An Excursion in Chern Groups

A Thesis

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This is to certify that this thesis entitled "*An Excursion in Chern Groups*" submitted towards the partial fulfilment of the Mathematics M.Sc Degree Program at the Indian Institute of Science Education and Research Pune represents work carried out by `Sutirtha Datta` under the supervision of `Steven Spallone`.

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Declaration

I hereby declare that the contents in the report titled "An Excursion in Chern Groups" are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune under the supervision of Professor Steven Spallone and has not been submitted anywhere else for any other degree.

Sutirtha Datta

To my beloved Mam, who always wanted me to do well.

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Contents

Nomenclature

$\mathcal{C}$	Chern Group
$\mathcal{C}_1$	Group generated by the total Chern Class of line bundles
$\mathcal{G}(B)$	Unit group of $H^*(B, \mathbf{Z})$ with constant term 1
$\mathcal{P}_k$	k-element ordered subsets
$\mathcal{W}$	Stiefel-Whitney Group
Ω	Loop Space
π_n	n-th homotopy group
Σ	Reduced Suspension
BG	Classifying Space for G
c	Total Chern Class
H^*_{even}	$\prod_n H^{2n}$
$K(G,n)$	Eilenberg-Maclane Spaces
S^n	n-sphere
T^n	n-torus : $S^1 \times \cdots \times S^1$, n times
w	Total Stiefel-Whitney Class
$\widetilde{KO}$	Reduced real K-group
$\widetilde{K}$	Reduced complex K-group

Introduction

We start with a manifold B. In this project, we are interested to figure out the multiplicative group generated by the total Chern Classes of all complex vector bundles over B. We call it the "Chern Group" of B and denote it by $\mathcal{C}(B)$. It is a subgroup of $H^*_{\text{even}}(B, \mathbf{Z})^\times$, the multiplicative group of the even part of the integral cohomology ring of B. When we restrict $H^*_{\text{even}}(B, \mathbf{Z})^\times$ to have 1 as the constant term, we denote that group by $\mathcal{G}(B)$. We describe these groups for familiar spaces, for example, spheres and tori.

We will also inspect the analogous subgroup $\mathcal{W}(B)$, generated by the total Stiefel-Whitney Classes.

The key results of the thesis are:

Theorem 0.0.1. *The Chern Groups of the even dimensional spheres are cyclic groups generated by* $1+(n-1)!\alpha$ *where* α *is a generator of* $H^{2n}(S^{2n}, \mathbf{Z})$*. The index of* $\mathcal{C}(S^{2n})$ *in* $\mathcal{G}(S^{2n})$ *is* $(n-1)!$.

Theorem 0.0.2. *The Chern Groups of odd dimensional spheres are trivial.*

Theorem 0.0.3. $\mathcal{C}(T^n) = \mathcal{G}(T^n)$ *for* $n \leq 5$.

Theorem 0.0.4. *The Stiefel-Whitney groups of the n-spheres are given by:*

$$\mathcal{W}(S^n) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & n = 1 \\ 1 + H^n(S^n, \mathbf{Z}/2\mathbf{Z}) & n = 2, 4 \\ 1 & n = 3, 5, 6, 7 \text{ and } n \geq 9 \end{cases}$$

For $n = 1, 2$ *and* 4*,* $\mathcal{W}(S^n)$ *is the multiplicative group of* $H^*(S^n, \mathbf{Z/2Z})$

Structure of the Thesis

This thesis predominantly focuses on the group generated by the total characteristic classes of vector bundles over a particular base space. The purpose is mainly three-folds : introducing some of the techniques for the classification of vector bundles in the first chapter and then computing a few Chern Groups and Stiefel-Whitney Groups in the remaining chapters. The classification problem requires several tools. As a consequence, Chapter 1 plays a pivotal role in the thesis, introducing Characteristic Classes, Classifying Spaces and K-theory Groups. For a more detailed treatment of these, one may refer to [Hat17], [Hat00], [MS74] and [Bre13].

Original Contributions

The computational results in Chapter 2 and 4 are new in the sense that, the results used to conclude them are pretty much familiar and easily found in literature, however the particular consequences we mentioned were not earlier discussed. The algebraic approach adopted in Chapter 3 and the results in section 3.4 and 3.5 are original.

Chapter 1

Preliminary Notions

1.1 Isomorphism Class of Vector Bundles

Let X be a topological space. Consider all vector bundles over X of rank k. We denote their isomorphism class by $Vect^n(X)$. For real vector bundles we use $Vect^n_{\mathbf{R}}(X)$ and for complex we write $Vect^n_{\mathbf{C}}(X)$. We have the following correspondence (cf. [Hat17])

$$\begin{Bmatrix}\text{Complex Vector Bundles of rank } k \\ \text{over paracompact base space } X\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{Homotopy Class of maps} \\ \text{from } X \ \text{ to } \mathrm{Grass}_k(\mathbf{C}^\infty)\end{Bmatrix}$$

In other words, $Vect^k_{\mathbf{C}}(X) \cong [X, \mathrm{Grass}_k(\mathbf{C}^\infty)]_\bullet$, where the $\bullet$ at the end denotes that these are based homotopy classes. When $X = S^n$, [Hat17] gives the following correspondence:

$$Vect^k_{\mathbf{C}}(S^n) \cong [S^{n-1}, GL(k, \mathbf{C})]_\bullet$$

Note that $[S^{n-1}, GL(k, \mathbf{C})]_\bullet := \pi_{n-1}(GL(k, \mathbf{C}))$. $U(k)$ being a maximal, compact subgroup of $GL(k, \mathbf{C})$. Hence we can write:

$$Vect_{\mathbf{C}^k(S^n)} \cong \pi_{n-1}(U(k))$$

For $n > 2$, we can write this as

$$Vect_{\mathbf{C}^k}(S^n) \cong \pi_{n-1}(SU(k))$$

Here is a table that compiles first few homotopy groups of some Special Unitary Groups. We know that $Vect^k_{\mathbf{C}}(S^n) \cong \pi_{n-1}(SU(k))$ [Wei13]

Homotopy Groups of Special Unitary Groups					
k	π_1	π_2	π_3	π_4	π_5
SU(1)	0	0	0	0	0
SU(2)	0	0	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
SU(3)	0	0	$\mathbf{Z}$	0	$\mathbf{Z}$
SU(4)	0	0	$\mathbf{Z}$	0	$\mathbf{Z}$
SU(5)	0	0	$\mathbf{Z}$	0	$\mathbf{Z}$
SU(6)	0	0	$\mathbf{Z}$	0	$\mathbf{Z}$
SU(7)	0	0	$\mathbf{Z}$	0	$\mathbf{Z}$

In fact a more general statement on this correspondence says that

$$Vect^k_{\mathbf{C}}(\Sigma X) \cong [X, U(k)]_\bullet$$

Where ΣX denotes the suspension of X.

