Eisenstein parts of homology and cohomology groups of Bianchi 3-fold

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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द्वारा / By

प्रांजल विश्वकर्मा / Pranjal Vishwakarma

पंजीकरण सं. / Registration No.:
20172024

शोध प्रबंध पर्यवेक्षक / Thesis Supervisor:

डॉ. देबरघा बनर्जी / Dr. Debargha Banerjee

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Dedicated to

Pandit Madan Mohan Malaviya

I dedicate my thesis to Pandit Madan Mohan Malviya, the founder of Banaras Hindu University (BHU). He was a respected educational reformer known for his innovative ideas and strong leadership. His ideas and leadership in creating BHU greatly influenced my thoughts. BHU's significant role in education enriches my studies, impacting academia and my life.

Certificate

Certified that the work incorporated in the thesis entitled "*Eisenstein parts of homology and cohomology groups of Bianchi* 3*-fold*", submitted by *Pranjal Vishwakarma* was carried out by the candidate under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: August 28, 2024

Dr. Debargha Banerjee
Thesis Supervisor

Declaration

Name of Student: Pranjal Vishwakarma
Reg. No.: 20172024
Thesis Supervisor(s): Dr. Debargha Banerjee
Department: Mathematics
Date of joining the program: 01/08/2017
Date of Pre-Synopsis Seminar: 14/12/2023
Title of Thesis: Eisenstein parts of homology and cohomology groups of Bianchi 3-fold

I declare that this written submission represents my ideas in my own words. Where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity. I have not misrepresented, fabricated, or falsified any idea, data, fact, or source in my submission. I understand that violation of the above will result in disciplinary action by the institute and may also lead to penal action if proper citation or permission has not been obtained from the sources used.

The work reported in this thesis is the original work done by me under the guidance of *Dr. Debargha Banerjee*.

Date: August 28, 2024 *Pranjal Vishwakarma*

Pranjal Vishwakarma

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Contents

Abstract

Let $K = \mathbb{Q}(\sqrt{-d})$ where $d(> 0)$ is a square-free integer. Let $\mathcal{O}_K$ be the ring of integers of K.

Consider the hyperbolic 3-space $\mathbb{H}_3$ (Upper half space),

$$\mathbb{H}_3 := \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}.$$

We define the extended 3- dimensional upper half space to be

$$\overline{\mathbb{H}}_3 := \mathbb{H}_3 \cup K \cup \{\infty\}.$$

We denote the full Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$ by G and choose Γ to be a subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ of finite index with no elements of finite order.

Let $Y_\Gamma = \Gamma \backslash \mathbb{H}_3$ be a hyperbolic 3-manifold. Consider the Baily-Borel-Satake compactification of Y_Γ, which is $X_\Gamma^{BB} = \Gamma \backslash \overline{\mathbb{H}}_3$, obtained by adding the set of cusps.

The Borel-Serre compactification of Y_Γ, which is X_Γ^{BS} obtained by adding a 2-torus to each cusp ∂X_Γ^{BS} (except for $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\sqrt{-3})$ for which we add spheres instead).

The first result of this thesis is related to the Eisenstein cycle and the Eisenstein part of homology. We explicitly write down the *Eisenstein cycles* (or we say *Eisenstein element*) in the first homology groups of quotients of hyperbolic 3-space as linear

combinations of Cremona symbols (a generalization of Manin symbols) for imaginary quadratic fields. These cycles generate the Eisenstein part of the homology groups.

Using Poincaré duality, we can relate cohomology and homology. We also studied the Eisenstein part of the cohomology groups. The second result of this thesis is related to the Eisenstein and cuspidal parts of the cohomology groups. We have calculated the trace of the first and second Eisenstein cohomology groups and the Lefschetz number. As an application of J.Rohlfs' result in §8.4.1, we find an asymptotic dimension formula (in the level aspect) for the cuspidal cohomology groups of congruence subgroups of the form $\Gamma_1(N)$ inside the full Bianchi groups.

Statement of originality

The main original research results presented in this thesis, along with their respective chapters and sections, are as follows:

In Chapter 4, Section 4.3, Proposition 4.3.1, and its Corollary 4.3.2 are discussed.

In Chapter 5, Section 5.3.1, the discussion revolves around Theorem 5.4.3, along with Propositions 5.4.5 and 5.4.4.

Furthermore, Chapter 6 discusses Proposition 6.1.4.

Additionally, Chapter 7 presents Lemma 7.3.2, Lemma 7.3.3, Lemma 7.3.4, Proposition 7.3.5, and Corollary 7.3.6.

Moreover, Chapter 8 contains Theorem 8.3, Theorem 8.3.2, and Proposition 8.4.2.

The majority of the results presented in this thesis are derived from the research paper referenced as [4].

Nomenclature

$\mathbb{H}_2$	$\Big\{z \in \mathbb{C} : Im(z) > 0\Big\}$
$\mathbb{H}_3$	$\Big\{(z,t) \in \mathbb{C} \times \mathbb{R} \mid z \in \mathbb{C}, t \in \mathbb{R}, t > 0\Big\}$
K	Imaginary quadratic field
$\overline{\mathbb{H}}_2$	$\mathbb{H}_2 \cup \mathbb{Q} \cup \{\infty\}$
$\overline{\mathbb{H}}_3$	$\mathbb{H}_3 \cup K \cup \{\infty\}$
$\mathrm{SL}_2(\mathbb{Z})$	$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}, ad - bc = 1\right\}$
$\mathrm{SL}_2(\mathcal{O}_K)$	$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathcal{O}_K, ad - bc = 1\right\}$
Γ	Congruence Subgroup
$\Gamma(N) \leq \mathrm{SL}_2(\mathbb{Z})$	$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\bmod N) \in \mathrm{SL}_2(\mathbb{Z})\right\}$
$\Gamma_0(N) \leq \mathrm{SL}_2(\mathbb{Z})$	$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} (\bmod N) \in \mathrm{SL}_2(\mathbb{Z})\right\}$

$\Gamma_1(N) \leq \mathrm{SL}_2(\mathbb{Z})$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} (\bmod N) \in \mathrm{SL}_2(\mathbb{Z}) \right\}$
$\Gamma(\mathfrak{a}) \leq \mathrm{SL}_2(\mathcal{O}_K)$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\bmod \mathfrak{a}) \in \mathrm{SL}_2(\mathcal{O}_K) \right\}$
$\Gamma_1(\mathfrak{a}) \leq \mathrm{SL}_2(\mathcal{O}_K)$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} (\bmod \mathfrak{a}) \in \mathrm{SL}_2(\mathcal{O}_K) \right\}$
$\Gamma_0(\mathfrak{a}) \leq \mathrm{SL}_2(\mathcal{O}_K)$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} (\bmod \mathfrak{a}) \in \mathrm{SL}_2(\mathcal{O}_K) \right\}$
$q = e(\tau)$	$\exp(2\pi i \tau)$
$Y(\Gamma)$	$\Gamma(N)\backslash\mathbb{H}_2$
$X(\Gamma)$	$\Gamma(N)\backslash\overline{\mathbb{H}}_2$
Y_Γ	$\Gamma\backslash\mathbb{H}_3$
X_Γ^{BB}	Bailey-Borel-Satake compactification of $\Gamma\backslash\mathbb{H}_3$
X_Γ^{BS}	Borel-Serre compactification of $\Gamma\backslash\mathbb{H}_3$
$L(-,-,-)$	Lefschetz number
tr	trace
β	$(\beta_0 = -dz/t, \beta_1 = dt/t, \beta_2 = d\bar{z}/t)$

1

Introduction

Eisenstein parts of the homology and cohomology groups for *classical* modular curves encode several arithmetic information, such as special values of L-functions. Eisenstein ideals were introduced by Mazur, and these ideals found applications in several important breakthroughs in the last few decades, including Fermat's last theorem and the main conjecture of Iwasawa theory.

Merel introduced Eisenstein cycles that are the basis of the *integral* homology groups of classical elliptic modular curves (space of modular symbols). Banerjee-Merel [2] investigated Eisenstein cycles for *congruence* subgroups of classical modular groups.

In the classical case for the congruence subgroups of odd level, we can take intersection with the principal congruence subgroup of level two. As a consequence, we can assume all the cusps are lying above $\{0, \infty, 1\}$ and cycles can be taken of the form $\{g1, g(-1)\}$ for $g \in \mathrm{SL}_2(\mathbb{Z})$. In turn, these can be written in terms of loops of the modular curves.

For any odd integer N, Banerjee-Merel [2] expressed Eisenstein cycles in the first homology groups of modular curves of level N as linear combinations over $\overline{\mathbb{Q}}$ of Manin symbols (generators of homology groups of elliptic modular curves). The coefficients are computed in terms of periods (integrals over cycles of complex differential 1 forms associated to the Eisenstein series). These periods can be computed in terms of Dedekind sums.

This formulation was extended to subgroups of finite indices (not necessarily congruence subgroups) within the full modular group as detailed in [3] with a more complicated notion of periods. It should be noted that these Eisenstein cycles are *not* necessarily $\overline{\mathbb{Q}}$ valued for general subgroups of finite index.

John Cremona and his collaborators initiated the study of modular symbols (integral homology groups of quotients of 3-dimensional hyperbolic space) for imaginary quadratic fields. They are mostly interested in the cuspidal part of homology groups of the corresponding topological space. Ito [22] first defines periods in this setting and shows that the periods are again described by Sczech's Dedekind sums [32] for $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$.

In Chapter 2, we have provided a short overview of classical modular forms, while in Chapter 3, we delve into the definition of the fundamental domain, differentials on $\mathbb{H}_3$, and the definitions of scalar-valued Bianchi modular forms and vector-valued Bianchi modular forms.

In Chapter 4, we discuss Hida's differential forms and Ito's differential forms for the full group. Following that, in Section 4.3, we generalize Ito's differentials for subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$.

In Chapter 5, we discuss Cremona symbols, which generalize Manin symbols for Bianchi groups, and we also present some results on classical modular symbols. Following that, we derive the inner product formula for Bianchi modular forms using quasi-periods as follows

Proposition 1.0.1. *Let $F, S \in M_2(\Gamma)$ be two Bianchi modular forms, with at least one of them being a cusp form with $F = (F_0, F_1, F_2)$ and $S = (S_0, S_1, S_2)$. Consider the function*

$$H := iF_0\overline{S_0} + \frac{i}{2}F_1\overline{S_1} + iF_2\overline{S_2}.$$

The inner product of these two modular forms is given by $< F, S >= I$ with

$$I = \frac{1}{12[G:\Gamma]} \sum_{g \in \Gamma \backslash G} \int_{g0}^{g\infty} \int_{\partial \mathfrak{F}_K} H(\beta_0 \wedge \beta_1 \wedge \beta_2)$$

where $\partial \mathfrak{F}_K$ is the boundary of fundamental domain $\mathfrak{F}_K$.

Additionally, in our work, we express the Eisenstein cycles of the homology group associated with the quotient of hyperbolic 3-space by subgroups of finite index of Bianchi modular groups in terms of Cremona symbols. These Cremona symbols serve as a generalization of Manin symbols for Bianchi groups.

Definition 1.0.2. The *Eisenstein classes* are modular symbols $\mathcal{E} \in \mathrm{H}_1(X_\Gamma^{BB}, \partial(X_\Gamma^{BB}); \mathbb{C})$ such that

$$\int_{\mathcal{E}} F \cdot \beta = 0$$

for all cusp forms F (see the definition of the cusp forms). The *Eisenstein cycles* (or we say *Eisenstein element*) are paths within the Eisenstein classes.

Cremona [8] demonstrated (see also [7, Theorem 4.3.2]) that the map η is a surjective mapping from $\Gamma\backslash\mathrm{SL}_2(\mathcal{O}_K)$ onto $\mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{Q})$ defined as

$$\eta : (g) \longmapsto \{g0, g\infty\}.$$

We define the generalized 2-periods as follows

$$F_E(g) := \int_{\partial \mathfrak{F}_K} \star(E \cdot \beta)[g],$$

where E is the Eisenstein series of weight 2.

Now, we state our first result: The Eisenstein cycles generated by these symbols contribute to the Eisenstein part of the homology groups.

Theorem 1.0.3. *For any imaginary field K with class number one that is also an Euclidean domain, the modular symbol*

$$\mathcal{E}_E = \sum_{g \in \Gamma\backslash G} F_E(g)\eta(g)$$

is the Eisenstein cycle corresponding to the Eisenstein series $E \in E_2(\Gamma)$.

These Eisenstein cycles are determined in terms of generalized periods (integrals of two forms over a real surface) that are hard to compute as of now.

In the present setting, results are difficult to obtain because the corresponding topological spaces do *not* carry the same algebraic geometric structure as in the cases for elliptic or Hilbert modular forms settings. In particular, there is no analogue of dessin d'enfant.

Until now, we have discussed everything related to the homology groups of Bianchi 3-folds. Using Poincaré duality, we see that the cohomology and homology groups are related. Now, we will discuss the Eisenstein and cuspidal cohomology groups of Bianchi 3-folds.

In Chapter 6, we provide definitions of Eisenstein cohomology and cuspidal cohomology, and we compute the dimension of Eisenstein cohomology. Following that, we discuss Szech cocycles and certain expectations.

Till date, there is no nice formula for the dimensions of the space of Bianchi cusps forms similar to elliptic cases. Note that we can not apply the Riemann-Roch theorem in this setting to obtain a dimension formula. Sengün and his collaborators [10] studied the dimensions of the cohomology of Bianchi groups for the *principal* congruence subgroups.

Now, we use the result of J.Rohlfs, which gives the asymptotic bounds on cuspidal cohomology

$$\dim \ \mathrm{H}^1_{cusp}(\Gamma; M) \geq \frac{1}{2}\left| L(\rho, \Gamma, M) + \mathrm{tr}\left(\rho^1_{Eis}\right) - \mathrm{tr}\left(\rho^2_{Eis}\right) - \mathrm{tr}\left(\rho^0\right)\right|$$

where $L(\cdot)$ denotes the Lefschetz number, $\mathrm{tr}(\rho^i_{Eis})$ denotes the trace on the i-th Eisenstein cohomology group, and ρ is an involution.

In Chapter 7, we discuss the definition of the Lefschetz number for the full group and subgroups of the form $\Gamma(N)$ based on the work of Sengün and Türkelli [10]. Subsequently, we calculate the Lefschetz number for subgroups of the form $\Gamma_1(N)$ in Section 7.3.

In Chapter 8, we investigate the trace of Eisenstein cohomology for the full group and subgroups of the form $\Gamma(N)$ based on the work of Sengün and Türkelli [10]. Subsequently, we calculate the trace of Eisenstein cohomology groups for subgroups of the form $\Gamma_1(N)$ in Section 8.3.

We explore the Eisenstein part of the cohomology groups. As an application, we derive an asymptotic dimension formula in the level aspect for the space of cuspidal cohomology groups.

A natural question arises regarding the generalization of these computations to other crucial congruence subgroups such as $\Gamma_1(\mathfrak{a})$ as defined below. To study Diophantine questions like modularity, Bianchi modular 3-fold associated with these subgroups are expected to play an important role. As an application of our study of Eisenstein part of the cohomology groups of subgroups of Bianchi modular groups, we give a *lower bound* on the dimension of corresponding cuspidal cohomology groups for the congruence subgroups of the form $\Gamma_1(p^n)$ as $n \to \infty$ in Section 8.4.

The *lower bound* on the dimension of corresponding cuspidal cohomology groups for the congruence subgroups of the form $\Gamma_1(p^n)$ as follows

Proposition 1.0.4. *Let p be a rational prime that is unramified in K and let $\Gamma_1(p^n)$ denote the subgroup of level $(p)^n$ of a Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$. Then*

1. $$\dim \mathrm{H}^1_{cusp}\left(\Gamma_1(p^n); M_k\right) \gg k$$

   as k increases and n is fixed.

2. *Assume further that the class number of K is one. We have the following asymptotic bound.*
   $$\dim \mathrm{H}^1_{cusp}\left(\Gamma_1(p^n); \mathbb{C}\right) \gg p^{3n}$$

   as n increases.

Note that this part contains corresponding Bianchi cusp forms by the Harder-Eichler-Shimura isomorphism theorem (generalized Matsushima's formula). These results show that at least these many Bianchi cusp forms will be there for the congruence subgroups of the form $\Gamma_1(p^n)$.

2

Classical Modular Forms

The group $\mathrm{SL}_2(\mathbb{Z})$ is known as the *Modular Group*. Let

$$\mathbb{H}_2 := \{\tau = x + iy \in \mathbb{C} \mid y > 0\}$$

be the upper half-space in $\mathbb{C}$.

We define an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}_2$. For $\tau \in \mathbb{H}_2$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Definition 2.0.1. Let k be an integer. A function $f : \mathbb{H}_2 \to \mathbb{C}$ is called *weakly modular* of weight k if for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}_2$ and $f(\alpha(\tau)) = (c\tau + d)^k f(\tau)$.

Considering $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, any weakly modular function of weight k becomes a periodic function with period 1, and it satisfies $f\left(\frac{-1}{\tau}\right) = \tau^k f(\tau)$.

Furthermore, since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, if k is an odd integer, then f must be identically zero.

Consider $D = \left\{\tau \in \mathbb{C} \mid |\tau| < 1\right\}$. The upper half-plane $\mathbb{H}_2$ is topologically equivalent to the punctured open unit disk $D' = D - \{0\}$ under the mapping $\tau \mapsto e(\tau) = e^{2\pi i\tau}$ and $q = e(\tau)$.

Thus, any function f defined on $\mathbb{H}_2$ can be represented as g on D', where $g(q) = f\left(\frac{\log q}{2\pi i}\right)$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the defined function g is well defined even though $\log$ is determined up to $2\pi i\mathbb{Z}$.

If f is a weakly modular function of weight k, we say f is holomorphic at ∞ if g extends analytically to $0 \in \mathbb{C}$. In such a case, f admits a Fourier expansion $f(\tau) = \sum_{n\geq 0} a_n(f)q^n$, where $q = e(\tau)$.

Definition 2.0.2. (**Modular form**). A function $f : \mathbb{H}_2 \to \mathbb{C}$ is called *modular form* of weight k if

(i) f is weakly modular of weight k,

(ii) f is holomorphic on $\mathbb{H}_2$,

(iii) f is holomorphic at ∞.

Remark 2.0.3. *There are no nonzero modular forms of odd-weight k for the modular group* $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.0.4 (**Cusp form**). A modular form f of weight k is called a cusp form of weight k if the Fourier expansion of f vanishes at ∞.

For $N \in \mathbb{N}$, let

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\bmod N) \right\}.$$

As the kernel of the natural morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, we see $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$, called the principal congruence subgroup of level N.

Definition 2.0.5. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\exists\, N$ such that $\Gamma(N) \subset \Gamma$, and the least such N is called the level of Γ.

Let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\bmod N) \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\bmod N) \right\}.$$

The subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$ are congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of level N.

From the definitions, it is evident that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$. Being the kernel of the natural homomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right),$$

the subgroup $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. In fact, the map is a surjection, inducing an isomorphism

$$\frac{\mathrm{SL}_2(\mathbb{Z})}{\Gamma(N)} \cong \mathrm{SL}_2\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right),$$

hence

$$\left|\mathrm{SL}_2\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)\right| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Therefore, any congruence subgroup has a finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Consider $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\tau \in \mathbb{H}_2$. The factor of automorphy at γ is defined as $j(\gamma, \tau) = c\tau + d$.

For $f : \mathbb{H}_2 \to \mathbb{C}$, we define the weight-$k$ operator $[\gamma]_k$ such that

$$(f[\gamma]_k)(\tau) = (\det(\gamma))^{k-1} j(\gamma, \tau)^{-k} f(\gamma(\tau)).$$

Since the factor of automorphy is neither zero nor infinity, f is meromorphic if and only if $f[\gamma]_k$ is meromorphic.

Definition 2.0.6. We define function $f : \mathbb{H}_2 \to \mathbb{C}$ to be a *weakly modular form* of weight

k with respect to Γ if $f[\gamma]_k = f$ for all $\gamma \in \Gamma$.

Definition 2.0.7. Let Γ be a congruence subgroup and k be an integer. A function $f : \mathbb{H}_2 \to \mathbb{C}$ is a modular form of weight k with respect to Γ if

(i) f is holomorphic,

(ii) it has weight k invariance under Γ,

(iii) $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 2.0.8. A modular form f of weight k with respect to Γ is termed a *cusp form* of weight k with respect to Γ if $f[\gamma]_k$ vanishes at infinity for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 2.0.9 (Fundamental domain). A fundamental domain for a group Γ acting on upper half space $\mathbb{H}_2$ is a region in the upper half-plane $\mathbb{H}_2$ that contains exactly one point from each Γ-orbit.

Proposition 2.0.10. *Let $\mathcal{D} = \{\tau \in \mathbb{C} \mid |\tau| \geq 1, |Re(\tau)| \leq 1\}$. Then, the set $\mathcal{D}$ is called a fundamental domain for* $\mathrm{SL}_2(\mathbb{Z})$.

Let Γ be a congruence subgroup. Then the modular curve for Γ, denoted by $Y(\Gamma)$, is the set $\{\Gamma\tau \mid \tau \in \mathbb{H}_2\} = \Gamma \backslash \mathbb{H}_2$.

We define $X(\Gamma) = \{\Gamma\tau \mid \tau \in \overline{\mathbb{H}}_2\}$, where $\overline{\mathbb{H}}_2 = \mathbb{H}_2 \cup \mathbb{Q} \cup \{\infty\}$.

For the topology on $X(\Gamma)$, we define a map $\phi : \mathbb{H}_2 \to Y(\Gamma)$ given by $\tau \mapsto \Gamma\tau$. The topology on $Y(\Gamma)$ is defined using the quotient topology, where $\mathbb{H}_2$ inherits the subspace topology of the Euclidean topology on $\mathbb{C}$.

We define the topology on $X(\Gamma)$ in such a way that $Y(\Gamma)$ becomes a dense subset.

Since the action of the congruence subgroup Γ on $\mathbb{H}_2$ is properly discontinuous, $Y(\Gamma)$ is a Hausdorff space.

The subgroup $\Gamma_\tau = \{\gamma \in \Gamma \mid \gamma(\tau) = \tau\}$ is known as the isotropy subgroup.

Definition 2.0.11. An element $\tau \in \mathbb{H}_2$ is termed an *elliptic point* if the isotropy subgroup Γ_τ is non-trivial as a group of transformations, that is, if the containment $\{\pm I\}\Gamma_\tau \supset \{\pm I\}$ of matrix groups is proper. The corresponding point $\Gamma\tau \in Y(\Gamma)$ is also referred to as an elliptic point.

Remark 2.0.12. *For any congruence subgroup Γ and any $\tau \in \mathbb{H}_2$, there exists a neighborhood $U \subset \mathbb{C}$ of τ such that for any $\gamma \in \Gamma$, if $\gamma(U) \cap U \neq \emptyset$, then $\gamma \in \Gamma_\tau$. Additionally, U contains no other elliptic points.*

Definition 2.0.13. For $\tau \in \mathbb{H}_2$, *the period of* τ, denoted by h_τ, is defined as $|\frac{\{\pm I\}\Gamma_\tau}{\{\pm I\}}|$.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ (where $\gamma \neq \pm I$) and $\tau \in \mathbb{H}_2$. Then $\gamma(\tau) = \tau$ if and only if the equation $c\tau^2 - (a-d)\tau - b = 0$ has a solution. The fact that $\tau \in \mathbb{H}_2$ implies that the characteristic polynomial of the matrix is either $x^2 + 1$ or $x^2 \pm x + 1$.

Furthermore, we observe that $\gamma^3 = I$, $\gamma^4 = I$, or $\gamma^6 = I$. Hence, h_τ is finite. Moreover, it can be verified that the isotropy subgroups are cyclic.

Additionally, $h_\tau = h_{\gamma(\tau)}$ for any $\gamma \in \Gamma$. It is straightforward to show that the imaginary part of an elliptic point is strictly less than 2.

Now we can describe the local structure at any point $\Gamma\tau \in Y(\Gamma)$ (which is well-defined). Let $h = h_\tau$. Consider $\delta = \delta_\tau = \begin{pmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$.

It follows that $\delta(\tau) = 0$ and $\delta(\overline{\tau}) = \infty$. Clearly, $h_{\delta(\tau)} = h$. We choose a neighborhood U of τ as in Corollary 2.0.12, and define $\psi : U \to \mathbb{C}$ by $\psi = \rho \circ \delta$, where $\rho(z) = z^h$.

Now we define $\varphi : \phi(U) \to \psi(U)$ such that $\varphi(\phi(z)) = \psi(z)$. This map is a homeomorphism because fractional linear transformations are homeomorphisms. It can be verified that if U_1 and U_2 are two neighbourhoods of τ and φ_1, φ_1 are two maps like φ above, the map between $\varphi_1(\phi(U_1) \cap \phi(U_2))$ and $\varphi_2(\phi(U_1) \cap \phi(U_2))$ is analytic. Thus, this provides a manifold structure on $Y(\Gamma)$.

