Topics in Riemann Surfaces

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This is to certify that this thesis entitled "*Topics in Riemann Surfaces*" submitted towards the partial fulfilment of the Mathematics M.Sc Degree Program at the Indian Institute of Science Education and Research Pune represents work carried out by *Joson Josh Martires Henriques* under the supervision of *Dr. Diganta Borah*.

Dr. Diganta Borah
Master's Thesis Supervisor

Declaration

I hereby declare that the matter embodied in the report entitled "*Topics in Riemann Surfaces*" are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of *Dr. Diganta Borah* and the same has not been submitted elsewhere for any other degree.

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Contents

1 Introduction

This thesis consists of a study of some topics in Riemann surfaces. Starting with a quick overview of definitions and properties of maps between Riemann surfaces, we move to a proof of the Uniformization theorem for Riemann surfaces. We give a brief overview of the properties of Green's functions on Riemann surfaces needed to prove the Uniformization theorem. We then describe the correspondence between divisors and line bundles on Riemann surfaces. The final section deals with some consequences of the Behnke-Stein Runge theorem and shows how the $\bar{\partial}$ - equation on an open Riemann surface can always be solved. We also use this result to show that every holomorphic line bundle on an open Riemann surface is trivial.

2 Riemann Surfaces

2.1 Definition and examples

Definition 2.1. Let X be a connected, second-countable, Hausdorff topological space. The pair $(X, \mathcal{A})$ is said to be a Riemann surface where $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is a maximal atlas for X i.e.,
(i) $\{U_\alpha\}_{\alpha \in A}$ is an open cover for X
(ii) Each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}$ is a homeomorphism.
(iii) Whenever $U_\alpha \cap U_\beta \neq \phi$, the map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is holomorphic.

Remark 2.1. 1) Since every holomorphic map is smooth when treated as a map from a subset of $\mathbb{R}^2$ to a subset of $\mathbb{R}^2$ we immediately obtain that every Riemann surface is a 2-dimensional real manifold.
2) Every Riemann surface as a real manifold is oriented because if we express the transition map $\varphi_\alpha \circ \varphi_\beta^{-1} = u + iv$ and compute the real Jacobian we get $u_x v_y - u_y v_x$. However, as a result of the Cauchy-Riemann equations we can see that this is the same as $u_x^2 + u_y^2$ which is always positive.

We now mention some examples of Riemann surfaces.

Example 2.1. Any domain $D \subset \mathbb{C}$ can be treated as a Riemann surface. In this case, the atlas consists of only one chart (D, φ) where φ is the identity map.

Example 2.2. The extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ can be given the structure of a Riemann surface. We cover $\mathbb{C}_\infty$ with two charts namely (U_0, φ_0) and (U_1, φ_1) where $U_0 = \mathbb{C}$, $\varphi_0(\zeta) = \zeta$ and $U_1 = \mathbb{C}_\infty \setminus \{0\}$, $\varphi_1(\zeta) = 1/\zeta$.

Example 2.3. The projective space $\mathbb{CP}^1$ is a Riemann surface. In this case $\mathbb{CP}^1$ is covered by two open sets $U_0 = \{[z_0, z_1] : z_0 \neq 0\}$ and $U_1 = \{[z_0, z_1] : z_1 \neq 0\}$. The coordinates are given by the maps $\varphi_0 : U_0 \to \mathbb{C}, \varphi_0([z_0, z_1]) = z_1/z_0$ and $\varphi_0 : U_0 \to \mathbb{C}, \varphi_1([z_0, z_1]) = z_0/z_1$.

Definition 2.2. A complex-valued function $F : V \subset X \to \mathbb{C}$ is said to be holomorphic (repectively meromorphic) on V if for every coordinate chart U_α that non-trivially intersects V, $F \circ \varphi_\alpha^{-1} : \varphi_\alpha(V \cap U_\alpha) \to \mathbb{C}$ is holomorphic (respectively meromorphic). We say two Riemann surfaces X and Y are conformally equivalent if there exists a bijective holomorphic map $F : X \to Y$, with F^{-1} also holomorphic. (We will soon see that we get the holomorphicity of F^{-1} for free.)

We now list out some important properties of maps between Riemann surfaces and provide a sketch of their proofs.

Theorem 2.1 (Open Mapping Theorem). *Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is an open map.*

Proof. Let O be any open subset of X. We claim that $f(O)$ must be open in Y. It suffices to show that every point $f(x) \in f(O)$ is an interior point of $f(O)$. We fix charts (U, φ) and (V, ψ) around x and $f(x)$ respectively.
Consider the map $\psi \circ f \circ \varphi^{-1}$ on $\varphi(U \cap f^{-1}(V))$. We push an open neighbourhood of $\varphi(x)$ under $\psi \circ f \circ \varphi^{-1}$ to get $A \subset \mathbb{C}$, which is open as a consequence of the open mapping theorem for holomorphic functions between open subsets of $\mathbb{C}$. It is easy to check that $\psi^{-1}(A)$ gives the required neighborhood of $f(x)$. $\square$

Theorem 2.2 (**Maximum Principle**). *Let X be a Riemann surface and $f : X \to \mathbb{C}$ be a non-constant holomorphic function. Then $|f|$ does not attain its maximum.*

Proof. Suppose there exists $a \in X$ such that

$$R = f(a) = \max\{|f(x)| : x \in X\}$$

Consequently, $f(X) \subset B[0, R]$, where $B[0, R]$ denotes the closed ball centered at 0 with radius R. However, the open mapping theorem guarantees that $f(X)$ is open and hence $f(a)$ must be an interior point of $f(X)$. This contradicts the fact that $f(a) \in \partial B[0, R]$. $\square$

We have the following corollaries which shall be of use in the subsequent sections.

Corollary 2.1. *Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Additionally, assume the X is compact. Then f is onto and Y is compact.*

Proof. Using Theorem 2.1 we know that $f(X)$ must be open. Compactness of X implies that $f(X)$ is compact and hence closed. Since Y is connected and $f(X)$ is a non-empty clopen set, we conclude that $f(X) = Y$, and hence f is surjective and Y is compact. $\square$

Corollary 2.2. *Every holomorphic map $f : X \to \mathbb{C}$ on a compact Riemann surface X is constant.*

Proof. Suppose f is non-constant. Then by the previous corollary we obtain that $f(X) = \mathbb{C}$. But this is absurd since $f(X)$ is supposed to be a compact subset of $\mathbb{C}$. Hence, f must be constant. $\square$

2.1.1 Harmonic functions on Riemann surfaces

Definition 2.3. Let X be a Riemann surface and f be a real-valued smooth function on X. We say that f is harmonic at $p \in X$ if for some chart $(U, z = x + iy)$ containing p, we have $\Delta(f \circ z^{-1})(z(p)) = 0$.

Remark 2.2. One can check that the above definition is independent of the choice of chart i.e., if $p \in U_\alpha \cap U_\beta$ with local coordinates z_α and z_β respectively, then $\Delta(f \circ z_\alpha^{-1})(z_\alpha(p)) = 0$ implies $\Delta(f \circ z_\beta^{-1})(z_\beta(p)) = 0$. We refer the reader to the calculation for the same given on page 57 of [6].

2.2 Properties of holomorphic maps between Riemann surfaces

We now prove some important properties of holomorphic maps between Riemann surfaces which shall be handy when we introduce the notion of divisors Riemann surfaces.

The next result on the local normal form shows that every non-constant holomorphic map 'looks like' a power map. We make this idea precise in the next result.

Theorem 2.3 (**Local Normal Form**). *Let $F : X \to Y$ be a non-constant holomorphic map. Then for each $p \in X$, there is a unique integer $m \geq 1$ such that for every chart $\varphi_2 : U_2 \to V_2$ on Y centered around $F(p)$, there exists a chart $\varphi_1 : U_1 \to V_1$ on X centered at p such that $\varphi_2(F(\varphi_1^{-1}(z))) = z^m$.*

A proof of the above result can be found in [5]. The unique integer m guaranteed by the theorem above is referred to as the *multiplicity* of F at p, denoted as $\operatorname{mult}_p(F)$.

We now provide a technique to compute $\operatorname{mult}_p(F)$, without having to explicitly find the charts around p and $F(p)$ which put F in the local normal form. If we fix any local coordinates around p and $F(p)$, then in terms of these local coordinates, F can be written as $w = h(z)$ where h is a holomorphic map. With the above notation $\operatorname{mult}_p(F) = 1 + \operatorname{ord}_{z_0}(dh/dz)$, where $\operatorname{ord}_{z_0}(dh/dz)$ denotes the order of zero of dh/dz at z_0.