Remark. From the above correspondences, one can actually realize that $Vect^n_{\mathbf{C}}$ as a functor is representable by $\mathrm{Grass}_n(\mathbf{C}^\infty)$.

Similar results hold for real vector bundles:

$$Vect^k_{\mathbf{R}}(S^n) \cong \pi_{n-1}(O(k))$$

1.2 Characteristic Classes

Characteristic Classes are certain cohomological invariants which can distinguish vector bundles.

Definition 1.2.1. *Characterestics Classes are natural transformations from $Vect^n$ to $H^*(\_, A)$ for an Abelian group A.*

Four impotant Characteristic Classes are Stiefel-Whitney Classes, Chern Classes, Euler Class and Pontryagin Classes. We will be interested in the first two. Axiomatic definitions and properties of them can be found in [MS74] or [Hat17].

Remark. When our base space is paracompact, characteristic classes are elements from $H^*(\mathrm{Grass}_n(\mathbf{K}^\infty), A)$. $(K = \mathbf{R}, \mathbf{C})$

Proposition 1.2.2. *If X has the homotopy type of a CW complex then $H^n(X, G) \approx [X, K(G, n)]$*

Proposition 1.2.3. *Complex Line Bundles over X are in one-one correspondence with $H^2(X, \mathbf{Z})$. This isomorphism is known as the first Chern Class $c_1(X)$.*

1.3 Chern Characters

We want to get a natural ring homomorphism from the K theory to the (singular) cohomology theory. Total Chern Classes are not additive, in the sense that, it takes direct sum of two vector bundles to a cup product. We will be rather interested to send a $\oplus$ to $+$ and $\otimes$ to $\smile$. Chern Character perfectly solves this purpose. It is defined in the level of line bundles, from which one can consider it for arbitrary vector bundles using direct sum and tensor products.

Definition 1.3.1. *For a complex line bundle $L \to B$, the Chern Character $ch(E)$ is defined as $e^{c_1(L)} = \sum_{k=0}^{\infty} \frac{(c_1(L))^k}{k!} \in H^{even}(B, \mathbf{Q})$*

Remark. $ch(L \otimes L') = e^{c_1(L \otimes L')} = e^{c_1(L) + c_1(L')} = ch(L)ch(L')$.

$ch(E)$ will turn out to be inside the rational cohomology ring $H^{\text{even}}(B, \mathbf{Q})$. But we will be interested in the case when it lands inside $H^{\text{even}}(B, \mathbf{Z})$, for example in 2.3.1.

1.4 Classifying Spaces and Eilenberg MacLane Spaces

Here by a topological space X or a topological group G, we always mean that it has the homotopy type of a CW-complex. For different constructions of Classifying spaces one can look at [May99], [Ste68] or [Seg68].

Definition 1.4.1. *A "universal" G-bundle is a principal G-bundle whose total space EG is weakly contractible[1], and with the property that any principal G-bundle over an arbitrary topological space B is a pullback of it by some "classifying map" $f : B \to BG$. BG is then called the "classifying space" of the topological group G[2].*

$$\begin{array}{ccc} E & \xrightarrow{f'} & EG \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{f} & BG \end{array}$$

[1] A topological space is "weakly contractible" if all of its homotopy groups are trivial. The notions weakly contractible and contractible are equivalent for CW complexes, thanks to Whitehead's Theorem

[2] Or more appropiately Classifying space for principal G-bundles

In other words BG is the quotient of the weakly contractible total space EG by a free action of G.

Example 1.4.1. *Some Examples of Classifying Spaces are:* $B\mathbf{Z} = S^1, BS^1 = \mathbf{CP}^\infty, BS^3 = \mathbf{HP}^\infty, B\mathbf{Z}^n = T^n$

Remark. Which spaces can arise as classifying spaces? This can be answered upto homotopy equivalence, since ΩBG is homotopy equivalent to G. (cf. [Hat00], [tD08]). Ω is the "loopspace" of BG.

Definition 1.4.2. *A connected topological space X is an "Eilenberg-MacLane Space of type $K(G,n)$" for $n > 1$ if*

$$\pi_k(X) = \begin{cases} G & k = n \\ 0 & k \neq n \end{cases}$$

Example 1.4.2. *Some examples of Eilenberg-MacLane Spaces are:*

- S^1 *is a* $K(Z, 1)$
- $\mathbf{RP}^\infty$ *is a* $K(\mathbf{Z}/2\mathbf{Z}, 1)$
- $\mathbf{CP}^\infty$ *is a* $K(\mathbf{Z}, 2)$

Remark. $K(G,n)$ behaves as "representing space" for the n-th singular cohomology with coefficients in G. Thus for any based CW complex X, the set $[X, K(G,n)]$ is in bijection with $H^n(X,G)$, naturally.

Proposition 1.4.3. *For a discrete group G, BG is an Eilenberg-MacLane Space of type $K(G,1)$.*

Proof. Consider the long exact sequence in homotopy:

$$\cdots \to \cancelto{0}{\pi_{n+1}(EG)} \to \pi_{n+1}(BG) \to \pi_n(G) \to \cancelto{0}{\pi_n(EG)} \to \cdots$$

We have $\pi_n(EG) = 0 \; \forall n$ since EG is contractible. Therefore $\pi_{n+1}(BG) = \pi_n(G) \; \forall n$. In other words, G being discrete $\pi_n(BG) = 0$ for $n \neq 1$ and $\pi_1(BG) = G$. Consequently BG is $K(G,1)$. □

Remark. The homotopy groups of BG and G are equal, upto a shift.

1.5 K-theory Groups

In the last section, we considered complex vector bundles of a particular rank over a base space and stated a bijection to the homotopy class of clutching functions. If we consider all complex vector bundles (without fixing the rank) over a given base X, then it is not a group with respect to $\oplus$. It is a semiring and denoted by $K(X)$ or $Vect_{\mathbf{C}}(X)$. The second notation is consistent to what we used in the previous sections. For all definition below we restrict X to compact Hausdorff topological spaces.

Definition 1.5.1. *Two vector bundles ξ and η are called to be "stably isomorphic" if there exists $m, n \in \mathbf{Z}_{\geq 0}$ such that $\xi \oplus \varepsilon^m_{\mathbf{C}} \cong \eta \oplus \varepsilon^n_{\mathbf{C}}$, where ε^k denotes trivial complex vector bundle of rank k. We denote $\xi \sim \eta$*

We state Bott Periodicity and one of its corollaries. cf. [Hat17] for proof.