Remark 2.0.14. *Let $\mathcal{D} = \{\tau \in \mathbb{C} \mid |\tau| \geq 1, |Re(\tau)| \leq 1\}$. Then the map $f : \mathcal{D} \to Y(\mathrm{SL}_2(\mathbb{Z}))$ defined by $f(\tau) = \mathrm{SL}_2(\mathbb{Z})\tau$ is a surjective mapping.*

Remark 2.0.15. *The modular curve $Y(\Gamma)$ is a connected but non-compact complex manifold of dimension 1. To render it compact, it is enough to compactify the domain of the map φ, as the continuous. Hence, we include $\mathbb{Q} \cup \{\infty\}$ in $\mathbb{H}_2$. It is evident that $\overline{\mathbb{H}}_2 = \mathbb{H}_2 \cup \mathbb{Q} \cup \{\infty\}$ is compact.*

Definition 2.0.16. An element in the set $\Gamma \backslash (\mathbb{Q} \cup \{\infty\})$ is called as a *cusp* for the congruence subgroup Γ.

Theorem 2.0.17. *For any congruence subgroup* Γ*,* $X(\Gamma)$ *is a compact, connected, and Hausdorff space. Furthermore,* $X(\Gamma)$ *is a complex manifold of dimension 1, making it a Riemann Surface.*

Remark 2.0.18. *For the classical case, where* $X(\Gamma)$ *is a Riemann surface, we can utilize the Riemann-Hurwitz formula, genus, and Riemann-Roch theorem to determine the dimension of the set of all modular forms, the set of all cusp forms, and the set of all Eisenstein forms.*

Definition 2.0.19 (Differential Forms). When k is even, the weight k modular form $f(\tau)$ can be interpreted as the differential form $f(\tau)(d\tau)^{k/2}$.

This form is fixed under the action of every element of Γ, meaning it transforms in a way that maintains its overall structure. This invariance under Γ indicates that $f(\tau)(d\tau)^{k/2}$ is a Γ-invariant differential form.

Let M_k denote the set of all modular forms of weight k, and S_k denote the set of all cusp forms of weight k. Both sets are vector spaces over $\mathbb{C}$.

Definition 2.0.20 (Petersson inner product). Let f and g be modular forms of weight k for $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, where at least one of them is a cusp form. Let $\mathcal{D}$ be a fundamental domain for Γ. The function

$$\langle , \rangle : M_k \times S_k \longrightarrow \mathbb{C},$$

$$\langle f, g\rangle = \iint\limits_{\mathcal{D}} f(\tau)\overline{g(\tau)}y^{k-2}dxdy$$

is called the Petersson inner product.

Remark 2.0.21. *The differential* $f(\tau)\overline{g(\tau)}y^{k-2}dxdy$ *is invariant under the action of* $\mathrm{SL}_2(\mathbb{Z})$*, which means that the inner product is independent of the fundamental domain chosen*

The set of all Eisenstein series $E_k(\Gamma)$ of weight k is the orthogonal complement of $S_k(\Gamma)$ inside $M_k(\Gamma)$. In other words, we have the decomposition

$$M_k(\Gamma) = S_k(\Gamma) \oplus E_k(\Gamma).$$

For proofs and more detailed theory of classical modular forms, we refer to [11].

3

Bianchi Modular Forms

A Bianchi modular form, roughly speaking, is a modular form that is defined over $\mathrm{SL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field K.

3.1 Hyperbolic 3-space

Let K be an imaginary quadratic field and $\mathcal{O}_K$ is the ring of integers of K.

Consider the hyperbolic 3-space $\mathbb{H}_3$ (Upper half space)

$$\mathbb{H}_3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid z \in \mathbb{C}, t \in \mathbb{R}, t > 0\}.$$

We can also view $\mathbb{H}_3$ as a subspace of the skew field of the quaternions $\mathbb{H}$ with the basis over $\mathbb{R}$ being given by $1,\ i,\ j,\ k$. The injective map from $\mathbb{H}_3$ to $\mathbb{H}$ is given by

$$\mathbb{H}_3 \to \mathbb{H}$$
$$(z,t) \to z + tj$$

where $z = x + iy \in \mathbb{C}$.

We define the extended 3-dimensional upper half space to be

$$\overline{\mathbb{H}}_3 := \mathbb{H}_3 \cup K \cup \{\infty\}.$$

Sometimes, we also write ∞ as $j\infty$. Here, $j\infty$ indicates the direction we are taking infinity in the t component.

The points $\{(k,0) \mid k \in K\}$ and the point at infinity, ∞, are called as the "cusps".

The geometry of $\overline{\mathbb{H}}_3$ is hyperbolic. Thus, geodesic lines are vertical half-lines and semicircles centered on the plane $\{(z,t) \mid t=0\}$, while geodesic surfaces are vertical half-planes and hemispheres centered on $\{(z,t) \mid t=0\}$.

There is a hyperbolic structure defined on $\mathbb{H}_3$ (the metric coming from the line element) by

$$ds^2 = \frac{dx^2+dy^2+dt^2}{t^2}$$

with $z = x+iy$.

Every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{H}_3$ as an orientation preserving isometry via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,t) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2}{|cz+d|^2+|c|^2t^2}, \frac{t}{|cz+d|^2+|c|^2t^2} \right).$$

We can also view $\mathbb{H}_3$ as a set of matrices. Consider the hyperbolic 3- space $\mathbb{H}_3$ as

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} z & -t \\ t & \bar{z} \end{pmatrix} \middle| \, z \in \mathbb{C}, t \in \mathbb{R}, t>0 \right\}.$$

Now, we can see the transitive action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}_3$ by generalized fractional linear transformations, via

$$\gamma \cdot u = (\rho(a)u + \rho(b))(\rho(c)u+\rho(d))^{-1}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, $u = \begin{pmatrix} z & -t \\ t & \bar{z} \end{pmatrix} \in \mathbb{H}_3$ and $\rho(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$.

Fix a point $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{H}_3$. Then the stabilizer of ϵ is $\mathrm{SU}_2(\mathbb{C})$ and so we may identify the symmetric spaces

$$\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C}) \simeq \mathbb{H}_3.$$

For more details, we refer to the work of Ghate [14, §2.2].

We consider the topology (Matrix topology) on $\mathrm{SL}_2(\mathbb{C})$ that is defined by the following norm

$$\|M\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 3.1.1. A *Bianchi group* is a group of the form $\mathrm{SL}_2(\mathcal{O}_K)$ or $\mathrm{PSL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field K.

3.2 Congruence subgroups

Let the full Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$ be denoted by G, and let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_K$. Then we define the *principal congruence subgroup* of G, of level $\mathfrak{a}$, as follows

$$\Gamma(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\bmod \mathfrak{a}) \right\}.$$

This can be expressed equivalently as

$$\Gamma(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \;\middle|\; a-1, b, c, d-1 \in \mathfrak{a} \right\}.$$

Thus, $\Gamma(\mathfrak{a})$ consists of matrices $\gamma \in G$ such that γ is "congruent to the identity matrix modulo $\mathfrak{a}$". The principal congruence subgroup $\Gamma(\mathfrak{a})$ is a normal subgroup of G.

The index of $\Gamma(\mathfrak{a})$ in G is given by

$$[G : \Gamma(\mathfrak{a})] = N(\mathfrak{a})^3 \prod_{\mathfrak{p}|\mathfrak{a}} (1 - N(\mathfrak{p})^{-2}),$$

where $N(\mathfrak{a})$ represents the norm of the ideal $\mathfrak{a}$, and the product is taken over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ that divide $\mathfrak{a}$.

Next, we define

$$\Gamma_0(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, c \in \mathfrak{a} \right\},$$

and

$$\Gamma_1(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, a-1, c, d-1 \in \mathfrak{a} \right\}.$$

The index of $\Gamma_0(\mathfrak{a})$ in G is given by

$$[G : \Gamma_0(\mathfrak{a})] = N(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} (1 + N(\mathfrak{p})^{-1}).$$

The group $\Gamma_0(\mathfrak{a})$ is a subgroup of G that contains $\Gamma(\mathfrak{a})$ as a normal subgroup. The index of $\Gamma(\mathfrak{a})$ in $\Gamma_0(\mathfrak{a})$ is given by

$$[\Gamma_0(\mathfrak{a}) : \Gamma(\mathfrak{a})] = N(\mathfrak{a})^2 \prod_{\mathfrak{p}|\mathfrak{a}} (1 - N(\mathfrak{p})^{-1}),$$

where the product is taken over all prime ideals $\mathfrak{p}$ dividing $\mathfrak{a}$.

Definition 3.2.1. A congruence subgroup Γ of G is one that contains $\Gamma(\mathfrak{a})$ for some nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_K$.

For further information, please see [7, §2.5].

Let $\mathcal{O}_K$ be the ring of integers of an imaginary quadratic field K of class number $h(K) = 1$. Let $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K) := G$ be a subgroup of finite index with no elements of finite order.

Let $Y_\Gamma = \Gamma \backslash \mathbb{H}_3$ be a hyperbolic 3-manifold. Consider the Baily-Borel-Satake compactification of Y_Γ, which is $X_\Gamma^{BB} = \Gamma \backslash \overline{\mathbb{H}}_3$, obtained by adding the set of cusps [26].

The Borel-Serre compactification X_Γ^{BS} of Y_Γ , see [35, appendix], is a compact 3-fold with boundary ∂X_Γ^{BS} and with interior homeomorphic to Y_Γ.

The discriminant of K is less than -4, that is, K is neither equal to $\mathbb{Q}(i)$ nor $\mathbb{Q}(\sqrt{-3})$.

The Borel-Serre compactification of Y_Γ, which is X_Γ^{BS} obtained by adding a 2-torus to each cusp ∂X_Γ^{BS} (except for $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\sqrt{-3})$ for which we add spheres instead).

When K is $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, due to the extra units, the cross-sections of the cusps, which are again parametrized by the class group, are 2-orbifolds whose underlying manifolds are 2- spheres (torus folded by an involution).

For more details, we refer to the work of D. Rahm and Sengün [28].

3.3 Fundamental domain

We can also view

$$\mathbb{H}_3 = \{(z,t) \mid z \in \mathbb{C}, t \in \mathbb{R}^+\} \simeq \mathrm{GL}_2(\mathbb{C})/Z \cdot \mathrm{SU}_2(\mathbb{C})$$

is the hyperbolic 3 space, where Z denotes the diagonal matrices in $\mathrm{GL}_2(\mathbb{C})$. An invariant metric for the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{H}_3$ is then given by

$$ds = \frac{dzd\bar{z} + (dt)^2}{t^2}.$$

Definition 3.3.1. We say that a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ acts discontinuously on $\mathbb{H}_3$ if every compact subset of $\mathbb{H}_3$ meets only finitely many elements of its Γ-orbit.

Remark 3.3.2. *A subgroup Γ acts discontinuously if and only if it is discrete (in the matrix topology).*

Definition 3.3.3 (Fundamental domain). We define a fundamental domain or region for the action of $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$ on $\mathbb{H}_3$. That is, a subset $\mathfrak{F}_K$ of $\mathbb{H}_3$ with the following properties [7]

(i) $\mathfrak{F}_K$ is open in $\mathbb{H}_3$,

(ii) Every orbit of $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$ in $\mathbb{H}_3$ intersects $\mathfrak{F}_K$ at most once and intersects the closure of $\mathfrak{F}_K$ at least once.

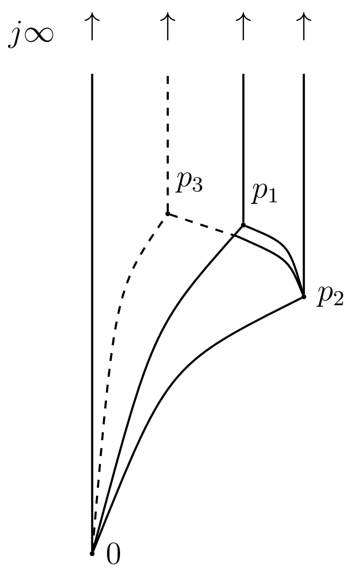

Figure 3.1: Fundamental domain for full group $\mathrm{SL}_2(\mathbb{Z}[i])$

For example, we are interested in the case $K = \mathbb{Q}(i)$, the ring of integers $\mathbb{Z}[i]$, and the full group $G = \mathrm{SL}_2(\mathbb{Z}[i])$.

This group is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \; R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

with the relations $TU = UT, S^2 = R^2 = (RS)^2 = (URS)^2 = (TS)^3 = (UR)^3 = 1$.

For the full group $\mathrm{SL}_2(\mathcal{O}_K)$, the fundamental domain $\mathfrak{F}_K$ is shown in Figure 3.1. For more details and for different imaginary quadratic fields, the fundamental domain $\mathfrak{F}_K$ is defined in [7, §2.3].

Consider three points on the unit sphere $P_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\sqrt{3}\right), P_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2}\right)$, and $P_3 = \left(0, \frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$. A fundamental domain for the action of G is given by vertices at $0, \infty$, and P_1, P_2, and P_3.

Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$. The fundamental domain $\mathfrak{F}_K$ for Γ is obtained by taking a union of fundamental domains for cosets of Γ within $\mathrm{SL}_2(\mathcal{O}_K)$. The stabilizer of P_2 is $\Gamma_{P_2} = \langle TS, UR \rangle$, which is of order 12 (explanation given in [7, p.59]).

Choose a fundamental domain $\mathfrak{F}_K$ for the action of Γ on $\overline{\mathbb{H}}_3$ with $\{0, j\infty\}$ as one of its edges. Let Γ_P be the stabilizer of a vertex P of $\mathfrak{F}_K$. Form a larger basic polyhedron by taking the union of the translates of $\mathfrak{F}_K$ by a finite subgroup Γ_P.

Call the union of the 12 translates of $\mathfrak{F}_K$ by Γ_P, the basic polyhedron, denoted B. It is a hyperbolic octahedron, and its 12 edges are precisely the images of $\{0, \infty\}$ under the action of Γ_P.

We refer to [7, page 59] for the diagram depicting the domain $\mathfrak{F}_K$ and the octahedron B. Note that B has a triangular face whose edges are the transforms of $\{0, \infty\}$ under I, TS, and $(TS)^2$, while the images of $\{0, \infty\}$ under $UR, (UR)^2$ are $\{\infty, i\}$ and $\{i, 0\}$ respectively.

For more details on the fundamental domain and basic polyhedron, we refer to the work of Cremona [7, §4.1].

The boundary of the fundamental domain, denoted as $\partial\mathfrak{F}_K$, is a combination of six faces given by $B_1 = \{0, P_1, \infty\}$, $B_2 = \{0, P_3, \infty\}$, $B_3 = \{P_2, P_3, \infty\}$, $B_4 = \{P_1, P_2, \infty\}$, $B_5 = \{0, P_1, P_2\}$, and $B_6 = \{0, P_2, P_3\}$. Integration over the boundary of the fundamental domain is the sum of integrations over each face.

The fundamental domain $\mathfrak{F}_K$ and the boundary of the fundamental domain $\partial\mathfrak{F}_K$ will be used in calculating the inner product formula given in §5.3.

3.4 Differentials on $\mathbb{H}_3$

The space of real 1-differential forms on $\mathbb{H}_3$ is given by the basis

$$\beta = (\beta_0, \beta_1, \beta_2) = (-\frac{dz}{t}, \frac{dt}{t}, \frac{d\bar{z}}{t}),$$

and denote the pullback of each β_i to $\mathrm{GL}_2(\mathbb{C})$ by ω_i.

Here, β is the standard basis of $\mathbb{H}_3$, and we also refer to β as the standard differential form.

A differential form on $\mathrm{GL}_2(\mathbb{C})$ is the inverse image of a differential form on $\mathbb{H}_3$ if and only if it can be written as $\varphi_0\omega_0 + \varphi_1\omega_1 + \varphi_2\omega_2$. Here, $\Phi = (\varphi_0, \varphi_1, \varphi_2)$ satisfies $\Phi(gkz) = \Phi(g)\rho(kz)$ for every $g \in \mathrm{GL}_2(\mathbb{C}), k \in \mathrm{SU}_2(\mathbb{C}), z \in Z$, and ρ is a fixed representation of $\mathrm{SU}_2(\mathbb{C})$.

Remark 3.4.1. *The space $\Omega^1(\mathbb{H}_3; \mathbb{C})$ is a 3-dimensional $C^\infty(\mathbb{H}_3)$-module spanned by the ele-*

ments dz/t, dt/t, and $d\bar{z}/t$.

3.4.1 The Hodge star operator and harmonicity

Let X be a Riemannian manifold of dimension n. There is a linear operator

$$\star : \Omega^r(X;\mathbb{C}) \longrightarrow \Omega^{n-r}(X;\mathbb{C})$$

on differential forms with the following properties. If α and β are r-forms, then

(i) $\star\star\alpha = (-1)^{r(n+1)}\alpha$,

(ii) $\alpha \wedge \star\beta = \beta \wedge \star\alpha$,

(iii) $\alpha \wedge \star\alpha = f dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, where $f > 0$ and $\{dx_1, \ldots, dx_n\}$ is a positively orientated orthogonal basis for $\Omega^n(X;\mathbb{C})$.

Definition 3.4.2. We define a Hermitian inner product on $\Omega^n(X;\mathbb{C})$ as

$$\langle \alpha, \beta \rangle := \int_X \alpha \wedge \star\beta.$$

The standard differentiation operator is defined as

$$d : \Omega^r(X;\mathbb{C}) \longrightarrow \Omega^{r+1}(X;\mathbb{C}),$$

is called the *exterior derivative*. Then, we define a map

$$\delta : \Omega^{r+1}(X;\mathbb{C}) \longrightarrow \Omega^r(X;\mathbb{C})$$

by using

$$\delta := (-1)^{n(r+1)+1} \star d \star .$$

It can be verified that δ acts as the adjoint of d with respect to the Hermitian inner product.

Definition 3.4.3. The *Laplace operator* is defined to be

$$\Delta = d\delta + \delta d.$$

Definition 3.4.4. A differential form ω is said to be *harmonic* if $\Delta\omega = 0$.

Proposition 3.4.5. *A differential form ω is harmonic if and only if it is closed and co-closed, that is, if $d\omega = d(\star\omega) = 0$.*

Proof. We know that d and δ are adjoint to each other under the Hermitian inner product. We see that

$$\langle \Delta\omega, \omega\rangle = \langle d\omega, d\omega\rangle + \langle \delta\omega, \delta\omega\rangle = \langle \omega, \Delta\omega\rangle.$$

This completes the proof as $\delta\omega = 0$ if and only if $d(\star\omega) = 0$. □

Let $F = (F_0, F_1, F_2)$ be a function on $\mathbb{H}_3$, and $F \cdot \beta = F_0\beta_0 + F_1\beta_1 + F_2\beta_2$ a 1-form on $\mathbb{H}_3$. Recall that $\star$ is the Hodge star operator for differential forms. By definition, we have

$$\star(F \cdot \beta) = -\frac{1}{2} i\bar{F}_1(\beta_0 \wedge \beta_2) + i\bar{F}_0(\beta_1 \wedge \beta_2) + i\bar{F}_2(\beta_0 \wedge \beta_1).$$

Observe that

$$d\left(\star(F \cdot \beta)\right) = iH_F\beta_0 \wedge \beta_1 \wedge \beta_2$$

with $H_F = t\left(\frac{\partial \bar{F}_1}{\partial z} + \frac{1}{2}\frac{\partial \bar{F}_0}{\partial t} + \frac{\partial \bar{F}_2}{\partial \bar{z}}\right)$.

Remark 3.4.6. *The differential form $F \cdot \beta$ is harmonic if and only if $F \cdot \beta$ and $\star(F \cdot \beta)$ are closed forms.*

Definition 3.4.7. We say that F is *slowly increasing* if $\exists\, N \geq 0$ such that

$$F\Bigg(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}(z,t)\Bigg) = O(|x|^N), \quad x \in \mathbb{R}$$

as $x \to \infty$ uniformly over compact sets in $\mathbb{H}_3$.

Definition 3.4.8 (Harmonic function). A function $F : \mathbb{H}_3 \to \mathbb{C}^3$ is said to be *harmonic* if

(i) $F \cdot \beta$ is a harmonic differential form,

(ii) F is slowly increasing.

3.5 Bianchi modular forms

3.5.1 Scalar valued Bianchi modular forms

We now define scalar-valued *Bianchi modular form* of weight 2. For, we wish to define a slashing operator such that $F \cdot \beta$ is an invariant differential under the action of a finite index subgroup Γ of $\mathrm{SL}_2(\mathcal{O}_K)$.

The space of left-invariant differential forms is now 3-dimensional, with a basis

$$\beta = (\beta_0, \beta_1, \beta_2) = \left(-\frac{dz}{t}, \frac{dt}{t}, \frac{d\bar{z}}{t}\right).$$

Calculating the Jacobian matrix of the transformation, we find that

$$\frac{d\left(z', t', \bar{z}'\right)}{d(z, t, \bar{z})} = \frac{1}{\left(|r|^2 + |s|^2\right)^2} \begin{pmatrix} r^2 & -2rs & s^2 \\ r\bar{s} & (r\bar{r} - s\bar{s}) & -\bar{r}s \\ \bar{s}^2 & 2\overline{rs} & \bar{r}^2 \end{pmatrix}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $r = \overline{cz + d}$ and $s = \bar{c}t$. In terms of the basis β for differentials, this becomes

$$\beta' = \left(|r|^2 + |s|^2\right)^{-1} \begin{pmatrix} r^2 & -2rs & s^2 \\ r\bar{s} & (r\bar{r} - s\bar{s}) & -\bar{r}s \\ \bar{s}^2 & 2\overline{rs} & \bar{r}^2 \end{pmatrix} \beta.$$

The complex modular group $\mathrm{SL}_2(\mathbb{C})$ acts on the space of differential 1-forms as $\beta' = \mathrm{J}(\gamma; (z, t))\beta$ with

$$\mathrm{J}(\gamma; (z, t)) = \frac{1}{\left(|r|^2 + |s|^2\right)^2} \begin{pmatrix} r^2 & -2rs & s^2 \\ r\bar{s} & (r\bar{r} - s\bar{s}) & -\bar{r}s \\ \bar{s}^2 & 2\overline{rs} & \bar{r}^2 \end{pmatrix}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $r = \overline{cz + d}$ and $s = \bar{c}t$.

Let $F : \mathbb{H}_3 \to \mathbb{C}^3$ be a harmonic function. Define the slash operator

$$(F \mid_\gamma)(z,t) = F(\gamma(z,t))\, \mathrm{J}(\gamma; (z,t)).$$

Definition 3.5.1 (Bianchi modular form). Let K be an imaginary quadratic field with class number 1 and the ring of integers $\mathcal{O}_K$. Let Γ be a subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ of finite index. Then, *Bianchi modular form of weight* 2 for Γ is a function $F : \mathbb{H}_3 \to \mathbb{C}^3$ satisfying

1. F is a harmonic function,

2. $F \mid_\gamma = F$ for all $\gamma \in \Gamma$.

   We denote the set of all Bianchi modular forms of weight 2 for Γ by $M_2(\Gamma)$.

Definition 3.5.2 (Bianchi cusp form). If $F \in M_2(\Gamma)$ satisfies the additional properties that $\int\limits_{\mathbb{C}/\mathcal{O}_K} F \mid_\gamma (z,t) dz = 0$ for every $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$, we call it a Bianchi cusp form.

3.5.2 Vector valued Bianchi modular forms

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and $u = z + jt \in \mathbb{H}_3$, let us define the multiplier system

$$j(\gamma, u) = \begin{pmatrix} cz + d & -ct \\ \bar{c}t & \overline{cz + t} \end{pmatrix}.$$

Given a function $F : \mathbb{H}_3 \to \mathbb{C}^{k+1}$ and $\gamma \in \mathrm{SL}_2(\mathbb{C})$, we define the *slash operator*

$$\left(F|_\gamma \right)(u) := \mathrm{Sym}^k \left(j(\gamma, u)^{-1} \right) F(\gamma u).$$

We now define vector-valued *Bianchi modular form* of weight 2. Let $F : \mathbb{H}_3 \to \mathbb{C}^3$ be a harmonic function. Define the slash operator

$$\left(F|_\gamma \right)(u) := \mathrm{Sym}^2 \left(j(\gamma, u)^{-1} \right) F(\gamma u),$$

where

$$j(\gamma, u) = \begin{pmatrix} cz + d & -ct \\ \bar{c}t & \overline{cz + t} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and Sym^2 is the 2-nd symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$.

Denote by $S_2(\Gamma)$ the space of all Bianchi cusp forms of weight 2 for Γ and the set of cusps of Γ (for both Satake and Bailey-Borel compactification) can be identified with the orbit space $\Gamma \backslash \mathbb{P}^1(K)$.

Since we have $F : \mathbb{H}_3 \to \mathbb{C}^3$ and

$$\mathrm{Sym}^2 \left(j(\gamma, u)^{-1} \right) = \frac{1}{|r|^2 + |s|^2} \begin{pmatrix} r^2 & 2r\bar{s} & \bar{s}^2 \\ -rs & |r|^2 - |s|^2 & \bar{r}\bar{s} \\ s^2 & -2\bar{r}s & \bar{r}^2 \end{pmatrix},$$

it follows that

$$(F|_\gamma)(u) = \frac{1}{|r|^2+|s|^2}\begin{pmatrix} r^2 & 2r\bar{s} & \bar{s}^2 \\ -rs & |r|^2-|s|^2 & \bar{r}\bar{s} \\ s^2 & -2\bar{r}s & \bar{r}^2 \end{pmatrix} F(\gamma u)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $r = \overline{cz+d}$ and $s = \bar{c}t$. The 1-forms $\beta_0 := -\frac{dz}{t}, \beta_1 := \frac{dt}{t}, \beta_2 := \frac{d\bar{z}}{t}$ form a basis of differential 1-forms on $\mathbb{H}_3$.

