Topics in Positivity of Line Bundles

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under the supervision of
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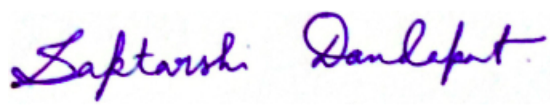

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This is to certify that this thesis entitled "*Topics in Positivity of Line Bundles*" submitted towards the partial fulfilment of the Mathematics M.Sc Degree Program at the Indian Institute of Science Education and Research Pune represents work carried out by *Saptarshi Dandapat* under the supervision of *Dr. Omprokash Das* and *Prof. Amit Hogadi*.

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This Master's thesis is dedicated to my Mother and Father.

Contents

1 Ample and Nef Line Bundles

The main goal of this report is to study the theory of positivity for line bundles and divisors on a projective algebraic variety over an algebraically closed field k. We discuss the basic theory of divisors, line bundles and linear series before proceeding into the classical theory of ample line bundles. We show how positivity can be recognised cohomologically and numerically. It can be also measured geometrically which is discussed in [Laz17], but we omit the discussion here. We describe the theory of $\mathbb{Q}$- and $\mathbb{R}$- divisors which is used for studying limits of ample bundles. Furthermore after defining Nef divisors, we discuss examples of ample cones and their structures with examples. The first section of this report ends with defining amplitude for a mapping and introduction towards the regularity theorem of Castelnuovo and Mumford.

1.1 Preliminaries

We start the section by recalling some basic facts and notations, and constructions involving divisors and line bundles. We follow the convention of [Har13]. After that we discuss intersection number of two divisors, numerical equivalence between divisors and asymptotic versions of the Riemann-Roch theorem. For the later, we follow the convention of [Laz17].

1.1.1 Line Bundles and Divisors

We start with the definition of Weil divisors, since these are easier to understand geometrically, but the constraint is that they can be defined only on cerntain types of Noetherian integral schemes. More generally on a scheme, we can define Cartier divisor, and explain connection between these two divisors and invertible sheaves, also known as vector bundles.

For this, we give an example as discussed in [Har13]. Let C is non-singular projective curve in $X = \mathbb{P}^2_k$, the projective plane over k, an algebraically closed field. For each line L in X, we have $L \cap C$ as finite set of points. If C is a curve of degree A, the intersection $L \cap C$ has exactly A points with proper multiplicity. We denote $L \cap C = \sum n_i P_i$, where $P_i \in C$ are points in the curve and n_i are multiplicities. This formal sum is defined as a divisor on C. In other words, *divisor* on a curve is an element of the free abelian group on the points of X. As L varies, we obtain a family of divisors on C, which are parameterised by a set of lines in X, which is the dual projective space $(\mathbb{P}^2)^*$. This set is known as *linear system of divisors on C* as we will define later. Now, the embedding $C \subset X$ can be reconstructed from its linear system in the following way. If $P \in C$ is point on the non-singular curve, the set of divisors in the linear system containing the point P, corresponds to lines in the dual space of $\mathbb{P}^2$, $L \in (\mathbb{P}^2)^*$ which passes through P, which in turn determines P as unique point in X. We generalise this notion of linear system and relations with embeddings in projective space later in detail.

If f is a rational function on X (a non-singular projective variety) then we can associate to it a divisor as $\operatorname{div}(f) = Z_0(f) - Z_\infty(f)$, which is the formal difference of the set of points where f is 0 and the set of points where f has a pole, counted with multiplicities. Such a divisor is called *principal divisor*.

Consider L_1 and L_2 be two lines in the projective surface $X = \mathbb{P}^2$, say they are given by homogeneous equations $f_1 = 0, f_2 = 0$ in X, and $D_1 = L_1 \cap C$ and $D_2 = L_2 \cap C$ are corresponding divisors. Now f_1/f_2 is a rational function on X, which can be restricted to a rational function h on C. Now, h has zeroes at points of D_1, and poles at the points of D_2 by definition. The existence of such a rational function can be taken as an intrinsic definition of linear equivalence between two divisors divisors D_1 and D_2.

Before formally defining the Weil divisors, we have the following definition from [Har13].

Definition 1.1.1. A scheme X is called *regular in codimension 1* if every local ring $\mathcal{O}_x$ of X of dimension 1 is regular.

Examples of such schemes are as follows. On a non-singular variety, all the local rings are regular. Local ring of every closed point is regular. Now, all local rings can be seen as localisation of local rings of closed points. Hence non-singular varieties of a field are regular in codimension 1. Also, on Noetherian normal scheme, all local rings of dimension 1 are integrally closed domain, so a scheme of this type are regular in codimension 1.

We consider schemes which satisfies following property:

(*) X is an integral Noetherian seperated scheme, regular in co-dimension 1. Then the Weil divisors are defined as follows.

Definition 1.1.2. ([Har13]) A *prime divisor* on a scheme X satisfying (*) is a closed integral subscheme T of co-dimension 1. A *Weil divisor* is element of $\operatorname{Div} X$, the free abelian group generated by the prime divisors. We denote, $D = \sum n_i T_i$, where T_i are prime divisors and $n_i \in \mathbb{Z}$, where finitely n_i's are non-zero. When all $n_i \geq 0$, then we define D to be *effective* divisor.

Now we define the notion of principal divisors in schemes of (*) type as discussed earlier on a curve. We use valuations to determine the multiplicities

n_i's in the definition of an Weil divisor. Consider T, a prime divisor on X, and $t \in T$ is a generic point (closure of $\{t\}$ is Y). Then the local ring $\mathcal{O}_{t,X}$ is a DVR with its quotient field K which is function field of X. The corresponding valuation, denoted by v_T, to be the valuation of T. Since X is seperated scheme, T can be uniquely determined by its valuation due to the following: If X is seperated scheme over the field k, then center of all valuations K/k on X is unique. (A valuation of K/k has *center* x on X (integral scheme of finite type over an algebraically closed field k) if its valuation ring A dominates the local ring $\mathcal{O}_{x,X}$).

Now if $g \in K^*$ be a non-zero rational function on X, $v_T(g) \in \mathbb{Z}$. If $v_T(g) > 0$, we say that g has zero along T of $v_T(g)$ order and a pole of order $-v_T(g)$ if it is negative. We have the following final lemma from [Har13].

Lemma 1.1.1. ([Har13]) Consider X is a scheme satisfying (*) and $g \in K^*$ is a non-zero function on X, then $v_T(g) = 0$ for all but finitely many prime divisors T.

This lemma allows us to make the following definition.

Definition 1.1.3. ([Har13]) The *divisor* of g denoted by (g) is defined by

$$(g) = \sum v_T(g) \cdot T,$$

where the sum is taken over all prime divisors of X. Any divisor which is equal to divisor of a function is called a *principal* divisor.

From the properties of valuations we have if $f_1, f_2 \in K^*$, $(f_1/f_2) = (f_1) - (f_2)$. So the map sending $f \mapsto (f)$ will give homomorphism from the group K^* to $\operatorname{Div} X$, where its image consisting of principal divisors is subgroup of the free group $\operatorname{Div} X$.

For a scheme of type (*) we generalise the notion of linear equivalence as described earlier.

Definition 1.1.4. ([Har13]) Two Weil divisors D_1, D_2 are defined as *linearly equivalent* ($D_1 \sim D_2$) if we have $D_1 - D_2$ is principal divisor. The free group $\operatorname{Div} X$ modulo principle divisors is called the *Divisor Class Group* of X, denoted by $\operatorname{Cl} X$.

This divisor class group gives us an way of determining whether a Noetherian integral domain is UFD or not by the following result.

Proposition 1.1.1. ([Har13]) The Noetherian domain A is UFD if and only if $X = \operatorname{Spec} A$ is normal scheme and the class group, $\operatorname{Cl} X = 0$.

The following proposition from commutative algebra is useful for calculating the class group of projective spaces as we describe in following.

Proposition 1.1.2. ([Mat70]) Consider I is an integrally closed Noetherian domain, then

$$I = \bigcap_{\operatorname{ht}\mathfrak{p}=1} I_{\mathfrak{p}}$$

where intersection has been taken over all prime ideals of I of height one.

Using the equivalent condition of determining UFD using class groups, we can calculate the class group of an affine space.

Example 1.1.1. ([Har13]) Take $X = \mathbb{A}_k^n$, then the class group $\operatorname{Cl} X = 0$. Since $\operatorname{Spec} k[x_1, \ldots, x_n] = X$ is a UFD.

Example 1.1.2. ([Har13]) For a Dedekind domain, the ideal class group defined in Algebraic Number Theory coincides with the definition of divisor class group. If I is a Dedekind domain, then $\operatorname{Cl}(\operatorname{Spec} I)$ is ideal class group of I. Thus proposition 1.1.1 can be generalised to the fact that the Noetherian domain I is Unique Factorization Domain if and only if the ideal class group is trivial.

Now we define degree of divisor on the projective space over the algebraically closed field k.

Definition 1.1.5. ([Har13]) Let $X = \mathbb{P}_k^n$, consider a divisor $D = \sum n_i T_i$. We define the *degree* of divisor D by $\deg D = \sum n_i \deg T_i$, where $\deg T_i$ are degree of hypersurface T_i.

Let H denote the hypersurface $x_0 = 0$, then we have the following: any divisor of degree d is numerically equivalent to a multiple of hypersurface divisor. This multiple is given by the following proposition and hence we have an isomorphism to the group of integers.

Proposition 1.1.3. ([Har13]) Consider A, a divisor of degree n, then $A \sim nH$. For any $g \in K^*$, we have $\deg(g) = 0$. This degree function produces an isomorphism $\deg : \operatorname{Cl} X \to \mathbb{Z}$.

In the following proposition, we discuss that removing a co-dimension 2 subset from a variety, does not change its class group.

Proposition 1.1.4. ([Har13]) Let $S \subset X$ be a proper closed subset and $V = X - S$, then we have the following.

1. There exists surjective homomorphism $\operatorname{Cl} X \to \operatorname{Cl} V$ defined by

$$A = \sum n_i T_i \mapsto \sum n_i (T_i \cap V),$$

   where we ignore the terms where $T_i \cap V$ is empty.

2. If $\operatorname{codim}(S, X) \geq 2$, then $\operatorname{Cl} X \to \operatorname{Cl} V$ is isomorphism.

3. If S is a irreducible subset of co-dimension 1, then there exists the exact sequence

$$\mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl} V \to 0,$$

   where we define the first map by $1 \mapsto 1 \cdot S$.

Hence we have the following result on a projective curve which calculates the class group of the complement of a curve of degree A in a projective surface.

Example 1.1.3. ([Har13]) If $C \subset \mathbb{P}^n_k$ is irreducible projective curve of degree d, then $\operatorname{Cl}(\mathbb{P}^2 - C) = \mathbb{Z}/d\mathbb{Z}$.

We now discuss about divisor class group on a curve, we define the notion of degree of a divisor on a curve and explain how is invariant under linear equivalence. By a *curve over a field* k we mean an integral separated scheme X of finite type over k of dimension 1. If X is proper over k, it is called complete, and if all the local rings of X are regular, then X is said to be non-singular. In this setup We have the following proposition.

Proposition 1.1.5. ([Har13]) Consider C is non-singular curve over field k with function field K, then the following are equivalent.

1. C is a complete variety.

2. C is a projective variety.

3. $C \cong t(C_k)$, where C_k is non-singular curve and t is the functor from category of varieties to the category of schemes as described in [Har13].

We can classify the image of a morphism from a complete non-singular curve to another curve by the following proposition, which enables us to define degree of a morphism.

Proposition 1.1.6. ([Har13]) If C is a complete nonsingular curve over k, and A is any curve over field k, and let $g : C \to D$ be a morphism. Then the image $g(C)$ is a point, or $g(C) = D$. In the later case, $K(C)$ is a finite field extension of $K(D)$, and the map g is a finite morphism and also D is complete.

Thus we have the following definition.

Definition 1.1.6. ([Har13]) If $g : X \to Y$ be a finite morphism of curves, we define the *degree* of g to be the degree of the field extension $[K(X) : K(Y)]$.

So for a divisor on curve $A = \sum n_i P_i$, where P_i are closed points, we define the *degree* of A to be $\sum n_i$. Thus we are able to define a homomorphism between the divisor class groups of X and Y. Consider X, Y to be nonsingular curves.

Definition 1.1.7. ([Har13]) If $g : X \to Y$ is a morphism then define homomorphism $g^* : \operatorname{Div} Y \to \operatorname{Div} X$. Take $A \in Y$ and let $a \in \mathcal{O}_A$ (an element of $K(Y)$) is local parameter at A with $v_A(a) = 1$, where v_A is the valuation of the DVR $\mathcal{O}_A$. Define

$$g^*(A) = \sum_{g(P)=A} v_P(t) \cdot P.$$

Since f is a finite morphism, the sum is well defined, and we get a divisor on X. If $t' = ut$ (where u is an unit) is another local parameter at A, for any point $P \in X$ with $g(P) = A$, u is an unit in $\mathcal{O}_P$, hence $v_P(a) = v_P(t')$. So we can conclude that, g^*A is independent of the choice of local parameter. By linearity we extend it to a divisor on Y, and it is easy to check that it is preserves linear equivalence. Hence g^* induces a homomorphism $g^* : \operatorname{Cl} Y \to \operatorname{Cl} X$.

We can define the degree of a pullback of a divisor under a finite morphism using the following proposition.

Proposition 1.1.7. ([Har13]) Consider $g : X \to Y$, a finite morphism, then for each divisor A on Y, we have $\deg g^*A = \deg g \cdot \deg A$.

We conclude that degree of a principal divisor on complete nonsingular curve C is zero. Let $g \in K(C)^*$, if $g \in k$, then $(g) = 0$. If $g \notin k$, then the inclusion of fields $k(g) \subseteq K(C)$ induces the finite morphism $\psi : C \to \mathbb{P}^1$, which is an isomorphism and finite. Now $(g) = \psi^*(\{0\} - \{\infty\})$. Since $\{0\} - \{\infty\}$ is a divisor of degree 0 on $\mathbb{P}^1$, we can conclude that (g) has degree zero. Hence the degree function induces a surjective homomorphism from the class group of X to the integers given by $\deg : \operatorname{Cl} X \to \mathbb{Z}$.

Therefore we have an equivalent condition to determine whether a nonsingular projective curve C is rational, which by definition means birational to the projective line.

Example 1.1.4. ([Har13]) C is rational curve if and only if there exists two distinct points $P_1, P_2 \in C$ with $P_1 \sim P_2$, i.e. they are numerically equivalent as divisors. If C is rational, it has an isomorphism to $\mathbb{P}^1$ and on $\mathbb{P}^1$, every two points are linearly equivalent as proven in chapter 2 of [Har13]. Conversely, let C has two distinct points $P_1 \sim P_2$. Then there is a rational function $g \in K(C)$ with $(g) = P_1 - P_2$. Consider the morphism $\psi : C \to \mathbb{P}^1$ determined by (g) as above. We have $\psi^*(\{0\}) = P_1$, hence ψ is a morphism with degree 1, that is ψ is birational and C is rational.

Now we wish to extend our definition of divisor to divisors on arbitrary schemes. The idea is: a divisor should be locally act like divisor of a rational function. We define the concept of Cartier divisor on a scheme.

Definition 1.1.8. ([Har13]) For each open affine set $U = \operatorname{Spec} A$, let S be collection of elements of A which are not zero divisors, and $K(U)$ is the localization of A by the multiplicative system S. $K(U)$ is called the *total quotient ring* of A. For each open set U, we denote by $S(U)$ the elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_x$ for all $x \in U$. Therefore, the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ forms a presheaf, whose associated sheaf of rings $\mathcal{K}$ is called the *sheaf of total quotient rings* of $\mathcal{O}$. On an arbitrary scheme, this $\mathcal{K}$ generalizes the concept of function field on an integral scheme.

The cartier divisor on X is now defined to be a global section of the following quotient sheaf of units.

Definition 1.1.9. ([Har13]) A *Cartier Divisor* on a scheme X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$.

A Cartier divisor on X can be described as giving an open cover $\{U_i\}$ of X, and for each i, an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that for each i, j, we have $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$. A Cartier divisor is said to be *principal* if it is in the image of the natural map $\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. Similarly as Weil divisors, two Cartier divisors are said to be linearly equivalent if their difference is principal.

Now we establish the relation between Weil and Cartier divisors by the following proposition from [Har13].

Proposition 1.1.8. ([Har13]) Let X be an integral, seperated Noetherian scheme, all of whose local rings are unique factorization domains. Then the group of Weil divisors $\operatorname{Div} X$ is isomorphic to the group of Cartier divisors $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$, also principal Weil divisors corresponds to principal Cartier divisors in this isomorphism.

On a ringed space X, a locally free $\mathcal{O}_X$-module of rank 1 is called an *Invertible sheaf*, or a Line bundle. In the following we show that the isomorphism classes of line bundles forms a group. Firstly, we guarantee the existance of inverse by the following proposition.

Proposition 1.1.9. ([Har13]) If $\mathcal{L}$ and $\mathcal{M}$ are line bundles on a ringed space X, then so is $\mathcal{L} \otimes \mathcal{M}$. If $\mathcal{L}$ is a line bundle then there exists a line bundle $\mathcal{L}^{-1}$ such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Hence we define the group of isomorphism classes of line bundles, the *Picard group* as follows.

Definition 1.1.10. ([Har13]) For a ringed space X, the *Picard group* of $X, \operatorname{Pic} X$ is defined to be group of isomorphism classes of line bundles under $\otimes$.

In what follows, a divisor always corresponds to Cartier divisor unless otherwise stated. We can associate a line bundle to a divisor by the following. We follow the convention of [Laz17].

Definition 1.1.11. ([Laz17]) Consider A is Cartier divisor, the subsheaf $\mathcal{O}_X(A)$ of the sheaf of total quotient rings $\mathcal{K}$ is defined by taking $\mathcal{O}_X(A)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by g_i^{-1} on U_i. This is well-defined since g_i/g_j is invertible on $U_i \cap U_j$ so g_i^{-1} and g_j^{-1} generate the same $\mathcal{O}_X$-module. We say $\mathcal{O}_X(A)$ to be the *sheaf associated to* A. Here $\{U_i\}$ is an affine cover of X.

Now for a scheme X we have the following relations which follows the above definition.

Proposition 1.1.10. ([Laz17])

1. For a divisor A, the map sending $A \mapsto \mathcal{O}_X(A)$ gives an injection between divisors and line bundles.
2. $\mathcal{O}_X(A_1 - A_2) \cong \mathcal{O}_X(A_1) \otimes \mathcal{O}_X(A_2)^{-1}$.
3. $A_1 \sim A_2$ if and only if $\mathcal{O}_X(A_1) \cong \mathcal{O}_X(A_2)$ as line bundles.