As a result of the above observation and the bijective correspondence between holomorphic maps from X to the Riemann sphere and meromorphic maps on a Riemann surface we obtain the following lemma.

Lemma 2.1. *Let f be a meromorphic function on a Riemann surface X, with associated holomorphic map $F : X \to \mathbb{C}_\infty$.*
1. If p is a zero of f, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f)$.
2. If p is a pole of f, then $\operatorname{mult}_p(F) = -\operatorname{ord}_p(f)$.
3. If p is neither a zero nor a pole of f, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f - f(p))$.

We now prove the main result of this section which shows that for any non-constant holomorphic map between compact Riemann surfaces $F : X \to Y$, the number of preimages of any point in Y counted with multiplicities is always constant.

Theorem 2.4. *Let $F : X \to Y$ be a non-constant holomorphic function between compact Riemann surfaces. Let $g : Y \to \mathbb{Z}$ be given by $g(y) = \Sigma_{p \in F^{-1}(y)} \operatorname{mult}_p(F)$, then g is constant. This constant value of g is called the degree of F and is denoted by* $\deg(F)$.

Proof. It suffices to show that g is a locally constant function. Fix $y \in Y$. Let $\{x_1, \dots, x_n\}$ be the inverse images of y. Fix a local coordinate w on Y around y. By Local normal form, we are guaranteed the existence of local coordinates z_i on X centered at x_i such that $w = z_i^{m_i}$.

We show that the map g is constant on the chart around y with local coordinate w. It suffices to show that for points near y, all the preimages are captured in the charts around x_i, guaranteed by the Local Normal Form. Suppose this is not the case, we now get a sequence of points (y_n) converging to y and a sequence of points (t_n) in X such that $F(t_n) = y_n$ for all n and (t_n) lies outside the neighbourhoods around x_i guaranteed by the Local Normal Form.

Since X is compact, we extract a convergent subsequence of (t_n) say (p_n) such that $p_n \to p$. It is clear that p also lies outside the neighbourhoods around x_i guaranteed by the Local Normal Form. However, the continuity of F implies that $F(p) = y$ which is absurd. Thus, our initial assumption was false and g is indeed locally constant. □

As an immediate consequence of the above result we have the following proposition which says that for a non-constant meromorphic function f on a compact Riemann surface X, the sum of the order of its poles and zeroes is 0.

Proposition 2.1. *Let f be a non-constant meromorphic function on a compact Riemann surface X. Then*

$$\Sigma \operatorname{ord}_p(f) = 0.$$

Proof. Let $F : X \to \mathbb{C}_\infty$ denote the holomorphic map associated to $f, \{x_i\}$ denote the zeroes of f and $\{y_j\}$ denote the poles of f.

$$\begin{aligned} \Sigma \operatorname{ord}_p(f) &= \Sigma_i \operatorname{ord}_{x_i}(f) + \Sigma_i \operatorname{ord}_{y_j}(f) \\ &= \Sigma_i \operatorname{mult}_{x_i}(F) + \Sigma_i \operatorname{mult}_{y_j}(F) \\ &= \deg(F) - \deg(F) \\ &= 0. \end{aligned}$$

□

3 Uniformization Theorem

In this section, we give a proof of the uniformization theorem which characterizes simply connected Riemann surfaces up to conformal equivalence. We first go over some preliminaries regarding Green's functions on Riemann surfaces. These results are crucial in the proof of the Uniformization theorem which is proved later in this section. We shall state the results regarding Green's functions here without proof. The interested reader may refer to [2] for a proof of these results.

Lemma 3.1 (Harnack's estimate)**.** *Let X be a Riemann surface. For every compact subset $F \subset X$, there exists a constant $K > 0$ such that*

$$\frac{1}{K} \leq \frac{v(p)}{v(q)} \leq K$$

for all $p, q \in F$ and every positive harmonic function v on X.

Definition 3.1 (Subharmonic functions on a Riemann surface)**.** Let X be a Riemann surface and $U \subset X$ be open. An upper semicontinuous function $v : U \to [-\infty, \infty)$ is said to be subharmonic on U, if for every $p \in U$, there is a coordinate patch with local coordinate z around p such that $v \circ z^{-1}$ is subharmonic at $z(p)$.

Theorem 3.1 (Maximum Principle for subharmonic functions)**.** *Let X be a Riemann surface and u be a subharmonic function on X. Suppose u attains its maximum at some point on X, then u must be constant on X.*

Corollary 3.1. *Let X be a Riemann surface and let $F \subset X$ be compact. If v is a subharmonic function on X satisfying $v(z) \leq c$ for every $z \in R \setminus F$, then $v(z) \leq c$, for each $z \in X$.*

Definition 3.2 (Perron families of subharmonic functions)**.** Let $\mathcal{F}$ be a non-empty collection of subharmonic functions defined on an open subset U of a Riemann surface X. We say that $\mathcal{F}$ is a Perron family if
(i) If $f, g \in \mathcal{F}$, then $\max\{f, g\} \in \mathcal{F}$.
(ii)If $f \in \mathcal{F}$, then for every coordinate disk D contained in W such that f is finite on ∂D, the function defined to be f on $W \setminus D$, and the harmonic extension of $f|_{\partial D}$ on D is in $\mathcal{F}$.

We now explain the definition of Green's function for a Riemann surface.
Let X be a Riemann surface and $q \in X$. Let z be a local coordinate around q such that $z(q) = 0$. We look at the collection $\mathcal{F}_q$ given by

$\mathcal{F}_q := \{u : u$ is subharmonic on $X \setminus \{q\}$ and u vanishes outside some compact subset of $X\}$

One checks that $\mathcal{F}_q$ is a Perron family of subharmonic functions. We say that the Green's function for X with a pole at q exists if the upper envelope of the above family is finite, this is denoted by

$$g(p,q) := \sup\{u(p) : u \in \mathcal{F}_q\}, \quad p \in X \setminus \{q\}$$

Otherwise, the upper envelope of $\mathcal{F}_q$ is $+\infty$ for all points of $X \setminus \{q\}$ and we say that the Green's function for X with a pole at q does not exist.

The next theorem characterizes Green's function in terms of positive harmonic functions on the Riemann surface X.

Theorem 3.2. *Let X be a Riemann surface with $q \in X$ such that the Green's function for X with a pole at q exists. Let z be a local coordinate around q with z(q)=0 then g is positive harmonic on $X \setminus \{q\}$ and $g + \log|z|$ extends to a harmonic function at q. Moreover, if f is any harmonic function on $X \setminus \{q\}$ such that $f + \log|z|$ extends to a harmonic function at q, then $g \leq h$ on $X \setminus \{q\}$.*

Theorem 3.3. *Let X be a Riemann surface such that $g(p, q_0)$ exists for some $q_0 \in X$, then $g(p,q)$ exists for every $q \in X$.*

Theorem 3.4 (Symmetry of Green's function)**.** *Let X be a Riemann surface for which the Green's function exists. Then*

$$g(p,q) = g(q,p), \qquad p,q \in X, p \neq q$$

Since not every Riemann surface has a Green's function, we define the notion of a Bipolar Green's function. It is known that every Riemann surface has a Bipolar Green's function.

Let X be a Riemann surface and a, b be two distinct points on X. We fix centered coordinate disks (D_1, z_1) around a and (D_2, z_2) around b. A bipolar Green's function with poles at a, b is defined as any harmonic function $G(p, a, b)$ on $X \setminus \{a, b\}$ such that

- $G + \log|z_1|$ extends to a harmonic function at a.
- $G - \log|z_2|$ extends to a harmonic function at b.
- G is bounded on $X \setminus (D_1 \cup D_2)$.

A Bipolar Green's function for X is not uniquely determined. It is however unique up to adding a bounded harmonic function.

Theorem 3.5. *Let X be Riemann surface and a, b be any two distinct points on X, Then a Bipolar Green's function for X with poles at a, b always exists.*

Theorem 3.6 (Uniformization theorem). *Every simply connected Riemann surface X is conformally equivalent to either $\mathbb{D}, \mathbb{C}$ or $\mathbb{C}_\infty$.*

Proof. We split this proof into two cases. We know that of the three Riemann surfaces mentioned above, $\mathbb{D}$ is the only one that has a Green's function. Hence in the case when the Green's function for X exists, we shall use it to conformally map X onto $\mathbb{D}$. If the Green's function for X does not exist, the results from the last section guarantee the existence of a Bipolar Green's function. In this case, we shall use Biploar Green's function to map X onto $\mathbb{C}$ or $\mathbb{C}_\infty$.