The modular group $\mathrm{SL}_2(\mathbb{C})$ acts on the space of differential 1 -forms as

$$\gamma \cdot {}^t(\beta_0, \beta_1, \beta_2) = \mathrm{Sym}^2(j(\gamma, z))^t(\beta_0, \beta_1, \beta_2).$$

Here Sym^2 is the 2-nd symmetric power representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ and ${}^t(\beta_0, \beta_1, \beta_2)$ is transpose of $(\beta_0, \beta_1, \beta_2)$.

With both the definitions of weights, the differential $F \cdot \beta$ is Γ invariant. We can also generalize vector-valued *Bianchi modular form* to arbitrary weight k by changing Sym^2 to Sym^k.

Definition 3.5.3 (Bianchi modular form). Let K be an imaginary quadratic field with class number 1 and the ring of integers $\mathcal{O}_K$. Let Γ be a subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ of finite index. Then, *Bianchi modular form of weight* 2 for Γ is a function $F : \mathbb{H}_3 \to \mathbb{C}^3$ satisfying

1. F is a harmonic function,

2. $F|_\gamma = F$ for all $\gamma \in \Gamma$.

We denote the set of all Bianchi modular forms of weight 2 by $M_2(\Gamma)$.

Combining both the definitions of vector-valued and scalar-valued Bianchi modular forms, we can say: A *Bianchi modular form* of weight 2 for a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$ is a real analytic, Γ-invariant function $F : \mathbb{H}_3 \to \mathbb{C}^3$. In other words, the function F satisfies the invariance property $F|_\gamma = F$ for all $\gamma \in \Gamma$.

Definition 3.5.4 (Bianchi cusp form). If $F \in M_2(\Gamma)$ satisfies the additional properties that $\int_{\mathbb{C}/\mathcal{O}_K} F|_\gamma (z,t)dz = 0$ for every $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$, we call it a Bianchi cusp form.

The weights of Bianchi modular forms can be scalar-valued [7] or, more generally, vector-valued [40] (see also [14]). We can also define Bianchi modular forms using the Adelic setting (i.e., as Automorphic forms). For the definition of Automorphic forms, we refer to the work of Ghate [14].

The last condition is equivalent to stating that the constant coefficient in the Fourier-Bessel expansion of $F|_\gamma$ is equal to zero for every $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$. The Γ-invariance implies that F has a Fourier-Bessel expansion of the form

$$F(z,t) = \sum_{\alpha \in \mathcal{O}_K, \alpha \neq 0} c(\alpha) t^2 \mathbf{K}\left(\frac{4\pi|\alpha|t}{\sqrt{d_K}}\right) \psi\left(\frac{\alpha z}{\sqrt{d_K}}\right),$$

where $\psi(z) = e^{2\pi(z+\bar{z})}$ and $\mathbf{K}(t) = \left(-\frac{i}{2}K_1(t), K_0(t), \frac{i}{2}K_1(t)\right)$, with K_0 and K_1 being the modified Bessel functions satisfying the differential equation

$$\frac{dK_j}{dt^2} + \frac{1}{t}\frac{dK_j}{dt} - \left(1 + \frac{1}{t^{2j}}\right) K_j = 0, \quad j = 0, 1,$$

and decreasing rapidly at infinity.

Let $S_2(\Gamma)$ be the space of all Bianchi cusp forms of weight 2 for a subgroup Γ of $\mathrm{SL}_2(\mathcal{O}_K)$. Let $F = (F_0, F_1, F_2)$ and $S = (S_0, S_1, S_2)$ with at least one of them being a cusp form. According to [25, p. 549], the inner product is given by

$$\langle F, S \rangle = \frac{1}{12i[G:\Gamma]} \int_{\Gamma\backslash\mathbb{H}_3} F \cdot \beta \wedge \star(S \cdot \beta).$$

The set of all Eisenstein Bianchi modular forms $E_2(\Gamma)$ is the orthogonal complement of $S_2(\Gamma)$ inside $M_2(\Gamma)$. In other words, we have a decomposition

$$M_2(\Gamma) = S_2(\Gamma) \oplus E_2(\Gamma).$$

If Γ is a congruence subgroup of the form $\Gamma_1(N)$, the dimension of $E_2(\Gamma)$ can be computed using [34, Proposition 4.4].

4

Eisenstein differential forms

Let F be a Bianchi modular form for a subgroup $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$, i.e., $F \in M_2(\Gamma)$. We can attach a differential form to the Bianchi modular form. For example, the differential form attached to F is denoted as $F \cdot \beta$, where β is a standard differential form.

For an explicit exposition of the differential form attached to Bianchi modular forms, refer to the work of Ghate [14](also for CM fields in [15]).

A differential form associated with an Eisenstein series E, denoted as $E \cdot \beta$, where β is a standard differential form, can be considered as an Eisenstein differential form. This product results in a differential form that retains the modular properties of the original Eisenstein series.

4.1 Eisenstein series of weight 0

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$ be a discrete group and $\zeta \in \mathbb{P}^1(\mathbb{C})$ be a cusp of Γ. If $M \in \Gamma$ and $u = z + tj \in \mathbb{H}_3$, then $M \cdot u = z_M + t_M j \in \mathbb{H}_3$.

We have the stabilizer subgroup

$$\Gamma_\zeta := \{M \in \Gamma : M\zeta = \zeta\}$$

and its maximal unipotent subgroup

$$\Gamma'_\zeta := \left\{M \in \Gamma_\zeta : \mathrm{tr}\, M = \pm 2\right\}.$$

Let $A \in \mathrm{PSL}_2(\mathbb{C})$ be such that $A\zeta = \infty$. Then

$$\left(A\Gamma A^{-1}\right)_\infty = A\Gamma_\zeta A^{-1}, \quad \left(A\Gamma A^{-1}\right)'_\infty = A\Gamma'_\zeta A^{-1}.$$

We consider the series

$$E_A^*(u,s) := \sum_{M \in \left(A\Gamma A^{-1}\right)_\infty \backslash A\Gamma A^{-1}} t_M^{1+s}.$$

Eisenstein series E_A^* converges absolutely and uniformly on compact subsets of $\mathbb{H}_3 \times \{s : \mathrm{Re}\, s > 1\}$, and clearly $E_A^*(\cdot, s)$ is an $A\Gamma A^{-1}$ -invariant function on $\mathbb{H}_3$.

Hence, the Eisenstein series of weight 0 is

$$E_A(u,s) := E_A^*(Au,s) = \sum_{M \in \Gamma'_\zeta \backslash \Gamma} (t_{AM})^{1+s}$$

converges absolutely and uniformly on compact subsets of $\mathbb{H}_3 \times \{s : \mathrm{Re}\, s > 1\}$ and is a $\Gamma-$ invariant function on $\mathbb{H}_3$. We call E_A the Eisenstein series for cusp ζ.

4.2 Ito's differential forms [22] for full group $\mathrm{SL}_2(\mathcal{O}_K)$

Let us denote the elements of $\mathbb{H}_3$ as quaternion numbers $u = z + jt \in \mathbb{H}_3$, where $j^2 = -1$ and $ij = -ji$. Here, $z(u) = z$ and $t(u) = t$ if $u = z + jt$. A matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K)$ acts on $\mathbb{H}_3$ as

$$Mu := (au+b)(cu+d)^{-1},$$

where the right-hand side is taken in the skew field of quaternions. For complex numbers $(m,n) \neq (0,0)$, consider a matrix $M = \begin{pmatrix} * & * \\ m & n \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K)$ and define

$$t(m,n;u) = t(Mu),$$
$$J(m,n;u) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{z}}\right) z(Mu).$$

The Eisenstein series is defined as

$$E(u;s) := \sum_{m,n \in \mathcal{O}_K}' J(m,n;u) t(m,n;u)^s, \quad \mathrm{Re}(s) > 1.$$

It can be analytically continued to the entire s-plane. Here, the prime on the summation symbol means to omit the meaningless terms, i.e., the term corresponding to $m = n = 0$.

Let ω_E be a complex-valued differential form on $\mathbb{H}_3$ given by

$$\omega_E = E(u,0) \begin{pmatrix} dz \\ dt \\ d\bar{z} \end{pmatrix}.$$

According to [22], this differential form is closed and invariant under $\mathrm{SL}_2(\mathcal{O}_K)$.

Consider $D(\mathcal{O}_K) := w_1\bar{w}_2 - \bar{w}_1 w_2$ if $\mathcal{O}_K = \mathbb{Z}w_1 + \mathbb{Z}w_2$ with $\mathrm{Im}\,(w_1/w_2) > 0$, and $\mathcal{O}'_K = D(\mathcal{O}_K)^{-1}\overline{\mathcal{O}_K}$. The indefinite integral of ω_E can be expressed as a Fourier expansion. These integrals produce periods of the function

$$\widetilde{H}(u) = G_2(0)(z - \bar{z}) - \frac{4\pi}{D(\mathcal{O}_K)} t \sum_{m \in \mathcal{O}_K, n \in \mathcal{O}'_K}' \frac{\bar{m}n}{|mn|} K_1(4\pi|mn|t) e(mnz).$$

For each non-negative integer k, Ito's Eisenstein series is defined by generalized Hecke's summation tricks

$$G_k(x) := \sum_{w \in \mathcal{O}_K}' (w+x)^{-k}|w+x|^{-s}\Big|_{s=0}.$$

Here, the value at $s = 0$ is understood in the sense of analytic continuation. The prime in the summation symbol omits the terms corresponding to $m = n = 0$. Furthermore, $e(z) = \exp(2\pi i(z + \bar{z}))$ and $K_1(t)$ denotes the modified Bessel function of the second kind.

Recall the following important theorem regarding the periods of the Eisenstein series in the context of Bianchi modular forms. The function $\widetilde{H}$ is harmonic with respect to the Riemannian structure of $\mathbb{H}_3$ given by the $\mathrm{SL}_2(\mathbb{C})$-invariant Riemannian metric $\frac{1}{t^2}\left(dx^2 + dy^2 + dt^2\right) \quad (u = z + jt \in \mathbb{H}_3, z = x + iy)$.

Theorem 4.2.1. *Define the map* $\Psi : \mathrm{SL}_2(\mathcal{O}_K) \to \mathbb{C}$ *as*

$$\Psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} G_2(0)I(\frac{a+d}{c}) - D(a,c), & c \neq 0, \\ G_2(0)I\left(\frac{b}{d}\right), & c = 0 \end{cases}$$

with $I(z) := z - \bar{z}$ *and*

$$D(a,c) := \frac{1}{c}\sum_{r \in K/cK} G_1\left(\frac{ar}{c}\right) G_1\left(\frac{r}{c}\right).$$

Then

$$\widetilde{H}(Au) = \widetilde{H}(u) + \Psi(A)$$

for every A *in* $\mathrm{SL}_2(\mathcal{O}_K)$.

Let ω_E be the complex-valued differential form defined as

$$\omega_E = E(u,0)\begin{pmatrix} dz \\ dt \\ d\bar{z} \end{pmatrix}.$$

To compute the $\star$ operator of ω_E, we apply the operator $\star$ as defined in Section 3.4 to ω_E.

Until now, we have considered Ito's differential forms for the full group $\mathrm{SL}_2(\mathcal{O}_K)$. Now, we are generalizing Ito's differential forms for subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$.

4.3 Generalization of Ito's differential forms [22] for subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$

Let Γ be a subgroup of finite index in $\mathrm{SL}_2(\mathcal{O}_K)$ and $\{\kappa_1, \ldots, \kappa_h\}$ be a complete set of representatives of the cusps of Γ. For each i, let Γ_i denote the stabilizer of κ_i in Γ, and Γ_i' its maximal unipotent subgroup. Also, let κ_i be a cusp of Γ such that $\kappa_i = \sigma_i^{-1}\infty$. Define Γ_i as the stabilizer of κ_i in Γ, and Γ_i' as its maximal unipotent subgroup

$$\Gamma_i' = \{M \in \Gamma : M\kappa_i = \kappa_i, M = I \text{ or parabolic}\}.$$

We can express $\sigma_i \Gamma_i \sigma_i^{-1}$ as

$$\sigma_i \Gamma_i \sigma_i^{-1} = \left\{ \pm \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in L_i \right\},$$

where L_i is a lattice in $\mathbb{C}$.

Now, we define the weight-zero Eisenstein series for Γ at the cusp κ_i as

$$E_i(u, s) = \sum_{\sigma \in \Gamma_i' \backslash \Gamma} t(\sigma_i \sigma(u))^{1+s}.$$

Here, $\sigma_i \in \Gamma$ such that $\sigma_i(\kappa_i) = \infty$ and $t(u)$ denotes the t component of $u = (z, t)$. The summation over $\Gamma_i' \backslash \Gamma$ can be replaced with $\Gamma_i \backslash \Gamma$ by dividing by the finite index $[\Gamma_i' : \Gamma_i]$. According to [13], this function satisfies the following properties

1. It is invariant under Γ, i.e., $E_i(\gamma(u), s) = E_i(u, s)$ for all $\gamma \in \Gamma$.

2. It satisfies the differential equation $\Delta E_i(u, s) = (s^2 - 1)E_i(u, s)$, where Δ is the Laplace operator on $\mathbb{H}_3$.

   For $u = z + tj \in \mathbb{H}_3$, and $z = x + iy \in \mathbb{C}$, Δ represents the Laplace operator on $\mathbb{H}_3$ given by
$$\Delta = t^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} \right) - t\frac{\partial}{\partial t}.$$

3. It has a Fourier expansion of the form

$$E_i(\sigma_j^{-1}u, s) = \delta_{ij}a_0t^{1+s} + b_0t^{1-s} + \sum_{0\neq\alpha\in L_i'} c(\alpha, s)tK_s(2\pi|\alpha|t)e^{2\pi i\langle\alpha,z\rangle},$$

where L_i' denotes the dual lattice of L_i and $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{C}$. The coefficients a_0, b_0 are independent of i and j, and δ_{ij} is the Kronecker delta function. Additionally, $K_s(t)$ denotes the modified Bessel function of the second kind.

4. It has a meromorphic continuation to $s \in \mathbb{C}$, without any pole in the half plane $\mathrm{Re}(s) > 0$ except for possibly finitely many simple poles in $(0, 1]$. It has a simple pole at $s = 1$ with residue equal to $|L_i|\mathrm{vol}(\Gamma)^{-1}$. Here $|L_i|$ is the Euclidean area of a fundamental parallelogram of L_i, and $\mathrm{vol}(\Gamma)$ is the covolume of Γ.

The last condition can be expressed as

$$\lim_{s\to 1}(s-1)E_i(u, s) = C$$

where C is a constant independent of i.

4.3.1 Eisenstein differential forms of weight 2

Let

$$J(\sigma; u) = \left(\frac{\partial}{\partial z}, \ \frac{\partial}{\partial t}, \ \frac{\partial}{\partial \bar{z}}\right) z(\sigma(u)), \quad u \in \mathbb{H}_3, \sigma \in \Gamma.$$

Define Eisenstein series of the weight 2 for Γ at the cusp κ_i as

$$G_i(u, s) = \sum_{\sigma\in\Gamma_i'\backslash\Gamma} J(\sigma; u)t(\sigma_i\sigma(u))^{1+s},$$

which converges absolutely for $\mathrm{Re}(s)$ large enough. If κ_j is another cusp of Γ, the function $u \mapsto G_i(\sigma_j u, s)$ is invariant under the action of the lattice L_j corresponding to

$\sigma_j \Gamma'_j \sigma_j^{-1}$. Define the Eisenstein differential to be the complex-valued differential form

$$\omega_i(s) = G_i(u,s) \begin{pmatrix} dz \\ dt \\ d\bar{z} \end{pmatrix}$$

for $\mathrm{Re}(s)$ large enough.

Proposition 4.3.1. *The differential form $\omega_i(s)$ is invariant with respect to Γ, has meromorphic continuation to $s \in \mathbb{C}$ and is closed at $s = 0$.*

Proof. Let φ_γ be the automorphism $u \mapsto \gamma(u)$ of $\mathbb{H}_3$. From the relation

$$J(\sigma; \gamma(u))\mathfrak{J}(\gamma, u) = J(\sigma\gamma; u), \quad \gamma \in \Gamma,$$

where $\mathfrak{J}(\gamma, u)$ denotes the Jacobian matrix

$$\mathfrak{J}(\gamma, u) = \frac{\partial(z(\gamma u), t(\gamma u), \bar{z}(\gamma u))}{\partial(z, t, \bar{z})}.$$

It follows that

$$G_i(\gamma(u), s)\mathfrak{J}(\gamma, u) = G_i(u, s).$$

Then

$$\begin{aligned} \varphi_\gamma^* \omega_i(s) &= G_i(\gamma(u), s)\beta(\gamma(u)) \\ &= G_i(u, s)\mathfrak{J}(\gamma, u)^{-1}\mathfrak{J}(\gamma, u)\beta(u) \\ &= \omega_i(s), \end{aligned}$$

where we have used the transformation law, hence $\omega_i(s)$ is invariant under Γ.

Writing component-wise $G_i = (G_i^{(1)}, G_i^{(2)}, G_i^{(3)})$ and $J = (J^{(1)}, J^{(2)}, J^{(2)})$, for each $1 \leq j \leq 3$ we have that

$$G_i^{(j)}(u, s) = \sum_{\sigma \in \Gamma'_i \backslash \Gamma} J^{(j)}(\sigma; u) t(\sigma_i \sigma(u))^{1+s}.$$

More explicitly, if $\sigma = \left(\begin{smallmatrix} * & * \\ m & n \end{smallmatrix}\right)$, then the formulas

$$t(\sigma(u)) = (|mz+n|^2 + |mt|^2)^{-1}t$$

$$J(\sigma; u) = (|mz+n|^2 + |mt|^2)^{-2}((\overline{mz+n})^2, 2(\overline{mz+n})\bar{m}t, -(\bar{m}t)^2),$$

imply that

$$J(\sigma; u) = t(\sigma(u))^2\Big(\frac{(\overline{mz+n})^2}{t^2}, \frac{2(\overline{mz+n})\bar{m}}{t}, -\bar{m}^2\Big)$$

and so, for $t \gg 0$ we have

$$G_i^{(j)}(u,s) \ll \sum_{\sigma \in \Gamma_i' \backslash \Gamma} \frac{t(\sigma_i\sigma(u))^{3+s}}{t^{3-j}} = \frac{1}{t^{3-j}} E_i(u, s+2)$$

for each $1 \leq j \leq 3$ with the implied constant depending on u. Recall that the right-hand side is a scalar-valued Eisenstein series of weight zero for Γ at cusp κ_i.[1] It follows then from Corollary 3.2.4 of [13] that $G_A^{(j)}$ has polynomial growth at all cusps κ_n of Γ in the sense that

$$G_i^{(j)}(\sigma_n^{-1}u, s) = O(t^K)$$

for some constant $K > 0$ uniformly with respect to u. Moreover, by the same argument as Proposition 3.2.5 of [13] we see that each $G_A^{(j)}(u,s)$ satisfies the differential equation

$$\Delta G_i^{(j)}(u,s) = (s^2-1)G_i^{(j)}(u,s)$$

and are thus real analytic functions of u, holomorphic in s for $\mathrm{Re}(s)$ large enough. Then from the theory of Eisenstein series [13, §6.1] it follows that each $G_i^{(j)}$ has meromorphic continuation to $s \in \mathbb{C}$ and in particular holomorphic at $s = 0$, hence so is $\omega_i(s)$. Then by a well-known result of Harder the differential form $\omega_i(0)$ is closed [19, §4.2] (see also [22, §2]). □

Corollary 4.3.2. *There exists a function $H_i(u,s)$ such that $dH_i(u,0) = \omega_i(0)$ for each i. Also, one has $\Delta H_i(u,0) = 0$.*

Proof. The existence of $H_i(u,s)$ is clear. The second assertion follows from the Bessel

[1] This is denoted $E_A(P,s)$ in [13].

differential equation (3.5.2). □

Note that $0 = \Delta H_i(u,0) = \star d \star dH_i(u,0) = \star d \star d\omega_i(0)$. If we express the Fourier expansion of $H_i(u,s)$ as simply

$$a_0 t^{1+s} + b_0 t^{1-s} + \sum_{0 \neq \alpha \in L'} c(\alpha,s) t K_s(2\pi|\alpha|t) e^{2\pi i \langle \alpha, z\rangle}, \quad u = (z,t)$$

and for $s = 0$, the term $b_0 t^{1-s}$ must be replaced with $b_0 t \log t$ [13, §3]. Then, by using the relations

$$\frac{d}{dr} r^s K_s(r) = -r^s K_{s-1}(r),$$

for $\mathrm{Re}(s) > -\frac{1}{2}$, if we set $c(\alpha) = c(\alpha, 0)$, we can express the Fourier coefficients of $G_i(u,0)$ as simply

$$\begin{aligned} c_1(\alpha) &= c(\alpha)\frac{\partial}{\partial z} e^{2\pi i\langle\alpha,z\rangle} = 2\pi i c(\alpha)\frac{\partial}{\partial z}\langle\alpha, z\rangle, \\ c_2(\alpha) &= c(\alpha)\frac{\partial}{\partial t} tK_s(2\pi|\alpha|t) = 2\pi|\alpha|c(\alpha), \\ c_3(\alpha) &= c(\alpha)\frac{\partial}{\partial \bar{z}} e^{2\pi i\langle\alpha,z\rangle} = 2\pi i c(\alpha)\frac{\partial}{\partial \bar{z}}\langle\alpha, z\rangle. \end{aligned}$$

We refer to the recent paper of Miao-Nguyen-Wong [27] about explicit H_i.

We can compute $d(\star\omega_i)$ using the formula given in § 3.4 and considering the cusp $\kappa_i = \infty$. In this case

$$(F_0, F_1, F_2) = (G_i^{(1)}, G_i^{(2)}, G_i^{(3)}).$$

We have

$$d(\star(F \cdot \beta)) = iH_F \beta_0 \wedge \beta_1 \wedge \beta_2$$

with the function H_F given by

$$H_F = t\left(\frac{\partial \overline{G_i^{(2)}}}{\partial z} + \frac{1}{2}\frac{\partial \overline{G_i^{(1)}}}{\partial t} + \frac{\partial \overline{G_i^{(3)}}}{\partial z}\right).$$

We expect that $\int_{\Gamma\backslash\mathbb{H}_3} d(\star(F\cdot\beta))$ can be computed using Rankin-Selberg "unfolding".

4.4 Hida's differential forms

Consider Hida's differential form at the cusp $\kappa = \infty$ [21, §10]

$$\omega_E := \omega_\kappa(u; s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{t^s \cdot e_1 \cdot \rho_2(j(\gamma, u)^t)^{-1}}{(|cz + d|^2 + |ct|^2)^s} du$$

where

- $e_1 = (1, 0, 0)$ is the unit vector,
- $du = (dz, -dt, -d\bar{z})$ is a differential 1 form,
- $\rho_2 = \mathrm{Sym}^2(\mathbb{C}^2)$ is the second symmetric tensor representation of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$,
- for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\xi \backslash \Gamma$ and $u = \begin{pmatrix} z & -t \\ t & \bar{z} \end{pmatrix} \in \mathbb{H}_3$, the modular factor is defined by
$$j(\gamma; u) = \begin{pmatrix} cz + d & -ct \\ \bar{c}t & \overline{cz + d} \end{pmatrix}.$$

In particular, Hida is using $\mathrm{Sym}^2(j(\gamma, u))$ rather than $\mathrm{Sym}^2(j(\gamma, u)^{-1})$. By applying the Hodge $\star$ operator (cf. § 3.4) to the differential form ω_E, we can compute the $\star\omega_E$ for Hida's Eisenstein differential form at $s = 0$.

In this chapter, we have discussed Ito's differential forms of weight 2 for the full group $\mathrm{SL}_2(\mathcal{O}_K)$ and generalized these to Ito's differential forms for subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$. We have also examined Hida's differential forms. Both Ito's and Hida's differential forms are examples of Eisenstein differential forms.

In the formula given in §5.4, we are discussing the differential form $E \cdot \beta$, denoted by the differential form ω_E, and we are integrating $\star\omega_E$. Thus, we can consider ω_E as Ito's and Hida's differential form.

To compute the $\star$ operator of ω_E, we apply the operator $\star$ as defined in Section 3.4 to ω_E.

5

Modular symbols

5.1 Classical modular symbols

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. The topological space $X(\Gamma) = \Gamma \backslash \overline{\mathbb{H}}_2$ has a natural structure of a smooth, compact complex space of topological dimension 2. Any smooth path from a to b in $\mathbb{H}_2$ projects to a closed path in the quotient space $X(\Gamma) = \Gamma \backslash \overline{\mathbb{H}}_2$, and its homology class in $\mathrm{H}_1(X(\Gamma); \mathbb{Z})$ solely depends on a and b, independent of the chosen path due to the simply connectedness of $\mathbb{H}_2$. We denote this homology class by $\{a, b\}_\Gamma$, or simply $\{a, b\}$ when the group Γ is evident from the context.

