Principal Bundles in Generalized Complex Geometry

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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द्वारा / By

देवजीत पाल / Debjit Pal

पंजीकरण सं. / Registration No.:
20193689

शोध प्रबंध पर्यवेक्षक / Thesis Supervisor:
Prof. Mainak Poddar

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान पुणे

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Dedicated to

Maa and Baba
(my parents)

Certificate

Certified that the work incorporated in the thesis entitled "*Principal Bundles in Generalized Complex Geometry*", submitted by *Debjit Pal* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.

Mainak Poddar

Date: July 16, 2024

Prof. Mainak Poddar
Thesis Supervisor

Declaration

Name: Debjit Pal

Reg. No.: 20193689

Thesis Supervisor: Prof. Mainak Poddar

Department: Mathematics

Date of joining program: August 01, 2019

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I declare that this written submission represents my idea in my own words and where other's ideas have been included; I have adequately cited and referenced the original sources. I declare that I have acknowledged collaborative work and discussions wherever such work has been included. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

The work reported in this thesis is the original work done by me under the guidance of *Prof. Mainak Poddar*.

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Debjit Pal

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Contents

Abstract

In this thesis, we introduce the notion of a strong generalized holomorphic (SGH) fiber bundle and develop connection and curvature theory for an SGH principal G-bundle over a regular generalized complex (GC) manifold, where G is a complex Lie group. We also develop a de Rham cohomology for regular GC manifolds, and a Dolbeault cohomology for SGH vector bundles. Moreover, we establish a Chern-Weil theory for SGH principal G-bundles under certain mild assumptions on the leaf space of the GC structure. We also present a Hodge theory along with associated dualities and vanishing theorems for SGH vector bundles. Several examples of SGH fiber bundles are given. Additionally, we describe a family of regular generalized complex structures (GCS) on a principal torus bundle over a complex manifold with even dimensional fiber and characteristic class of type $(1, 1)$. The leaves of the associated symplectic foliation are exactly the fibers of the bundle. The speciality of such principal torus bundles is that they are not always SGH principal bundles. We show that such a GCS is equivalent to the product of the complex structure on the base and the symplectic structure on the fiber in a tubular neighborhood of an arbitrary fiber if and only if the bundle is flat, impacting the generalized Dolbeault cohomology of the bundle with a Künneth formula. Moreover, if a principal bundle over a complex manifold with a symplectic structure group admits a GCS with the fibers of the bundle as leaves of the associated symplectic foliation, and the GCS is equivalent to a product GCS in a neighborhood of every fiber, then we show that the bundle is flat and symplectic. This is similar to the behavior of SGH principal bundles over a complex manifold with symplectic fibers. This thesis is based on the following two articles:

1. Debjit Pal, Mainak Poddar; *Generalized complex structure on certain principal torus bundles [126].*

2. Debjit Pal, Mainak Poddar; *Strong generalized holomorphic principal bundles [127].*

Chapter 1

Introduction

Generalized complex (GC) geometry presents a unified framework for a range of geometric structures whose two extreme cases are complex and symplectic structures. The notion was introduced by Hitchin [81] and developed to a large extent by his doctoral students Gualtieri [70, 72] and Cavalcanti [36]. Intuitively, GC manifolds can be conceptualized as objects within a "category" that bridges the gap between symplectic and holomorphic categories.

A generalized complex structure (GCS) induces a (possibly singular) symplectic foliation. This phenomenon can be observed by the local characterization of a GC manifold; see [1,10,11,13]. Assuming a well-defined leaf space, one can study the transverse geometry (cf. [117,118,145,146]) in the context of regular GC manifolds. Furthermore, by using blowups in GC geometry, one can obtain numerous non-trivial examples of GC manifolds; see [14, 39, 41]. There are also well-defined stable GC structures that are equivalent to complex log symplectic structures (cf. [42–44]). Within the framework of GC manifolds, there exists a well-developed elliptic deformation theory and a Kuranishi-type moduli space; see [72, Section 5].

In addition to foliation theory and deformation theory, GC geometry has played a crucial role in extending classical theories within geometric analysis. In the works of Gualtieri [71, 73], the concept of generalized Kähler structures, and their associated generalized connection was introduced, thus extending the classical Kähler geometry. Subsequently, researchers delved into the realm of geometric analysis within the framework of GC geometry, particularly focusing on the Ricci flow direction. Analogous to classical concepts such as Ricci curvature, Kähler-Ricci flow, and Levi-Civita connection, notions like *generalized Ricci curvature, generalized Kähler-Ricci flow, and generalized connection* have been established in this framework; see [5, 58, 59, 141–143, 151]. Additionally, investigations into pseudo-Kähler geometry in the context of GC geometry have also been conducted; see [52]. For further study of generalized Kähler geometry in various fields, we refer to [33, 53, 60, 61, 74, 83, 92, 155–158].

Another aspect of GC geometry involves the study of pseudoholomorphic curves within its framework. In [128], Paleani introduced the concept of *generalized pseudo-holomorphic curves* for exact Courant algebroids, serving as an extension to the classical theory of pseudoholomorphic curves, revolutionized by Gromov (cf. [69]). Paleani's work primarily focuses on characterizing the local behavior and deformation theory of these curves. This may have significant implications for the extension of Floer homology and the Fukaya category (cf. [56, 57, 114, 122, 123]) to GC geometry.

Moreover, GC geometry finds applications in mathematical physics. Specifically, generalized Kähler geometry provides an alternative approach to studying certain bi-Hermitian geometries in supersymmetric sigma models, initially discovered by physicists (cf. [32]). Additionally, GC geometry aids in understanding the behavior of D-branes within complex and symplectic manifold settings; see [70, Chapter 7], [100, 101]. Generalized complex geometry provides new insights into phenomena like T-duality (cf. [40,66]) and Mirror Symmetry (cf. [70, Chapter 8]). It also plays an important role in the understanding of physical string theory, including supersymmetric flux compactifications that relate 10-dimensional physics to 4-dimensional worlds (see [35,63–65,104,162]) and sigma model (cf. [163, 164]).

Beyond the aforementioned aspects of GC geometry, another important area of application involves exploring GCS within bundles, focusing on vector bundles. These specialized vector bundles, termed generalized holomorphic (GH) vector bundles, were introduced by Gualtieri (cf. [70, Definition 4.27]). This generalization of holomorphic bundles in complex geometry to GC geometry has received much attention [62, 70, 72, 82, 84, 103, 154].

Principal bundles are essential in topology, geometry, and mathematical gauge theory, providing a framework for studying differential equations involving connections such as the Yang-Mills equations and quantum principal bundles; see [79, 137, 138]. Generally, the study of principal bundles or vector bundles involves the following four fundamental differential geometric aspects:

1. The exploration of connection and curvature;
2. Chern-Weil theory and characteristic classes;
3. Hodge theory and its associated dualities and vanishing theorems;
4. Deformation theory.

Notably, in the case of holomorphic principal bundles or vector bundles, these four aspects become even more interesting. Holomorphic structures introduce additional principles while also opening up new possibilities to explore, revealing rich geometric properties specific to the complex analytic setting; see [7, 34, 86, 95, 98, 99, 150]. Hence, given that GC geometry extends holomorphic geometry, it is only natural to pose the following question:

Question 1.0.1.

1. *What kind of vector or principal bundle theory arises within GC geometry?*
2. *How are these four classical geometric components represented within the framework of GC geometry?*

In [70, 72], by definition, generalized holomorphic (GH) vector bundles are complex vector bundles defined over a GC manifold equipped with a Lie algebroid connection. Wang further extended this in [152, 154], introducing GH principal bundles by extending the structure group action to an exact Courant algebroid. These provide an answer to (1) in Question 1.0.1. Additionally, in [153], Wang explored the deformation of GH vector bundles, covering one of the four geometric aspects. However, the absence of a generalized complex structure (GCS) on the total space of the bundle in these notions is a hindrance to the investigation of the remaining three components. Recently, in [103], Lang et al. tried to address this by considering a new concept of GH vector bundles equipped with a GCS on the total space. Their GCS on the total space is locally a product the GCS on the base and the fiber. They also introduced the Atiyah sequence of such bundles and defined its splitting as a generalized holomorphic (GH) connection. This new notion of GH vector bundle is more rigid than the notion due to Gualtieri [70, 72] and Hitchin [82], and yields a strict subclass. But, it has the potential of being more amenable to methods from complex geometry.

In this thesis, motivated by Question 1.0.1, we generalize the work of [103] on vector bundles to fiber and principal bundles. To distinguish these bundles from the earlier notions due to Gualtieri and Wang [154], we refer to them as *strong generalized holomorphic* (or *SGH*) bundles. A regular GCS induces a regular foliation with symplectic leaves and a transverse complex structure. The SGH bundles are intuitively characterized by the fact they are flat along the leaves and transversely holomorphic. However, they form a bigger category than the category of holomorphic bundles on the leaf space (when the leaf space is a manifold or an orbifold), see Examples 3.4.5, and 3.4.6.

Both the base and fiber of an SGH fiber bundle are GC manifolds, and the total space admits a GCS that is locally a product GCS derived from the base and the fiber, see Definition 3.1.1. In the context of vector bundles, SGH vector bundles correspond precisely to the GH vector bundles of Lang et al. (cf. [103]). Similarly, in the realm of principal bundles, they are a subclass of the GH principal bundles analyzed by Wang (cf. [154, Example 4.2]).

We also examine scenarios distinct from SGH principal bundles, where the generalized complex structure (GCS) of the total space of a principal bundle is locally equivalent (via B-transformations and diffeomorphisms) to a product GCS, though not necessarily identical to a product GCS as in SGH principal bundles. We provide a characterization for principal G-bundles over complex manifolds whose total space admits a GCS locally equivalent (via B-transformations and diffeomorphisms) to a product GCS, with G being a symplectic manifold. Note that this result applies to SGH principal G-bundles with G as a symplectic manifold. Furthermore, for torus principal bundles over a complex manifold M with a characteristic class of type $(1, 1)$, we establish that the flatness property is equivalent to the condition of being locally equivalent (via B-transformations and diffeomorphisms) to a product GCS.

The main contribution of this thesis is twofold. Firstly, adapting the methods of complex geometry, we introduce a suitable Dolbeault cohomology theory for SGH vector bundles and in using it to develop suitable generalizations of Chern-Weil theory and Hodge theory for these bundles. Typically, this requires assuming that the leaf space of the symplectic foliation is a complex (Kähler) orbifold. Secondly, we describe the local characteristics of the GCS on the total space for certain principal bundles that do not fall under the SGH category. Through this description, we determine their flatness property in terms of the GCS. Following earlier work of Angella (cf. [4]), we derive a spectral sequence for the generalized Dolbeault cohomology of the total space, together with a related Künneth formula. A more detailed outline of the thesis is given in the following section.

1.1 Structure of the thesis

The thesis is organised as follows.

1.1.1 GC Structures and related notions

In Chapter 2, we review the fundamental concepts regarding generalized complex structures (GCS) and generalized holomorphic (GH) maps. We commence by exploring linear GCS in Section 2.1, elucidating their isotropic and spinorial descriptions.

Subsequently, in Section 2.2, we discuss GC linear maps, which serve as the linear counterparts of GH maps. Here, we provide a comprehensive characterization, particularly when the GCS of the codomain space is induced from a complex structure.

To conclude, in Section 2.3, we revisit the notions of GCS and GH maps for smooth manifolds, and further explore their implications.

Additionally, we describe the associated cohomology theories, namely, the generalized Dolbeault cohomology and the corresponding Lie algebroid cohomology in Section 2.4.

1.1.2 Strong GH fiber bundles and Foliations on GC manifolds

Chapter 3 introduces the concept of SGH fiber bundles offering several illustrative examples. Additionally, we establish an analogue of the holomorphic Picard group and exponential sequence in Section 3.3.

We provide a comprehensive description of GH tangent and GH cotangent bundles in Section 3.2, which serve as the foundational elements of this thesis.

Furthermore, we discuss the leaf space associated to the regular symplectic foliation $\mathscr{S}$ with a transverse complex structure of a regular GCS. In general, the leaf space $M/\mathscr{S}$ might lack the Hausdorff property, as illustrated in Example 3.4.2. Nonetheless, assuming $M/\mathscr{S}$ is a smooth orbifold, we provide a structured description of $\mathscr{S}$ in Theorem 3.4.1.

In Subsection 3.4.1, we give some criteria on the GCS so that the leaf space of the associated symplectic foliation is a smooth torus, and therefore, satisfies the hypothesis that the leaf space be an orbifold, used in most of our results. This is a generalization of a result of Bailey et al. [12, Theorem1.9].

Subsection 3.4.2 presents a complete characterization of the leaf space of a left invariant GCS on a simply connected nilpotent Lie group and its associated nilmanifolds. Finally, examples of nontrivial SGH bundles on the Iwasawa manifolds are constructed, illustrating that the category of SGH bundles is in general different from the category of holomorphic bundles on the leaf space; see Examples 3.4.5-3.4.6.

1.1.3 Strong GH principal bundles and GH connections

Chapter 4 is the core of this thesis. We start by describing SGH principal bundles. Then, we follow Atiyah's approach to defining a holomorphic connection of a holomorphic principal bundle [7], to construct the Atiyah sequence of an SGH principal G-bundle P over a regular GC manifold M, where G is a complex Lie group:

$$0 \longrightarrow Ad(P) \longrightarrow At(P) \longrightarrow \mathcal{G}M \longrightarrow 0\,.$$

Here, $\mathcal{G}M$ is the GH tangent bundle of M, $Ad(P)$ is the adjoint bundle of P, and $At(P)$ is the Atiyah bundle of P. A GH connection on P is a GH splitting of the above the short exact sequence (cf. (4.2.8)), and the Atiyah class is the obstruction to such a splitting; see Definition 4.2.1 and Theorem 4.2.2.

Furthermore, in Section 4.3, a la Atiyah, we establish that the Atiyah class of an SGH vector bundle and the Atiyah class of its associated SGH principal bundle agree up to a sign in Theorem 4.3.1.

Utilizing Theorem 3.4.1, we develop the de Rham cohomology $H^{\bullet}_{D}(M)$ for regular GC manifolds in Proposition 4.4.3, and the Dolbeault cohomology $H^{\bullet,\star}_{d_L}(M,E)$ of an SGH vector bundle E in Corollary 4.4.2.

This leads to a notion of curvature of a smooth generalized connection (see Definition 4.2.3) on an SGH principal bundle in Subsection 4.4.2, and also, provides a crucial relationship between the curvature and the Atiyah class in Theorem 4.4.2.

In Section 4.5, we establish the generalized Chern-Weil homomorphism for SGH principal bundles in Definition 4.5.1 using the generalized connection of Theorem 4.4.2, and define the generalized characteristic classes.

1.1.4 Connections on SGH vector bundles and Hodge theory

In Chapter 5, we develop the theory of smooth generalized connection and its curvature for an SGH vector bundle. We also introduce a notion of transverse connection and its curvature in Definition 5.1.5 and present a related Chern-Weil theory for SGH vector bundles similar to Section 4.5.

Applying Theorem 4.3.1, we demonstrate in Theorem 5.1.3 that the existence of a GH connection on an SGH bundle is the same as the existence of a GH connection on its associated SGH principal bundle.

Section 5.2 develops generalized versions of classical results such as Serre duality and Poincaré duality. Moreover, we introduce a Hodge decomposition for the D-cohomology and d_L-cohomology (see Subsection 4.4.1) of a regular GC manifold in Theorem 5.2.2.

Extending Theorem 5.2.2 for $H^{\bullet,\star}_{d_L}(M,E)$ where E is an SGH vector bundle, we establish a generalized Hodge decomposition in Theorem 5.2.3 and provide a generalized Serre duality in Theorem 5.2.4, under the assumption of Theorem 5.2.2. Additionally, we establish analogues of Kodaira and Serre vanishing theorems in Theorem 5.2.5.

1.1.5 GCS on torus principal bundles

In Chapter 6, we study a family of GC structures on an even-dimensional torus principal bundle over a complex manifold with the characteristic class of type $(1,1)$.

In Section 6.1, we present a detailed argument regarding the existence of a family of regular GC structures on the total space of the torus principal bundle in Proposition 6.1.1. Then, in Theorem 6.1.1, we show the existence of such structures on the total space of more general principal bundles. Using Theorem 6.1.1, we provide an example demonstrating this concept (see Example 6.1.1).

In Section 6.2, we establish the following result (Theorem 6.2.1): In a trivializing neighborhood of a torus fiber, any GCS belonging to the family in Proposition 6.1.1 is equivalent to the product of the symplectic structure on the fiber and the complex structure on the base up to diffeomorphisms and B-transforms if and only if the principal bundle is flat.

More generally, in Theorem 6.2.2, we show that if a principal G-bundle, where G is a Lie group with a symplectic structure, admits a GCS which is locally equivalent to a product GCS in a neighborhood of every fiber, then the bundle is flat and symplectic. Using Theorem 6.2.2, we deduce a stronger version of Theorem 6.2.1, namely, Theorem 6.2.3 which says that a principal torus bundle over a complex manifold is symplectic and flat if and only if it admits a GCS which is equivalent to a product GCS in a neighborhood of each torus fiber.

In Section 6.3, an application of Theorem 6.2.1 arises when employing the spectral sequence developed by Angella et al. [4] to describe the generalized Dolbeault cohomology of the total space of the bundle. This application is discussed within a broader framework in Theorems 6.3.1-6.3.2 that encompasses symplectic fiber bundles, with certain assumptions regarding the GCS, slightly broadening the requirements outlined in [4]. The case of principal torus bundles is stated in Corollary 6.3.1, and a Künneth formula for the generalized Dolbeault cohomology of these bundles is given in Corollary 6.3.2.

This thesis relies on the content presented in the following pair of manuscripts:

1. Debjit Pal, Mainak Poddar,
   Generalized complex structure on certain principal torus bundles [126],
   available at https://arxiv.org/abs/2303.07835.

2. Debjit Pal, Mainak Poddar,
   Strong generalized holomorphic principal bundles [127],
   available at https://arxiv.org/abs/2404.18113.

Notations:

- Let M denote a smooth manifold and E denote a smooth fiber bundle over M. For any open set $U \subseteq M$,
  - $C^\infty(U)$ or $C^\infty(U, \mathbb{C})$ denotes the ring of $\mathbb{C}$-valued smooth functions on U and C^∞_M denotes the sheaf of $\mathbb{C}$-valued smooth functions over M.
  - $C^\infty(U, \mathbb{R})$ denotes the ring of $\mathbb{C}$-valued smooth functions on U and $C^\infty_{M,\mathbb{R}}$ denotes the sheaf of $\mathbb{R}$-valued smooth functions over M.
  - $C^\infty(E)$ denotes the corresponding sheaf of smooth sections of E, and we denote the set of smooth sections of E over U by $C^\infty(U, E)$ or by $C^\infty(E)(U)$. In particular, if E is a complex vector bundle, then $C^\infty(E)$ is the sheaf of $\mathbb{C}$-valued sections. Similarly, if E is only a real vector bundle, then $C^\infty(E)$ is the sheaf of $\mathbb{R}$-valued sections.
- Let M is a (regular) GC manifold and E denotes an SGH fiber bundle over M.
  - $\mathcal{O}_M$ denotes the sheaf of $\mathbb{C}$-valued GH functions on M
  - $\mathcal{G}M$ and $\mathcal{G}^*M$ denote the GH tangent bundle and the GH cotangent bundle, respectively.
  - $\mathbf{E}$ denotes the corresponding sheaf of GH sections of E, and we denote the set of GH sections of E over U by $\Gamma(U, \mathbf{E})$ or by $\mathbf{E}(U)$.
  - $A^\bullet$ denotes the sheaf of transverse generalized forms of degree $\bullet$. Similarly, $A^{\bullet,\bullet}$ denotes the sheaf of transverse generalized forms of bi-degree $(\bullet, \bullet)$.
  - F_M denotes the sheaf of smooth $\mathbb{C}$-valued functions over M which are constant along the leaves of the symplectic foliation associated with the GCS.
  - For any vector bundle E over M whose transition maps are leaf-wise constant, $F_M(E)$ denotes the sheaf of smooth leaf-wise constant sections of E.
  - $H^\bullet_D(M)$ denotes the D-cohomology, and $H^{\bullet;\star}_{d_L}(M, E)$ denotes the d_L-cohomology with coefficients in E.

Chapter 2

An Introduction to Generalized Complex Structures and related notions

Generalized Complex Structures (GCS) represent a broader framework encompassing both complex and symplectic structures. Essentially, both complex and symplectic structures, aside from their integrability condition, are significantly influenced by the behaviour of the pointwise cotangent bundle. In simpler terms, the linear version of these structures provides substantial insight and understanding. So, we first revisit the framework of generalized geometry in the linear case before progressively delving into GCS on smooth manifolds. For a comprehensive exploration of linear GCS and GCS on smooth manifolds, we refer to [70, 72]. Additionally, we explore the concept of generalized complex (GC) maps, which align with GCS and play a significant role in defining generalized holomorphic (GH) maps in subsequent chapters. More details regarding GC maps and GH maps can be found in [125, 149].

When considering a GCS on a smooth manifold, it gives rise to two significant perspectives: one in cohomology (cf. [37]) and the other in foliation theory. Cohomological aspects play a crucial role in examining the principal bundle viewpoint over a GC manifold, while foliation theory aids in grasping the transverse structure of a GCS. In favourable scenarios, such as when the leaf space of the induced foliation forms a smooth manifold, a wealth of information can be derived. For a detailed study of foliation and its transverse structures, we refer to [6, 117]. In this chapter, we will focus exclusively on delving into the cohomology theory in detail. The discussion of foliation theory will be reserved for Chapter 3. This chapter is split into four sections:

1. Linear generalized complex structures (Section 2.1).

2. Generalized complex linear maps (Section 2.2).

3. GC Structures and generalized holomorphic map on manifolds (Section 2.3).

4. Related cohomologies for GC manifolds (Section 2.4).

2.1 Linear generalized complex structures

Understanding generalized complex structures (GCS) on vector spaces is notably simpler compared to their manifold counterparts. Their classification is more direct, as the type of a GCS remains constant across the entire space, facilitating a straightforward categorization. In other words, linear GCS presents a simpler scenario, serving as a starting point in this chapter. In this section, we recall some fundamental notions of linear generalized complex (in short, GC) geometry. We will primarily refer to [70, 72] and the references therein, for most of the definitions and results.

Consider a finite dimensional real vector space V. The direct sum of V and its dual space, denoted as $V \oplus V^*$, is endowed with a natural symmetric bilinear form of signature (n, n)

$$\langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y)+\eta(X)) \quad \forall\ X+\xi, Y+\eta \in V \oplus V^*\,. \tag{2.1.1}$$

Definition 2.1.1. *([70, Chapter 4]) A generalized complex structure (GCS), denoted by $\mathfrak{J}_V$, on V is a linear automorphism of $V \oplus V^*$ satisfying the following two conditions*

1. *(complex condition)* $\mathfrak{J}_V^2 = -1$;

2. *(symplectic condition)* $\mathfrak{J}_V^* = -\mathfrak{J}_V$.

Here $(V \oplus V^*)^*$ *is identified with* $V \oplus V^*$ *via the bilinear form, as defined in* (2.1.1). *The pair* $(V, \mathfrak{J}_V)$ *is called GC vector space, and* $\mathfrak{J}_V$ *is called a linear GCS on* V.

Example 2.1.1. *Let V be any $2n$-dimensional vector space and consider the following two linear GCS on V*

$$\mathcal{J}_{V,I} := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \quad \text{and} \quad \mathcal{J}_{V,\omega} := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

where I and ω are the usual complex and symplectic structures, respectively.

Remark 2.1.1. *From Example 2.1.1, we can see that any even-dimensional vector space is a GC vector space. The converse is due to Gualtieri (cf. [72, Proposition 3.3]) that V is a GC vector space if and only if V is an even-dimensional real vector space. Put differently, a prerequisite for discussing linear GCS on V is that V must be of even dimension.*

Given any GC subspace $(V, \mathcal{J}_V)$, we can naturally complexify it, and consider it as an automorphism of $(V \oplus V^*) \otimes \mathbb{C}$. This leads to the following two complex subspaces of $(V \oplus V^*) \otimes \mathbb{C}$

$$\begin{aligned} L_V &:= \{x \in (V \oplus V^*) \otimes \mathbb{C} \,|\, \mathcal{J}x = ix\}; \\ \overline{L}_V &:= \{x \in (V \oplus V^*) \otimes \mathbb{C} \,|\, \mathcal{J}x = -ix\}, \end{aligned} \tag{2.1.2}$$

where $\overline{L}_V$ denotes the complex conjugation of L_V. L_V and $\overline{L}_V$ are the $+i$ and $-i$ eigenspaces, respectively. Since $\mathcal{J}_V$ is a real operator on $(V \oplus V^*) \otimes \mathbb{C}$ (that is, $\overline{\mathcal{J}_V x} = \mathcal{J}_V \overline{x}$), both L_V and $\overline{L}_V$ have identical characteristics. Below are some essential properties of L_V.

1. Because $\mathcal{J}_V$ is orthogonal with respect to the bilinear form, defined as in (2.1.1), $\langle x, y \rangle = 0$ for all $x, y \in L_V$. Here we are identifying $\langle\, ,\, \rangle$ with its complexification.

2. $\dim_{\mathbb{C}} L_V = \dim_{\mathbb{R}} V$ and $L_V \cap \overline{L}_V = \{0\}$.

Conversely, given any complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ with the preceding properties, we can define a linear GCS on $(V \oplus V^*) \otimes \mathbb{C}$ whose $+i$-eigenspace is L, as defined in (2.1.2). Therefore, the following proposition is available due to Guatieri (cf. [70, Proposition 4.3]).

Proposition 2.1.1. *Any linear GCS on V is equivalent to the specification of a complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ with the following properties*

1. *$\langle x, y \rangle = 0$ for all $x, y \in L$ where $\langle\, ,\, \rangle$, as defined in (2.1.1), is identified with its complexification.*

2. $\dim_{\mathbb{C}} L = \dim_{\mathbb{R}} V$ *and* $L \cap \overline{L} = \{0\}$.

The subspace L in Proposition 2.1.1 is known as *maximal isotropic subspace of real index zero* (cf. [70, Section 2.2]). In order to gain a deeper understanding of linear generalized complex structure on V, it is necessary to thoroughly examine maximal isotropic subspaces of $(V \oplus V^*) \otimes \mathbb{C}$. In the following section, we will delve into this topic extensively. Before proceeding further, it's important to address another aspect of linear GCS known as *B-field transformation* (cf. [70, Example 2.1]). It can also be seen as a method for obtaining a new linear GCS from an older one.

Let $\mathcal{J}_V$ be a linear GCS on V. Let $B \in \wedge^2 V^*$ and view it as a linear map $V \longrightarrow V^*$ via interior product $v \mapsto i_v B = B(v, \cdot)$. Then, We can deform $\mathcal{J}_V$ by B and get another linear GCS,

$$(\mathcal{J}_V)_B := e^{-B} \mathcal{J}_V e^B \quad \text{where} \quad e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \tag{2.1.3}$$

Definition 2.1.2. $(\mathcal{J}_V)_B$ *is called a B-field transformation or B-transformation of* $\mathcal{J}_V$.

The $+i$-eigenspace of $(\mathcal{J}_V)_B$ is just

$$(L_V)_B = \{X + \xi - B(X, \cdot) \,|\, X + \xi \in L_V\}. \tag{2.1.4}$$

2.1.1 Isotropic viewpoint of a linear GCS

In this subsection, we provide a formal introduction to the concept of a maximal isotropic subspace and review its properties. We also explore a characterization of these subspaces. The isotropic viewpoint of a linear GCS will prove highly beneficial in our future endeavours to characterize foliation aspects induced by a GCS on a manifold. Furthermore, it will provide a thorough understanding of GC maps on a GC manifold at a later stage. Throughout our discussion, we confine our discussion to real vector subspaces only. However, we will mainly utilize its complexification version, as we will soon observe.

Definition 2.1.3. *([70, Section 2.2]) Let* V *be a real vector space, and let* $L < V \oplus V^*$ *be a vector subspace.*

1. *L is called an isotropic subspace if* $\langle x, y \rangle = 0$ *for all* $x, y \in L$ *where* $\langle \, , \, \rangle$ *is defined as in* (2.1.1).