So, on X, $A \mapsto \mathcal{O}_X(A)$ is an injective homomorphism of the divisor class group of divisors to $\operatorname{Pic} X$. But this map is not in general surjective. Because there are line bundles which are not isomorphic to any invertible subsheaf of $\mathcal{K}$. Nakai has shown ([Laz17]) that the map is infact an isomorphism when X is projective scheme over a field. It is infact an isomorphism if X is integral scheme. Therefore, if X is noetherian, seperated, integral scheme whose all local rings are UFDs then we have the isomorphism $\operatorname{Cl} X \cong \operatorname{Pic} X$.

We have the classifying theorem for line bundles on a projective space as follows.

Proposition 1.1.11. ([Har13]) Every line bundle on $\mathbb{P}_k^n$ (for some field k) is isomorphic to $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$.

1.1.2 Linear Series

X be a projective variety which is non-singular and defined over algebraically closed field k. For line bundle $\mathcal{L}$ of X (which is in one-one correspondence with the group of divisors, same in this case), global section $\Gamma(X, \mathcal{L})$ form a finite-dimensional k-vector space due to the following theorem.

Theorem 1.1.1. ([Har13]) Let k be a field, and A is a finitely generated k-algebra and X is a projective scheme over A, and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\Gamma(X, \mathcal{F})$ is a finitely generated A-module. In particular, if $A = k$, then $\Gamma(X, \mathcal{F})$ is a finite dimensional k-vector space.

We now show how a line bundle gives rise to a divisor on a scheme X. Let $\mathcal{L}$ be a line bundle on X, and $s \in \Gamma(X, \mathcal{L})$ be a non-zero section of $\mathcal{L}$. We define an effective divisor $A = (s)_0$, the *divisor of zeros of* s as follows. Over an open set $U \subseteq X$ where $\mathcal{L}$ is trivial, let $\psi : \mathcal{L}|_U \to \mathcal{O}_U$ be an isomorphism. Then $\psi(s) \in \Gamma(U, \mathcal{O}_U)$. Thus $\{U, \psi(s)\}$ gives effective Cartier divisor A on X as U varies over a cover of X. Then for a projective variety (non-singular) we have the following.

Proposition 1.1.12. ([Har13]) Let A_0 be a divisor on X and let $\mathcal{L} \cong \mathcal{O}_X(A_0)$ be the corresponding line bundle. Then

1. for each nonzero $s \in \Gamma(X, \mathcal{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to A_0.

2. every effective divisor linearly equivalent to A_0 is $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$.

3. two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor of zeroes if and only if there is a $\lambda \in k^*$ such that $s = \lambda s'$.

The proposition allows us to make the definition of *complete linear system* as the following.

Definition 1.1.12. ([Har13]) A *complete linear system* on non-singular projective scheme or variety is defined to be set of effective divisors linearly equivalent to a given divisor A_0 and is denoted by $|A_0|$.

From the proposition, we get that, $|A_0|$ has an injection with $(\Gamma(X, \mathcal{L}) - \{0\})/k^*$. Hence, $|A_0|$ has a structure of set of closed points of a projective space over the field k.

Let $\mathcal{L}$ is a line bundle on a scheme X, and $W \subseteq H^0(X, \mathcal{L})$ a nonzero subset with finite dimension. Denote $|W| = \mathbb{P}_{\text{sub}}(W)$ the projective space of one dimensional subspaces of W. If X is a complete scheme or variety, $|W|$ is identified with the linear series of divisors of sections of W in the above sense. We take $W = H^0(X, \mathcal{L})$ (which is finite dimensional if X is complete, ([Har13])) - yields a *complete linear series* $|\mathcal{L}|$. Given a divisor A, we write $|A|$ to be the linear series corresponding the line bundle $\mathcal{O}_X(A)$.

Evaluation of sections in V gives morphism $\mathrm{eval}_V : V \otimes_{\mathbf{C}} \mathcal{O}_X \to \mathcal{L}$ of vector bundles on X. Using the vector bundle morphism we define base ideal and base locus of a linear system.

Definition 1.1.13. ([Laz17]) The *base ideal* of $|W|$, denoted as

$$\mathfrak{b}(|W|) = \mathfrak{b}(X, |W|) \subseteq \mathcal{O}_X,$$

is image of the map $W \otimes_{\mathbb{C}} L^* \to \mathcal{O}_X$ determined by eval_W. The *base locus*

$$\mathrm{Bs}(|W|) \subseteq X$$

of $|W|$ is the closed subset of X cut out by base ideal $\mathfrak{b}(|V|)$. To emphasize on scheme structure on $\mathrm{Bs}(|W|)$ determined by the base ideal $\mathfrak{b}(|W|)$ we see $\mathrm{Bs}(|W|)$ as the base scheme of $|W|$. When $W = H^0(X, \mathcal{L})$ or $W = H^0(X, \mathcal{O}_X(A))$ is finite-dimensional, base ideals of the complete linear series are written as $\mathfrak{b}(|\mathcal{L}|)$ and $\mathfrak{b}(|A|)$. Then $\mathrm{Bs}(|W|)$ is set of points where all sections in W vanish and $\mathfrak{b}(|W|)$ is ideal sheaf spanned by those sections.

Example 1.1.5. ([Laz17]) Assume X is projective (or complete), for fixed Cartier divisor A on X, for all $n_1, n_2 \in \mathbb{Z}$, greater than 1, we have inclusion

$$\mathfrak{b}(|n_1 A|) \cdot \mathfrak{b}(|n_2 A|) \subseteq \mathfrak{b}(|(n_1 + n_2)A|)$$

from the natural homomorphism

$$H^0(X, \mathcal{O}_X(n_1 A)) \otimes H^0(X, \mathcal{O}_X(n_2 A)) \to H^0(X, \mathcal{O}_X((n_1 + n_2)A))$$

determined by the multiplication of the sections.

Now we define the notion of globally generated line bundles, which are also known as basepoint-free.

Definition 1.1.14. ([Laz17]) We say that $|W|$ is *free* or *basepoint-free* if its base locus is empty, that is, $\mathfrak{b}(|W|) = \mathcal{O}_X$. Divisor A or a line bundle $\mathcal{L}$ is said to be free if the corresponding complete linear series to it is so. For line bundles, we say that $\mathcal{L}$ is *globally generated* or *generated by its global sections.*

That is, a linear series $|W|$ is said to be free if and only if for each $x \in X$ there exists section $t = t_x \in W$ such that $t(x) \neq 0$.

Now we describe how a linear series gives rise to a morphism of an irreducible variety into a projective space, as described in [Laz17]. If $\dim W \geq 2$ and $Y = \mathrm{Bs}(|W|)$, then the linear series $|W|$ determines a morphism

$$\psi = \psi_{|W|} : X - Y \to \mathbb{P}(V)$$

from complement of the base locus to projective space of one-dimensional subsets of W. So, for given $x \in X$, $\psi(x)$ is a hyperplane in W consisting those sections which vanishes at x. We choose basis $s_0, \ldots, s_r \in V$, which means that ψ is given by homogeneous coordinates by

$$\psi(x) = [t_0(x), \ldots, t_r(x)] \in \mathbb{P}^r.$$

If X is irreducible variety, then we can ignore the base locus and view the map $\psi_{|W|}$ as a rational mapping $\psi : X \dashrightarrow \mathbb{P}(W)$. If $|W|$ is free then the map $\psi_{|W|} : X \to \mathbb{P}(W)$ is a globally defined morphism.

The converse ([Laz17]) is also true that - a morphism to a projective space gives rise to a linear series when $Y = \emptyset$. Suppose given a morphism

$$\psi : X \to \mathbb{P}(W)$$

from X to the projective space of a one-dimensional subset of a vector space W also assume that $\psi(X)$ does not lie on any hyperplane. Then the pullback of the sections via ψ realizes $W = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ as subspace of $H^0(X, \psi^*\mathcal{O}_{\mathbb{P}}(1))$, and $|W|$ is a free linear series on X and ψ is identified with the corresponding morphism $\psi_{|W|}$.

1.1.3 Intersection Theory

In this section we discuss the intersection theory of two divisors or line bundles. As the report proceeds it turns out to be one of the most important aspect of positivity theory. In this regard, we quote the following theorem from [Har13].

Theorem 1.1.2. (Bertini's Theorem). ([Har13]) Let X be a non-singular closed subvariety of $\mathbb{P}^n_k$ for an algebraically closed field k. Then there exists a hyperplane $H \subseteq \mathbb{P}^n_k$, not containing X, and such that the scheme $H \cap X$ is regular at every point. If $\dim X \geq 2$, then $H \cap X$ is connected, hence irreducible and so $H \cap X$ is a non-singular variety. Also, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$, considered as a projective space.

Let X be a surface as in above theorem. We define the intersection number $D_1 \cdot D_2$ for any two divisors D_1, D_2 on X. If D_1 and D_2 are curves on X and if $P \in D_1 \cap D_2$ is a point of intersection, we say D_1 and D_2 meet *transversely* at P if the local equations f_1, f_2 of D_1, D_2 at P generate the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_{P,X}$. Which implies that, D_1, D_2 are non-singular at P since f_1 generates the maximal ideal of P in $\mathcal{O}_{P,D_1} = \mathcal{O}_{P,X}/(f_2)$.

As proven in [Har13], if C, D are non-singular curves meeting transversely at finite number of points $P_1, \ldots, P_r$, then the intersection number $C \cdot D$ should be r.

According to the definition of intersection numbers, it satisfies the following properties.

Theorem 1.1.3. ([Har13]) There is a unique pairing $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ denoted by $D_1 \cdot D_2$ for any two Weil divisors D_1, D_2 such that

1. if D_1, D_2 are non-singular curves which meet transversely, then $D_1 \cdot D_2 = \#(D_1 \cap D_2)$, the number of points in the intersection $D_1 \cap D_2$.
2. the pairing is symmetric: $D_1 \cdot D_2 = D_1 \cdot D_2$.
3. the pairing is additive: $(D_1^1 + D_1^2) \cdot D_2 = D_1^1 \cdot D_2 + D_1^2 \cdot D_2$ and,
4. the pairing depends only on linear equivalence classes of divisoirs: if $D_1^1 \sim D_1^2$ then $D_1^1 \cdot D_2 = D_1^2 \cdot D_2$.

If we have a collection of irreducible curves, and a very ample divisor then we have following lemma from [Har13].

Lemma 1.1.2. ([Har13]) Let $C_1, \ldots, C_r$ be irreducible curves on a surface X, and let A be a very ample divisor (defined later). Then almost all curves A' in the complete linear system $|A|$ are irreducible, non-singular, and meet each of the C_i transversely.

The number of points in the intersection can be found by the following formula.

Lemma 1.1.3. ([Har13]) Consider C_1, an irreducible non-singular curve on X and let C_2 is any curve which meets C_1 transversely, then

$$\#(C_1 \cap C_2) = \deg_{\mathbf{C}}(\mathcal{O}_X(C_2) \otimes \mathcal{O}_{C_1}).$$

Intersection number of the two curves can be determined by taking a sum of local intersection numbers over the points in the intersection.

Proposition 1.1.13. ([Har13]) If C_1, C_2 are curves on X having no common irreducible component, then

$$C_1 \cdot C_2 = \sum_{P \in C_1 \cap C_2} (C_1 \cdot C_2)_P$$

Now we give two examples of calculating intersection numbers explicitly. For curves in a projective surface we see that the intersection number is the product of their degrees.

Example 1.1.6. ([Har13]) Let $X = \mathbb{P}^2$, then the Picard group, $\operatorname{Pic} X \cong \mathbb{Z}$, take class l of a line as a generator. Since any two straight lines are equivalent and any two distinct straight lines meet only at one point, we have $l^2 = 1$. Thus if C_1, C_2 are curves of degrees m_1, m_2 respectively, we have $C_1 \sim m_2 h, C_2 \sim m_1 h$. So, $C_1 \cdot C_2 = m_1 m_2$.

The following is an example of calculating intersection numbers of two lines in a quadratic surface.

Example 1.1.7. ([Har13]) Consider X is a non-singular quadratic surface in $\mathbb{P}^3$. Then $\operatorname{Pic} X = \mathbb{Z} \oplus \mathbb{Z}$, we take generators the lines n_1 of type $(1, 0)$ and n_2 of type $(0, 1)$. Then $n_1^2 = 0, n_2^2 = 0, n_1 \cdot n_2 = 1$ since two straight lines in same family are skew, and straight lines in opposite family meet at one point. So if C_1 is a curve of type (p, q) and C_2 has type (p', q') then $C_1 \cdot C_2 = pq' + p'q$.

Example 1.1.8. ([Har13]) Let $\Omega_{X/k}$ is sheaf of differentials of X/k, and $\omega_X = \bigwedge^2 \Omega_{X/k}$ be the *canonical sheaf*. Any divisor K in the linear equivalence class corresponding to ω_X is called a *canonical divisor*. For example, if $X = \mathbb{P}^2, K = -2h$, so $K^2 = 4$.

The following proposition gives a formula for calculating genus of a non-singular curve V.

Proposition 1.1.14. (Adjunction Formula) ([Har13]). If V has genus g on surface X, and K is canonical divisor on X, then

$$2g - 2 = V \cdot (V + K).$$

We also get the similar formula for calculating genus of a curve in a quadratic surface as the following.

Example 1.1.9. ([Har13]) If V is a curve of type (p, q) on a quadratic surface, then $V + K$ has type $(p - 2, q - 2)$, so,

$$2g - 2 = p(q - 2) + (p - 2)q,$$

so, $g = pq - p - q + 1$.

For any divisor A on surface X, let $l(A)$ be the dimension of the 0-th cohomology of the line bundle corresponding to the divisor A. That is, $l(A) = \dim_k H^0(X, \mathcal{O}_X(A))$, thus we have $l(A) = \dim|A|+1$, where $|A|$ is the complete linear system of A. The *superabundance* $s(A)$ is defined to be the dimension of the first cohomology group $\dim H^1(X, \mathcal{O}_X(A))$. The *arithmetic genus*, p_a of X is defined using the Euler characteristic if the structure sheaf, by $p_a = \chi(\mathcal{O}_X) - 1$. Using these convention, we have the following theorem.

Theorem 1.1.4. (Riemann-Roch) ([Har13]). If A is any divisor on a surface X, then

$$l(A) - s(A) + l(K - A) = \frac{1}{2}A \cdot (A - K) + 1 + p_a.$$

The intersection theory on surface is relatively as compared to that in higher dimension. It can be summarised by an unique bilinear map $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$. In higher dimension, to develop intersection theory we introduce some functorial mappings g_* and g^* associated to a morphism $g : X \to X'$ as in [Har13].

Let $g : X \to X'$ be morphism of varieties and T is a subvariety of X. The pushforward of the morphism is defined as follows. If the dimensions, $\dim g(T) < \dim T$, we set $g_*(T) = 0$. If $\dim g(T) = \dim T$, then the function field $K(T)$ is a finite extension field of $K(g(T))$ and we set

$$g_*(T) = [K(T) : K(g(T))] \cdot \overline{g(T)}.$$

Extending by linearity defines a group homomorphism g_* from the group of cycles on X to the group of cycles on X'.

Let U be any subvariety of X, consider $g : \tilde{U} \to U$ be normalization of U. Then $\tilde{U}$ satisfies the condition $(*)$, so Weil divisors and linear equivalence can be discussed for the normalization. When $D_1 \sim D_2$ are linearly equivalent, we say g_*D_1 and g_*D_2 are *rationally equivalent* as cycles on X. If X is a normal scheme then the notion of rational equivalence coincides with usual linear equivalence of Weil divisors.

Next, we discuss the axioms of intersection theory from [Har13]. For each p, define $A^p(X)$ to be the group of cycles of co-dimension p on X modulo rational equivalence. Denote $A(X)$ to be the graded group $\bigoplus_{p=0}^{n} A^p(X)$, where $n = \dim X$. Now $A^0(X) = \mathbb{Z}$ and $A^p(X) = 0$ for $p > \dim X$. If X is complete, we have a natural group homomorphism given by the *degree* from $A^n(X)$ to $\mathbb{Z}$, defined by $\deg\left(\sum n_i Z_i\right) = \sum n_i$, where Z_i are points.

Intersection theory is a set of axioms on a given class of varieties $\mathfrak{V}$. It consists of a pairing $A^{r_1}(X) \times A^{r_2}(X) \to A^{r_1+r_2}(X)$ for all r_1, r_2 and for all $X \in \mathfrak{V}$ which satisfies following axioms. For $Y \in A^{r_1}(X), Z \in A^{r_2}(X)$, denote the intersection class by $Y \cdot Z$.

For a given morphism $g : X \to X'$ of varieties in $\mathfrak{V}$, we assume $X \times X' \in \mathfrak{V}$, we define a homomorphism, the pushforward, $g^* : A(X') \to A(X)$ as follows. For a sub-variety $V' \subseteq X'$ we define

$$g^*(V') = p_{1*}(\Gamma_g \cdot p_2^{-1}(V')).$$

where p_1 and p_2 are projections into first and second component of $X \times X'$, and Γ_g is graph of g, considered as a cycle on $X \times X'$.

The axioms are as follows as in [Har13].

1. The intersection pairing makes $A(X)$ into a commutative associative graded ring with identity for every $X \in \mathfrak{V}$, and is defined as the *Chow ring* of X.

2. For morphism of varieties $g : X \to X'$ in $\mathfrak{V}$, $g^* : A(X') \to A(X)$ is a homomorphism of graded groups. If $h : X' \to X''$ is another morphism then $g^* \circ h^* = (h \circ g)^*$.

3. For any proper morphism $g : X \to X'$ of varieties in $\mathfrak{V}$, $g_* : A(X) \to A(X')$ is a homomorphism of graded groups, and $h : X' \to X''$ is another morphism then $h_* \circ g_* = (h \circ g)_*$.

