\frac{1}{s}\right) \log\left(-\frac{1}{t}\right) + 2s^3 t \log\left(\mu^2\right) \log\left(-\frac{1}{u}\right) - 2s^3 t \log\left(\mu^2\right) \log\left(-\frac{1}{t}\right) + 2s^3 t \log\left(-\frac{1}{s}\right) \log\left(-\frac{1}{u}\right) \\
& - 2s^3 t \log\left(-\frac{1}{s}\right) \log\left(-\frac{1}{t}\right) + 2\pi^2 s^2 tu + s^2 tu + \pi^2 stu^2 + 2t^3 u \log^2\left(-\frac{1}{t}\right) + 2t^3 u \log^2\left(-\frac{1}{u}\right) \\
& + t^3 u \log\left(-\frac{1}{t}\right) - t^3 u \log\left(-\frac{1}{u}\right) - 4t^3 u \log\left(-\frac{1}{t}\right) \log\left(-\frac{1}{u}\right) + 3t^2 u^2 \log^2\left(-\frac{1}{t}\right) + 3t^2 u^2 \log^2\left(-\frac{1}{u}\right) \\
& - 6t^2 u^2 \log\left(-\frac{1}{t}\right) \log\left(-\frac{1}{u}\right) + \pi^2 tu^3 + 2tu^3 \log^2\left(-\frac{1}{t}\right) + 2tu^3 \log^2\left(-\frac{1}{u}\right) - tu^3 \log\left(-\frac{1}{t}\right) \\
& + tu^3 \log\left(-\frac{1}{u}\right) - 4tu^3 \log\left(-\frac{1}{t}\right) \log\left(-\frac{1}{u}\right) \Bigg) - \frac{9\Delta g^4 s \left(s \log\left(-\frac{1}{s}\right) + t \log\left(-\frac{1}{t}\right) + u \log\left(-\frac{1}{u}\right)\right)}{8\pi^2 tu},
\end{aligned}
\tag{4.2}
$$

where ($\Delta = \frac{1}{\epsilon} - \gamma + \log(4\pi)$).

This diagram is divergent in the $t \to 0$ limit. Hence, as before, this spoils the analysis. And again, first carrying out the integration over the arc and then taking the limit does no good.

4.2 Mass regulator

Since the amplitudes obtained using dimensional regularization are very big and unwieldly to work with, we try to use a mass regulator instead in this section. We give the gluons and ghosts a small mass m in the propagators, then asymptotically as $m \to 0$, we have

$$\begin{aligned}\mathcal{M}_{++++} &= \frac{9g^4 s \left(\log\left(-\frac{m^2}{s}\right) + \log\left(\frac{s}{m^2}\right)\right)}{16\pi^2 m^2} \\ &\quad - \frac{9g^4\left(\log^2\left(-\frac{m^2}{s}\right) - \log^2\left(\frac{s}{m^2}\right) + 7\log\left(-\frac{m^2}{s}\right) + 2\log\left(\frac{s}{m^2}\right) - 3\right)}{16\pi^2}\end{aligned} \tag{4.3}$$

$$\mathcal{M}_{++--} = -\frac{9g^4}{16\pi^2} \tag{4.4}$$

$$\begin{aligned}\mathcal{M}_{+-+-} &= \frac{9g^4\left(\log^2\left(-\frac{m^2}{s}\right) - \log^2\left(\frac{s}{m^2}\right) + 2\log\left(-\frac{m^2}{s}\right) + 7\log\left(\frac{s}{m^2}\right) + 3\right)}{16\pi^2} \\ &\quad + \frac{9g^4 s \left(\log\left(-\frac{m^2}{s}\right) + \log\left(\frac{s}{m^2}\right)\right)}{16\pi^2 m^2}\end{aligned} \tag{4.5}$$

As expected, the amplitudes still diverge in the $m \to 0$ limit. Let us again try integrating first and taking the limit later.

The integral over the arc of radius R comes out to be

$$\frac{9g^4\left(4R - 5m^2\right)}{16\pi m^2 R^2}, \tag{4.6}$$

which still diverges. Taking this limit is very important since gauge theories are unitary only if the gauge bosons have zero mass. If we also had the EFT operators in the Lagrangian, this would have given us the result $+\infty +$ Wilson coefficients > 0, which doesn't tell us anything about the coefficients.

Chapter 5

Conclusion and Outlook

We saw that positivity constraints seem to be spoilt in the Euler–Heisenberg Lagrangian when we add a light fermion or a light scalar. These bounds almost reappear when we take the limits $s/m^2 \to 1$ and $m^2/\Lambda^2 \to 1$. But the bounds we get from considering the theory without these extra particles should still hold because at the end of the day, the EFT operators are obtained after integrating out the exact same heavy particles.

Another way to think about this is by excluding this light particle theoretically, which also yields a perfectly good EFT. Carrying out the analysis tells us about the heavy particles that have been integrated out. In the examples considered, the light particle can have no bearing on this since it does not interact with the heavy particle. But perhaps more complicated and contrived models can be thought of where the light and heavy particles interact and thus excluding it from the theory would no longer be applicable. It might therefore still be worthwhile to investigate this matter further.

Next, we saw that on considering one-loop amplitudes for gluon scattering, the amplitude is divergent in the $t \to 0$ limit. We cannot even ignore this problem for some time, first integrate the amplitude over the arc and then take the limit (as we are able to in some other conditions, such as where the amplitude goes as s/t) because that limit too is divergent. Using mass regulator and trying the same thing is also of no help. This also seems to be a limitation of this analysis and requires further consideration since the subluminality argument still continues to hold.

It is possible that the Froissart–Martin bound does not apply here because gluons are massless, and giving them a small mass breaks unitarity. Unitarity is not broken if we give a small mass to a scalar particle for regulatory purposes. In that case, we would be able to utilize the bound even if our theory had a massless particle.

In conclusion, more work needs to be done to fully understand the issue, and maybe modify the arc analysis method so that we are able to derive more useful information about the Wilson coefficients.

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