Proposition 1.5.2. *The homomorphism $\theta : \tilde{K}(X) \to \tilde{K}(\Sigma^2 X)$ given by the external product $\theta(a) = (H-1) * a$ is an isomorphism, where H is the canonical line bundle over $\mathbf{CP}^1 \cong S^2$.*

Corollary 1.5.3. *The reduced K-groups $\tilde{K}(S^{2n+1}) = 0$ and $\tilde{K}(S^{2n}) \cong \mathbf{Z}$, which is generated by $(H-1)^{*n}$, the n- fold reduced external product.*

Now we switch to the real $K-$ theory groups which are denoted by KO[3]. In fact, real $K-$ theory is more complicated than the complex $K-$ theory. One of the most striking difference is in Bott Periodicity. The period for the $\widetilde{KO}$ groups are 8, under $\widetilde{K}$ groups, where the period was 2. In the introductory section of [Hat17], the first few $\widetilde{K}$ groups of S^n are given:

Real K theory Groups of the Spheres							
S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$

1.6 Shuffles and Permutations

In the game of cards, a riffle is a standard method to shuffle the cards. One divides the whole deck of cards into two smaller decks and interleaves them. A riffle shuffle permutation is a particular permutation of the entire deck of cards that can be achieved by a single riffle shuffle. A (p, q)-shuffle is a special case of the riffle shuffle permutation where the two smaller decks have cardinality p and q (cf. [Mac12], [May92]).

[3] The notation subsumes the O of Orthogonal groups. Ideally complex K theory should be written KU, for the corresponding unitary groups.

Definition 1.6.1. *A* (p, q) *shuffle is an element* $\sigma \in S_{p+q}$ *with*

$$\sigma(1) < \cdots < \sigma(p) \quad \textit{and} \quad \sigma(p+1) < \cdots < \sigma(p+q)$$

Remark. Signature of the shuffle is defined to be the sign of σ.

Let X, X' be two disjoint subsets of $\{1, \ldots, n\}$ with $|X| = p$ and $|X'| = q$. Write $(X \mid X')$ for the permutation of $X \cup X'$ given by listing the members of X in order, and then listing the members of X' in order. When $X \cup X' = \{1, \ldots, n\}$ it is a (p, q)-shuffle.

Example 1.6.1. *Let* $X = \{1, 3\}$ *and* $X' = \{2, 5\}$*. The permutation is then* (1325)*, and with one swap we get to* (1235)*. Therefore*

$$\operatorname{sgn}(\{1, 3\} \mid \{2, 5\}) = -1.$$

Example 1.6.2. *Let* $X = \{2, 4\}$ *and* $X' = \{1, 3, 5, 6\}$*. The permutation is* (241356)*. We can use the following sequence of swaps to get the identity permuatation.*

$$(241356) \xrightarrow{\text{2 swaps to move 1}} (124356) \xrightarrow{\text{1 swap to move 4}} (123456)$$

The total number of swaps is 3. Therefore

$$\operatorname{sgn}(\{2, 4\} \mid \{1, 3, 5, 6\}) = -1.$$

We will use $\operatorname{sgn}(X \mid X')$ in Chapter 3 for the tori.

Chapter 2

Chern Groups for Spheres

2.1 Cohomology Ring of Sphere

The integral cohomology ring for the spheres are given by $H^*(S^n, \mathbf{Z}) = \frac{\mathbf{Z}[a]}{(a^2)}$, where a is of degree n. The units are of the form $\{\pm 1 \pm a\mathbf{Z}\}$. We will often use the notation $\mathcal{G}(S^{2n})$ for the units of the form $\{1 + a\mathbf{Z}\}$.

We start inspecting the cases with first few spheres of lower dimensions.

2.2 n-Spheres for n=1,2,3

Proposition 2.2.1. *The Chern Group $\mathcal{C}(S^1)$ is trivial*

Proof. $H^{2k}(S^1, \mathbf{Z}) = 0 \ \forall \ \ k \in \mathbf{Z} \implies c_k(\xi) = 0$ for all bundles ξ over S^1 $\implies \mathcal{C}(S^1)$ is **trivial.** □

Proposition 2.2.2. *The Chern Group $\mathcal{C}(S^2)$ is the entire unit group $\mathcal{G}(S^2)$*

Proof. Given $a \in H^2(S^2, \mathbf{Z})$, by the correspondence result 1.2.3, there is a complex line bundle X^a over S^2 such that $c_1(X^a) = a$. Hence $\mathcal{C}(S^2)$ is the entire unit group. □

Proposition 2.2.3. *The Chern Group $\mathcal{C}(S^3)$ is trivial*

Proof. Since $\pi_2(G) = 0$ for any Lie Group G, $\pi_2(SU(k)) = 0$ cf. [BTD13], and hence $\mathcal{C}(S^3)$ is **trivial.** □

2.3 4-Sphere

We first state the following two statements from [Hat17]. For a proof cf. chapter 4 of [Hat17] or section 9, chapter 20 of [Hus66].

Proposition 2.3.1. *The map* $ch : \tilde{K}(S^{2n}) \to H^{2n}(S^{2n}, \mathbf{Q})$ *is injective with image equal to the subgroup* $H^{2n}(S^{2n}, \mathbf{Z}) \subset H^{2n}(S^{2n}, \mathbf{Q})$.

Now it follows from the definition of Chern Characters that:

Corollary 2.3.2. *A class in* $H^{2n}(S^{2n}, \mathbf{Z})$ *occurs as a Chern Class if it is divisible by* $(n-1)!$

Proof. Since $H^k(S^{2n}, \mathbf{Z}) = 0$ for $k \neq 2n$, we have $c_1(\xi) = \cdots = c_{n-1}(\xi) = 0$ for a complex vector bundle ξ over S^{2n}. Hence

$$ch(\xi) = \dim(\xi) + \frac{s_n(c_1, \cdots, c_n)}{n!}$$

Now by the recursion formula for s_n (cf. section 2.3 of [Hat17]), we have $s_n = \sigma_1 s_{n-1} - \sigma_{@} s_{n-2} + \cdots + (-1)^{n-1} n\sigma_n$, so

$$\frac{s_n(c_1, \cdots, c_n)}{n!} = \frac{nc_n(\xi)}{n!} = \frac{c_n(\xi)}{(n-1)!}$$

Combining this with the previous proposition, we get the corollary. □

Remark. We used the crucial fact (again, cf. [Hat17]) that for any symmetric polynomial[1] of degree k in the variables $x_1, \cdots, x_n$ can be expressed as a unique polynomial s_k in $\sigma_1, \cdots, \sigma_k$. In other words, there is a polynomial such that

$$s_k(\sigma_1, \cdots, \sigma_k) = \sum_{i=1}^{n} x_i^k$$

From the last corollary, we can deduce that the analogous maps

$$\pi_{2n}(BSU(n), \mathbf{Z}) \to H^{2n}(S^{2n}, \mathbf{Z})$$

are maps given by multiplication by $(n-1)!$. Hence for $n = 2$ the map is an isomorphism.