4. If $g : X \to X'$ be a proper morphism, if $p \in A(X)$ and $q \in A(X')$, then

   $$g_*(p \cdot g^* q) = g_*(p) \cdot q.$$

   This is called the *Projection formula.*

5. If Z_1, Z_2 are cycles on X, and $\Delta : X \to X \times X$ is diagonal morphism, then

   $$Z_1 \cdot Z_2 = \Delta^*(Z_1 \times Z_2).$$

6. If Z_1, Z_2 are sub-varieties on X intersecting properly (i.e., every irreducible component of $Z_1 \cap Z_2$ has co-dimension $\operatorname{codim} Z_1 + \operatorname{codim} Z_2$), then we can write

   $$Z_1 \cdot Z_2 = \sum i(Z_1, Z_2; W_j) W_j,$$

   where the sum is over irreducible components W_j of $Z_1 \cap Z_2$, and the integer $i(Z_1, Z_2; W_j)$ depends only on a neighborhood of the generic point of W_j on X. Also $i(Z_1, Z_2; W_j)$ is called *local intersection multiplicity* of Z_1 and Z_2 along W_j.

7. If V is a subvariety of X and D is effective cartier divisor which meets V properly, then $V \cdot D$ is cycle associated to Cartier divisor $V \cap D$ on V, defined by restriction of the local equation of D to V.

The following theorem guarantees that an unique intersection theory exists for cycles in non-singular projective varieties.

Theorem 1.1.5. ([Har13]) Consider $\mathfrak{V}$ is class of non-singular quasi-projective varieties over an algebraically closed field k. Then there exists an unique intersection theory for cycles modulo rational equivalence on $X \in \mathfrak{V}$ which satisfies above 7 axioms.

Example 1.1.10. ([Har13]) This example shows that intersection theory can not be expected for singular varieties unlike above. For example, consider an intersection theory on quadratic cone $T : xy = z^2$ in $\mathbb{P}^3$, which is singular as proved in [Har13]. Let U be the ruling $z = x = 0$ and V be $z = y = 0$. Then $2V$ is linearly equivalent to hyperplane section which can be taken as conic C on T which meets U, V transversely in one point. So

$$1 = U \cdot C = U \cdot (2V).$$

Using linearity, we get $U \cdot V = \frac{1}{2} \notin \mathbb{Z}$. Hence the intersection theory does not exist in the quadratic cone.

We now follow the convention of [Laz17] to denote intersection numbers. Let X be a complete irreducible complex variety. For given Cartier divisors $A_1, \dots, A_n \in \operatorname{Div} X$ and a irreducible subvariety $Y \subseteq X$ of dimension n, we denote the intersection number

$$(A_1 \cdot \ldots \cdot A_n \cdot Y) \in \mathbb{Z}.$$

We can also define the intersection number topologically as follows using characteristic classes.

Each of the line bundles $\mathcal{O}_X(A_i)$ has its first Chern class

$$c_1(\mathcal{O}_X(A_i)) \in H^2(X; \mathbb{Z}),$$

The cup product of these gives an element

$$c_1(\mathcal{O}_X(A_1)) \cdot \ldots \cdot c_1(\mathcal{O}_X(A_n)) \in H^{2n}(X; \mathbb{Z}).$$

Denoting by $[W] \in H_{2n}(X; \mathbb{Z})$ the fundamental class of W, cap product finally leads to an integer:

$$(c_1(\mathcal{O}_X(A_1)) \cdot \ldots \cdot c_1(\mathcal{O}_X(A_n))) \cap [V] \in H_0(X; \mathbb{Z}) = \mathbb{Z}$$

For r Cartier divisors on an r-dimensional irreducible projective (or complete) variety X, we have the following properties of the intersection numbers from [Laz17]. It matches with the definition of intersection number of two curves on a surface as defined earlier. Here the intersection number is symmetric and multi-linear likewise the bi-linear case earlier.

1. The integer $(A_1 \cdot \ldots \cdot A_r)$ is symmetric and multi-linear in its arguments.
2. Similarly as previous, it depends only on linear equivalence class of A_i.
3. If $A_1, \ldots, A_r$ are effective divisors which meet transversely at smooth points of X, then intersection number is given by

$$(A_1 \cdot \ldots \cdot A_r) = \#\{A_1 \cap \ldots \cap A_r\}.$$

Given an irreducible subvariety $U \subseteq X$ of dimension m, the intersection number

$$(A_1 \cdot \ldots \cdot A_m \cdot U) \in \mathbb{Z}$$

is defined ([Laz17]) by replacing each A_i with linearly equivalent divisors A'_i whose support don't contain U and intersecting the restrictions of A'_i on U. If A_r is irreducible, reduced and effective then we can compute $(A_1 \cdot \ldots \cdot A_r)$ by taking $U = A_r$.

The intersection number satisfies *projection formula*: if $g : Y \to X$ is a generically finite proper map which is surjective, then

$$(g^* A_1 \cdot \ldots \cdot g^* A_r) = (\deg g) \cdot (A_1 \cdot \ldots \cdot A_r)$$

This matches with proposition 1.1.7 for a single divisor A and a finite morphism.

We now introduce the notion of numerical equivalence likewise linear equivalence.

Definition 1.1.15. ([Laz17]) Two cartier divisors D, and $D' \in \mathrm{Div}(X)$ are *numerically equivalent* if their intersection number is same for all irreducible curve $T \subseteq X$, i.e.

$$(D \cdot T) = (D' \cdot T)$$

and it is written as $D \equiv_{\mathrm{num}} D'$.

This discussion allows us to defined the quotient group of the class group by numerically trivial divisors, i.e. a divisor or a line bundle which is numerically equivalent to zero.

Definition 1.1.16. ([Laz17]) The *Neron-Severi group* of a scheme X is defined to be the group

$$N^1(X) = \operatorname{Div} X / \operatorname{Num} X$$

where $\operatorname{Num} X$ is the subgroup of $\operatorname{Div} X$ consisting of all numerically trivial (numerically equivalent to 0) divisors.

The following proposition gives more information about the group structure of $N^1(X)$.

Proposition 1.1.15. ([Laz17]) The $N^1(X)$ is a free abelian group with finite rank.

Definition 1.1.17. ([Laz17]) The rank of $N^1(X)$ is called the *Picard number* of X, denoted by $\rho(X)$.

The following lemma allows us to talk about intersection number of classes in $N^1(X)$ with a subvariety.

Lemma 1.1.4. ([Laz17]) Consider X is complete scheme, and

$$A_1, \ldots, A_n, A'_1, \ldots, A'_n \in \operatorname{Div} X$$

be n-Cartier divisors on X. If $A_j \equiv_{\text{num}} A'_j$ for each j, then

$$(A_1 \cdot \ldots \cdot A_n \cdot [U]) = (A'_1 \cdot \ldots \cdot A'_n \cdot [U])$$

for all subscheme $U \subseteq X$ of pure dimension n.

Hence we have the following definition of intersection number of a representative of classes in the group $N^1(X)$.

Definition 1.1.18. ([Laz17]) Given the classes $d_1, \ldots, d_k \in N^1(X)$, we denote by $(d_1, \ldots, d_k \cdot [U])$ the intersection number of a representative of the classes in discussion.

1.1.4 Riemann-Roch

We discuss asymptotic forms of the Riemann-Roch theorem here. It gives us an asymptotic formula for Euler characteristic of a line bundle on a irreducible projective variety X of dimension n. Before stating the theorem, we need to make sense of the *cycle* and *rank* of a coherent sheaf $\mathcal{F}$ on X.

Definition 1.1.19. ([Laz17]) The *rank* rank($\mathcal{G}$) of $\mathcal{G}$ is defined to be the length of the stalk of $\mathcal{G}$ at the generic point of X. If X is reduced, then we have

$$\operatorname{rank}(\mathcal{G}) = \dim_{\mathbf{C}(X)} \mathcal{G} \otimes \mathbf{C}(X)$$

where $\mathbf{C}(X)$ is the (constant) sheaf of rational functions on X. If X is reducible (of dimension n), then we define the rank of $\mathcal{G}$ along a n-dimensional irreducible component U of X : $\operatorname{rank}_U(\mathcal{G}) = \operatorname{length}_{\mathcal{O}_u} \mathcal{G}_u$, where $\mathcal{G}_u$ is the stalk of $\mathcal{G}$ at the generic point u of U. The *cycle* of $\mathcal{G}$ is defined to be the n-cycle

$$Z_n(\mathcal{G}) = \sum_U \operatorname{rank}_U(\mathcal{G}) \cdot [U],$$

the sum is taken over all n-dimensional components of X.

Then we have the following theorem.

Theorem 1.1.6. (**Asymptotic Riemann-Roch, I**. [Laz17]) Take A a divisor on X, then the Euler characteristic $\chi(X, \mathcal{O}_X(kA))$ is a polynomial whose degree is not more than n in k with

$$\chi(X, \mathcal{O}_X(kA)) = \frac{(A^n)}{n!} k^n + O(k^{n-1}).$$

More generally, we have for all coherent sheaf $\mathcal{G}$ on X,

$$\chi(X, \mathcal{G} \otimes \mathcal{O}_X(kA)) = \operatorname{rank}(\mathcal{G}) \cdot \frac{(A^n)}{n!} k^n + O(k^{n-1}).$$

This theorem gives us an estimate about the dimension of the i-th sheaf cohomology groups of line bundles over X. As the following corollary states -

Corollary 1.1.1. ([Laz17]) If $H^i(X, \mathcal{G} \otimes \mathcal{O}_X(kA)) = 0$ for $i > 0$ and $k >> 0$ then

$$h^0(X, \mathcal{G} \otimes \mathcal{O}_X(kA)) = \operatorname{rank}(\mathcal{G}) \cdot \frac{(A^n)}{n!} k^n + O(k^{n-1})$$

for large k. More generally, the above holds provided that

$$h^i(X, \mathcal{G} \otimes \mathcal{O}_X(kA)) = O(k^{n-1})$$

for $i > 0$.

If we have a finite etale covering of X, then we can also calculate the Euler characteristic of the pullback of any coherent sheaf on X via the covering map.

Proposition 1.1.16. ([Laz17]) Consider $g : Y \to X$ is a finite etale covering of complete schemes, and $\mathcal{G}$ is a coherent sheaf on X then

$$\chi(Y, g^*\mathcal{G}) = \deg(Y \to X) \cdot \chi(X, \mathcal{G}).$$

Proposition 1.1.17. ([Laz17]) Take A is a divisor with the property that the dimension of the i-th sheaf cohomology, $h^i(X, \mathcal{O}_X(kA)) = O(k^{n-1})$ for $i > 0$. We fix $\alpha \in \mathbb{Q}$, $\alpha > 0$, with

$$0 < \alpha^n < (A^n).$$

Then for $k >> 0$, for all smooth points $x \in X$ there exists a divisor $A = A_x \in |kA|$ with

$$\text{mult}_x(D) \geq k \cdot \alpha.$$

Here $\text{mult}_x(D)$ denotes the usual multiplicity of D at the point x, that is order of vanishing at the point x of local equation for D. This demonstrates a construction of what is known as *singular divisor*.

1.2 The Classical Theory

Suppose we have a divisor A on a projective variety X, we want to make sense of the fact that A is a *positive* divisor. Intuitively we ask if A is a hyperplane section under a projective embedding of X - we then define A is very ample. Since it is technically difficult to work with very ample divisors even on curves, we work with certain positive multiples of A to be very ample - in this case A is said to be ample. In this section we discuss about how on a projective variety, amplitude of a divisor or a line bundle can be characterised cohomologically and numerically.

Now we define formally the ample and very ample line bundles.

Definition 1.2.1. ([Laz17]) Consider X is complete scheme and V is a line bundle on X.

1. V is defined as *very ample* if there is a closed embedding $X \subseteq \mathbb{P}$ of X into a projective space $\mathbb{P}^N$ such that

$$V = \mathcal{O}_X(1) =_{\text{def}} \mathcal{O}_{\mathbb{P}^N}(1)|_X.$$

2. V is *ample* if $V^{\otimes k}$ is very ample for some $k > 0$.

A Cartier divisor A on X is ample or very ample if the corresponding line bundle $\mathcal{O}_X(A)$ is so.

We have a characterisation for the ampleness of line bundles over an irreducible curve X in the following.

Example 1.2.1. ([Laz17]) Consider X, an irreducible curve and V is line bundle, then V is ample if and only if its degree, $\deg(V) > 0$.

1.2.1 Cohomological Properties

The amplitude of a divisor or line bundle can be determined cohomologically by the following theorem due to Cartan, Serre and Grothendieck. It deals with the ampleness of a line bundle and its cohomologies after twisting (the amount of twisting required will be determined later) and taking the tensor product with a coherent sheaf.

Theorem 1.2.1. ([Laz17]) Consider V is a line bundle on complete scheme X, and then the following are equivalent.

1. V is an ample line bundle.

2. **(Serre vanishing).** Given a coherent sheaf $\mathcal{G}$ on X, there exists $k_1 = k_1(\mathcal{G}) \in \mathbb{N}$ with property that

$$H^i(X, \mathcal{G} \otimes V^{\otimes k}) = 0 \quad \text{for every } i > 0, k \geq k_1(\mathcal{G}).$$

3. Given a coherent sheaf $\mathcal{G}$ on X, there exists $k_2 = k_2(\mathcal{F}) \in \mathbb{N}$ such that $\mathcal{G} \otimes V^{\otimes k}$ is generated by global sections for every $k \geq k_2(\mathcal{G})$.

4. There exists $k_3 \in \mathbb{N}$ such that $V^{\otimes k}$ is a very ample line bundle for every $k \geq k_3$.

For non-complete schemes, property 3 is taken as a definition of amplitude.

The following examples give a way to construct more ample and very ample divisors from a given ample divisor.

Example 1.2.2. ([Laz17]) Consider D_1, D_2 are Cartier divisors on the projective scheme X; if D_1 is ample, so is $kD_1 + D_2$ for every $k >> 0$. Also, $kD_1 + D_2$ is a very ample Cartier Divisor if $k >> 0$.

If we have two ample line bundles on two schemes, we can define an ample line bundle on the product scheme by the following example.

Example 1.2.3. ([Laz17]) Consider V_1 and V_2 are ample line bundles on two given projective schemes A and B, then $\mathrm{pr}_1^* V_1 \otimes \mathrm{pr}_2^* V_2$ is ample line bundle on $A \times B$.

Now consider two complete schemes, A and B, and a finite morphism between them. Then, we can construct an ample line bundle on B from the same given on A by the following proposition.

Proposition 1.2.1. ([Laz17]) Let $g : B \to A$ as above, and V is ample line bundle on A. Then g^*V is ample on B. Particularly, if $B \subseteq A$ is sub-scheme, the restriction $V|_B$ of V to B is an ample line bundle.

The ampleness of a globally generated line bundle can be determined by embedding the underlying scheme into a projective space determined by the complete linear system corresponding to the line bundle, as follows.

Corollary 1.2.1. ([Laz17]) Suppose V is globally generated. Let

$$\psi = \psi_{|V|} : A \to \mathbb{P}H^0(A, V)$$

be resulting map to the projective space determined by complete linear system $|V|$. Then V is ample line bundle if and only if ψ is a finite mapping, or equivalently if and only if

$$\int_Z c_1(V) > 0$$

for each irreducible curves $Z \subseteq A$.

The following proposition gives us a method for checking the ampleness of a line bundle on a complete scheme A, without checking on the whole scheme.

Proposition 1.2.2. ([Laz17]) Consider V, line bundle on scheme A, then

1. V is an ample line bundle if and only if V_{red} is ample line bundle on A_{red}.

2. V is an ample line bundle if and only if restricting V to all irreducible component of A is ample.

The following theorem says if we have a proper morphism of schemes $g : A \to B$, we can get a family of ample line bundles on the domain scheme from a given line bundle on the sub-scheme of A which is obtained by preimage of a fixed point in the codomain. The family of line bundles are on restriction of preimages of an open neighborhood around that fixed point. To state precisely, we have the following.

Theorem 1.2.2. ([Laz17]) Consider V a line bundle on A. Given $p \in P$, write

$$A_p = g^{-1}(p), V_p = V|_{A_p}.$$

Assume, V_0 is ample line bundle on A_0 for some point $0 \in P$. Then there exists open neighborhood V of 0 in P such that V_p is ample on A_p for all $p \in V$.

In the cohomological criterion for determining ampleness, we discussed the Serre Vanishing. One useful application of Serre vanishing is the following. We develop the aymptotic Riemann-Roch theorem for ample cartier divisors.

Example 1.2.4. (**Asymptotic Riemann-Roch, II** [Laz17]) Let A be an ample Cartier divisor on an irreducible projective variety X which is n-dimensional. Then

$$h^0(X, \mathcal{O}_X(kA)) = \frac{(A^n)}{n!} \cdot k^n + O(k^{n-1}).$$

More generally, for $\mathcal{G}$, any coherent sheaf on X then the dimension of the 0-th sheaf cohomology group

$$h^0(X, \mathcal{G} \otimes \mathcal{O}_X(kA)) = \text{rank}(\mathcal{G})\frac{(A^n)}{n!} \cdot k^n + O(k^{n-1}).$$

The following gives an upper bound on the dimension of 0-th cohomology group of a line bundle on a irreducible projective variety X of dimension n.

Example 1.2.5. ([Laz17]) If A is any divisor on X, then there exists positive constant $C > 0$ such that the dimension

$$h^0(X, \mathcal{O}_X(kA)) \leq Ck^n \quad \text{for every } k.$$

1.2.2 Numerical Properties

In this section we discuss Numerical properties to determine amplitude of a line bundle or a divisor on a projective scheme X.

Theorem 1.2.3. (**Nakai-Moishezon-Kleiman criterion**. [Laz17]) Consider V, line bundle X. Then V is ample if and only if its $\dim W$ times self intersection is positive, i.e.

$$\int_W c_1(V)^{\dim(W)} > 0$$

for all positive irreducible subvariety $W \subseteq X$ with positive dimension.

It can be shown [Laz17] that ampleness is invariant in an numerical equivalence class. Which enables us to define *ample class* in $N^1(X)$.

Corollary 1.2.2. ([Laz17]) Consider $A, A' \in \text{Div}X$ are two numerically equivalent Cartier divisors on X, then A is ample if and only if A' is.

Definition 1.2.2. ([Laz17]) A numerical equivalence class $d \in N^1(X)$ is *ample* if it is the class of ample line bundle or divisor.

If we know the Picard number, that is the rank of $N^1(X)$ of a projective variety, we can conclude the following.

Example 1.2.6. ([Laz17]) Consider T, projective variety with $\rho(T) = 1$, (Picard number), then all non-zero effective divisors on T are ample divisors.

Now using the Nakai's criterion we can conclude about the ampleness of a line bundle or divisor over a projective scheme using its pullback via a surjective, finite map as the following.

Corollary 1.2.3. ([Laz17]) Consider $g : B \to A$, a surjective, finite map, V a line bundle on A. If g^*V is an ample line bundle on B then V is ample on X.