Suppose that a and b are cusps that are equivalent to mod Γ. We can use them to construct a homology class. We take any reasonable oriented path between a and b on $\mathbb{H}_2$, say the geodesic directed from a to b, and then take the image mod Γ. Since a and b are equivalent mod Γ, the image becomes a closed oriented 1-curve on $X(\Gamma)$, i.e. a 1-cycle. Thus we get a class in $\mathrm{H}_1(X(\Gamma); \mathbb{Z})$. Let us denote this class by $\{a, b\}$. Note that this notation looks a lot like the set $\{a, b\}$, but it is not. It really represents an ordered pair since if we change the roles of a and b, we reverse the orientation on the cycle and thus get the opposite class $\{b, a\} = -\{a, b\}$. This can be confusing, but the notation is traditional.

Now consider the pairing $S_2(\Gamma) \times \mathrm{H}_1(X(\Gamma); \mathbb{Z}) \to \mathbb{C}$ given by integration

$$(f, \{a, b\}) \longmapsto 2\pi i \int_a^b f(z)dz := \langle \{a, b\}, f \rangle$$

where $S_2(\Gamma)$ is the space of all classical cusp forms of weight 2.

This is independent of the path between a and b since f is holomorphic (essentially, this boils down to Cauchy's theorem from complex analysis). Note also that f has to be a cusp form for the integral to make sense. If f is nonvanishing at the cusp, say when f is an Eisenstein series, and the integral diverges. We can extend from integral homology to real homology to get a pairing

$$S_2(\Gamma) \times \mathrm{H}_1 (X(\Gamma); \mathbb{R}) \to \mathbb{C}.$$

This is done in an obvious way. First choose an integral basis of $\mathrm{H}_1 (X(\Gamma); \mathbb{Z})$. Any class in $\mathrm{H}_1 (X(\Gamma); \mathbb{R})$ can be written as a linear combination of this basis with real coefficients, so we can extend the pairing using linearity.

Definition 5.1.1. The modular symbol attached to the pair of cusps a, b is the real homology class $\{a, b\} \in \mathrm{H}_1 (X(\Gamma); \mathbb{R})$.

Here are some basic properties of modular symbols

1. $\{a, b\} = -\{b, a\}$, (2-term relation)
2. $\{a, b\} = \{a, c\} + \{c, b\}$, (3-term relation)
3. $\{ga, gb\} = \{a, b\}$ for all $g \in \Gamma$, (Γ-action)
4. $\{a, ga\} \in \mathrm{H}_1 (X(\Gamma); \mathbb{Z})$,
5. $\{a, ga\} = \{b, gb\}$.

These are all easy to verify. The 2-term relation just says that reversing the limits of integration introduces a minus sign. The 3-term relation says that we can divide an integral into two integrals by introducing a common new endpoint. Perhaps the last is the most complicated. It can be proved by considering the square in Figure 5.1.

Properties (4) and (5) imply that we have constructed a map

$$\begin{aligned} \Gamma &\longrightarrow \mathrm{H}_1 (X(\Gamma); \mathbb{Z}) \\ g &\longmapsto \{a, ga\} \end{aligned}$$

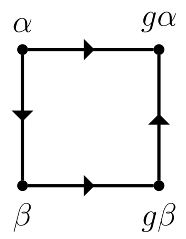

Figure 5.1: $\{\alpha, g\alpha\} = \{\beta, g\beta\}$

that is independent of a. By the way, our construction of modular symbols means that all we can say a priori is that $\{a, b\} \in \mathrm{H}_1(X(\Gamma); \mathbb{R})$, i.e., $\{a, b\}$ is a real homology class. However, the theorem of Manin-Drinfeld tells us that this class often lies in the rational homology $\mathrm{H}_1(X(\Gamma); \mathbb{Q}) = \mathrm{H}_1(X(\Gamma); \mathbb{Z}) \otimes \mathbb{Q}$.

Now recall that, the map $\phi : \overline{\mathbb{H}}_2 \to X(\Gamma)$ is the natural quotient map.

Proposition 5.1.2 (Manin). *Let $a \in \overline{\mathbb{H}}_2$. The map*

$$\Gamma \to \mathrm{H}_1(X(\Gamma); \mathbb{Z})$$

is defined as

$$g \longmapsto \{a, ga\}$$

is a surjective group homomorphism that does not depend on the choice of a. This kernel of this homomorphism is generated together by the commutators, the elliptic elements, and the parabolic elements of the group Γ.

Proposition 5.1.3 (Distinguished classes). *Let $J = \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ be the set of right cosets. We define the map*

$$\xi : J \to \mathrm{H}_1(X(\Gamma); \mathbb{R})$$

as follows: if $j \in J$ and g is any representative of the class j, then

$$\xi(j) = \{g(0), g(i\infty)\}.$$

Obviously, this class does not depend on the choice of the representative g. We have thereby defined a finite family of homology classes $\xi(J)$, and we shall call its elements distinguished classes.

Theorem 5.1.4 (Manin-Drinfeld). *If Γ is a congruence subgroup, and a and b are cusps of Γ then $\{a, b\} \in \mathrm{H}_1(X(\Gamma); \mathbb{Q})$.*

Let $\mathbb{M}_2(\Gamma)$ denote the $\mathbb{Q}$-vector space generated by the $\{a, b\}$, modulo the 2-term and 3-term relations and Γ-action.

Theorem 5.1.5 (Manin). *We have an isomorphism*

$$\mathbb{M}_2(\Gamma) \xrightarrow{\sim} \mathrm{H}_1\left(X(\Gamma), \partial X(\Gamma); \mathbb{Q}\right).$$

Let $X_0(11) = \Gamma_0(11)\backslash\overline{\mathbb{H}}_2$. For example - the modular curve $X_0(11)$ has genus one and has two cusps. Thus $X_0(11)$ is topologically a torus, and the usual homology group $\mathrm{H}_1\left(X_0(11); \mathbb{Q}\right)$ has dimension 2.

We claim the relative homology $\mathrm{H}_1\left(X_0(11), \partial X_0(11); \mathbb{Q}\right)$ is 3-dimensional. Indeed, we still have the two closed 1-cycles, giving our two dimensions from before, and now there is an additional class, which can be represented by a path from one cusp to the other.

Let $\mathbb{B}_2(\Gamma)$ be the $\mathbb{Q}$-vector space generated by the cusps of $X(\Gamma)$, equipped with the obvious Γ-action. Define

$$\partial : \mathbb{M}_2(\Gamma) \longrightarrow \mathbb{B}_2(\Gamma)$$

by

$$\{a, b\} \longmapsto b - a.$$

Put $\mathbb{S}_2(\Gamma) = \ker(\partial)$. Classes in $\mathbb{S}_2(\Gamma)$ are called cuspidal modular symbols. Manin proved that cuspidal modular symbols exactly capture the homology of $X(\Gamma)$.

Theorem 5.1.6 (Manin). *We have an isomorphism*

$$\mathbb{S}_2(\Gamma) \xrightarrow{\sim} \mathrm{H}_1\left(X(\Gamma); \mathbb{Q}\right).$$

For more details, we refer to the notes of Paul E. Gunnells [17].

5.2 Modular symbols over imaginary quadratic fields

The modular symbols over imaginary quadratic fields, defined by John Cremona in [7], are also called Cremona symbols.

Let $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$ be a subgroup of finite index. Let a and b represent two points within the extended upper half-space $\overline{\mathbb{H}}_3 = \mathbb{H}_3 \cup K \cup \{\infty\}$, which are equivalent under the action of Γ. This equivalence means there exists $\gamma \in \Gamma$ such that $\gamma(a) = b$. Consequently, any smooth path from a to b in $\mathbb{H}_3$ projects to a closed path in the quotient space $X_\Gamma^{BB} = \Gamma\backslash\overline{\mathbb{H}}_3$, and its homology class in $\mathrm{H}_1(X_\Gamma^{BB}; \mathbb{Z})$ solely depends on a and b, independent of the chosen path due to the simply connectedness of $\overline{\mathbb{H}}_3$. We denote this homology class by $\{a, b\}_\Gamma$, or simply $\{a, b\}$ when the group Γ is evident from the context.

Extending this definition to points a and b not equivalent under Γ, we denote their homology class by $\{a, b\}$ when identifying homology classes with functionals on the space of differentials.

The real homology class identified with the functional $\omega \to \int_A^B \varphi^*\omega$, where ω is a differential on X_Γ^{BB} and $\varphi : \overline{\mathbb{H}}_3 \to X_\Gamma^{BB}$ is the natural projection.

The modular symbols provide a concrete approach to the group $\mathrm{H}_1(X_\Gamma^{BB}; \mathbb{Z})$.

Modular symbols $\{a, b\}$ have the following properties, whose proof is immediate

(i) $\{a, a\} = 0$,

(ii) $\{a, b\} + \{b, a\} = 0$,

(iii) $\{a, b\} + \{b, c\} + \{c, a\} = 0$,

(iv) $\{\gamma a, \gamma b\} = \{a, b\}$ if $\gamma \in \Gamma$,

(v) $\{a, \gamma a\} = \{b, \gamma b\}$ if $\gamma \in \Gamma$, for any a and b in $\overline{\mathbb{H}}_3$.

Proof: $\{a, \gamma a\} = \{a, b\} + \{b, \gamma b\} + \{\gamma b, \gamma a\}$

$$= \{a, b\} + \{b, \gamma b\} + \{b, a\}$$
$$= \{b, \gamma b\}.$$

(vi) $\{a, \gamma a\} \in \mathrm{H}_1\left(X_\Gamma^{BB}; \mathbb{Z}\right)$ if $\gamma \in \Gamma$.

Any element of $\mathrm{H}_1\left(X_\Gamma^{BB}; \mathbb{Z}\right)$ can in fact be written as $\{a, \gamma a\}$ for some $\gamma \in \Gamma$, and $a \in \mathbb{P}^1(K) = K \cup \{\infty\}$.

5.3 Inner product formula for Bianchi modular forms

We assume in this section that K is an imaginary quadratic field of class number *one* that is also an Euclidean domain. Recall the fundamental domains described in [13, Chapter 7], [1, Chapter 3], [7].

5.3.1 Quasi-periods

Definition 5.3.1 (Quasi-periods of Bianchi modular forms). Choose an edge of the Fundamental domain connecting ∞ to two different points P_1 and P_2 of the floor of the Bianchi domain [1, p. 70]. Now write $\{P_i, j\infty\}$ as the translate $h_i\{0, \infty\}$ with $h_i \in \Gamma$ as in [7]. This is very crucial for our computation and the reason we assume that K is an Euclidean domain. For a Bianchi modular form $F = (F_0, F_1, F_2) : \mathbb{H}_3 \to \mathbb{C}^3$ and $u := z + tj \in \mathbb{H}_3$, For example choose the point $P_1 = (\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$ for $K = \mathbb{Q}(i)$. We integrate over simply connected domain $\mathbb{H}_3$. When integrating with respect to dz and $d\bar{z}$, we consider the t component as constant. Similarly, when integrating with respect to dt, we take z component as constant, and then we obtain $z = \frac{1}{2}$, $\bar{z} = \frac{1}{2}$ and $t = \frac{\sqrt{3}}{2}$

$$\pi_{F_0}(z) := \int_{\frac{1}{2}}^{z} F_0(z', t)\frac{-dz'}{t} \quad \text{for arbitrary variable point } (z', t) \in \mathbb{H}_3,$$

$$\pi_{F_1}(t) := \int_{\frac{\sqrt{3}}{2}}^{t} F_1(z, t')\frac{dt'}{t'} \quad \text{for arbitrary variable point } (z, t') \in \mathbb{H}_3,$$

$$\pi_{F_2}(\bar{z}) := \int_{\frac{1}{2}}^{\bar{z}} F_2(z', t)\frac{d\bar{z}'}{t} \quad \text{for arbitrary variable point } (z', t) \in \mathbb{H}_3,$$

and

$$\pi_F := \pi_{F_0}(z) + \pi_{F_1}(t) + \pi_{F_2}(\bar{z}).$$

Then, we have

$$d\pi_F = d\pi_{F_0}(z) + d\pi_{F_1}(t) + d\pi_{F_2}(\bar{z}).$$

The π_{F_i} are the quasi-periods of Bianchi modular forms.

We have chosen z' and t' arbitrarily. For the sake of better notation, we replace z' with z and t' with t, i.e., $u = (z,t) \in \mathbb{H}_3$, then

$$d\pi_F = F_0(u)\frac{-dz}{t} + F_1(u)\frac{dt}{t} + F_2(u)\frac{d\bar{z}}{t}$$

with P_1, P_2, and P_3 are points in the fundamental domain defined in § 3.3.

Choose functions π_{F_i} such that we have

$$d(\pi_{F_i}) = F_i(u)\beta_i.$$

We prove the following formula that works for any arbitrary subgroup $\Gamma \leq G := \mathrm{SL}_2(\mathbb{Z}[i])$ of finite index. This is a generalization of the inner product formula of Banerjee-Merel [3] for imaginary quadratic fields.

Proposition 5.3.2. *Let $F, S \in M_2(\Gamma)$ be two Bianchi modular forms, with at least one of them being a cusp form with $F = (F_0, F_1, F_2)$ and $S = (S_0, S_1, S_2)$. Consider the function*

$$H := iF_0\overline{S_0} + \frac{i}{2}F_1\overline{S_1} + iF_2\overline{S_2}.$$

The inner product of these two modular forms is given by $\langle F, S\rangle = I$ with

$$I = \frac{1}{12[G:\Gamma]} \sum_{g\in\Gamma\backslash G} \int_{g0}^{g\infty} \int_{\partial\mathfrak{F}_K} H(\beta_0 \wedge \beta_1 \wedge \beta_2)$$

where $\partial\mathfrak{F}_K$ is the boundary of the fundamental domain $\mathfrak{F}_K$.

Proof. By § 3.5, the inner product is given by

$$\begin{aligned}\langle F, S\rangle &:= \frac{1}{12i[G:\Gamma]} \int_{\Gamma\backslash\mathbb{H}_3} F\cdot\beta \wedge \star(S\cdot\beta) \\ &= \frac{1}{12i[G:\Gamma]} \sum_{g\in\Gamma\backslash G} \int_{\mathfrak{F}_K} F|_g\cdot\beta \wedge \star(S|_g\cdot\beta).\end{aligned}$$

In the above equations, $\mathfrak{F}_K$ is the fundamental domain as in [7, diagram 4.2] with boundary $\partial\mathfrak{F}_K$. Integration over the boundary of the fundamental domain $\partial\mathfrak{F}_K$, as defined in § 3.3. Now write $\{P_1, j\infty\}$ as the translate $h\{0,\infty\}$ with $h \in \Gamma$ as in [1].

Consider the integral

$$\int_{\mathfrak{F}_K} F|_g \cdot \beta \wedge \star(S|_g \cdot \beta).$$

Choose a quasi-period such that $d(\pi_F) = F \cdot \beta$.

Note that $d(\pi_F \cdot (\star(S \cdot \beta)) = d(\pi_F) \wedge \star(S \cdot \beta) + \pi_F \cdot d(\star(S \cdot \beta))$. We know that $d(\star(S \cdot \beta)) = 0$ because S is a harmonic function. Observe that

$$\begin{aligned}
d(\pi_F \cdot (\star(S.\beta))) &= d(\pi_F) \wedge \star(S \cdot \beta) + \pi_F \cdot d(\star(S \cdot \beta)),\\
d(\pi_F \cdot (\star(S.\beta))) &= d(\pi_F) \wedge \star(S \cdot \beta) + 0,\\
d(\pi_F \cdot (\star(S.\beta))) &= F \cdot \beta \wedge \star(S \cdot \beta).
\end{aligned}$$

By Stokes theorem, we have

$$\begin{aligned}
\int_{\Gamma\backslash\overline{\mathbb{H}}_3} F \cdot \beta \wedge \star(S \cdot \beta) &= \int_{\Gamma\backslash\overline{\mathbb{H}}_3} d\left(\pi_F \cdot (\star(S \cdot \beta))\right)\\
&= \sum_{g \in \Gamma\backslash G} \int_{\mathfrak{F}_K} d\left(\pi_{F_{|g}} \cdot (\star(S|_g \cdot \beta))\right)\\
&= \sum_{g \in \Gamma\backslash G} \int_{\partial\mathfrak{F}_K} \pi_{F_{|g}} \cdot (\star(S|_g \cdot \beta)).
\end{aligned}$$

Hence, we deduce that

$$\sum_{g \in \Gamma\backslash G} \left(\int_{\partial\mathfrak{F}_K} \pi_{F_{|g}} \cdot (\star(S|_g \cdot \beta) \right).$$

This is equal to

$$\sum_{g\in\Gamma\backslash G}\left(\int\limits_{\partial\mathfrak{F}_K}\pi_{F_{|g}}\cdot\Big(-\frac{1}{2}i\bar{S}_{1|_g}(\beta_0\wedge\beta_2)+i\bar{S}_{0|_g}(\beta_1\wedge\beta_2)+i\bar{S}_{2|_g}(\beta_0\wedge\beta_1)\Big)\right)$$
$$=\sum_{g\in\Gamma\backslash G}\left(\int\limits_{\partial\mathfrak{F}_K}-\frac{1}{2}i\pi_{F_{1|_g}}\bar{S}_{1|_g}(\beta_0\wedge\beta_2)+i\pi_{F_{0|_g}}\bar{S}_{0|_g}(\beta_1\wedge\beta_2)+i\pi_{F_{2|_g}}\bar{S}_{2|_g}(\beta_0\wedge\beta_1)\right)$$
$$=\sum_{g\in\Gamma\backslash G}\Bigg(-\frac{1}{2}i\int_{\frac{\sqrt{3}}{2}}^{\infty}F_{1|_g}\beta_1\int\limits_{\partial\mathfrak{F}_K}\bar{S}_{1|_g}(\beta_0\wedge\beta_2)+i\int\limits_{\frac{1}{2}}^{\infty}F_{0|_g}\beta_0\int_{\partial\mathfrak{F}_K}\bar{S}_{0|_g}(\beta_1\wedge\beta_2)$$
$$+i\int_{\frac{1}{2}}^{\infty}F_{2|_g}\beta_2\int_{\partial\mathfrak{F}_K}\bar{S}_{2|_g}(\beta_0\wedge\beta_1)\Bigg)$$
$$=\sum_{g\in\Gamma\backslash G}\Bigg(-\frac{1}{2}i\int_{0}^{\infty}F_{1|_{gh}}\beta_1\int_{\partial\mathfrak{F}_K}\bar{S}_{1|_g}(\beta_0\wedge\beta_2)+i\int_{0}^{\infty}F_{0|_{gh}}\beta_0\int_{\partial\mathfrak{F}_K}\bar{S}_{0|_g}(\beta_1\wedge\beta_2)$$
$$+i\int_{0}^{\infty}F_{2|_{gh}}\beta_2\int_{\partial\mathfrak{F}_K}\bar{S}_{2|_g}(\beta_0\wedge\beta_1)\Bigg).$$

We know that $\{P_1,\infty\}$ is the translation of $\{0,\infty\}$ with $h\in\Gamma$ as in [1], so we can change the limit of quasi-periods from $\{P_1,\infty\}$ to $\{0,\infty\}$ using h. After h acts on the fundamental domain $\mathfrak{F}_K$, it will only translate, and $h\cdot\mathfrak{F}_K$ will remain the fundamental domain; integration on the boundary will remain the same. As a result, we deduce that

$$I=\frac{1}{12[G:\Gamma]}\sum_{g\in\Gamma\backslash G}\left(-\frac{1}{2}\int_0^{\infty}F_{1|_{gh}}\beta_1\int_{\partial\mathfrak{F}_K}\bar{S}_{1|_{gh}}(\beta_0\wedge\beta_2)+\int_0^{\infty}F_{0|_{gh}}\beta_0\int_{\partial\mathfrak{F}_K}\bar{S}_{0|_{gh}}(\beta_1\wedge\beta_2)\right)$$
$$+\frac{1}{12[G:\Gamma]}\sum_{g\in\Gamma\backslash G}\left(\int_0^{\infty}F_{2|_{gh}}\beta_2\int_{\partial\mathfrak{F}_K}\bar{S}_{2|_{gh}}(\beta_0\wedge\beta_1)\right).$$

As g varies in $\Gamma\backslash G$ so does gh for a fixed $h\in G_{P_1}$ and we get

$$I = \frac{1}{12[G:\Gamma]} \sum_{g \in \Gamma \backslash G} \left(-\frac{1}{2} \int_{g0}^{g\infty} F_1 \beta_1 \int_{\partial \mathfrak{F}_K} \bar{S}_1(\beta_0 \wedge \beta_2) + \int_{g0}^{g\infty} F_0 \beta_0 \int_{\partial \mathfrak{F}_K} \bar{S}_0(\beta_1 \wedge \beta_2) \right)$$
$$+ \frac{1}{12[G:\Gamma]} \sum_{g \in \Gamma \backslash G} \int_{g0}^{g\infty} F_2 \beta_2 \int_{\partial \mathfrak{F}_K} \bar{S}_2(\beta_0 \wedge \beta_1).$$

By collecting the terms with respect to the volume form $\beta_0 \wedge \beta_1 \wedge \beta_2$, we deduce that

$$I = \frac{1}{12[G:\Gamma]} \sum_{g \in \Gamma \backslash G} \int_{g0}^{g\infty} \int_{\partial \mathfrak{F}_K} H(\beta_0 \wedge \beta_1 \wedge \beta_2).$$

The expression for H is provided as follows

$$H := iF_0\overline{S_0} + \frac{i}{2} F_1 \overline{S_1} + iF_2 \overline{S_2}.$$

□

5.4 Eisenstein classes related to modular symbols

Theorem 5.4.1 (Cremona). *The Cremona symbols $g \cdot \{0, \infty\}$ for $g \in \mathrm{GL}_2(\mathcal{O}_K)$ generate $H_1\left(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{Z}\right)$.*

Definition 5.4.2 (Eisenstein classes). The *Eisenstein classes* are modular symbols, denoted by $\mathcal{E}$, belonging to the homology group $\mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{C})$, and satisfying the condition

$$\int_{\mathcal{E}} F \cdot \beta = 0$$

for all cusp forms F (refer to Definition 3.5.3 for the definition of cusp forms and β is defined in §3.4). The *Eisenstein cycle* is also called an *Eisenstein element*.

These classes are associated with *Eisenstein cycles*, which are paths within the Eisenstein classes; i.e., the *Eisenstein cycles* are paths in the Eisenstein classes.

We prove that the boundaries of Eisenstein cycles are non-zero under certain assumptions for a restricted class of subgroups. It is expected that Eisenstein classes will have certain Hecke equivariant properties for congruence subgroups similar to classical situation (see also [2, Theorem 2]).

The main objective of this work is to explicitly determine Eisenstein classes that serve as generators of the Eisenstein part within the Bianchi modular symbols. In other words, we aim to explicitly construct an Eisenstein cycle $\mathcal{E}_E$ corresponding to a given Eisenstein series E.

Cremona [8] demonstrated (see also [7, Theorem 4.3.2]) that the map η is a surjective mapping from $\Gamma\backslash \mathrm{SL}_2(\mathcal{O}_K)$ onto $\mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{Q})$ defined as

$$\eta : (g) \longmapsto \{g0, g\infty\}.$$

The kernel of this map can be computed as described in the referenced work (as in [7, Theorem 4.3.2]).

Let $\star$ denote the Hodge star operator acting on differential forms, and let $\mathfrak{F}_K$ be the fundamental domain for the imaginary number field K with boundary $\partial\mathfrak{F}_K$ [1]. We define the generalized 2-periods as follows

$$F_E(g) := \int_{\partial\mathfrak{F}_K} \star(E \cdot \beta)[g].$$

Here, $E \cdot \beta$ can be considered as Ito's and Hida's differential form defined in Chapter 4.

Let $E_2(\Gamma)$ be the space of all Eisenstein modular forms of weight 2, as defined in § 3.5. For the next result, we assume that $K = \mathbb{Q}(\sqrt{-d})$ is an Euclidian domain and has the class number one. In other words, we assume $d \in \{1, 2, 3, 7, 11\}$.

The next theorem is a generalization of the result by Banerjee and Merel. In [2], they calculated Eisenstein cycles as modular symbols for the classical case.

Theorem 5.4.3. *For any imaginary field K with class number one that is also an Euclidean domain, the modular symbol*

$$\mathcal{E}_E = \sum_{g \in \Gamma\backslash G} F_E(g)\eta(g)$$

is the Eisenstein cycle corresponding to the Eisenstein series $E \in E_2(\Gamma)$.

Here, "corresponding " means that for every Eisenstein series, we obtain an Eisenstein cycle (or we can say we obtain the Eisenstein element).

Proof. For an Eisenstein series $E \in E_2(\Gamma)$, define

$$F_E(g) := \int_{\partial \mathfrak{F}_K} \star(E \cdot \beta)[g].$$

Using [8, theorem 1], consider the modular symbol $\mathcal{E}_E \in \mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{C}) := \mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{Q}) \otimes \mathbb{C}$ as in the statement of the theorem

$$\mathcal{E}_E := \sum_{g \in \Gamma \backslash G} F_E(g) \eta(g).$$

We compute the integral as follows

$$I = \frac{1}{12[G:\Gamma]} \int_{\mathcal{E}_E} F \cdot \beta = \frac{1}{12[G:\Gamma]} \sum_{g \in \Gamma \backslash G} F_E(g) \int_{\eta(g)} F \cdot \beta.$$

Consider two Bianchi modular forms of weight 2, $F = (F_0, F_1, F_2)$ and $E = (E_0, E_1, E_2)$ in $M_2(\Gamma)$. Consider the function

$$H := iF_0\overline{E_0} + \frac{i}{2} F_1 \overline{E_1} + iF_2\overline{E_2}$$

as in Proposition 5.3.2.