[1]Symmetric Polynomial in n variables is a polynomial such that if one of the variables is interchanged, then one obtains the exact same polynomial. For example $x + y$ is symmetric, however $x - y$ is not a symmetric polynomial.

Proposition 2.3.3. $\mathcal{C}(S^4) = \mathcal{G}(S^4)$.

Proof. We focus on the rank 2 bundles $\xi : E \to S^4$. We have $Vect^2_{\mathbf{C}}(S^4) \cong \pi_3(SU(2))$. The second Chern class $c_2(\xi) \in H^4(S^4, \mathbf{Z})$. It can also be viewed as an element of $H^4(BSU(2), \mathbf{Z}) \cong [BSU(2), K(\mathbf{Z}, 4)]_\bullet$. Now:

$$H^4(S^4, \mathbf{Z}) \cong [S^4, K(\mathbf{Z}, 4)]_\bullet \text{ and } [S^4, BSU(2)]_\bullet \cong \pi_4(BSU(2))$$

Consider the map $BSU(2) \to K(\mathbf{Z}, 4)$ which classifies c_2. But we showed that it induces isomorphisms at the homotopy group level. Note that

$$\pi_4(BSU(2)) \cong \pi_3(SU(2)) \cong Vect^2_{\mathbf{C}}(S^4)$$

Thus vector bundles of rank 2 are indeed classified[2] by $c_2(\xi)$. ☐

Note that for $n > 2$, 2.3.2 is not isomorphism anymore. However, we still get an injection. For the other even dimensional spheres the ch map is injective. Thus we can classify rank n complex vector bundles over S^{2n} are classified by c_n, in other words, the Chern Group is non-trivial and is given by $1 + (n-1)!\ H^{2n}(S^{2n}, \mathbf{Z})$.

Thus we are done with all even dimensional spheres. Now the odd dimensional spheres are left, except S^1 and S^3 whose Chern Groups were found to be trivial. We first produce a specific kind of vector bundle over S^5.

2.4 5-Sphere

Before going to prove a lemma about S^5, we recall that An Euler Section of $\tau\mathbf{C}^n$ is a vector field of the form $\sum x_i(p)\frac{\partial}{\partial x_i}|_p$, where $p \in \mathbf{C}^3$ and x_i are the coordinate functions for $i = 1, \cdots, n$. If we consider the Euler section for the tangent bundle $\tau\mathbf{C}^3$, then it is given by $\sum_{i=1}^{3} x_i(p)\frac{\partial}{\partial x_i}|_p$, which is nowhere vansihing on S^5.

Lemma 2.4.1. *There exists a complex vector bundle E over S^5 of rank 2 which is stably trivial but non-trivial.*

Proof. Consider the restriction of the tangent bundle of $\mathbf{C}^3$ to S^5 : $\tau\mathbf{C^3}|_{S^5}$. Take the trivial complex line subbundle of $\tau\mathbf{C^3}|_{S^5}$, spanned by the Euler section σ, which is non-vanishing on S^5. Now one can take the orthogonal

[2] This method is applicable to S^4. For the remaining cases Corollary 2.3.2 is needed.

complement of $\varepsilon^1_{\mathbf{C}}$ using the Hermitian metric of $\mathbf{C}^3$ and construct a total space. Then there exists a stably trivial vector bundle $E \to S^5$ of rank 2.

$$\tau\mathbf{C^3}|_{S^5} = E \oplus \varepsilon^1_{\mathbf{C}}$$

Recall that S^5 is not parallelizable. Regarding $E \to S^5$ as a $\mathbf{R}$-vector subbundle of τS^5 of rank 4, one can write

$$\tau S^5 = E \oplus \varepsilon^1_{\mathbf{R}}$$

Where the trivial line bundle $\varepsilon^1_{\mathbf{R}}$ is spanned by $i\sigma$, σ is the previous Euler section[3]. Had E been trivial, one would have had τS^5 is trivial, a contradiction. □

Remark. In the similar manner, one can construct such stably trivial, but non-trivial vector bundles over S^{2n+1}, except the case of 7-sphere, which is parallelizable.

Corollary 2.4.2. *There exists a non-trivial complex vector bundle $E \to S^5$ whose Chern classes are trivial.*

Proof. Because E is stably trivial, we can realte the total Chern Classes as:

$$c(\tau\mathbf{C}|_{S^5}) = c(E) \smile c(\varepsilon^1_{\mathbf{C}})$$

Hence the total Chern class of E is trivial and $c_i(E) = 0$ for $i = 1, 2$. □

Recall the fact that $Vect^n_{\mathbf{C}}(S^5) \cong \pi_4(SU(n))$. Now, $\pi_4(SU(n)) = 0$ for $n \neq 2$ and $\pi_4(SU(2)) \cong \mathbf{Z/2Z}$. Consequently, there are only one non-trivial complex vector bundle of rank 2 over S^5, upto bundle isomorphism. Now combining this with 2.4.2, we conclude that $\mathcal{C}(S^5)$ is trivial.

One can use the same procedure to produce complex vector bundles over odd dimensional spheres as 2.4.2, except the case of S^7, whose tangent bundle is trivial. Also the corresponding homotopy group is not always $\mathbf{Z}/2\mathbf{Z}$, so the same approach often fails. However, thanks to complex $K-$theory, we can state a general result for all odd dimensional spheres.

[3]This is multiplication by the complex number i. For example if σ is the vector field given by $\begin{bmatrix} x \\ y \end{bmatrix}$, then $i\sigma$ will be the vector field $\begin{bmatrix} -y \\ x \end{bmatrix}$

Proposition 2.4.3. *All complex vector bundles over odd dimensional spheres are stably trvial.*

Proof. 1.5.3 says that $\widetilde{K}(S^{2n+1}) = 0$. In other words, there is only one stable isomorphism class. A non-trivial vector bundle ξ is stably isomorphic to the trivial vector bundle ε^{ℓ} implies that there are integers m, n such that $\xi \oplus \varepsilon^m = \varepsilon^{\ell} \oplus \varepsilon^n$. Thus $\xi \oplus \varepsilon^m = \varepsilon^{\ell+n}$ and ξ is stably trivial. □

Now stably trivial bundle implies its total Chern Class is trivial. The last proposition then says $\mathcal{C}(S^{2n+1})$ is **trivial.**

We summarize the outcome of this chapter in the following theorem.