The Nakai's Criterion is still valid for all complete scheme without assuming that it is projective scheme. The projectivity hypothesis has been used in the proof for writing given divisor A as difference of 2 very ample divisors. However, we can modify this step in case of a complete scheme.

As an example of the above corollary, we have a way to determine whether a line bundle corresponding to a divisor on a smooth projective surface A is basepoint-free. Recall that, a line bundle is called basepoint-free if there are enough sections to give a morphism into a projective space. We use the morphism to a projective space from the surface and use the corollary as follows.

Example 1.2.7. ([Laz17]) Consider $T \subseteq A$ is an irreducible curve with positive self intersection, that is, $(T^2) > 0$. Then the line bundle, $\mathcal{O}_X(kT)$ is basepoint-free for $k >> 0$.

This criterion can also be used to give an estimate of the growth of dimension of the cohomology groups of a coherent sheaf on a projective scheme A, as we vary the number of twists.

Example 1.2.8. ([Laz17]) Consider a divisor E on A which has dimension r. $\mathcal{G}$ a coherent sheaf on A, then for all i we have,

$$h^i(A, \mathcal{G}(kE)) = O(k^r).$$

The higher sheaf cohomology groups might have maximal growth. If A is smooth and $-E$ is ample, then $h^r(A, \mathcal{O}_A(kE)) = h^0(A, \mathcal{O}_A(K_A - kE))$ (where K_A is the canonical divisor) by the Serre duality, and the later group grows in the order of k^r. For an example, consider A is the blowing up $\mathrm{Bl}_x(\mathbb{P}^2)$ of the projective plane $\mathbb{P}^2$ at the point x, with the exceptional divisor as E. Then we have $h^1(A, \mathcal{O}_A(kE)) = \binom{k}{2}$ has a quadratic growth. Thus the growth of the sheaf cohomology groups can be determined.

We can also detect the growth of the sheaf cohomology groups of the pull-backs of a divisor on projective, locally finite type schemes via a surjective, finite map.

Example 1.2.9. ([Laz17]) Consider

$$\tau : A' \to A$$

is such a map and A, A' are schemes as mentioned above with dimension r. E is a divisor on A and $E' = \tau^* E$. Then we have for all $i \geq 0$,

$$h^i(A', \mathcal{O}_{A'}(kE')) = h^i(A, (\tau_* \mathcal{O}_{A'}) \otimes \mathcal{O}_A(kE)) + O(k^{r-1}).$$

For a divisor A on a projective scheme X we have the following estimation.

Theorem 1.2.4. ([Laz17]) Consider A is r-dimensional projective scheme, E a divisor on A with the following property:

$$(E^{\mathrm{dim}W} \cdot W) \geq 0 \quad \text{for every irreducible sub-varieties } W \subseteq A,$$

then the dimension of the i-th sheaf cohomology group,

$$h^i(A, \mathcal{O}_A(kE)) = O(k^{r-1}) \qquad \text{for } i \geq 1.$$

1.3 Theory of the $\mathbb{Q}$-Divisors and the $\mathbb{R}$-Divisors

In the discussion of positivity, The discussion of small perturbations of a given class is very useful. So we formalise the notion of $\mathbb{Q}$- and $\mathbb{R}$-divisors.

1.3.1 Definitions for $\mathbb{Q}$-Divisors

Definition 1.3.1. ([Laz17]) Consider A is a scheme or algebraic variety. A $\mathbb{Q}$-divisor on the scheme A is defined as an element of $\mathbb{Q}$-vector space

$$\mathrm{Div}_{\mathbb{Q}} A =_{\mathrm{def}} \mathrm{Div} A \otimes_{\mathbb{Z}} \mathbb{Q}.$$

A $\mathbb{Q}$-divisor E is *integral* if it is in the image of natural map from $\mathrm{Div}A \to \mathrm{Div}_{\mathbb{Q}} A$, the $\mathbb{Q}$-divisor is said to be *effective* if it can be written of the form $E = \sum c_i A_i$ with $c_i \geq 0$ and A_i is effective.

Now we define the support of a divisor as analogous to support of a function in complex analysis.

Definition 1.3.2. ([Laz17]) Consider $E \in \mathrm{Div}_{\mathbb{Q}}A$, a co-dimension one subset $T \subseteq A$ *supports* E, or is a *support of* E if union of supports of E_i is contained in T.

Now, for a complete scheme or variety A, we have the following generalisations of the definitions from previous discussions

Definition 1.3.3. ([Laz17]) Let A be complete.

1. For a given sub-scheme or sub-variety $T \subseteq A$ of pure dimension n, we define the $\mathbb{Q}$-valued intersection product as
$$\mathrm{Div}_{\mathbb{Q}}A \times \ldots \times \mathrm{Div}_{\mathbb{Q}}A \to \mathbb{Q},$$
$$(A_1, \ldots, A_n) \mapsto (A_1 \cdot \ldots \cdot A_k \cdot [T])$$
via extension of the scalars from the product we had on $\mathrm{Div}A$.

2. The $\mathbb{Q}$-divisors $E_1, E_2 \in \mathrm{Div}_{\mathbb{Q}}(A)$ are *numerically equivalent*, denoted by $E_1 \equiv_{\mathrm{num},\, \mathbb{Q}} E_2$, if
$$(E_1 \cdot Z) = (E_2 \cdot Z)$$
for all curves $Z \subseteq A$. We denote $N^1(A)_{\mathbb{Q}}$ the finite-dimensional Q-vector space consisting of numerical equivalence classes of the $\mathbb{Q}$-divisors.

3. The $\mathbb{Q}$-divisors $E_1, E_2 \in \mathrm{Div}_{\mathbb{Q}}\, A$ are linearly equivalent, and written as $E_1 \equiv_{\mathrm{lin}\, \mathbb{Q}} E_2$ if there is an integer m such that mE_1 and mE_2 are integral and linearly equivalent in the usual sense.

4. Consider $g : B \to A$ is morphism such that image of all associated subvarieties of B meet support of $E \in \mathrm{Div}_{\mathbb{Q}}\, A$ properly, then $g^*E \in \mathrm{Div}_{\mathbb{Q}}\, B$ is defined via extension of the scalars from the pullback on the integral divisors.

5. Consider $g : B \to A$ is morphism of projective schemes or complete varieties, extension of the scalars give functorially an induced homomorphism $g^* : N^1(A)_{\mathbb{Q}} \to N^1(B)_{\mathbb{Q}}$ which is compatible with divisor level pullback as in 4.

Similarly, we have the notion of ampleness of a divisor on $\mathbb{Q}$-divisors as the following.

Definition 1.3.4. ([Laz17]) Given a $\mathbb{Q}$-divisor $E \in \operatorname{Div}_{\mathbb{Q}} A$ is said to be ample if any one of the following equivalent conditions are satisfied.

1. E can be written as the form $E = \sum m_i E_i$ where $m_i \in \mathbb{Q}$ and $m_i > 0$ for each i, and each E_i are ample divisor.

2. There exists $n \in \mathbb{N}$ satisfying $n \cdot E$ is an ample and integral divisor.

3. E satisfies statement of the Nakai's criterion, that is

$$(E^{\dim W} \cdot W) > 0$$

for all irreducible sub-varieties $W \subseteq X$ with $\dim W > 0$.

The following proposition asserts that small perturbation of a ample divisor by any arbitrary divisor still remains ample.

Proposition 1.3.1. ([Laz17]) Consider A is projective variety, D is ample $\mathbb{Q}$-divisor on A and F is arbitrary $\mathbb{Q}$-divisor. Then $D+\alpha F$ is ample $\mathbb{Q}$-divisor for every sufficiently small $0 \leq |\alpha| << 1, \alpha \in \mathbb{Q}$. More generally, if we are given a finite family of $\mathbb{Q}$-divisors $F_1, \ldots, F_n$ on A,

$$D + \alpha_1 F_1 + \ldots + \alpha_n F_r$$

is an ample divisor for every sufficiently small $0 \leq |\alpha_i| << 1, \alpha \in \mathbb{Q}$.

1.3.2 $\mathbb{R}$-Divisors and Their Amplitude

We define similarly the vector space over $\mathbb{R}$ as in [Laz17] by

$$\operatorname{Div}_{\mathbb{R}} A = \operatorname{Div} A \otimes \mathbb{R}$$

of the $\mathbb{R}$-divisors on a scheme A. Effective $\mathbb{R}$-divisors, Pullbacks and supports are likewise defined as before for $\mathbb{Q}$-divisor.

Example 1.3.1. We have the following isomorphism from the group $N^1(X)_{\mathbb{R}}$ to the original $N^1(X)$

$$N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Now we define amplitude for a $\mathbb{R}$-divisor on a complete scheme A as following.

Definition 1.3.5. ([Laz17]) Consider E, a $\mathbb{R}$-divisor on A. It is defined to be ample divisor if it can be written as finite sum

$$E = \sum n_i E_i$$

where $n_i > 0, n_i \in \mathbb{R}$ and E_i are ample Cartier divisors for each i.

Similar to Nakai's criterion we have the following method to check whether a $\mathbb{R}$-divisor is ample.

Proposition 1.3.2. ([Laz17]) Consider A, an ample $\mathbb{R}$-divisor then

$$(A^{\dim W} \cdot W) > 0$$

for each irreducible sub-varieties $W \subseteq A$ with $\dim W > 0$.

As seen previously, we have

Proposition 1.3.3. ([Laz17]) *Amplitude* of $\mathbb{R}$-divisor depends upon the numerical equivalence class only.

As in the case of $\mathbb{Q}$-divisors on a projective variety A, we have the openness property of amplitude for a $\mathbb{R}$-divisor. Which says that small perturbations of ample $\mathbb{R}$-divisors remain ample.

Example 1.3.2. ([Laz17]) Consider D, an ample $\mathbb{R}$-divisor on A. Given a finite family of $\mathbb{R}$-divisors $F_1, \ldots, F_n$, the following $\mathbb{R}$-divisor

$$D + \alpha_1 F_1 + \ldots + \alpha_n F_n$$

is an ample divisor for each suffciently small $0 \leq |\alpha_i| << 1, \alpha \in \mathbb{R}$.

1.4 Nef Divisors and Line Bundles

It has been already seen that for a projective scheme A, a class $d \in N^1(A)_{\mathbb{Q}}$ is an ample class if and only if the Nakai's inqualities

$$\int_W d^{\dim W} > 0 \text{ for every irreducible } W \subseteq A \text{ with } \dim W > 0$$

are satisfied. From this, we can argue that the limits of amples classes should be characterised by the corresponding weak inequalities

$$\int_W d^{\dim W} \geq 0 \quad \text{for every } W \subseteq A.$$

1.4.1 Definition and Formal Properties

We define Nef Line bundles on a complete scheme A, and discuss its properties as follows.

Definition 1.4.1. ([Laz17]) A line bundle V on A is defined as *numerically effective*, or *nef*, if the self intersection

$$\int_T c_1(V) \geq 0$$

for all irreducible curves $T \subseteq A$. Similarly, for a Cartier divisor E on A (with $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ coefficients), we define it to be nef if

$$(E \cdot T) \geq 0$$

for all irreducible curves $T \subseteq A$.

The following lemma is useful for determining nefness of a line bundle.

Proposition 1.4.1. ([Laz17]) Chow's lemma reduces statements about complete schemes or varieties to projective schemes. Consider A, a complete scheme, then there is a projective scheme A' along with surjective morphism $g : A' \to A$ which is isomorphism over dense open subset of the scheme A.

The properties of nefness of a line bundle on a complete variety are straightforward.

Example 1.4.1. ([Laz17]) Consider V, line bundle on complete scheme A.

1. Consider $g : B \to A$, a proper mapping. If the line bundle V is nef then g^*V is nef line bundle on B, Particularly, restricting a nef line bundles to a sub-scheme we get a nef line bundle.

2. For 1, if g is a surjective map and g^*V is a nef line bundle on B, then V is itself nef on A.

3. V is a nef line bundle if and only if V_{red} is a nef line bundle on A_{red}.

4. V is nef line bundle if and only if restricting it to each irreducible component of A remains nef.

In the following two examples we get more methods for determining nefness of a line bundle.

Example 1.4.2. ([Laz17]) Consider A, a complete variety, V a globally generated line bundle on A, then V is nef.

For an irreducible curve T on a projective surface S nefness can be determined by calculating the self-intersection number.

Example 1.4.3. ([Laz17]) If the self intersection number is positive, that is $(T^2) \geq 0$, then we have T is a nef divisor.

1.4.2 Theorem of Kleiman

The fundamental theorem of the nef divisors were given by Kleiman. It gives a sufficient condition for a $\mathbb{R}$-divisor E on a complete scheme A to be nef.

Theorem 1.4.1. ([Laz17]) Consider E, a nef $\mathbb{R}$-divisor on A, then

$$(E^n \cdot W) \geq 0$$

for all irreducible n-dimensional sub-variety $W \subseteq A$. Similarly, in case of a line bundle,

$$\int_W c_1(V)^{\dim W} \geq 0$$

for all nef line bundle V on A.

Like the openness criterion for ampleness, we have similar result for nefness of $\mathbb{R}$-divisors on a projective scheme A. That is small perturbation of a nef divisor by an ample divisor gives an ample divisor. Note that the condition $0 < |\alpha| << 1$ previously, is here replaced by for any $\alpha > 0$ using nef divisors.

Corollary 1.4.1. ([Laz17]) Consider E is nef $\mathbb{R}$-divisor on A. For any given ample $\mathbb{R}$-divisor D on A, we have

$$E + \alpha \cdot D$$

is an ample divisor for all $\alpha > 0$. Conversely, for any two divisors E and D such that $E + \alpha D$ is ample divisor for every sufficiently small $\alpha > 0$, then E is a nef divisor.

The following gives a necessary and sufficient condition for a $\mathbb{R}$-divisor on a projective scheme A, to be ample divisor.

Corollary 1.4.2. ([Laz17]) Consider D an ample $\mathbb{R}$-divisor on A. Given a $\mathbb{R}$-divisor E on A. Then we have E is an ample divisor if and only if there is $\alpha > 0$ such that

$$\frac{(E \cdot T)}{(D \cdot T)} \geq \alpha$$

for all irreducible curves $T \subseteq A$.

Using the above result, we can get the following estimate of intersection numbers of ample divisors on a projective variety A and irreducible curves on it.

Example 1.4.4. ([Laz17]) Consider E_1 and E_2 be ample divisors A, then there exists $m, M \in \mathbb{Q}, m, M > 0$ such that

$$m \cdot (E_1 \cdot T) \leq (E_2 \cdot T) \leq M \cdot (E_1 \cdot T)$$

for all irreducible curves $T \subseteq A$.

Using the following theroem due to Seshadri, we get a necessary and sufficient condition for a divisor on a projective variety A to be ample divisor. This is known as *Seshadri's criterion*.

Theorem 1.4.2. ([Laz17]) Consider E is divisor on A. Then E is an ample divisor if and only if there is $\alpha > 0$ such that

$$\frac{(E \cdot T)}{\operatorname{mult}_t T} \geq \alpha$$

for all points $t \in A$ and all irreducible curves $T \subseteq A$ which passes through t.

For a given proper map which is surjective between two algebraic varieties, for a given line bundle on the domain, we can get a family of nef line bundles if the restriction of the line bundle to a base space which is pre-image of a fixed point under the map.

Proposition 1.4.2. ([Laz17]) Consider $g : A \to B$ is proper surjective morphism, V a line bundle on A. For given $x \in B$, we put

$$A_x = g^{-1}(x), V_x = V|_{A_x}.$$

If V_0 is given to be nef line bundle for some fixed point $0 \in B$, then there exists countable union $I \subset B$ of proper sub-varieties of B which does not contain 0 such that the restriction V_x is a nef line bundle for every $x \in B - I$.

For nef classes on a complete scheme A, their intersection number is non-negative.

Example 1.4.5. ([Laz17]) Consider $d_1, \ldots, d_n \in N^1(X)_{\mathbb{R}}$ be nef classes on A. Then

$$\int_A d_1 \cdot \ldots \cdot d_n \geq 0.$$

Example 1.4.6. ([Laz17]) Consider A, smooth projective surface which has non-negative Kodaira dimension, that is $|nK_A| \neq$ for some $n > 0$, where K_A is the canonical divisor. Then the scheme A is minimal - that is it does not contain any smooth rational curve which has self intersection number (-1) - if and only if canonical divisor K_A is a nef divisor. These surfaces are defined as *Minimal Surfaces*.

Example 1.4.7. ([Laz17]) In the above example 1.4.6, we point out a notion of *minimality* in case of a higher dimensional algebraic variety. Non-singular projective variety A is said to be *minimal* if the canonical divisor K_A is a nef divisor. Generally, a *minimal variety* is a normal projective variety which has only canonical singularities with the canonical divisor K_A to be nef.

1.4.3 Cones

Consider M, a finite-dimensional vector space over $\mathbb{R}$. A *cone* in M is defined by a set $C \subseteq M$, which is stable under the multiplication by positive scalars. In this section we discuss cones in $N^1(A)_{\mathbb{R}}$ for a scheme A.

Definition 1.4.2. ([Laz17]) The *ample cone*

$$\mathrm{Amp}(A) \subset N^1(A)_{\mathbb{R}}$$

of the scheme A is convex cone consisting of every ample $\mathbb{R}$-divisor classes on A. The *nef cone*

$$\mathrm{Nef}(A) \subset N^1(A)_{\mathbb{R}}$$

is convex cone which consists of all the nef $\mathbb{R}$-divisor classes.

By the following theorem due to Kleiman we have a topological relation between Ample and Nef cones of a projective scheme A.

Theorem 1.4.3. ([Laz17])

1. The nef cone $\mathrm{Nef}(A)$ is closure of ample cone, that is

$$\mathrm{Nef}(A) = \overline{\mathrm{Amp}(A)}.$$

2. The ample cone $\mathrm{Amp}(A)$ is interior of nef cone, that is

$$\mathrm{Amp}(A) = \mathrm{int}(\mathrm{Nef}(A)).$$

Now we introduce the following definition of *one-cycles* on a complete variety A for defining the *Cone of curves* or the *Mori cone.*

Definition 1.4.3. ([Laz17]) We denote the $\mathbb{R}$-vector space by $V_1(A)_{\mathbb{R}}$ of the *real one-cycles* on A which consists of all the finite $\mathbb{R}$-linear combinations of irreducible curves on the variety A. Elements $\beta \in V_1(A)_{\mathbb{R}}$ are written as formal sum

$$\beta = \sum b_i \cdot T_i$$

where $b_i \in \mathbb{R}$ and $T_i \subset A$ are irreducible curves for each i. Two given one-cycles $\beta_1, \beta_2 \in V_1(A)_{\mathbb{R}}$ are defined as *numerically equivalent* when

$$(E \cdot \beta_1) = (E \cdot \beta_2)$$

for all $E \in \operatorname{Div}_{\mathbb{R}} A$. Corresponding vector space consisting of numerical equivalence classes of one cycles is denoted by $N_1(A)_{\mathbb{R}}$. Hence we have the pairing

$$N^1(A)_{\mathbb{R}} \times N_1(A)_{\mathbb{R}} \to \mathbb{R}, \quad (\eta, \beta) \mapsto (\eta \cdot \beta) \in \mathbb{R}.$$

Also we have that the vector space $N_1(A)_{\mathbb{R}}$ is finite dimensional vector space over $\mathbb{R}$. We induce the standard Euclidean topology on this real vector space.