By Proposition 5.3.2, we know that

$$\langle F, E \rangle = \frac{1}{12[G:\Gamma]} \sum_{g \in \Gamma \backslash G} \int_{\mathfrak{F}_K} H|_g \beta_0 \wedge \beta_1 \wedge \beta_2.$$

We have

$$\begin{aligned} \int_{\mathcal{E}_E} F \cdot \beta &= \sum_{g \in \Gamma \backslash G} F_E(g) \int_{\eta(g)} F \cdot \beta \\ &= \sum_{g \in \Gamma \backslash G} \int_{\eta(g)} \int_{\partial \mathfrak{F}_K} H|_g \beta_0 \wedge \beta_1 \wedge \beta_2. \end{aligned}$$

By Proposition 5.3.2, the right-hand side is equal to

$$\int_{\mathcal{E}_E} F \cdot \beta = -\, 12i[G : \Gamma]\langle F, E\rangle$$

and we know that

$$-12i[G : \Gamma]\langle F, E\rangle = 0.$$

Since, by definition, Eisenstein modular forms are defined to be the complement of cusp forms.

$$\int_{\mathcal{E}_E} F \cdot \beta = 0$$

for all cusp forms F.

□

We now compute the integral a bit explicitly under certain assumptions. Recall that by Corollary 4.3.2, we have

$$dH_E = \omega_E$$

for a function (degree zero differential form) H_E.

Proposition 5.4.4. *Let $\delta(H_E\omega)$ be a non-vanishing function on the Riemannian 2-manifold $\partial\mathfrak{F}_K$. Then we have $F_E(I) \neq 0$.*

Proof. We then have $\star\omega_E = \star dH_E$. Let ω be the non-vanishing top form on the Riemannian manifold Y_Γ. By our assumption on H_E, we have $H_E\omega$ is a top form on the Riemannian 3-manifold. Observe that

$$\delta(H_E\omega) = -\star d \star (H_E\omega) = -\star dH_E = -\star \omega_E.$$

By assumption, we have $\delta(H_E\omega)$ is a volume form on 2 dimensional submanifold $\partial\mathfrak{F}_K$. Hence, we have

$$\int_{\partial\mathfrak{F}_K} \star\omega_E = -\int_{\partial\mathfrak{F}_K} \delta(H_E\omega) \neq 0.$$

□

Note that the assumption is reasonable since $\eta(z) \neq 0$ for all z in the upper half plane.

5.4.1 Eisenstein elements for $\Gamma_0(p)$

Consider the subgroup $\Gamma = \Gamma_0(p)$ for a prime p of $\mathcal{O}_K$ that is inert. In this case, there are only two cusps $[0]$ and $[\infty]$ similar to the classical case and the corresponding modular curve is denoted by $Y_0(p)$. There are two Eisenstein series. Define

$$\partial\mathfrak{F}_K(p) := \int\limits_{\partial\mathfrak{F}_K} \star dH_E.$$

We compute the boundary of the Eisenstein element explicitly in this case.

Proposition 5.4.5. *Consider the congruence subgroup $\Gamma_0(p)$ with p as above. Under the assumption on H_E as above, the boundary of the Eisenstein element is non-zero and determined by $\partial\mathfrak{F}_K(p)$.*

Proof. It follows that

$$\begin{aligned}
\delta(\mathcal{E}_E) &= \sum_{g\in\Gamma\backslash \mathrm{SL}_2(\mathcal{O}_K)} \int_{\partial\mathfrak{F}_K} \star\omega_E[g]([g0]-[g\infty]) \\
&= \left(\sum_{g\in\Gamma\backslash \mathrm{SL}_2(\mathcal{O}_K)} \int_{\partial\mathfrak{F}_K} \star\omega_E[g]\right)([0]-[\infty]) \\
&= \left(\int\limits_{Y_0(p)} \star\omega_E\right)([0]-[\infty]) \\
&= \left(\int\limits_{Y_0(p)} \star dH_E\right)([0]-[\infty])
\end{aligned}$$

for a function H_E as in Corollary 4.3.2. In particular, it shows that $\delta(\mathcal{E}_E) \neq 0$ if $\int\limits_{Y_0(p)} \star dH_E = [\mathrm{SL}_2(\mathcal{O}_K) : \Gamma_0(p)]\partial\mathfrak{F}_K(p) \neq 0$ by Proposition 5.4.4. $\square$

5.4.2 Different pairings of homology and cohomology groups

Let Γ be a torsion-free subgroup such as $\Gamma_1(N)$ with N sufficiently large. Consider the compactifications $X \in \{X_\Gamma^{BB}, X_\Gamma^{BS}\}$ of the Riemannian 3-manifold Y_Γ. We have isomorphisms

$$S_2(\Gamma) \simeq \mathrm{H}^1(X;\mathbb{C}) \simeq \mathrm{H}_1(X;\mathbb{C})$$

The first isomorphism is obtained by the map $F \mapsto F \cdot \beta$. The second isomorphism is given by the duality induced by the evaluation pairing $(\omega, \gamma) \mapsto \int_\gamma \omega$.

We also have a pairing [21, p. 474]

$$\langle , \rangle : \mathrm{H}_c^1(Y_\Gamma;\mathbb{C}) \times \mathrm{H}^2(Y_\Gamma;\mathbb{C}) \to \mathbb{C}.$$

We also have the following isomorphism (cf. [26, p. 288])

$$\mathrm{H}_1(X, \partial X;\mathbb{C}) \simeq \mathrm{H}_c^1(Y_\Gamma;\mathbb{C}) \simeq \mathrm{H}^2(Y_\Gamma;\mathbb{C}).$$

With the identification $\mathrm{H}_{dR}^1(Y_\Gamma) \simeq \mathrm{H}^1(Y_\Gamma;\mathbb{C})$, we can make this pairing explicit

$$\mathrm{H}_c^1(Y_\Gamma;\mathbb{C}) \times \mathrm{H}_1(X, \partial X;\mathbb{C}) \to \mathbb{C}.$$

The map is given again by the same evaluation pairing $(\omega, \gamma) \mapsto \int_\gamma \omega$.

Recall the formulation of Poincaré-Lefschetz duality [16, p. 53]. In this case, the intersection pairing will be given by

$$\circ : \mathrm{H}_1(X, \partial X;\mathbb{Z}) \times \mathrm{H}_2(Y_\Gamma;\mathbb{Z}) \to \mathbb{Z}.$$

The Eisenstein element $\mathcal{E}_E$ corresponding to the Eisenstein modular form E of weight 2, is the unique element such that $\mathcal{E}_E \circ c = \int_c \star(E \cdot \beta)$ for all $c \in \mathrm{H}_2(Y_\Gamma;\mathbb{Z})$. We expect these two definitions of Eisenstein elements to coincide.

Using the Eichler-Shimura-Harder correspondence, we know that $S_2(\Gamma)$ is isomorphic to $\mathrm{H}_{cusp}^1(X;\mathbb{C})$ via the map F to $F \cdot \beta$. We know $\mathrm{H}^1(X;\mathbb{C}) = \mathrm{H}_{cusp}^1(X;\mathbb{C}) \oplus \mathrm{H}_{Eis}^1(X;\mathbb{C})$, and if E is an Eisenstein series, then $E \cdot \beta$ is a differential form belonging to $\mathrm{H}_{Eis}^1(X;\mathbb{C})$.

For a good description and explicit computation of the Eichler-Shimura-Harder isomorphism, we refer to the work of Ghate [14, page 20](also for CM fields in [15]).

5.4.3 Eisenstein elements are retractions

Since we are using the Bailey-Borel-Satake compactification for homology groups, we have a short exact sequence

$$0 = \mathrm{H}_1(\partial X_\Gamma^{BB}; \mathbb{Z}) \to \mathrm{H}_1(X_\Gamma^{BB}; \mathbb{Z}) \to \mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{Z}) \xrightarrow{\delta} \mathbb{Z}[\partial X_\Gamma^{BB}] \to \mathbb{Z} \to 0$$

where the map $\mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{Z}) \xrightarrow{\delta} \mathbb{Z}[\partial X_\Gamma^{BB}]$ is obtained from the boundary map $\delta : \mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{Z}) \to \mathrm{H}_0(\partial X_\Gamma^{BB}; \mathbb{Z})$.

The above exact sequence splits over the field $\mathbb{C}$. We have a retraction map

$$R : \mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{C}) \to \mathrm{H}_1(X_\Gamma^{BB}; \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^1(X_\Gamma^{BB}; \mathbb{C}), \mathbb{C})$$

given by $R(c)(\omega) = \int_c \omega$.

Note that this is a section of the inclusion map. Hence, if we tensor with $\mathbb{C}$, we have $\delta(x) = 0$ if and only if $R(x) = x$. This is the same as $x \in \mathrm{H}_1(X_\Gamma^{BB}; \mathbb{C})$.

By the definition of Eisenstein element, we have $R(\mathcal{E}_E) = 0$. Note that $\mathcal{E}_E \neq 0$ as $F_E(I) \neq 0$ for I, the identity element, by Proposition 5.4.4 under certain assumptions. Hence, we deduce that $\delta(\mathcal{E}_E) \neq 0$ with the same assumptions.

6

Eisenstein cohomology

6.1 Definition of Eisenstein cohomology

For the definition of Eisenstein cohomology groups, we are following Sengün and Türkelli [10]. We compute the Eisenstein cohomology groups of Borel-Serre compactification of Y_Γ. This is a compactifcation obtained by adding a 2 torus to every cusp (except for $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$).

The Borel-Serre compactification [36, appendix] X_Γ^{BS} of Y_Γ is a compact, real 3-dimensional manifold with boundary ∂X_Γ^{BS} whose interior is homeomorphic to Y_Γ.

Given $k \geqslant 0$, let the space of homogeneous polynomials of degree k on the variables x, y with complex coefficients be denoted by $\mathbb{C}[x,y]_k$. The modular group $\mathrm{SL}_2(\mathbb{C})$ acts on this space in the obvious way permitted by the two variables. Consider the $\mathrm{SL}_2(\mathbb{C})$-module

$$M_k := \mathbb{C}[x,y]_k \otimes_{\mathbb{C}} \overline{\mathbb{C}[x,y]_k}$$

where the overline on the second factor indicates that the action on this factor is twisted with complex conjugation, that is, first, we apply complex conjugation to the coefficients of the matrix and then apply the matrix to the polynomial. Considered as a module, M_k gives rise to a locally constant sheaf $\mathcal{M}_k$ on Y_Γ whose stalks are isomorphic to M_k.

In Sengün-Türkelli [10], they use the notation $E_{k,k}$ instead of M_k and for the sheaf $\mathcal{E}_k$ instead of $\mathcal{M}_k$.

Considered as a module, M_k gives rise to a locally constant sheaf $\mathcal{M}_k$ on Y_Γ whose stalks are isomorphic to M_k. Consider the long exact sequence

$$\ldots \to \mathrm{H}^i_c\left(Y_\Gamma; \mathcal{M}_k\right) \xrightarrow{incl^i} \mathrm{H}^i\left(X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) \xrightarrow{res^i} \mathrm{H}^i\left(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) \to \ldots$$

where H^i_c denotes compactly supported cohomology and $\overline{\mathcal{M}}_k$ is a certain sheaf on X_Γ^{BS} that extends $\mathcal{M}_k$.

The image of the compactly supported cohomology by the inclusion map $incl$ inside the cohomology group is called the cuspidal cohomology group. Note that this is the same as the kernel of the restriction map res.

The kernel of the restriction map gives a subspace of $\mathrm{H}^i(Y_\Gamma; \mathcal{M}_k)$ which is called the cuspidal cohomology, denoted $\mathrm{H}^i_{cusp}(Y_\Gamma; \mathcal{M}_k)$.

By the Eichler-Shimura-Harder isomorphism [19], this consists of all cohomological cuspidal automorphic representations (cusp forms) [34, p. 409].

Note that the long exact sequence associated with the pair $\left(X_\Gamma^{BS}, \partial X_\Gamma^{BS}\right)$ as above is compatible with the action of the involution τ from Section 7.1.

Definition 6.1.1 (Eisenstein cohomology groups). The kernel of the restriction map is known as *cuspidal cohomology* H^i_{cusp}. The complement of subspace cuspidal cohomology within H^i is the *Eisenstein cohomology* H^i_{Eis}. This is isomorphic to the image of the restriction map inside the cohomology of the boundary.

Let Γ be a subgroup of the Bianchi group, and let the associated orbifold (or manifold, if Γ is torsion-free) $\Gamma \backslash \mathbb{H}_3$ have finite volume. In this case, we say that Γ has finite covolume.

If Γ is torsion-free, then Y_Γ is manifold. The whole point of the Borel-Serre compactification is to show that Y_Γ has the homotopy type of a manifold with a boundary. Therefore, its cohomology is finitely generated, which is not obvious for an open manifold.

The decomposition $\mathrm{H}^i = \mathrm{H}^i_{cusp} \oplus \mathrm{H}^i_{Eis}$ respects the Hecke action. By construction, the embedding $Y_\Gamma \hookrightarrow X_\Gamma^{BS}$ is a homotopy equivalence. We have the following isomorphisms

$$\mathrm{H}^i\left(X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) \simeq \mathrm{H}^i\left(Y_\Gamma; \mathcal{M}_k\right) \simeq \mathrm{H}^i\left(\Gamma; M_k\right).$$

Note that the last one is the group cohomology. Via the above isomorphisms, we define the cuspidal (respectively Eisenstein parts) of the cohomology groups $\mathrm{H}^i(\Gamma; M_k)$.

The cuspidal parts (respectively, the Eisenstein part) consist of elements that are zero (non-zero) on parabolic elements of Γ, as they are complements of each other within the cohomology group, and their direct sum constitutes the entire cohomology group. In other words, these are in kernel (not in kernel) of res^i.

We denote the Eisenstein part of the cohomology group by $\mathrm{H}^1_{Eis}(\Gamma; M_k)$.

The boundary ∂X_Γ^{BS} is a disjoint union of 2-tori, each corresponds to a cusp of Y_Γ (except for $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$). The set of cusps of Γ can be identified with the orbit space $\Gamma\backslash\mathbb{P}^1(K)$. It is well known that when Γ is the full Bianchi group, the number of cusps is $h(K)$, the class number of K. Recall that we assumed $h(K) = 1$.

Let $\mathcal{C}_\Gamma$ be the set of cusps of the congruence subgroups of the form Γ. We are grateful to Professor Sengün for the outline of the proof (see also [28] for full subgroups).

Proposition 6.1.2. *[10, Proposition 4.1. p. 247]. Let Γ be a congruence subgroup of a Bianchi group. Then*

$$\dim \mathrm{H}^0\left(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) = \dim \mathrm{H}^2\left(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) = \#\mathcal{C}_\Gamma$$
$$\dim \mathrm{H}^1\left(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) = 2 \cdot \#\mathcal{C}_\Gamma.$$

It follows from algebraic topology that for $k > 0$, the image of the restriction map

$$\mathrm{H}^i\left(X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) \to \mathrm{H}^i\left(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right)$$

is surjective when $i = 2$ and its image has half the rank of the target space when $i = 1$. Hence, we have the following result.

Proposition 6.1.3. *Let Γ be a congruence subgroup as above. Then*

if $k = 0$,

1. $\dim \mathrm{H}^0_{Eis}\left(X_\Gamma^{BS}; \mathbb{C}\right) = 1$,
2. $\dim \mathrm{H}^1_{Eis}\left(X_\Gamma^{BS}; \mathbb{C}\right) = \#\mathcal{C}_\Gamma$,
3. $\dim \mathrm{H}^2_{Eis}\left(X_\Gamma^{BS}; \mathbb{C}\right) = \#\mathcal{C}_\Gamma - 1$.

If $k > 0$,

1. $\dim \mathrm{H}^0_{Eis}\left(X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) = 0,$

2. $\dim \mathrm{H}^i_{Eis}\left(X_\Gamma^{BS}; \overline{\mathcal{M}}_k\right) = \#\mathcal{C}_\Gamma$ *for* $i = 1, 2.$

Proof. We start by considering the long exact sequence associated with the pair $(X_\Gamma^{BS}, \partial X_\Gamma^{BS})$

$$\ldots \to \mathrm{H}^1_c(Y_\Gamma; \mathcal{M}_k) \xrightarrow{incl^1} \mathrm{H}^1(X_\Gamma^{BS}; \overline{\mathcal{M}}_k) \xrightarrow{res^1} \mathrm{H}^1(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k) \to \mathrm{H}^2_c(Y_\Gamma; \mathcal{M}_k)$$

$$\xrightarrow{incl^2} \mathrm{H}^2(X_\Gamma^{BS}; \overline{\mathcal{M}}_k) \xrightarrow{res^2} \mathrm{H}^2(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k) \to \mathrm{H}^3_c(Y_\Gamma; \mathcal{M}_k) \xrightarrow{incl^3} \mathrm{H}^3(X_\Gamma^{BS}; \overline{\mathcal{M}}_k) = 0.$$

The subscript "$_c$" denotes compactly supported cohomology. Using the Poincaré duality [20, p. 133]

$$\mathrm{H}^i_c(Y_\Gamma; \mathcal{M}_k) \times \mathrm{H}^{3-i}(X_\Gamma^{BS}; \overline{\mathcal{M}}_k) \to \mathbb{C}, \qquad \text{for i} \in \{0, 1, 2, 3\},$$

we have $\mathrm{H}^3_c(Y_\Gamma; \mathcal{M}_k)$ is isomorphic to $\mathrm{H}^0(X_\Gamma^{BS}; \overline{\mathcal{M}}_k)$.

Consider the map

$$res^2 : \mathrm{H}^2(X_\Gamma^{BS}; \overline{\mathcal{M}}_k) \to \mathrm{H}^2(\partial X_\Gamma^{BS}; \overline{\mathcal{M}}_k),$$

if $k = 0$ then $\mathcal{M}_k = \mathbb{C}$ is a constant sheaf and we get

$$\mathrm{H}^0(X_\Gamma^{BS}; \mathbb{C}) = \mathbb{C}.$$

Using Proposition 6.1.2, we deduce that $\dim_\mathbb{C} \mathrm{H}^0_{Eis}(X_\Gamma^{BS}; \mathbb{C}) = 1$. By Poincaré duality, we have $\mathrm{H}^3_c(Y_\Gamma; \mathbb{C}) \simeq \mathrm{H}^0(X_\Gamma^{BS}; \mathbb{C})$, this implies $\mathrm{H}^3_c(Y_\Gamma; \mathbb{C}) \simeq \mathbb{C}$.

When $\mathcal{M}_k \simeq \mathbb{C}$ is a trivial sheaf, the image of the restriction map res^2 has one dimensional cokernel by Harder [18, Proposition 4.7.1].

Hence, we have $\dim_\mathbb{C} \mathrm{H}^2_{Eis}(X_\Gamma^{BS}; \mathbb{C})$ is $\#\mathcal{C}_\Gamma - 1$. We now consider the long exact sequence associated with $(X_\Gamma^{BS}, \partial X_\Gamma^{BS})$

$$0 \to \mathrm{H}^0(X_\Gamma^{BS}; \mathbb{C}) \xrightarrow{res^0} \mathrm{H}^0(\partial X_\Gamma^{BS}; \mathbb{C}) \xrightarrow{\partial^0} \mathrm{H}^1_c(Y_\Gamma; \mathbb{C}) \xrightarrow{incl^1} \mathrm{H}^1(X_\Gamma^{BS}; \mathbb{C}) \xrightarrow{res^1} \mathrm{H}^1(\partial X_\Gamma^{BS}; \mathbb{C})$$

$$\xrightarrow{\partial^1} \mathrm{H}^2_c(Y_\Gamma; \mathbb{C}) \xrightarrow{incl^2} \mathrm{H}^2(X_\Gamma^{BS}; \mathbb{C}) \xrightarrow{res^2} \mathrm{H}^2(\partial X_\Gamma^{BS}; \mathbb{C}) \xrightarrow{\partial^2} \mathrm{H}^3_c(Y_\Gamma; \mathbb{C}) \xrightarrow{incl^3} \mathrm{H}^3(X_\Gamma^{BS}; \mathbb{C}) = 0.$$

By successive applications of the rank-nullity theorem, we obtain the following

1. $\dim_\mathbb{C} \mathrm{H}^0(\partial X_\Gamma^{BS}; \mathbb{C}) = \dim \mathrm{Im}\, res^0 + \dim \mathrm{Im}\, \partial^0 \implies \#\mathcal{C}_\Gamma = 1 + \dim \mathrm{Im}\, \partial^0,$

2. $\dim_{\mathbb{C}} \mathrm{H}^1_c(Y_\Gamma; \mathbb{C}) = \dim \mathbf{Im}\, \partial^0 + \dim \mathbf{Im}\, incl^1$,

3. $\dim_{\mathbb{C}} \mathrm{H}^1(X_\Gamma^{BS}; \mathbb{C}) = \dim \mathbf{Im}\, incl^1 + \dim \mathbf{Im}\, res^1$,

4. $\dim_{\mathbb{C}} \mathrm{H}^1(\partial X_\Gamma^{BS}; \mathbb{C}) = \dim \mathbf{Im}\, res^1 + \dim \mathbf{Im}\, \partial^1 \implies 2 \cdot \#\mathcal{C}_\Gamma = \dim \mathbf{Im}\, res^1 + \dim \mathbf{Im}\, \partial^1$,

5. $\dim_{\mathbb{C}} \mathrm{H}^2_c(Y_\Gamma; \mathbb{C}) = \dim \mathbf{Im}\, \partial^1 + \dim \mathbf{Im}\, incl^2$,

6. $\dim_{\mathbb{C}} \mathrm{H}^2(X_\Gamma^{BS}; \mathbb{C}) = \dim \mathbf{Im}\, incl^2 + \dim \mathbf{Im}\, res^2$,

7. $\dim_{\mathbb{C}} \mathrm{H}^2(\partial X_\Gamma^{BS}; \mathbb{C}) = \dim \mathbf{Im}\, res^2 + \dim \mathbf{Im}\, \partial^2 \implies \#\mathcal{C}_\Gamma = \dim \mathbf{Im}\, res^2 + \dim \mathbf{Im}\, \partial^3$
   $\implies \#\mathcal{C}_\Gamma = \dim \mathbf{Im}\, res^2 + 1$ (because $\mathrm{H}^3_c(Y_\Gamma; \mathbb{C}) = \mathbb{C}$)
   $\implies \dim \mathbf{Im}\, res^2 = \#\mathcal{C}_\Gamma - 1$.

Using Poincaré duality, we get

$$\begin{gathered} \mathrm{H}^1_c(Y_\Gamma; \mathbb{C}) \simeq \mathrm{H}^2(X_\Gamma^{BS}; \mathbb{C}) \\ \dim \mathbf{Im}\, \partial^0 + \dim \mathbf{Im}\, incl^1 = \dim \mathbf{Im}\, incl^2 + \dim \mathbf{Im}\, res^2 \\ \#\mathcal{C}_\Gamma - 1 + \dim \mathbf{Im}\, incl^1 = \dim \mathbf{Im}\, incl^2 + \#\mathcal{C}_\Gamma - 1 \\ \implies \dim \mathbf{Im}\, incl^1 = \dim \mathbf{Im}\, incl^2 \end{gathered}$$

and

$$\begin{gathered} \mathrm{H}^2_c(Y_\Gamma; \mathbb{C}) \simeq \mathrm{H}^1(X_\Gamma^{BS}; \mathbb{C}); \\ \dim \mathbf{Im}\, \partial^1 + \dim \mathbf{Im}\, incl^2 = \dim \mathbf{Im}\, incl^1 + \dim \mathbf{Im}\, res^1; \\ \implies \dim \mathbf{Im}\, \partial^1 = \dim \mathbf{Im}\, res^1. \end{gathered}$$

We know that $\dim \mathrm{H}^1(\partial X_\Gamma^{BS}; \mathbb{C}) = \dim \mathbf{Im}\, res^1 + \dim \mathbf{Im}\, \partial^1$

$$\implies 2 \cdot \#\mathcal{C}_\Gamma = \dim \mathbf{Im}\, res^1 + \dim \mathbf{Im}\, \partial^1$$

$$\implies 2 \cdot \#\mathcal{C}_\Gamma = \dim \mathbf{Im}\, res^1 + \dim \mathbf{Im}\, res^1$$

$$\implies 2 \cdot \#\mathcal{C}_\Gamma = 2 \cdot \dim \mathbf{Im}\, res^1$$

$$\implies \dim \mathbf{Im}\, res^1 = \#\mathcal{C}_\Gamma$$

$$\implies \dim_{\mathbb{C}} \mathrm{H}^1_{Eis}(X_\Gamma^{BS}; \mathbb{C}) = \#\mathcal{C}_\Gamma.$$

If $k > 0$ then $\mathcal{M}_k$ is a locally constant sheaf, and we get

$$\mathrm{H}_c^3(Y_\Gamma;\mathcal{M}_k) = \mathrm{H}^0(X_\Gamma^{BS};\overline{\mathcal{M}}_k), \quad \mathrm{H}^0(X_\Gamma^{BS};\overline{\mathcal{M}}_k) \simeq \mathrm{H}^0(\Gamma; M_k).$$

We know that $\mathrm{H}^0(\Gamma; M_k) = 0$, which implies $\mathrm{H}^0(X_\Gamma^{BS};\overline{\mathcal{M}}_k) = 0$ and $\mathrm{H}_c^3(Y_\Gamma;\mathcal{M}_k) = 0$.