Theorem 2.4.4. *The Chern Group of every odd dimensional spheres are trivial. For even dimensional spheres, $\mathcal{C}(S^{2n})$ is the cyclic group generated by the element $1 + (n-1)!\alpha$ where α is a generator of $H^{2n}(S^{2n}, \mathbf{Z})$.*

Remark. One can take a fundamental class of S^{2n}, say $\mu_{S^{2n}}$ and dualize it to get a generator of the top cohomology, in other words, $Hom(\mu_{S^{2n}}, \mathbf{Z})$ will be a generator of $H^{2n}(S^{2n}, \mathbf{Z})$.

One can also note that the index of $\mathcal{C}(S^{2n})$ in $\mathcal{G}(S^{2n})$ is in fact $(n-1)!$. Therefore only for S^2 and S^4, the Chern Group exhausts the unit group $\mathcal{G}$. For the other even dimensional spheres, the Chern Group will be cyclic of even index in $\mathcal{G}(S^{2n})$.

Chapter 3

Chern Groups for Tori

3.1 Cohomology Ring of Tori

Denote $T^n = S^1 \times \cdots \times S^1$ (n times). First, we will consider T^2. It is well-known ([Hat00]) that:

$$H^k(T^2, \mathbf{Z}) = \begin{cases} \mathbf{Z} & k = 0, 2 \\ \mathbf{Z} \oplus \mathbf{Z} & k = 1 \\ 0 & k > 2 \end{cases}$$

In particular $H^k(T^n, \mathbf{Z}) = \mathbf{Z}^{\binom{n}{k}}$. The cohomology ring for T^n is the exterior algebra $\Lambda_{\mathbf{Z}}[v_1, \cdots, v_n]$ with each v_i having degree 1 and $v_i v_j = -v_j v_i \ \forall i \neq j$

We write down a general algebraic setup to take care of the Chern Groups of T^4 onwards. Let R be the integral cohomology ring $\Lambda_{\mathbf{Z}}[v_1, \cdots, v_n]$ with each $|v_i| = 1$ with $v_i v_j = -v_j v_i$ and R^i is the homogeneous ith degree terms. Suppose

$$\mathcal{G}(T^n) := 1 + \sum_{i \text{ even}} R^i$$

Finally we define two subgroups of $\mathcal{G}$, namely,

$$\mathcal{C}_1(T^n) := \langle 1 + R^2 \rangle$$

and $K \subseteq \mathcal{G}$ generated by the elements $\{1 + v_i v_j\}$, with $1 \leq i \neq j \leq n$.

Remark. Note that the Chern Group $\mathcal{C}(T^n)$ lies between $\mathcal{G}(T^n)$ and $\mathcal{C}_1(T^n)$.

An immediate consequence is the following:

Proposition 3.1.1. *The subgroup K is a free Abelian group of rank $\binom{n}{2}$.*

Proof. Essentially, we show that $\{1 + v_i v_j\}_{1 \leq i \neq j \leq n}$ is a basis of K. The spanning part is clear from the definition of K. Now suppose we have

$$\prod_{1 \leq i \neq j \leq n} (1 + v_i v_j)^{a_{ij}} = 1.$$

Indeed if some $a_{ij} \neq 0$ the corresponding $1 + v_i v_j$ element will end up being in the final product. Hence each a_{ij} has to be zero. □

3.2 Torus for n=2,3

Proposition 3.2.1. $\mathcal{C}(T^2) = \{1 + \alpha; \alpha \in H^2(T^2)\}$

Proof. Start with the correspondence between the isomorphism class of complex line bundles over X and $H^2(X, \mathbf{Z})$. $H^2(T^2, \mathbf{Z}) = \mathbf{Z}$ gives existence of non-trivial line bundles over the Torus. Now, combine the facts that $c_i \in H^{2i}(T^2)$ and $H^k(T^2)$ vanishes for $k > 2$. □

Proposition 3.2.2. $\mathcal{C}(T^3) = \{1 + \alpha; \alpha \in H^2(T^3)\}$

Proof. Existence of non-trivial complex line bundle over T^3 follows by non-triviality of second cohomology. $H^k(T^3, \mathbf{Z}) = 0$ for $k > 3$, shows that there is no c_2. □

3.3 4-Torus

We attempt to solve this algebraically. Here R is $\Lambda_{\mathbf{Z}}[v_1, v_2, v_3, v_4]$ with each $|v_i| = 1$ for $i = 1, 2, 3, 4$ and R^i is the homogeneous ith degree terms. The subgroups of $\mathcal{G}(T^4)$, which were discussed in the general setup are $\mathcal{C}_1(T^4) := \langle 1 + R^2 \rangle$ and $K \subseteq \mathcal{G}$[1] generated by $\{1 + v_i v_j\}$, with $1 \leq i \neq j \leq 4$

Remark. The subgroup $K \subsetneq \mathcal{C}_1(T^4)$

To see that, consider the element $1 + v_1 v_2 + v_2 v_3 + v_3 v_4 \in \mathcal{C}_1(T^4)$. If it were also in K, then it must have been of the form $(1 + v_1 v_2)(1 + v_2 v_3)(1 + v_3 v_4)$. So there must be the product term $v_1 v_2 v_3 v_4$.

We can write an element y of $\mathcal{G}$ is the form $1 + a v_1 v_2 + b v_2 v_3 + c v_3 v_4 + d v_4 v_1 + e v_1 v_3 + f v_2 v_4 + x \prod_{i=1}^{4} v_i$ $(a, b, c, d, e, f, x \in \mathbf{Z})$.

[1] Often we will just write $\mathcal{G}$ for $\mathcal{G}(T^n)$, if it is clear from the context.

Proposition 3.3.1. *An element $y \in K$ if and only if $x = ac - bd - ef$.*

Proof. The proposition follows from the factorization below:

$$(1 + av_1v_2)\,(1 + bv_2v_3)\,(1 + cv_3v_4)\,(1 + dv_4v_1)\,(1 + ev_1v_3)\,(1 + fv_2v_4)$$
$$= 1 + av_1v_2 + bv_2v_3 + cv_3v_4 + dv_4v_1 + ev_1v_3 + fv_2v_4 + (ac - bd - ef)\prod_{i=1}^{4} v_i$$

□

Thus, we can consider the map $\theta : \mathcal{G} \longrightarrow (\mathbf{Z}, +)$, given by sending

$$y \mapsto x - ac + bd + ef$$

Proposition 3.3.2. *The map θ is a group homomorphism*

Proof. Consider two elements from $\mathcal{G}$ and their product:

$$y = 1 + av_1v_2 + bv_2v_3 + cv_3v_4 + dv_4v_1 + ev_1v_3 + fv_2v_4 + x\prod_{i=1}^{4} v_i \quad \text{and}$$

$$y' = 1 + a'v_1v_2 + b'v_2v_3 + c'v_3v_4 + d'v_4v_1 + e'v_1v_3 + f'v_2v_4 + x'\prod_{i=1}^{4} v_i$$