Now we are ready to define the Mori cone of curves over a complete variety A.

Definition 1.4.4. ([Laz17]) Cone of curves denoted as

$$\mathrm{NE}(A) \subseteq N_1(A)_{\mathbb{R}}$$

is cone which is spanned by classes of every effective one-cycles on the variety A. Precisely we define,

$$\mathrm{NE}(A) = \left\{ \sum n_i[Z_i] \mid Z_i \subset X \text{ is irreducible curve, } n_i \geq 0 \right\}.$$

The topological closure of this cone

$$\overline{\mathrm{NE}}(A) \subseteq N_1(A)_{\mathbb{R}}$$

is called the *closed cone of curves* on the variety A.

The cone defined above is not always closed, so we need to take its closure in usual topology. An example where $\mathrm{NE}(A)$ is not closed is given later. The following proposition builds a relation between the closed cone of effective curves with the nef cone as follows.

Proposition 1.4.3. ([Laz17]) The closed cone $\overline{\mathrm{NE}}(X)$ is dual to the nef cone, $\mathrm{Nef}(X)$, that is,

$$\overline{\mathrm{NE}}(A) = \{\beta \in N_1(A)_{\mathbb{R}} \mid (\eta \cdot \beta) \geq 0 \text{ for each } \eta \in \mathrm{Nef}(A)\}.$$

For a fixed divisor $E \in \operatorname{Div}_{\mathbb{R}} A$ on a complete scheme A, whic is not trivial numerically. It is denoted by

$$\psi_E : N_1(A)_{\mathbb{R}} \to \mathbb{R}$$

linear functional given by intersection with E, we define:

$$E^{\perp} = \{\beta \in N_1(A)_{\mathbb{R}} \mid (E \cdot \beta) = 0\}$$

$$E_{>0} = \{\beta \in N_1(A)_{\mathbb{R}} \mid (E \cdot \beta) > 0\}.$$

An important use of this Cone of curves is to determine ampleness of a $\mathbb{R}$-divisor on a projective variety. It is done using the following theorem.

Theorem 1.4.4. ([Laz17]) Consider E, a $\mathbb{R}$-divisor on the variety A, then the divisor E is an ample divisor if and only if

$$\overline{\mathrm{NE}}(A) - \{0\} \subseteq E_{>0}.$$

In other words, we choose norm on the real vector space $N_1(A)_{\mathbb{R}}$, denote it by

$$B = \{\beta \in N_1(A)_{\mathbb{R}} \mid \|\beta\| = 1\}$$

which is "unit sphere" of classes in the real vector space $N_1(A)_{\mathbb{R}}$ having length 1. Then the divisor E is an ample divisor if and only if

$$\left(\overline{\mathrm{NE}}(A) \cap S\right) \subseteq \left(E_{>0} \cap B\right).$$

The result is known as the Kleiman's criterion to determine amplitude of a $\mathbb{R}$-divisor.

Now we look at some examples and applications of the theorem.

Example 1.4.8. ([Laz17]) Closed cone of curves $\overline{\mathrm{NE}}(A) \subset N_1(A)_{\mathbb{R}}$ on A doesn't contain any infinite straight line. Equivalently, if $\beta \in N_1(A)_{\mathbb{R}}$ is class with both $\beta, -\beta \in \overline{\mathrm{NE}}(A)$, then $\beta = 0$.

Example 1.4.9. ([Laz17]) Consider M, smooth projective surface then one-cycles are same as divisors, so we have the equality

$$N^1(A)_{\mathbb{R}} = N_1(A)_{\mathbb{R}}.$$

1. We have inclusion of the cones

   $$\mathrm{Nef}(A) \subseteq \overline{\mathrm{NE}}(A),$$

   equality holds if and only if $(T^2) \geq 0$ for each irreducible curves $T \subset A$.

2. Assume $T \subset A$, irreducible curve for which, $(T^2) \leq 0$, then $\overline{\mathrm{NE}}(A)$ is spanned by $[T]$, also the sub-cone

   $$\overline{\mathrm{NE}}(A)_{T \geq 0} =_{\mathrm{def}} T_{\geq 0} \cap \overline{\mathrm{NE}}(A).$$

3. In case 2, $[T]$ lies on boundary of the cone of curves $\overline{\mathrm{NE}}(A)$. Also if, $(T^2) < 0$ then $[T]$ spans extremal ray in that cone.

1.4.4 Fujita's Vanishing Theorem

The theorem by Fujita suggests, the Serre type vanishing theorems could be operated uniformly with respect to some twists by the nef divisors. The following is the statement of the theorem.

Theorem 1.4.5. ([Laz17]) Consider A, which is a complex projective scheme and E is ample divisor on A. For all coherent sheaves $\mathcal{G}$ on A, there exits $n(\mathcal{G}, E) \in \mathbb{Z}$ satisfying,

$$H^i(A, \mathcal{G} \otimes \mathcal{O}_A(nE + N)) = 0 \quad \text{for every } i > 0, n \geq n(\mathcal{G}, E)$$

and a nef divisor N on A.

Fujita showed that using an argument with the Frobenius that the theorem also holds over algebraically closed ground fields with positive characteristic.

The following proposition gives a characteristic of line bundles over a projective scheme A, using a finite type scheme.

Proposition 1.4.4. ([Laz17]) There exists scheme F (which is of finite type) along with line bundle V on $A \times F$ with property that all numerically trivial line bundles V on scheme A arise as restriction

$$V_p = V|_{A_p} \quad \text{for } p \in F,$$

where $A_p = A \times \{p\}$.

Therefore we have the following corollary:

Corollary 1.4.3. ([Laz17]) The line bundle V on A is numerically trivial if and only if there exists $n \in \mathbb{N}$ with the property that $V^{\otimes n}$ is deformation of trivial line bundle.

As previously discussed the growth of higher cohomology groups of ample divisors, we dicuss the same for the nef divisors on a projective scheme A, in the following theorem.

Theorem 1.4.6. ([Laz17]) Consider A has dimension k, E is a nef divisor on scheme A. Then for all coherent sheaves $\mathcal{G}$ on A we have,

$$h^i(A, \mathcal{G}(rE)) = O(r^{k-i}).$$

Hence, we similarly have the asymptotic form of Riemann-Roch theroem for nef divisors on a d-dimensional irreducible projective variety A.

Corollary 1.4.4. ([Laz17]) Consider E is nef divisor on A. Then

$$h^0(A, \mathcal{O}_A(mE)) = \frac{(E^d)}{d!} \cdot m^d + O(m^{d-1}).$$

Generally we have,

$$h^0(A, \mathcal{G} \otimes \mathcal{O}_A(mE)) = \text{rank}(\mathcal{G}) \cdot \frac{(E^d)}{d!} \cdot m^d + O(m^{d-1})$$

for all coherent sheaves $\mathcal{G}$ on A.

1.5 Examples and Complements

In this section we provide some examples of the ample cones and the nef cones on surfaces.

1.5.1 Ruled Surfaces

Consider S, a smooth projective surface with genus g, V is vector bundle over S with rank 2. Denote $P = \mathbb{P}(V)$ with

$$\psi : \mathbb{P}(V) \to S$$

be bundle projection. For simplicity of calculation, consider V is with even degree. After the twist by suitable divisor without loss of generality we can assume that $\deg(V) = 0$.

In this setup the group $N^1(P)_{\mathbb{R}}$ can be generated by two of the classes

$$\eta = c_1(\mathcal{O}_P(1)) \text{ and } t = [T]$$

where T is fibre of ψ. Intersection forms of P are determined by relations

$$(\eta^2) = \deg(V) = 0, (\eta \cdot t) = 1, (t^2) = 0.$$

Particular we have, $((pt + q\eta)^2) = 2pq$. Now representing the class $(pt + q\eta)$ by (p, q) in $t - \eta$ plane, we get that, nef cone $\text{Nef}(P)$ lies inside first quadrant $p, q \geq 0$. Also, fibre T is evidently nef. Hence, non-negative "t-axis" forms a boundary of two, of nef cone. In other words, t lies on boundary of $\overline{\text{NE}}(P)$.

For second ray on boundary of $\text{Nef}(P)$, depends on geometry of the bundle V. The following two cases are possible.

Case I: V is unstable. A rank 2 vector bundle V with degree 0 is said to be unstable if V has line bundle quotient E with negative degree $e = \deg(E) < 0$. If we assume this quotient exists,

$$T = \mathbb{P}(E) \subset \mathbb{P}(V) = P$$

is effective curve in class $pt + \eta$. We have $(T^2) = 2p < 0$, it follows from previous example - ray spanned by $[T]$ bounds the closed cone $\overline{\text{NE}}(P)$. Hence $\text{Nef}(P)$ is bounded by dual ray which is generated by $(-pt + \eta)$.

Case II: U **is semistable.** A vector bundle with degree 0 is called semi-stable if it doesn't have quotients with negative degree. If V is semi-stable vector bundle then all symmetric powers of V, S^nV are same too. Which implies, if D is line bundle with degree d satisfying $H^0(S, S^nV \otimes D) \neq 0$, then $d \geq 0$. Now, consider $T \subset P$ is effective curve. Then T arises as section of $\mathcal{O}_P(n) \otimes \psi^*D$ for $m \in \mathbb{N}$ and line bundle D on S. Now,

$$H^0(P, \mathcal{O}_P(n) \otimes \psi^*D) = H^0(S, S^nU \otimes D),$$

Hence we have $d = \deg(D) \geq 0$. Equivalently, $(pt + q\eta)$ class lies in first quadrant. So, in this case, $\text{Nef}(P) = \overline{\text{NE}}(P)$.

The following examples shows us why we need to take closure of the cone of curves, that is, it is not closed in usual topology. We take an example of a ruled surface from [Laz17].

Example 1.5.1. ([Laz17]) For the second case in the previous example, we determine if "positive η-axis", $\mathbb{R}_+ \cdot \eta$ lies in $\text{NE}(X)$ of the effective cones or in the closure. Equivalently, if there is irreducible curve $T \subset P$ such that $[T] = n\eta$ for $n \geq 1$. Existence of this curve is same as existence of some line bundle D with degree 0 on S with $H^0(S, S^nV \otimes D) \neq 0$, this implies S^nV is a semi-stable vector bundle but is not strictly stable. By the virtue of theorem by Narasimhan and Seshadri, describing stable bundles using unitary representations of fundamental group of the surface, $\pi_1(S)$, Hartshorne has checked when the surface S is with genus $g(S) \geq 2$, there are bundles V with degree 0 on S with the following property:

$$H^0(S, S^nV \otimes D) = 0 \text{ for every } n \geq 1$$

when $\deg(D) \leq 0$. It holds for "sufficiently general" semi-stable vector bundle V. Hence there are no effective curves T on resulting surface $P = \mathbb{P}(V)$ along with the class $[T] = n\eta$, hence positive η-axis is absent in the cone of the effective curves. The above example was given by Mumford.

Next, we give an example of a non-ample line bundle which is positive on every irreducible curves.

Example 1.5.2. ([Laz17]) The above example by Mumford yields example of surface P with line bundle on it V which satisfies $\int_T c_1(V) > 0$ for all irreducible curves $T \subset P$, but V is not ample. Consider W is a vector bundle which satisfies that $H^0(S, S^nW \otimes D) = 0$ for every $n \geq 1$, let $P = \mathbb{P}(W)$

and $V = \mathcal{O}_P(1)$. From this, we can see that it is not sufficient to check the intersections with only curves in the Nakai's criterion. This gives us an example where linear functional ψ_η which is determined by the intersection with η is positive on cone of curves $\mathrm{NE}(P)$ for non-ample vector bundle η. It explains why it is needed to take the closed cone $\overline{\mathrm{NE}}(P)$ in Theorem 1.4.4.

1.5.2 Product of Curves

Consider S, smooth irreducible projective curve over $\mathbb{C}$ with genus $g = g(S)$. Say $A = S \times S$, $p_1, p_2 : A \to E$ are projection maps. We fix the point $Q \in S$. In the group $N^1(A)_{\mathbb{R}}$ consider three classes

$$g_1 = [\{Q\} \times S], g_2 = [S \times \{Q\}], d = [\Delta],$$

where $\Delta \subset A$ is diagonal of A. Provided that the genus $g(S) \geq 1$, these three classes are independent, if the surface S has general moduli then the classes span $N^1(A)_{\mathbb{R}}$. Intersections among the classes are given by:

$$(d \cdot g_1) = (d \cdot g_2) = (g_1 \cdot g_2) = 1,$$

$$((g_1^2)) = ((g_2^2)) = 0,$$
$$(d^2) = 2 - 2g(S).$$

Elliptic curves. Let $g(S) = 1$ then $A = S \times S$ is abelian surface and we have the following lemma.

Lemma 1.5.1. ([Laz17]) All effective curves in the product space A is nef, so

$$\mathrm{Nef}(A) = \overline{\mathrm{NE}}(A).$$

The class $\zeta \in N^1(X)_{\mathbb{R}}$ is a nef class if and only if

$$(\zeta^2) \geq 0, (\zeta \cdot l) \geq 0$$

for an ample class l. Particularly, if

$$\zeta = u \cdot g_1 + v \cdot g_2 + w \cdot d,$$

then the class ζ is a nef class if and only if

$$uv + vw + wu \geq 0$$

$$u + v + w \geq 0.$$

1.5.3 Abelian Varieties

Consider a k-dimensional abelian variety A. M, fixed ample divisor on the variety A. Then the following proposition which gives an equivalent condition of being ample and nef divisor.

Proposition 1.5.1. ([Laz17]) The $\mathbb{R}$-divisor E on A is an ample divisor if and only if

$$(E^i \cdot M^{k-i}) > 0$$

for every $0 \leq i \leq k$, and E is a nef divisor if and only if $(E^k \cdot M^{k-i}) \geq 0$ for every i.

Hence we have the equivalent condition for nef class which is not ample class.

Corollary 1.5.1. [Laz17]0 Consider $d \in N^1(A)_{\mathbb{R}}$ is nef class which is not ample, we have $(d^k) = 0$.

This gives rise to examples in which the nef cone is locally bounded by polynomial hypersurfaces of large degree.

1.5.4 The Cone Theorem

Consider A, smooth complex projective variety, K_A a canonical divisor on A. For any divisor E on A write

$$\overline{\mathrm{NE}}(A)_{E \geq 0} = \overline{\mathrm{NE}}(A) \cap E_{\geq 0},$$

for subset of $\overline{\mathrm{NE}}(A)$ which lies in non-negative half-space determined by divisor E. The following theorem is called *Cone theorem.*

Theorem 1.5.1. ([Laz17]) Consider A is a r-dimensional surface, the canonical divisor K_A is not a nef divisor.

1. There exist countably many rational curves $T_i \subseteq A$, such that

$$0 \leq -(T_i \cdot K_A) \leq r + 1,$$

which along with $\overline{\mathrm{NE}}(A)_{K_A \geq 0}$ generate the closed cone $\overline{\mathrm{NE}}(A)$, that is

$$\overline{\mathrm{NE}}(A) = \overline{\mathrm{NE}}(A)_{K_A \geq 0} + \sum_i \mathbb{R}_+ \cdot [T_i].$$

2. Suppose L is a given ample divisor. For any $\alpha > 0$, there exist finitely many such curves - say $T_1, \ldots, T_k$ such that their classes lie in region $(K_A + \alpha \cdot L)_{\leq 0}$. Hence,

$$\overline{\mathrm{NE}}(A) = \overline{\mathrm{NE}}(A)_{(K_A+\alpha L)\geq 0} + \sum_{j}^{k} \mathbb{R}_+ \cdot [T_i].$$

Example 1.5.3. ([Laz17]) Suppose A is smooth projective variety for which $-K_A$ is ample, where K_A is the canonical divisor. Such variety is known as *Fano Variety* in literature. Then the closed cone $\overline{\mathrm{NE}}(A) \subseteq N^1(A)_{\mathbb{R}}$ is finite rational polytope, which is spanned by classes of rational curves.

Example 1.5.4. ([Laz17]) Consider A, smooth projective d-dimensional variety, E is ample integral divisor on A. Then $K_A + (d+1)E$ is a nef divisor, $K_A + (d+2)E$ is an ample divisor. Generally, for H, any ample divisor with

$$(H \cdot T) \geq d+1 \quad \text{(respectively } (H \cdot T \geq d+2))$$

for all irreducible curves $T \subseteq A$, then $K_A + H$ is a nef (respectively $K_A + H$ is an ample) divisor.

1.6 Amplitude of a Mapping

Now we define the notion of amplitude relative to some mapping and some facts related to it. Let $g : A \to B$ is a proper map of schemes and take coherent sheaf $\mathcal{G}$ on A, then the push-forward $g_*\mathcal{G}$ is coherent sheaf on B, therefore we form B-scheme:

$$\mathbb{P}(\mathcal{G}) =_{\text{def}} \mathrm{Proj}_{\mathcal{O}_B}(\mathrm{Sym}(\mathrm{g}_*\mathcal{G})) \to B$$

its fibre over a fixed point $b \in B$ is projective space of 1-dimensional quotients of fibre $g_*(\mathcal{G}) \otimes \mathbf{C}(t)$. It is analogue of projective space of sections of a sheaf on complete variety, in relative setting. Moreover, there exists natural mapping $g^*g_*\mathcal{G} \to \mathcal{G}$. Its surjectivity is analogue of global generation of the sheaf $\mathcal{G}$ in absolute situation.

These motivates to define *Amplitude for scheme map.*

Definition 1.6.1. ([Laz17]) Consider a proper map of schemes $g : A \to B$, and V, a line bundle on A.