For $k > 0$, the restriction map is surjective. We analyze the image of the restriction maps following Harder [18, Proposition 4.7.1] and, the restriction map

$$\mathrm{H}^i\left(X_\Gamma^{BS};\overline{\mathcal{M}}_k\right) \to \mathrm{H}^i\left(\partial X_\Gamma^{BS};\overline{\mathcal{M}}_k\right)$$

is onto when $i = 2$ and its image has half the rank of the target space when $i = 1$. So, we have

$$\begin{aligned} \dim \mathrm{H}^1_{Eis}\left(X_\Gamma^{BS};\overline{\mathcal{M}}_k\right) &= \tfrac{1}{2}\dim \mathrm{H}^1\left(\partial X_\Gamma^{BS};\overline{\mathcal{M}}_k\right) \\ \dim \mathrm{H}^2_{Eis}\left(X_\Gamma^{BS};\overline{\mathcal{M}}_k\right) &= \dim \mathrm{H}^2\left(\partial X_\Gamma^{BS};\overline{\mathcal{M}}_k\right). \end{aligned}$$

We know that $\dim \mathrm{H}^1\left(\partial X_\Gamma^{BS};\overline{\mathcal{M}}_k\right) = 2 \cdot \#\mathcal{C}_\Gamma$ and $\dim \mathrm{H}^2\left(\partial X_\Gamma^{BS};\overline{\mathcal{M}}_k\right) = \#\mathcal{C}_\Gamma$. This implies that $\dim \mathrm{H}^i_{Eis}\left(X_\Gamma^{BS};\overline{\mathcal{M}}_k\right) = \#\mathcal{C}_\Gamma$ for $i = 1, 2$.

□

This just shows that the dimension of the Eisenstein cohomology is determined by the number of cusps.

6.1.1 Computation of number of cusps for $\Gamma_1(N)$

By [12, p. 165], the set of cusps for $\mathrm{SL}_2(\mathcal{O}_K)$ can be identified with the class group of K. Recall that we assumed that the class number of the imaginary number field K is one.

We are interested in congruence subgroups of the form $\Gamma = \Gamma_1(\mathfrak{a})$ and denote the corresponding set of cusps by $\mathcal{C}_N$ for the ideal $\mathfrak{a} = (N)$.

Let $P := \left\{\pm\begin{bmatrix} 1 & j \\ 0 & 1\end{bmatrix} : j \in \mathcal{O}_K\right\}$ be the parabolic subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$. Note that $\bar{P} = \left\{\pm\begin{bmatrix} 1 & j \\ 0 & 1\end{bmatrix} : j \in \mathcal{O}_K/N\mathcal{O}_K\right\}$ and $\bar{P}_+ = \left\{\begin{bmatrix} 1 & j \\ 0 & 1\end{bmatrix} : j \in \mathcal{O}_K/N\mathcal{O}_K\right\}$ is the "positive" half of $\bar{P}$.

Here, the overbar signifies reduction modulo (N). For $N \in \mathbb{N}$, let $\mathcal{C}_N$ be the set of cusps of congruence subgroups of the form $\Gamma_1(N)$.

Proposition 6.1.4.

$$\#\mathcal{C}_N = |\bar{P}_+\backslash \mathrm{SL}_2(\mathcal{O}_K/N\mathcal{O}_K)/\bar{P}|$$

Proof. The group $\mathrm{SL}_2(\mathcal{O}_K)$ acts transitively on the set of cusps. Recall that $P = \mathrm{SL}_2(\mathcal{O}_K)_\infty$ is the subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ fixing the cusp ∞. The map

$$\Gamma_1(N)\backslash \mathrm{SL}_2(\mathcal{O}_K)/P \longrightarrow \{\text{ cusps of } \Gamma_1(N)\}$$

given by

$$\Gamma_1(N)\alpha P \mapsto \Gamma_1(N)\alpha(\infty)$$

is a bijection [11] (the proof works for imaginary quadratic fields also since the class number is one). We know that $\mathrm{SL}_2(\mathcal{O}_K)/P$ identifies with $K \cup \{\infty\}$, so that the double coset space $\Gamma_1(N)\backslash \mathrm{SL}_2(\mathcal{O}_K)/P$ gets identified with the cusps $\Gamma_1(N)\backslash(K \cup \{\infty\})$.

The double coset space $\Gamma(N)\backslash \mathrm{SL}_2(\mathcal{O}_K)/P$ is naturally viewed as $\mathrm{SL}_2(\mathcal{O}_K/N\mathcal{O}_K)/\bar{P}$ where $\bar{P}$ denotes the projected image of P in $\mathrm{SL}_2(\mathcal{O}_K/N\mathcal{O}_K)$, that is,

$$\bar{P} = \left\{ \pm \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} : j \in \mathcal{O}_K/N\mathcal{O}_K \right\}.$$

We have a decomposition $\Gamma_1(N) = \bigcup_j \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \Gamma(N)$, the double coset space is naturally viewed as $\bar{P}_+\backslash \mathrm{SL}_2(\mathcal{O}_K/N\mathcal{O}_K)/\bar{P}$ where $\bar{P}_+ = \left\{ \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} : j \in \mathcal{O}_K/N\mathcal{O}_K \right\}$ is the "positive" half of $\bar{P}$ and again the overbar signifies reduction modulo (N).

We deduce that

$$|\bar{P}_+\backslash \mathrm{SL}_2(\mathcal{O}_K/N\mathcal{O}_K)/\bar{P}| = \#\mathcal{C}_N.$$

□

The cusps of $\Gamma_1(p^n)$ are in bijection with the sets $\{\pm(\bar{x}, \bar{y})\} \subset (\mathcal{O}/(p^n))^2$ such that the order of $(\bar{x}, \bar{y})$ is p^n. The bijection is defined via the map $\frac{x}{y} \mapsto (y, -x)$ with

x modulo $gcd(y, N)$.

Corollary 6.1.5. $\#\mathcal{C}_p = p^2 - 1$.

Proof. We know that $\mathcal{O}_K/p\mathcal{O}_K$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and number of non-zero choices of $y \in \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ is $p^2 - 1$ with $gcd(y, p) = 1$. Since the possibilities of x and $gcd(y, p) = 1$, we get the possibilities of x is x modulo 1. Hence, the number of choices of x is 1, and the number of choices for (x, y) is $p^2 - 1$. □

6.1.2 Szech cocycles and some expectations

The classical Dedekind sums arise from the homomorphism

$$\phi : \Gamma \to \mathbb{C}$$

for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, given by periods of Eisenstein series

$$\phi = \phi_E : \gamma \mapsto \int_{[\gamma]} \omega = \int_{z_0}^{\gamma z_0} E(z)dz$$

where $E(z)$ is the unique Eisenstein series (non-holomorphic modular form) of weight 2 on Γ.

Mazur's work uses the analogous Rademacher homomorphism for $\Gamma = \Gamma_0(N)$ and $\Gamma(N)$, while Merel studies the case $\Gamma = \Gamma_0(N)$, where in each case $E(z)$ is replaced by the holomorphic Eisenstein series of weight 2 on $\Gamma(N)$ and $\Gamma_0(N)$ respectively. These periods are described by Dedekind sums and, therefore, are seen to be integral. The work of Banerjee and Merel shows that the Eisenstein cycle can be written as a linear combination of Manin symbols ξ

$$\mathcal{E}_E = \sum_{x \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} F_E(x)\xi(x) = \sum_{x \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} F_E(x)\xi(x)$$

where the coefficients are shown to be a scalar multiple of ϕ_E, and therefore can be given in terms of Dedekind sums, thus Bernoulli numbers and special values of L functions.

Over an imaginary quadratic field, Ito [22] showed that for $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$, the period is again described by elliptic Dedekind sums introduced by Sczech [32] as discussed above. In the case of congruence subgroups of the form $\Gamma_1(N)$, however, the analogue of elliptic Dedekind sums has not yet been defined to the best of our knowledge. The explicit Sczech 1-cocycles [33] (see also Sengn̈-Türkelli [10, §4.2.3, p. 252]) can be used to produce a basis for group cohomology $\mathrm{H}^1_{Eis}(\Gamma;\mathbb{C})$ for some subgroup $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$.

Szech cocycles are expected to be computed using the cocycle ϕ_E associated to the Bianchi Eisenstein modular form E of weight 2. For a lattice $L = \mathcal{O}_K$, consider the subgroup

$$\Gamma(u,v) = \{A \in \mathrm{SL}_2(\mathcal{O}_K) | (u,v)A = (u,v)\} \text{ for } (u,v) \in (\mathbb{C}/L)^2\}.$$

Note that $\Gamma_1(N)$ is a subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ of the form $\Gamma(u,v)$ for $(u,v) = (0,1) \in \left(\frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\right)^2$. On the other hand, it is easy to see that the congruence subgroup $\Gamma_0(\mathfrak{P})$ is not a subset of the form $\Gamma(u,v)$ for any $(u,v) \in (\mathbb{C}/L)^2$ for a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$.

We are interested in subgroups of the form $\Gamma_1(N)$ for the rest of the thesis in the setting of [33, Theorem 9, p. 101].

We utilize the explicit 1-cocycles defined by Sczech in [33] to construct a basis for $\mathrm{H}^1_{Eis}(\Gamma_1(N);\mathbb{C})$. Considering $\mathcal{O}_K$ as a lattice in $\mathbb{C}$, for $k \in \{0,1,2\}$ and $u \in \mathbb{C}$, we define

$$E_k(u) = E_k(u,\mathcal{O}_K) = \sum_{\substack{w\in\mathcal{O}_K \\ w\neq -u}} (w+u)^{-k}|w+u|^{-s}\Bigg|_{s=0}$$

where the notation $..\,|_{s=0}$ means that the corresponding value is determined by analytic continuation to $s = 0$. We further define $E(u)$ by setting

$$2E(u) = \begin{cases} 2E_2(0), & u \in \mathcal{O}_K \\ \wp(u) - E_1(u)^2, & u \notin \mathcal{O}_K \end{cases}$$

where $\wp(u)$ denotes the Weierstrass $\wp$-function.

Let N be a positive integer. Given $u, v \in \frac{1}{N}\mathcal{O}_K$, Sczech introduced certain homomorphisms

$$\Psi(u,v) : \Gamma_1(N) \to \mathbb{C}$$

that depend solely on the classes of $(u, v) \in (\frac{1}{N}\mathcal{O}_K/\mathcal{O}_K)^2$. For $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_1(N)$, we have a simple expression

$$\Psi(u, v)(A) = -\left(\frac{\bar{b}}{d}\right) E(u) - \frac{b}{d} E_0(u) E_2(v)$$

where

$$\left(\frac{t}{s}\right) = -1 + \#\left\{y \bmod s\mathcal{O}_K \mid y^2 \equiv t \mod s\mathcal{O}_K\right\}$$

represents the Legendre symbol. For non-parabolic $A \in \Gamma_1(N)$, there exists a similar but more intricate description involving the Eisenstein series E_k's. This generalizes the classical Dedekind sums. It's noteworthy that $E(-u) = E(u)$ and $E_k(-u) = E_k(u)$ for even k, implying $\Psi(-u, -v) = \Psi(u, v)$ on parabolic elements. Indeed, by examining the definition in [33, Section 4], it becomes evident that $\Psi(-u, -v) = \Psi(u, v)$.

Sczech shows that if the congruence subgroup is of the form $\Gamma(u, v)$ for a *fixed* $(u, v) \in \left(\frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\right)^2$ like $\Gamma_1(N)$, we can define the collection of homomorphisms $\Psi(u, v)$ with $(u, v) \in \left(\frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\right)^2$ residing in the Eisenstein part of the cohomology. Remarkably, the number of linearly independent homomorphisms in this set equals the number of cusps of Γ. Thus they generate $\mathrm{H}^1_{Eis}(\Gamma_1(N); \mathbb{C})$. Note that the above Eisenstein series associated with different cusps are linearly independent because they are non-vanishing only at their associated cusp. This implies that the cohomology classes of Sczech cocycles, which are associated with the cusps of $\Gamma_1(N)$ form a basis of $\mathrm{H}^1_{Eis}(\Gamma_1(N); \mathbb{C})$.

We can define the analogue of the Rademacher homomorphism for Γ

$$\phi_E : \gamma \mapsto \int_{\gamma \mathfrak{F}_K} d\left(\star \omega_E\right)$$

where E is an Eisenstein series of weight 2 on Γ. We strongly believe there are connections with Szech cocycles as discussed in § 6.1.2. These are generalizations of period functions defined for the full subgroup (cf. § 3.5 Theorem 4.2.1). Our belief comes from the inner product formula proved in Proposition 5.3.2.

We now list some properties of the Eisenstein series that we believe are connections between generalized Dedekind sums and period integrals of the Eisenstein series.

1. Ito [22] showed that (see also Weselmann [39]) up to a coboundary, the cocycles of Sczech are integrals of closed harmonic differential forms.

2. These differential forms are given by certain Eisenstein series defined on the hyperbolic space $\mathbb{H}_3$.

3. Following Ito, we form an Eisenstein series $E_{(u,v)}(\tau, s)$ for $(\tau, s) \in \mathbb{H}_3 \times \mathbb{C}$ with values in $\mathbb{C}^3$ associated to each cusp of $\Gamma_1(N)$. As a function of s, $E_{(u,v)}(\tau, s)$ can be analytically continued to all of $\mathbb{C}$.

4. Harder [18] showed that the differential 1-form on the hyperbolic 3-space induced by $E_{(u,v)}(\tau, s)$ is closed for $s = 0$. Ito [22] showed that the cocycle given by the integral of this closed differential 1-form differs from the cocycle $\Psi(u, v)$ of Sczech by a coboundary.

7

Lefschetz number

Lefschetz number, denoted by $L(f)$, is a numerical invariant associated with a continuous map f from a topological space X to itself. It is calculated by summing over certain traces of the induced linear maps on the homology groups of X. The Lefschetz number, which connects the topological properties of X to the fixed points of the map f, is an essential tool in algebraic topology and has applications in numerous other mathematical fields.

Recall the study of the Lefschetz fixed point theorem as presented by Sengñ-Türkelli [10].

Consider G as the Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$, viewed as a lattice inside the real Lie group $\mathrm{SL}_2(\mathbb{C})$, thereby being a discrete group of isometries of hyperbolic 3-space, denoted $\mathbb{H}_3$. Let $\rho \in Aut(G)$ be an involution, and $\mathfrak{g} = \{1, \rho\}$ be a subgroup of the automorphism group $Aut(G)$ (note that $Out(G)$ is a finite elementary abelian 2-group explicitly determined by Smillie-Vogtmann [38]). Consider Γ as a $\mathfrak{g}$-stable torsion-free finite index subgroup of G.

Let M_k be a Γ-module with a $\mathfrak{g}$-action, and $\mathcal{M}_k$ be the locally constant sheaf on Y_Γ induced by M_k. Ensure that the action of $\mathfrak{g}$ on the module is compatible with the action on Γ, i.e., ${}^\rho(g \cdot e) = {}^\rho g \cdot {}^\rho e$. Then, $\mathfrak{g}$ acts on the cohomology groups $\mathrm{H}^i\left(Y_\Gamma; \mathcal{M}_k\right)$.

Define the *Lefschetz number* as follows

$$L(\rho, \Gamma, M_k) := \sum_i (-1)^i \operatorname{tr}\left(\rho \mid \mathrm{H}^i\left(Y_\Gamma; \mathcal{M}_k\right)\right).$$

Let Y_Γ^ρ be the set of fixed points of the ρ-action on Y_Γ. Denote by $\mathcal{M}_k^\rho$ the restriction of the sheaf $\mathcal{M}_k$ to Y_Γ^ρ. Then, ρ acts on the stalk of $\mathcal{M}_k^\rho$, and $L\left(\rho, Y_\Gamma^\rho, \mathcal{M}_k^\rho\right)$ is defined. As per [31, p.152], it holds that

$$L(\rho, \Gamma, M_k) = L\left(\rho, Y_\Gamma^\rho, \mathcal{M}_k^\rho\right).$$

The connected components of Y_Γ^ρ can be parametrized by the first non-abelian (Galois) cohomology $\mathrm{H}^1(\mathfrak{g}; \Gamma)$. If γ is a cocycle for $\mathrm{H}^1(\mathfrak{g}; \Gamma)$, we have a γ-twisted ρ-action on $\mathbb{H}_3$ given by $x \mapsto {}^\rho x\gamma^{-1}$. The fixed point set $\mathbb{H}_3(\gamma)$ of the γ-twisted action on $\mathbb{H}_3$ is non-empty, and its image in Y_Γ, denoted $F(\gamma)$, is a locally symmetric subspace of Y_Γ^ρ.

There is also a γ-twisted ρ-action on Γ given by $g \mapsto \gamma^\rho g\gamma^{-1}$ for $g \in \Gamma$. Let $\Gamma(\gamma)$ denote the set of fixed points of this action. When Γ is torsion-free, the canonical map

$$\pi_\gamma : \Gamma(\gamma)\backslash\mathbb{H}_3(\gamma) \to Y_\Gamma$$

is injective. The image of π_γ is homeomorphic to $F(\gamma)$. There is a twisted ρ-action on M_k as well, given by $m \mapsto {}^\rho m\gamma$ for $m \in M_k$. The trace of this action on M_k does not depend on the choice of the cocycle γ in its class and can therefore be written as $\operatorname{tr}\left(\rho_\gamma \mid M_k\right)$.

We can express the Lefschetz trace formula for the torsion-free case in a geometric manner as follows:

Theorem 7.0.1. (**J**.**Rohlfs**). *Assuming that Γ is torsion-free, we have*

$$L(\rho, \Gamma, M_k) = \sum_{\gamma \in \mathrm{H}^1(\mathfrak{g};\Gamma)} \chi(F(\gamma)) \operatorname{tr}\left(\rho_\gamma \mid M_k\right)$$

where $\chi(F(\gamma))$ is Euler characteristic of $F(\gamma)$. A more generalized version of the theorem, accommodating Γ with possibly torsion elements, is provided by Blume-Nienhaus [6, I.1.6].

Let σ represent the complex conjugation. The action of σ on $\mathbb{H}_3$ is defined as $(z, t) \mapsto (\bar{z}, t)$, where $\bar{z}$ denotes the complex conjugate of z.

It also acts on $\mathrm{SL}_2(\mathbb{C})$ by operating on the entries of a matrix in an obvious manner. If $A \in \mathrm{SL}_2(\mathbb{C})$, then we denote its image under the action of σ as ${}^\sigma A$, or simply $\bar{A}$.

We now introduce the concept of the *twisted complex conjugation*, denoted by τ. The action of τ on $\mathbb{H}_3$ is given by $(z,t) \mapsto (-\bar{z},t)$, where $\bar{z}$ denotes the complex conjugate of z.

Its action on $\mathrm{SL}_2(\mathbb{C})$ is defined as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix}$, where the bar in the notation denotes the complex conjugation. We can regard τ as the composition $\alpha \circ \sigma = \sigma \circ \alpha$, where

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \beta$$

and $\beta := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, with $\alpha(z,t) = (-z,t)$ for every $(z,t) \in \mathbb{H}_3$.

Both σ and τ are orientation-reversing, and they can be naturally extended to the Borel-Serre compactification (see [30, section 1.4]). The action of σ on M_k can be described as follows: $\sigma(P \otimes Q) = Q \otimes P$. Similarly, for τ, we have $\tau(P \otimes Q) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Q \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P$. These actions are compatible with those on $\mathrm{SL}_2(\mathbb{C})$.

We now discuss the Lefschetz numbers for these two involutions. We use the symbol ρ to denote these two involutions in the results that apply to both of them. We begin with a useful lemma (see [6, I.4.3] for proof).

Lemma 7.0.2. *Let $\gamma \in \Gamma$ and $x = (\gamma\rho)^2$. Then* $\mathrm{tr}\,(\gamma\rho \mid M_k) = \mathrm{tr}\,(x \mid M_k)$.

7.1 Lefschetz numbers for full group

Recall that G denotes the full Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$. For $k = 0$, i.e., when $M_k = \mathbb{C}$, the Lefschetz numbers for σ and τ were computed by Krämer [24]. For general M_k, these numbers were computed by Blume-Nienhaus [6].

For a rational prime p, which ramifies in K, and an integer a, let $(a \mid p)$ denote the corresponding Hilbert symbol. By definition, $(a \mid p)$ equals 1 if there is an element in some finite extension of K_p, the completion of K at the unique prime ideal over p, whose norm is equal to a, and it equals -1 otherwise. Equivalently, $(a \mid p)$ is the value at a of the quadratic character associated with the local extension $\mathbb{Q}_p(\sqrt{d})/\mathbb{Q}_p$. Note that if $p \neq 2$, then $(a \mid p)$ equals the Legendre symbol $\left(\frac{a}{p}\right)$.

Theorem 7.1.1. *(**Blume-Nienhaus**, [6]). Let D be the discriminant of $K/\mathbb{Q}$, with D_2 being its 2-part. Let ρ represent either τ or σ. Also, let $q = 1$ or $q = -1$ depending on whether $\rho = \tau$ or $\rho = \sigma$, respectively. Then*

$$\begin{aligned}(-1)^k L(\rho, G, M_k) = & \frac{-q}{12} \prod_{\substack{p|D \\ p\neq 2}} \left(p + \left(\frac{q}{p}\right)\right) \prod_{\substack{p|D \\ p=2}} (D_2 + (q \mid 2))\,(k+1) \\ & + \frac{q}{12} \prod_{\substack{p|D \\ p\neq 2}} \left(1 + \left(\frac{-q}{p}\right)\right) \prod_{\substack{p|D \\ p=2}} (4 + (-q \mid 2))(-1)^k (k+1) \\ & + \frac{1}{2} \prod_{\substack{p|D \\ p\neq 2}} \left(1 + \left(\frac{-2q}{p}\right)\right) \left(\frac{k+1}{4}\right) \\ & + \frac{1}{3} \left(\prod_{\substack{p|D \\ p\neq 3}} (1 + (-3q \mid p)) + (-1)^k \prod_{p|D} (1 + (-q \mid p)) \right) \left(\frac{k+1}{3}\right).\end{aligned}$$

Here, products over empty sets are understood to be equal to 1.

7.2 Lefschetz number of σ for $\Gamma(N)$

Let $\Gamma = \Gamma(N) \subseteq \mathrm{SL}_2(\mathcal{O}_K)$, a congruence subgroup of level N. Its image in $\mathrm{PSL}_2(\mathcal{O}_K)$ is denoted by $\bar{\Gamma}$. For $N > 2$, both Γ and $\bar{\Gamma}$ are torsion-free. We employ Theorem 7.0.1 to compute the Lefschetz numbers $L(\sigma, \Gamma(N), M_k)$.

Firstly, we analyze the fixed point set Y_Γ^σ. Let $H(1)$ be the subset of $\mathrm{H}^1(\sigma; \bar{\Gamma})$ consisting of cocycles $\gamma \in \bar{\Gamma}$ with $\det(\gamma^\sigma \gamma) = 1$, and $H(2)$ be the subset with $\det(\gamma^\sigma \gamma) = -1$. We have $\mathrm{H}^1(\sigma; \bar{\Gamma}) = H(1) \cup H(2)$. If Γ is torsion-free, then $\mathrm{H}^1(\sigma; \bar{\Gamma}) = H(1)$.

Consider the matrices

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_1' = \begin{pmatrix} 1 & \sqrt{d} \\ 0 & 1 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\gamma_2' = \begin{cases} \begin{pmatrix} 1+\sqrt{d} & (2-d)/2 \\ -2 & -1+\sqrt{d} \end{pmatrix} & \text{if } d \equiv 2 \bmod 4, \\ \begin{pmatrix} \sqrt{d} & (d-1)/2 \\ 2 & \sqrt{d} \end{pmatrix} & \text{if } d \equiv 1 \bmod 4. \end{cases}$$

Note that γ_1 and γ_1' belong to $H(1)$, while γ_2 and γ_2' belong to $H(2)$. The locally symmetric space $F(\gamma)$ is a surface if $\gamma \in H(1)$ and is a point if $\gamma \in H(2)$. J.Rohlfs [29] provides the count of translations for the surfaces corresponding to γ_1 and γ_1', and for the points corresponding to γ_2 and γ_2'.