Then we can write their product as $yy' = 1 + (a + a')v_1v_2 + ... + (f + f')v_2v_4$
$$+(ac' - bd' + a'c - b'd - ef' - e'f + x + x')\prod v_i$$

Applying θ to yy' we get $\theta(yy') = (ac' - bd' + a'c - b'd - ef' - e'f + x + x')$
$$-(ac + a'c + a'c' + ac' + ... - ef - e'f - ef' - e'f')$$
$$= x + x' + ac + a'c' + bd + b'd' + ef + e'f' = x - (ac + bd + ef)$$
$$+x' - (a'c' + b'd' + e'f') = \theta(y) + \theta(y')$$

Hence θ is indeed a group homomorphism. □

By Proposition 3.3.1 the kernel of θ is the subgroup K. Now we will check surjectivity of the map. Observe that, $\theta(1 + v_1v_2 + v_3v_4) = -1$ and $\theta(1 + v_2v_3 + v_4v_1) = 1$. One can now increase the coefficients and obtain any integer as the value of θ. Therefore, we get the following exact sequence, where K is free abelian group of rank $\binom{4}{2} = 6$

$$1 \to K \to \mathcal{G} \to \mathbf{Z} \to 0$$

This exact sequence in fact splits and hence $\mathcal{G} \cong K \oplus \mathbf{Z}$. Consequently, $\mathcal{G}$ is free abelian group of rank 7. By correspondence theorem subgroups of $\mathcal{G}$

containing K are in one-one correspondence with subgroups of $\mathcal{G}/K \cong \mathbf{Z}$.

Since, $z = 1 + v_2v_3 + v_4v_1$ is an element of $\mathcal{C}_1(T^4)$ such that $\theta(z) = 1$, we must have the entire $\mathbf{Z}$ as the image of $\mathcal{C}_1(T^4)$ and hence $\boxed{\mathcal{G} = \mathcal{C}_1(T^4)}$ in this case.

The conclusion of the section is the following

Theorem 3.3.3. *Chern Group of* 4*-Torus is generated by the total Chern Classes of Line Bundles and it is free abelian group of rank* 7*.*

3.4 5-Torus

In this case K is free of rank $\binom{5}{2} = 10$. Also $K \subsetneq \mathcal{C}_1(T^5)$ because of the element $1 + v_1v_2 + v_3v_4$. An arbitrary element of $\mathcal{G}$ can be written as:

$$y = 1 + \sum_{i<j} a_{ij}v_iv_j + \sum_{i<j<k<l} b_{ijkl}v_iv_jv_kv_l$$

Using the ordered subset notation, we can write this as (regarding $v_{\{ij\}} := v_iv_j$ and $v_{\{ijkl\}} := v_iv_jv_kv_l$)

$$y = 1 + \sum_{\sigma\in\mathcal{P}_2} a_\sigma v_\sigma + \sum_{\tau\in\mathcal{P}_4} b_\tau v_\tau$$

To clarify the notation we expand the two sums for this case:

$$\sum_{\sigma\in\mathcal{P}_2} a_\sigma v_\sigma = a_1v_1v_2 + a_2v_2v_3 + a_3v_3v_4 + a_4v_4v_5 + ... + a_9v_2v_4 + a_{10}v_4v_1$$

$$\sum_{\tau\in\mathcal{P}_4} b_\tau v_\tau = b_1v_1v_2v_3v_4 + b_2v_2v_3v_4v_5 + b_3v_3v_4v_5v_1 + b_4v_4v_5v_1v_2 + b_5v_5v_1v_2v_3$$

We define a map $\theta : \mathcal{G} \to \text{Maps}(\mathcal{P}_4, \mathbf{Z})$ as follows:

$$y \mapsto (\tau \mapsto \theta_\tau(y))$$

For $\tau \in \mathcal{P}_4$ we define θ_τ as follows:

$$y \mapsto b_\tau - \sum_{\sigma\neq\sigma'\in\mathcal{P}_2, \sigma\sigma'=\tau} \text{sgn}(\sigma \mid \sigma')a_\sigma a_{\sigma'}$$

Where $\sigma\sigma'$ denotes the union of σ and σ'. Since $\text{Maps}(\mathcal{P}_4, \mathbf{Z})$ is a group under pointwise addition and it is isomorphic to $\mathbf{Z}^5$, we can actually take the co-domain to be $\mathbf{Z}^5$ and get a surjective group homomorphism.

Proposition 3.4.1. *θ is a Group Homomorphism and its kernel is K*

Proof. In each component, the given map behaves like the θ in the T^4 case. The second statement holds because one can use the expansion of $(1 + a_1v_1v_2)(1 + a_2v_2v_3)...(1 + a_{10}v_4v_1)$, similar to the proof of 3.3.1 □

Let us denote the basis of $\mathbf{Z}^5$ by $\{e_i\}$, which is a 5-tuple, with 1 at the i-th position and rest all zero.

- $e_1 = \theta(1 + v_2v_3 + v_4v_1) = \theta(1 + v_1v_3 + v_2v_4)$
- $e_2 = \theta(1 + v_3v_4 + v_5v_2) = \theta(1 + v_3v_5 + v_2v_4)$
- $e_3 = \theta(1 + v_4v_5 + v_1v_3) = \theta(1 + v_3v_5 + v_4v_1)$
- $e_4 = \theta(1 + v_5v_2 + v_4v_1) = \theta(1 + v_5v_1 + v_2v_4)$
- $e_5 = \theta(1 + v_5v_2 + v_1v_3) = \theta(1 + v_1v_2 + v_3v_5)$

Evidently, θ is surjective and K being its kernel, we obtain $\mathcal{G} \cong \mathbf{Z}^5 \oplus K$.

Since K is free of rank 10, $\mathcal{G}$ is free of rank 15. Now again by correspondence theorem and the observation that in the above list, each e_i are written as image of an element of $\mathcal{C}_1(T^5)$, we conclude that $\boxed{\mathcal{G} = \mathcal{C}_1(T^5)}$ in this case.