1. v is a very ample line bundle relative to the map g , or said to be g-very ample, when canonical map

$$\sigma : g^*g_*V \to V$$

is surjective map and defines embedding

$$\begin{array}{ccc} A & \xrightarrow{i} & \mathbb{P}(g_*V) \\ & \searrow^{g} & \downarrow \\ & & B \end{array}$$

of schemes over B.

2. V is an ample bundle relative to g , or g-ample, when $V^{\otimes n}$ is g-very ample for $n \in \mathbb{N}$.

Cartier divisor E on A is called very ample for the map g if corresponding line bundle is g-very ample, and g-amplitude for Cartier $\mathbb{Q}$-divisors are defined by clearing the denominators.

Example 1.6.1. ([Laz17]) Consider W vector bundle on scheme B, then Serre line bundle $\mathcal{O}_{\mathbb{P}(W)}(1)$ on $S = \mathbb{P}(W)$ is an ample bundle for natural map $p : S \to B$.

All properties in definition are local properties at the scheme B. Equivalently, all conditions hold for $g : A \to B$ if and if only they hold for all restrictions $g_i : A_i = g^{-1}(W_i) \to W_i$ of the map g to inverse images of open sets of open covering $\{W_i\}$ of B.

The first condition of definition is same as existence of coherent sheaf $\mathcal{G}$ on B with embedding $j : A \hookrightarrow \mathbb{P}(\mathcal{G})$ over B with $V = \mathcal{O}_{\mathbb{P}(W)}(1)|_A$. Such embeddings give rise to surjection $\zeta : g^*\mathcal{G} \to V$. It determines homomorphism $\beta : \mathcal{G} \to g_*V$ with factorization

$$\begin{array}{ccc} g^*\mathcal{G} & \xrightarrow{\zeta} & V \\ & \searrow^{g^*\beta} & \uparrow \sigma \\ & & g^*g_*V \end{array}$$

of ζ. Therefore we get that σ is a surjective map. Also, given embedding j is seen as composition of morphism $i : A \to \mathbb{P}(g_*V)$ which arises from σ with linear projection

$$(\mathbb{P}(g_*V) - \mathbb{P}(\operatorname{coker} \zeta)) \to \mathbb{P}(\mathcal{G})$$

of B-schemes.

From this we conclude: B is *affine* - such that g_*V is globally generated line bundle - then V is a very ample line bundle for g if and only if there exists embedding $t : A \hookrightarrow \mathbb{P}^N \times B$ such that $V = t^*\mathcal{O}_{\mathbb{P}^N \times B}(1)$. This property can also be taken as an alternative definition for a very ample line bundle relative to some mapping. Now we have the following equivalent statements for amplitude with respect to a map.

Theorem 1.6.1. ([Laz17]) Consider a proper morphism $g : A \to B$ of schemes, V a line bundle on A. Then following statements are equivalent:

1. V is an ample line bundle for the map g.

2. For all coherent sheaves $\mathcal{G}$ on A, there is $n_1 = n_1(\mathcal{G}) \in \mathbb{N}$ with

$$R^i g_*(\mathcal{G} \otimes V^{\otimes n}) = 0 \quad \text{for every } i > 0, n \geq n_1(\mathcal{G}).$$

3. For all coherent sheaves $\mathcal{G}$ on A, there is $m_2 = m_2(\mathcal{G}) \in \mathbb{N}$ with the canonical mapping

$$g^* g_*(\mathcal{G} \otimes V^{\otimes n}) \to \mathcal{G} \otimes V^{\otimes n}$$

is a surjective map when $n \geq n_2$.

4. There exists $n_3 \in \mathbb{N}$ with $V^{\otimes n}$ is g-very ample bundle for all $n \geq n_3$.

The following theorem gives us a way to determine ampleness fibre-wise.

Theorem 1.6.2. ([Laz17]) Consider the proper map $g : A \to B$ of schemes, V line bundle on A, for $b \in B$, say

$$A_b = g^{-1}(b), V_b = V|_{A_b}$$

Then V is an ample bundle for g if and only if V_b is an ample bundle on A_b for each $b \in B$.

Similarly, we have in following the Nakai's criterion for determining amplitude for a scheme map.

Corollary 1.6.1. ([Laz17]) $\mathbb{Q}$-divisor E on A is an ample divisor with respect to map g if and only if $A^{\dim W} \cdot W > 0$ for all irreducible sub-varieties $W \subset A$ with $\dim W > 0$, which maps to a fixed point in B.

Following result gives a summary of connection between the relative and the global amplitude.

Proposition 1.6.1. ([Laz17]) Let $g : A \to B$ is a map of projective schemes, V line bundle on A, suppose E is ample line bundle on B. Then V is g-ample if and only if $V \otimes g^*(E^{\otimes n})$ is ample on A for every $n >> 0$.

Similar to amplitude we define nefness with respect to a map between schemes.

Definition 1.6.2. ([Laz17]) Take $g : A \to B$, a proper morphism, V line bundle on A is a nef bundle with respect to the map g if restriction $V_b = V|_{A_b}$ of V to all fibres are nef line bundles, or in other words, if $(c_1(V) \cdot T) \geq 0$ for all curve $T \subset A$ which maps to a fixed point in B.

However if a divisor is nef with respect to some map, then it need not be a nef divisor. The following is an example for the same.

Example 1.6.2. ([Laz17]) Assume $A = C \times C$ is product of an elliptic curve with itself and $g : A \to C$ is projection to first co-ordinate. Say $F = C \times \{\text{point}\}$ and $\Delta \subset A$ is diagonal of the product space. Then $F - \Delta$ is a g-nef divisor. But, for all divisors H on C, the new divisor $F - \Delta + g^*H$ has a self-intersection number -2, hence it cannot be a nef divisor.

Everything discussed in the section is applicable for varieties which are defined over arbitrary algebraically closed field with any characteristic.

1.7 Castelnuovo–Mumford Regularity

Theorems by Cartan–Serre–Grothendieck imply - all cohomological properties associated to the coherent sheaf $\mathcal{G}$ on projective space $\mathbb{P}$ disappear after a twist by sufficiently high multiple of hyperplane bundle. The Castelnuovo–Mumford regularity gives quantitative measure that how much twist is required for this.

1.7.1 Definitions, its Formal Properties, and its Variants

Suppose W is a vector space over $\mathbb{C}$ with dimension $d + 1$, and we denote $\mathbb{P} = \mathbb{P}(W)$ which is corresponding d-dimensional projective space. Then the Castelnuovo-Mumford regularity of a coherent sheaf $\mathcal{G}$ on $\mathbb{P}$, is defined as follows.

Definition 1.7.1. ([Laz17]) Define the sheaf $\mathcal{G}$ is *n-regular* ($n \in \mathbb{Z}$) in sense of *Castelnuovo–Mumford* when

$$H^i(\mathbb{P}, \mathcal{G}(n - i)) = 0 \quad \text{for each } i > 0.$$

Following are few examples of sheaves of above kind.

Example 1.7.1. ([Laz17])

1. The bundle $\mathcal{O}_{\mathbb{P}}(l)$ is $(-l)$-regular.

2. Ideal sheaf $\mathcal{J}_S \subseteq \mathcal{O}_{\mathbb{P}}$ of linear subspace $S \subseteq \mathbb{P}$ is 1-regular.

3. Suppose $A \subseteq \mathbb{P}$ is hypersurface with degree a, the structure sheaf of A, that is $\mathcal{O}_A$, viewed as extension by zero as coherent sheaf on the projective space $\mathbb{P}$ is $(a-1)$-regular.

The following theroem is due to Mumford which determines regularity of a sheaf on $\mathbb{P}$.

Theorem 1.7.1. ([Laz17]) Suppose $\mathcal{G}$ is n-regular sheaf. Then for all $i \geq 0$:

1. The line bundle $\mathcal{G}(n+i)$ is generated by global sections of it.
2. Natural maps

$$H^0(\mathbb{P}, \mathcal{G}(n)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(i)) \to H^0(\mathbb{P}, \mathcal{G}(n+i))$$

   are surjective maps.

3. The sheaf $\mathcal{G}$ is $(n+i)$-regular.

Hence we define regularity with respect to a globally generated ample line bundle.

Definition 1.7.2. ([Laz17]) V, ample line bundle on a projective variety A is generated by its global sections. $\mathcal{G}$ is a coherent sheaf on A. It is n-regular with respect to V if the sheaf cohomology group

$$H^i(A, \mathcal{G} \otimes V^{\otimes(n-i)}) = 0 \text{ for } i > 0.$$

Then we have the modified theorem by Mumford as follows.

Theorem 1.7.2. ([Laz17]) If $\mathcal{G}$ is n-regular sheaf on A relative to V. Then for all $l \geq 0$:

1. $\mathcal{G} \otimes V^{\otimes(n+l)}$ can be generated by the global sections.
2. Natural map

$$H^0(A, \mathcal{G} \otimes V^{\otimes n}) \otimes H^0(A, V^{\otimes l}) \to H^0(A, \mathcal{G} \otimes V^{\otimes(n+l)})$$

   is a surjective maps.

3. The sheaf $\mathcal{G}$ is $(n+l)$-regular relative to B.

Example 1.7.2. ([Laz17]) Extension of two n-regular sheaves on projective space $\mathbb{P}$ is still n-regular.

Example 1.7.3. ([Laz17]) Take a resolution of the coherent sheaf $\mathcal{G}$ on $\mathbb{P}$ by long exact sequence

$$\ldots \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0 \to \mathcal{G} \to 0$$

of coherent sheaves on the projective space $\mathbb{P}$. When $\mathcal{G}_j$'s are $(n+j)$-regular for all $j \geq 0$, then $\mathcal{G}$ is n-regular. Additionally, the map $H^0(\mathbb{P}, \mathcal{G}_0(n)) \to H^0(\mathbb{P}, \mathcal{G}(n))$ is a surjective map.

Partial converse of the above example gives an useful characterization of the n-regularity. Now the following proposition gives an equivalent criterion for n-regularity of a coherent sheaf $\mathcal{G}$ on $\mathbb{P}$.

Proposition 1.7.1. ([Laz17]) $\mathcal{G}$ is n-regular if and only if $\mathcal{G}$ can be resolved by long exact sequence

$$\ldots \to \oplus\ \mathcal{O}_{\mathbb{P}}(-n-2) \to \oplus\ \mathcal{O}_{\mathbb{P}}(-n-1) \to \oplus\ \mathcal{O}_{\mathbb{P}}(-n) \to \mathcal{G} \to 0$$

the terms are the direct sums of indicated line bundles.

Now we are able to define the *Regularity of a coherent sheaf* $\mathcal{G}$ *on projective space.*

Definition 1.7.3. ([Laz17]) Castelnuovo–Mumford regularity, denoted by $\operatorname{reg}(\mathcal{G})$ of $\mathcal{G}$ is defined to be least $n \in \mathbb{Z}$ such that $\mathcal{G}$ is n-regular.

The following example illustrates a notion of regularity relative to a vector bundle V on an irreducible projective k-dimensional variety A.

Example 1.7.4. (**Regularity with respect to a Vector Bundle.**) Suppose V is vector bundle on A with property - for each points $a \in A$, there exists section of V with zero locus being finite set which contains a. (For example, take: $V = W \oplus \cdots \oplus W$ (k times), where W is globally generated ample line bundle on A.) If $\mathcal{G}$ is coherent sheaf on A with

$$H^i(A, \wedge^i V^* \otimes \mathcal{G}) = 0 \text{ for } i > 0,$$

then the sheaf $\mathcal{G}$ is globally generated.

The regularity property helps us to determine amplitude of a line bundle in the following way.

Proposition 1.7.2. ([Laz17]) Suppose V is line bundle on A which is 0-regular relative to globally generated ample line bundle W. Then product bundle $V \otimes W$ is a very ample line bundle.

Now, we define the regularity relative to a proper surjective map, g of schemes A to B.

Example 1.7.5. (**Regularity with respect to a mapping.**) For given fixed line bundle V on A which satisfies:

1. the bundle V is an ample bundle for the map g,
2. Canonical map $g^*g_*V \to V$ is a surjective map.

For given fixed coherent sheaf $\mathcal{G}$ on A, define $\mathcal{G}$ is n-regular relative to V and g when

$$R^i g_*(\mathcal{G} \otimes V^{\otimes(n-i)}) = 0 \quad \text{for } i > 0.$$

When $\mathcal{G}$ is n-regular relative to V and g . For each $l \geq 0$:

1. Homomorphism
$$g^* g_*(\mathcal{G} \otimes V^{\otimes(n+l)}) \to \mathcal{G} \otimes V^{\otimes(n+l)}$$
is a surjective homomorphism.
2. Also the following map
$$g_*(\mathcal{G} \otimes V^{\otimes n}) \otimes g_*(V^{\otimes l}) \to g_*(\mathcal{G} \otimes V^{\otimes(n+l)})$$
is a surjective map.
3. $\mathcal{G}$ is $(n+l)$-regular relative to V and g .

In the following example, we introduce the notion of regularity on some projective bundle.

Example 1.7.6. ([Laz17]) Suppose V is vector bundle on a scheme or variety A, having projectivization $\sigma : \mathbb{P}(V) \to A$. Coherent sheaf $\mathcal{G}$ on $\mathbb{P}(V)$ is n-regular relative to the map σ when

$$R^i \sigma_*(\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}(V)}(n-i)) = 0$$

for $i > 0$. When the condition holds, we have:

1. $\sigma^*\sigma_*(\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}(V)}(n))$ maps surjectively to $\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}(V)}(n))$.
2. Map
$$\sigma_*(\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}(V)}(n)) \otimes \sigma_*\mathcal{O}_{\mathbb{P}(V)}(l)) \to \sigma_*(\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}(V)}(n+l))$$
is a surjective map, for $l \geq 0$.
3. The sheaf $\mathcal{G}$ is $(n+1)$-regular for the map σ.

2 Linear Series

In the final chapter of this report, we illustrate linear series on projective variety P, using theory developed in previous sections to analyze complete linear series $|nA|$ associated to divisor A on P that need not be an ample divisor or a nef divisor. We first focus on the asymptotic theory in the following.

2.1 Asymptotic Theory

Consider V, a line bundle on projective variety P. Here we discuss asymptotic behavior of linear series $|V^{\otimes n}|$ as $n \to \infty$. Here P will always denote irreducible projective variety over $\mathbb{C}$, unless otherwise specified. To start with we give some of the basic definitions.

2.1.1 Basic Definitions

We define the exponent and Semigroup for a line bundle V on P.

Definition 2.1.1. ([Laz17]) By definition, *Semigroup* of the bundle V has those non-negative powers of V which have non-zero section, precisely:

$$\mathbf{N}(V) = \mathbf{N}(P, V) = \{n \geq 0 \mid H^0(P, V^{\otimes n}) \neq 0\}.$$

Consider $\mathbf{N}(V) \neq (0)$, then all sufficiently large elements in $\mathbf{N}(P, V)$ are the multiples of largest $e = e(V) \geq 1, e \in \mathbb{N}$, defined as *exponent* of the bundle V, each sufficiently large multiples of exponent, $e(V)$ comes in $\mathbf{N}(P, V)$. Semigroup $\mathbf{N}(P, E)$, exponent $e = e(E)$ of Cartier divisor E are constructed similarly after passing it to the line bundle $V = \mathcal{O}_P(E)$.

The following gives an example in a particular case for the above defined semi-group and exponents.

Example 2.1.1. ([Laz17]) Consider A, $d \geq 1$ dimensional projective variety with non-trivial torsion line bundle ζ, having order f in the picard group $\mathrm{Pic}(A)$. S, another l- dimensional projective variety, V, very ample line bundle on S. Say

$$W = S \times A, Z = p_1^*(V) \otimes p_2^*(\zeta).$$

Then we have $e(Z) = f$, $\mathbf{N}(V) = \mathbf{N}f$. Here, $V^{\otimes n}$ is a globally generated bundle if $n \in \mathbf{N}(V)$, again by definition $H^0(P, V^{\otimes n}) = 0$ otherwise.

For given element $n \in \mathbf{N}(P, V)$, consider rational map

$$\psi_n = \psi_{|V^{\otimes n}|} : P \dashrightarrow \mathbb{P}H^0(P, V^{\otimes n})$$

which is associated to complete linear series $|V^{\otimes n}|$. Denote

$$Z_n = \psi_n(P) \subseteq \mathbb{P}H^0(P, V^{\otimes n})$$

is closure of image of graph of the map ψ_n. Our aim is to understand asymptotic birational nature of such maps when $n \to \infty$.

In the following, we define some birational invariants of a variety or scheme.

Definition 2.1.2. ([Laz17]) Consider P is a normal scheme. *Iitaka dimension* of line bundle V is defined as

$$\kappa(V) = \kappa(P, V) = \max_{n \in \mathbf{N}(V)} \{\dim \psi_n(P)\},$$

given that $\mathbf{N}(P, V) \neq 0$. When $H^0(P, V^{\otimes n}) = 0$ for each $n > 0$, we say $\kappa(P, V) = -\infty$. When P is a non-normal scheme, we pass it to normalization $\mu : P' \to P$, set

$$\kappa(P, V) = \kappa(P', \mu^* V).$$

For given Cartier divisor E, take $\kappa(P, E) = \kappa(P, \mathcal{O}_P(E))$.

Hence, either $\kappa(P, V) = -\infty$ or $0 \leq \kappa(P, V) \leq \dim P$.

Example 2.1.2. ([Laz17]) Consider canonical divisor K_P on P which is smooth projective variety. $\kappa(P) = \kappa(P, K_P)$ is defined as *Kodaira dimension* of variety P: which is most basic birational invariant of varieties. Kodaira dimension of singular variety is defined as Kodaira dimension of any smooth model.

In the following example we introduce Kodaira dimension in case of singular varieties.

Example 2.1.3. ([Laz17]) When the variety P is a smooth variety, $\mathcal{O}_P(K_P) = \omega_P$ is dualizing line bundle on P. But when P is singular, it may happen dualizing sheaf ω_P exists as line bundle on P, but $\kappa(P, \omega_P) > \kappa(P)$. (It happens for example when $P \subseteq \mathbb{P}^3$ is cone over smooth plane curve with large degree.) Impact of the singularities on the Kodaira dimension, other birational invariants play important role in minimal model program.

In the following example we see that Iitaka dimension of line bundles does not behave consistently after restriction to some sub-variety.