Theorem 7.2.1. *(**J.Rohlfs**, [29, Theorem 4.1]). Let D be the discriminant of $K/\mathbb{Q}$ and t be the number of distinct prime divisors of D. Let $(N) = \prod_{p|D} \mathfrak{p}_p^{j_p} \prod_{p\nmid D} (p)^{j_p}$ be an ideal with $N > 2$, and let $\Gamma = \Gamma(N)$ be the congruence subgroup of level (N). Let $s = \#\{p \text{ prime} \mid \ p|D, p \neq 2 \text{ and } j_p \neq 0\}$.*

Then Y_Γ^σ consists of translations of surfaces $F(\gamma_1)$ and $F(\gamma_1')$ and the number of transla-

tions of these surfaces are denoted by A and B respectively in the table below.

d	j_2	A	B
$d \equiv 1(4)$	≥ 0	2^{t-s}	0
$d \equiv 2(4)$	0	2^{t-s}	2^{t-s-1}
	1	2^{t-s}	2^{t-s-1}
	2	$8 \cdot 2^{t-s}$	0
	≥ 3	$8 \cdot 2^{t-s-1}$	0
$d \equiv 3(4)$	0	2^{t-s}	2^{t-s-1}
	1	2^{t-s}	0
	2	$8 \cdot 2^{t-s}$	0
	$j_2 = 2n+1 \geq 3$	2^{t-s-1}	0
	$j_2 = 2n \geq 4$	$8 \cdot 2^{t-s-1}$	0

Theorem 7.2.2. ***(Sëngun- Türkelli** [10]). Let $\Gamma(N)$, A, B be as in the theorem above. Then*

$$L(\sigma, \Gamma(N), M_k) = \begin{cases} (A+2B) \cdot \frac{-N^3}{12} \prod_{p|N} \left(1-p^{-2}\right) \cdot (k+1) & \text{if } N \text{ is even,} \\ (A+3B) \cdot \frac{-N^3}{12} \prod_{p|N} \left(1-p^{-2}\right) \cdot (k+1) & \text{if } N \text{ is odd.} \end{cases}$$

Proof. For the proof, we refer to the work of Sëngun and Türkelli [10, page 7]. □

7.3 Lefschetz number of σ for $\Gamma_1(N)$

In this section, we use [10, Theorem 2.1] to calculate the Lefschetz numbers for the congruence subgroups of the form $\Gamma_1(N)$.

Lemma 7.3.1. (**J.Rohlfs**). *Assume that Γ is torsion-free. Then*

$$(X_\Gamma^{BS})^\sigma = \bigcup_{\gamma \in \mathrm{H}^1(\sigma;\Gamma)} F(\gamma)$$

where $(X_\Gamma^{BS})^\sigma$ the set of fixpoints of σ in X_Γ^{BS}.

Let $\Gamma = \Gamma_1(N)$. Then $X_\Gamma = X_1(N)$, and we know that

$$(X_\Gamma^{BS})^\sigma = \bigcup_{\gamma \in \mathrm{H}^1(\sigma;\Gamma)} F(\gamma)$$

with

$$(X_\Gamma^{BS})^\sigma = \#\mathrm{H}^1(\sigma;^{\gamma_1}\Gamma)\cdot F(\gamma_1) \cup \#\mathrm{H}^1(\sigma;^{\gamma_1'}\Gamma)\cdot F(\gamma_1').$$

Then $(X_\Gamma^{BS})^\sigma$ consists of translations of surfaces $F(\gamma_1)$ and $F(\gamma_1')$ and the number of translations of these surfaces are denoted by $A_1(N)$ and $B_1(N)$.

The set of places of K will be denoted by V, and V_∞ (resp. V_f) refers to the set of archimedean (resp. non-archimedean) places of K.

For the following lemma, we use $\mathcal{O}$ instead of $\mathcal{O}_K$ for convenience.

For all finite places $v \in V_f$, let $\mathfrak{p}_v$ be the prime ideal corresponding to v and $N_v := [K_v : \mathbb{Q}_p]$ if $v|p$. Consider the congruence subgroup $\Gamma = \Gamma_1(N) \subset \mathrm{SL}_2(\mathcal{O})$ for an ideal $N\mathcal{O} = \prod \mathfrak{p}_v^{j_v}$. The completion $\Gamma_v(j_v)$ of $\Gamma(= \Gamma_1(\mathrm{N}))$ in $\mathrm{SL}_2(\mathcal{O}_v)$, $v \in V_f$, is a subgroup of $\mathrm{SL}_2(\mathcal{O}_v)$. Then one has $\mathfrak{p}_v \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p, and $p\mathcal{O}_v = \mathfrak{p}_v^{e_v}$. Define

$$s_v := \left[\frac{e_v}{p-1}\right] + 1.$$

Let $M(s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $s \in \mathbb{N}$, denote the set of (2×2)-matrices with a, c, d in $\mathfrak{p}_v^s$ and b in $\mathcal{O}_v$. For all $s \geqslant s_v$, the exponential map defines a bijection exp: $M(s) \mapsto 1 + M(s)$. Then, the map exp induces for any $s \geqslant s_v$ a bijection

$$\mathfrak{sl}_2(\mathcal{O}_v) \cap M(s) \xrightarrow{\sim} \Gamma_v(s)$$

where $\mathfrak{sl}_2(\mathcal{O}_v)$ denotes the $\mathcal{O}_v$-Lie algebra of $\mathrm{SL}_2(\mathcal{O}_v)$.

Now, we are generalizing the results mentioned in the work of J.Rohlfs-Schwarmer [31] from subgroups of symplectic groups to subgroups of the Bianchi group $\mathrm{SL}_2(\mathcal{O})$.

Lemma 7.3.2. *Local cohomology groups for a subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O})$ can be computed as follows:*

1. *If $\mathfrak{p}_v \mid p$ and if $p > 2$ then $\mathrm{H}^1(\sigma;\Gamma_v(j_v)) = \{1\}$ for $j_v = 0, 1, 2, \ldots$*

2. *If* $\mathfrak{p}_v \mid 2$ *and if* $j_v \geqslant s_v + e_v$ *then cardinality of* $\mathrm{H}^1(\sigma; \Gamma_v(j_v)) = 2^{N_v}$ *for some* $N_v \in \mathbb{N}$. *The inclusion* $\Gamma_v(j_v) \to \Gamma_v(j_v - e_v)$ *induces the trivial map:*

$$\mathrm{H}^1(\sigma; \Gamma_v(j_v)) \to \mathrm{H}^1(\sigma; \Gamma_v(j_v - e_v)).$$

Proof. (1) First consider the case $j_v \geqslant s_v$. Let x be a cocycle for $\langle\sigma\rangle$ in $\Gamma_v(j_v)$ i.e., $x.^{\sigma}x = 1$, thus, ${}^{\sigma}x = \bar{x}$. So there exists an X in $\mathfrak{sl}_2(\mathcal{O}_v) \cap M(j_v)$ with $\exp(X) = x$ and $\exp(-X) = {}^{\sigma}x$. Define $y := \exp(-X/2)$; note that $y \in \Gamma_v(j_v) \cap \sigma(\Gamma_v(j_v)) = \Gamma_v(j_v)$. We have ${}^{\sigma}y = \exp(-\sigma(X)/2) = \exp(X/2)$. Thus, we obtain

$$y^{-1\sigma}y = x$$

i.e., x is equivalent to 1 , and $\mathrm{H}^1(\sigma; \Gamma_v(j_v)) = \{1\}$.

Next, the quotient group $\Gamma_v(j_v)/\Gamma_v(j_v+1)$ is a commutative p-group, and since the order $|\langle\sigma\rangle| = 2$ is coprime to p, we get $\mathrm{H}^1(\sigma; \Gamma_v(j_v)/\Gamma_v(j_v+1)) = \{1\}$. Using the long cohomology sequence attached to the sequence

$$1 \to \Gamma_v(j_v+1) \to \Gamma_v(j_v) \to \Gamma_v(j_v)/\Gamma_v(j_v+1) \to 1$$

we get $\mathrm{H}^1(\sigma; \Gamma_v(j_v)) = \{1\}$ for $j_v \geqslant 1$.

For $j_v = 0$, i.e. $\Gamma_v(j_v) = \mathrm{SL}_2(\mathcal{O}_v)$. From [29, Collary 2.7], we get

$$\mathrm{H}^1(\sigma; \mathrm{SL}_2(\mathcal{O}_v)) = \{1\}.$$

This proves our assertion for all j_v.

(2) As in the proof of (1), given a cocycle $\langle\sigma\rangle$ in $\Gamma_v(j_v), j_v \geqslant s_v + e_v$, there exists an element X in $\mathfrak{sl}_2(\mathcal{O}_v) \cap M(j_v)$ with $\exp(X) = x$ and $\exp(-X) = {}^{\sigma}x$.

Define $y := \exp(-X/2)$. Then $y \in \Gamma_v(j_v - e_v)$. By using ${}^{\sigma}y = \exp(X/2)$ we obtain $\bar{y}^{\sigma}y = x$, that is, x is equivalent to 1 viewed as an element in $\mathrm{H}^1(\sigma; \Gamma_v(j_v - e_v))$. This proves the second assertion in (2). This fact implies that we have an inclusion (after shifting the parameter j_v by e_v)

$$1 \to \mathrm{H}^1(\sigma; \Gamma_v(j_v)) \to \mathrm{H}^1(\sigma; \Gamma_v(j_v)/\Gamma_v(j_v + e_v)).$$

The quotient group $\Gamma_v(j_v)/\Gamma_v(j_v+e_v)$ can be identified with the abelian group

$$R := \mathfrak{sl}_2(\mathcal{O}_v) \cap M(j_v)/\mathfrak{sl}_2(\mathcal{O}_v) \cap M(j_v+e_v) \cong U(\mathcal{O}_v/2\mathcal{O}_v)$$

where $U(\mathcal{O}_v/2\mathcal{O}_v) := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathcal{O}_v) \,\middle|\, a, c \in \mathcal{O}_v/2\mathcal{O}_v \text{ and } b \in \mathcal{O}_v \right\}$.

Note that, for $s = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in U(\mathcal{O}_v/2\mathcal{O}_v)$, we get

$$\sigma(s) = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix}.$$

Such an element s represents a cocycle for $\langle\sigma\rangle$ in R if and only if $s \equiv \bar{s}^{-1} \bmod 2$. Coboundaries for $\langle\sigma\rangle$ in R are represented by elements

$$s + {}^{\sigma}s = \begin{pmatrix} a+\bar{a} & b+\bar{b} \\ c+\bar{c}, & -(a+\bar{a}) \end{pmatrix} \quad \text{where} \quad a, c \in \mathcal{O}_v/2\mathcal{O}_v \quad \text{and} \quad b \in \mathcal{O}_v.$$

Hence $\mathrm{H}^1(\sigma; R)$ can be identified with the set

$$\Delta = \{\mathrm{diag}\,(t, -t) \mid t \in \mathcal{O}_v/2\mathcal{O}_v\}$$

of diagonal matrices, and these classes are in the image of $\mathrm{H}^1(\sigma; \Gamma_v(j_v))$ under the map above. Thus, (2) holds.

□

Lemma 7.3.3. *Let σ and Γ be as above. For $\gamma \in \{\gamma_1, \gamma_1'\}$, the cohomology groups $\mathrm{H}^1(\sigma; {}^{\gamma}\Gamma)$ are 2 groups.*

Proof. We recall the relation from J.Rohlfs [29, page 201]

$$\mathrm{H}^1(\sigma; \Gamma) = \prod_v \mathrm{H}^1(\sigma; \Gamma_v(j_v)).$$

Using $\#\mathrm{H}^1(\sigma; \Gamma_v(j_v)) = 2^{N_v}$, we get

$$\#\mathrm{H}^1(\sigma;\Gamma) = \#\prod_v \mathrm{H}^1(\sigma;\Gamma_v(j_v)) = \prod_v 2^{N_v}.$$

The γ- twisted σ action on Γ produces the following short exact sequence

$$0 \to {}^{\gamma}\Gamma \to \Gamma \to \Gamma/{}^{\gamma}\Gamma \to 0.$$

This induces an exact sequence

$$\ldots \to \mathrm{H}^0\,(\sigma;\Gamma/{}^{\gamma}\Gamma) \to \mathrm{H}^1\,(\sigma;{}^{\gamma}\Gamma) \xrightarrow{\delta} \mathrm{H}^1\,(\sigma;\Gamma) \to \mathrm{H}^1\,(\sigma;\Gamma/{}^{\gamma}\Gamma) \to \ldots$$

From basic group theory, we have

$$\frac{\mathrm{H}^1(\sigma;{}^{\gamma}\Gamma)}{\mathrm{Ker}\delta} \simeq \mathrm{Im}\delta.$$

Since σ is an involution, so $\mathrm{H}^0\,(\sigma;\Gamma/{}^{\gamma}\Gamma)$ is a 2 group. We also note that $\mathrm{Im}\delta \subset \mathrm{H}^1(\sigma;\Gamma)$ and hence a 2 group. We deduce that that $\#\mathrm{H}^1(\sigma;{}^{\gamma}\Gamma) = 2^r$ for some $r \in \mathbb{N}\cup\{0\}$. □

We define the following important quantities that we use in our computations

$$A_1(N) := \#\mathrm{H}^1(\sigma;{}^{\gamma_1}\Gamma) = 2^a \tag{7.1}$$

$$B_1(N) := \#\mathrm{H}^1(\sigma;{}^{\gamma_1{}'}\Gamma) = 2^b \tag{7.2}$$

for $a, b \in \mathbb{N}\cup\{0\}$.

Using [10, Theorem 2.1] and proposition 7.3.1, we now calculate the Lefschetz number for $\Gamma_1(N)$. Consider the subgroup $\Gamma(u,v) = \{A \in \mathrm{SL}_2(\mathcal{O}_K)|(u,v)A = (u,v)\}$ for $u, v \in \mathbb{C}/\mathcal{O}_K$. Note that $\Gamma_1(N)$ is a subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ of the form $\Gamma(u,v)$.

Lemma 7.3.4. *The congruence subgroup $\Gamma_0(\mathfrak{P})$ is not a subset of $\Gamma(u,v)$ for any non-zero $(u,v) \in (\mathbb{C}/\mathcal{O}_K)^2$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{P})$ such that $a \neq 1, d \neq 1(\bmod\ \mathfrak{P})$, and assume that $A \in \Gamma(u,v)$ for some $0 \neq (u,v) \in (\mathbb{C}/\mathcal{O}_K)^2$. We then have $c \equiv 0 \pmod{\mathfrak{P}}$, and

$$
\begin{gathered}
(u,v)A = (u,v)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Big(au+cv, bu+dv\Big), \\
\text{and } \Big(au+cv, bu+dv\Big) = \Big(u,v\Big) \\
\implies \Big((a-1)u+cv, bu+(d-1)v\Big) = (0,0) \\
\implies (a-1)u+cv = 0, bu+(d-1)v = 0, \text{and we know that } c \equiv 0 \pmod{\mathfrak{P}} \\
\implies (a-1)u = 0 \text{ and } bu+(d-1)v=0 \pmod{\mathfrak{P}} \\
(a-1)u = 0 \implies u = 0 \text{ and } bu+(d-1)v = 0 \implies v = 0
\end{gathered}
$$

and $a \neq 1, d \neq 1 (\text{mod } \mathfrak{P})$. Hence, we do not get any non-zero solution and deduce that $\Gamma_0(\mathfrak{P}) \not\subset \Gamma(u,v)$ for all non-zero $(u,v) \in (\mathbb{C}/\mathcal{O}_K)^2$. □

Because of the above Lemma, we are interested in the congruence subgroups of the form $\Gamma_1(N)$ in this thesis. It will be really interesting to see similar results for the congruence subgroups of the form $\Gamma_0(N)$.

From now on, we use the notation $\Gamma_1(N)^{\gamma\sigma}$ to denote that we first apply the σ action to $\Gamma_1(N)$ and then apply the γ action. This action is equivalent to the action obtained by γ- twisted ρ-action when ρ is σ. We write this to specify that we are using σ.

Proposition 7.3.5. *Consider the subgroup $\Gamma_1(N)$ and the quantities $A_1(N), B_1(N)$ be as above. Consider the quantity*

$$
C_1(N) := \frac{-N^2}{12} \prod_{p|N} \left(1 - p^{-2}\right) \cdot (k+1).
$$

The Lefschetz number is given by

$$
L\left(\sigma, \Gamma_1(N), M_k\right) = \begin{cases} \left(A_1(N) + (\frac{N+2}{2}) B_1(N)\right) C_1(N) & \text{if } N \text{ is even} \\ \left(A_1(N) + (\frac{N+1}{2}) B_1(N)\right) C_1(N) & \text{if } N \text{ is odd.} \end{cases}
$$

Proof. For each $\gamma \in H(1)$, we have $\operatorname{tr}(\gamma\sigma \mid M_k) = \operatorname{tr}(\mathbf{1} \mid M_k) = (k+1)$ (cf. Lemma 7.0.2). By [10, Theorem 2.1], we need to calculate the Euler Poincaré characteristics $\chi\left(\Gamma_1(N)^{\gamma\sigma}\right)$ with $\gamma \in \{\gamma_1, \gamma_1'\}$.

Let $\Gamma_1^e(N)$ be the congruence subgroup of the elliptic modular group $\mathrm{SL}_2(\mathbb{Z})$ of level N. An easy calculation shows that $\Gamma_1(N)^{\gamma_1\sigma} = \Gamma_1^e(N)$. Let $Y_1(N)$ denotes the hyperbolic surface associated to $\Gamma_1^e(N)$. The compactification $X_1(N)$ is obtained from $Y_1(N)$ by adding the cusps. It is well-known that $Y_1(N)$ has $\frac{1}{2}\sum_{d|N}\phi(d)\cdot\phi\left(\frac{N}{d}\right)$ cusps. By [37, 1.6 .4], we have

$$\chi\left(X_1(N)\right) = (-1/12)N^2\prod_{p|N}\left(1-p^{-2}\right) + \frac{1}{2}\sum_{d|N}\phi(d)\cdot\phi\left(\frac{N}{d}\right).$$

We deduce that

$$\begin{aligned}\chi\left(\Gamma_1^e(N)\right) := \chi\left(Y_1(N)\right) &= \chi\left(X_1(N)\right) - \#\{\text{ cusps of } Y_1(N)\}\\ &= (-1/12)N^2\prod_{p|N}\left(1-p^{-2}\right).\end{aligned}$$

We now calculate the Euler-Poincaré characteristics $\chi\left(\Gamma_1(N)^{\gamma_1'\sigma}\right)$. For $h = \begin{pmatrix} 1 & \sqrt{d} \\ 0 & 2 \end{pmatrix}$, we have the following description

$$\Gamma_1(N)^{\gamma_1'\sigma} = \left\{\begin{pmatrix} x+z\sqrt{d} & y+\frac{w-x}{2}\sqrt{d} \\ 2z & w-z\sqrt{d}\end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K) \mid \right.$$
$$\left. x-1\equiv w-1\equiv z\equiv 0 \pmod{N} \text{ and } w\equiv x \pmod{2N}\right\}.$$

For $d\equiv 2 \pmod 4$, the condition that $w\equiv x \pmod{2N}$ directly follows from the condition that the determinant is 1 for a matrix in $\mathrm{SL}_2(\mathcal{O}_K)$. An easy calculation shows that

$$h^{-1}\Gamma_1(N)^{\gamma_1'\sigma}h = \left\{\begin{pmatrix} x & 2y+zd \\ z & w\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid x\equiv w\equiv 1, \quad z\equiv 0 \pmod{N}\right\}.$$

Thus, we have a relation

$$h^{-1}\Gamma_1(N)^{\gamma_1'\sigma}h = \Gamma_1^e(N)\cap\Gamma'.$$

Here, $\Gamma^{'} \leq \mathrm{SL}_2(\mathbb{Z})$ is the group of matrices that are of type $\left\{ \begin{pmatrix} x & 2y \\ z & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}$. A small check shows that the index is given by

$$\left[\Gamma_1^e(N) : \Gamma_1^e(N) \cap \Gamma^{'}\right] = \begin{cases} \frac{N+2}{2} & \text{if } N \text{ is even,} \\ \frac{N+1}{2} & \text{if } N \text{ is odd.} \end{cases}$$

If N is odd, then the cosets of the quotient spaces are represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & N-4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & N-2 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, for even N, they are represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & N-3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & N-1 \\ 0 & 1 \end{pmatrix}.$$

Similarly for when $d \equiv 1, 3 \pmod 4$, the index $\left[\Gamma_1^e(N) : h^{-1}\Gamma_1(N)^{\gamma_1^{\prime}\sigma} h\right]$ is $\frac{N+2}{2}$ if N is even, and it is $\frac{N+1}{2}$ if N is odd. This implies that

$$\chi\left(\Gamma_1(N)^{\gamma_1^{\prime}\sigma}\right) = \begin{cases} \left(\frac{N+2}{2}\right) \chi\left(\Gamma_1^e(N)\right) & \text{if } N \text{ is even,} \\ \left(\frac{N+1}{2}\right) \chi\left(\Gamma_1^e(N)\right) & \text{if } N \text{ is odd.} \end{cases}$$

This completes the proof of the proposition (using Theorem 7.0.1 and Lemma 7.0.2). □

Corollary 7.3.6. *Let p be an odd rational prime that is unramified over K. Let t be the number of distinct prime divisors of D. Then, for $n > 0$, we have*

$$L\left(\sigma, \Gamma_1\left(p^n\right), M_k\right) = -(2^a + (p^n + 1)2^{b-1}) \cdot \frac{\left(p^{2n} - p^{2n-2}\right)}{12} \cdot (k+1).$$

Proof. Since p is odd, recall that p is the only prime divisor of the level. Since the prime

p is unramified, we have

$$\begin{aligned}
&\left(A_1(N) + \left(\frac{N+1}{2}\right) B_1(N)\right) \cdot \frac{-N^2}{12} \prod_{p|N} \left(1 - p^{-2}\right) \cdot (k+1) \\
&= \left(2^a + \left(\frac{p^n+1}{2}\right) 2^b\right) \cdot \frac{-p^{2n}}{12} \left(1 - p^{-2}\right) \cdot (k+1) \\
&= \left(2^a + (p^n+1)2^{b-1}\right) \cdot \frac{-1}{12} \left(p^{2n} - p^{2n-2}\right) \cdot (k+1) \\
&= -\left(2^a + (p^n+1)2^{b-1}\right) \cdot \frac{\left(p^{2n} - p^{2n-2}\right)}{12} \cdot (k+1).
\end{aligned}$$

□

8

Trace on the Eisenstein cohomology

For any $k \in \mathbb{N} \cup \{0\}$, let $\mathbb{C}[x, y]_k$ represent the space of homogeneous polynomials of degree k in variables x and y with complex coefficients. The modular group $\mathrm{SL}_2(\mathbb{C})$ acts naturally on this space. Now, define the $\mathrm{SL}_2(\mathbb{C})$-module M_k as the tensor product of $\mathbb{C}[x, y]_k$ and its complex conjugate. This module, denoted as M_k, induces a locally constant sheaf $\mathcal{M}_k$ on Y_Γ, where the stalks of $\mathcal{M}_k$ are isomorphic to M_k, and the action on the conjugate factor is twisted with complex conjugation.

Let σ represent the *complex conjugation*, and let the *twisted complex conjugation* be denoted by τ. The actions of both σ and τ on $\mathbb{H}_3$ and $\mathrm{SL}_2(\mathcal{O}_K)$ are defined in Section 7.1.

For $N \in \mathbb{N}$, let $\mathcal{C}_N$ be the set of cusps of the congruence subgroups of the form $\Gamma_1(N)$.

8.1 Trace on the Eisenstein cohomology for full group

Recall the following Proposition by Sengün-Türkelli [10].

Proposition 8.1.1. *(**Sëngun- Türkelli** [10]). Let $G = \mathrm{SL}_2(\mathcal{O}_K)$. Then, the image of the restriction map*

$$\mathrm{H}^1\left(X_G^{BS}; \mathbb{C}\right) \to \mathrm{H}^1\left(\partial X_G^{BS}; \mathbb{C}\right)$$

is inside the -1-eigenspace of complex conjugation acting on $\mathrm{H}^1\left(\partial X_G^{BS}; \mathbb{C}\right)$.

Let us note that this result is extended to all maximal orders of $M_2(K)$ (with complex conjugation twisted accordingly) by Blume-Nienhaus [6, V.5.7] and by Berger [5, Section 5.2].

Corollary 8.1.2. *(Sëngun- Türkelli [10]). Let σ^i_{Eis} be the involution on $\mathrm{H}^i_{Eis}(\mathrm{SL}_2(\mathcal{O}_K);\mathbb{C})$ given by complex conjugation. Then*

$$\mathrm{tr}\left(\sigma^0_{Eis}\right) = 1, \quad \mathrm{tr}\left(\sigma^1_{Eis}\right) = -h(K), \quad \mathrm{tr}\left(\sigma^2_{Eis}\right) = -2^{t-1}+1$$

where t is the number of primes that ramify in K and $h(K)$ is the class number of K.

Proof. Let us denote $X = X^{BS}_{\mathrm{SL}_2(\mathcal{O}_K)}$ for convenience.

The claim for σ^0_{Eis} follows straightforwardly from the fact that $\mathrm{H}^0_{Eis}(X;\mathbb{C}) = \mathrm{H}^0(X;\mathbb{C}) = \mathbb{C}$. The action of σ on the latter is trivial. It's a well-known result that the set of cusps of $\mathrm{SL}_2(\mathcal{O}_K)$ is in bijection with the class group of K, and the action of complex conjugation σ on the cusps corresponds to taking the inverse in the class group. See [36, Theorem 9] for further details.

Thus, an element of the class group is fixed by σ if and only if it is of order 2. Genus Theory tells us that the number of elements of order 2 in the class group is 2^{t-1}, implying that the trace of the involution induced by σ on $\mathrm{H}^0(\partial X;\mathbb{C})$ is 2^{t-1}.