The conclusion of the section is the following:

Theorem 3.4.2. *Chern Group of 5-Torus is generated by the total Chern Classes of Line Bundles and it is free abelian group of rank 15.*

3.5 6-Torus

Here K is free of rank $\binom{6}{2} = 15$. Also $K \subsetneq \mathcal{C}_1(T^6)$ because of the element $1 + v_1v_2 + v_3v_4 + v_5v_6$. An element of $\mathcal{G}$ will look like

$$y = 1 + \sum_{\sigma \in \mathcal{P}_2} a_\sigma v_\sigma + \sum_{\tau \in \mathcal{P}_4} b_\tau v_\tau \ + \ x \prod_{i=1}^{6} v_i$$

Consider the group homomorphism $\theta_4 : \mathcal{G} \to H^4(T^6) \cong \mathbf{Z}^{15}$ given by :

$$y \mapsto (\theta_{4,1}(y), ..., \theta_{4,15}(y)) \text{ where}$$

$$\theta_{4,i} : y \mapsto b_{\tau_i} - \sum_{\sigma \neq \sigma' \in \mathcal{P}_2, \sigma\sigma' = \tau_i} \operatorname{sgn}(\sigma \mid \sigma') a_\sigma a_{\sigma'}$$

Note that $K \subsetneq \operatorname{Ker}(\theta_4)$. Denote $K_4 := \operatorname{Ker}(\theta_4)$

We define the map $\theta_6 : K_4 \to H^6(T^6, \mathbf{Z}) \cong (\mathbf{Z}, +)$ as follows:

$$y \mapsto x - \sum_{\sigma\sigma'\sigma'' = [\mathbf{n}] \sigma, \sigma', \sigma'' \in \mathcal{P}_2} \operatorname{sgn}(\sigma \mid \sigma' \mid \sigma'') a_\sigma a_{\sigma'} a_{\sigma''}$$

Proposition 3.5.1. *The map $\theta_6 : K_4 \to \mathbf{Z}$ is a group homomorphism.*

Proof. Consider two elements from $\mathcal{G}$ and their product:

$$y = 1 + \sum_{\sigma \in \mathcal{P}_2} a_\sigma v_\sigma + \sum_{\tau \in \mathcal{P}_4} b_\tau v_\tau \ + \ x \prod_{i=1}^{6} v_i$$

$$y' = 1 + \sum_{\sigma \in \mathcal{P}_2} a'_\sigma v_\sigma + \sum_{\tau \in \mathcal{P}_4} b'_\tau v_\tau \ + \ x' \prod_{i=1}^{6} v_i$$

$$\text{their product } yy' = 1 + \sum_{\sigma \in \mathcal{P}_2} (a_\sigma + a'_\sigma) v_\sigma + \sum_{\tau \in \mathcal{P}_4} \overline{b_\tau} v_\tau \ + \ \overline{x} \prod_{i=1}^{6} v_i$$

$$\text{where } \overline{b_\tau} = b_\tau + b'_\tau + \sum_{\sigma\sigma' = \tau, \sigma, \sigma' \in \mathcal{P}_2} \operatorname{sgn}(\sigma \mid \sigma')(a_\sigma a'_{\sigma'} + a_{\sigma'} a'_\sigma)$$

$$\text{and } \overline{x} = x + x' + \sum_{\sigma\tau = [\mathbf{n}] \sigma \in \mathcal{P}_2, \tau \in \mathcal{P}_4} \operatorname{sgn}(\sigma \mid \tau) \left(a'_\sigma b_\tau + a_\sigma b'_\tau\right)$$

$$\theta(yy') = x + x' + \sum_{\sigma\tau = [\mathbf{n}] \sigma \in \mathcal{P}_2, \tau \in \mathcal{P}_4} \operatorname{sgn}(\sigma \mid \tau)(a'_\sigma b_\tau + a_\sigma b'_\tau) -$$
$$\sum_{\sigma\sigma'\sigma'' = [\mathbf{n}] \sigma, \sigma', \sigma'' \in \mathcal{P}_2} \operatorname{sgn}(\sigma \mid \sigma' \mid \sigma'')(a_\sigma + a'_\sigma)(a_{\sigma'} + a'_{\sigma'})(a_{\sigma''} + a'_{\sigma''})$$

Since the domain of θ_6 is K_4, we have

$$\sum_{\sigma\sigma'\sigma'' = [\mathbf{n}] \sigma, \sigma', \sigma'' \in \mathcal{P}_2} \operatorname{sgn}(\sigma \mid \sigma' \mid \sigma'')(a_\sigma + a'_\sigma)(a_{\sigma'} + a'_{\sigma'})(a_{\sigma''} + a'_{\sigma''})$$
$$= \sum \operatorname{sgn}(\sigma \mid \sigma' \mid \sigma'') \left(a_\sigma a_{\sigma'} a_{\sigma''} + a'_\sigma a'_{\sigma'} a'_{\sigma''} + a_\sigma \sum (a'_{\sigma'} a'_{\sigma''}) + a'_\sigma \sum (a_{\sigma'} a_{\sigma''}) \right)$$
$$= \sum \operatorname{sgn}(\sigma \mid \sigma' \mid \sigma'') \left(a_\sigma a_{\sigma'} a_{\sigma''} + a'_\sigma a'_{\sigma'} a'_{\sigma''} + a_\sigma b'_\tau + a'_\sigma b_\tau\right)$$

In other words, $\theta(yy') = \theta(y) + \theta(y') \ \forall \ y, y' \in K_4$. $\square$

However, the approach for the 4-torus did not work here since the map θ is no longer a group homomorphism when its domain is $\mathcal{G}$. It is interesting question to ask about the surjectivity of the map θ_6 and whether $\mathcal{C}_1(T^6)$ is equal to the Chern Group. This has not been resolved yet.

We note down the conclusion of the chapter as follows:

Theorem 3.5.2. *For $n \leq 5$ the Chern Group $\mathcal{C}(T^n)$ equals $\mathcal{G}(T^n)$.*

Chapter 4

Stiefel-Whitney Groups

Just like Chern Classes a similar analysis can be done on Stiefel Whitney Classes, for real vector bundles. In fact, analogous subgroups were mentioned in [MS23] for representations, which turned out to be cyclic for $SL(2,q)$.

Definition 4.0.1. *Given a base space X, the Stiefel Whitney Group $\mathcal{W}(X)$ is defined to be the subgroup of of $H^\bullet(B, \mathbf{Z}/2\mathbf{Z})^\times$ generated by the total Stiefel Whitney Classes of all real vector bundles over X.*

To begin with we consider $B = S^n$. The following results will make their appearances frequently (See [Hat17] or [Wei13] for proof):

Proposition 4.0.2. *Real Vector Bundles of rank k over ΣX are in bijective correspondence with the based homotopy class $[X, O(k)]$*

Corollary 4.0.3. *Real Vector Bundles of rank k over S^n are in bijection with $\pi_{n-1}(O(k)) \cong \pi_{n-1}(SO(k))$ for $n > 2$*

$$\boxed{Vect^k_{\mathbf{R}}(S^n) \cong \pi_{n-1}(SO(k))}$$

Here is a table that compiles first few homotopy groups of some Special Orthogonal Groups.