Case 1: Green's function for X exists.
Fix $q_0 \in X$ and consider the green's function $g(p, q_0)$. We know that $g(p, q_0)$ has a logarithmic pole at q_0. The following lemma guarantees the existence of an analytic function φ which shall be crucial in the construction of the conformal map from X to $\mathbb{D}$.

Lemma 3.2. *There exists an analytic function $\varphi : X \to \mathbb{C}$ such that $|\varphi(p)| = e^{-g(p,q_0)}$.*

Proof. Let $\mathcal{A} = \{(\Delta_\alpha, \varphi_\alpha) : \alpha \in A\}$ be an atlas of coordinate disks for X. We first define the required function on each coordinate disc Δ_α.
Suppose Δ_α is a coordinate disk such that $q_0 \notin \Delta_\alpha$. Since $g(p, q_0)$ is harmonic in Δ_α, $g \circ \varphi_\alpha^{-1}$ is a real-valued harmonic function on $\mathbb{D}$. Define $G_\alpha : \mathbb{D} \to \mathbb{C}$ such that,

$$\mathrm{Re}(G_\alpha) = g \circ \varphi_\alpha^{-1}.$$

Let $H_\alpha : \Delta_\alpha \to \mathbb{C}$ be given by $H_\alpha = G_\alpha \circ \varphi_\alpha$. Thus we have,

$$\mathrm{Re}(H_\alpha) = \mathrm{Re}(G_\alpha \circ \varphi_\alpha) = \mathrm{Re}(G_\alpha) \circ \varphi_\alpha = g$$

Now if we consider the function $F_\alpha : \Delta_\alpha \to \mathbb{C}$, given by $F_\alpha(p) = e^{-H_\alpha(p)}$, we can check that $|F_\alpha(p)| = e^{-g(p,q_0)}$.
We now consider charts of the form $(\Delta_\alpha, \varphi_\alpha)$ such that $q_0 \in \Delta_\alpha$. In this case, the function f given by $f(p) = g(p) + \log|\varphi_\alpha(p) - \varphi_\alpha(q)|$ is real harmonic in Δ_α. We repeat the same procedure as in the previous case to obtain a function $H_\alpha : \Delta_\alpha \to \mathbb{C}$ such that $\mathrm{Re}(H_\alpha) = f$. One can check that in this case if we define $F_\alpha : \Delta_\alpha \to \mathbb{C}$ as $F_\alpha(p) = (\varphi_\alpha(p) - \varphi_\alpha(q))e^{-H_\alpha(p)}$, then $|F_\alpha| = e^{-g}$ as required.

Thus, we have constructed the required function at least locally on every coordinate chart. We further observe that if $\Delta_\alpha \cap \Delta_\beta \neq \phi$ then $|F_\alpha| = |F_\beta|$ on $\Delta_\alpha \cap \Delta_\beta$ and as a result $F_\alpha = e^{i\theta} F_\beta$ and F_α works as an analytic continuation of F_β from $\Delta_\alpha \cap \Delta_\beta$ to $\Delta_\alpha \cup (\Delta_\alpha \cap \Delta_\beta)$. We now use the monodromy theorem to get the desired function defined on all of X and the proof for the lemma is complete. □

We now wish to show that the map φ guaranteed by the above lemma is injective. It suffices to show that for each $q_1 \in X, \varphi(q_1)$ is attained only once at q_1. Let A be an automorphism of the unit disc which exchanges 0 and $\varphi(q_1)$ and $\Psi = A \circ \varphi$ i.e.,

$$\Psi(p) = \frac{\varphi(p) - \varphi(q_1)}{1 - \overline{\varphi(q_1)}\varphi(p)}$$

Clearly, Ψ is analytic on X with $\Psi(q_1) = 0$. It suffices to show that Ψ has no zeroes on $X \setminus \{q_1\}$. We will show that for each p in $X \setminus \{q_1\}$, $\log|\Psi(p)| = -g(p,q)$ and as a result $|\Psi(p)| = e^{-g(p.q_1)}$ which is never zero.
Consider any $u \in \mathcal{F}_{q_1}$. We know that $u + \log|\Psi|$ is subharmonic on X. Since u vanishes outside a compact subset of X and $|\Psi(p)| < 1$, we obtain that $u + \log|\Psi| < 0$ outside a compact subset of X.
By maximum principle, for every $p \in X$,

$$u(p) + \log|\Psi(p)| < 0$$

We now take supremum over $u \in \mathcal{F}_q$ to obtain,

$$g(p, q_1) + \log|\Psi(p)| \leq 0$$

We would like to obtain equality in the previous inequality. If we show equality at any one point, we are done by maximum principle. We check at the point $q_0 \in X$.

$$\begin{aligned} g(q_0, q_1) + \log|\Psi(q_0)| &= g(q_0, q_1) - \log|-\varphi(q_1)| \\ &= g(q_0, q_1) - g(q_1, q_0) \\ &= 0. \end{aligned}$$

Thus, $\log|\Psi(p)| = -g(p, q_1)$ for all p, and our proof for the injectivity of φ is complete. As a result, φ maps X conformally into $\mathbb{D}$. The Riemann mapping theorem guarantees that this region can be mapped into all of $\mathbb{D}$ conformally and we obtain that X is conformally equivalent to $\mathbb{D}$.

Case 2: Green's function for X does not exist.

We first prove a lemma which will be of use in this case.

Lemma 3.3. *If the Green's function for a Riemann surface X does not exist, then every bounded analytic function on X is constant.*

Proof. We prove the contrapositive of the above statement. Let f be a non-constant bounded analytic function on X. We show that for every $q \in X, g(p,q)$ exists.
Consider any $q \in X$. After composing with a suitable biholomorphic map, we may assume

$f : X \to \mathbb{D}$ with $f(q) = 0$.
Consider any $u \in \mathcal{F}_q$, We know that $u(p)$+$\log|f(p)|$ is subharmonic on X and $u(p) + \log|f(p)| < 0$ outside some compact subset of X.
By Maximum Principle, $u(p) + \log|f(p)| < 0$ on X, and as a consequence $u(p) < \ln|f(p)|$. This gives an upper bound on members of the Perron family $\mathcal{F}_q$. Thus $g(p, q)$ exists. □

We now return to the main proof where we use the Biploar Green's function on X to construct a suitable conformal map.
Fix any $q_1, q_2 \in X$ with $q_1 \neq q_2$. Similar to the construction in Case 1, we find a meromorphic function φ on X such that for all $p \in X$, $\varphi(p) = e^{-G(p.q_1,q_2)}$, where $G(p.q_1, q_2)$ is the Bipolar Green's function with poles at q_1 and q_2. This function φ has a simple zero at q_1 and a simple pole at q_2. Furthermore, since G is bounded outside disks B_1, B_2 centered around q_1, q_2 we obtain a $C > 0$ such that,

$$\tfrac{1}{C} \leq |\varphi(p)| \leq C, \text{ for every } p \in X \setminus (B_1 \cup B_2)$$

We now claim that the map φ constructed above is injective. Consider any $q_0 \in X$. We show that $\varphi(q_0)$ is attained only once at q_0.
Let $G(p, q_0, q_2)$ be the Bipolar Green's function with poles at q_0 and q_2 and φ_0 be a meromorphic function such that $|\varphi_0(p)| = e^{-G(p,q_0,q_2)}$. Consider the function Ψ is given by,

$$\Psi(p) = \frac{\varphi(p) - \varphi(q_0)}{\varphi_0(p)}$$

The poles at q_2 cancel and Ψ is a bounded analytic function. By the previous lemma, it must be constant. Since Ψ does not vanish at q_2, we conclude that φ attains the value $\varphi(q_0)$ only once at q_0. As a result, φ maps X conformally onto a simply connected region of $\mathbb{C}_\infty$.
If $\varphi(X)$ is all of $\mathbb{C}_\infty$, we conclude that X is conformally equivalent to $\mathbb{C}_\infty$. If $\mathbb{C}_\infty \setminus \varphi(X)$ consists of one point, we use a fractional linear transformation to map it to the complex plane conformally. We note that $\mathbb{C}_\infty \setminus \varphi(X)$ cannot consist of more than one point, since if that were the case, we could map $\varphi(X)$ conformally to $\mathbb{D}$ and then the existence of the Green's function on $\mathbb{D}$ would imply the existence of Green's function for X, which is absurd.