2. L is called a maximal isotropic subspace if it is isotropic and $\dim_{\mathbb{R}} L = \dim_{\mathbb{R}} V$.

Example 2.1.2. *V and V^* are two natural maximal isotropic subspaces.*

Example 2.1.3. *Let E be any subspace of V, and let $\sigma \in \wedge^2 E^*$ be considered as an map from E to E^*. Consider the subspace of $V \oplus V^*$*

$$L(E, \sigma) := \{X + \xi \in E \oplus V^* \,|\, \xi|_E = \sigma(X)\} . \tag{2.1.5}$$

By [70, Example 2.5], $L(E, \sigma)$ is a maximal isotropic subspace.

The following theorem, due to Gualtieri (cf. [70, Proposition 2.6]), provides a complete classification of maximal isotropic subspaces of $V \oplus V^*$.

Theorem 2.1.1. *Every maximal isotropic subspace of $V \oplus V^*$ can be expressed in the form $L(E, \sigma)$, where $L(E, \sigma)$ is defined as in* (2.1.5)*.*

Note that, the integer $k = \mathrm{Codim}_{\mathbb{R}}(E) = \dim_{\mathbb{R}} \mathrm{Ann}(E)$ is an invariant associated to a maximal isotropic subspace $L = L(E, \sigma)$ where E is the projection of L onto V.

Definition 2.1.4. *The type of a maximal isotropic subspace L of $V \oplus V^*$ is the codimension of the projection of L onto V and it is denoted by* $\mathrm{Type}(L)$*.*

Remark 2.1.2. *Note that* $\mathrm{Type}(L) \in \{0, \ldots, \dim_{\mathbb{R}} V\}$ *because* $\dim_{\mathbb{R}} E \in \{0, \ldots, \dim_{\mathbb{R}} V\}$.

Let $B \in \wedge^2 V^*$. Consider the B-transformation $e^B : V \oplus V^* \longrightarrow V \oplus V^*$, as defined in (2.1.3). One can see that $e^B(X+\xi) = X+\xi+B(X, \cdot)$ which implies that B-transformation does not effect the V-component. In particular, given any maximal isotropic subspace $L = L(E, \sigma)$ with the inclusion map $i : E \hookrightarrow V$, the maximal isotropic subspace

$$e^B(L) = L(E, \sigma + i^*B) = \{X + \xi + B(X, \cdot) \,|\, X + \xi \in L\} , \tag{2.1.6}$$

shows that $\mathrm{Type}(L)$ is invariant under any B-transformation. Now, the inclusion map $i : E \hookrightarrow V$ induces an onto map $i^* : V^* \longrightarrow E^*$, and so, there exists $B' \in \wedge^2 V^*$ such that $i^*B' = \sigma$. This implies

$$L = e^{B'}(L(E, 0)) = e^{B'}(E \oplus \mathrm{Ann}(E)) .$$

To summarize the preceding discussion, we arrive at the following proposition.

Proposition 2.1.2. *([70, Section 2.2]) Any maximal isotropic subspace $L = L(E, \sigma)$ is a B-transformation of $L(E, 0)$ for B chosen such that $i^*B = \sigma$, that is*

$$L = e^B(L(E, 0)) .$$

Remark 2.1.3. *Note that, after a B-transformation of a linear GCS $\mathfrak{J}_V$ on V, the corresponding $+i$-eigensubspace $(L_V)_B$, defined in* (2.1.4)*, is simply $(L_V)_B = e^{-B}(L_V)$ by* (2.1.6) *where L_V denotes the $+i$-eigensubspace of the linear GCS.*

This subsection primarily discusses the maximal isotropic subspace of $V \oplus V^*$ using linear algebra, mainly focusing on its projection onto V . However, a more sophisticated algebraic approach involves describing it through Clifford algebra of $V \oplus V^*$, specifically utilizing pure spinors. This refined approach offers a deeper understanding of the maximal isotropic subspace. The subsequent section delves into this with greater elaboration.

2.1.2 Spinorial viewpoint of a linear GCS

Consider a finite dimensional real vector space, denoted as V. The maximal isotropic subspaces within $V \oplus V^*$ can alternatively be characterized by their associated pure spinor lines. Viewing them from a spinorial perspective enables a more nuanced examination of the linear generalized complex structures. This approach lays the groundwork for describing GCS on manifolds through differential forms, facilitating a differential geometric understanding and capturing cohomological aspects in a more refined manner. Below, we revisit this connection. For a detailed study of Clifford algebras and spinors, we refer to [49].

Consider the action of $V \oplus V^*$ on $\wedge^\bullet V^*$ defined by

$$(X + \xi) \cdot \varphi = i_X\varphi + \xi \wedge \varphi .$$

This action can be extended to the Clifford algebra of $V \oplus V^*$ corresponding to the natural pairing (2.1.1). This gives a natural choice for spinors, namely, the elements of the exterior algebra on V^*, $\wedge^\bullet V^*$.

Definition 2.1.5. *([70, Section 2.5]) Let $\phi \in \wedge^\bullet V^*$ be a nonzero spinor.*

1. *Null space of ϕ, denoted as $L_\phi < V \oplus V^*$, is defined as $L_\phi := \{x \in V \oplus V^* \,|\, x \cdot \phi = 0\}$.*

2. *The spinor ϕ is called pure if L_ϕ is a maximal isotropic subspace.*

3. *The* 1*-dimensional subspace in $\wedge^\bullet V^*$, denoted as U_{L_ϕ}, generated by the spinor ϕ, is called spinor line. U_{L_ϕ} is called a pure spinor line if ϕ is a pure spinor.*

Remark 2.1.4. *The null space L_ϕ is by definition a isotropic subspace of $V \oplus V^*$. So, ϕ is a pure spinor, if* $\dim_{\mathbb{R}} L_\phi = \dim_{\mathbb{R}} V$.

Notice that, if ϕ is a pure spinor then $e^B\phi := e^B \wedge \phi$ is also a pure spinor for any $B \in \wedge^2 V^*$. In particular, $L_{e^B\phi} = e^{-B}(L)$. Therefore, in order to characterize the pure spinor associated with a maximal isotropic subspace $L(E, \sigma)$, it suffices to describe the pure spinor associated with $L(E, 0)$ according to Proposition 2.1.2. The following proposition attributed to Gualtieri (cf. [70, Lemma 2.23]) effectively resolves our issue.

Proposition 2.1.3. *Given any subspace $E \leq V$ of* $\mathrm{Codim}_{\mathbb{R}}(E) = k$*, the maximal isotropic subspace $L(E, 0)$ is associated to the pure spinor line* $\det(\mathrm{Ann}(E)) < \wedge^k V^*$.

This leads to a comprehensive characterization of the relationship between a maximal isotropic subspace and its corresponding pure spinor line. Put differently, it provides a clear depiction of the pure spinor line associated with any given maximal isotropic subspace, as outlined below.

Theorem 2.1.2. *([72, Proposition 1.3]) Let $L = L(E, \sigma) \leq V \oplus V^*$ be a maximal isotropic subspace with* $\mathrm{Codim}_{\mathbb{R}}(E) = k$*. Let $B \in \wedge^2 V^*$ be a* 2*-form such that $i^*B = -\sigma$ where $i : E \hookrightarrow V$ is the inclusion map. Let $\{\theta_1, \dots, \theta_k\} \in V^*$ be a basis for* $\mathrm{Ann}(E)$*. Then, the pure spinor line U_L associated with L, is generated by*

$$\phi = e^B \theta_1 \wedge \cdots \wedge \theta_k \, .$$

Corollary 2.1.1. *There is a bijection between the set of maximal isotropic subspaces of $V \oplus V^*$ and the set of pure spinor lines in $\wedge^\bullet V^*$.*

Define a linear map α on $\wedge^\bullet V^*$ which acts on decomposable forms by

$$\alpha(a_1 \wedge \ldots \wedge a_i) = a_i \wedge \ldots \wedge a_1 \, .$$

Definition 2.1.6. *Given two forms of mixed degree $\sigma_i = \sum \sigma_i^j$, $i = 1, 2$, where* $\deg(\sigma_i^j) =$ *j, in an n-dimensional vector space, we define their pairing, $(\sigma_1 \, , \, \sigma_2)$ by*

$$(\sigma_1 \, , \, \sigma_2) = (\alpha(\sigma_1) \wedge \sigma_2)_{Top}, \tag{2.1.7}$$

where Top indicates the degree n component of the wedge product.

The forthcoming result will play a pivotal role in establishing the linear GCS in the subsequent subsection. In simpler terms, this outcome delineates the intersection of maximal isotropic subspaces in relation to the pure spinor utilizing the pairing defined in (2.1.7).

Proposition 2.1.4. *([70, Proposition 2.21]) Given any two maximal isotropic subspace $L_1, L_2 \leq V \oplus V^*$, $\dim_{\mathbb{R}}(L_1 \cap L_2) = \{0\}$ if and only if $(\phi_1, \phi_2) \neq 0$ where ϕ_i is the pure spinor associated with L_i for $i = 1, 2$, and $(\, , \,)$ is the pairing as defined in* (2.1.7).

2.1.3 Complexification and classification

The inherent bilinear form $\langle \, , \, \rangle$, as defined in (2.1.1), extends naturally from $V \oplus V^*$ to $(V \oplus V^*) \otimes \mathbb{C}$. By complexifying $V \oplus V^*$, all of the aforementioned results and definitions regarding maximal isotropic subspaces and spinors for $V \oplus V^*$ can be naturally broadened to $(V \oplus V^*) \otimes \mathbb{C}$. We consolidate the preceding results within this new framework.

Theorem 2.1.3. *([70, Proposition 2.25]) Let V denote a real vector space of finite dimension. A maximal isotropic subspace L of $(V \oplus V^*) \otimes \mathbb{C}$, of $\mathrm{Type}(L) = k \in \{0, \ldots, \dim_{\mathbb{R}} V\}$, can be characterized equivalently by the following information:*

1. *A complex subspace E of $\dim_{\mathbb{C}} E = \dim_{\mathbb{R}} V - k$ within the vector space $V \otimes \mathbb{C}$, along with a complex form $\sigma \in \wedge^2 E^*$.*

2. *A complex pure spinor line U_L is generated by*

$$\phi = c' e^{B + i\omega} \theta_1 \wedge \cdots \wedge \theta_k \, ,$$

where $B + i\omega \in \wedge^2 V^* \otimes \mathbb{C}$, $c' \in \mathbb{C} \backslash \{0\}$, and $\{\theta_1, \ldots, \theta_k\} \in V^* \otimes \mathbb{C}$ are linearly independent complex 1-forms.

3. *A complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ with maximal isotopic property with respect to $\langle \, , \, \rangle$ such that $E = \rho(L)$ with $\dim_{\mathbb{C}} E = \dim_{\mathbb{R}} V - k$ where $\rho : (V \oplus V^*) \otimes \mathbb{C} \longrightarrow V \otimes \mathbb{C}$ is the natural projection map onto $V \otimes \mathbb{C}$.*

The Proposition 2.1.1 demonstrates that the maximal isotropic property alone within a subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ does not invariably guarantee the presence of a linear generalized complex structure (GCS) on V. This is due to the fact that if L is a maximal isotropic subspace, then its complex conjugate $\overline{L}$ also forms a maximal isotropic subspace, and the intersection $L\cap\overline{L}$ may not always be a zero subspace. Specifically, $L\cap\overline{L}=\tilde{L}\otimes\mathbb{C}$ for some real subspace $\tilde{L}<V\oplus V^*$. To illustrate, one can take any maximal isotropic subspace within $V\oplus V^*$ and complexify it, yielding another maximal isotropic subspace within $(V\oplus V^*)\otimes\mathbb{C}$, denoted as L'. Consequently, $L'\cap\overline{L'}$ represents the complexification of the initial maximal isotropic subspace. Hence, the complex dimension of $L\cap\overline{L}$ assumes significance in the analysis of linear GCS. Thus, we introduce the following definition.

Definition 2.1.7. *([70, Definition 2.26]) The real index of a maximal isotopic subspace $L<(V\oplus V^*)\otimes\mathbb{C}$, denoted by $r(L)$, is defined as the complex dimension of $L\cap\overline{L}$, that is,*

$$r(L):=\dim_{\mathbb{C}} L\cap\overline{L}=\dim_{\mathbb{R}}\tilde{L}\,,$$

where $L\cap\overline{L}=\tilde{L}\otimes\mathbb{C}$ for some real subspace $\tilde{L}<V\oplus V^*$.

According to Proposition 2.1.1, if we have a maximal isotropic subspace $L<(V\oplus V^*)\otimes\mathbb{C}$ with real index zero, it will induce a linear generalized complex structure (GCS) on V such that L becomes the $+i$-eigenspace of that linear GCS. In simpler terms, if we consider a GC vector space V, we can decompose $(V\oplus V^*)\otimes\mathbb{C}$ as $(V\oplus V^*)\otimes\mathbb{C}=L\oplus\overline{L}$ where L is the maximal isotropic subspace of real index zero, and projecting this decomposition onto $V\otimes\mathbb{C}$, we obtain $V\otimes\mathbb{C}=E+\overline{E}$ where $E=\rho(L)$. Thus, the following relationship holds.

$$\begin{aligned}
&2\dim_{\mathbb{C}}E-\dim_{\mathbb{C}}(E\cap\overline{E})=\dim_{\mathbb{C}}V\otimes\mathbb{C}\\
\Longrightarrow\ &2\dim_{\mathbb{C}}V\otimes\mathbb{C}-2\dim_{\mathbb{C}}E=\dim_{\mathbb{C}}V\otimes\mathbb{C}-\dim_{\mathbb{C}}(E\cap\overline{E})\\
\Longrightarrow\ &2\,\mathrm{Codim}_{\mathbb{C}}(E)=\mathrm{Codim}_{\mathbb{C}}(E\cap\overline{E})\\
\Longrightarrow\ &\mathrm{Type}(L)=\frac{1}{2}\mathrm{Codim}_{\mathbb{C}}(E\cap\overline{E})\,.
\end{aligned}$$

Since $\mathrm{Codim}_{\mathbb{C}}(E\cap\overline{E})\in\{0,\dots,2n\}$, $\mathrm{Type}(L)\in\{0,\dots,n\}$ where $\dim_{\mathbb{R}}V=2n$ (cf. Remark 2.1.1).

Definition 2.1.8. *Let $(V,\mathfrak{J}_V)$ be a GC vector space. Then, the type of the linear GCS $\mathfrak{J}_V$, denoted by $\mathrm{Type}(\mathfrak{J}_V)$, is defined to be the type of the corresponding $+i$-eigen space of $\mathfrak{J}_V$, that is,*

$$\mathrm{Type}(\mathfrak{J}_V):=\mathrm{Type}(L_V)\in\left\{0,\dots,\frac{\dim_{\mathbb{R}}V}{2}\right\}\,,$$

where L_V *is the corresponding* $+i$*-eigen space (maximal isotropic subspace of real index zero) of* $\mathfrak{J}_V$ *as defined in Proposition 2.1.1.*

Similar to Theorem 2.1.3, a maximal isotropic subspace of real index zero can be described by its projection onto $V \otimes \mathbb{C}$ and also its spinorial viewpoint using Theorem 2.1.2 and Proposition 2.1.4, respectively. This characterization not only aids in a deeper understanding of GC maps but also illuminates the vector bundle viewpoint and cohomological perspective of a GCS on smooth manifolds. This characterization, credited to Gualtieri (cf. [70, Proposition 4.4, Theorem 4.8]), is formalized in the subsequent theorem.

Theorem 2.1.4. *Let* V *real vector space with* $\dim_{\mathbb{R}} V = 2n$ *(cf. Remark 2.1.1). Let* $L = L(E, \sigma)$ *be a maximal isotropic subspace of* $(V \oplus V^*) \otimes \mathbb{C}$ *with* $\mathrm{Type}(L) = k$ *. Let* U_L *be the corresponding complex pure spinor line, generated by*

$$\phi = e^{B+i\omega}\Omega \,,$$

where $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ *and* $B + i\omega$*,* $\{\theta_1, \ldots, \theta_k\}$ *are as in Theorem 2.1.3. Then the real index of* L *vanishes, that is,* $r(L) = 0$ *if and only if one of the following conditions is satisfied.*

1. $E + \overline{E} = V \otimes \mathbb{C}$ *and the real* 2*-form* $\Omega_\Delta := \mathrm{Im}(\sigma|_{\Delta \otimes \mathbb{C}})$ *is non-degenerate on* $\Delta \otimes \mathbb{C}$ *where* $E \cap \overline{E} = \Delta \otimes \mathbb{C}$ *with* $\Delta \leq V$ *as a real subspace.*

2. $(\phi, \overline{\phi}) = \omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0$ *where* $\overline{\phi}$ *is the generator for the pure spinor line* $U_{\overline{L}}$ *. In other words,*

    - $\{\theta_1, \ldots, \theta_k \,, \overline{\theta_1}, \ldots, \overline{\theta_k}\}$ *are linearly independent, and*
    - $\omega|_\Delta$ *is non-degenerate on* Δ *where* $\Delta \leq V$ *is a real* $(2n - 2k)$*-dimensional subspace defined as* $\Delta \otimes \mathbb{C} = \ker(\Omega \wedge \overline{\Omega})$ *.*

Remark 2.1.5. *In Theorem 2.1.4, we used the same notation for both* $\ker(\Omega \wedge \overline{\Omega})$ *and* $E \cap \overline{E}$*. This is because, according to Theorem 2.1.2, the set* $\{\theta_1, \ldots, \theta_k\} \in V^* \otimes \mathbb{C}$ *in Theorem 2.1.4 forms a basis for* $\mathrm{Ann}(E)$*, implying that* $E \cap \overline{E}$ *is a subspace of* $\ker(\Omega \wedge \overline{\Omega})$*. Therefore, the equality of dimensions, that is,*

$$\dim_{\mathbb{C}} \ker(\Omega \wedge \overline{\Omega}) = \dim_{\mathbb{C}} E \cap \overline{E} = 2n - 2k \,,$$

indicates that $\ker(\Omega \wedge \overline{\Omega})$ *and* $E \cap \overline{E}$ *are identical, that is,*

$$\ker(\Omega \wedge \overline{\Omega}) = E \cap \overline{E} \,.$$

Example 2.1.4. *([70, pp 45-46])*

1. *Let W be a symplectic dimensional vector with a symplectic structure ω. Then $\mathcal{J}_{W,\omega}$, as defined in Example 2.1.1, is a linear GCS of type 0 with the corresponding pure spinor line generated by $e^{i\omega}$.*

2. *Let V be a complex n-dimensional vector with a complex structure I. Then $\mathcal{J}_{V,I}$, as defined in Example 2.1.1, is a linear GCS of type n with the corresponding pure spinor line $\wedge^{n,0}V^*$ generated by any nonzero $(n,0)$-form $\Omega^{n,0} \in \wedge^{n,0}V^*$.*

3. *The product linear GCS, denoted by $\mathcal{J}_{W\times V} := \mathcal{J}_{W,\omega} \oplus \mathcal{J}_{V,I}$, on $(W, \mathcal{J}_{W,\omega}) \oplus (V, \mathcal{J}_{V,I})$ is of type n and can be described by the pure spinor line in $\wedge^\bullet(W \oplus V)^* \otimes \mathbb{C}$ generated by $e^{i\omega} \wedge \Omega^{n,0}$.*

Example 2.1.4 The third example in Example 2.1.4 demonstrates that when provided with any two complex and symplectic vector spaces, it's possible to obtain a linear GCS of a type that does not fall into the extremal cases. On the other hand, for an even-dimensional real vector space, it's always possible to achieve a linear GCS of both extremal types. This leads to the question of describing a linear GCS of a type between these extremal cases. It has been established that such a linear GCS closely resembles the product linear GCS as outlined in Example 2.1.4. In other words, any linear GCS of a fixed type can be completely characterized. The following theorem presents this characterization.

Theorem 2.1.5. *([70, Theorem 4.13]) A linear GCS of type k over a real vector space of dimension $2n$ can be represented as a B-transformation applied to the direct sum of a symplectic structure, having a real dimension of $2n - 2k$, and a complex structure with a complex dimension of k. In other words, it can be viewed as the B-transformation of the product linear GCS, $\mathcal{J}_{W\times V}$, as illustrated in Example 2.1.4 where $(V^k, \mathcal{J}_{V,I})$ represents a suitable complex vector space of complex dimension k, and $(W, \mathcal{J}_{W,\omega})$ denotes a suitable symplectic vector space of real dimension $2n - 2k$.*

2.2 Generalized complex linear maps

When considering any geometric structure imposed on a smooth manifold, the central inquiry pertains to the types of mappings that preserve the structure. In the context of complex manifolds, the natural choice for such mappings is holomorphic maps. However, in the case of symplectic manifolds, defining analogous mappings is not immediately evident since the pullback of a symplectic form may not always be a non-degenerate form. Nevertheless, by regarding a symplectic manifold as a Poisson manifold with the naturally induced Poisson bivector, we can establish a framework for defining such mappings. For a detailed study of the geometry of complex and Poisson manifolds, we refer to [50, 86, 148]. Because GC manifolds encompass both complex and symplectic manifolds, the fundamental requirement for maps on such manifolds must adhere to the criteria set forth for mappings in complex and Poisson manifolds.

In the context of GC manifolds, such mappings are termed *generalized holomorphic (GH) maps*. It will become apparent in subsequent sections that GH maps rely on the point-wise GC vector space structure of the tangent bundle of GC manifolds. Thus, before delving into the manifold case, it is imperative to examine such mappings within the framework of GC vector spaces. This section serves as an introduction to the linear version of GH maps, also known as *generalized complex (GC) maps*. We will primarily rely on [125, 149] and the references contained therein for most of the definitions and results. We might also recommend referring to the article [127, Section 2] for a comprehensive overview of the topics discussed in this section, conveniently gathered in one place.

Let $(V, \mathcal{J}_V)$ be a GC linear space with $+i$-eigenspace L_V. Consider the projection map

$$\rho : (V \oplus V^*) \otimes \mathbb{C} \longrightarrow V \otimes \mathbb{C}\,.$$

Let $\rho(L_V) = E_V$ and let $E_V \cap \overline{E_V} = \Delta_V \otimes \mathbb{C}$ where $\Delta_V \leq V$ is a real subspace. Then by Theorem 2.1.3 and Theorem 2.1.4, we have

1. $L_V = L(E_V, \sigma)$ for some $\sigma \in \wedge^2 E_V^*$;

2. $E_V + \overline{E_V} = V \otimes \mathbb{C}$ with a non-degenerate form Ω_{Δ_V} on $\Delta_V \otimes \mathbb{C}$, defined as

$$\Omega_{\Delta_V} := \mathrm{Im}(\sigma|_{\Delta_V \otimes \mathbb{C}})\,.$$

Since Ω_{Δ_V} is non-degenerate, following [125, Section 3], $\widetilde{P}_V := L(\Delta_V \otimes \mathbb{C}, \Omega_{\Delta_V})$ is called the associated *linear Poisson structure* of $\mathfrak{J}_V$ on $V \otimes \mathbb{C}$.

Definition 2.2.1. *([125]) Let $\psi : (V, \mathfrak{J}_V) \longrightarrow (W, \mathfrak{J}_W)$ be a linear map between two GC linear spaces. Then ψ is called a generalized complex (GC) map if*

1. $\psi(E_V) \subseteq E_W$ *;*

2. $\psi_\star(\widetilde{P}_V) = \widetilde{P}_W$ *where $\psi_\star$ denotes the pushforward of a Dirac structure, as in [125, Section 1], namely,*

$$\psi_\star(\widetilde{P}_V) = \{\psi(Y) + \eta \in (W \oplus W^*) \otimes \mathbb{C} \,|\, Y + \psi^*(\eta) \in \widetilde{P}_V\}\,.$$

Remark 2.2.1. *Given a B-field transformation of $\mathfrak{J}_V$, we see that*

$$\rho((L_V)_B) = \rho(L_V)\,,$$

where $(L_V)_B$ is as in (2.1.4). Since the imaginary part of σ is also preserved, the associated linear Poisson structures are the same for both GCS. This shows that the notion of GC map is insensitive to B-field transformations.

Let $(V, \mathfrak{J}_V)$ be a generalized complex (GC) linear space. Then $\mathfrak{J}_V$ can be written as

$$\mathfrak{J}_V = \begin{pmatrix} -J_V & \beta_V \\ B_V & J_V^* \end{pmatrix}$$

where $J_V \in \mathrm{End}(V)$, $B_V \in \mathrm{Hom}_{\mathbb{R}}(V, V^*)$ and $\beta_V \in \mathrm{Hom}_{\mathbb{R}}(V^*, V)$. Using $\mathfrak{J}_V^* = -\mathfrak{J}_V$ (cf. Definition 2.1.1), we get $B_V \in \wedge^2 V^*$ and $\beta_V \in \wedge^2 V$.

Lemma 2.2.1. *Let $\psi : V \longrightarrow W$ be a GC map between two GC linear spaces. Then $\psi(E_V \cap \overline{E_V}) = E_W \cap \overline{E_W}$.*

Proof. Let $w \in \Delta_W \otimes \mathbb{C} = E_W \cap \overline{E_W}$. Then $w + \Omega_{\Delta_W}(w) \in \widetilde{P}_W$. Since $\psi_\star(\widetilde{P}_V) = \widetilde{P}_W$, there exist $v \in \Delta_V \otimes \mathbb{C} = E_V \cap \overline{E_V}$ and $\eta \in W^* \otimes \mathbb{C}$ such that

$$w + \Omega_{\Delta_W}(w) = \psi(v) + \eta\,.$$

This shows that $\psi(v) = w$ and $E_W \cap \overline{E_W} \subseteq \psi(E_V \cap \overline{E_V})$.

For the converse part, let $v \in E_V \cap \overline{E_V}$ be a non-zero element. Let $\psi(v) = w \in W \otimes \mathbb{C}$, and let $\widetilde{\Omega}_{\Delta_W} : W \otimes \mathbb{C} \longrightarrow \Delta_W^* \otimes \mathbb{C}$ be an extension of $\Omega_{\Delta_W} : \Delta_W \otimes \mathbb{C} \longrightarrow \Delta_W^* \otimes \mathbb{C}$ such that $\widetilde{\Omega}_{\Delta_W} \in \wedge^2 W^* \otimes \mathbb{C}$. Then, we have $\psi^*(\widetilde{\Omega}_{\Delta_W}(w)) \in V^* \otimes \mathbb{C}$ and

$$\Omega_{\Delta_V}^{-1}(\psi^*(\widetilde{\Omega}_{\Delta_W}(w))|_{\Delta_V \otimes \mathbb{C}}) + \psi^*(\widetilde{\Omega}_{\Delta_W}(w)) \in \widetilde{P}_V \,. \tag{2.2.1}$$

Denote $\Omega_{\Delta_V}^{-1}(\psi^*(\widetilde{\Omega}_{\Delta_W}(w))|_{\Delta_V \otimes \mathbb{C}}) \in \Delta_V \otimes \mathbb{C}$ by v'. Then,

$$\begin{aligned}
& v' = \Omega_{\Delta_V}^{-1}(\psi^*(\widetilde{\Omega}_{\Delta_W}(w))|_{\Delta_V \otimes \mathbb{C}}) \\
\Longrightarrow\ & \Omega_{\Delta_V}(v') = \psi^*(\widetilde{\Omega}_{\Delta_W}(w))|_{\Delta_V \otimes \mathbb{C}} \\
\Longrightarrow\ & \Omega_{\Delta_V}(v')(v) = \psi^*(\widetilde{\Omega}_{\Delta_W}(w))(v) \\
\Longrightarrow\ & \Omega_{\Delta_V}(v')(v) = \widetilde{\Omega}_{\Delta_W}(w)(\psi(v)) \\
\Longrightarrow\ & \Omega_{\Delta_V}(v', v) = 0 \quad (\text{as } \ \psi(v) = w) \\
\Longrightarrow\ & v = kv' \quad (\text{as } \ \Omega_{\Delta_V} \ \text{ is non-degenerate on } E_V \cap \overline{E_V} \text{ and } \ k \in \mathbb{C}\backslash\{0\}) \\
\Longrightarrow\ & v = k\,\Omega_{\Delta_V}^{-1}(\psi^*(\widetilde{\Omega}_{\Delta_W}(w))|_{\Delta_V \otimes \mathbb{C}}) \,.
\end{aligned}$$

Note that $\psi(\Omega_{\Delta_V}^{-1}(\psi^*(\widetilde{\Omega}_{\Delta_W}(w))|_{\Delta_V \otimes \mathbb{C}}) + (\widetilde{\Omega}_{\Delta_W}(w)) \in \widetilde{P}_W$ by (2.2.1), and also

$$\psi(\Omega_{\Delta_V}^{-1}(\psi^*(\widetilde{\Omega}_{\Delta_W}(w))|_{\Delta_V \otimes \mathbb{C}}) = \frac{1}{k}\psi(v) = \frac{1}{k}w \,.$$

Thus $w \in E_W \cap \overline{E_W}$ and $\psi(E_V \cap \overline{E_V}) \subseteq E_W \cap \overline{E_W}$. This proves the lemma. □

Remark 2.2.2. *The assertion made in the statement of Lemma 2.2.1 is claimed in the proof of [125, Proposition 3.2]. However, the argument given there is not very explicit.*

Remark 2.2.3. *The inclusion $\psi(E_V \cap \overline{E_V}) \subseteq E_W \cap \overline{E_W}$ can also be understood as follows: Let $v \in E_V \cap \overline{E_V}$ be a non-zero element. Note that, ψ is a real linear map, that is,*

$$\overline{\psi(v')} = \psi(\overline{v'}) \,,$$

for all $v' \in V \otimes \mathbb{C}$. Let $v = \overline{b}$ for some $b \in E_V$. Then, by Definition 2.2.1, $\psi(v)$, $\psi(b) \in E_W$ and so, $\overline{\psi(b)} \in \overline{E_W}$. Since ψ is a real map, $\psi(v) = \psi(\overline{b}) \in \overline{E_W}$. Therefore,

$$\psi(v) \in E_W \cap \overline{E_W} \,.$$

The proofs of the next two lemmas are modelled on similar arguments in [103].