Example 2.1.4. ([Laz17]) Consider $S = \text{Bl}_Q(\mathbb{P}^2)$ is blowing up projective plane $\mathbb{P}^2$ at point Q, say F and J are exceptional divisor and pullback of hyperplane divisor, respectively. Now, $\mathcal{O}_S(J)$, $\mathcal{O}_S(J+F)$ are with maximal Iitaka dimension, but after restrictions of them to F Iitaka dimensions goes to $0, -\infty$ respectively. Thus we see, Iitaka dimension decreases after restriction. In contrary, say $T = \mathbb{P}^1 \times \mathbb{P}^1$, $V = p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, so that $\kappa(T, V) = -\infty$. If $Z = \{\text{point}\} \times \mathbb{P}^1$, the Iitaka dimension $\kappa(Z, V|_Z) = 1$, hence Iitaka dimension is increased after restriction.

We now discuss the nature of Iitaka dimension of a line bundle after deformation. We see that it is not a invariant after deformation.

Example 2.1.5. ([Laz17]) If line bundle $V \in \text{Pic}^0(P)$, then $\kappa(P, V) = 0$ when V is torsion or trivial, $\kappa(P, V) = -\infty$ in other cases. By taking product like the Example 2.1.1 we get similar examples where κ is arbitrarily large.

We want to see when the induced homomorphism from a morphism between to irreducible projective varieties to their Picard group is a injective homomorphism. For this we define the notion of *Algebraic fibre space*.

Definition 2.1.3. ([Laz17]) We define *algebraic fibre space* to be surjective projective map of irreducible, reduced varieties $g : A \to B$ with the property that $g_*\mathcal{O}_A = \mathcal{O}_B$.

The following example illustrates the relation between the fibre spaces and the function fields of the corresponding varieties.

Example 2.1.6. ([Laz17]) Suppose A, B are normal varieties. $g : A \to B$ is projective surjective morphism,

$$\mathbf{C}(B) \subseteq \mathbf{C}(A)$$

are corresponding finitely generated extension of function fields. Now, g is fibre space if and only if $\mathbf{C}(B)$ is an algebraically closed in field $\mathbf{C}(A)$. Therefore algebraic fibre spaces make sense in birational category: a dominant rational map $A \dashrightarrow B$ induces algebraically closed extension of function fields.

By the following lemma, we get an way to determine the Iitaka dimension of pull back of a line bundle for algebraic fibre space.

Lemma 2.1.1. ([Laz17]) Consider $g : A \to B$, algebraic fibre space, V is line bundle on B. We have

$$H^0(A, g^*V^{\otimes n}) = H^0(B, V^{\otimes n}) \text{ for each } n \geq 0.$$

Particularly, $\kappa(B, V) = \kappa(A, g^*V)$.

Now we are ready to discuss the injectivity of the picard groups of projective irreducible varieties induced from algebraic fibre space.

Example 2.1.7. ([Laz17]) Say $g : A \to B$ be algebraic fibre space. Then induced homomorphism

$$g^* : \operatorname{Pic} B \to \operatorname{Pic} A$$

is an injective homomorphism. Say V is line bundle on V with $g^*V \cong \mathcal{O}_A$. We have $H^0(B, V) = H^0(A, g^*V) \neq 0$ by previous lemma, and similarly $H^0(B, V^*) = H^0(A, g^*V^*) \neq 0$. Therefore $V = \mathcal{O}_B$.

The algebraic fibre space carry forwards Normality of varieties as in the following example.

Example 2.1.8. ([Laz17]) Let $g : A \to B$ algebraic fibre space. If A is a normal variety, then B is also normal. ($\mu : B' \to B$ is normalization of B. So g factors through μ, as g is fibre space this makes μ an isomorphism.)

We define the *Section ring* associated to given line bundle V on prpjective variety A.

Definition 2.1.4. ([Laz17]) The graded ring or the section ring associated to the line bundle V is graded $\mathbb{C}$-algebra

$$R(V) = R(A, V) = \bigoplus_{n \geq 0} H^0(A, V^{\otimes n}).$$

For a projective space the section ring associated to the line bundle $\mathcal{O}_{\mathbb{P}^m}(1)$ is as follows.

Example 2.1.9. ([Laz17]) Say $V = \mathcal{O}_{\mathbb{P}^m}(1)$, then we have $R(V) = \mathbb{C}[Z_0, \ldots, Z_m]$ is homogeneous coordinate ring of the projective space $\mathbb{P}^m$.

We now define finitely generated line bundle or divisors using the section ring.

Definition 2.1.5. ([Laz17]) V is a line bundle on projective variety A is said to be finitely generated when the section ring $R(A, V)$ is finitely generated as $\mathbb{C}$-algebra. Divisor E is said to be finitely generated when corresponding line bundle $\mathcal{O}_A(E)$ is same.

Definition 2.1.6. ([Laz17]) *Stable base locus* of a divisor E is algebraic set

$$\mathbf{B}(E) = \bigcap_{n \geq 1} \operatorname{Bs}(|nE|).$$

Stable base locus is defined only as closed subset of A: we do not view right side as intersection of schemes.

Proposition 2.1.1. Stable base locus $\mathbf{B}(E)$ as defined above, is unique minimal element of family of the algebraic sets $\{\mathrm{Bs}(|nE|)\}_{n\geq 1}$. There is $n_0 \in \mathbb{Z}$ with

$$\mathbf{B}(E) = \mathrm{Bs}(|ln_0E|) \quad \text{for each } l >> 0.$$

2.1.2 Semiample Line Bundles

Now we analyze asymptotic behavior of the maps ψ_m determined by $|V^{\otimes m}|$ for a large value of $m \in \mathbf{N}(A,V)$. To start with we define Semi-ample bundles or divisors.

Definition 2.1.7. ([Laz17]) V, a line bundle on complete scheme is defined to be semi-ample when $V^{\otimes n}$ is globally generated for a $n > 0$. Divisor E is said to be semi-ample if corresponding line bundle to it is also so.

For given semi-ample line bundle V, say $M(A,V) \subseteq \mathbf{N}(A,V)$ is sub-semigroup

$$M(A,V) = \{n \in \mathbb{N} \mid V^{\otimes n} \text{ is free}\}.$$

Denote by $h = h(V)$ the "exponent" of the sub-semigroup $M(A,V)$, that is largest positive integer for which each elements in $M(A,V)$ is multiple of the exponent h.

Given an element $n \in M(A,V)$, we write $Z_n = \psi_n(A)$ for image of morphism

$$\psi_n = \psi_{|V^{\otimes n}|} : A \to Z_n \subseteq \mathbb{P}H^0(A, V^{\otimes n})$$

constructed by complete linear series $|V^{\otimes n}|$. By an abuse of the notation, we say ψ_n as map from A to Z_m rather than to the projective space $\mathbb{P}$.

Now we have the theorem about semi-ample fibrations for a semi-ample bundle on a projective normal variety A.

Theorem 2.1.1. ([Laz17]) Say V is semi-ample bundle on A. There exists algebraic fibre space $\psi : B \to A$ with property - for each sufficiently large $n \in M(A,V) \in \mathbb{Z}$,

$$Z_n = Z \text{ and } \psi_n = \psi.$$

Moreover, there exists ample line bundle E on Z with $\psi^*E = V^{\otimes h}$ where $h = g(V)$ is exponent of $M(A,V)$.

Now we develop some lemma and theorems required for proving the Iitaka Fibration theorem.

Lemma 2.1.2. ([Laz17]) Given $n \in M(A,V)$, for each sufficiently large $l >> 0, k \in \mathbb{Z}$, composition map

$$A \xrightarrow{\psi_{ln}} Z_{ln} \xrightarrow{\mu_l} Z_n$$

where $\mu_l : Z_{ln} \to Z_n$ is finite map and $\psi_n = \mu_l \circ \psi_{ln}$, this gives Stein factorization of the map ψ_n, that is ψ_{ln} is algebraic fibre space. Particularly, Z_{ln}, ψ_{ln} are not dependent on l for $l >> 0$.

The following example illustrates the surjectivity of the multiplication maps for a line bundle V on some projective normal variety A.

Example 2.1.10. ([Laz17]) Suppose V is generated by its global sections. There is $n_0 = n_0(V) \in \mathbb{Z}$ for which the maps

$$H^0(A, V^{\otimes p}) \otimes H^0(A, V^{\otimes q}) \to H^0(A, V^{\otimes (p+q)})$$

those are determined by the multiplications are surjective maps when $p, q \geq n_0$. Generally, for given coherent sheaf $\mathcal{G}$ on A,

$$H^0(A, \mathcal{G} \otimes V^{\otimes p}) \otimes H^0(A, \mathcal{G} \otimes V^{\otimes q}) \to H^0(A, \mathcal{G} \otimes V^{\otimes (p+q)})$$

is a surjective map for $p, q >> 0$.

For a semi-ample line bundle V on projective normal algebraic variety, we have the following theorem.

Theorem 2.1.2. ([Laz17]) Line bundle V is finitely generated, that is by definition the section ring $R(A, V)$ of V, is finitely generated as $\mathbb{C}$-algebra.

If the base locus of the complete linear series of a line bundle is finite, then the theorem by Zariski and Fujita concludes about the semi-ampleness of the line bundle (as in the set up of the previous theorem).

Theorem 2.1.3. ([Laz17]) If the base locus $\mathrm{Bs}\,|V|$ is finite. The bundle V is semi-ample, that is $V^{\otimes n}$ is free for a $n > 0$.

2.1.3 Iitaka Fibration

We now state one of the most important theorems of the current section. It is known as the *Iitaka Fibrations*. We take a line bundle V on a projective normal variety A as always.

Theorem 2.1.4. ([Laz17]) If the bundle V satisfies $\kappa(A, V) > 0$. Then, for each sufficiently large $l \in \mathbf{N}(A, V)$, rational maps $\psi_l : A \dashrightarrow Z_l$ are birationally equivalent to fixed algebraic fibre space

$$\psi_\infty : A_\infty \to Z_\infty$$

of normal varieties, restriction of the bundle V to general fibre of ψ_∞ has its Iitaka dimension $= 0$. Specifically, for a large $l \in \mathbf{N}(A, V)$ the diagram of rational maps and morphisms commutes.

$$
\begin{array}{ccc}
A & \xleftarrow[v_\infty]{} & A_\infty \\
{\scriptstyle \psi_l}\big\downarrow & & \big\downarrow{\scriptstyle \psi_\infty} \\
Z_l & \xleftarrow[\mu_l]{} & Z_\infty
\end{array}
$$

Here horizontal maps are birational, v_∞ is morphism. We get $\dim Z_\infty = \kappa(A, V)$. Also, when $V_\infty = v_\infty^* V$, $G \subseteq A_\infty$ is very general fibre of ψ_∞ , then

$$\kappa(G, V_\infty|_G) = 0.$$

Now we define Iitaka fibrations associated to given line bundle, and that of a variety.

Definition 2.1.8. ([Laz17]) Say $\psi_{infty} : A_\infty \to Z_\infty$ is Iitaka fibration associated to V. Then the fibration is unique up to birational equivalence. This fibration of divisor E is defined by passing it to the line bundle $\mathcal{O}_A(E)$.

Definition 2.1.9. (**Iitaka fibration of a variety.**) Iitaka fibration of an irreducible variety A is defined to be Iitaka fibration which is associated to canonical bundle on any non-singular model of variety A. A very general fibre G of Iitaka fibration satisfies its Iitaka dimension $\kappa(G) = 0$.

2.2 Big Line Bundles and Divisors

Here we discuss about an important class of line bundles in the positivity theory, those of maximal Iitaka dimension.

2.2.1 Basic Properties and definition of Big Divisors

Definition 2.2.1. ([Laz17]) Line bundle V on projective variety A is said to be *big* if the Iitaka dimension $\kappa(A, V) = \dim A$. Cartier divisor E on A is said to be big if $\mathcal{O}_A(E)$ is also so.

As an example, pullback of ample line bundle by generically finite morphism is a big line bundle. For a normal variety A, Iitaka fibration theorem implies - V is big if and only if map $\psi_n : A \dashrightarrow \mathbb{P}H^0(A, V^{\otimes n}$ constructed by $V^{\otimes n}$ is surjective and birational to its image for a $n > 0$.

Example 2.2.1. ([Laz17]) Smooth projective variety A is defined to be of *general type* if and only if the canonical divisor K_A is a big divisor.

For big line bundles the following lemma gives an estimation of the dimension of the 0-th sheaf cohomology group of a divisor on r-dimensional projective variety.

Lemma 2.2.1. ([Laz17]) Divisor E on A is a big divisor if and only if there exists constant $Q > 0$ with

$$h^0(A, \mathcal{O}_A(nE)) \geq Q \cdot n^r$$

for each sufficiently large $n \in \mathbf{N}(A, E)$.

Now we discuss some examples of Big divisors as follows. First we look at blow-ups of a projective space at finitely may points.

Example 2.2.2. ([Laz17]) Say $Z \subseteq \mathbb{P}^k$ is finite set, we view it as reduced scheme,

$$\nu : P = \mathrm{Bl}_Z(\mathbb{P}^k) \to \mathbb{P}^k$$

is blowing up of $\mathbb{P}^n$ along the points in Z. Say $F = F_Z$, $J = J_Z$ are exceptional divisor, pullback of hyperplane class respectively, put $K = pJ - qF$. Also,

$$\nu_* \mathcal{O}_A(nK) = \mathcal{O}_{\mathbb{P}^k}(nK) \otimes \mathcal{I}^{nq},$$

$\mathcal{I} = \mathcal{I}_{Z/\mathbb{P}}$ is ideal sheaf of Z in $\mathbb{P}^k$, so $H^0(P, \mathcal{O}_P(nK)$ can be identfied with space of hypersurfaces with degree np which vanishes to order $\geq nq$ at every point in Z. Suppose $\#Z = z$. Then B is a big divisor provided $p^k > z \cdot q^k$. Also, it is evident that B is not nef: if $p < 2q$ then proper transform of line through 2 points of Z have negative intersection with the divisor B.

We have the following lemma useful for characterising big divisors on irreducible projective variety A. It is known as the *Kodaira's lemma.*

Proposition 2.2.1. ([Laz17]) Consider B is big Cartier divisor and E, be another effective Cartier divisor on A. Then

$$H^0(A, \mathcal{O}_A(nB - E)) \neq 0$$

for each sufficiently large $n \in \mathbf{N}(A, B)$.

As a corollary of the theorem, the following characterizes big divisors on A as above.

Corollary 2.2.1. ([Laz17]) The following statements are equivalent:

1. B is a big divisor.

2. For each ample integral divisor E on A, there is $n \in \mathbb{N}$, effective divisor M on A with $nB \equiv_{\text{lin}} E + M$.

3. Same statement as in 2nd, replaced by *for some* against *for each* ample divisor E.

4. There is ample divisor E, $n \in \mathbb{N}$, effective divisor M with $nB \equiv_{\text{num}} E + M$.

Therefore, bigness of the divisor B depends on only the numerical equivalence class. Next we discuss some cohomological characterization for big divisors on projective variety A.

Example 2.2.3. ([Laz17]) Divisor B is big divisor if and only if following holds: For coherent sheaf $\mathcal{G}$ on A, there is $n = n(\mathcal{G}) \in \mathbb{N}$ with $\mathcal{G} \otimes \mathcal{O}_A(nB)$ is generically generated by global sections, that is such that natural map

$$H^0(A, \mathcal{G} \otimes \mathcal{O}_A(nB)) \otimes_{\mathbb{C}} \mathcal{O}_A \to \mathcal{G} \otimes \mathcal{O}_A(nB)$$

is a generically surjective map.

Therefore, when B is a big divisor then the exponent $f(B) = 1$, that is each sufficiently large multiples of the divisor B are effective divisors. In the following corollary we discuss the behaviour of big divisors after restrictions to sub-variety.

Corollary 2.2.2. ([Laz17]) Consider B, which is big line bundle on projective variety A. There exists proper Zariski-closed subset $U \subsetneq A$ satisfying if $U \subseteq A$ is sub-variety of A which is not contained inside U, then restriction $B_U = B|_U$ is big line bundle on the sub-variety U. Particularly, take D, a general member of very ample linear series, then the restriction B_D is big.

The following example shows that arbitrary restriction to a sub-variety of big divisor is not necessarily a big divisor.

Example 2.2.4. ([Laz17]) Consider $S = \mathrm{Bl}_Q \mathbb{P}^2$ is blowing up projective plane at point Q, let F, J denote exceptional divisor, pullback of line, respectively. Then, $\mathcal{O}_S(J + F)$ is a big line bundle, but $\mathcal{O}_F(J + F) = \mathcal{O}_{\mathbb{P}^1}(-1)$ is not a big line bundle.

The following example shows that the analogue of openness property for ample and nef divisors is not true for big divisors.

Example 2.2.5. ([Laz17]) We can find family of the line bundles V_p on varieties A_p, parameterized by curve Z , with $\kappa(A_p, V_p) = -\infty$ for a general $p \in Z$, but V_0 is big for a fixed point $0 \in Z$. To give example, we start with finite set $W \subset \mathbb{P}^2$ consisting of points in projective plane, consider blow-up

$$A_W = \mathrm{Bl}_W(\mathbb{P}^2) \to \mathbb{P}^2$$

of $\mathbb{P}^2$ along the points in W, we put $E_W = 2J_W - F_W$. After taking W consisting sufficiently general points in large number, we assume there does not exist any curves with degree $2n$ in $\mathbb{P}^2$ with multiplicity $\geq n$ at all points of W. Which is

$$H^0(A_W, \mathcal{O}_{A_W}(nE_W)) = 0 \text{ for each } n > 0.$$

Also, if points of W are co-linear, then the divisor $E_W - J_W$ is an effective divisor, so E_W is a big divisor. Now the family (A_p, V_p) can be obtained by varying the set W in suitable one-parameter family W_p. Note, on fixed variety, bigness is an invariant under any deformation.

B, which is integral divisor, is big if and only if every positive multiple of it is a big divisor. Hence we define the following:

Definition 2.2.2. ([Laz17]) $\mathbb{Q}$-divisor B is a big divisor if there exists $n \in \mathbb{N}$ with nB is big integral divisor.

Bigness is numerical property of the $\mathbb{Q}$-divisors. Now we extend this definition to the $\mathbb{R}$-divisors and describe the corresponding cone in the group $N^1(A)_{\mathbb{R}}$. The following theorem gives a numerical criterion to determine bigness.

Theorem 2.2.1. ([Laz17]) Consider P and Q are nef $\mathbb{Q}$-divisors on a d-dimensional projective variety A. Let,

$$(P^d) > d \cdot (P^{d-1} \cdot Q),$$

then the divisor $P - Q$ is a big divisor.

We end this subsection by a theorem determining the bigness of nef divisors.

Theorem 2.2.2. ([Laz17]) Consider N is nef divisor on some d-dimensional irreducible projective variety A. Then the divisor N is a big divisor if and only if the top self-intersection is a strictly positive number, that is $(N^d) > 0$.