This completes the proof for this case too. □

4 Complex Line Bundles

Definition 4.1. Let M be a manifold. A line bundle L is a manifold together with a smooth map $\pi : L \to M$ such that
(i) For each $p \in M, \pi^{-1}(p)$ has a $\mathbb{C}$- vector space structure.
(ii) There exists an open cover $\{U_\alpha\}_{\alpha \in A}$ and diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{C} \\ & {}_{\pi}\searrow \quad \swarrow_{p_1} & \\ & U_\alpha & \end{array}$$

Remark 4.1. 1. Thus, for each $p \in U_\alpha$ we have a map

$$\phi_{\alpha,p} : \pi^{-1}(p) \longrightarrow \{p\} \times \mathbb{C}$$

which is a $\mathbb{C}$-linear isomorphism.
2. Whenever $U_\alpha \cap U_\beta \neq \phi$ we have maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$
$$p \mapsto \phi_{\alpha,p} \circ \phi^{-1}_{\beta,p}(1)$$

which are called transition functions.
3. In light of (ii) in Definition 1.1, every line bundle is locally trivial.
4. We observe that $g_{\alpha\alpha} = 1$ on U_α and $g_{\alpha\beta} \cdot g_{\beta_\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. These are referred to as cocycle conditions. In fact, a line bundle is completely determined by the following data - an open cover $\{U_\alpha\}_{\alpha \in A}$, a collection of transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$.

Definition 4.2. Let $\pi : L \to M$ be a line bundle. A section is a smooth map $s : M \to L$ such that $\pi \circ s = Id_M$. The vector space of all sections of L, is denoted by $\Gamma(L)$.

As a result, every section carries a point p on the manifold to a member of its fiber. Furthermore, every section being injective provides a copy of M inside L. In fact, every line bundle admits a section, namely the *zero section.* The zero section carries each point $p \in M$ to the 0 of $\pi^{-1}(p)$. It is easy to check that the map so defined is a smooth map from M to L, and hence a smooth section.

4.1 Local frames.

We now introduce the notions of local sections and local frames and then use them to get a necessary and sufficient condition for a line bundle to be trivial.

If $L \to M$ is a complex line bundle and $U \in M$ is an open set, then $\pi^{-1}(U) \to M$ is itself a line bundle, denoted by $L|_U$. A *frame* for L over U is a member of $\Gamma(L|_U)$ having no zeroes.

Theorem 4.1. *A line bundle is trivial iff it has a global frame.*

Proof. We make the following observation: If L has a frame over U say ζ, then the map

$$v \mapsto (\pi(v), \frac{v}{\zeta(\pi(v)})$$

yields an isomorphism $L|_U \to U \times \mathbb{C}$. and establishes the triviality of $L|_U$. In particular, if L has a global frame over M, we get that L is the trivial bundle.
Conversely, if L is the trivial line bundle over M, we have a map $\phi : L \to M \times \mathbb{C}$. we use this map to construct a global frame for L. Consider the section given by $\zeta(p) = \phi^{-1}(p, 1)$. Since $\phi_p : \pi^{-1}(p) \to \{p\} \times \mathbb{C}$ is a linear isomorphism, we obtain that ζ is nowhere vanishing and hence a frame. □

Remark 4.2. It is now clear that every section can be locally expressed as some smooth function times the frame on every trivializing chart i.e., $s = s|_U \zeta_U$, where s_U is smooth.

Definition 4.3. Let L and X be complex manifolds and $\pi : L \to X$ be a complex line bundle. We say L is a holomorphic line bundle if π is holomorphic and the local trivialization maps are all holomorphic.

Note that given line bundles L, L' on M, we can construct new line bundles on M given by $L \otimes L'$ and L^*. The reader may refer to [4] for an explicit description of these line bundles. However, we note that the operation of taking tensor products of line bundles gives a group structure to the set of all holomorphic line bundles on M. We shall henceforth refer to this group at the Picard group of M, denoted by $\text{Pic}(M)$. We fix some notation here before moving on to examples of line bundles.

Definition 4.4. A section of a holomorphic line bundle is said to be holomorphic if it is holomorphic as a map from M to L. The collection of all holomorphic sections of L is denoted by $\Gamma_{\mathcal{O}}(L)$.

One can similarly define the notion of meromorphic sections of a line bundle, denoted by $\Gamma_{\mathcal{M}}(L)$.

4.2 Examples of line bundles

4.2.1 Complex tangent bundle of a Riemann surface

The complex tangent bundle of a Riemann surface X, is denoted by $TX = \sqcup_{p \in X} T_pX$. Here T_pX refers to the space of all derivations on smooth functions around p. We have a natural map $\pi : TX \to X$. Suppose f is a smooth function around p and (U, z) is a chart around p, the derivation $\frac{\partial}{\partial z}|_p \in T_pX$. Here n $\frac{\partial}{\partial z}|_p f = \frac{\partial}{\partial w}|_p (f \circ z^{-1})$ where $w = x + iy$ and $\frac{\partial}{\partial w}$ is the first Wirtingler operator.

TX admits local trivializations over any atlas of X. Let $\{U_\alpha, z_\alpha\}_{\alpha \in A}$ be an atlas for X. The maps $\phi_\alpha : \pi^{-1}(U_\alpha) : U_\alpha \times \mathbb{C}$ given by $\phi_\alpha(v) = (\pi(v), v(z_\alpha))$ endow TX with a line bundle structure over X.

4.2.2 Cotangent bundle of a Riemann surface

The dual of the tangent bundle over a Riemann surface is referred to as the cotangent bundle. This line bundle is also referred to as the canonical bundle and is denoted by K_X.

4.2.3 Tautological bundle over $\mathbb{P}^n$

Consider the set given by

$$\mathbb{E} = \{(z, l) \in \mathbb{C}^{n+1} \times \mathbb{P}^n : z \in l\}$$

We have a natural map $\pi : \mathbb{E} \to \mathbb{P}^n$ which is the restriction of the usual projection map $p_2 : \mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$. For each $l \in \mathbb{P}^n, \pi^{-1}(l)$ is the set of points that lie on l and can be naturally given a vector space structure. We give $\mathbb{E}$ the structure of a complex line bundle over $\mathbb{P}^n$. $\mathbb{E}$ admits local trivializations over the canonical atlas of $\mathbb{P}^n$, where $\mathbb{P}^n$ has an open cover given by $\{U_i : 1 \leq i \leq n+1\}$ with $U_i = \{[x_0 : \cdots : x_n + 1] : x_i \neq 0\}$. These local trivializations are given by

$$\begin{aligned} \varphi_i : \pi^{-1}(U_i) &\to U_i \times \mathbb{C} \\ (z, l) &\mapsto (l, z_i) \end{aligned}$$

where $z = (z_1, \ldots, z_{n+1})$.
This line bundle is commonly referred to as the Tautological line bundle over $\mathbb{P}^n$.

4.2.4 Hyperplane bundle

The hyperplane bundle, denoted by $\mathbb{H}$ is defined as the dual of the Tautological bundle i.e., $\mathbb{H} = \mathbb{E}^*$. In particular, we are interested in the space of sections of the hyperplane bundle.
Let $\tau : \mathbb{H} \to \mathbb{P}^n$ denote the projection map. It is clear that for each $l \in \mathbb{P}^n, \tau^{-1}(l)$ consists

of linear functionals on the one-dimensional subspace of $\mathbb{C}^{n+1}$ represented by the line l. Any linear functional on $\mathbb{C}^{n+1}$, yields a section of $\mathbb{H}$.
In fact, any global section of $\mathbb{H}$ comes from a linear functional on $\mathbb{C}^{n+1}$. A proof of the same can be found in [6]. As a result we have

$$\mathbb{C}^{n+1} \cong \Gamma_{\mathcal{O}}(\mathbb{H}).$$

5 Divisors

We now introduce the notion of divisors on a Riemann surface which attempt to at least locally capture the information regarding zeroes and poles of a meromorphic function or 1-form on a Riemann surface. A formal definition of divisors is as under.