Lemma 2.2.2. *Let* $(V, \mathfrak{J}_V)$ *and* $(W, \mathfrak{J}_W)$ *are GC linear spaces with*

$$\mathfrak{J}_V = \begin{pmatrix} -J_V & \beta_V \\ B_V & J_V^* \end{pmatrix} \quad and \quad \mathfrak{J}_W = \begin{pmatrix} -J_W & 0 \\ 0 & J_W^* \end{pmatrix}$$

where J_W *is a complex structure on* W*. Then* $\psi : V \longrightarrow W$ *is a GC map if and only if*

$$\psi \circ J_V = J_W \circ \psi \quad , \quad \psi \circ \beta_V = 0\,.$$

Proof. Let $\dim V = 2n$ and let the type of $\mathfrak{J}_V$ be $k \in \mathbb{N} \cup \{0\}$. Since the definition of GC map is invariant under a B-transformation, we can assume without loss of generality that

$$(V, \mathfrak{J}_V) = (V_1, J_1) \oplus (V_2, J_2)\,,$$

where $(V_1, J_1) = (\mathbb{R}^{2k}, J_{\mathbb{R}^{2k}}) = \mathbb{C}^k$ and $(V_2, J_2) = (\mathbb{R}^{2n-2k}, \omega_0)$. Here, $J_{\mathbb{R}^{2k}}$ and ω_0 denote the standard complex and symplectic structures on the corresponding spaces.

It follows that $E_V = V_1^{0,1} \oplus (V_2 \otimes \mathbb{C})$. Since $E_V \cap \overline{E_V} = V_2 \otimes \mathbb{C}$, the Poisson bivector on V is

$$\widetilde{\beta}_V = \begin{cases} 0, & \text{on } V_1^* \otimes \mathbb{C}\,, \\ \omega_0^{-1}, & \text{on } V_2^* \otimes \mathbb{C}\,. \end{cases}$$

Hence,

$$\widetilde{P}_V = L(V_2 \otimes \mathbb{C}, \omega_0) = L(V^* \otimes \mathbb{C}, \widetilde{\beta}_V)\,.$$

Similarly, as W is a complex vector space, we have $E_W = W^{0,1}$ and so, $E_W \cap \overline{E_W} = \{0\}$. Thus, β_W, the Poisson bivector on W is 0 and we get

$$\widetilde{P}_W = W^* \otimes \mathbb{C} = L(W^* \otimes \mathbb{C}, 0)\,.$$

Then, by Lemma 2.2.1, $\psi_\star(\widetilde{P}_V) = \widetilde{P}_W$ if and only if $\psi \circ \omega_0^{-1} = 0$. Thus, ψ is a GC map if and only if

$$\psi(V_1^{0,1} \oplus (V_2 \otimes \mathbb{C})) \subset W^{0,1} \text{ and } \psi \circ \omega_0^{-1} = 0\,.$$

Hence, for any $v_1 \in V_1$ and $v_2 \in V_2$, we have

$$\psi((-J_1(v_1))) = (-J_W)(\psi(v_1)) \text{ and } \psi(v_2) = 0\,.$$

This implies

$$\psi \circ J_V = J_W \circ \psi \quad \text{and} \quad \psi \circ \beta_V = 0$$

where

$$J_V = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \beta_V = -\widetilde{\beta}_V = \begin{pmatrix} 0 & 0 \\ 0 & -\omega_0^{-1} \end{pmatrix}.$$

□

Let ψ be any complex valued linear function on V. Considered as an element of $V^* \otimes \mathbb{C}$, ψ has two components corresponding to the decomposition $(V \oplus V^*) \otimes \mathbb{C} = L_V \oplus \overline{L_V}$, namely ψ_{L_V} and $\psi_{\overline{L_V}}$.

Lemma 2.2.3. *A linear map $\psi : (V, \mathfrak{J}_V) \longrightarrow \mathbb{C} = (\mathbb{R}^2, J_{\mathbb{R}^2})$ between two GC linear spaces, is a GC map if and only if $\psi \in (L_V \cap (V^* \otimes \mathbb{C}))$ that is, $\psi_{\overline{L_V}} = 0$.*

Proof. Let $\mathfrak{J}_V$ be written as

$$\mathfrak{J}_V = \begin{pmatrix} -J_V & \beta_V \\ B_V & J_V^* \end{pmatrix}.$$

Suppose $\psi : (V, \mathfrak{J}_V) \longrightarrow (\mathbb{R}^2, J_{\mathbb{R}^2})$ is linear a GC map. Let

$$\begin{aligned} \psi_{L_V} &= Y_1 + \eta_1\,, \\ \psi_{\overline{L_V}} &= Y_2 + \eta_2\,. \end{aligned} \tag{2.2.2}$$

where $Y_1, Y_2 \in V \otimes \mathbb{C}$ and $\eta_1, \eta_2 \in V^* \otimes \mathbb{C}$. Then, considering ψ as an element of $V^* \otimes \mathbb{C}$, we have $Y_1 + Y_2 = 0$ and $\eta_1 + \eta_2 = \psi$. Since $\psi_{L_V} \in L_V$ and $\psi_{\overline{L_V}} \in \overline{L_V}$, we have the following equations,

$$\begin{aligned} -J_V(Y_1) + \beta_V(\eta_1) = iY_1\,, \quad B_V(Y_1) + J_V^*(\eta_1) = i\eta_1\,, \\ -J_V(Y_2) + \beta_V(\eta_2) = -iY_2\,, \quad B_V(Y_2) + J_V^*(\eta_2) = -i\eta_2\,. \end{aligned} \tag{2.2.3}$$

Now, by Lemma 2.2.2, $\psi \circ \beta_V = 0$ which implies $\beta_V(\psi) = 0$. By adding the equations in the first column in (2.2.3), we get $\beta_V(\psi) = i(Y_1 - Y_2)$ which implies $Y_1 = Y_2$. Since $Y_1 + Y_2 = 0$, we derive $Y_1 = Y_2 = 0$. Then, the second column in (2.2.3) yields $J_V^*(\eta_1) = i\eta_1$ and $J_V^*(\eta_2) = -i\eta_2$. Adding these, we obtain

$$\begin{aligned} & J_V^*(\psi) = i(\eta_1 - \eta_2) \\ \Longrightarrow\ & \psi \circ J_V = i(\eta_1 - \eta_2) \\ \Longrightarrow\ & J_{\mathbb{R}^2} \circ \psi = i(\eta_1 - \eta_2) \quad \text{(by Lemma 2.2.2)} \\ \Longrightarrow\ & i(\eta_1 + \eta_2) = i(\eta_1 - \eta_2) \\ \Longrightarrow\ & \eta_2 = 0\,. \end{aligned}$$

It follows that $\psi_{\overline{L_V}} = 0$.

Conversely, Suppose $\psi_{\overline{L_V}} = 0$. Then $\psi \in L_V$ and so $\mathcal{J}_V(\psi) = i\psi$. This implies that

$$\beta_V(\psi) = 0\,, \quad J_V^*(\psi) = i\psi\,,$$
$$\Longrightarrow \psi \circ \beta = 0\,, \quad \psi \circ J_V = J_{\mathbb{R}^2} \circ \psi\,.$$

Thus, by Lemma 2.2.2, ψ is a GC map. □

2.3 Generalized complex structures and generalized holomorphic maps on manifolds

The notion of a Generalized Complex Structure (GCS) can be visualized as a linear GCS that varies point by point on the tangent bundle of a smooth manifold. Our current endeavour extends our previous work to encompass the tangent bundle, maintaining a pointwise perspective throughout. Since we are dealing with smooth manifolds, we aim to ensure that all operations vary smoothly from point to point. This necessitates an integrable condition similar to that found in complex or symplectic manifolds. This integrability is defined using the Courant bracket.

In this section, we formally define GCS on smooth manifolds along with the notion of GH map. Additionally, we provide some elementary examples to enhance understanding of GCS and GH maps. To provide context, we revisit some concepts introduced in Section 2.1 and Section 2.2, framing them in terms of the tangent bundle. We will primarily refer to [70,72,125] and the references therein, for most of the definitions and results regarding GCS and GH maps on smooth manifolds. We might also recommend referring to Section 2 of the articles [126, 127] for a comprehensive overview of the topics discussed in this section, conveniently gathered in one place.

2.3.1 Generalized complex structures

To define a GCS on an even dimensional smooth manifold M, we need three key ingredients. Firstly, given any $2n$-dimensional smooth manifold M, the direct sum of the tangent and cotangent bundles of M, which we denote by $TM \oplus T^*M$, is endowed with a natural symmetric bilinear form of signature $(2n, 2n)$

$$\langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y)+\eta(X))\,. \tag{2.3.1}$$

Secondly, we need the *Courant Bracket* on the smooth sections of $TM \oplus T^*M$ which is defined as follows.

Definition 2.3.1. *The Courant bracket is a skew-symmetric bracket defined on smooth sections of $TM \oplus T^*M$, given by*

$$[X+\xi, Y+\eta] := [X,Y]_{Lie} + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi), \tag{2.3.2}$$

where $X, Y \in C^\infty(TM)$, $\xi, \eta \in C^\infty(T^*M)$, $[\,,\,]_{Lie}$ is the usual Lie bracket on vector fields, and $\mathcal{L}_X$, i_X are the Lie derivative and the interior product of forms with respect to the vector field X, respectively.

For the third ingredient, consider the action of $TM \oplus T^*M$ on $\wedge^\bullet T^*M$ defined by

$$(X+\xi)\cdot\varphi = i_X\varphi + \xi\wedge\varphi\,,$$

where $X+\xi \in TM \oplus T^*M$ and $\varphi \in \wedge^\bullet T^*M$. This action can be extended to the Clifford algebra of $TM \oplus T^*M$ corresponding to the natural pairing (2.3.1). This gives a natural choice for spinors, namely, the exterior algebra of cotangent bundle, $\wedge^\bullet T^*M$. Define a linear map α on $\wedge^\bullet T^*M$ which acts on decomposable forms by

$$\alpha(a_1\wedge\ldots\wedge a_i) = a_i\wedge\ldots\wedge a_1\,.$$

Definition 2.3.2. *Given two forms of mixed degree $\sigma_i = \sum\sigma_i^k$, $i = 1, 2$, where $\deg(\sigma_i^k) = k$, in an n-dimensional vector space, we define their pairing, $(\sigma_1\,,\,\sigma_2)$ by*

$$(\sigma_1\,,\,\sigma_2) = (\alpha(\sigma_1)\wedge\sigma_2)_{Top}, \tag{2.3.3}$$

where Top indicates the degree n component of the wedge product.

Now, we are ready to present the notion of the generalized complex structure (GCS) on a $2n$-dimensional smooth manifold M in three equivalent ways.

Definition 2.3.3. *([70], [72]) A generalized complex structure or GCS on M is determined by any of the following three equivalent sets of data:*

1. *A subbundle* L_M *of* $(TM \oplus T^*M) \otimes \mathbb{C}$ *which is maximal isotropic with respect to the natural bilinear form* (2.3.1)*, and involutive with respect to the Courant bracket* (2.3.2)*, and also satisfies* $L_M \cap \bar{L}_M = \{0\}$.

2. *A bundle automorphism* $\mathfrak{J}_M$ *of* $TM \oplus T^*M$ *which satisfies the following conditions:*

   (a) $\mathfrak{J}_M^2 = -1$

   (b) $\mathfrak{J}_M^* = -\mathfrak{J}_M$*, i.e.,* $\mathfrak{J}_M$ *is orthogonal with respect to the natural pairing* (2.3.1)

   (c) $\mathfrak{J}_M$ *has vanishing Nijenhuis tensor, i.e.,*

   $$N(A, B) := \mathfrak{J}_M[\mathfrak{J}_M A, B] + \mathfrak{J}_M[A, \mathfrak{J}_M B] + [A, B] - [\mathfrak{J}_M A, \mathfrak{J}_M B] = 0$$

   for all $A, B \in C^\infty(TM \oplus T^*M)$.

3. *A line subbundle* U_M *of* $\wedge^\bullet T^*M \otimes \mathbb{C}$ *which generated locally at each point by a form of the form* $\phi = e^{(B+i\omega)} \wedge \Omega$*, such that the pairing* (2.3.3)

   $$(\phi\,,\,\bar{\phi}) = \omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0, \quad (\textit{non-degenerate condition})$$

   where B *and* ω *are real 2-forms and* Ω *is a decomposable complex* k*-form, and* ρ *satisfies*

   $$d\phi = u \cdot \phi, \quad (\textit{integrability condition}) \tag{2.3.4}$$

   for some $u \in C^\infty((TM \oplus T^*M) \otimes \mathbb{C})$*, where* d *is the exterior derivative.*

The pair $(M, \mathfrak{J}_M)$ *is called a generalized complex (in short, GC) manifold.*

At each point, the degree of Ω is same as the *type* of the GCS at that point, as we have seen in Theorem 2.1.4.

Definition 2.3.4. *Let* $(M, \mathfrak{J}_M)$ *be a GC manifold.*

1. *A point near which the type is locally constant is called a regular point. If every point of* M *is regular, we say that the GCS is regular.*

2. M *is called a regular GC manifold if the GCS,* $\mathfrak{J}_M$ *is regular.*

3. *The line bundle* U_M *that defines the GCS is called the canonical line bundle.*

Remark 2.3.1. *Note that equations* (2.1.1) *and* (2.1.7) *are identical to equations* (2.3.1) *and* (2.3.3)*, respectively, when considering GC linear spaces. Transitioning from GC linear spaces to GC manifolds, the Courant bracket provides the integrability condition for $\mathcal{J}_M$, which can be described in terms of the involutivity of L_M, the vanishing Nijenhuis tensor, or satisfying* (2.3.4)*, as defined in Definition 2.3.3.*

Remark 2.3.2. *In Definition 2.3.3, the equivalent conditions* 1 *and* 2 *are related to each other by the fact that the subbundle L_M can be obtained as the $+i$-eigenbundle of the automorphism $\mathcal{J}_M$. This precisely corresponds to Proposition 2.1.1 for linear GCS.*

Let $f : (M, \mathcal{J}_M) \longrightarrow (N, \mathcal{J}_N)$ be a smooth map between two GC manifolds. Then, following [70, Section 2.7], we can define the pullback and push forward of the $+i$-eigenbundles, denoted by $f^\star L_N$ and $f_\star L_M$, respectively, as follows.

$$\begin{aligned} f^\star L_N :=& \{X + f^*\eta \in (TM \oplus T^*M) \otimes \mathbb{C} \,|\, f_*X + \eta \in L_N\}\,; \\ f_\star L_M :=& \{f_*X + \eta \in (TN \oplus T^*N) \otimes \mathbb{C} \,|\, X + f^*\eta \in L_M\}\,. \end{aligned} \tag{2.3.5}$$

Let us consider some simple examples of GCS, similar to Example 2.1.1.

Example 2.3.1. *Let $(M,\, J_M)$ is a complex manifold with a complex structure J_M. Then the natural GCS, of type $\frac{\dim_{\mathbb{R}} M}{2}$, on M is given by the bundle automorphism*

$$\mathcal{J}_M := \begin{pmatrix} -J_M & 0 \\ 0 & J_M^* \end{pmatrix} : TM \oplus T^*M \longrightarrow TM \oplus T^*M\,.$$

Its corresponding $+i$-eigen bundle is

$$L_M = T^{0,1}M \oplus (T^{1,0}M)^*\,.$$

Example 2.3.2. *Let $(M,\, \omega)$ be a symplectic manifold with a symplectic structure ω. Then, the bundle automorphism*

$$\mathcal{J}_M := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} : TM \oplus T^*M \longrightarrow TM \oplus T^*M\,,$$

gives a natural GCS of type 0 *on M. The $+i$-eigen bundle of this GCS is*

$$L_M = \{X - i\omega(X) \,|\, X \in TM \otimes \mathbb{C}\}\,.$$

Example 2.3.3. *([70, Section 4.8]) Consider the following differential form.*

$$\rho = z_1 + dz_1 \wedge dz_2 \quad on \quad \mathbb{C}^2 = (\mathbb{R}^4, J_{\mathbb{R}^4})\,.$$

- *When* $z_1 = 0$*, we have*

$$\rho = dz_1 \wedge dz_2 \quad \text{and} \quad d\rho = 0\,.$$

- *When* $z_1 \neq 0$*, we have*

$$\rho = z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}} \quad \text{and} \quad d\rho = (-\frac{\partial}{\partial z_2}) \cdot \rho\,.$$

Therefore, from Definition 2.3.3, we can see that ρ *induces a GCS on* $\mathbb{C}^2$.

Example 2.3.4. *([70, Example 4.12]) Consider two GC manifolds denoted as* $(M_1, \mathcal{J}_{M_1})$ *and* $(M_2, \mathcal{J}_{M_2})$*. Then, the product GCS, denoted as* $\mathcal{J}_{M_1 \times M_2}$*, is characterized by the maximal isotropic subbundle* $\bigoplus_{j=1}^{2} \mathrm{Pr}_j^{\star}(L_{M_j})$*, with the corresponding canonical line bundle locally given as* $\bigwedge_{j=1}^{2} \mathrm{Pr}_j^{*}\, \phi_j$*. Here, for* $j = 1, 2$*,* L_{M_j} *and* ϕ_j *denote the respective* $+i$*-eigenbundle and local generator of the canonical line bundle for* $\mathcal{J}_{M_j}$*, while the map* $\mathrm{Pr}_j : M_1 \times M_2 \longrightarrow M_j$ *is the natural projection map onto the* j*-th component and* $\mathrm{Pr}_j^{\star}(L_{M_j})$ *is defined similarly as in* (2.3.5)*.*

Remark 2.3.3. *Note that, the GCS in Example 2.3.3 is of type* 0 *outside a codimension* 2 *hypersurface and type* 2 *along the hypersurface.*

Before delving deeper, it's essential to discuss B-transformations of a GCS on smooth manifolds. Let B be an element in $\Omega^2(M)$, interpreted as a map $B : TM \longrightarrow T^*M$ defined via the interior product such that $X \mapsto i_X B = B(X, \cdot)$. This allows us to construct a natural orthogonal, with respect to the natural bilinear form as in 2.3.1, bundle automorphism

$$e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : TM \oplus T^*M \longrightarrow TM \oplus T^*M\,.$$

When transitioning from linear GCS to GCS on smooth manifolds, one might initially assume that for any real smooth 2-form B, the bundle automorphism $(\mathcal{J}_M)_B$ of $TM \oplus T^*M$, defined analogously to (2.1.3), , yields a GCS on M. However, this assumption is not entirely accurate. Although $(L_M)_B$, defined similarly as in (2.1.4), is a maximal isotropic subbundle, there remains a requirement for involutivity of $(L_M)_B$ with respect to the Courant bracket, as defined in (2.3.2). This condition is equivalent to ensuring that e^B is an automorphism of the Courant bracket, expressed as

$$[e^B(C), e^B(D)] = e^B([C, D])\,,$$

for all $C, D \in (TM \oplus T^*M)$. The following proposition addresses this issue and provides a necessary and sufficient condition for e^B to be an automorphism of the Courant bracket.

Proposition 2.3.1. *([70, Proposition 3.23]) For a real smooth 2-form $B \in \Omega^2(M)$, e^B is an automorphism of the Courant bracket if and only if dB vanishes, that is, $dB = 0$ if and only if*

$$[e^B(C), e^B(D)] = e^B([C, D]) \quad \textit{for all} \quad C, D \in (TM \oplus T^*M) .$$

To sum up, when considering any GC manifold $(M, \mathfrak{J}_M)$, a B-field transformation (in short, B-transformation) of $\mathfrak{J}_M$ only involves deformation by a real closed 2-form B, resulting in another GCS on M, denoted as

$$(\mathfrak{J}_M)_B := e^{-B} \circ \mathfrak{J}_M \circ e^B \quad \text{where} \quad e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} , \tag{2.3.6}$$

similar to the definition in equation (2.1.3), with the $+i$-eigenbundle, denoted as

$$(L_M)_B := \{X + \xi - B(X, \cdot) \,|\, X + \xi \in L_M\} . \tag{2.3.7}$$

Definition 2.3.5. *Let $(M, \mathfrak{J}_M)$ be a GC manifold. Let $\mathrm{Type}(\mathfrak{J}_{M,x})$ denote the type of $\mathfrak{J}_{M,x}$ at $x \in M$, as defined in Definition 2.1.8.*

1. *If M is a regular GC manifold, the type of $\mathfrak{J}_M$ is denoted by $\mathrm{Type}(\mathfrak{J}_M)$, that is, $\mathrm{Type}(\mathfrak{J}_M) := \mathrm{Type}(\mathfrak{J}_{M,x})$ for all $x \in M$.*

2. *$\mathfrak{J}_M$ is called a GCS of symplectic type if $\mathfrak{J}_M$ is regular and $\mathrm{Type}(\mathfrak{J}_M) = 0$. Consequently, the manifold M is then referred to as a GC manifold of symplectic type.*

3. *If $\mathfrak{J}_M$ is regular and its type equals $\frac{\dim_{\mathbb{R}} M}{2}$, then M is called a GC manifold of complex type. In such cases, $\mathfrak{J}_M$ is called a GCS of complex type.*

The subsequent proposition provides a complete description of the GC manifold for both complex and symplectic types.

Proposition 2.3.2. *([70, Examples 4.10-4.11])*

1. *Any GCS of complex type is a B-transformation of a GCS, induced by a complex structure, as in Example 2.3.1.*

2. *Any GCS of symplectic type is a B-transformation of a GCS, induced by a symplectic form, as in Example 2.3.2.*

We conclude this subsection by revisiting the generalized Darboux theorem, which provides a local characterization of a GCS around any regular point in the GC manifold.

Theorem 2.3.1. *([72, Theorem 4.3]) For a regular point $x \in (M, \mathfrak{J}_M)$ of* $\text{Type}(\mathfrak{J}_{M,x}) = k$*, there exists an open neighborhood $U_x \subset M$ of x such that, after a B-transformation, U_x is GH homeomorphic to $U_1 \times U_2$, where $U_1 \subset (\mathbb{R}^{2n-2k}, \omega_0), U_2 \subset \mathbb{C}^k$ are open subsets with ω_0 being the standard symplectic structure.*

2.3.2 Generalized holomorphic maps

As outlined in Section 2.2, GC maps depend on $\rho(L_M)$ for a general GC manifold M with $+i$-eigenbundle L_M. The primary reason for this behaviour can be seen in Definition 2.2.1. However, in the case of smooth GC manifolds, the pointwise dimension of $\rho(L_M)$ is not constant; thus, the type is not constant across a general GC manifold, as illustrated in Example 2.3.3. Consequently, except around a regular point, the concept of GC maps cannot smoothly vary from point to point. Therefore, the most suitable and natural notion for GH maps is to rely on the pointwise GC vector space structure of the tangent bundle of GC manifolds.

Definition 2.3.6. *([125]) A smooth map $\psi : (M, \mathfrak{J}_M) \longrightarrow (M', \mathfrak{J}_{M'})$ between two GC manifolds is called a generalized holomorphic (GH) map if for each $x \in M$,*

$$(\psi_*)_x : T_xM \longrightarrow T_{\psi(x)}M'$$

is a GC map, as in Definition 2.2.1. Let $J_{\mathbb{R}^2}$ be the standard complex structure on $\mathbb{R}^2$ so that $(\mathbb{R}^2, J_{\mathbb{R}^2})$ is identified with $\mathbb{C}$. Consider $(M', \mathfrak{J}_{M'}) = (\mathbb{R}^2, \mathfrak{J}_{\mathbb{R}^2})$ where $\mathfrak{J}_{\mathbb{R}^2}$ is as in Example 2.3.1. In this case, a GH map ψ is called a GH function.

Let L_M be the $+i$-eigenbundle of $\mathfrak{J}_M$ so that we have,

$$(TM \oplus T^*M) \otimes \mathbb{C} = L_M \oplus \overline{L_M}\,.$$

Let d be the exterior derivative on M.

Lemma 2.3.1. *Given an open set $U \subseteq M$, a smooth map $\psi : (U, \mathcal{J}_U) \longrightarrow \mathbb{C} = (\mathbb{R}^2, J_{\mathbb{R}^2})$ is a GH function if and only if for each $x \in U$, we have*

$$d\psi_x \in (L_M \cap (T^*M \otimes \mathbb{C}))_x .$$

Proof. Follows from Lemma 2.2.3 and Definition 2.3.6. □

Let $\mathcal{O}_M$ be the sheaf of $\mathbb{C}$-valued GH functions on M. By Lemma 2.3.1, $\mathcal{O}_M$ is a subsheaf of the sheaf of smooth $\mathbb{C}$-valued functions on M. To begin with, we consider some simple examples of $\mathcal{O}_M$.

1. When (M, J_M) is a complex manifold with J_M as its complex structure. Then the induced natural GCS is, as given in Example 2.3.1,

$$\mathcal{J}_M := \begin{pmatrix} -J_M & 0 \\ 0 & J_M^* \end{pmatrix}$$

   with its $+i$-eigenbundle

$$L_M = T^{0,1}M \oplus (T^{1,0}M)^* .$$

   By Lemma 2.3.1, we can see that, given any GH map ψ, $d\psi \in \Omega^{1,0}(M)$ that is, ψ is a holomorphic function. So $\mathcal{O}_M$ will be the sheaf of holomorphic functions on M.

2. When (M, ω) is a symplectic manifold with a symplectic structure ω. The induced GCS, as given in Example 2.3.2,

$$\mathcal{J}_M := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

   with its $+i$-eigenbundle

$$L_M = \{X - i\omega(X) \,|\, X \in TM \otimes \mathbb{C}\}$$

   which is naturally identified with $TM \otimes \mathbb{C}$. So, $d\psi = 0$ for any GH map ψ, which implies ψ is locally constant. Hence $\mathcal{O}_M$ is a sheaf of locally constant functions.