By Poincaré duality and the orientation-reversing nature of complex conjugation, we deduce that the trace of the involution induced by σ on $\mathrm{H}^2(\partial X;\mathbb{C})$ is -2^{t-1}. The long exact sequence associated with the pair $(X,\partial X)$ tells us that $\mathrm{H}^2(\partial X;\mathbb{C}) \simeq \mathrm{H}^2_{Eis}(X;\mathbb{C}) \oplus \mathrm{H}^3(X,\partial X;\mathbb{C})$. Here, the last summand is isomorphic to $\mathbb{C}$, and σ acts on it as -1, which follows from the fact that the action of σ on $\mathrm{H}^0(X;\mathbb{C})$ is trivial. This completes the proof for σ^2_{Eis}. □

8.2 Trace formula on the Eisenstein cohomology for $\Gamma(N)$

Theorem 8.2.1.*(Sëngun- Türkelli [10]). Assume that the class number of K is one and let p be a rational prime that is inert in K. Let ρ be the complex conjugation acting on the* *cohomology group. Then we have*

$$\operatorname{tr}\left(\sigma \mid \mathrm{H}^1_{Eis}\left(\Gamma\left(p^n\right);\mathbb{C}\right)\right) = \begin{cases} -(p^2+1) & \text{if } n = 1, \\ -\left(p^{2n}-p^{2n-2}\right) & \text{if } n > 1. \end{cases}$$

Proof. For a proof, we refer to the work of Sëngun and Türkelli [10, page 15]. □

Theorem 8.2.2. *(**Sëngun- Türkelli** [10]). Let t be the number of rational primes ramifying in K and σ be a complex conjugation acting on the cohomology group. Let $N = p_1^{n_1}\dots p_r^{n_r}$ be a positive odd number whose prime divisors p_i are unramified in K and let $\Gamma(N)$ be the congruence subgroup of the Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$ of level ideal (N). Then*

$$\operatorname{tr}\left(\sigma \mid \mathrm{H}^2_{Eis}\left(\Gamma(N);M_k\right)\right) = -2^{t-r-1}\cdot\prod_{i=1}^{r}\left(p_i^{2n_i}-p_i^{2n_i-2}\right)+\delta(0,k),$$

where δ is the Kronecker δ-function.

In particular, the trace of σ^2_{Eis} on $\mathrm{H}^2_{Eis}\left(\mathrm{SL}_2(\mathcal{O}_K);M_k\right)$ is $-2^{t-1}+\delta(0,k)$.

Proof. For a proof, we refer to the work of Sëngun and Türkelli [10, page 11]. □

8.3 Trace on the Eisenstein cohomology for $\Gamma_1(N)$

We compute the trace of complex conjugation on the Eisenstein cohomology following Sengün-Türkelli [10] for the subgroups of the form $\Gamma_1(N)$ inside the Bianchi modular groups.

Theorem 8.3.1. *Assume that the class number of K is one and let p be a rational prime that is inert in K. Let σ denote involution induced on the Eisenstein cohomology of $\Gamma_1(N)$ by a nontrivial an automorphism of K. Then we have*

$$\operatorname{tr}\left(\sigma \mid \mathrm{H}^1_{Eis}\left(\Gamma_1\left(p^n\right);\mathbb{C}\right)\right) = \begin{cases} -2 & \text{if } n = 1 \\ -\#\mathcal{C}_{p^n}\cdot\left(\frac{-2}{p^2-1}\right) & \text{if } n > 1 \end{cases}$$

where $\#\mathcal{C}_{p^n}$ is the number of cusps of $\Gamma_1(p^n)$.

Proof. This proof is motivated by the work of Sengün-Türkelli [10].

By Ito's result [23], we establish the following relation

$$\Psi(0,0)(\bar{A}) = -\Psi(0,0)(A).$$

Here, the bar indicates taking the complex conjugates of the entries of the matrix A. More generally, Ito demonstrates that

$$\Psi(u,v)(\bar{A}) = \frac{-1}{N^2} \sum_{s,t \in \frac{1}{N}\mathcal{O}_K/\mathcal{O}_K} \phi(s\bar{v} - t\bar{u})\Psi(s,t)(A)$$

where $\phi(z) := \exp(2\pi i(z - \bar{z})/D)$ and D is the discriminant of K. Observe that when $(s,t) = (u,v)$ or $(s,t) = (0,0)$, we have $\phi(s\bar{v} - t\bar{u}) = 1$. Using this, we can rewrite this summation more suggestively:

$$\Psi(u,v)(\bar{A}) = \frac{-1}{N^2}\left[\Psi(0,0)(A) + \Psi(u,v)(A) + \left(\sum_{\substack{s,t \in \frac{1}{N}\mathcal{O}_K/\mathcal{O}_K \\ (s,t)\neq(u,v) \\ (s,t)\neq(0,0)}} \phi(s\bar{v} - t\bar{u})\Psi(s,t)(A)\right)\right].$$

The action of complex conjugation σ on the Sczech cocycles is given by

$$\sigma(\Psi(u,v))(A) := \Psi(u,v)(\bar{A}).$$

We observe that $\sigma(\Psi(u,v))$ is expressed as a summation over all the Sczech cocycles. Let us consider the operator σ as a linear operator on the formal space $\mathbb{C}[\Psi_N]$, where the Sczech cocycles serve as a basis.

The pair $(0,0) \in \left(\frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\right)^2$ never corresponds to a cusp of $\Gamma_1(N)$, so we eliminate the term $\Psi(0,0)$ from the summation. Applying Ito's summation formula for the case $(u,v) = (0,0)$, we obtain

$$\Psi(0,0)(\bar{A}) = \frac{-1}{N^2}\left[\Psi(0,0)(A) + \left(\sum_{\substack{s,t\in \frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\\(s,t)\neq(0,0)}} \Psi(s,t)(A)\right)\right].$$

Given the relation $\Psi(0,0)(\bar{A}) = -\Psi(0,0)(A)$, we have

$$\Psi(0,0)(A) = \frac{1}{N^2-1}\sum_{\substack{s,t\in \frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\\(s,t)\neq(0,0)}} \Psi(s,t)(A).$$

Now, for $(u,v) \neq (0,0)$, we find

$$\Psi(u,v)(\bar{A}) = \frac{-1}{N^2}\left[\Psi(0,0)(A) + \left(\sum_{\substack{s,t\in \frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\\(s,t)\neq(0,0)}} \phi(s\bar{v}-t\bar{u})\Psi(s,t)(A)\right)\right].$$

Substituting $\Psi(0,0)(A)$, we obtain the expression

$$\Psi(u,v)(\bar{A}) = \frac{-1}{(N^2)\,(N^2-1)}\sum_{\substack{s,t\in \frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\\(s,t)\neq(0,0)}} \Psi(s,t)(A) + \frac{-1}{N^2}\sum_{\substack{s,t\in \frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\\(s,t)\neq(0,0)}} \phi(s\bar{v}-t\bar{u})\Psi(s,t)(A).$$

As $\Psi(-u,-v) = \Psi(u,v)$, we define $\mathbb{C}\left[\Psi_N^*\right]$ to be the formal vector space generated by the basis

$$\left\{\Psi(u,v) \mid (u,v) \in \left(\frac{1}{N}\mathcal{O}_K/\mathcal{O}_K\right)^2 / \pm 1, (u,v)\neq(0,0) \quad \text{and } u \text{ modulo } gcd(v,N)\right\}.$$

Having eliminated $\Psi(0,0)$ from the above equation, we now regard σ as a linear operator on the formal space $\mathbb{C}\left[\Psi_N^*\right]$.

Recall that $\Psi(-u,-v) = \Psi(u,v)$ and $\phi(-u\bar{v}+v\bar{u}) = \phi(u\bar{v}-v\bar{u}) = 1$. We notice that the coefficient of the summand $\Psi(u,v)(A)$ on the right-hand side of the equality is

$$\frac{-1}{(N^2)(N^2-1)}+\frac{-1}{N^2}=\frac{-2}{N^2-1}.$$

This implies that the trace of σ on $\mathbb{C}\left[\Psi_N^*\right]$ is $(\#\mathcal{C}_N)\cdot\frac{-2}{N^2-1}$.
For $N=p$, we have $\#\mathcal{C}_p=p^2-1$. Hence, the trace of σ on $\mathbb{C}\left[\Psi_p^*\right]$ is $\left(p^2-1\right)\cdot\frac{-2}{p^2-1}=-2$.

For $N=p^n$ with $n>1$, we have $\#\mathcal{C}_{p^n}=|\bar{P}_+\backslash \mathrm{SL}_2(\mathcal{O}_K/p^n\mathcal{O}_K)/\bar{P}|$. The trace of σ on $\mathbb{C}\left[\Psi_p^*\right]$ is $\left(\#\mathcal{C}_{p^n}\right)\cdot\frac{-2}{p^2-1}$.

□

Theorem 8.3.2. *Consider an imaginary quadratic field denoted by K, and let t be the number of distinct prime divisors of the discriminant of $K/\mathbb{Q}$. For a positive odd number $N=p_1^{n_1}\dots p_r^{n_r}$ with prime divisors p_i that are unramified in K, let $\Gamma_1(N)$ be the congruence subgroup of the Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$ with a level ideal of (N). Let ρ be the involutions induced in the Eisenstein cohomology of $\Gamma_1(N)$ by non-trivial automorphisms of K. Then*

$$\operatorname{tr}\left(\sigma \mid \mathrm{H}^2_{Eis}\left(\Gamma_1(N);M_k\right)\right)=\alpha\cdot-2^{t-r-1}\cdot\prod_{i=1}^{r}\left(p_i^{2n_i}-p_i^{2n_i-2}\right)+\delta(0,k)$$

where $\alpha=\#\left((\Gamma_1(N)/\Gamma(N))^{\rho}\right)$ and δ is the Kronecker δ-function.

In particular, the trace of ρ^2_{Eis} on $\mathrm{H}^2_{Eis}\left(\mathrm{SL}_2(\mathcal{O}_K);M_k\right)$ is $-2^{t-1}+\delta(0,k)$.

Proof. This proof is motivated by the work of Sengün-Türkelli [10]

Let us assume that $k>0$ and $\Gamma=\Gamma_1(N)$. In this scenario, the restriction map $\mathrm{H}^2\left(X_\Gamma^{BS};\mathcal{M}_k\right)\to\mathrm{H}^2\left(\partial X_\Gamma^{BS};\mathcal{M}_k\right)$ is onto. Hence, it suffices to compute the trace of ρ^2 on $\mathrm{H}^2\left(\partial X_\Gamma^{BS};\mathcal{M}_k\right)$. Using Poincaré duality and considering that ρ reverses the orientation, we can reduce the problem to computing the trace of ρ^0 on $\mathrm{H}^0\left(\partial X_\Gamma^{BS};\mathcal{M}_k\right)$.

The cohomology of the boundary can be seen as a direct sum of the cohomology of the boundary components X_c, which are 2-tori:

$$\mathrm{H}^0\left(\partial X_\Gamma^{BS};\mathcal{M}_k\right)\simeq\bigoplus_{c\in\mathcal{C}_\Gamma}\mathrm{H}^0\left(X_c^{BS};\mathcal{M}_k\right)\simeq\bigoplus_{c\in\mathcal{C}_\Gamma}\mathrm{H}^0\left(\Gamma_c;M_k\right).$$

If c is a cusp, then ρ maps Γ_c to $\Gamma_{\rho(c)}$. If $c\neq\rho(c)$, then

$$\mathrm{H}^0\left(\Gamma_c;M_k\right)\oplus\mathrm{H}^0\left(\Gamma_{\rho(c)};M_k\right)$$

forms a ρ^0-invariant subspace of $\bigoplus_{c\in\mathcal{C}_\Gamma} \mathrm{H}^0\left(\Gamma_c; M_k\right)$. Since ρ^0 takes the basis of the first summand to the basis of the second summand, the trace of ρ^0 on this subspace is 0.

If $c = \rho(c)$, then ρ^0 acts on the one-dimensional space $\mathrm{H}^0\left(\Gamma_c; M_k\right)$ and hence has the trace equal to 1. Therefore,

$$\mathrm{tr}\left(\sigma \mid \mathrm{H}^0\left(\partial X_\Gamma^{BS}; \mathcal{M}_k\right)\right) = \#\left(\mathcal{C}_\Gamma\right)^\rho$$

which means that the trace of ρ^0 equals the number of cusps of Γ that are invariant under the action of ρ.

Recall that $\mathcal{C}_\Gamma = \bigsqcup_{x\in\mathcal{C}_G} \Gamma\backslash Gx$, where G is $\mathrm{SL}_2(\mathcal{O}_K)$. Clearly, a cusp $c = gx\Gamma \in \mathcal{C}_\Gamma$ is ρ-invariant only if $x \in \mathcal{C}_G$ is also invariant under ρ. Thus, we have

$$(\mathcal{C}_\Gamma)^\rho = \bigsqcup_{x\in(\mathcal{C}_G)^\rho} (\Gamma\backslash Gx)^\rho.$$

According to [10, Corollary 4.4], we deduce that $\#\left(\mathcal{C}_G\right)^\rho = 2^{t-1}$. By [10, Lemma 4.5], it is enough to compute $\#\left(\Gamma\backslash G\infty\right)^\rho$.

Now, let us denote

$$R := \mathcal{O}_K/(N), \quad U^+(R) := \begin{pmatrix} \pm 1 & R \\ 0 & \pm 1 \end{pmatrix}, \quad U(R) := \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}.$$

Following our discussion on the cusps, we get

$$\begin{aligned} \#((\Gamma/\Gamma(N)) \cdot (\Gamma(N)\backslash G\infty))^\rho &= \#\Big((\Gamma/\Gamma(N))^\rho \cdot (\Gamma(N)\backslash G\infty)^\rho\Big) \\ &= \alpha \cdot \#\left(U^+(R)\backslash \mathrm{SL}_2(R)\right)^\rho. \end{aligned}$$

Now, using Sengün-Türkelli [10] and above observation, it is enough to compute $\#\left(U^+(R)\backslash \mathrm{SL}_2(R)\right)^\rho$. Observe that σ fixes the ideal (N). Hence, the action of σ on $\mathcal{O}_K$ descends to an action on R. The action of σ and τ on $\mathrm{SL}_2(R)$ are defined as follows

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \sigma_a & \sigma_b \\ \sigma_c & \sigma_d \end{pmatrix}, \quad \tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \sigma_a & -\sigma_b \\ -\sigma_c & \sigma_d \end{pmatrix},$$

where σ_a is the action of σ on a.

Let us first handle the special case when $N = p^n$ for a prime p. Consider the bijections of sets of matrices

$$U^+(R)\backslash \mathrm{SL}_2(R) \simeq U^+(R)\backslash B(R) \times B(R)\backslash \mathrm{SL}_2(R)$$

where $B(R)$ is the subgroup of upper-triangular matrices in $\mathrm{SL}_2(R)$. There are well-known bijections

$$B(R)\backslash \mathrm{SL}_2(R) \leftrightarrow \mathbb{P}^1(R), \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a:c)$$

where $\mathbb{P}^1(R)$ denotes the projective line over R, and

$$U(R)\backslash B(R) \leftrightarrow R^*, \qquad \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mapsto a.$$

These bijections lead to the identification

$$U^+(R)\backslash \mathrm{SL}_2(R) \simeq R^*/\{\pm 1\} \times \mathbb{P}^1(R).$$

It is straightforward to transfer the action of σ and τ to the right-hand side. We immediately see that

$$\left(U^+(R)\backslash \mathrm{SL}_2(R)\right)^\rho \simeq (R^*/\{\pm 1\})^\rho \times \mathbb{P}^1(R)^\rho.$$

We first compute $\#\mathbb{P}^1(R)^\rho$. It can be seen that $\mathbb{P}^1(R^\sigma) \hookrightarrow \mathbb{P}^1(R)$ and in fact $\mathbb{P}^1(R^\sigma) = \mathbb{P}^1(R)^\sigma$. Note that $\mathbb{P}^1(R^\sigma) \simeq \mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z})$ and thus has the cardinality $p^n + p^{n-1}$. Computation shows that $\mathbb{P}^1(R)^\tau$ has the same number of elements.

Now, let us compute $\#(R^*/\{\pm 1\})^\rho$. The actions of σ and τ are same on R^*. Clearly, we have $\#(R^*/\{\pm 1\})^\rho = (1/2) \cdot \#(R^*)^\rho$. We now consider the following cases.

- Let p be a prime that splits in K. We then have

$$(R)^\rho = (\mathcal{O}_K/p^n\mathcal{O}_K)^\rho = \left(\mathbb{Z}\sqrt{-d}/p^n\mathbb{Z}\sqrt{-d}\right)^\rho = \left(\mathbb{Z}\sqrt{-d}\right)^\rho / \left(p^n\mathbb{Z}\sqrt{-d}\right)^\rho.$$

  1. When $\rho = \sigma$, the automorphism σ acts on $\mathbb{Z}\sqrt{-d}$ by $\sigma(a+b\sqrt{-d}) = a-b\sqrt{-d}$.

Let $(a+b\sqrt{-d}) \in \left(\mathbb{Z}\sqrt{-d}\right)^{\sigma}$. Then $(a+b\sqrt{-d}) = (a-b\sqrt{-d})$, which implies $b=0$ and $a \in \mathbb{Z}$.

2. On the other hand if $\rho = \tau$, the automorphism τ acts on $\mathbb{Z}\sqrt{-d}$ by $\tau(a+b\sqrt{-d}) = -a+b\sqrt{-d}$. Let $(a+b\sqrt{-d}) \in \left(\mathbb{Z}\sqrt{-d}\right)^{\tau}$. Then $(a+b\sqrt{-d}) = (-a+b\sqrt{-d})$, which implies $a=0$ and $b \in \mathbb{Z}$. We have $R \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$ and hence $\#\,(R^*)^{\rho} = \#\,(\mathbb{Z}/p^n\mathbb{Z})^* = p^n - p^{n-1}$.

- When p is inert in K, we can view R as the quadratic extension $(\mathbb{Z}/p^n\mathbb{Z})\,[\omega]$ of the ring $\mathbb{Z}/p^n\mathbb{Z}$. It follows that $R^* = \{a + b\cdot\omega \in (\mathbb{Z}/p^n\mathbb{Z})^*\,[\omega]\}$. We now calculate $(R)^{\rho}$ when p is split in K. We deduce that $(R^*)^{\rho}$ is given by

$$\{a+b\cdot\omega \in (\mathbb{Z}/p^n\mathbb{Z})\,[\omega] \mid p \nmid a, b=0 \text{ for } \rho=\sigma \text{ or } p \nmid b, a=0 \text{ for } \rho=\tau\}.$$

The cardinality of this set is $p^n - p^{n-1}$.

In both inert and split cases, we get the quantity

$$\#\left(U^+(R)\backslash \mathrm{SL}_2(R)\right)^{\rho} = \frac{1}{2}\cdot\left(p^{2n} - p^{2(n-1)}\right).$$

To complete the proof, let us assume that $N = p_1^{n_1}\dots p_r^{n_r}$ is a positive number whose prime divisors p_i are unramified in K. The result in this general case follows from the simple fact that

$$\mathrm{SL}_2(\mathcal{O}_K/(N)) \simeq \mathrm{SL}_2\left(\mathcal{O}_K/\left(p_1\right)^{n_1}\right) \times \cdots \times \mathrm{SL}_2\left(\mathcal{O}_K/\left(p_r\right)^{n_r}\right).$$

□

8.4 Lower bounds for the cohomology via Lefschetz number and trace

Recall the following from Sengün-Türkelli [10, Proposition 5.2].

Assuming ρ is an orientation-reversing involution, we aim to establish a lower bound for the dimension of the cuspidal cohomology in relation to the Lefschetz number of ρ. Assuming ρ extends its action from Y_Γ to X_Γ^{BS}, it induces involutions on the terms of the long exact sequence. Denoting ρ^i as the action of ρ on the cohomology group H^i, we have

$$\operatorname{tr}\left(\rho^i\right) = \operatorname{tr}\left(\rho^i_{cusp}\right) + \operatorname{tr}\left(\rho^i_{Eis}\right).$$

Poincaré duality implies $\mathrm{H}^1_{cusp} \simeq \mathrm{H}^2_{cusp}$. Since ρ is an orientation-reversing involution, it follows that $\operatorname{tr}\left(\rho^1_{cusp}\right) = -\operatorname{tr}\left(\rho^2_{cusp}\right)$. Hence, we obtain

$$L(\rho, \Gamma, M_k) = \operatorname{tr}\left(\rho^0\right) - 2\operatorname{tr}\left(\rho^1_{cusp}\right) - \operatorname{tr}\left(\rho^1_{Eis}\right) + \operatorname{tr}\left(\rho^2_{\mathrm{Eis}}\right) \tag{8.1}$$

This leads to the following proposition:

Proposition 8.4.1 (J.Rohlfs). *We have*

$$\dim \mathrm{H}^1_{cusp}(\Gamma; M) \geq \frac{1}{2}\left|L(\rho, \Gamma, M) + \operatorname{tr}\left(\rho^1_{Eis}\right) - \operatorname{tr}\left(\rho^2_{Eis}\right) - \operatorname{tr}\left(\rho^0\right)\right|.$$

Proof. Since ρ is an involution, the eigenvalues of ρ^1_{cusp} are ± 1, and so

$$\dim \mathrm{H}^1(\Gamma; M) \geq \left|\operatorname{tr}\left(\rho^1_{cusp}\right)\right|.$$

The result follows from Equation 8.1. □

Note that when $M = M_k$ with $k > 0$, $\operatorname{tr}\left(\rho^0\right) = 0$ since M is an irreducible Γ-representation.

Proposition 8.4.2. *Let p be a rational prime that is unramified in K and let $\Gamma_1(p^n)$ denote the subgroup of level (p^n) of a Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$. Then*

1. $$\dim \mathrm{H}^1_{cusp}\left(\Gamma_1(p^n); M_k\right) \gg k$$

   as k increases and n is fixed.

2. *Assume further that the class number of K is one. We have the following asymptotic*

bound:

$$\dim \mathrm{H}^1_{cusp}\left(\Gamma_1\left(p^n\right);\mathbb{C}\right) \gg p^{3n}$$

as n increases.

Proof. From Proposition 8.4.1, we get

$$\dim \mathrm{H}^1_{cusp}\left(\Gamma; M_k\right) \geq \frac{1}{2}\left(L(\sigma, \Gamma, k) + \mathrm{tr}\left(\sigma^1_{Eis}, \Gamma, k\right) - \mathrm{tr}\left(\sigma^2_{Eis}, \Gamma, k\right)\right).$$

Fix the congruence subgroup Γ. According to Proposition 6.1.3, the dimension of the Eisenstein part of the cohomology remains consistent for every weight $k > 0$. Thus, the weight-dependent aspect of the asymptotic behavior can be inferred from Corollary 7.3.6 in the preceding section.

The assertion in (2) is a direct consequence of Theorem 8.3.2 and Theorem 8.3.1, in conjunction with the Lefschetz number formula provided in Corollary 7.3.6. □

9

Future research plans on the Bianchi modular forms

In this section, we would like to explore some potential future prospects to pursue.

Problem 9.0.1. *Can we find the exact value of the integration by integrating over all the faces of the imaginary quadratic field using the inner product formula Proposition 5.3.2?*

The approach will be to find all the faces of the fundamental domain §3.3 with exact limits.

Problem 9.0.2. *Can we generalize the definition of quasi-periods in Section 5.3.1 for different imaginary quadratic fields?*

The approach in this direction would be to first find the fundamental domain of that field and carefully choose the limits of integration.

Problem 9.0.3. *After solving problem 9.0.2, can we also generalize the inner product formula Proposition 5.3.2 for different imaginary quadratic fields?*

The approach in this direction would be to find the fundamental domain of that field and identify all the faces of the fundamental domain.

Problem 9.0.4. *We know from Section 5.4.2 that* $\mathcal{E}_E \in \mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \mathbb{C})$. *Can we show that* $\mathcal{E}_E \in \mathrm{H}_1(X_\Gamma^{BB}, \partial X_\Gamma^{BB}; \overline{\mathbb{Q}})$*?*

Problem 9.0.5. *Can we find the exact values of a and b given in 2^a and 2^b as defined in §7.1 and §7.2?*

Problem 9.0.6. *Can we generalize Proposition 6.1.4, Theorem 8.2.1, and Theorem 8.3.1 for imaginary quadratic fields with class number greater than one?*

Problem 9.0.7. *In this thesis, we are considering $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$, as a subgroup of finite index with no elements of finite order. Can we generalize our work for $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$ to be a subgroup of the finite index with elements of finite order (i.e., having some torsion) or a subgroup of the infinite index?*

Problem 9.0.8. *Can we determine the upper bound on the dimension of cuspidal cohomology mentioned in Proposition 8.4.2 and also calculate the dimension of cuspidal cohomology?*

One can also attempt to solve the conjecture of cuspidal cohomology in Bianchi modular forms as proposed in the work of Haluk Sengün [9].

Problem 9.0.9. *Can we determine the Eisenstein part of the cohomology groups for subgroups of the form $\Gamma_0(N)$ and generalize the Sczech cocycle technique for subgroups of the form $\Gamma_0(N)$?*

Problem 9.0.10. *Can we generalize Chapters 7 and 8 for congruence subgroups Γ of the symplectic group $\mathrm{Sp}_{2n}(\mathcal{O}_K)$ (i.e., for Siegel modular forms)?*

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