Definition 5.1. A divisor on X is a map $D : X \to \mathbb{Z}$ which vanishes outside a discrete subset of X. The set of points of X where D is non-zero is referred to as the support of F. We shall henceforth use the formal notation for a divisor given by

$$D = \Sigma_{p \in X} D(p)p.$$

Definition 5.2. Let X be a compact Riemann surface and D be a divisor on X, then the support of D is a finite subset of X and we define the degree of D, denoted by $\deg(D) := \Sigma_{p \in X} D(p)$.

We observe that the set of all divisors on X, denoted by $\mathrm{Div}(X)$ forms an abelian group under pointwise addition. As a result, we obtain a group homomorphism given by

$$\begin{aligned} \varphi : \mathrm{Div}(X) &\to \mathbb{Z} \\ D &\mapsto \deg(D) \end{aligned}$$

5.1 Examples of Divisors

5.1.1 Principal Divisors

Let f be a non-zero meromorphic function on X. Then we have the following divisor on X given by $\mathrm{div}(f)$

$$\mathrm{div}(f) = \Sigma_{p \in X} \mathrm{ord}_p(f)p.$$

Any divisor of this form is called a Principal divisor on X. We denote the collection of all principal divisors on X as $\mathrm{PDiv}(X)$.

It is easy to verify the following properties of principal divisors on X.
Let f, g be non-zero meromorphic functions on X. Then,
1. $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.
2. $\mathrm{div}(f/g) = \mathrm{div}(f) - \mathrm{div}(g)$.

Proposition 3.1 can now be rephrased as - For any non-zero meromorphic function f on a Riemann surface X, we have $\deg(\mathrm{div}(f)) = 0$.

5.1.2 Canonical Divisors

Let ω be a non-zero meromorphic 1-form on X. Then we have the following divisor on X given by

$$\operatorname{div}(\omega) = \Sigma_{p\in X} \operatorname{ord}_p(\omega)p$$

Any divisor of this form is called a canonical divisor on X and we denote the collection of canonical divisors on X by $\operatorname{KDiv}(X)$.

Remark 5.1. If f is a meromorphic function and ω is a meromorphic 1-form, then $f\omega$ is a meromorphic 1-form. Moreover, $\operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega)$. This shows that the sum of a canonical divisor with a principal divisor is again a canonical divisor. This leads us to the question if the difference of two canonical divisors is always principal. The answer is in the affirmative. The reader may refer to [5] for a proof of the same. In the next section we shall introduce the notion of linear equivalence of divisors and return back to this observation.

5.1.3 Inverse Image Divisors

Let $F : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Let $q \in Y$. Then $F^*(q)$ is a divisor on X given by

$$F^*(q) = \Sigma_{p\in F^{-1}(q)} \operatorname{mult}_p(F)p$$

We observe that if X, Y are compact then $\deg(F^*(q))$ is independent of q and is the same as $\deg(F)$.

We can in fact generalize this construction further to pullback any divisor on Y to obtain a new divisor on X. Let $D = \Sigma_{q\in Y} D(q)q$ be a divisor on Y, then $F^*(D)$ is the divisor on X given by

$$F^*(D) = \Sigma_{q\in Y} D(q)F^*(q).$$

If we think of divisors as functions, we can rewrite this as

$$F^*D(p) = \operatorname{mult}_p(F)D(F(p)).$$

Proposition 5.1. *Let $F : X \to Y$ be a non-constant holomorphic function between Riemann surfaces. Then*
1. $F^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ is a group homomorphism.
2. Principal divisors pull back to principal divisors i.e.,

$$F^*(\operatorname{div}(f) = \operatorname{div}(f \circ F))$$

Proof. 1. Consider any $p \in X$

$$\begin{aligned} F^*(D+D')(p) &= \text{mult}_p(F)(D(p)+D'(p)) \\ &= \text{mult}_p(F)D(p) + \text{mult}_p(F)D'(p) \\ &= F^*D(p) + F^*D'(p). \end{aligned}$$

Thus $F^*(D+D') = F^*D + F^*D'$, which proves 1.

2. Consider any $p \in X$

$$\begin{aligned} F^*(\text{div } f)(p) &= \text{mult}_p(F) \,\text{div}\, f(F(p)) \\ &= \text{mult}_p(F) \,\text{ord}_{F(p)}(f) \\ &= \text{ord}_p(f \circ F) \\ &= \text{div}(f \circ F)(p) \end{aligned}$$

□

5.2 Linear Equivalence of Divisors

Definition 5.3. Let D_1, D_2 be divisors on X. We say that D_1 is linearly equivalent to D_2 if $D_1 - D_2 \in \text{PDiv}(X)$.

Remark 5.2.
- 1. It is easy to see that linear equivalence is an equivalence relation on $\text{Div}(X)$. In the next section, we try to identify the collection of divisors on X, modulo linear equivalence.
- $D \sim 0$ if and only if D is a Principal Divisor.
- As a consequence of Proposition 3.1, on a compact Riemann surface $X, D_1 \sim D_2$ implies that $\deg(D_1) = \deg(D_2)$.

5.3 Examples of linearly equivalent divisors

Example 5.1. Let f be a non-zero meromorphic function on X. Then we can look at the divisors of zeroes and poles of f, denoted by $\text{div}_0(f)$ and $\text{div}_\infty(f)$, respectively. It is easy to see that these two divisors are linearly equivalent since their difference is the Principal Divisor given by $\text{div}(f)$.

Example 5.2. Remark 4.1 says that any two canonical divisors on X are linearly equivalent. In fact, any divisor linearly equivalent to a canonical divisor must itself be canonical.

Example 5.3. Any two point divisors on $\mathbb{C}_\infty$ are linearly equivalent. (Here by a point divisor we mean the divisor which associates the integer 1 to a fixed point and 0 to every other point on the Riemann surface.) We give a quick justification for this - Let D_1, D_2 be point divisors corresponding to the points λ_1 and λ_2. Suppose neither of them is equal to ∞, then $f(z) = (z - \lambda_1)/(z - \lambda_2)$ is a meromorphic function such that $\mathrm{div}(f) = D_1 - D_2$. If one of the points, say λ_1 is ∞, we use $f(z) = z - \lambda_2$.

Example 5.4. Let $F : X \to Y$ be a holomorphic map. Then F pulls back linearly equivalent divisors on Y, to linearly equivalent divisors on X i.e., if $D_1 \sim D_2$ on Y, then $F^*D_1 \sim F^*D_2$ on X. This is immediate since if $D_1 - D_2 = \mathrm{div}(f)$, for some meromorphic function f on Y, then $F^*(D_1) - F^*(D_2) = F^*(D_1 - D_2) = F^*(\mathrm{div}(f)) = \mathrm{div}(f \circ F)$.

5.4 Line Bundle - Divisor correspondence

In this section, we shall explore the links between Line Bundles and Divisors on a Riemann surface. The main result of this section will be to identify the collection of divisors on X modulo linear equivalence.

5.4.1 Line bundle associated to divisor.

Let $D \in \mathrm{Div}(X)$. We fix an atlas $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ such that $U_\alpha \subset\subset X$, for each $\alpha \in A$. Now, for each α we can fix a meromorphic function $f_\alpha \in \mathcal{M}(U_\alpha)$ such that

$$\mathrm{Ord}(f_\alpha) = D|_{U_\alpha} = \Sigma_{p \in U_\alpha} D(p) \cdot p$$

For instance we could take $f_\alpha = \prod_{p \in U_\alpha} (z - p)^{D(p)}$.

Thus we get a collection of functions $\{f_\alpha\}_{\alpha \in A}$ such that for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \phi$ we obtain a holomorphic function $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ given by $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$. (Here the notation $\mathcal{O}(V)$ denotes the collection of all holomorphic functions on V.) It is easy to see that $g_{\alpha\beta}$ satisfy the cocycle condition. By (4) in Remark 2.1 we know that there exists a line bundle L_D admitting local trivializations over $\{U_\alpha\}_{\alpha \in A}$. with transition functions given by $g_{\alpha\beta}$.
Moreover, since $g_{\alpha\beta} f_\beta = f_\alpha$, we can use the functions $\{f_\alpha\}_{\alpha \in A}$ to obtain a meromorphic section of L_D, say $s_D \in \Gamma_{\mathcal{M}}(L_D)$. We define $s_D = f_\alpha \zeta_\alpha$ on each U_α, where ζ_α is the local frame over U_α. It is easy to see that $\mathrm{ord}(s_D) = D$. Thus, every divisor on a Riemann surface X is a canonical divisor with respect to some line bundle on X. We summarize the discussion here in the form of the following lemma.