Note that, in a simpler terms, Theorem 2.3.1 implies that, for some real closed form 2-form $B_\phi \in \Omega^2(U_1 \times U_2)$, there exists a GH homeomorphism

$$\phi : (U_x, \mathcal{J}_{U_x}) \longrightarrow (U_1 \times U_2, (\mathcal{J}_{U_1 \times U_2})_{B_\phi})$$

where $(\mathcal{J}_{U_1\times U_2})_{B_\phi}$ is a the B-transformation, as in (2.3.6), of the product GCS, denoted by $\mathcal{J}_{U_1\times U_2}$. Let $p = (p_1, \ldots, p_{2n-2k})$ and $z = (z_1, \ldots, z_k)$ represent coordinate systems for $\mathbb{R}^{2n-2k}$ and $\mathbb{C}^k$, respectively, and consider the corresponding local coordinates around x

$$(U_x, \phi, p, z) := (U_x, \phi\,; p_1, \ldots, p_{2n-2k}, z_1, \ldots, z_k)\,. \tag{2.3.8}$$

We note that the subspaces $(E_M)_x$ and $(\overline{E_M})_x$ admit the following description,

$$\begin{aligned}(E_M)_x &= \operatorname{Span}\left\{\partial_{p_i}|_x, \partial_{\overline{z_j}}|_x \colon 1 \le i \le 2n-2k, 1 \le j \le k\right\},\\ (\overline{E_M})_x &= \operatorname{Span}\left\{\partial_{p_i}|_x, \partial_{z_j}|_x \colon 1 \le i \le 2n-2k, 1 \le j \le k\right\}.\end{aligned} \tag{2.3.9}$$

Using Theorem 2.3.1 in the case of a regular GC manifold, It has been established that we can obtain a concise depiction of coordinate transformations using local coordinates, as in (2.3.8). This is demonstrated by the following corollary.

Corollary 2.3.1. *([103, Proposition 2.7]) Let $(M, \mathcal{J}_M)$ be a regular GC manifold of type k. Let's assume that (U, ϕ, p, z) and (U', ϕ', p', z') are two local coordinates, as in* (2.3.8)*, with $U \cap U' \neq \emptyset$. Then, the coordinate transformation satisfies the following condition*

$$\frac{\partial z'_i}{\partial \overline{z_j}} = \frac{\partial z'_i}{\partial p_l} = 0 \quad \textit{for all} \quad i, j \in \{1, \ldots, k\}\,, l \in \{1, \ldots, 2n-2k\}\,.$$

Furthermore, local coordinates, in (2.3.8), provide a requisite and complete condition for GH functions on a regular GC manifold, as demonstrated below.

Proposition 2.3.3. *([103, Example 2.8]) Let $(M, \mathcal{J}_M)$ be a regular type k GC manifold. Then, $f : M \longrightarrow \mathbb{C}$ is a GH function if and only if at every point on M, expressed in terms of local coordinates, as shown in* (2.3.8)*, f satisfies the following*

$$\frac{\partial f}{\partial \overline{z_j}} = \frac{\partial f}{\partial p_l} = 0 \quad \textit{for all} \quad i, j \in \{1, \ldots, k\}\,, l \in \{1, \ldots, 2n-2k\}\,.$$

Definition 2.3.7. *(cf. [103]) A diffeomorphism $\phi : (M, \mathcal{J}_M) \longrightarrow (N, \mathcal{J}_N)$ between two GC manifolds is called a generalized holomorphic (GH) homeomorphism if*

$$\begin{pmatrix}\phi_* & 0\\ 0 & (\phi^{-1})^*\end{pmatrix} \circ \mathcal{J}_M = \mathcal{J}_N \circ \begin{pmatrix}\phi_* & 0\\ 0 & (\phi^{-1})^*\end{pmatrix}. \tag{2.3.10}$$

When $N = M$, ϕ is called GH automorphism.

For any GC manifold $(M, \mathcal{J}_M)$, let $\operatorname{Diff}_{\mathcal{J}_M}(M)$ be the subgroup of $\operatorname{Diff}(M)$ defined by

$$\operatorname{Diff}_{\mathcal{J}_M}(M) := \{\phi \in \operatorname{Diff}(M)\,|\, \phi \text{ is a GH automorphism of } (M, \mathcal{J}_M)\}\,. \tag{2.3.11}$$

Remark 2.3.4. *Note that a GH homeomorphism* ϕ *and its inverse* ϕ^{-1} *are both GH maps. This can be observed as follows. Let* L_M, L_N *denote the* $+i$*-eigen bundles of* $\mathcal{J}_M, \mathcal{J}_N$*, respectively. For every point* p *in* M*, consider the subset of* $(T_{\phi(p)}N \oplus T^*_{\phi(p)}N) \otimes \mathbb{C}$

$$\phi_\star(L_M\,|_p) := \{\phi_*(X) + \eta \,|\, X + \phi^*\eta \in L_M\,|_p\}\,.$$

Then, for any $X \in T_pM \otimes \mathbb{C}$ *and* $\eta \in T^*_{\phi(p)}N \otimes \mathbb{C}$

$$J_N(\phi_*(X) + \eta) = \mathcal{J}_N\left(\begin{pmatrix}\phi_* & 0\\ 0 & (\phi^{-1})^*\end{pmatrix}(X + \phi^*\eta)\right)\,.$$

By (2.3.10)*, we get* $Y + \xi \in \phi_\star(L_M)$ *if and only* $Y + \xi \in L_N$*, that is*

$$\phi_\star(L_M) = L_N\,,$$

and using [125, Corollary 3.3], we conclude that both ϕ *and* ϕ^{-1} *are GH maps. But the converse is not true always. A GH map which is a diffeomorphism may not always be a GH homeomorphism. The reason is that a GH map is defined up to a B-transformation whereas a GH homeomorphism between two GC manifolds shows that their GC structures are the same.*

2.4 Related cohomologies for generalized complex manifolds

A generalized complex structure (GCS) defined on a smooth manifold gives rise to two distinct cohomology theories. The first one arises from the Clifford action of the $-i$-eigenbundle of the GCS on the canonical line bundle, while the second is induced by the $\pm i$-eigenbundles. The former is commonly referred to as generalized Dolbeault cohomology, while the latter is known as the Lie algebroid cohomology associated with the $\pm i$-eigenbundles. We will mainly refer to [4, 37, 70, 72] and the references therein for details on generalized Dolbeault cohomology. For a detailed study of Lie algebroid and its cohomology, we suggest referring to [109, 111] and the references therein.

2.4.1 Generalized Dolbeault cohomology

Given a GCS, $\mathcal{J}_M$ on a $2n$-dimensional manifold M, we get a decomposition of the complex of differential forms as follows: Let $U_M \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ be the canonical line bundle of $\mathcal{J}_M$, as in Definition 2.3.3. Then the $+i$-eigenbundle L_M of $\mathcal{J}_M$ in $(TM \oplus T^*M) \otimes \mathbb{C}$ can be obtained as

$$L_M = \operatorname{Ann}(U_M) = \{u \in (TM \oplus T^*M) \otimes \mathbb{C} \,|\, u \cdot U_M = 0\}\,.$$

For each $i \in \mathbb{Z}$, define

$$U_M^i := \wedge^{n-i}\overline{L_M} \cdot U_M \ \subset \wedge^\bullet T^*M \otimes \mathbb{C}. \tag{2.4.1}$$

Note that $U_M^i = 0$ for each $i < -n$ and $i > n$, and U_M^n is the canonical line bundle U_M. We have

$$\wedge^\bullet T^*M \otimes \mathbb{C} = \bigoplus_{i=-n}^{n} U_M^i\,.$$

Denote by $C^\infty(U_M^i)$ the vector space of smooth sections of U_M^i. Let d be the exterior derivative on M. Then [70, Theorem 4.23] implies that

$$d : C^\infty(U_M^i) \longrightarrow C^\infty(U_M^{i+1}) \oplus C^\infty(U_M^{i-1}). \tag{2.4.2}$$

decomposes into two differential operators as

$$d = \partial_M + \bar{\partial}_M\,.$$

The $\bar{\partial}_M$ and ∂_M operators are defined by composing d with the projections onto $C^\infty(U_M^{i-1})$ and $C^\infty(U_M^{i+1})$, respectively,

$$\begin{aligned} \bar{\partial}_M &: C^\infty(U_M^i) \longrightarrow C^\infty(U_M^{i-1}),\\ \partial_M &: C^\infty(U_M^i) \longrightarrow C^\infty(U_M^{i+1})\,. \end{aligned} \tag{2.4.3}$$

Thus, we obtain a $\mathbb{Z}$-graded differential complex $\{C^\infty(U_M^i), \bar{\partial}\}_M$ and the cohomology of this complex is called the *generalized Dolbeault cohomology* of M (cf. [37], [72, Proposition 3.15]),

$$GH_{\bar{\partial}}^\bullet(M) := \frac{\ker\left(\bar{\partial}_M : C^\infty(U_M^\bullet) \longrightarrow C^\infty(U_M^{\bullet-1})\right)}{\operatorname{img}\left(\bar{\partial}_M : C^\infty(U_M^{\bullet+1}) \longrightarrow C^\infty(U_M^\bullet)\right)}\,. \tag{2.4.4}$$

To get an idea about generalized Dolbeault cohomology, it is useful to consider it first for some simple cases as follows.

(a) When M is a complex manifold, The canonical line bundle of the GCS, as defined in Example 2.3.1, is just $\wedge^{(n,0)}T^*M$ and $\overline{L_M} = T^{1,0}M \oplus (T^*M)^{0,1}$. One can see that

$$U_M^\bullet = \oplus_{p-q=\bullet} \wedge^{(p,q)} T^*M.$$

So in this case, the generalized Dolbeault cohomology is just

$$\begin{aligned} GH^\bullet_{\bar\partial}(M) &= \oplus_{p-q=\bullet} H^q(M, \Omega^p(M))\,, \\ &= \oplus_{p-q=\bullet} H^{p,q}(M). \end{aligned} \tag{2.4.5}$$

(b) When (M, ω) is a symplectic manifold, the canonical bundle of the GCS, as defined in Example 2.3.2, is generated by $e^{i\omega}$ and its null space is

$$L_M = \{X - i\omega(X, \cdot)|\, X \in TM \otimes \mathbb{C}\}\,.$$

By [37, Theorem 2.2], one can see that

$$U_M^\bullet = \{e^{i\omega}(e^{\frac{\Lambda}{2i}}\eta)|\, \eta \in \wedge^{n-\bullet}T^*M \otimes \mathbb{C}\},$$

where Λ is the interior product with the bivector $-\omega^{-1}$. Hence, the generalized Dolbeault cohomology is isomorphic to the complex de Rham cohomology of M

$$GH^\bullet_{\bar\partial}(M) = H^{n-\bullet}(M; \mathbb{C}). \tag{2.4.6}$$

Note that, after a B-field transformation of a GCS $\mathfrak{J}_M$ on a smooth manifold M, a local section of the canonical line bundle of $(\mathfrak{J}_M)_B$ is of the form $e^B \wedge \phi$ where ϕ is a local section of the canonical line bundle of $\mathfrak{J}_M$. Hence, the canonical line bundle of the deformed structure is

$$(U_M)_B = e^B \cdot U\,,$$

with the $+i$-eigenbundle $(L_M)_B$ as defined in 2.3.7. So, for each $i \in \mathbb{Z}$, we get another decomposition

$$\wedge^\bullet T^*M \otimes \mathbb{C} = \bigoplus_{i=-n}^{n} (U_M)^i_B\,, \quad \text{where} \quad (U_M)^i_B = e^B\, U^i_M.$$

Then for $\beta \in C^\infty(U^i_M)$,

$$d(e^B\beta) = e^B d\beta = e^B\partial\beta + e^B\bar\partial\beta,$$

where $e^B\partial\beta \in C^\infty((U_M)^{i+1}_B)$ and $e^B\bar\partial\beta \in C^\infty((U_M)^{i-1}_B)$. Hence,

$$(\bar\partial_M)_B = e^B\bar\partial_M e^{-B} \tag{2.4.7}$$

and

$$(\partial_M)_B = e^B \partial_M e^{-B} \,. \tag{2.4.8}$$

The cohomology of the $\mathbb{Z}$-graded complex $\{C^\infty((U_M)^i_B), (\bar{\partial}_M)_B\}$, denoted by $GH_{(\bar{\partial}_M)_B}(M)$, is defined as

$$GH^\bullet_{\bar{\partial}_B}(M) := \frac{\ker\Big((\bar{\partial}_M)_B : C^\infty((U_M)^\bullet_B) \longrightarrow C^\infty((U_M)^{\bullet-1}_B)\Big)}{\mathrm{img}\Big((\bar{\partial}_M)_B : C^\infty((U_M)^{\bullet+1}_B) \longrightarrow C^\infty((U_M)^\bullet_B)\Big)} \,. \tag{2.4.9}$$

Hence, by equation (2.4.7), a B-field transformation preserves the generalized Dolbeault cohomology of M up to isomorphism

$$GH^\bullet_{\bar{\partial}_B}(M) \cong GH^\bullet_{\bar{\partial}}(M). \tag{2.4.10}$$

2.4.2 Associated Lie algebroid cohomology

Consider the $+i$-eigenbundle L_M over the GC manifold M. Both the triplets $(L_M, [\,,\,], \rho)$ and $(\overline{L_M}, [\,,\,], \rho)$ define the structure of a Lie algebroid. Here, $[\,,\,]$ denotes the Courant bracket as defined in (2.3.2), and $\rho : (TM \oplus T^*M) \otimes \mathbb{C} \longrightarrow TM \otimes \mathbb{C}$ represents the projection map. By identifying L_M^* with $\overline{L_M}$ via the symmetric bilinear form defined in (2.3.1), we then obtain two differential operators as follows:

$$d_L : C^\infty(\wedge^\bullet L_M^*) \longrightarrow C^\infty(\wedge^{\bullet+1} L_M^*)\,; \tag{2.4.11}$$

$$d_{\overline{L}} : C^\infty(\wedge^\bullet \overline{L_M}^*) \longrightarrow C^\infty(\wedge^{\bullet+1} \overline{L_M}^*)\,. \tag{2.4.12}$$

In particular, for any $\omega \in C^\infty(\wedge^n \overline{L_M})$ and $X_i \in C^\infty(L_M)$ for all $i \in \{1, \cdots, n+1\}$, we have

$$\begin{aligned} d_L\omega(X_1, \cdots, X_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \rho(X_i)(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{n+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \cdots, \hat{X}_j, \cdots, X_{n+1})\,, \end{aligned}$$

The associated Lie algebroid cohomology corresponding to the complexes $\{C^\infty(\wedge^\bullet L_M^*), d_L\}$ and $\{C^\infty(\wedge^\bullet \overline{L_M}^*), d_{\overline{L}}\}$, are denoted by $H^\bullet(L_M)$, and $H^\bullet(\overline{L_M})$, respectively, and are defined as follows:

$$\begin{aligned} H^\bullet(L_M) &:= \frac{\ker(d_L : \wedge^\bullet L_M^* \longrightarrow \wedge^{\bullet+1} L_M^*)}{\mathrm{img}(d_L : \wedge^{\bullet-1} L_M^* \longrightarrow \wedge^\bullet L_M^*)}\,. \\ H^\bullet(\overline{L_M}) &:= \frac{\ker(d_{\overline{L}} : \wedge^\bullet \overline{L_M}^* \longrightarrow \wedge^{\bullet+1} \overline{L_M}^*)}{\mathrm{img}(d_{\overline{L}} : \wedge^{\bullet-1} \overline{L_M}^* \longrightarrow \wedge^\bullet \overline{L_M}^*)}\,. \end{aligned} \tag{2.4.13}$$

To understand the associated Lie algebroid cohomology, it is helpful to start with some simple examples. Let us explore two cases outlined in [72, Section 3.2].

1. In the case of a complex structure, such as in Example 2.3.1, the Lie algebroid L_M is given by $T^{0,1}M \oplus (T^*M)^{1,0}$ while $d_L = \bar{\partial}$ is the usual $\bar{\partial}$-operator for a complex manifold. The associated Lie algebroid complex is a sum of usual Dolbeault complexes, resulting in

$$H^\bullet(L_M) = \bigoplus_{p+q=\bullet} H^p(M, \wedge^q T^{1,0}M)\,.$$

2. In the case of a symplectic structure, as in Example 2.3.2, the Lie algebroid L_M is the graph of $i\omega$, and is, therefore, isomorphic to $TM \otimes \mathbb{C}$ as a Lie algebroid. Consequently, its associated Lie algebroid cohomology is simply the complex de Rham cohomology, given by

$$H^\bullet(L_M) = H^\bullet(M, \mathbb{C})\,.$$

2.4.3 Relation between these two cohomologies

Let $(M, \mathfrak{J}_M)$ denote a $2n$-dimensional GC manifold with the canonical bundle U_M and the $+i$-eigenbundle L_M. In accordance with the integrability condition mentioned in (3) of Definition 2.3.3, the exterior derivative d induces the map

$$d : C^\infty(U_M) \longrightarrow C^\infty(\overline{L_M} \cdot U_M)\,.$$

With the identification $L_M^* = \overline{L_M}$, we can establish $U_M^i = \wedge^{n-i} L_M^* \otimes U_M$ where U_M^i is defined as in (2.4.1). Thus, the differential operator $\bar{\partial}_M$ can be understood as

$$\bar{\partial}_M : C^\infty(\wedge^{n-i} L_M^* \otimes U_M) \longrightarrow C^\infty(\wedge^{(n-i)+1} L_M^* \otimes U_M)\,, \quad \text{for } -n \le i \le n$$

extending from $d : C^\infty(U_M) \longrightarrow C^\infty(L_M^* \otimes U_M)$ via the rule

$$\bar{\partial}_M(\alpha \otimes \beta) = d_L \alpha \otimes \beta + (-1)^{|\alpha|} \alpha \otimes d\beta\,,$$

where $\alpha \in C^\infty(\wedge^{n-i} L_M^*)$ and $\beta \in C^\infty(U_M)$.

This demonstrates that the differential operator $\bar{\partial}_M$ is effectively derived from the Lie algebroid deRham operator d_L. Put differently, we can interpret the generalized

Dolbeault cohomology as a Lie algebroid cohomology with coefficients in the canonical line bundle of the GC manifold. With this understanding, we conclude this chapter.

Chapter 3

Strong Generalized Holomorphic Fiber Bundles and Induced Foliations on Generalized Complex Manifolds

Generalized complex (GC) geometry encompasses both complex and symplectic structures, as previously discussed. An elementary example illustrating a generalized complex structure (GCS) neither of complex nor symplectic type is the product of a complex and a symplectic manifold, demonstrated in Example 2.1.4. More broadly, the product GCS, as outlined in Example 2.3.4, is a straightforward instance derived from two given GC manifolds. However, for more intricate examples, a natural approach should involve considering fiber bundle theory over a GC manifold, where the fiber itself is a GC manifold, thus locally inducing the GC structure from the two given GC manifolds. For a detailed study of general fiber bundle theory, we refer to [85, 140] and the references therein.

The main objective of this chapter is twofold. Firstly, we introduce the concept of strong generalized holomorphic (SGH) fiber bundles and explore various examples of SGH fiber bundles. In this framework, we also establish the notion generalized holomorphic (GH) Picard group. Our focus is particularly on providing a comprehensive description of GH tangent and GH cotangent bundles, which serve as the foundational elements of this thesis. Secondly, we delve into the foliation induced by a GCS and provide a connection between transverse geometry and GH tangent and cotangent bundles. In particular, we illustrate that by imposing certain conditions on the leaf space of the base GC manifold, SGH fiber bundles provide a further understanding of the transverse geometry of the base GC manifold. We also provide examples to demonstrate that not all SGH fiber bundles derive from the leaf space, even if the leaf space is a smooth manifold. For more details on the transverse geometry of a foliation and its leaf space, we refer to [6,117,118,121,145,146] and the references therein. This chapter is based on [127, Sections 3-5, 10 and 12-13] and splits into four sections:

1. Strong generalized holomorphic fiber bundles (Section 3.1).
2. Generalized holomorphic tangent and generalized holomorphic cotangent bundles (Section 3.2).
3. Generalized holomorphic Picard groups (Section 3.3)
4. Induced foliations on GC manifolds (Section 3.4).

3.1 Strong generalized holomorphic fiber bundles

In this section, we define strong generalized holomorphic (SGH) fiber bundles and explore various examples of SGH fiber bundles. As a fiber bundle, the total space of an SGH fiber bundle admits a GCS that is locally a product GCS, induced from both the base and fiber; see Definition 3.1.1. In the context of vector bundles, SGH vector bundles are precisely the GH vector bundles defined by Gualtieri and Lang et al. ([70,103]). Similarly, in the realm of principal bundles, they are the GH principal bundles analyzed by Wang ([154, Example 4.2]). We elucidate SGH fiber bundles by analyzing their transition maps in Theorem 3.1.1. Additionally, we investigate special cases, such as when the GCS of the fiber or the base manifold is induced by symplectic structures (see Lemma 3.1.1 and Lemma 3.1.2) or complex structures (see Lemma 3.1.3).

Let $(M, \mathcal{J}_M)$ be a generalized complex (GC) manifold. Then $\mathcal{J}_M$ can be written as

$$\mathcal{J}_M = \begin{pmatrix} -J_M & \beta_M \\ B_M & J_M^* \end{pmatrix} \tag{3.1.1}$$

where $J_M \in \operatorname{End}(TM)$, $B_M \in \Omega^2(M)$ and $\beta_M \in \mathfrak{X}^2(M)$. Let $\operatorname{Diff}_{\mathcal{J}_M}(M)$ denote the subgroup of $\mathrm{Diff}(M)$ defined by

$$\mathrm{Diff}_{\mathcal{J}_M}(M) := \{\phi \in \mathrm{Diff}(M) \,|\, \phi \text{ is a GH automorphism of } (M, \mathcal{J}_M)\}\,. \tag{3.1.2}$$

Definition 3.1.1. *Let G be a connected Lie group. A smooth fiber bundle $F \hookrightarrow E \xrightarrow{\pi} M$ over a GC manifold $(M, \mathcal{J}_M)$ with a typical fiber $(F, \mathcal{J}_F)$ and structure group G is called an strong generalized holomorphic (SGH) fiber bundle if*

1. *E is a GC manifold,*

2. *there is an open cover $\{U_\alpha\}$ of M and a family of local trivializations $\{\phi_\alpha\}$ of E*

$$\{\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F\}$$

   such that every ϕ_α is a GH homeomorphism when $U_\alpha \times F$ is endowed with the standard product GC structure.

In addition, if F is a vector space and G is a subgroup of $GL(F)$, then we say that E is an SGH vector bundle over M.

Example 3.1.1. *Let M_1 be a complex manifold and V_1 be a holomorphic vector bundle over M_1. Let M_2 be a symplectic manifold and V_2 be a flat vector bundle over M_2. Then $\otimes_i \mathrm{Pr}_i^*(V_i) \longrightarrow M_1 \times M_2$ is an SGH vector bundle where $\mathrm{Pr}_i : M_1 \times M_2 \longrightarrow M_i$ is the natural projection map onto i-th component. Here $M_1 \times M_2$ is considered with the product GCS.*

The following theorem is a generalization of [103, Proposition 3.2], providing a characterization of SGH fiber bundles in terms of their transition maps.

Theorem 3.1.1. *Let E be a fiber bundle over $(M, \mathcal{J}_M)$ with typical fiber $(F, \mathcal{J}_F)$ and structure group G. Let $\{U_\alpha, \phi_\alpha\}$ be a family of local trivializations with transition functions $\phi_{\alpha\beta}$ as follows,*

$$\{\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F\}\,,\; \phi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \longrightarrow G\,,$$

where $\phi_{\alpha\beta}(x) = \phi_\alpha|_{\pi^{-1}(x)} \circ \phi_\beta^{-1}(x, \cdot)$ for all $x \in U_{\alpha\beta}$. Then, E is SGH fiber bundle over M with local trivializations $\{U_\alpha, \phi_\alpha\}$ if and only if

1. *$\phi_{\alpha\beta}(m) \in \mathrm{Diff}_{\mathcal{J}_F}(F)$ for all $m \in U_{\alpha\beta}$,*

2. For each $(m,f)\in M\times F$, the following equations hold:

$$(\rho_f)_*\circ(\phi_{\alpha\beta})_{*m}\circ J_{U_{\alpha\beta}} \;=\; J_F\circ(\rho_f)_*\circ(\phi_{\alpha\beta})_{*m}\,,$$

$$(\rho_f)_*\circ(\phi_{\alpha\beta})_{*m}\circ\beta_{U_{\alpha\beta}} \;=\; 0\,,$$

$$B_F\circ(\rho_f)_*\circ(\phi_{\alpha\beta})_{*m} \;=\; 0\,,$$

for all $(m,f)\in M\times F$, where $J_{U_{\alpha\beta}}$, J_F, $\beta_{U_{\alpha\beta}}$, B_F are as in equation (3.1.1)*, and the map $\rho_f : G\longrightarrow F$ is defined as $\rho_f(g):=g\cdot f$.*

Proof. Consider the map

$$\psi_{\alpha\beta}=\phi_\alpha\circ\phi_\beta^{-1} : U_{\alpha\beta}\times F\longrightarrow U_{\alpha\beta}\times F\,. \tag{3.1.3}$$

Note that $\psi_{\alpha\beta}(m,f)=(m,\phi_{\alpha\beta}(m)\cdot f)$ for all $(m,f)\in U_{\alpha\beta}\times F$.

First, we claim that E is an SGH fiber bundle if and only if $\psi_{\alpha\beta}$ is a GH automorphism for any fixed α,β. Indeed, if $\psi_{\alpha\beta}$ is a GH automorphism, then

$$\begin{pmatrix}(\psi_{\alpha\beta})_* & 0\\ 0 & (\psi_{\beta\alpha})^*\end{pmatrix}\circ\mathcal{J}_{U_{\alpha\beta}\times F}=\mathcal{J}_{U_{\alpha\beta}\times F}\circ\begin{pmatrix}(\psi_{\alpha\beta})_* & 0\\ 0 & (\psi_{\beta\alpha})^*\end{pmatrix}\,, \tag{3.1.4}$$

where $\mathcal{J}_{U_{\alpha\beta}\times F}=(J_{ij})_{2\times2}$ is the product GC structure on $U_{\alpha\beta}\times F$. Then

$$\begin{pmatrix}(\phi_\alpha^{-1})_* & 0\\ 0 & (\phi_\alpha)^*\end{pmatrix}\circ(J_{ij})_{2\times2}\circ\begin{pmatrix}(\phi_\alpha)_* & 0\\ 0 & (\phi_\alpha^{-1})^*\end{pmatrix} \tag{3.1.5}$$

is an endomorphism of $T\pi^{-1}(U_\alpha)\oplus T^*\pi^{-1}(U_\alpha)$ that produces the GC structure on $\pi^{-1}(U_\alpha)$. By equation (3.1.4) this structure is independent of the choice of ϕ_α. Hence, we obtain a GC structure on E such that ϕ_α becomes GH homeomorphism. The converse is obvious.

Now, it is enough to show that $\psi_{\alpha\beta}$ is a GH automorphism if and only if (1) and (2) are satisfied.