2.2.2 Pseudoeffective and Big Cones

We start the subsection with defining big $\mathbb{R}$-divisors. Like in previous sections, A is d-dimensional projective variety.

Definition 2.2.3. ([Laz17]) A $\mathbb{R}$-divisor $R \in \mathrm{Div}_{\mathbb{R}}(A)$ is said to be big divisor if it could be written of the form

$$R = \sum r_i \cdot R_i$$

where all R_i's are big integral divisors, $r_i > 0, r_i \in \mathbb{R}$ for each i.

The following proposition guarantees that bigness depend on numerical equivalence class.

Proposition 2.2.2. ([Laz17]) Consider R, R' are $\mathbb{R}$-divisors on A.

1. When $R \equiv_{\text{num}} R'$, then R is big if and only if R' is a big divisor.

2. R is a big divisor if and only if $R \equiv_{\text{num}} E+M$ where E is ample divisor, M is effective $\mathbb{R}$-divisor.

The following example gives a relation between ample $\mathbb{R}$-divisors and $\mathbb{R}$-divisors which are nef and big.

Example 2.2.6. ([Laz17]) Consider N is nef and big $\mathbb{R}$-divisor. There is effective $\mathbb{R}$-divisor E with $N - \frac{1}{l}E$ is ample $\mathbb{R}$-divisor for each sufficiently large $l \in \mathbb{N}$.

Corollary 2.2.3. ([Laz17]) Assume $B \in \mathrm{Div}_{\mathbb{R}}(A)$ is big $\mathbb{R}$-divisor, suppose $F_1, \ldots, F_p \in \mathrm{Div}_{\mathbb{R}}(A)$ are arbitrary $\mathbb{R}$-divisors. Then

$$B + \alpha_1 F_1 + \cdots + \alpha_p F_p$$

is big for each sufficiently small $0 \leq |\alpha_i| << 1, \alpha_i \in \mathbb{R}$, for each i.

Now we are ready to define the big and pseudoeffective cones.

Definition 2.2.4. ([Laz17]) The big cone is defined as

$$\mathrm{Big}(A) \subseteq N^1(A)_{\mathbb{R}}$$

convex cone consisting of every big $\mathbb{R}$-divisor classes on A. The pseudo-effective cone is defined as

$$\overline{\mathrm{Eff}}(A) \subseteq N^1(A)_{\mathbb{R}}$$

closure of convex cone spanned by classes of every effective $\mathbb{R}$-divisors. Divisor $E \in \mathrm{Div}_{\mathbb{R}}(A)$ is pseudo-effective if the class of E lies in the pseudo-effective cone.

The following theorem gives a relation between big cone and pseudo-effective cone.

Theorem 2.2.3. ([Laz17]) Big cone is interior of pseudo-effective cone, also, pseudo-effective cone is closure of big cone.

2.2.3 Volume of a Big Divisor

One of the most useful things in the study of positivity of line bundles, is the concept of volume of a divisor or line bundle. So we start this subsection by defining volume for line bundles. Again A is d-dimensional projective variety.

Definition 2.2.5. ([Laz17]) Take V, a line bundle on A. *Volume* of the bundle V is defined as

$$\mathrm{vol}_A(V) = \limsup_{n\to\infty} \frac{h^0(A, V^{\otimes n})}{n^d/d!}.$$

Note, $\mathrm{vol}(V) > 0$ if and only if the line bundle V is a big line bundle. If V is a nef line bundle, then from asymptotic Riemann–Roch theorem, it follows -

$$\mathrm{vol}(V) = \int_A c_1(V)^d$$

is top self-intersection of V.

The following example gives an estimate of volume on blow-up of a projective space.

Example 2.2.7. ([Laz17]) Consider $S = \mathrm{Bl}_Q(\mathbb{P}^n)$ and $\#Q = q$. Say $E = aJ - bF$, where F and J are exceptional divisor and pullback of hyperplane divisor, respectively. Then the volume

$$\mathrm{vol}(E) \geq a^d - s \cdot b^d.$$

Volume of big line bundle may also be an irrational number as shown in a later example. The following proposition gives some formula for calculating volume.

Proposition 2.2.3. ([Laz17]) Say B is big divisor on XA.

1. For fixed $t \in \mathbb{N}$, the volume,

$$\mathrm{vol}(tB) = t^d \,\mathrm{vol}(B).$$

2. For given fixed divisor M on X, given $\alpha > 0$ there is $q_0 = q_0(M, \alpha) \in \mathbb{Z}$ with

$$\frac{1}{q^d} \cdot |\,\mathrm{vol}(qB - M) - \mathrm{vol}(qB)| < \alpha$$

for each $q > q_0$.

The following lemma ensures that volume is invariant in a complete linear series.

Lemma 2.2.2. ([Laz17]) Suppose V is big line bundle, E is very ample divisor on A. If $R, R' \in |E|$ are two very general divisors, then the volume

$$\mathrm{vol}_R(V|_R) = \mathrm{vol}_{R'}(V|_{R'}).$$

Lemma 2.2.3. ([Laz17]) Suppose E is divisor on A, $t \in \mathbb{N}$ is a fixed number. Then

$$\limsup_n \frac{h^0(A, \mathcal{O}_A(nE))}{n^d/d!} = \limsup_l \frac{h^0(A, \mathcal{O}_A(tlE))}{(tl)^d/d!}.$$

For finitely generated divisor E on normal projective variety A, the volume $\mathrm{vol}(E) \in \mathbb{Q}$.

Proposition 2.2.4. ([Laz17]) Take two numerically equivalent divisors D_1, D_2 on A, then

$$\mathrm{vol}(D_1) = \mathrm{vol}(D_2).$$

This tells about numerical nature of volume.

Lemma 2.2.4. ([Laz17]) For each numerically trivial divisor T, there is fixed divisor F with property:

$$H^0(A, \mathcal{O}_A(F + T)) \neq 0.$$

Particularly, if the divisor T_0 is numerically trivial divisor, then we have for each $n \in \mathbb{Z}$, $F \pm nT_0$ divisor is linearly equivalent to some effective divisor.

The following proposition illustrates birational invariance of the volume function. Suppose A, A' are n-dimensional irreducible varieties.

Proposition 2.2.5. ([Laz17]) Take a birational projective mapping $\mu : A' \to A$. Given integral or a $\mathbb{Q}$-divisor E on A, put $E' = \mu^* E$. Then the volume of the pullback

$$\mathrm{vol}_{A'}(E') = \mathrm{vol}_A(E).$$

In the following theorem we view volume as a continuous function on finite dimensional vector space $N^(A)_{\mathbb{R}}$, after inducing usual topology by fixing a norm.

Theorem 2.2.4. ([Laz17]) We fix norm $\| \ \|$ on $N^1(A)_{\mathbb{R}}$ which induces usual topology on finite-dimensional vector space. Then there exists constant $K > 0$ which satisfies

$$|\,\mathrm{vol}(\zeta) - \mathrm{vol}(\zeta')| \leq K \cdot (\max(\|\zeta\|, \|\zeta'\|))^{d-1 \cdot \|\zeta - \zeta'\|}$$

for any arbitrary two classes $\zeta, \zeta' \in N^1(A)_{\mathbb{Q}}$.

Corollary 2.2.4. Since the constant K is independent of the elements in $N^1(A)_{\mathbb{Q}}$, the volume function is infact uniformly continuous. Since $\mathbb{Q}$ is dense $\mathbb{R}$ in usual topology, the function sending $\zeta \to \mathrm{vol}(\zeta)$ on $N^1(A)_{\mathbb{Q}}$ can be extended uniquely to continuous function

$$\mathrm{vol} : N^1(A)_{\mathbb{R}} \to \mathbb{R}$$

The following example shows volume after blow-up of a projective space at a point.

Example 2.2.8. ([Laz17]) Say $S = \mathrm{Bl}_Q(\mathbb{P}^k)$, denote $f, j \in N^1(S)_{\mathbb{R}} = \mathbb{R}^2$ classes of exceptional divisor F, pullback J of hyperplane, respectively. Nef cone of A is then generated by the classes j and $j - f$. In the plane spanned by them, volume is:

$$\mathrm{vol}(p \cdot j - q \cdot f) = ((p \cdot j - q \cdot f)^k) = p^k - q^k.$$

Now, if $a, b \geq 0$, linear series of $|aJ + bF|$ contains the divisor bF as fixed component. Hence in region spanned by the classes j and f - corresponding to effective divisors which are not nef, volume is given by

$$\mathrm{vol}(p \cdot j - q \cdot f) = ((p \cdot j)^k) = p^k.$$

Otherwise, volume is 0.

The following example illustrates the volume of the difference of two nef divisors.

Example 2.2.9. ([Laz17]) Consider $\zeta, \chi \in \mathrm{Nef}(A)_{\mathbb{R}}$ are two real nef classes on projective variety A with dimension d. Then, the volume

$$\mathrm{vol}(\zeta - \chi) \geq (\zeta^d) - d \cdot (\zeta^{d-1} \cdot \chi).$$

Example 2.2.10. ([Laz17]) In this example we show that the volume increases towards the direction of effective divisors. Take $\zeta \in N^1(A)_{\mathbb{R}}$ is a big divisor class and $f \in N^1(A)_{\mathbb{R}}$ is an effective class then

$$\mathrm{vol}(\zeta) \leq \mathrm{vol}(\zeta + f).$$

Example 2.2.11. ([Laz17]) In this example we describe the birational invariance of volume in real case. Take $\mu : A' \to A$ is birational morphism. Then the volume,

$$\mathrm{vol}_A(\zeta) = \mathrm{vol}_{A'}(\mu^* \zeta)$$

for each classes $\zeta \in N^1(A)_{\mathbb{R}}$.

2.3 Examples and Complements

Now we go through some actual examples and results related to linear series.

2.3.1 Cutkosky's Construction

Consider Z is irreducible projective variety, with its dimension $\dim Z = z$, we fix k integral (Cartier) divisors $B_0, B_1, \ldots, B_k$ on the variety Z. Say

$$\mathcal{F} = \mathcal{O}_Z(B_0) \oplus \ldots \oplus \mathcal{O}_Z(B_k),$$

such that $\mathcal{F}$ is a vector bundle which has rank $k+1$ on Z . We say

$$A = \mathbb{P}(\mathcal{F}) \text{ and } V = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1).$$

Hence A is irreducible $n = z + k$-dimensional projective variety. Cutkosky's idea was that after careful choices of the B_i's, we lead to some interesting behaviors of linear series associated to the line bundle V.

Lemma 2.3.1. ([Laz17]) Say $A = \mathbb{P}(\mathcal{F})$, and by abuse of notation we write $\mathcal{O}_A(l)$ for denoting the bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(l)$ on A.

1. $$H^0(A, \mathcal{O}_A(l)) = \bigoplus_{p_0+\ldots+p_k=l} H^0(Z, \mathcal{O}_Z(p_0B_0 + \ldots + p_kB_k)).$$

2. $\mathcal{O}_A(1)$ is an ample bundle if and only if all divisors B_i are ample divisors on Z.

3. $\mathcal{O}_A(1)$ is a nef bundle if and only if all divisors B_i are nef divisors on Z.

4. $\mathcal{O}_A(1)$ is a big line bundle if only only if non-negative $\mathbb{Z}$-linear combination of B_i's is big divisor on Z.

5. Given $n \in \mathbb{N}$, $\mathcal{O}_A(n)$ is free line bundle if and only if nB_j follows its path in free linear series on Z for every $0 \le j \le k$

In the following example we construct an explicit example where a line bundle has irrational volume.

Example 2.3.1. ([Laz17]) Say $S = C \times C$ is product of elliptic curve with itself. We have already seen: $\mathrm{Nef}(S)$ is circular cone $\mathcal{N}$, which consists of classes which have non-negative self-intersection (also, non-negative intersection with ample divisor). We choose two integral divisors which are ample

$$E_1, E_2 \in \mathrm{Div}(S) \text{ with two classes } e_1, e_2 \in N^1(S)_{\mathbb{R}}$$

so that ray in $N^1(S)_{\mathbb{R}}$ emerging from e_1 in direction of $-e_2$ meets boundary of $\mathcal{N}$ at some irrational point. Equivalently,

$$\zeta =_{\text{def}} \max\{x \mid e_1 - x \cdot e_2 \text{ is a nef class}\} \notin \mathbb{Q}.$$

Precisely, ζ is smallest root of quadratic polynomial

$$p(t) = ((e_1 - x \cdot e_2)^2).$$

hence "most" of the choices of E_1 and E_2 will lead to choice of irrational ζ.

Now we apply the Cutkosky's construction with

$$B_0 = E_1, B_1 = -E_2$$

Then we have

$$h^0(A, \mathcal{O}_A(l)) = \sum_{p+q=l} h^0(S, \mathcal{O}_S(aE_1 - bE_2)).$$

Now divisor $aE_1 - bE_2$ which appears in sum is a nef divisor, precisely if $\frac{b}{a} < \zeta$, and doesn't have sections when $\frac{b}{a} > \zeta$. Applying Riemann-Roch on S , we get: for $a, b \geq 0$

$$h^0(S, \mathcal{O}_S(aE_1 - bE_2)) = \begin{cases} \frac{1}{2}((ae_1 - be_2)^2) & \text{when } \frac{b}{a} < \zeta, \\ 0 & \text{when } \frac{b}{a} > \zeta \end{cases}$$

Now after combining the both results, we substitute $(l - a)$ for b, and after dividing by $\frac{l^3}{3!}$ we have,

$$\frac{h^0(A, \mathcal{O}_A(l))}{l^3/3!} = \frac{3!}{2} \cdot \sum_{a \geq \frac{l}{1+\zeta}}^{l} \left(\left(\frac{a}{l} e_1 - \left(1 - \frac{a}{l}\right) e_2 \right)^2 \right) \cdot \frac{1}{l}.$$

Now, right-hand side of the equation is Riemann sum for integral of quadratic function:

$$h(t) = \left((te_1 - (1-t)e_2)^2\right).$$

Now as $l \to \infty$ we get

$$\text{vol}_A(\mathcal{O}_A(1)) = \frac{3!}{2} \cdot \int_{\frac{1}{1+\sigma}}^{1} h(t)dt.$$

Now let

$$E_1 = G_1 + G_2, E_2 = 3 \cdot (G_2 + \Delta)$$

where $G_1, G_2, \Delta \subseteq S$ are the fibres of projections $S = C \times C \to C$, and diagonal respectively. Then $\zeta = \frac{3-\sqrt{5}}{6}$ is an irrational number. We get $h(t) = 38t^2 - 54t + 18$, and by calculating integral we get an irrational volume.

2.3.2 Base Loci of Nef and Big Linear Series

The following theorem is known as Wilson's theorem. Also in this subsection A is d-dimensional projective variety.

Theorem 2.3.1. ([Laz17]) Take E which is nef and big divisor on A. There is $n_0 \in \mathbb{N}$ with effective divisor M so that

$$|nE - M|$$

is a free linear series for each $n \geq n_0$.

Now we define multiplicity of linear series at some given point on A

Definition 2.3.1. ([Laz17]) Take $a \in A$ a fixed point. Given the divisor E on A and the linear series $|W| \subseteq |E|$, we denote $\text{mult}_a|W|$ multiplicity at point a of general divisor in linear series $|W|$. In other words,

$$\text{mult}_a|W| = \min_{E' \in |W|} \{\text{mult}_a E'\}.$$

This integer is defined as multiplicity of the linear series $|W|$ at point a.

The following result gives us a bound on the multiplicity of linear series.

Corollary 2.3.1. ([Laz17]) With the hypothesis of Wilson's Theorem, there is constant $K > 0$ independent of n and a so that $\text{mult}_a|nE| \leq K$ for each $a \in A$.

Theorem 2.3.2. ([Laz17]) If A is normal, E a big nef divisor on A. Then the section ring $R(A, E)$ is finitely generated if and only if the divisor E is semi-ample, that is $|kE|$ is a free linear series for a $k > 0$.

The following is counter example for a divisor which is not big.

Example 2.3.2. ([Laz17]) Consider C is smooth curve with genus ≥ 2, we run Cutkosky's construction by taking $B_0 = 0$, $B_1 = Q$ where Q is divisor having degree 0, which is non-torsion in $\text{Pic}^0(C)$. Then, $\mathcal{O}_S(1)$ is a nef bundle on $S = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(Q))$, but the divisor at the infinity $\mathbb{P}(\mathcal{O}_C(Q)) \subseteq S$ comes with multiplicity n in the base locus of $\mathcal{O}_A(n)$.

2.3.3 Theorem by Campana and Peternell

In the following we have the Nakai's criterion for the $\mathbb{R}$-divisors. We denote A for projective scheme.

Theorem 2.3.3. ([Laz17]) Take $d \in N^1(A)_{\mathbb{R}}$, a class which has positive intersection with all irreducible subvarieties of A. Equivalently,

$$\left(d^{\dim W} \cdot W\right) > 0$$

for each $W \subseteq A$ with $\dim W > 0$. Then the class d is ample class.

2.3.4 Zariski Decompositions

We end this report by discussion of Zariski decomposition of a pseudoeffective integral divisor. It is useful since the volume of such divisors can be determined by only looking at positive part of the decomposition.

Theorem 2.3.4. ([Laz17]) Consider A, smooth projective surface, E is pseudoeffective integral divisor on A. Then the divisor E could be written uniquely as a sum of a positive part and negative part

$$E = P + N$$

of $\mathbb{Q}$-divisors which satisfies the following properties:

1. the positive divisor part P is nef divisor.
2. the negative divisor part $N = \sum_{j=1}^{k} c_j F_j$ is effective divisor, if $N \neq 0$ intersection matrix
   $$\|(F_i \cdot F_j)\|$$
   determined by the components of the negative divisor part N is a negative definite matrix.
3. P part is orthogonal with every components of N, that is $(P \cdot F_j) = 0$ for each $1 \leq j \leq k$.

The following corollaries gives the desired formula for calculating volume and to determine whether the section ring is finitely generated.

Corollary 2.3.2. ([Laz17]) Volume of a integral divisor on some surface is rational number. In particular,

$$\text{vol}(E) = (P^2).$$

Corollary 2.3.3. ([Laz17]) Consider B is big divisor on smooth surface A, section ring $R(A, B)$ is then finitely generated if and only if the positive divisor part P of B is a semiample divisor.

It has been shown by Cutkosky and Srinivas that for non-singular projective surface A over some algebraic closure of a finite field, E, which is effective on A, its section ring $R(A, E)$ is finitely generated.

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