Lemma 5.1. *Let $D \in \mathrm{Div}(X)$, then there exists $L_D \in \mathrm{Pic}(X)$ and $s_D \in \Gamma_{\mathcal{M}}(L_D)$ such that $D = \mathrm{ord}(s_D)$.*

5.4.2 Divisor associated to a line bundle with a meromorphic section

Let L be now a line bundle that admits a non-zero meromorphic section s. We have a divisor given by $\text{Ord}(s)$. We now ask a question motivated from the discussion in the previous section. If we construct $L_{\text{Ord}(s)}$, is it the same as L? The answer is in the affirmative. The interested reader can find a proof of this in [6]. We also have a relation between s and $s_{\text{Ord}(s)}$. We summarize the same in the following lemma.

Lemma 5.2. *Let X be a Riemann surface and $L \to X$ be a line bundle over X with a non-zero meromorphic section $s \in \Gamma_{\mathcal{M}}(L)$, then $L_{\text{Ord}(s)} = L$ and $s_{\text{Ord}(s)}$ is a nowhere zero multiple of s.*

The discussion from above shows that we have a map

$$\begin{aligned} \psi : \text{Div}(X) &\longrightarrow \text{Pic}(X) \\ D &\longmapsto L_D \end{aligned}$$

It is easy to check that $L_{D+D'} = L_D \otimes L'_D$ and $L_{-D} = L^*_D$ which shows that ψ is a group homomorphism.
We now investigate the kernel of ψ.

$$\ker(\psi) = \{D \in \text{Div}(X) : L_D \cong X \times \mathbb{C}.\}$$

The triviality of L_D implies that $D = \text{Ord}(f)$ for $f \in \mathcal{M}(X)$. A proof of this can be found in [6]. This shows that the kernel of ψ is exactly the group of Principal Divisors of X.

We now check if the map ψ is surjective. In the light of Lemma 4.2, it suffices to know if every line bundle on X admits a non-zero meromorphic section. The answer to this question is in the affirmative, although its proof is beyond the scope of this report.

The first isomorphism theorem for groups now yields that

$$\frac{\text{Div}(X)}{\text{PDiv}(X)} \cong \text{Pic}(X).$$

Theorem 5.1. *On a Riemann surface X, the set of holomorphic line bundles is in bijective correspondence with the set of divisors modulo linear equivalence.*

6 Projective maps

In this section, we define projective maps. For the rest of this section let $H \to X$ be a holomorphic line bundle over a Riemann surface X, and let W be a finite-dimensional subspace of $\Gamma_{\mathcal{O}}(H)$. The collection of basepoints of W is defined as under

$$\mathrm{Bs}(W) := \{p \in X : s(p) = 0, \forall s \in W\}$$

For every $x \in X \setminus \mathrm{Bs}(W)$, we have a proper subspace of W given by

$$\varphi_W(x) := \{s \in W : s(x) = 0\}$$

In fact, $\varphi_W(x)$ is a hyperplane in W. One can show this by establishing $\varphi_W(x)$ as the kernel of non-trivial functional on W. In particular, if we fix $f \in H_x^* \setminus \{0\}$, the map given by

$$s \mapsto f(s(x))$$

shows that $\varphi_W(x)$ is a hyperplane in W. There is a bijective correspondence between hyperplanes in W and members of $\mathbb{P}(W^*)$, where $\mathbb{P}(V)$ is the notation for the projectivization of a vector space V. The reader may refer to [6] for an elaborate description of this bijective correspondence. As a result, we obtain a map

$$\varphi_W : X \setminus \mathrm{Bs}(W) \to \mathbb{P}(W^*)$$

We refer to such maps as projective maps.

6.1 Describing projective maps using a basis of W

We now fix a basis for W, say $s_0, \ldots, s_n$ and we use it to give a better description of the map φ_W. We recall that for any finite-dimensional vector space V, if we fix a basis $f_1, \ldots, f_m$ of V^*, we can use it to give coordinates on V. via the map

$$v \mapsto (f(v_1), \ldots, f(v_n))$$

So now if we treat s_i as members of W^{**}, we have coordinates on W^* and we can use them to identify $\mathbb{P}(W^*)$ with $\mathbb{P}^n$. If we fix $\zeta \in H_x \setminus \{0\}$, we obtain the following functional on W.

$$Ev : W \to \mathbb{C}$$
$$s \mapsto s(x)/\zeta$$

It is easy to see that the kernel of Ev is the same as the subspace φ_W described above. By our earlier observation on the induced coordinates on W^*, the map Ev has coordinates given by $[s_0(x)/\zeta, \ldots, s_n(x)/\zeta]$. Hence, it is customary to use the notation

$$\varphi_W(x) = [s_0(x), \ldots, s_n(x)]$$

We now make an important observation about projective maps, namely that every holomorphic map from X to $\mathbb{P}^n$ can be thought of as a projective map.

Theorem 6.1. *Let $\psi : X \to \mathbb{P}^n$ be a holomorphic map. Then $\psi = \varphi_W$ for some $W \subset \Gamma_{\mathcal{O}}(H)$, where H is a holomorphic line bundle over X.*

Proof. Let $H := \psi^*\mathbb{H}$, where $\mathbb{H}$ denotes the hyperplane bundle over $\mathbb{P}^n$ as described in section 4. Let $s_j := \psi^*(z_j)$ where z_j denotes the homogeneous coordinates on P_n. Let $W =$span$\{s_0, \ldots, s_n\}$. One can check that $\psi = [s_0, \ldots, s_n] = \varphi_W$. □

7 Solving $\bar{\partial}$ for smooth data

In this section, we wish to show that every $(0,1)$ form on an open Riemann surface X is always of the form $\bar{\partial} f$ for some smooth function f on X. We shall then see some consequences of this result. The reader is suggested to go through Appendix II which reviews notation regarding complex differential forms. A local version of this result, stated below can be proved using Green's functions. We refer the reader to Theorem 7.1.18 of [6] for a proof of the same.

Theorem 7.1. *Let X be a Riemann surface and $Y \subset\subset X$ be an open, proper subset of X. Then for every $(0,1)$ form ω on X, there exists a smooth function g on X such that $\bar{\partial} g = \omega$ on Y.*

In order to prove a global version of the above result, we need to develop some machinery regarding holomorphic hulls and Runge domains. We devote the initial part of this section to doing the same.

7.1 Regular Exhaustion on open Riemann surfaces

Definition 7.1. Let $Y \subset X$. The holomorphic hull of Y in X, denoted by $\hat{Y}_{\mathcal{O}(X)}$ is defined as under

$$\hat{Y}_{\mathcal{O}(X)} = Y \cup_{C \in \mathcal{C}} C$$

where $\mathcal{C}$ is the collection of connected components of $X \setminus Y$ which are precompact in X.

Remark 7.1. It is crucial to note that the holomorphic hull depends on the choice of the ambient Riemann surface that Y lives in, For instance if we take $Y = S^1 \subset \mathbb{C}$, then $\hat{Y}_{\mathcal{O}(\mathbb{C})} = \overline{\mathbb{D}}$, where $\mathbb{D}$ denotes the open unit disk. On the other hand, $\hat{Y}_{\mathcal{O}(\mathbb{C}\setminus\{0\})} = S^1$. If an open set Y equals its holomorphic hull in X, we say Y is *Runge* in X. The main result of this section is to prove that an open Riemann surface can be 'exhausted' by relatively compact, open, Runge sets. To prove this result, we first look at some properties of holomorphic hulls.

Theorem 7.2. *If $Y \subset X$ is closed (respectively compact), then so is $\hat{Y}_{\mathcal{O}(X)}$.*

Proof. We first prove the assertion about closedness. If Y is closed, $X \setminus Y$, being an open subset of a manifold, admits a manifold structure. Hence $X \setminus Y$ is locally connected, which implies that each of its connected components is open. The complement of $\hat{Y}_{\mathcal{O}(X)}$ consists of those components of $X \setminus Y$ which are not precompact in X. Since each of these components is open, the complement of $\hat{Y}_{\mathcal{O}(X)}$ is open, and consequently, $\hat{Y}_{\mathcal{O}(X)}$ is closed.