The product GC structure on $U_{\alpha\beta}\times F$ can be expressed as

$$J_{11}=\begin{pmatrix}-J_{U_{\alpha\beta}} & 0\\ 0 & -J_F\end{pmatrix}\,,\; J_{21}=\begin{pmatrix}B_{U_{\alpha\beta}} & 0\\ 0 & B_F\end{pmatrix}\,,$$

$$J_{12}=\begin{pmatrix}\beta_{U_{\alpha\beta}} & 0\\ 0 & \beta_F\,,\end{pmatrix}\,,\; J_{22}=\begin{pmatrix}J^*_{U_{\alpha\beta}} & 0\\ 0 & J^*_F\end{pmatrix}\,.$$

Upon simplification, the expression for equation (3.1.4), at $(m, f) \in U_{\alpha\beta} \times F$, can be represented as:

$$(\psi_{\alpha\beta})_{*(m,f)} \circ J_{11} = J_{11} \circ (\psi_{\alpha\beta})_{*(m,f)} \,, \tag{3.1.6}$$

$$(\psi_{\alpha\beta})_{*(m,f)} \circ J_{12} = J_{12} \circ (\psi_{\beta\alpha})^*_{(m,f)} \,, \tag{3.1.7}$$

$$(\psi_{\beta\alpha})^*_{(m,f)} \circ J_{21} = J_{21} \circ (\psi_{\alpha\beta})_{*(m,f)} \,, \tag{3.1.8}$$

$$(\psi_{\beta\alpha})^*_{(m,f)} \circ J_{22} = J_{22} \circ (\psi_{\beta\alpha})^*_{(m,f)} \,. \tag{3.1.9}$$

Since $\psi_{\alpha\beta}(m, f) = (m, \phi_{\alpha\beta}(m) \cdot f)$ where $\phi_{\alpha\beta}(m) \in G$, the map

$$(\psi_{\alpha\beta})_{*(m,f)} : T_m U_{\alpha\beta} \oplus T_f F \longrightarrow T_m U_{\alpha\beta} \oplus T_{\phi_{\alpha\beta}(m)\cdot f} F$$

can be expressed as

$$(\psi_{\alpha\beta})_{*(m,f)} = \begin{pmatrix} Id_{U_{\alpha\beta}} & 0 \\ (\rho_f)_* \circ (\phi_{\alpha\beta})_{*m} & (\phi_{\alpha\beta}(m))_* \end{pmatrix} , \tag{3.1.10}$$

and the map

$$(\psi_{\beta\alpha})^*_{(m,f)} : T^*_m U_{\alpha\beta} \oplus T^*_f F \longrightarrow T^*_m U_{\alpha\beta} \oplus T^*_{\phi_{\alpha\beta}(m)\cdot f} F$$

can be expressed as

$$(\psi_{\beta\alpha})^*_{(m,f)} = \begin{pmatrix} Id_{U_{\alpha\beta}} & (\phi_{\beta\alpha})^*_m \circ \rho^*_{\phi_{\alpha\beta}(m)\cdot f} \\ 0 & (\phi_{\beta\alpha}(m))^* \end{pmatrix} . \tag{3.1.11}$$

From equations (3.1.6) and (3.1.10), we have

$$(\phi_{\alpha\beta}(m))_* \circ (-J_F) = (-J_F) \circ (\phi_{\alpha\beta}(m))_* \tag{3.1.12}$$

and

$$(\rho_f)_* \circ (\phi_{\alpha\beta})_{*m} \circ (-J_{U_{\alpha\beta}}) = (-J_F) \circ (\rho_f)_* \circ (\phi_{\alpha\beta})_{*m} \,. \tag{3.1.13}$$

Using equations (3.1.7), (3.1.10) and (3.1.11), we get

$$(\phi_{\alpha\beta}(m))_* \circ \beta_F = \beta_F \circ (\phi_{\beta\alpha}(m))^* \,, \tag{3.1.14}$$

$$(\rho_f)_* \circ (\phi_{\alpha\beta})_{*m} \circ \beta_{U_{\alpha\beta}} = 0 \,, \tag{3.1.15}$$

and

$$\beta_{U_{\alpha\beta}} \circ (\phi_{\beta\alpha})^*_m \circ \rho^*_{\phi_{\alpha\beta}(m)\cdot f} = 0 \,. \tag{3.1.16}$$

From equations (3.1.8), (3.1.10) and (3.1.11), we have

$$(\phi_{\beta\alpha}(m))^* \circ B_F = B_F \circ (\phi_{\alpha\beta}(m))_*\,, \tag{3.1.17}$$

$$(\phi_{\beta\alpha})^*_m \circ \rho^*_{\phi_{\alpha\beta}(m)\cdot f} \circ B_F = 0\,, \tag{3.1.18}$$

and

$$B_F \circ (\rho_f)_* \circ (\phi_{\alpha\beta})_{*m} = 0\,. \tag{3.1.19}$$

From equations (3.1.9) and (3.1.11), we have

$$(\phi_{\beta\alpha}(m))^* \circ J^*_F = J^*_F \circ (\phi_{\beta\alpha}(m))^* \tag{3.1.20}$$

and

$$(\phi_{\beta\alpha})^*_m \circ \rho^*_{\phi_{\alpha\beta}(m)\cdot f} \circ J^*_F = J^*_{U_{\alpha\beta}} \circ (\phi_{\beta\alpha})^*_m \circ \rho^*_{\phi_{\alpha\beta}(m)\cdot f}\,. \tag{3.1.21}$$

Now, we can see that equations (3.1.12), (3.1.14), (3.1.17) and (3.1.20) hold if and only if

$$\begin{pmatrix} (\phi_{\alpha\beta}(m))_* & 0 \\ 0 & (\phi_{\beta\alpha}(m))^* \end{pmatrix} \circ \mathcal{J}_F = \mathcal{J}_F \circ \begin{pmatrix} (\phi_{\alpha\beta}(m))_* & 0 \\ 0 & (\phi_{\beta\alpha}(m))^* \end{pmatrix} \tag{3.1.22}$$

where $\mathcal{J}_F = \begin{pmatrix} -J_F & \beta_F \\ B_F & J^*_F \end{pmatrix}$ as in (3.1.1). Therefore, equations (3.1.12), (3.1.14), (3.1.17) and (3.1.20) hold if and only if $\phi_{\beta\alpha}(m) \in \mathrm{Diff}_{\mathcal{J}_F}(F)$. Note that, by skew-symmetry, $\beta^*_{U_{\alpha\beta}} = -\beta_{U_{\alpha\beta}}$ and $B^*_F = -B_F$. Since (m, f) is arbitrary, considering duals, we observe that

- equation (3.1.15) holds if and only if equation (3.1.16) holds,
- equation (3.1.19) holds if and only if equation (3.1.18) holds,
- equations (3.1.13) and (3.1.21) are equivalent to each other.

Therefore, $\psi_{\alpha\beta}$ is a GH automorphism if and only if equations (3.1.13), (3.1.15), (3.1.19) and (3.1.22) hold. Hence, $\psi_{\alpha\beta}$ is a GH automorphism if and only if (1) and (2) are satisfied as desired. □

Definition 3.1.2. *Let E be an SGH fiber bundle over a GC manifold $(M, \mathcal{J}_M)$ and $U \subseteq M$ be an open set. A smooth section $s : U \longrightarrow E$ is called a GH section if s is a GH map from $(U, \mathcal{J}_M|_U)$ to $(E, \mathcal{J}_E)$.*

Definition 3.1.3. *Given any two SGH fiber bundles E and E' over M, a smooth map $\phi : E \longrightarrow E'$ is called an SGH bundle homomorphism if*

1. *ϕ is a bundle homomorphism between E and E' as smooth fiber bundles.*

2. *ϕ is a GH map.*

If, in addition, ϕ is a GH homeomorphism, then $\phi : E \longrightarrow E'$ is called SGH bundle isomorphism.

3.1.1 The case when either the fiber or the base manifold is a symplectic manifold

In this subsection, we consider the special case of SGH fiber bundles, where the generalized complex structure (GCS) on either the fiber or the base manifold is induced by a symplectic structure. Our objective is to provide a characterization of this case.

Let E be an SGH fiber bundle over a GC manifold $(M, \mathcal{J}_M)$. Let a symplectic manifold (F, ω) be its typical fiber. The GC structure on F can be expressed as

$$\mathcal{J}_F = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} .$$

Note that $J_F = J_F^* = 0$ and $B_F = -\beta_F^{-1} = \omega$. Since ω is non-degenerate and $f \in F$ is arbitrary, for each $m \in M$, equation (3.1.19) holds if and only if $(\phi_{\alpha\beta})_{*m} = 0$, i.e., $\phi_{\alpha\beta}$ is a locally constant map on $U_{\alpha\beta}$. From equation (3.1.17), for $X, Y \in TM$, we have

$$\begin{aligned} \omega &= (\phi_{\alpha\beta}(m))^* \circ \omega \circ (\phi_{\alpha\beta}(m))_* \\ \Longrightarrow \omega(X) &= (\phi_{\alpha\beta}(m))^*(\omega((\phi_{\alpha\beta}(m))_*(X))) \\ \Longrightarrow \omega(X, Y) &= \omega((\phi_{\alpha\beta}(m))_*(X), (\phi_{\alpha\beta}(m))_*(Y)) \\ \Longrightarrow \omega(X, Y) &= (\phi_{\alpha\beta}(m))^*\omega(X, Y) . \end{aligned}$$

Lemma 3.1.1. *Any SGH fiber bundle E over a GC manifold M with a symplectic fiber (F, ω) is a smooth symplectic fiber bundle with a flat connection.*

Let E be an SGH fiber bundle over a symplectic manifold (M, ω) with a typical fiber

$(F, \mathcal{J}_F)$. The GC structure on the base is given by

$$\mathcal{J}_M = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Note that $J_M = J_M^* = 0$ and $B_M = -\beta^{-1} = \omega$. Since ω is nondegenarate, by equation (3.1.15), for any $(m, f) \in U_{\alpha\beta} \times F$, $(\rho_f)_* \circ (\phi_{\alpha\beta})_{*m} = 0$. Thus, $(\phi_{\alpha\beta})_{*m} = 0$, i.e., $\phi_{\alpha\beta}$ is locally constant on $U_{\alpha\beta}$. So, equations (3.1.13) and (3.1.19) are also satisfied. Hence, we have the following.

Lemma 3.1.2. *E be a smooth fiber bundle over a symplectic manifold (M, ω) with a typical fiber $(F, \mathcal{J}_F)$. Then, E is SGH fiber bundle over M with local trivializations $\{U_\alpha, \phi_\alpha\}$ if and only if*

1. *$\phi_{\alpha\beta}(m) \in \mathrm{Diff}_{\mathcal{J}_F}(F)$ for all $m \in U_{\alpha\beta}$,*

2. *$\phi_{\alpha\beta}$ is constant on $U_{\alpha\beta}$, that is, E admits a flat connection.*

3.1.2 The case when the fiber is a complex manifold

In this subsection, we examine a specific instance of SGH fiber bundles, focusing on the case where the GCS on the fiber is induced by a complex structure. Our goal is to precisely characterize this scenario.

Let E be an SGH fiber bundle over a GC manifold $(M, \mathcal{J}_M)$ with typical fiber a complex manifold (F, J_F) where J_F is a complex structure on F. Then the naturally induced GC structure on F can be written as

$$\mathcal{J}_F = \begin{pmatrix} -J_F & 0 \\ 0 & J_F^* \end{pmatrix}.$$

We can see that $B_F = \beta_F = 0$ and also by equation (3.1.22), for any $m \in U_{\alpha\beta}$, $\phi_{\alpha\beta}(m)$ is a biholomorphic automorphism.

Proposition 3.1.1. *Let $(M, \mathcal{J}_M)$ be a GC manifold and (N, J_N) be a complex manifold with a complex structure J_N. Then, for any smooth map $\psi : M \longrightarrow N$, the following are equivalent.*

1. *ψ is a GH map.*

2. *For any open set $U \subset M$, $\psi : U \longrightarrow N$ is a GH map.*

3. *$\psi_* \circ J_M = J_N \circ \psi_*$, $\quad \psi_* \circ \beta_M = 0$.*

Here $\mathfrak{J}_M$ is as in (3.1.1) *and N is considered as a GC manifold with the natural GC structure induced by J_N.*

Proof. Follows from Lemma 2.2.2. □

By Proposition 3.1.1 and equation (3.1.15), for any $f \in F$, we can show that $\rho_f \circ \phi_{\alpha\beta}$ is a GH map. Thus, we have the following result.

Lemma 3.1.3. *Let E be a smooth fiber bundle over a GC manifold $(M, \mathfrak{J}_M)$ with typical fiber a complex manifold (F, J_F). Let $\{U_\alpha, \phi_\alpha\}$ be a family of trivial localization. Then E is an SGH fiber bundle over M with local trivialization $\{U_\alpha, \phi_\alpha\}$ if and only if*

1. *for each $m \in U_{\alpha\beta}$, $\phi_{\alpha\beta}(m)$ is a biholomorphic map,*

2. *for any $f \in F$, $\rho_f \circ \phi_{\alpha\beta}$ is a GH map.*

Example 3.1.2. *Let M be a GC manifold and $\tilde{M}$ be a covering space. Let $K \leq \pi_1(M)$ be a subgroup corresponding to $\tilde{M}$ such that $\tilde{M}/K \cong M$. Note that $K \hookrightarrow \tilde{M} \xrightarrow{\pi} M$ is a principal K-bundle where π is the covering map. Since π is a local diffeomorphism, M induces a GC structure (of the same type) on $\tilde{M}$ such that π becomes a GC map. Let $\rho : K \longrightarrow GL_l(\mathbb{C})$ be a representation. Define*

$$\tilde{M} \times_\rho \mathbb{C}^l := \tilde{M} \times \mathbb{C}^l / \sim ,$$

where $(m, z) \sim (n, w)$ if and only if $n = m \cdot g^{-1}$ and $w = \rho(g) \cdot z$ for some $g \in K$. Since K is discrete, the transition maps of the associated vector bundle $\tilde{M} \times_\rho \mathbb{C}^l \longrightarrow M$ are locally constant. Hence, by Lemma 3.1.3, $\tilde{M} \times_\rho \mathbb{C}^l \longrightarrow M$ is a (flat) SGH vector bundle over M.

Remark 3.1.1. *Note that, when E denotes a vector bundle over a GC manifold M, using Lemma 3.1.3, we can see that E is an SGH vector bundle if and only if it is a GH vector bundle in the sense described by Lang et al. ([103, Definition 3.1]).*

3.2 Generalized holomorphic tangent and generalized holomorphic cotangent bundles

In this section, we demonstrate the existence of two natural SGH vector bundles associated with any regular GC manifold. These bundles are referred to as the *generalized holomorphic (GH) tangent bundle* and the *generalized holomorphic (GH) cotangent bundle*. Essentially, they depict the tangent and cotangent bundles of a regular GC manifold in the transverse direction, evident from their local description.

Let M be a GC manifold. Let E be an SGH vector bundle of real rank $2l$ over M. Consider the sheaf $\mathbf{E}$ of GH sections of E over M, that is, for any open set $U \subseteq M$,

$$\Gamma(U, \mathbf{E}) := \{ s \in C^\infty(U, E) \,|\, s \, \text{is a GH section of } E \text{ over } U \} \,.$$

Note that $\mathbf{E}$ is a sheaf of $\mathcal{O}_M$-modules. On a trivializing neighborhood U,

$$E|_U \cong U \times \mathbb{C}^l \,,$$

so that $\Gamma(U, \mathbf{E}) \cong \bigoplus_l \mathcal{O}_M(U)$. This implies that $\mathbf{E}$ is a locally free sheaf of complex rank l over M. (We will henceforth follow the convention of denoting the sheaf of GH sections of a GH vector bundle by the corresponding bold letter.)

Conversely, given any locally free sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules of rank l, one can construct an SGH vector bundle in the following manner. Let $\{U_\alpha\}$ be a covering of M such that $\mathcal{F}|_{U_\alpha}$ is free and

$$\widehat{\psi_\alpha} : \mathcal{F}|_{U_\alpha} \longrightarrow \bigoplus_l \mathcal{O}_{U_\alpha} \,,$$

is the corresponding isomorphism. Now consider the isomorphism of sheaves of modules

$$\widehat{\psi_{\alpha\beta}} = \widehat{\psi_\alpha} \circ (\widehat{\psi_\beta}^{-1}) : \bigoplus_l \mathcal{O}_{U_\alpha \cap U_\beta} \longrightarrow \bigoplus_l \mathcal{O}_{U_\alpha \cap U_\beta} \,.$$

Since every endomorphism of $\bigoplus_l \mathcal{O}_{U_\alpha}$ is represented by an $l \times l$ matrix, $\widehat{\psi_{\alpha\beta}}$ defines an $l \times l$ matrix $(\phi_{\alpha\beta})$ whose elements are GH functions over $U_\alpha \cap U_\beta$. One can check that the matrices satisfy the cocycle conditions and thus they can be regarded as the transition maps of an SGH vector bundle $E_{\mathcal{F}}$ of real rank $2l$ over M such that

$$\mathbf{E}_{\mathcal{F}} \cong \mathcal{F} \quad \text{as } \mathcal{O}_M\text{-modules} \,.$$

Hence, we get the following.

Proposition 3.2.1. *Let M be a GC manifold and $l \in \mathbb{N}$. Consider the following two sets*

$$\mathscr{E}_l := \textit{Set of all isomorphism classes of SGH vector bundles of real rank } 2l \textit{ over } M\,,$$

and

$$\mathscr{S}_l := \textit{Set of all isomorphism class of locally free } \mathcal{O}_M\textit{-modules of complex rank } l\,.$$

Then the association $E \longrightarrow \mathbf{E}$ induces an one to one correspondence between $\mathscr{E}_l$ and $\mathscr{S}_l$. The inverse map is given by the association $\mathcal{F} \longrightarrow E_{\mathcal{F}}$.

Now, let $(M, \mathcal{J}_M)$ be a regular GC manifold of type $k \in \mathbb{N} \cup \{0\}$. Let L_M and $\overline{L_M}$ are its corresponding $+i$ and $-i$-eigenspace sub-bundles of $(TM \oplus T^*M) \otimes \mathbb{C}$ respectively. Define

$$\mathcal{G}^*M := L_M \cap (T^*M \otimes \mathbb{C})\,. \tag{3.2.1}$$

By [103, Proposition 3.13], $\mathcal{G}^*M$ is an SGH vector bundle over M. It is called the *generalized holomorphic (GH) cotangent bundle*. The GH sections of $\mathcal{G}^*M$ are called GH 1-forms. Since $\mathcal{G}^*M$ is B-field transformation invariant, locally (cf. (2.3.8), (2.3.9)), the space of GH 1-forms is of the form

$$\mathrm{Span}_{\mathcal{O}_U}\{dz_1 \cdots dz_k\}\,.$$

This shows that $\mathcal{G}^*\mathbf{M}$, the sheaf of GH sections of $\mathcal{G}^*M$, is a locally free sheaf of $\mathcal{O}_M$-modules of finite rank k. Define

$$\mathcal{G}\mathbf{M} := \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}^*\mathbf{M}, \mathcal{O}_M)\,. \tag{3.2.2}$$

Since $\mathcal{G}^*\mathbf{M}$ is a locally free sheaf of $\mathcal{O}_M$-modules of rank k, $\mathcal{G}\mathbf{M}$ will also be a locally free sheaf with the same rank. Then, by Proposition 3.2.1, the corresponding SGH vector bundle is defined as

$$\mathcal{G}M := E_{\mathcal{G}\mathbf{M}}\,. \tag{3.2.3}$$

Here $\mathcal{G}M$ as given in Proposition 3.2.1. It is called *generalized holomorphic (GH) tangent bundle*. The GH sections of $\mathcal{G}M$ are called GH vector fields. Since $\mathcal{G}^*M$ is B-transformation invariant, $\mathcal{G}M$ is also invariant under B-transformation. Thus, locally (cf. (2.3.8)), the space of GH vector fields is of the form

$$\mathrm{Span}_{\mathcal{O}_U}\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_k}\}$$

Note that $\mathcal{G}M$ and $\mathcal{G}^*M$ are dual to each other as $\mathcal{O}_M$-modules of their GH sections. But, we can say more. Observe that $C^\infty(\mathcal{G}M) = \mathcal{G}\mathbf{M} \otimes_{\mathcal{O}_M} C^\infty_M$. Then

$$\begin{aligned} C^\infty(\mathcal{G}^*M) &= \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} C^\infty_M \\ &\cong \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}, \mathcal{O}_M) \otimes_{\mathcal{O}_M} C^\infty_M \\ &\cong \mathrm{Hom}_{\mathcal{O}_M \otimes_{\mathcal{O}_M} C^\infty_M}(\mathcal{G}\mathbf{M} \otimes_{\mathcal{O}_M} C^\infty_M\,, \mathcal{O}_M \otimes_{\mathcal{O}_M} C^\infty_M) \\ &\quad \text{(by [30, Proposition 7, Section 5, Chapter II])} \\ &= \mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M), C^\infty_M) \\ &= C^\infty((\mathcal{G}M)^*)\,. \end{aligned} \tag{3.2.4}$$

Here, $(\mathcal{G}M)^*$ is the dual SGH vector bundle of $\mathcal{G}M$. This shows that $\mathcal{G}M$ and $\mathcal{G}^*M$ are also dual to each other as C^∞_M-modules of their smooth sections, that is, they are dual to each other as complex vector bundles over M.

Remark 3.2.1. *We note that the definition of $\mathcal{G}M$, given in [103, p.16], as $\mathcal{G}M := \overline{L} \cap TM \otimes \mathbb{C}$ is flawed as it varies with B-transformations. In other words, it is not always the case that $\mathcal{G}M$ and $e^B(\overline{L}_M) \cap TM \otimes \mathbb{C}$ are same, while $\mathcal{G}^*M = e^B(L_M) \cap T^*M \otimes \mathbb{C}$ for any B-transformation. Therefore, this does not guarantee duality with respect to $\mathcal{G}^*M$.*

3.3 Generalized holomorphic Picard groups

In this section, we introduce the GH Picard group and present a generalized form of the holomorphic short exact sequence in Theorem 3.3.1. Essentially, we focus on SGH vector bundles of real rank 2 and establish a group structure for the set of isomorphism classes of SGH vector bundles of real rank 2 (see Theorem 3.3.2). We follow a similar approach as outlined in [68] for the holomorphic Picard group.

Let M be a smooth manifold. Let $C^{\infty,*}_M$ be the sheaf of smooth $\mathbb{C}^*$-valued functions on M, and let $\{\mathbb{Z}\}$ denote the locally constant sheaf over M whose stalk at each point is $\mathbb{Z}$. Consider the exponential map $\mathbb{C} \longrightarrow \mathbb{C}^*$ defined by $\exp(z) = e^{2\pi i\, z}$. Then for any open set $U \subseteq M$ and $f \in C^\infty_M(U)$, the map exp induces a map, again denoted by exp,

$$\exp : C^\infty_M(U) \longrightarrow C^{\infty,*}_M(U) \tag{3.3.1}$$

defined by $\exp(f)(x) = e^{2\pi i\, f(x)}$ for all $x \in U$. This induces a morphism of sheaves, again denoted by exp,

$$\exp : C^{\infty}_{M} \longrightarrow C^{\infty,*}_{M} \, . \tag{3.3.2}$$

Note that any $k \in \{\mathbb{Z}\}(U)$ is in the kernel of exp, that is, $\{\mathbb{Z}\}(U) \subseteq \ker(\exp)$ where exp is as in (3.3.1). To show the other side, consider $f = w + iv \in C^{\infty}_{M}(U)$ such that

$$\exp(w + iv) = 1 \, .$$

Here $v, w : U \longrightarrow \mathbb{R}$ are smooth maps. Then one observes that, for all $x \in U$,

$$\begin{aligned} & e^{-2\pi v(x)}(\cos 2\pi w(x) + i \sin 2\pi w(x)) = 1 \, , \\ \Longrightarrow \; & e^{-2\pi v(x)}(\cos 2\pi w(x)) = 1 \;\text{ and }\; e^{-2\pi v(x)}(\sin 2\pi w(x)) = 0 \, ; \\ \Longrightarrow \; & \cos 2\pi w(x) > 0 \;\text{ and }\; \sin 2\pi w(x) = 0 \;\; (\text{as } e^{-2\pi v(x)} > 0) \, . \end{aligned}$$

Then there exists a smooth map $g : U \longrightarrow \mathbb{Z} \subset \mathbb{R}$ such that $2w(x) = g(x)$ for all $x \in U$. Thus w is a locally constant function. Now for any $x \in U$, $g(x)$ is even. Thus, for all $x \in U$, $w(x) \in \mathbb{Z}$ and $\cos 2\pi w(x) = 1$. This implies $e^{-2\pi v(x)} = 1$ and so $v(x) = 0$. This shows that $f \in \{\mathbb{Z}\}(U)$, and hence

$$\ker(\exp) = \{\mathbb{Z}\}(U) \, .$$

Thus we get the following exact sequence of sheaves over M,

$$0 \longrightarrow \{\mathbb{Z}\} \hookrightarrow C^{\infty}_{M} \xrightarrow{\exp} C^{\infty,*}_{M} \quad . \tag{3.3.3}$$

To show that exp is a surjective map of sheaves, it is enough to show that for any $x \in M$, the map $C^{\infty}_{M,x} \xrightarrow{\exp} C^{\infty,*}_{M,x}$ is onto. For that, it is enough to show that, for any simply connected open set $U \subset M$, the map exp, in (3.3.1), is onto.

Let $U \subset M$ be a simply connected open set and $g \in C^{\infty,*}_{M}(U)$. Note that the map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$ is a holomorphic covering map and $\mathbb{C}$ is the universal cover. Since U is simply connected, there exists a unique smooth map $f \in C^{\infty}_{M}(U)$ such that the following diagram commutes

$$\begin{array}{ccc} & & \mathbb{C} \\ & \overset{f}{\nearrow} & \\ U & & \Big\downarrow \exp \\ & \underset{g}{\searrow} & \\ & & \mathbb{C}^* \end{array} \quad ; \text{ that is, } g = \exp(f) \, .$$

Thus the map in (3.3.2) is onto. This implies the following

Proposition 3.3.1. *Let M be a smooth manifold. Then we have the following short exact sequence of sheaves*

$$0 \longrightarrow \{\mathbb{Z}\} \hookrightarrow C^\infty_M \xrightarrow{\exp} C^{\infty,*}_M \longrightarrow 0 \ . \tag{3.3.4}$$

where C^∞_M, $C^{\infty,*}_M$ and $\{\mathbb{Z}\}$ are sheaves over M with their usual meanings.

Now, let M be a GC manifold and let $\mathcal{O}^*_M$ be the sheaf of $\mathbb{C}^*$-valued GH functions over M. One can see that $\mathcal{O}^*_M$ is a subsheaf of $\mathcal{O}_M$ which is again subsheaf of C^∞_M. Note that, given any open set $U \subseteq M$ and any smooth map $f : U \longrightarrow \mathbb{C}$, we have $d(\exp(f)) = \exp(f)\, df$. This implies

$$d_L(\exp(f)) = \exp(f)\, d_L f$$

where d_L as in (2.4.11). By Lemma 2.3.1, we can see that

$$f \in \mathcal{O}_M(U) \ \text{ if and only if } \ \exp(f) \in \mathcal{O}^*_M(U)\,.$$

This shows that we can restrict the short exact sequence in Proposition 3.3.1 to $\mathcal{O}_M$ which gives us the following.

Theorem 3.3.1. *Let M be GC manifold. Then we have the following short exact sequence of sheaves over M*

$$0 \longrightarrow \{\mathbb{Z}\} \hookrightarrow \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}^*_M \longrightarrow 0 \ . \tag{3.3.5}$$

3.3.1 GH Picard Groups

Let E be an SGH line bundle over a GC manifold M with local trivializations $\{U_\alpha, \phi_\alpha\}$. The transition functions $\phi_{\alpha\beta}$, as defined in Theorem 3.1.1, are clearly non-vanishing GH functions by Lemma 3.1.3, that is, $\phi_{\alpha\beta} \in \mathcal{O}^*_M(U_\alpha \cap U_\beta)$, and satisfy

$$\begin{aligned} \phi_{\alpha\beta} \cdot \phi_{\beta\alpha} &= 1\,; \\ \phi_{\alpha\beta} \cdot \phi_{\beta\gamma} \cdot \phi_{\gamma\alpha} &= 1\,. \end{aligned} \tag{3.3.6}$$

For any collection of nonzero GH functions $\{g_\alpha \in \mathcal{O}^*_M(U_\alpha)\}$, we can define an alternative trivialization of E over $\{U_\alpha\}$ by

$$\phi'_\alpha = g_\alpha \cdot \phi_\alpha\,;$$

and the corresponding transition functions $\{\phi'_{\alpha\beta}\}$ will then be given by

$$\phi'_{\alpha\beta} = \frac{g_\alpha}{g_\beta} \cdot \phi_{\alpha\beta} \,. \tag{3.3.7}$$

One can see that any other trivialization of E over $\{U_\alpha\}$ can be obtained in this way.