Homotopy Groups of Special Orthogonal Groups					
k	π_1	π_2	π_3	π_4	π_5
SO(2)	$\mathbf{Z}$	0	0	0	0
SO(3)	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
SO(4)	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}^2$	$(\mathbf{Z}/2\mathbf{Z})^2$	$(\mathbf{Z}/2\mathbf{Z})^2$
SO(5)	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
SO(6)	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$
SO(7)	$\mathbf{Z}/2\mathbf{Z}$	0	$\mathbf{Z}$	0	0

Proposition 4.0.4. *Real Line bundles over X are in one one correspondence with $H^1(X, \mathbf{Z}/\mathbf{2Z})$. This assignment is essentially the first Stiefel-Whiteny Class $w_1(X)$.*

4.1 Spheres of Dimension 1,2 and 3

We digress a bit about the singular cohomology groups and cohomology ring of n-sphere.

$$H^k(S^n, \mathbf{Z}) = \begin{cases} \mathbf{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

The cohomology ring of of sphere $H^*(S^n, \mathbf{Z}) = \frac{\mathbf{Z}[x]}{(x^2)}$, with $|x| = n$ One can now re-write these with $\mathbf{Z}/2\mathbf{Z}$ coefficients.

$$H^k(S^n, \mathbf{Z}/2\mathbf{Z}) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

$H^*(S^n, \mathbf{Z}/\mathbf{2Z}) = \frac{\mathbf{Z}/2\mathbf{Z}[a]}{(a^2)}$, with $|a| = n$

Proposition 4.1.1. *Stiefel Whitney Group of S^1 is cyclic of order 2.*

Proof. $H^*(S^1, \mathbf{Z}/\mathbf{2Z}) = \frac{\mathbf{Z}/\mathbf{2Z}[a]}{(a^2)} = \{0, 1, a, 1 + a\}$ with $a \in H^1(S^1, \mathbf{Z}/\mathbf{2Z})$
Now $\left(\frac{\mathbf{Z}/\mathbf{2Z}[a]}{(a^2)}\right)^\times = \{1, 1 + a\} \cong \mathbf{Z}/\mathbf{2Z}$.
Therefore $\mathcal{W}(S^1) = H^*(S^1, \mathbf{Z}/\mathbf{2Z})^\times \cong \mathbf{Z}/\mathbf{2Z}$ □

Proposition 4.1.2. *Stiefel Whitney Group of S^2 exhausts the entire unit group $H^\bullet(S^2, \mathbf{Z}/2\mathbf{Z})^\times$*

Proof. Consider the map induced by the coefficient map $\mathbf{Z} \to \mathbf{Z}/\mathbf{2Z}$

$$\kappa : H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z}/\mathbf{2Z})$$

1.2.3 says, given $a \in H^2(S^2, \mathbf{Z})$, there exists a complex line bundle X^a over S^2, such that the first Chern Class $c_1(X^a) = a$.

By Prop 3.8 of [Hat17] one has $\kappa c_1(X^a) = w_2(X^a_{\mathbf{R}})$, where $\xi_{\mathbf{R}}$ denotes the **R**ealization of a complex vector bundle ξ. Consequently, w_2 is non-trivial. Hence the SW group $\mathcal{W}(S^2)$ exhausts the entire unit group $H^\bullet(S^2, \mathbf{Z}/\mathbf{2Z})^\times$. □

Proposition 4.1.3. *Stiefel Whitney Group of* S^3 *is trivial.*

Proof. For any Lie Group G, the second homotopy group is always trivial (cf.[BTD13]). In particular $\pi_2(SO(k)) = 0$. Now 4.0.3 says that there is no non-trivial vector bundles over S^3. Consequently, $\mathcal{W}(S^3)$ is **trivial**. □

4.2 Spheres of Dimension ≥ 9

We will refer to the following result from the celebrated paper [AH61].

Proposition 4.2.1. *Let* Y *be a finite CW complex, not necessarily connected. Then the total Stiefel Whitney Class* $w(\xi) = 1$ *for any real vector bundle* ξ *over the* 9*-fold suspension of* Y*.*

The previous proposition gives the Stiefel Whitney groups of S^9 and higher.

Proposition 4.2.2. $\mathcal{W}(S^n)$ *are trivial for* $n \geq 9$*.*

Proof. Since $S^{n+1} = \Sigma S^n$, 4.2.1 says that the Stiefel Whitney classes of ξ vanish for any ξ over S^n for $n \geq 9$. Hence $\mathcal{W}(S^n)$ is trivial for $n \geq 9$ □

4.3 Remaining Cases

Recall that, $\mathcal{C}(S^4)$ turned out to be $1 + H^4(S^4, \mathbf{Z})$. Composing the multiplication by $(n-1)!$ map by the coefficient induced homomorphism κ, we associate an element of $H^4(S^4, \mathbf{Z}/2\mathbf{Z})$ with a complex vector bundle.

$$\widetilde{K}(S^4) \xrightarrow{\times(2-1)!} H^4(S^4, \mathbf{Z}) \xrightarrow{\kappa} H^4(S^4, \mathbf{Z}/2\mathbf{Z})$$

By Prop 3.8 of [Hat17], it turns out that $\kappa(c_2) = w_4$. Therefore, $\mathcal{W}(S^4)$ is non trivial and equals $1 + H^4(S^4, \mathbf{Z}/2\mathbf{Z})$

Now one cannot use the same κ map for the other spheres, as for $n > 2$, we start getting even values for $(n-1)!$. Consequently that becomes zero under the coefficient map $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$. To resolve this impasse, we take help of the real $K-$ theory groups. Recall that $\widetilde{KO}(S^n)$ is trivial for $n = 3, 5, 6, 7$.

Proposition 4.3.1. *Every real vector bundle over S^n is stably trivial for $n \equiv 3, 5, 6, 7(\mod 8)$*

Proof. As $\widetilde{KO}(S^n)$ is trivial for $n \equiv 3, 5, 6, 7(\mod 8)$, there is only one stable isomorphism class of real vector bundles over these S^n. Consequently, every vector bundle is stably trivial. □

Just like Chern Classes, Stiefel Whitney classes also satisfy the Whitney product axiom. In other words:

$$w(\xi \oplus \eta) = w(\xi) \smile w(\eta)$$

It follows that, stably trivial vector bundles have Stiefel Whitney Classes trivial. Consequently $\mathcal{W}(S^n)$ is trivial for $n = 3, 5, 6, 7$

Remark. The case for S^8 has not been resolved in this thesis.

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