We now prove the assertion about compactness. By the previous argument for any compact set $Y \subset X$, we already know that $\hat{Y}_{\mathcal{O}(X)}$ is closed. In order to show that $\hat{Y}_{\mathcal{O}(X)}$

is compact, it suffices to show that it is a subset of some precompact set.

We fix a precompact neighborhood V of Y and let V_j denote the components of $X \setminus Y$. We first claim that each V_j intersects $\overline{V}$ non-trivially. Suppose this were not the case. Then for some V_j,

$$V_j \subset X \setminus \overline{V} \subset X \setminus V$$

We take closure in X on both sides to obtain that $\bar{V_j} \subset X \setminus V \subset X \setminus Y$. But since V_j is a connected component we obtain $\bar{V_j} = V_j$ and V_j is closed in X. On the other hand, V_j is also open in X since it is a component of $X \setminus Y$, which is locally connected. This is not possible since X is connected and hence our initial claim that each V_j meets $\overline{V}$ is justified.

Now, since ∂V is compact and all of the V_j's are disjoint, we conclude that only finitely many of them cover ∂V, while the remaining must be contained in V, since they must anyway meet $\overline{V}$ by our earlier observation.

Let I_0 now be the set of all those $i \in I$ for which V_i is precompact. We now know that there exists a finite set $F \subset I_0$ such that for all $k \in F$, $V_k \cap \partial V \neq \phi$, and for all $k \in I_0 \setminus F$, $V_k \subset V$. Consequently,

$$\hat{Y}_{\mathcal{O}(X)} \subset V \cup_{k \in F} V_k$$

Since the set on the right side is precompact, the proof is complete. □

We know that every manifold, by virtue of being a second-countable topological space, admits a compact exhaustion. We use this compact exhaustion, coupled with the above properties to move a step closer towards the promised result of exhaustion by means of precompact Runge open sets.

Theorem 7.3. *On any open Riemann surface X, there exists a sequence $M_k \subset X$ of compact subsets such that $\hat{M}_{k\mathcal{O}(X)} = M_k$ for all k, $M_i \subset$ Interior(M_{i+1}) for all i and $X = \cup_i M_i$.*

Proof. Let $\{F_i : i \in \mathbb{N}\}$ be the collection that provides a compact exhaustion for X. We use this collection to construct the required collection M_i inductively. Set $M_0 = \phi$ and $M_1 = \hat{F}_{1\mathcal{O}(X)}$. Now suppose $M_2, \ldots, M_k$ having desired properties have been chosen. Let J_{k+1} be a compact set that contains $M_k \cup F_k$ in its interior. We set $M_{k+1} = \hat{J}_{k+1\mathcal{O}(X)}$ and that completes the proof. □

In order to achieve an exhaustion of X by means of *open* Runge, precompact sets we would like to 'refine' the exhaustion obtained by the above theorem. The next lemma provides us with the necessary tools for the same.

Lemma 7.1. *Let M_1, M_2 be compacts subsets of X with $M_1 \subset Interior(M_2)$, and $\hat{M}_{i\mathcal{O}(X)} = M_i$ for $i = 1, 2$. Then there exists an open subset $U \subset X$ such that $\hat{U}_{\mathcal{O}(X)} = U$ with $M_1 \subset U \subset M_2$ and U has a smooth boundary.*

Proof. We lay out a rough sketch of this proof. The reader may refer to ?? for complete details. We cover ∂M_2 by finitely many closed disks $D_1, \ldots, D_N$ which are chosen in such a way that $D_i \cap M_1 = \phi$ for all i. We choose $U = M_2 \setminus (D_1 \cup \cdots \cup D_N)$. Since the disks cover ∂M_2, it is easy to see that $U = Int(M_2) \setminus (D_1 \cup \cdots \cup D_N)$, and consequently U is open. We leave it to the reader to check that U satisfies the requirements of the lemma. □

As an immediate consequence, we obtain the following exhaustion of open Riemann surfaces promised at this section's start.

Corollary 7.1. *Let X be an open Riemann surface. Then there exist relatively compact, open Runge subsets $U_1 \subset\subset U_2 \subset\subset \ldots$ such that each U_i has a smooth boundary and $\cup_i U_i = X$.*

7.2 Consequences of Behnke-Stein Runge Theorem

In this section, we first state the Behnke-Stein Runge Theorem. A proof of this result is beyond the scope of this project and we shall not see a proof of it. Nevertheless, we shall see some consequences of this result.

Theorem 7.4 (Behnke-Stein Runge theorem)**.** *For any open Riemann surface X and any $Y \subset\subset X$ where Y is a Runge open set, $\mathcal{O}(X)|_Y$ is dense in $\mathcal{O}(Y)$, where $\mathcal{O}(Y)$ is given the compact-open topology.*

As a first consequence of this theorem, we use it to give a function-theoretic description of the holomorphic hulls described in the previous section.

Proposition 7.1. *Let F be a compact subset of an open Riemann surface X. Then*

$$\hat{F}_{\mathcal{O}(X)} = \{x \in X : |f(x)| \leq \sup_K |f|, \forall f \in \mathcal{O}(X)\} := B$$

Proof. We show mutual containment for the two sets described above. Let $x \in \hat{F}_{\mathcal{O}(X)}$. If $x \in F$ we are done. If not, $x \in U$ where U is a relatively compact component of $X \setminus F$. Consequently $\partial U \subset F$. We now use the Maximum Principle to conclude that for each $f \in \mathcal{O}(X)$, we have $|f(x)| \leq \sup_{\partial U} |f| \leq \sup_K |f|$. This shows one-way containment.
For the other way containment, we show that $(\hat{F}_{\mathcal{O}(X)})^C \subset B^C$, (where A^C denotes the complement of a set A). Let $x \notin \hat{F}_{\mathcal{O}(X)}$. Let U be a component of $X \setminus F$ which contains x and is not precompact in X. We now choose a coordinate disk D centered at x which

is contained in V. We leave it to the reader to check that $\hat{K}_{\mathcal{O}(X)} \cup D$ is Runge with $\hat{K}_{\mathcal{O}(X)} \cap D = \phi$. Now consider the function g, given by

$$g(x) = \begin{cases} 0, & \text{if } x \in \hat{K}_{\mathcal{O}(X)} \\ 1, & \text{if } x \in D \end{cases}$$

The Behnke-Stein Runge theorem allows us to approximate g by means of globally defined holomorphic functions. Consequently, there exists $G \in \mathcal{O}(X)$ such that,

$$|G(x)| > 1/2 > \sup_K |G|$$

This shows that $x \notin B$ and our proof is complete. □

The Behnke-Stein Runge theorem also allows us to strengthen Theorem 7.1. We state and give a proof of the same here.

Theorem 7.5. *Let X be an open Riemann surface and ω be a $(0,1)$ form on X. Then there exists a smooth function $f : X \to \mathbb{C}$ such that $\omega = \bar{\partial} f$.*

Proof. We fix a normal exhaustion of X given by $\{U_i\}_{i\in\mathbb{N}\cup\{0\}}$. Theorem 7.1 guarantees us the existence of functions $\{h_i\}_{i\in\mathbb{N}\cup\{0\}}$ such that $\bar{\partial} g_i = w$ on each U_i. We use these functions to construct a sequence $\{f_i : U_i \to \mathbb{C}\}_{\mathbb{N}} \cup \{0\}$ such that

$$\bar{\partial} f_i = \omega \text{ on } U_{i-1} \qquad \text{and} \qquad \sup_{U_{i-1}} |f_{i+1} - f_i| < \frac{1}{2^i}.$$

We take $f_0 = 0$ and $f_1 : U_1 \to \mathbb{C}$ to be any smooth function such that $\bar{\partial} f_1 = \omega$ on Y_0. Now suppose functions $f_2, \ldots, f_n$ having the desired properties have been chosen, we make a choice for f_{n+1}.

We observe that $f_n - h_n \in \mathcal{O}(U_i)$, since $\bar{\partial}(f_i - h_i) = \bar{\partial}(f_n) - \bar{\partial}(h_n) = \omega - \omega = 0$. By Behnke-Stein Runge theorem, there exists a global holomorphic function $h \in \mathcal{O}(X)$ such that $\sup_{U_i} |f_n - h_n - h| < \frac{1}{2^n}$. We now choose $f_{n+1} = h_n - h$. It is easy to see that $\bar{\partial}(f_{n+1}) = \omega$ on U_n and f_n satisfies the required estimate with f_{n+1}.