On the other hand, given any collection of GH functions $\{\phi_{\alpha\beta} \in \mathcal{O}^*_M(U_\alpha \cap U_\beta)\}$, satisfying (3.3.6), we can construct an SGH line bundle E with transition functions $\{\phi_{\alpha\beta}\}$ by taking the union of $U_\alpha \times \mathbb{C}$ overall α and identifying $z \times \mathbb{C}$ in $U_\alpha \times \mathbb{C}$ and $U_\beta \times \mathbb{C}$ via multiplication by $\phi_{\alpha\beta}(z)$. Any two such collections of GH function $\{\phi_{\alpha\beta}\,,\, \phi'_{\alpha\beta} \in \mathcal{O}^*_M(U_\alpha \cap U_\beta)\}$, satisfying (3.3.6), define isomorphic SGH line bundles over $\{U_\alpha\}$ if and only if there exists a collection of nonzero GH functions $\{g_\alpha \in \mathcal{O}^*_M(U_\alpha)\}$, satisfying (3.3.7).

Note that the transition functions $\{\phi_{\alpha\beta} \in \mathcal{O}^*_M(U_\alpha \cap U_\beta)\}$ of E over $\{U_\alpha\}$ represents a Čech 1-cochain on M with coefficients in $\mathcal{O}^*_M$ and the relations in (3.3.6) show that $\{\phi_{\alpha\beta}\}$ is indeed a Čech 1-cocycle. Moreover, by the last two paragraphs, we can see that any two cocycles $\{\phi_{\alpha\beta}\}$ and $\{\phi'_{\alpha\beta}\}$ define isomorphic SGH line bundles if and only if $\{\phi_{\alpha\beta} \cdot (\phi'_{\alpha\beta})^{-1}\}$ is a Čech co-boundary. This implies that any SGH bundle isomorphism class of an SGH line bundle over M defines a unique element in $H^1(M, \mathcal{O}^*_M)$ and vice versa.

Consider the set $\mathscr{E}_1$ as defined in Proposition 3.2.1. We can give a group structure, denoted by τ, on $\mathscr{E}_1$ where multiplication is given by tensor product and inverses by dual bundles. Denote the group $(\mathscr{E}_1\,,\,\tau)$ by $\mathcal{G}\,\mathrm{Pic}(M)$, that is,

$$\mathcal{G}\,\mathrm{Pic}(M) := (\mathscr{E}_1\,,\,\tau)\,. \tag{3.3.8}$$

By the last paragraph, we have proved the following.

Theorem 3.3.2. *For any GC manifold M, $\mathcal{G}\,\mathrm{Pic}(M) \cong H^1(M, \mathcal{O}^*_M)$ as groups.*

Definition 3.3.1. *$\mathcal{G}\,\mathrm{Pic}(M)$ is called the generalized holomorphic (GH) Picard Group of M.*

3.4 Induced foliations on GC manifolds

In this section, we discuss the leaf space associated with the regular symplectic foliation $\mathscr{S}$ with a transverse complex structure of a GCS. In general, the leaf space $M/\mathscr{S}$ might lack the Hausdorff property, as illustrated in Example 3.4.2. Nonetheless, assuming $M/\mathscr{S}$ is a smooth orbifold, we provide a structured description of $\mathscr{S}$ in Theorem 3.4.1. For more details on the general theory of orbifolds and related notions, we refer to [2, 8, 15, 16, 31, 46, 75–78, 87–91, 93, 94, 107, 117, 131, 135, 159] and the references therein.

Let $(M, \mathcal{J}_M)$ be a GC manifold with L_M representing its $+i$-eigenbundle of $\mathcal{J}_M$. We consider the natural projection $\rho : (TM \oplus T^*M) \otimes \mathbb{C} \longrightarrow TM \otimes \mathbb{C}$ and $\rho(L_M) = E_M$. Then, $\Delta_M \otimes \mathbb{C} = E_M \cap \overline{E_M}$ forms a real distribution of variable dimension within $TM \otimes \mathbb{C}$. According to Theorem 2.1.3 and Theorem 2.1.4, Δ_M integrates (in the sense of Sussmann, cf. [70, Theorem 3.9]) to a singular symplectic foliation $\mathscr{S}$, implying that $\mathcal{J}_M$ induces a generalized symplectic foliation.

However, around a regular point of type k, as indicated by (2.3.9), Δ_M induces a regular symplectic foliation of real codimension $2k$ with a transverse complex structure (cf. [70, Proposition 3.11-3.12]), that is $\mathscr{S}$ becomes a regular symplectic foliation around a regular point. Particularly, $\omega|_{\Delta_M \otimes \mathbb{C}}$ acts as a symplectic form on $\Delta_M \otimes \mathbb{C} = \ker(\Omega \wedge \overline{\Omega})$, where Ω defines a complex structure transverse to Δ_M. Here, ω and Ω are defined in Definition 2.3.3. With this characterization, the following proposition naturally follows.

Proposition 3.4.1. *([72, Proposition 4.2]) On a regular neighborhood of a GC manifold, the leaf space $M/\mathscr{S}$ of the regular symplectic foliation $\mathscr{S}$ admits a canonical integrable complex structure.*

Proposition 3.4.1 simplifies the understanding of the leaf space around a regular point. Therefore, grasping the induced symplectic foliation on regular GC manifolds becomes more straightforward.

So, consider M to be a regular GC manifold of dimension $2m$ and type k. Let $\mathscr{S}$ denote the associated symplectic foliation of complex codimension k which is transversely holomorphic. Let $T\mathscr{S}$ be the corresponding involutive subbundle of TM of rank $2m-2k$, called the tangent bundle of the foliation. The normal bundle of the foliation, denoted by $\mathcal{N}$, is defined by

$$\mathcal{N} := TM/T\mathscr{S} .$$

By Proposition 3.4.1, $\mathcal{N}$ is an integrable subbundle with a complex structure. Then $\mathcal{N}$ has a decomposition given by the complex structure

$$\mathcal{N} \otimes \mathbb{C} = \mathcal{N}^{1,0} \oplus \mathcal{N}^{0,1} . \tag{3.4.1}$$

As the exact sequence

$$0 \longrightarrow T\mathscr{S} \longrightarrow TM \longrightarrow \mathcal{N} \longrightarrow 0$$

splits smoothly, $\mathcal{N}$ may be regarded as a subbundle of TM complementary to $T\mathscr{S}$, and we may identify $\mathcal{N}^{1,0}$ with $\mathcal{G}M$. Define

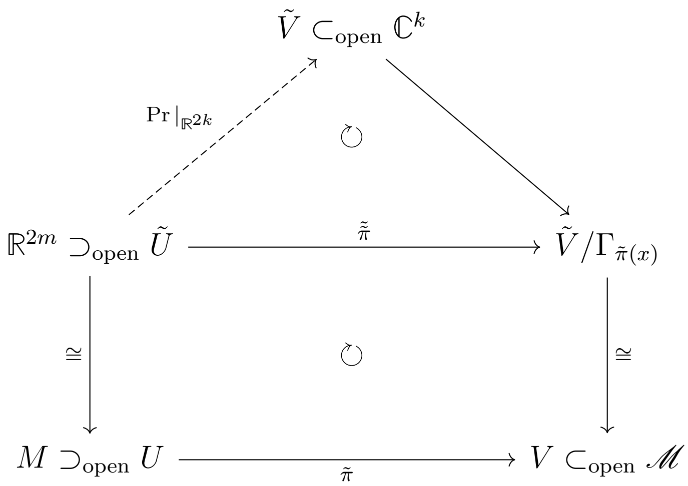
$$\mathscr{M} := M/\mathscr{S} \tag{3.4.2}$$

to be the leaf space of the foliation $\mathscr{S}$. This is a topological space that has the quotient topology induced by the quotient map

$$\tilde{\pi} : M \longrightarrow \mathscr{M}\,. \tag{3.4.3}$$

The map $\tilde{\pi}$ is open (cf. [117, Section 2.4]).

In general, $\mathscr{M}$ could be rather wild. To have a reasonable theory, we assume that $\mathscr{M}$ admits a smooth orbifold structure. Since $\mathscr{S}$ is transversely holomorphic, $\mathscr{M}$ then becomes a complex orbifold. Moreover, observe that $\tilde{\pi}$ is a smooth complete orbifold map (cf. [27, Definition 3.1]): Namely, for any point $x \in M$ and $\tilde{\pi}(x) \in \mathscr{M}$, there exist orbifold charts $\tilde{U}$ and $(\tilde{V}, \Gamma_{\tilde{\pi}(x)})$ corresponding to x and $\tilde{\pi}(x)$, respectively, such that the following diagram commutes.



Here $\Gamma_{\tilde{\pi}(x)}$ is the isotropy group corresponding to $\tilde{\pi}(x)$. The rows of the commutative diagram are diffeomorphisms of smooth orbifolds and $\tilde{\tilde{\pi}}$ is the lift of $\tilde{\pi}$. One can see from this diagram that each point $y \in \mathscr{M}$ is a regular value of $\tilde{\pi}$. Thus, by the preimage theorem for orbifolds (cf. [27, Theorem 4.2]), $\tilde{\pi}^{-1}(y)$ is an embedded submanifold of real dimension $2m - 2k$ for all $y \in \mathscr{M}$. Hence, each leaf is not only an immersed but also a closed embedded submanifold of M.

Definition 3.4.1. *An open set in M is called a transverse open set if it is a union of leaves. An open cover $\mathcal{U} = \{U_\alpha\}$ is called a transverse open cover of M if each U_α is a transverse open subset of M.*

Let S be a leaf of $\mathscr{S}$. By the Tubular Neighborhood Theorem, there exists a transverse neighborhood (tubular neighborhood) of S which is diffeomorphic to the normal bundle $\mathcal{N}_S$ of S. One can see that $\mathcal{N}_S$ is just the pullback of $\mathcal{N}$ via the inclusion map $S \hookrightarrow M$. Due to the transverse complex structure, $\mathcal{N}$, as well as $\mathcal{N}_S$, can be thought of as a complex vector bundle of complex rank k. Consider the partial connection, known as the Bott connection (cf. [29, Section 6]), on $\mathcal{N}$ which is flat along the leaves. Then its pullback on $\mathcal{N}_S$ gives a flat connection. Thus, considering $\mathcal{N}_S$ as a complex vector bundle, by [95, Proposition 1.2.5]

$$\mathcal{N}_S \cong \tilde{S} \times_\rho \mathbb{C}^k$$

where $\rho : \pi_1(S) \longrightarrow GL_k(\mathbb{C})$ is the linear holonomy representation of $\pi_1(S)$ and $\tilde{S}$ is the universal cover of S.

Definition 3.4.2. *A $2k$-dimensional embedded submanifold of M is called a transversal section if it is transversal to the leaves of $\mathscr{S}$.*

Note that by [117, Proposition 2.20], $\mathscr{M}$ admits a Riemannian metric which makes $\mathscr{S}$ into a Riemannian foliation. Since S is an embedded submanifold, $T \cap S$ is discrete for any transversal section T. Then, following the proof of [117, Theorem 2.6], one can show that the holonomy group of S, $\mathrm{Hol}(S)$ is finite. By the differentiable slice theorem, we can indeed assume that the action of $\mathrm{Hol}(S)$ on T is linear, that is,

$$\mathrm{Hol}(S) = \mathrm{img}(\rho)\,.$$

We summarise our observations as follows.

Theorem 3.4.1. *Let M be a regular GC manifold and let $\mathscr{S}$ be the induced symplectic foliation. Assume that $M/\mathscr{S}$ has a smooth orbifold structure. Then, we have the following.*

1. *Each leaf of $\mathscr{S}$ is an embedded closed submanifold of M.*

2. *The holonomy group of each leaf is finite.*

3. *$(M, \mathscr{S})$ is a regular Riemannian foliation.*

4. *Around each leaf S, there exists a tubular neighborhood U such that U is diffeomorphic to $\tilde{S} \times_{\mathrm{Hol}(S)} \mathbb{C}^k$ where $\tilde{S}$ is the universal cover of S and $\mathrm{Hol}(S)$ is the holonomy group of S. Here, $\mathrm{Hol}(S)$ acts on $\mathbb{C}^k$ via a linear holonomy representation.*

Example 3.4.1. *Let F be a symplectic manifold and $\tilde{F}$ be its universal cover. Then as in Example 3.1.2, $\tilde{F} \times_\rho \mathbb{C}^l$ is a regular GC manifold of type l. The induced symplectic foliation $\mathscr{S}$ is the foliation of F-parameter submanifolds, that is, sets of the form*

$$S_x = \left\{[\tilde{m}, y] \,|\, \tilde{m} \in \tilde{F}, y \in [x]\right\} \quad \textit{where} \quad [x] := \{\rho(g) \cdot x \,|\, g \in \pi_1(F)\} \subset \mathbb{C}^l \,.$$

This implies that the leaf space $\tilde{F} \times_\rho \mathbb{C}^l / \mathscr{S}$ is exactly

$$\mathbb{C}^l/\rho := \{[x] \,|\, x \in \mathbb{C}^l\} \,.$$

The isotropy group at 0 is $\mathrm{img}(\rho)$ which is the linear holonomy group. Therefore, we get that $\tilde{F} \times_\rho \mathbb{C}^l / \mathscr{S}$ is a smooth orbifold if and only if the linear holonomy group is finite.

Remark 3.4.1. *It is tempting to think that the leaf space of a regular GCS is either manifold or an orbifold. But, it may not be even Hausdorff. The following example demonstrates this.*

Example 3.4.2. *Consider the product GCS on $M \times F$ where M is a complex manifold and F is a symplectic manifold. Let $N \subset F$ be a closed submanifold such that $F \backslash N$ is disconnected. Fix $m \in M$, Consider the open submanifold*

$$X_m = M \times F \backslash \{m \times N\} \,.$$

Consider the natural regular GCS on X_m induced from $M \times F$. Let $(x, f) \in X_m$. Then, the leaf of the induced foliation $\mathscr{S}_m$, through (x, f), is of the following form

$$S_{(x,f)} = \begin{cases} F & \textit{if } x \neq m \,, \\ (F \backslash N)_\alpha & \textit{if } x = m \,, \end{cases}$$

where $(F \backslash N)_\alpha$ denotes the connected component of $F \backslash N$ that contains f for $x = m$. One can see that the leaf space $X_m / \mathscr{S}_m$ is not Hausdroff. Thus, we obtain an infinite family of regular GC manifolds with non-Hausdorff leaf space.

3.4.1 Strong generalized Calabi-Yau manifolds and its leaf spaces

In this subsection, we give some criteria on the GCS so that the leaf space of the associated symplectic foliation is a smooth torus, and therefore, satisfies the hypothesis that the leaf space be an orbifold, used in most of our results in this manuscript. This is a generalization of a result of Bailey et al. [12, Theorem1.9].

Let M^{2n} be a GC manifold with $+i$-eigenbundle L of $(TM \oplus T^*M) \otimes \mathbb{C}$. Consider the bundle $\bigwedge^\bullet T^*M \otimes \mathbb{C}$ as a spinor bundle for $(TM \oplus T^*M) \otimes \mathbb{C}$ with the following Clifford action

$$(X + \eta) \cdot \rho = i_X(\rho) + \eta \wedge \rho \quad \text{for } X + \eta \in (TM \oplus T^*M) \otimes \mathbb{C}\,.$$

Then, there exits a unique line subbundle U_M of $\bigwedge^\bullet T^*M \otimes \mathbb{C}$, called the *canonical line bundle* associated to the GCS, which is annihilated by L under the above Clifford action.

At each point of M, U_M is generated by a

$$\rho = e^{B+i\omega}\Omega\,,$$

where B, ω are real 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ is a complex decomposable k-form where k is the type of the GCS at that point. By [70, 72], the condition $L \cap \overline{L} = \{0\}$ is equivalent to the non-degeneracy condition

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0\,. \tag{3.4.4}$$

The involutivity of L, with respect to the Courant bracket, is equivalent to the following condition on any local trivialization ρ of U_M,

$$d\rho = (X + \eta) \cdot \rho\,,$$

for some $X + \eta \in C^\infty((TM \oplus T^*M) \otimes \mathbb{C})$.

Definition 3.4.3. *A GC manifold M of type k is said to be a generalized Calabi-Yau manifold if its canonical bundle U_M is a trivial bundle admitting a nowhere-vanishing global section ρ such that $d\rho = 0$ (cf. [70, 72]). M is called a strong generalized Calabi-Yau manifold if, in addition, ρ is such that $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ is globally decomposable and $d\theta_j = 0$ for $1 \leq j \leq k$.*

Remark 3.4.2. *Note that any generalized Calabi-Yau manifold is orientable because we get a global nowhere-vanishing volume form* $\omega^{n-k} \wedge \Omega \wedge \overline{\Omega}$.

Example 3.4.3.

1. *Any type* 1 *generalized Calabi-Yau manifold is a strong generalized Calabi-Yau manifold.*

2. *Any* 6*-dimensional nilmanifold with* $(b_1, b_2) \in \{(4,6),(4,8),(5,9),(5,11),(6,15)\}$ *admits a type* 2 *strong generalized Calabi-Yau structure (cf. [38, Table 1]) where* b_1 *and* b_2 *are the first and second betti numbers, respectively.*

Let M^{2n} be a compact connected strong generalized Calabi-Yau manifold of type k. Under some assumptions on the leaves of the induced foliation, we show that the foliation is simple. To show this, we need to use an extended version of the techniques used in [12, Section 1.2].

Let ρ be a nowhere-vanishing closed section of the corresponding canonical line bundle U_M. We can express ρ in the following form

$$\rho = e^{B+i\omega} \wedge \Omega$$

where B, ω are real 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ is a complex decomposable k-form with $d\theta_j = 0$ for $1 \leq j \leq k$. Fix $p \in \{1, \ldots, k\}$. Let $\theta_p = \operatorname{Re}(\theta_p) + i \operatorname{Im}(\theta_p)$ where $\operatorname{Re}(\theta_p)$ and $\operatorname{Im}(\theta_p)$ denote the real and imaginary parts of θ_p, respectively. First, we show that $[\operatorname{Re}(\theta_p)]$ and $[\operatorname{Im}(\theta_p)]$ are linearly independent in $H^1_{dR}(M, \mathbb{R})$.

If possible, let there exist a nontrivial linear combination, say

$$\lambda_R[\operatorname{Re}(\theta_p)] + \lambda_I[\operatorname{Im}(\theta_p)] = 0\,.$$

Fix $m_0 \in M$ and define the map $f : M \longrightarrow \mathbb{R}$ as

$$f(m) = \int_{[m_0,m]} (\lambda_R \operatorname{Re}(\theta_p) + \lambda_I \operatorname{Im}(\theta_p))\,,$$

where integral is taken over any path connecting m_0 to $m \in M$. The function f is well-defined as $\lambda_R \operatorname{Re}(\theta_p) + \lambda_I \operatorname{Im}(\theta_p)$ is exact. Without loss of generality, let $\lambda_R \neq 0$. Note that $\theta_p \wedge \overline{\theta_p} = -2i \operatorname{Re}(\theta_p) \wedge \operatorname{Im}(\theta_p)$. Then the non-degeneracy condition

$$\omega^{n-k} \wedge \left(\bigwedge_{j=1}^{k} \theta_j \wedge \overline{\theta_j} \right) \neq 0$$

implies that

$$\omega^{n-k} \wedge \left(\bigwedge_{\substack{j=1 \\ j \neq p}}^{k} \theta_j \wedge \overline{\theta_j} \right) \wedge df \neq 0 \, .$$

This shows that df is nowhere vanishing and so, f is a submersion. Therefore, $f(M)$ is open. However, $f(M)$ is also closed since M is compact. Thus $f(M) = \mathbb{R}$ which is a contradiction. Hence $[\mathrm{Re}(\theta_p)]$ and $[\mathrm{Im}(\theta_p)]$ are linearly independent.

Now, the non-degeneracy condition (3.4.4) is an open condition that gives us the freedom to choose $\theta_j \in \Omega^1(M, \mathbb{C})$ $(1 \leq j \leq k)$ such that $[\mathrm{Re}(\theta_j)]$ and $[\mathrm{Im}(\theta_j)]$ are still linearly independent in $H^1(M, \mathbb{Q})$. Then, we can consider the map

$$\tilde{f} : M \longrightarrow \mathbb{C}^k/\Gamma \cong \prod_j \mathbb{T}^2 \quad \text{defined as} \quad \tilde{f}(m) = \bigoplus_j \int_{[m_0, m]} \theta_j \, ,$$

where the integral is taken over any path connecting m_0 to m and

$$\Gamma = \oplus_j [\theta_j](H_1(M, \mathbb{Z}))$$

is a co-compact lattice in $\mathbb{C}^k$. As before, using the non-degeneracy condition, one can show that $\tilde{f}$ is a surjective submersion.

Suppose S is a leaf of the induced foliation $\mathscr{S}$ which is closed. Then S is a compact embedded submanifold in M. Let X_j be a complex vector field on M such that

$$\theta_l(X_j) = \delta_{lj} \quad \text{and} \quad \overline{\theta_l}(X_j) = 0 \, ,$$

where δ_{lj} is Kronecker delta and $l, j \in \{1, \ldots, k\}$. This is possible since the normal bundle $\mathcal{N}$ of the foliation is trivial and the transverse holomorphic structure induces an integrable complex structure on $\mathcal{N}$ so that $C^\infty(\mathcal{N}^{1,0*}) = \langle \theta_j \,|\, j = 1, \ldots, k \rangle$ and $C^\infty(\mathcal{N}^{0,1*}) = \langle \overline{\theta_j} \,|\, j = 1, \ldots, k \rangle$ where $\mathcal{N} \otimes \mathbb{C} = \mathcal{N}^{1,0} \oplus \mathcal{N}^{0,1}$ as defined in (3.4.1). Let $\mathrm{Re}(X_j)$ and $\mathrm{Im}(X_j)$ be the real and imaginary parts of X_j, respectively. Note that $\mathrm{Re}(X_j)$ and $\mathrm{Im}(X_j)$ are pointwise linearly independent and $\mathcal{L}_Y \Omega = \mathcal{L}_Y \overline{\Omega} = 0$ where $Y \in \{\mathrm{Re}(X_j), \mathrm{Im}(X_j) : 1 \leq j \leq k\}$. Therefore, these vector fields preserve the foliation $\mathscr{S}$ which is determined by $\ker(\Omega \wedge \overline{\Omega})$.

Consider the map

$$\psi_S : S \times \mathbb{R}^{2k} \longrightarrow M \, ,$$

defined by

$$\psi_S(s, \lambda_1, \ldots, \lambda_{2k}) = \exp \left(\sum_{j=1}^{k} \left(\lambda_{2j-1} \, \mathrm{Re}(X_j) + \lambda_{2j} \, \mathrm{Im}(X_j) \right) \right) (s) \, .$$

ψ_S is a local diffeomorphism as $(\psi_S)_*(TS \oplus \mathbb{R}^{2k}) = TM$. Since $\mathrm{Re}(X_j)$ and $\mathrm{Im}(X_j)$ preserve the foliation $\mathscr{S}$, $\sum_{j=1}^{k}\Big(\lambda_{2j-1}\,\mathrm{Re}(X_j) + \lambda_{2j}\,\mathrm{Im}(X_j)\Big)$ also preserves $\mathscr{S}$. We conclude that all leaves in a neighborhood of S are diffeomorphic to S. More precisely, since S is compact, ψ_S provides a leaf preserving local diffeomorphism between a tubular neighborhood of S in M and $S \times \prod_j \mathbb{D}^2$. Here $\mathbb{D}^2 \subset \mathbb{R}^2$ is an open disk.

Let V be the set of points in M that lie in leaves that are diffeomorphic to S. Then V is an open subset of M. Let $q \in \overline{V}$. Let $\alpha : \mathbb{D}^{2n-2k} \longrightarrow M$ be a local parametrization of the leaf through q such that $\alpha(0) = q$. Then, the map $\psi : \mathbb{D}^{2n-2k} \times \prod_j \mathbb{D}^2 \longrightarrow M$ defined by

$$\psi(s, \lambda_1, \dots, \lambda_{2k}) = \exp\left(\sum_{j=1}^{k}\Big(\lambda_{2j-1}\,\mathrm{Re}(X_j) + \lambda_{2j}\,\mathrm{Im}(X_j)\Big)\right)(\alpha(s))$$

is a again a leaf preserving local diffeomorphism. Here, $\mathbb{D}^{2n-2k} \times \prod_j \mathbb{D}^2$ is considering with the product GCS. So, $\mathrm{img}(\psi) \cap V \neq \emptyset$. Let $q^{'} \in \mathrm{img}(\psi) \cap V$ and $S^{'}$ be the compact leaf through it. For some $(s, \lambda_1, \dots, \lambda_{2k}) \in \mathbb{D}^{2n-2k} \times \prod_j \mathbb{D}^2$, we have

$$q^{'} = \psi(s, \lambda_1, \dots, \lambda_{2k})\,.$$

By taking the inverse of exp, we can shows that $\psi_{S'}(q^{'}, -\lambda_1, \dots, -\lambda_{2k}) \in \mathrm{img}(\alpha)$. Therefore, $\mathrm{img}(\alpha) \cap \mathrm{img}(\psi_{S'}) \neq \emptyset$. Thus, $S^{'}$ is diffeomorphic to the leaf through q via $\psi_{S'}$. This is because, α is a local parametrization of the leaf through q, and $\psi_{S'}(q^{'}, -\lambda_1, \dots, -\lambda_{2k})$ lies in a leaf which is diffeomorphic to $S^{'}$ and it is also in $\mathrm{img}(\alpha)$. This implies that the leaf through q must be diffeomorphic to $S^{'}$, since any two leaves are either disjoint or identical. Hence $q \in V$.

Since V is both open and closed and M is connected, we have $V = M$. This conclude that M is a fibration (fibre bundle) $M \longrightarrow B$ over a compact connected $2k$-dimensional smooth manifold. Now, θ_j $(1 \leq j \leq k)$ vanishes when restricted to a leaf by [72, Corollary 2.8]. Since θ_j is also closed, it is basic for this fibration, that is, B has $2k$-number of linearly independent nowhere-vanishing closed real 1-forms.

Proposition 3.4.2. *Let B be any smooth compact connected $2k$-dimensional manifold. Suppose B has $2k$ linearly independent nowhere-vanishing closed real 1-forms. Then B is diffeomorphic to a product of 2-dimensional tori $\prod_{j=1}^{k} \mathbb{T}^2$.*

Proof. Let $\{\theta_1, \dots, \theta_{2k}\}$ be linearly independent nowhere-vanishing closed 1-forms on B. Note that $\wedge_j \theta_j$ is a volume form for B, which is an open condition. So, we can choose

θ_j's such that θ_j's are linearly independent in $H^1(B, \mathbb{Q})$. Fix $m_0 \in B$ and consider the following map

$$\phi : B \longrightarrow \mathbb{R}^{2k}/\Gamma \cong \prod_{j=1}^{k} \mathbb{T}^2 \quad \text{defined as} \quad \phi(m) = \bigoplus_{j=1}^{2k} \int_{[m_0,m]} \theta_j \,,$$

where integral is taken over any path connecting m_0 to m. Let $\Gamma = \oplus_{j=1}^{2k}[\theta_j](H_1(B,\mathbb{Z}))$. Then Γ is a co-compact lattice in $\mathbb{R}^{2k}$. One can easily see that

$$\left(\bigwedge_{\substack{j=1 \\ j \neq p}}^{2k} \theta_j \right) \wedge d\phi_p \neq 0 \,,$$

where $\phi_p : B \longrightarrow \mathbb{R}/[\theta_p](H_1(B,\mathbb{Z})) \cong S^1$ is the natural projection of ϕ onto the p-th component. This implies that ϕ_j $(1 \leq j \leq 2k)$ is a submersion. Hence, ϕ is a submersion.

It follows that ϕ is a local diffeomorphism. Since B is compact, ϕ is a proper map. Therefore, $\phi : B \longrightarrow \prod_{j=1}^{k} \mathbb{T}^2$ is a covering map and it induces an injective map

$$\pi_1(\phi) : \pi_1(B) \longrightarrow \pi_1(\prod_{j=1}^{k} \mathbb{T}^2) \cong \bigoplus_{2k} \mathbb{Z} \,.$$

Then $\pi_1(B) \cong \bigoplus_l \mathbb{Z}$ for some $l \leq 2k$. Using the de Rham isomorphism and the universal coefficient theorem, we have

$$H^1_{dR}(B,\mathbb{R}) \cong \mathrm{Hom}(H_1(B,\mathbb{R}),\mathbb{R}) \quad \text{and} \quad H_1(B,\mathbb{R}) \cong H_1(B,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}\,, \ \text{respectively}\,.$$

Since the rank of $H^1_{dR}(B,\mathbb{R})$ is $2k$, $\mathrm{Rank}(H_1(B,\mathbb{Z})) = 2k$. As $\pi_1(B) \cong H_1(B,\mathbb{Z})$ (since $\pi_1(B)$ is abelian), we get

$$\pi_1(B) \cong \bigoplus_{2k} \mathbb{Z} \,.$$

So, there exists a smooth covering map $\tilde{\phi} : \mathbb{R}^{2k} \longrightarrow B$, such that B is diffeomorphic to $\mathbb{R}^{2k}/\pi_1(B) \cong \prod_{j=1}^{k} \mathbb{T}^2$. □

We have proved the following result.

Theorem 3.4.2. *Let M be a compact connected strong generalized Calabi-Yau manifold of type k. Let $\mathscr{S}$ be the induced foliation. Then, the following statements hold.*

1. *There exists a smooth surjective submersion $\tilde{f} : M \longrightarrow \prod_{j=1}^{k} \mathbb{T}^2$.*

2. *Suppose $\mathscr{S}$ has a closed leaf. Then, we have:*

(a) All leaves are diffeomorphic and compact. Their holonomy group is trivial.

(b) The leaf space $M/\mathscr{S}$ is a smooth manifold.

(c) The submersion $\tilde{f}$ can be chosen so that the components of the fibers of $\tilde{f}$ are the symplectic leaves of $\mathscr{S}$.

3.4.2 Nilpotent Lie groups, nilmanifolds, and SGH bundles

In this subsection, we give a complete characterization of the leaf space of a left invariant GCS on a simply connected nilpotent Lie group and its associated nilmanifolds. Finally, we construct some examples of nontrivial SGH bundles on the Iwasawa manifolds which show that the category of SGH bundles is in general different from the category of holomorphic bundles on the leaf space.

Let G^{2n} be a simply connected nilpotent Lie group and $\mathfrak{g}$ be its real lie algebra. Suppose G has a left-invariant GCS. Since G is diffeomorphic to $\mathfrak{g}$ via the exponential map, any left-invariant GCS is regular of constant type, say k . The canonical line bundle, corresponding to a left-invariant GCS, is trivial as G is contractible. So, we can choose a global trivialization of the form

$$\rho = e^{B+i\omega} \wedge \Omega \,, \tag{3.4.5}$$

where B, ω are real left invariant 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ is a complex decomposable k-form with left invariant complex 1-forms θ_j $(1 \leq j \leq k)$.

Let $\mathscr{S}$ be the induced foliation and $\mathcal{N}$ be its normal bundle. Then we know that $T\mathscr{S} = \ker(\Omega \wedge \overline{\Omega})$. Using [3, Theorem 4], the left-invariant GCS corresponds to a real Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{s} \cong T_{id}\mathscr{S}$ where $id \in G$ is the identity element. Since any (simply connected) nilpotent Lie group is solvable, $S = \exp(\mathfrak{s})$ is a closed simply connected Lie subgroup of G by [48, Section II]. Note that, by the closed subgroup theorem, S is an embedded submanifold of G and $T_{id}S = T_{id}\mathscr{S}$. Note that

$$TG \cong G \times \mathfrak{g} \quad \text{and} \quad T\mathscr{S} \cong G \times \mathfrak{s} \,.$$

Thus any leaf of $\mathscr{S}$ is diffeomorphic to S via the left multiplication map. This implies that G is foliated by the left cosets of S, that is, the leaf space $G/\mathscr{S}$ is G/S . Since S is closed,

G/S is a smooth homogeneous manifold such that the quotient map $\pi_S : G \longrightarrow G/S$ is a smooth submersion.

Contractibily of G also implies that $\mathcal{N} \cong G \times \mathbb{R}^{2k}$. Let $\langle\, ,\rangle$ be an inner product on $\mathbb{R}^{2k}$. Consider the metric $\langle\, ,\rangle'$ on $G \times \mathbb{R}^{2k}$ defined as

$$\langle (g,v),(h,w)\rangle' = \langle v,w\rangle \quad \text{for all } (g,v),(h,w) \in G \times \mathbb{R}^{2k}.$$

Note that $\langle\, ,\rangle'$ is G-invariant. Then there exists a left-invariant metric, say h on $\mathcal{N}$ such that $(\mathcal{N}, h)$ is isometric to $(G \times \mathbb{R}^{2k}, \langle\, ,\rangle')$. This h is a left-invariant transverse metric on G.

Hence, we have established the following result.

Theorem 3.4.3. *Let G be a simply connected nilpotent Lie group with $\mathfrak{g}$ as its real lie algebra. Suppose G has a left-invariant GCS. Let $\mathscr{S}$ be the foliation induced by the GCS. Then, the following hold.*

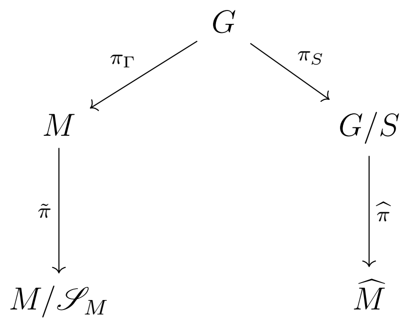
1. *All leaves of $\mathscr{S}$ are diffeomorphic to the leaf through the identity element of G.*
2. *$\mathscr{S}$ is a Riemannian foliation. In particular, G admits a transverse left-invariant metric.*
3. *G is foliated by the left cosets of S where $S \subset G$ is a closed simply connected Lie subgroup. The leaf space $G/\mathscr{S}$ is the homogeneous manifold G/S.*

Let $\Gamma \subset G$ be a maximal lattice (that is, cocompact, discrete subgroup). Malcev (cf. [112]) showed that such a lattice exists if and only if $\mathfrak{g}$ has rational structure constants in some basis. Let $M^{2n} = \Gamma\backslash G$ be a nilmanifold with a left-invariant GCS. Using [38, Theorem 3.1], we can say that this left-invariant GCS is generalized Calabi-Yau. This GCS on M is induced from a left-invariant GCS on G. Let ρ be a global trivialization for the canonical line bundle of the left-invariant GCS on G as defined in (3.4.5). It also induces a global trivialization for the canonical line bundle of the left-invariant GCS on M. Let $\mathscr{S}_M$ be the induced foliation corresponding to this GCS on M and S_M be the leaf through the coset $\Gamma \in M$. Note that, since $\mathscr{S}$ is Γ-invariant, $\mathscr{S}_M$ is just induced by $\mathscr{S}$, that is, $\mathscr{S}_M = \Gamma\backslash\mathscr{S}$.

Now, the quotient map $\pi_\Gamma : G \longrightarrow M$ is a principal Γ-bundle as well as a covering map. It induces a principal $(S\cap\Gamma)$-bundle $\pi_\Gamma|_S : S \longrightarrow S_M$ and so the fundamental group

$\pi_1(S_M) = S \cap \Gamma$. Therefore, we can identify $S_M = (S \cap \Gamma)\backslash S$. Note that $\pi_\Gamma^{-1}(S_M) = \Gamma S$. By [132, Theorem 1.13], $S \cap \Gamma$ is a maximal lattice in S if and only if ΓS is closed. Thus, $S_M = (S \cap \Gamma)\backslash S \subset M$ is a compact leaf if and only if ΓS is closed. The transverse left-invariant metric on G (see Theorem 3.4.3), is preserved by Γ-action. Therefore, it induces a transverse metric on M. This implies that $\mathscr{S}_M$ is a Riemannian foliation.

Consider the natural left Γ-action on G/S defined as $\eta \cdot gS = (\eta g)S$ and its quotient space $\widehat{M} := \Gamma\backslash G/S$ with the quotient topology such that $\widehat{\pi} : G/S \longrightarrow \widehat{M}$ is continuous. Note that this map is also open. So, we have the following diagram,

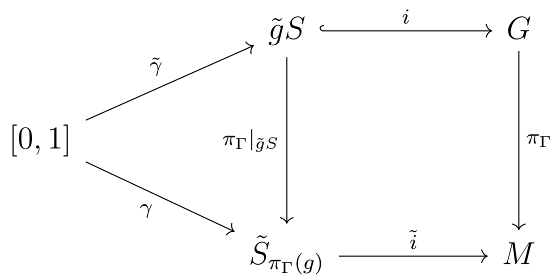


where $M/\mathscr{S}_M$ is the leaf space and $\tilde{\pi}$ is the quotient map as defined in (3.4.3). We will use $\tilde{S}_x$ to denote the leaf through $x \in M$. Let $g \in G$ and consider the map

$$\Phi : M/\mathscr{S}_M \longrightarrow \widehat{M} \quad \text{defined as} \quad \Phi(\tilde{S}_{\pi_\Gamma(g)}) = \widehat{\pi}(\pi_S(g))\,. \tag{3.4.6}$$

Let $g, g' \in G$ such that $\pi_\Gamma(g)$ and $\pi_\Gamma(g')$ are in the same leaf, that is, $\tilde{S}_{\pi_\Gamma(g)} = \tilde{S}_{\pi_\Gamma(g')}$. To show Φ is well-defined, we need to show that $\widehat{\pi}(\pi_S(g)) = \widehat{\pi}(\pi_S(g'))$.

Let $\gamma : [0,1] \longrightarrow \tilde{S}_{\pi_\Gamma(g)}$ be a path such that $\gamma(0) = \pi_\Gamma(g)$ and $\gamma(1) = \pi_\Gamma(g')$. Since G is the universal cover of M, the path γ lifts to a unique path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = g$ and $\tilde{\gamma}(1) = g''$ with $\pi_\Gamma(g') = \pi_\Gamma(g'')$. Now the path $\tilde{\gamma}$ is contained in one of the connected components of $\pi_\Gamma^{-1}(\tilde{S}_{\pi_\Gamma(g)})$, which is a leaf of $\mathscr{S}$, say, $\tilde{g}S$ for some $\tilde{g} \in G$, such that $\pi_\Gamma|_{\tilde{g}S} : \tilde{g}S \longrightarrow \tilde{S}_{\pi_\Gamma(g)}$ is a universal covering map. So, we get the following commutative diagram,



where i is inclusion and $\tilde{i}$ is injective immersion. Since $g, g'' \in \tilde{g}S$, we have $\pi_S(g) = \pi_S(g'')$. Now g' and g'' are in the same fiber of the universal covering map π_Γ, and as

$\pi_1(M) = \Gamma$, there exists $\eta \in \Gamma$ such that $\eta \cdot g^{''} = g^{'}$. Note that Γ preserves the foliation $\mathscr{S}$ and so, $\pi_S(\eta \cdot g^{''}) = \eta \cdot \pi_S(g^{''})$. Therefore, we can see that $\pi_S(g^{'}) = \eta \cdot \pi_S(g)$ which implies that $\widehat{\pi}(\pi_S(g)) = \widehat{\pi}(\pi_S(g^{'}))$. Hence, the map Φ is well defined.

Now define the inverse of Φ as

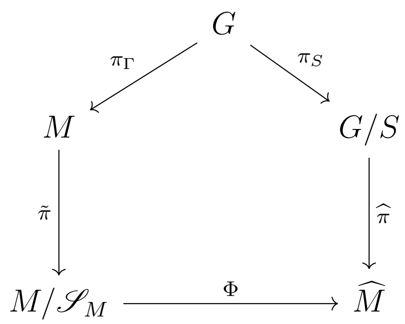
$$\Phi^{-1}\Big(\widehat{\pi}(\pi_S(g))\Big) = \tilde{S}_{\pi_\Gamma(g)} \,.$$

We need to show that Φ^{-1} is well-defined. For that, let $g, g^{'} \in G$ with the condition that $\widehat{\pi}(\pi_S(g)) = \widehat{\pi}(\pi_S(g^{'}))$. It is enough to show that $\tilde{S}_{\pi_\Gamma(g)} = \tilde{S}_{\pi_\Gamma(g^{'})}$. The given condition on $\pi_S(g)$ and $\pi_S(g^{'})$ implies that there exists $\eta \in \Gamma$ such that

$$\pi_S(g) = \eta \cdot \pi_S(g^{'}) = \pi_S(\eta \cdot g^{'}) \,.$$

Then, there exists $\tilde{g} \in G$ such that $g, \eta \cdot g^{'} \in \tilde{g}S$, that is, they belong to the same leaf of $\mathscr{S}$. This implies that $\pi_\Gamma(g) = \pi_\Gamma(\eta \cdot g^{'}) = \pi_\Gamma(g^{'})$. Hence $\tilde{S}_{\pi_\Gamma(g)} = \tilde{S}_{\pi_\Gamma(g^{'})}$. So Φ^{-1} is well-defined.

Note that $\pi_\Gamma, \tilde{\pi}, \pi_S$ and $\widehat{\pi}$ are open maps and we have the following commutative diagram:



This implies both Φ and Φ^{-1} are continuous and so, Φ is a homeomorphism.

Let $x, y \in \widehat{M}$. There exist $g, g^{'} \in G$ such that $\widehat{\pi}^{-1}(x) = \Gamma g S$ and $\widehat{\pi}^{-1}(y) = \Gamma g^{'} S$. Now the map $\Gamma g S \longrightarrow \Gamma g^{'} S$ defined as $\eta g s \longmapsto \eta g^{'} s$ is a diffeomorphism. In particular, both orbits are diffeomorphic to ΓS. Suppose ΓS is closed. Set

$$\ker(\widehat{\pi}) := \Big\{(gS, g^{'}S) \,|\, \widehat{\pi}(gS) = \widehat{\pi}(g^{'}S)\Big\} \subset G/S \times G/S \,.$$

To show $\widehat{M}$ is Hausdroff, it is enough to show that $\ker(\widehat{\pi})$ is closed, because $\widehat{\pi}$ is an open surjection.

Let $\{(g_n^1 S, g_n^2 S)\}_n \in \ker(\widehat{\pi})$ be a sequence converging to $(g^1 S, g^2 S)$. Then $\{g_n^j S\}_n$ is converging to $g^j S$ for $j = 1, 2$. By the assumption on $\{g_n^j S\}_n$ $(j = 1, 2)$, they belong to the same Γ-orbit, in other words, there exist $\tilde{g}$ such that $g_n^j S \in \Gamma \tilde{g} S$ $(j = 1, 2)$. Since any two Γ-orbits are diffeomorphic, and ΓS is closed, $\Gamma\tilde{g}S$ is also closed. This implies that $g^j S \in \Gamma\tilde{g}S$ for $j = 1, 2$. Therefore, $\widehat{\pi}(g^1 S) = \widehat{\pi}(g^2 S)$. This implies $(g^1 S, g^2 S) \in \ker(\widehat{\pi})$ and $\ker(\widehat{\pi})$ is closed. Hence, $M/\mathscr{S}_M$ is Hausdroff. So, each leaf is closed as well as compact in M. Since $\mathscr{S}_M$ is a Riemannian foliation, the holonomy group of any leaf is finite, and $M/\mathscr{S}_M$ is a smooth orbifold. Hence we have proved the following.

Theorem 3.4.4. *Let $M = \Gamma\backslash G$ be a nilmanifold with a left-invariant GCS. Let $\mathscr{S}_M$ be the induced foliation. Then, the following hold.*

1. *$\mathscr{S}_M$ is a Riemannian foliation.*

2. *$M/\mathscr{S}_M$ is homeomorphic to $\Gamma\backslash G/S$ where $S \subset G$ is a closed simply connected Lie subgroup.*

3. *$M/\mathscr{S}_M$ is a compact smooth orbifold $\iff$ ΓS is closed $\iff$ $(S\cap\Gamma)\backslash S$ is compact.*

Example 3.4.4. *Consider the complex Heisenberg group*

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \Bigg| \, z_j \in \mathbb{C}\,(j = 1, 2, 3) \right\}.$$

Here z_j $(j = 1, 2, 3)$ is a holomorphic co-ordinate of $\mathbb{C}^3 = \{(z_1, z_2, z_3)\}$. G is a 6-dimensional simply connected nilpotent lie group. Consider a maximal lattice $\Gamma \subset G$ defined as

$$\Gamma = \left\{ \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \Bigg| \, a_j \in \mathbb{Z} \oplus i\mathbb{Z}\,(j = 1, 2, 3) \right\}.$$

Then, Γ acts on G by left multiplication and the corresponding nilmanifold $M = \Gamma\backslash G$ is known as the Iwasawa manifold. Let $\mathfrak{g}$ be the real lie algebra of G. Choose a basis $\{e_1, e_2, \ldots, e_6\} \in \mathfrak{g}^*$ by setting

$$dz_1 = e_1 + ie_2\,, \quad dz_2 = e_3 + ie_4\,, \quad \textit{and} \quad z_1 dz_2 - dz_3 = e_5 + ie_6\,.$$

These real 1-forms are pullbacks of the corresponding 1-forms on M, which we denote by the same symbols. They satisfy the following equations:

$$de_j = 0 \quad \forall\, 1 \le j \le 4\,.$$

$$de_5 = e_{13} + e_{42} \quad \textit{and} \quad de_6 = e_{14} + e_{23}\,.$$

Here, we make use of the notation $e_{jl} = e_j \wedge e_l$ *for all* $j, l \in \{1, \dots, 6\}$.

Consider the mixed complex form

$$\rho = e^{i(e_{56})}(e_1 + ie_2) \wedge (e_3 + ie_4) \quad \text{on } M\,.$$

Note that $de_5 \wedge de_6 = 0$ *and* $(e_1 + ie_2) \wedge (e_3 + ie_4) = de_5 + i\, de_6$. *Then, we have,*

$$\begin{aligned} d\rho &= e^{i(e_{56})} \wedge d(ie_{56}) \wedge (e_1 + ie_2) \wedge (e_3 + ie_4) \\ &= e^{i(e_{56})} \wedge d(ie_{56}) \wedge (de_5 + i\, de_6) \\ &= -e^{i(e_{56})} \wedge (e_6 + ie_5) \wedge de_5 \wedge de_6 \\ &= 0\,, \end{aligned}$$

and

$$\begin{aligned} e_{56} \wedge (e_1 + ie_2) \wedge (e_3 + ie_4) \wedge (e_1 - ie_2) \wedge (e_3 - ie_4) &= e_{56} \wedge (de_5 + i\, de_6) \wedge (de_5 - i\, de_6) \\ &= -ie_{56} \wedge de_5 \wedge de_6 \\ &\neq 0\,. \end{aligned}$$

By [70, Theorem 3.38, Theorem 4.8], M *admits a type* 2 *strong generalized Calabi-Yau structure whose canonical line bundle is generated by* ρ. *It is straightforward to see that* ρ, *when considered as a mixed form on* G, *gives a left-invariant GCS on* G *which is a strong generalized Calabi-Yau structure.*

Let $f : G \longrightarrow \mathbb{C}^2$ *be the natural projection defined as*

$$\tilde{\pi}\left(\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}\right) = (z_1, z_2)\,.$$

One can see that Γ*-acts on* (z_1, z_2) *via left translation by* $\mathbb{Z} \oplus i\mathbb{Z}$. *This shows that* f *induces a surjective submersion*

$$\tilde{f} : M \longrightarrow \bigoplus_{j=1}^{2} \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z} \cong \mathbb{T}^2 \times \mathbb{T}^2\,,$$

that satisfies the following commutative diagram,

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & \mathbb{C}^2 \\ {\scriptstyle \pi_\Gamma}\big\downarrow & & \big\downarrow \\ M & \xrightarrow[\quad \tilde{f} \quad]{} & \mathbb{T}^2 \times \mathbb{T}^2 \end{array}$$

where $\mathbb{C}^2 \longrightarrow \mathbb{T}^2 \times \mathbb{T}^2$ *is the natural quotient map. So, there exist* $\theta_1, \theta_2 \in \Omega^2(\mathbb{T}^2 \times \mathbb{T}^2, \mathbb{C})$ *such that* $\tilde{f}^*(\theta_1) = e_1 + ie_2$ *and* $\tilde{f}^*(\theta_2) = e_3 + ie_4$ *. Now, each fiber of this submersion is diffeomorphic to* $\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z} \cong \mathbb{T}^2$ *. Therefore, the foliation induced by the strong generalized Calabi-Yau structure on* M *is simple with leaf space* $\mathbb{T}^2 \times \mathbb{T}^2$ *and with the fibers as leaves.*

3.4.3 When the leaf space is a manifold

Let M be a type k regular GC manifold such that the leaf space $M/\mathscr{S}$ of the induced foliation $\mathscr{S}$ is a smooth manifold. Then $\mathscr{M} = M/\mathscr{S}$, as defined in (3.4.2), becomes a complex manifold of complex dimension k and the quotient map $\tilde{\pi} : M \longrightarrow \mathscr{M}$, as defined in (3.4.3), becomes a smooth surjective submersion. In particular, $\tilde{\pi}$ is an open map.

For any open set $V \subseteq M$, consider the map $\tilde{\pi}^{\#} : \tilde{\pi}^{-1}\mathcal{O}_{\mathscr{M}} \longrightarrow \mathcal{O}_M$ defined as

$$\tilde{\pi}^{\#}(f) = f \circ \tilde{\pi} \quad \text{for } f \in \mathcal{O}_{\mathscr{M}}(\tilde{\pi}(V)) \,, \tag{3.4.7}$$

where $\mathcal{O}_{\mathscr{M}}$ is the sheaf of holomorphic functions on $\mathscr{M}$. To show $\tilde{\pi}^{\#}$ is an isomorphism, it is enough to show $\tilde{\pi}^{\#}_x : (\tilde{\pi}^{-1}\mathcal{O}_{\mathscr{M}})_x \longrightarrow (\mathcal{O}_M)_x$ is an isomorphism for any $x \in M$.

Let $x \in M$ and set $y = \tilde{\pi}(x)$. Let $\{U, \phi\}$ be a co-ordinate chart around y in $\mathscr{M}$, and let $S_x = \tilde{\pi}^{-1}(y)$ be the fiber (leaf) over y. Then, choosing U sufficiently small, we have the following commutative diagram by Theorem 3.4.1,

$$\begin{array}{ccc} x \in V = \tilde{\pi}^{-1}(U) & \xrightarrow{\quad \tilde{\phi} \quad} & \tilde{S}_x \times U' \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \mathrm{Pr}_2} \\ U \subset \mathscr{M} & \xrightarrow[\quad \phi \quad]{} & U' \subset \mathbb{C}^k \end{array}$$

where $\tilde{\phi}$ is a GH homeomorphism, $U' \subset \mathbb{C}^k$ is an open set, and $\tilde{S}_x$ is the universal cover of S_x. Note that $\mathcal{O}_V$ is isomorphic to $\tilde{\phi}^{-1}\mathcal{O}_{S_x \times \mathbb{C}^k}$ via $\tilde{\phi}^{\#}$, defined in a similar manner as in (3.4.7), and $\mathcal{O}_{S_x \times U'} = \mathrm{Pr}_2^{-1}\, \mathcal{O}_{U'}$. Using commutativity of the diagram and the fact that $\mathcal{O}_U$ is isomorphic to $\phi^{-1}\mathcal{O}_{U'}$ via $\phi^{\#}$, defined similarly as in (3.4.7), we can show that $\tilde{\pi}^{-1}\mathcal{O}_U$ is isomorphic to $\mathcal{O}_V$ via $\tilde{\pi}^{\#}$. Therefore, $\tilde{\pi}^{\#}_x$ is isomorphism and so is $\tilde{\pi}^{\#}$. Similarly one can show that $\tilde{\pi}^{\#}$ is also an isomorphism even when we replace $\mathcal{O}_M$ by F_M ,, that is,

$$\tilde{\pi}^{\#} : \tilde{\pi}^{-1}C^{\infty}_{\mathscr{M}} \longrightarrow F_M \quad \text{is an isomorphism} \,. \tag{3.4.8}$$

Let GM and G^*M be the GH tangent and GH cotangent bundle of M, as defined in (3.2.3) and (3.2.1), respectively. Let $\{U_\alpha\}$ be a coordinate atlas of $\mathscr{M}$ such that $\tilde{\pi}^{-1}U_\alpha \cong \tilde{S_\alpha} \times U_\alpha'$ via a GH homeomorphism for some leaf S_α and $U_\alpha' \subset \mathbb{C}^k$ open set where $\tilde{S_\alpha}$ is the universal cover of S_α. Note that,

$$F_M(G^*M)|_{\tilde{\pi}^{-1}U_\alpha} = \mathrm{Span}_{F_M(\tilde{\pi}^{-1}U_\alpha)}\{dz_1, \ldots, dz_k\}$$

where z_j $(1 \le j \le k)$ are holomorphic coordinates on U_α'. Then (3.4.8) naturally induces an isomorphism

$$\tilde{\tilde{\pi}}^{\#}|_{U_\alpha} : C^\infty_{\mathscr{M}}(T^{1,0*}\mathscr{M})|_{U_\alpha} \longrightarrow F_M(G^*M)|_{\tilde{\pi}^{-1}U_\alpha}\,,$$

which gives rise to a sheaf isomorphism

$$\tilde{\tilde{\pi}}^{\#} : \tilde{\pi}^{-1}C^\infty_{\mathscr{M}}(T^{1,0*}\mathscr{M}) \longrightarrow F_M(G^*M)\,,$$

where $T^{1,0}\mathscr{M}$ is the holomorphic tangent bundle of $\mathscr{M}$. Replacing $F_M(G^*M)$ by $F_M(\overline{G^*M})$, one can show, similarly, that

$$\tilde{\tilde{\pi}}^{\#} : \tilde{\pi}^{-1}C^\infty_{\mathscr{M}}(T^{0,1*}\mathscr{M}) \longrightarrow F_M(\overline{G^*M})\,.$$

Also, similarly, we can show that $F_M(GM) \cong \tilde{\pi}^{-1}C^\infty_{\mathscr{M}}(T^{1,0}\mathscr{M})$ because $F_M(GM) = Hom_{F_M}(F_M(G^*M), F_M)$. We summarize our results as follows.

Theorem 3.4.5. *Let M be a regular GC manifold such that the leaf space of the induced foliation is a smooth manifold $\mathscr{M}$. Let GM and G^*M be the GH tangent and GH cotangent bundle of M. Let $\tilde{\pi} : M \longrightarrow \mathscr{M}$ be the quotient map, and let $T^{1,0}_{\mathscr{M}}$ be the sheaf holomorphic sections of the holomorphic tangent bundle of $\mathscr{M}$. Then the following hold.*

1. $F_M \cong \tilde{\pi}^{-1}C^\infty_{\mathscr{M}}$ *and* $\mathcal{O}_M \cong \tilde{\pi}^{-1}\mathcal{O}_{\mathscr{M}}$.

2. $F_M(GM) \cong \tilde{\pi}^{-1}C^\infty_{\mathscr{M}}(T^{1,0}\mathscr{M})$ *and* $F_M(G^*M) \cong \tilde{\pi}^{-1}C^\infty_{\mathscr{M}}(T^{1,0*}\mathscr{M})$. *In particular,*

$$GM \cong \tilde{\pi}^*T^{1,0}\mathscr{M} \quad \textit{and} \quad G^*M \cong \tilde{\pi}^*T^{1,0*}\mathscr{M}\,.$$

3. $\mathcal{G}\mathbf{M} \cong \tilde{\pi}^{-1}T^{1,0}_{\mathscr{M}}$ *and* $\mathcal{G}^*\mathbf{M} \cong \tilde{\pi}^{-1}T^{1,0*}_{\mathscr{M}}$.