We now claim that for each member U_i of the normal exhaustion, $\{f_n\}$ is a uniformly Cauchy sequence with respect to the C^1-norm. We now write $f_n = f_0 + \Sigma_{j=0}^n f_j - f_{j+1}$. However, by Behnke-Stein Runge theorem, since on Y_{k-1} for $k < n$, $f_n = f_k + h_{k,n}$ where $h_{k,n} \in \mathcal{O}(X)$, we obtain that the sup norm of $h_{k,n}$ on Y_{k-1} must be at most $\frac{1}{2^{k-1}}$. This gives us that $\{f_n\}$ is uniformly Cauchy with respect to the C^1 norm for each member U_i in the normal exhaustion. Consequently, there exists an $f \in \mathcal{O}(X)$ such that $f_n \to f$ in C^1 norm on each member of the normal exhaustion. It is easy to now see that,

$$\bar{\partial} f = \bar{\partial} \lim f_n = \lim \bar{\partial} f_n = \omega$$

This completes the proof. □

The above theorem has some interesting consequences, some of which we shall discuss here.

Theorem 7.6. *Let X be an open Riemann surface and $L \to X$ be a holomorphic line bundle. Then L must be trivial.*

Proof. In order to show that L is trivial we produce a section of L that has no zeroes. We fix an open cover $\{U_i\}_{i\in I}$ of X such that each $U_i, U_j \cap U_k$ is simply connected and $L|_{U_i}$ is trivial. For every i, let $e_i \in \Gamma_{\mathcal{O}}(L|_{U_i})$ be a nowhere vanishing section. Thus, on every intersection $U_i \cap U_j$, we obtain the holomorphic functions given by

$$h_{ij} := e_i/e_j$$

Clearly, h_{ij} is nowhere zero. Since its domain is simply connected, we obtain $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ such that $e^{g_{ij}} = h_{ij}$.

We now fix a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$, given by $\{\Psi_i\}_{i\in I}$. We use this partition of unity to define new functions on each member of the open cover,

$$p_i := \Sigma_{j\in J} \Psi_j g_{ij} \quad \text{on } \ U_i$$

These functions defined on each U_i provide us with locally defined $(0,1)$ forms given by $\bar{\partial} p_i$ on U_i. We claim that these actually provide us with a globally defined $(0,1)$ form. To this end, it suffices to show $\bar{\partial} p_i = \bar{\partial} p_j$ on $U_i \cap U_j$. We now show that $p_i - p_j$ is holomorphic. This is clear since

$$p_i - p_j = \Sigma_k \Psi_k (g_{ik} - g_{jk}) = g_{ij}$$

Thus, we have a $(0,1)$ form on X given by ω such that $\bar{\partial} p_i = \omega$ on each U_i. Now let h be any smooth function satisfying $\bar{\partial} h = \omega$, and we set

$$q_i := p_i - h$$

It is easy to see that for each $U_i, q_i \in \mathcal{O}(U_i)$ and $q_i - q_j = g_{ij}$. We now define local sections of L given by

$$s_i = e^{-q_i} e_i \ \text{ on } \ U_i$$

Clearly, s_i has no zeroes. We claim that these s_i's provide us with a global section s such that $s = s_i$ on each U_i. This is immediate since on $U_i \cap U_j$,

$$s_i = e^{-q_i} e_i = e^{-g_{ij}} e^{-q_j} e_i = e^{-q_j} e_j = s_j$$

Since, $s \in \Gamma_{\mathcal{O}}(L)$ is nowhere zero, we conclude that L must be trivial. □

As an immediate fallout of the triviality of holomorphic line bundles on open Riemann surfaces, we obtain two important consequences which we state below.

Theorem 7.7 (Weierstrass Product Theorem)**.** *Every divisor on an open Riemann surface is principal.*

Proof. Let D be a divisor on an open Riemann surface X. Let L_D and s_D be its corresponding line bundle and meromorphic section of L_D respectively, where

$$\text{Ord}(s_D) = D$$

Theorem 7.6 guarantees the existence of $s \in \Gamma_{\mathcal{O}}(L)$ such that s has no zeroes. Consider the meromorphic section given by $f = s_D/s$. We have,

$$\text{Ord}(f) = \text{Ord}(s_D) - \text{Ord}(s) = \text{Ord}(s_D) = D$$

□

Theorem 7.8. *Let X be an open Riemann surface and f be a meromorphic function on X. Then there exist $g, h \in \mathcal{O}(X)$ such that $f = g/h$ on X.*

Proof. Consider the divisor of poles of f given by $D := (f)_\infty$. Let L and s be the line bundle and section corresponding to this divisor i.e., $\text{Ord}(s) = D$. Since $D \geq 0$, s must be holomorphic.
Theorem 7.6 guarantees the existence of $t \in \Gamma_{\mathcal{O}}(L)$ such that t has no zeroes. Now consider the holomorphic function given by $h = s/t$. We observe that,

$$\text{Ord}(fh) = \text{Ord}(f) + \text{Ord}(s) = (f)_0 - (f)_\infty + (f)_\infty = (f)_0$$

Since $\text{Ord}(fh) \geq 0, fh = g$ must be holomorphic. Thus $f = g/h$. □

8 Appendix

This appendix specifies the notation regarding complex differential forms used in this thesis. We record some important line bundles which were introduced in section 3.

8.1 Canonical bundle K_X

Refer to examples in section 3.2 for the definition of the canonical bundle. The sections of this bundle are referred to as $(1, 0)$ forms. An important method of constructing $(1, 0)$ forms from smooth functions on a Riemann surface is as under.

Let $f : X \to \mathbb{C}$ be a smooth function and $\{(U_\alpha, z_\alpha) : \alpha \in A\}$ be an atlas for X. Let

$$\omega = \frac{\partial f}{\partial z_\alpha} dz_\alpha \quad \text{on } U_\alpha$$

One can check that this gives a well-defined $(1, 0)$ form on all of X, which shall be referred to as ∂f

8.2 Conjugate of the canonical bundle

Given any vector space V, it is possible to construct a new vector space $\overline{V}$ such that for any vector space W, linear maps from V to W correspond to conjugate-linear maps from $\overline{V}$ to W. If $(V, +, \cdot)$ is the original vector space, $\overline{V}$ has the same underlying set as that of V. The '+' operation of $\overline{V}$ is also the same as that of V. It is however given a new scalar multiplication structure using the operation '*', where $z * v = \overline{z} \cdot v$.

It can be checked that if $L = \bigsqcup_{p \in X} L_p$ is a line bundle over X, then $\overline{L} = \bigsqcup_{p \in X} \overline{L_p}$ can be given the structure of a line bundle over X. With this notation in place we now look at $\overline{K_X}$

The smooth sections of this bundle are referred to as $(0, 1)$ forms. As in the previous subsection, we go over the method to construct $(0, 1)$ forms from smooth functions on the Riemann surface.
Let $g : X \to \mathbb{C}$ be a smooth map on a Riemann surface with an atlas given by $\{(U_\alpha, z_\alpha) : \alpha \in A\}$. Let

$$\alpha = \frac{\partial f}{\partial \bar{z_\alpha}} d\bar{z_\alpha} \quad \text{on } U_\alpha$$

One can check that this gives a well-defined $(0, 1)$ form on X. We refer to this $(0, 1)$ form as $\bar{\partial} f$.

8.3 1-forms

1-forms are elements of $\Gamma(K_X) \oplus \Gamma(\overline{K_X})$. Thus, locally every 1-form can be represented as under,

$$\omega = f_\alpha dz_\alpha + g_\alpha d\bar{z_\alpha} \quad \text{on } U_\alpha$$

If $f : X \to \mathbb{C}$ is a function we define the exterior derivative of f, given by the 1 form df as under

$$df := \partial f + \bar{\partial} f$$

If the local coordinate $z_\alpha = x_\alpha + iy_\alpha$, then one can check that

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

References

[1] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation.

[2] Theodore W. Gamelin. *Complex analysis*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[3] S. Kumaresan. *A course in differential geometry and Lie groups*, volume 22 of *Texts and Readings in Mathematics*. Hindustan Book Agency, New Delhi, 2002.

[4] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[5] Rick Miranda. *Algebraic curves and Riemann surfaces*, volume 5 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1995.

[6] Dror Varolin. *Riemann surfaces by way of complex analytic geometry*, volume 125 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.