Remark 3.4.3. *Theorem 3.4.5 implies that the pullback of any holomorphic vector bundle on the leaf space is an SGH vector bundle of M. A natural question is whether all SGH vector bundles arise in this way. The following two examples demonstrate that this is not always the case.*

Example 3.4.5. *Let M_1 be a complex manifold and M_2 be a symplectic manifold. Consider the natural product GCS on $M_1 \times M_2$. Consider the SGH vector bundle $\otimes_i \mathrm{Pr}_i^* V_i$ over $M_1 \times M_2$, as defined in Example 3.1.1 where V_1 is a holomorphic vector bundle over M_1 and V_2 is flat vector bundle over M_2. This bundle is not a pullback of a holomorphic vector bundle over M_1 unless V_2 is trivial.*

Example 3.4.6. *Let G be the Heisenberg group and $M = \Gamma \backslash G$ be the Iwasawa manifold with the left-invariant GCS as defined in Example 3.4.4. Let $\rho : \Gamma \longrightarrow GL_l(\mathbb{C})$ be a nontrivial (faithful) representation. Let $G \times_\rho \mathbb{C}^l$ be the SGH bundle over M as defined in Example 3.1.2. Let S ($\cong \mathbb{T}^2$) be a leaf of the induced foliation. Considering $\pi_1(S) < \Gamma$, $(G \times_\rho \mathbb{C}^l)|_S$ is isomorphic to $\mathbb{R}^2 \times_{\rho'} \mathbb{C}^l$ where $\rho' = \rho|_{\pi_1(S)}$ is a non-trivial representation. If possible let, there exist a holomorphic vector bundle W over $\mathbb{T}^2 \times \mathbb{T}^2$ such that $\tilde{f}^* W = G \times_\rho \mathbb{C}^l$. But, then the restriction of $\tilde{f}^* W$ to any of the fibers of $\tilde{f}$ is a trivial bundle which is not possible as the fibers of $\tilde{f}$ are the leaves of the induced foliation by the left-invariant GCS. Hence $G \times_\rho \mathbb{C}^l$ is an SGH vector bundle on M which is not a pullback of any holomorphic vector bundle on $\mathbb{T}^2 \times \mathbb{T}^2$.*

Chapter 4

Strong Generalized Holomorphic Principal Bundles and Generalized Holomorphic Connections

This chapter is the core of the thesis, delving into SGH principal bundles and elucidating their associated geometric properties. More precisely, our focus here is on establishing the theory of generalized holomorphic (GH) connections and their curvature, alongside associated geometric concepts such as Chern-Weil theory and characteristic classes, for SGH principal G-bundles over regular GC manifolds, where G denotes a complex Lie group.

We begin by delineating SGH principal bundles through the utilization of Theorem 3.1.1. Notably, we find that characterizing these bundles is more straightforward when the structure group is a complex Lie group (cf. Proposition 4.1.2). Subsequently, we construct the Atiyah sequence for SGH principal G-bundles over regular GC manifolds where G is a complex Lie group, employing an adaptation of Atiyah's approach for defining holomorphic connections on holomorphic principal bundles [7]. This endeavour yields the Atiyah class, serving as an obstruction to the aforementioned splitting of the Atiyah sequence, and lays the groundwork for the theory of GH connection; see Definition 4.2.1 and Theorem 4.2.2.

A fundamental result regarding the Atiyah class is the relationship between the Atiyah class of a holomorphic vector bundle and the Atiyah class of its associated holomorphic principal bundle, where these classes differ only by a sign. This prompts the question of whether such a relationship holds within the SGH setting, and Theorem 4.3.1 confirm this with a positive answer.

In our exploration of the theory of the curvature of a GH connection, we develop de Rham cohomology and Dolbeault cohomology, under the assumption that the leaf space $M/\mathscr{S}$ is an orbifold. This leads us to describe the Atiyah class through sheaf cohomology theory and present a related Chern-Weil theory, introducing a new type of characteristic class of such GH principal bundles. It is important to note that the Atiyah sequence is also studied for principal bundles over various geometric spaces; see [9,18,19,21,22,24–26,45,55,102,108,110,113,129,139,161]. This chapter is based on [127, Sections 4-6 and 8-9] and splits into five sections:

1. Strong generalized holomorphic principal bundles (Section 4.1).
2. Generalized Holomorphic Connections on SGH Principal bundles (Section 4.2).
3. Atiyah class of an SGH vector bundle (Section 4.3).
4. Dolbeault cohomology of SGH vector bundles (Section 4.4).
5. Generalized Chern-Weil Theory and characteristic classes (Section 4.5).

4.1 Strong generalized holomorphic principal bundles

In this section, we begin by formally defining SGH principal G-bundles for a real Lie group G that admits a GC structure. We employ Definition 3.1.1, wherein we essentially substitute "fiber bundle" with "principal G-bundle" to capture the essence. Consequently, we establish the Characterization presented in Proposition 4.1.1 for SGH principal G-bundles, akin to Theorem 3.1.1. However, this description becomes notably simpler since the mapping ρ_f for $f \in F$ in Theorem 3.1.1 reduces to the left translation by f, given $F = G$ and G acts on F via right translation.

Particularly, if we consider G to be a complex Lie group, this depiction becomes further simplified, as demonstrated in Proposition 4.1.2. Following this, we elaborate on the SGH version of the classical result regarding the one-to-one correspondence between the sets of isomorphic classes of vector bundles of a fixed rank, denoted as l, and isomorphic classes of principal GL_l-bundles, as detailed in Proposition 4.1.3.

Let G be a real (connected) Lie group with a GC structure $\mathcal{J}_G$. Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a smooth principal G-bundle over a GC manifold $(M, \mathcal{J}_M)$. Let $\{U_\alpha, \phi_\alpha\}$ be a family of local trivializations

$$\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G\,, \tag{4.1.1}$$

with transition functions

$$\phi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \longrightarrow G\,, \tag{4.1.2}$$

where $\phi_{\alpha\beta}(x) = \phi_\alpha|_{\pi^{-1}(x)} \circ \phi_\beta^{-1}(x, \cdot)$ for all $x \in U_{\alpha\beta}$.

Definition 4.1.1. *P is called an SGH principal G-bundle over $(M, \mathcal{J}_M)$ if*

1. *P is a GC manifold.*

2. *There exist local trivializations $\{U_\alpha, \phi_\alpha\}$ such that every ϕ_α is a GH homeomorphism when $U_\alpha \times G$ is endowed with the standard product GC structure.*

As in (3.1.1), $\mathcal{J}_M$ and $\mathcal{J}_G$ can be written in the following form

$$\mathcal{J}_M = \begin{pmatrix} -J_M & \beta_M \\ B_M & J_M^* \end{pmatrix} \quad \text{and} \quad \mathcal{J}_G = \begin{pmatrix} -J_G & \beta_G \\ B_G & J_G^* \end{pmatrix}, \text{ respectively}\,.$$

Remark 4.1.1. *Note that in the definition of an SGH principal G-bundle, we do not require that the group operations on G be GH maps, or that the left or right translations by elements of G be GH homeomorphisms. However, if we assume that G is a complex Lie group then these conditions hold.*

Proposition 4.1.1. *The following are equivalent.*

1. *P is an SGH principal G-bundle over $(M, \mathcal{J}_M)$ with local trivializations $\{U_\alpha, \phi_\alpha\}$ and transition functions $\{\phi_{\alpha\beta}\}$.*

2. *For all nonempty $U_{\alpha\beta} \subseteq M$ and $(m, f) \in U_{\alpha\beta} \times G$, the map*

$$\psi_{\alpha\beta} : U_{\alpha\beta} \times G \longrightarrow U_{\alpha\beta} \times G \quad \textit{defined as} \quad \psi_{\alpha\beta}(m, f) = (m, \phi_{\alpha\beta}(m) \cdot f)$$

   is a GH automorphism of $U_{\alpha\beta} \times G$.

3. *The transition functions satisfy the following: For all $m \in U_{\alpha\beta}$,*

(a) $\phi_{\alpha\beta}(m) \in \mathrm{Diff}_{\mathfrak{J}_G}(G)$,

(b) $(\phi_{\alpha\beta})_{*m} \circ J_{U_{\alpha\beta}} = J_F \circ (\phi_{\alpha\beta})_{*m}$,

(c) $(\phi_{\alpha\beta})_{*m} \circ \beta_{U_{\alpha\beta}} = 0$,

(d) $B_G \circ (\phi_{\alpha\beta})_{*m} = 0$.

Proof. Follows from Theorem 3.1.1. □

Proposition 4.1.2. *Let G be a (connected) complex Lie group. Then, the following are equivalent.*

1. *P is an SGH principal G-bundle over $(M, \mathfrak{J}_M)$ with local trivializations $\{U_\alpha, \phi_\alpha\}$ and transition functions $\{\phi_{\alpha\beta}\}$.*

2. *For all nonempty $U_{\alpha\beta} \subseteq M$ and $(m, f) \in U_{\alpha\beta} \times G$, the map*

$$\psi_{\alpha\beta} : U_{\alpha\beta} \times G \longrightarrow U_{\alpha\beta} \times G \quad \textit{defined as} \quad \psi_{\alpha\beta}(m, f) = (m, \phi_{\alpha\beta}(m) \cdot f)$$

is a GH automorphism of $U_{\alpha\beta} \times G$.

3. *The transition maps $\phi_{\alpha\beta}$ satisfy the following:*

(a) *$\phi_{\alpha\beta}(m)$ is a biholomorphic map on G $\forall\, m \in U_{\alpha\beta}$,*

(b) *each $\phi_{\alpha\beta}$ is a GH map.*

Proof. Follows from Proposition 4.1.1 and Lemma 3.1.3. □

Let M be a GC manifold and let E be an SGH vector bundle of real rank $2l$ over M with local trivializations $\{U_\alpha, \phi_\alpha\}$. Then, by Theorem 3.1.1 and [103, Proposition 3.2], we have

1. $\phi_{\alpha\beta}(m) \in GL_l(\mathbb{C})$, i.e., E is a complex vector bundle of of complex rank l,

2. each entry $B_{\lambda\gamma} : U_{\alpha\beta} \longrightarrow \mathbb{C}$ of $\phi_{\alpha\beta} = (B_{\lambda\gamma})_{l\times l}$ is a GH function,

where $\phi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_{2l}(\mathbb{R})$ is the transition map as in Theorem 3.1.1.

Following the standard associated principal bundle construction (cf. [140, Chapter 3]), we construct the principal bundle P_E associated to E as follows: Consider the disjoint union $\bigsqcup U_\alpha \times GL_l(\mathbb{C})$ where $U_\alpha \subset M$ varies over a trivializing open cover of E. Define an equivalence relation on this set by declaring elements $(b,h) \in U_\beta \times GL_l(\mathbb{C})$ and $(a,g) \in U_\alpha \times GL_l(\mathbb{C})$ to be equivalent if and only if $a = b$ and $g = \phi_{\alpha\beta}(b)h$,

$$(b,h) \sim (a,g) \iff a = b \text{ and } g = \phi_{\alpha\beta}(b)h\,.$$

Now define

$$P_E := \bigsqcup_\alpha U_\alpha \times GL_l(\mathbb{C}) \Big/ \sim\ . \tag{4.1.3}$$

For each $m \in U_{\alpha\beta}$,

$$\phi_{\alpha\beta}(m) \circ J_{\mathbb{R}^{2l}} = J_{\mathbb{R}^{2l}} \circ \phi_{\alpha\beta}(m)\,,$$

where $J_{\mathbb{R}^{2l}}$ denotes the natural complex structure on $\mathbb{R}^{2l}$, which implies $\phi_{\alpha\beta}(m)$ is biholomorphic. Considering $GL_l(\mathbb{C}) \subset GL_{2l}(\mathbb{R})$, note that the transition map $\phi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_l(\mathbb{C})$ is a GH map if and only if each entry

$$B_{\lambda\gamma} : U_{\alpha\beta} \longrightarrow \mathbb{C}$$

of $\phi_{\alpha\beta} = (B_{\lambda\gamma})_{l\times l}$ is a GH function. Hence, by Proposition 4.1.2, P_E is an SGH principal $GL_l(\mathbb{C})$-bundle.

Given an SGH principal $GL_l(\mathbb{C})$-bundle $\pi : P \longrightarrow M$ with local trivializations $\{U_\alpha, \phi_\alpha\}$, the associated vector bundle E_P is constructed as follows: Consider the right action of $GL_l(\mathbb{C})$ on $P \times \mathbb{C}^l$ defined by

$$(p,f)\cdot g = (p\cdot g, g^{-1}(f))\ \forall\ p \in P\,,\ f \in \mathbb{C}^l \text{ and } g \in GL_l(\mathbb{C})\,.$$

Define

$$E_P := P \times_{GL_l(\mathbb{C})} \mathbb{C}^l\,, \tag{4.1.4}$$

as the identification space of that right action. Denote by $[(p,f)]$ the equivalence class or orbit of $(p,f) \in P \times \mathbb{C}^l$ under the above action. Then, the map

$$\pi_P : E_P \longrightarrow M$$

defined by $\pi_P([p,f]) = \pi(p)$ gives the desired the vector bundle structure on E_P. Note that the transition map $\phi_{\alpha\beta}$ of E_P, as in Theorem 3.1.1, is a GH map by Proposition 4.1.2. Also, $\phi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_l(\mathbb{C})$ is a GH map if and only if each entry

$$B_{\lambda\gamma} : U_{\alpha\beta} \longrightarrow \mathbb{C}$$

of $\phi_{\alpha\beta} = (B_{\lambda\gamma})_{l\times l}$ is a GH function. Thus, by [103, Proposition 3.2], E_P is a GH vector bundle over M. The result below now follows using standard arguments.

Proposition 4.1.3. *Let $(M, \mathcal{J}_M)$ be a GC manifold and $l \in \mathbb{N}$. Consider the following set*

$$\begin{aligned}\mathscr{P}_{GL_l(\mathbb{C})} :=\ & \textit{Set of all isomorphism classes of SGH principal}\\ & GL_l(\mathbb{C})\textit{-bundles over } M\,.\end{aligned}$$

If P_E and E_P are as in the equations (4.1.3) *and* (4.1.4) *respectively, then the map*

$$\Phi : \mathscr{E}_l \longrightarrow \mathscr{P}_{GL_l(\mathbb{C})} \tag{4.1.5}$$

defined by $\Phi([E]) = [P_E]$ gives a bijective map between two sets with the inverse map defined as $\Phi^{-1}([P]) = [E_P]$ where $[E]$ and $[P]$ denotes the SGH bundle isomorphism classes of E and P, respectively, and $\mathscr{E}_l$ as given in Proposition 3.2.1.

4.2 Generalized Holomorphic Connections on SGH Principal bundles

The aim of this section is twofold. Firstly, we establish the Atiyah sequence for SGH principal G-bundles over regular GC manifolds, where G is a complex Lie group, and introduce the concept of a generalized holomorphic (GH) connection. Secondly, we compute the Atiyah class and provide a characterization of a GH connection using local trivializations.

4.2.1 SGH principal bundles with complex fibers and GH connections

There are some special properties of SGH principal bundles with a complex Lie group as a structure group which we similar to holomorphic principal bundles over complex manifolds. These properties do not hold in general. We list a few of them here that are important for our purposes.

Proposition 4.2.1. *Let $G \hookrightarrow P \xrightarrow{\pi} M$ be an SGH principal G-bundle over a regular GC manifold $(M, \mathcal{J}_M)$ where G is a complex Lie group. Then*

1. *P admits GH sections over any trivializing open set $U \subseteq M$.*

2. *If $s : V \to P$ is a GH section of P over an open subset $V \subseteq M$, then so is $s \cdot \phi$ for any GH map $\phi : V \to G$.*

3. *If s_1 and s_2 are any two GH sections of P over V, then there exists a unique GH map $\phi : V \to G$ such that $s_2 = s_1 \cdot \phi$.*

Proof. First, note that it suffices to prove the statement of the theorem for a local trivialization of P as a GH homeomorphism is a GH map. Additionally, by Proposition 3.1.1, any GH map between complex manifolds is simply a holomorphic map and vice versa.

Then, (1) follows from the fact that a constant map from $c : U \to G$ is GH by Proposition 3.1.1 as $c_* = 0$. This implies that the local trivialization of P over U admits a GH section.

Part (2) follows from the fact that the right action of G on itself is GH if G is a complex Lie group, and that the composition of GH maps is a GH map.

Part (3) follows from the fact that inversion operation in a complex Lie group, is a GH map, in fact, a GH homeomorphism. □

By Remark 4.1.1, G acts on P as a group of fiber preserving GH automorphisms, $P \times G \longrightarrow P$. The GCS induced by the complex structure on G is regular, which implies that P is a regular GC manifold. Let $\mathcal{G}^*P$ and $\mathcal{G}P$ denote the GH cotangent and GH tangent bundles of G as specified in (3.2.1) and (3.2.3), respectively. Since G acts on P, it has an induced action on $(TP \oplus T^*P) \otimes \mathbb{C}$. For any $g \in G$, we have

$$(X + \xi) \cdot g = \begin{pmatrix} g_*^{-1} & 0 \\ 0 & g^* \end{pmatrix} (X + \xi) \quad \text{for all} \quad X + \xi \in (TP \oplus T^*P) \otimes \mathbb{C}\,.$$

As $g : P \longrightarrow P$, $p \mapsto p \cdot g$, is a GH automorphism for every $g \in G$, it follows that G acts on i-eigen bundle L_P of $\mathcal{J}_P$. This implies that G acts on $\mathcal{G}^*P$ and hence on $\mathcal{G}P$. Define the *SGH Atiyah bundle* of P by

$$At(P) := \mathcal{G}P/G\,. \tag{4.2.1}$$

Then, a point of $At(P)$ is a field of GH tangent vectors, defined along one of the fibers of P, which is invariant under G. We shall show that $At(P)$ has a natural SGH vector bundle structure over M.

Let $m \in M$ and let $U \subset M$ be a sufficiently small open neighborhood of m such that there exist a GH section of P over U,

$$s : U \longrightarrow P\,. \tag{4.2.2}$$

Let $(\mathcal{G}P)_s$ be the restriction of $\mathcal{G}P$ to $s(U)$. Now since $s : U \longrightarrow s(U)$ is a diffeomorphism, $s(U)$ can be endowed with the structure of a regular GC manifold such that s becomes a GH homeomorphism between U and $s(U)$. Since s is a GH section, by [103, Example 3.3], $s^*(\mathcal{G}P)$ is an SGH vector bundle over U and so, $(s^{-1})^*(s^*(\mathcal{G}P))$ is also an SGH vector bundle over $s(U)$ which coincides with $(\mathcal{G}P)_s$ as a smooth bundle. This defines a canonical SGH bundle structure on $(\mathcal{G}P)_s$.

There is a natural one-to-one correspondence between $At(P)_U$ and $(\mathcal{G}P)_s$,

$$\gamma_s : At(P)_U \longrightarrow (\mathcal{G}P)_s\,, \tag{4.2.3}$$

where γ_s assigns to each invariant GH vector field along $\pi^{-1}(x) := P_x$ its value at $s(x)$. This is easily seen to be an isomorphism of smooth vector bundles. Then, the SGH vector bundle structure of $(\mathcal{G}P)_s$ defines an SGH vector bundle structure of $At(P)_U$.

It remains to show that this construction is independent of the choice of the GH section s. Let s_1 and s_2 be any two GH sections of P over U. Then, by Proposition 4.2.1, there exist a unique GH map $\phi : U \longrightarrow G$ such that

$$s_1(x) \cdot \phi(x) = s_2(x)\,, \quad \forall\ x \in U\,.$$

Note that the map $\psi : U \times G \longrightarrow U \times G$ defined as

$$\psi(x, g) = (x, \phi(x) \cdot g) \quad \text{for all } (x, g) \in U \times G$$

is a GH automorphism of $U \times G$ by Proposition 4.1.2. Therefore, ψ induces an isomorphism of SGH vector bundles, again denoted by ψ,

$$\psi : (\mathcal{G}P)_{s_1} \longrightarrow (\mathcal{G}P)_{s_2}\,,$$

satisfying

$$\gamma_{s_2} = \psi \circ \gamma_{s_1}\,.$$

Hence, the SGH vector bundle structure on $At(P)$ is well-defined.

Let $\mathcal{T}$ denote the sub-bundle of TP formed by vectors tangential to the fibers of P. Define $\mathcal{GT} = \mathcal{T} \cap \mathcal{G}P$. Since G acts on $\mathcal{T}$, it also acts on $\mathcal{GT}$. Define

$$\mathcal{R} = \mathcal{GT}/G\,. \tag{4.2.4}$$

If γ_s is defined as in (4.2.3), then restricting to $\mathcal{R}_U$, we get

$$\gamma_s|_{\mathcal{R}_U} := \gamma_s^{'} : \mathcal{R}_U \longrightarrow (\mathcal{GT})_s\,. \tag{4.2.5}$$

Note that $(\mathcal{GT})_s$ is also an SGH vector sub-bundle of $(\mathcal{G}P)_s$ as $(\mathcal{GT})$ is an SGH vector sub-bundle of $(\mathcal{G}P)$. Hence by above, $\mathcal{R}$ is also an SGH sub-bundle of $At(P)$.

We now examine $\mathcal{R}$ more closely. Let $\mathfrak{g}$ denote the complex Lie algebra of G. As a vector space, $\mathfrak{g}$ is the holomorphic tangent space of G at identity. In the SGH principal bundle P, for $x \in M$, each fiber P_x can be identified with G up to a left multiplication. Note that, each smooth tangent vector at the point $p \in P$, tangential to the fiber, defines a unique left-invariant smooth vector field on G. Since the left (respectively, right) multiplication is biholomorphic, the vector space of left (respectively, right) invariant holomorphic vector fields on G is then isomorphic with $\mathfrak{g}$ via left (respectively, right) multiplication. Note that by the locally product nature of the GCS on P, and the absence of a B transformation in a GH homeomorphism, any holomorphic tangent vector to a fiber of P is an element of $\mathcal{GT}$. Therefore, any holomorphic tangent vector to the fiber at the point $p \in P$ defines a unique left invariant GH tangent vector field on G. Thus, we have an SGH vector bundle isomorphism

$$\mathcal{GT} \cong P \times \mathfrak{g}\,.$$

Then, the action of G on $\mathcal{GT}$ induces an action on $P \times \mathfrak{g}$ as follows,

$$(p, l) \cdot g = (p \cdot g\,, \mathrm{Ad}(g^{-1}) \cdot l) \quad \forall \ (p, l) \in P \times \mathfrak{g}\,. \tag{4.2.6}$$

Let $P \times_G \mathfrak{g}$ be the identification space defined by the action in (4.2.6). The adjoint map is a biholomorphism due to the complex Lie group structure of G. Consequently, the transition maps of the complex vector bundle $P \times_G \mathfrak{g}$ are GH maps. Therefore, by Theorem 3.1.1 and [103, Proposition 3.2], $P \times_G \mathfrak{g}$ is an SGH vector bundle over M with fiber $\mathfrak{g}$ associated to P by the adjoint representation. Hence, $\mathcal{R} = \mathcal{GT}/G \cong P \times_G \mathfrak{g}$. We shall denote it by $Ad(P)$, that is,

$$Ad(P) := P \times_G \mathfrak{g}\,. \tag{4.2.7}$$

The projection $\pi : P \longrightarrow M$ induces a bundle map $\mathcal{G}\pi : At(P) \longrightarrow \mathcal{G}M$. Using Definition 3.1.1, we deduce that $\mathcal{G}\pi$ is an SGH vector bundle homomorphism.

Moreover, let T_s denote the tangent bundle and $\mathcal{G}T_s$ denotes the GH tangent bundle of $s(U)$ respectively where $s(U)$ is the image of the GH section s as in (4.2.2). Then we have $(\mathcal{G}P)_s = (\mathcal{G}\mathcal{T})_s \oplus (\mathcal{G}T_s)$ where $(\mathcal{G}P)_s$ and $(\mathcal{G}\mathcal{T})_s$ are as defined in (4.2.3) and (4.2.5), respectively. This implies the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R}_U & \xrightarrow{\xi} & At(P)_U & \xrightarrow{\mathcal{G}\pi} & \mathcal{G}U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_s'} & & \downarrow{\scriptstyle \gamma_s} & & \downarrow{\scriptstyle s^{\#}} & & \\ 0 & \longrightarrow & (\mathcal{G}\mathcal{T})_s & \longrightarrow & (\mathcal{G}P)_s & \longrightarrow & \mathcal{G}T_s & \longrightarrow & 0 \end{array}$$

where γ_s, γ_s' and ξ are as in (4.2.3), (4.2.5) and the natural inclusion map, respectively. Also, the map $s^{\#} : \mathcal{G}U \longrightarrow \mathcal{G}T_s$, induced by s, is an isomorphism of SGH vector bundles. We conclude that

$$0 \longrightarrow \mathcal{R} \xrightarrow{\xi} At(P) \xrightarrow{\mathcal{G}\pi} \mathcal{G}M \longrightarrow 0$$

is a short exact sequence of SGH vector bundles over M. We summarize our results in the following theorem.

Theorem 4.2.1. *Let P be an SGH principal G-bundle over a regular GC manifold $(M, \mathcal{J}_M)$ where G is a complex Lie group. Then, there exists a canonical short exact sequence $\mathcal{A}(P)$ of SGH vector bundles over M:*

$$0 \longrightarrow Ad(P) \longrightarrow At(P) \longrightarrow \mathcal{G}M \longrightarrow 0 \tag{4.2.8}$$

where $\mathcal{G}M$ is the GH tangent bundle of M as in (3.2.3)*, $Ad(P)$ is the SGH vector bundle associated to P by the adjoint representation of G as in* (4.2.7)*, and $At(P)$ is the SGH vector bundle of invariant GH tangent vector fields on P as in* (4.2.1)*.*

Definition 4.2.1. *Let P be an SGH principal G-bundle over a regular GC manifold M where G is a complex Lie group. A generalized holomorphic (GH) connection on P is a splitting of the short exact sequence $\mathcal{A}(P)$ in* (4.2.8) *such that the splitting map is a GH map.*

By [7, Proposition 2], the extension $\mathcal{A}(P)$ defines an element

$$a(P) \in H^1(M, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}\,,\, \mathbf{Ad}(\mathbf{P}))\,,$$

and $\mathcal{A}(P)$ is a trivial extension if and only if $a(P) = 0$.

Note that $\mathcal{G}^*\mathbf{M} = \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}\,,\,\mathcal{O}_M)$. Hence,

$$\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}\,,\,\mathbf{Ad}(\mathbf{P})) = \mathbf{Ad}(\mathbf{P}) \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M}\,.$$

Thus we have the following result.

Theorem 4.2.2. *An SGH principal G-bundle P over a regular GC manifold M defines an element*

$$a(P) \in H^1(M, \mathbf{Ad}(\mathbf{P}) \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M})\,.$$

P admits a GH connection if and only if $a(P) = 0$.

Definition 4.2.2. *The element $a(P)$ in Theorem 4.2.2 is called the Atiyah class of the SGH principal G-bundle P. The SGH vector bundle $At(P)$ in* (4.2.8) *is called the SGH Atiyah bundle of the SGH principal G-bundle P.*

Definition 4.2.3. *A smooth generalized connection in the principal bundle P is a smooth splitting of the short exact sequence $\mathcal{A}(P)$ in* (4.2.8)*.*

Remark 4.2.1. *In this case, when $\mathcal{A}(P)$ is considered as a short exact sequence of smooth vector bundles, again by [7, Proposition 2], the smooth extension $\mathcal{A}(P)$ defines an element*

$$a'(P) \in H^1(M, \mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M)\,,\,C^\infty(Ad(P)))\,,$$

and $\mathcal{A}(P)$ is a trivial smooth extension if and only if $a'(P) = 0$. But due to the partition of unity of smooth functions, we can see that $\mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M)\,,\,C^\infty(Ad(P))$ is a fine sheaf. Thus $H^1(M, \mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M)\,,\,C^\infty(Ad(P))) = 0$ and so $a'(P)$ will always be zero. This implies that a smooth generalized connection always exists.

Remark 4.2.2. *It may be feasible to omit the regularity assumption on M by utilizing sheaf theoretic language. In essence, within the context of sheaf theory, it might be possible to define the concept of the Atiyah class for an SGH principal bundle over a GC manifold without the need for regularity.*

Remark 4.2.3. *It is worth noting that the notion of a smooth generalized connection can be extended for any arbitrary real (connected) Lie group G that admits a regular GCS, by extending the approach developed for complex (connected) Lie groups in the preceding discussions. However, the concept of GH connection may not be extended to any such G*

using the same method. Because, when dealing with an SGH principal G-bundle over a regular GC manifold where G is an arbitrary real Lie group endowed with a regular GCS, the combination of Definition 2.3.6 and Theorem 3.1.1 implies that the transition maps may not be GH maps. So, there is a possibility that the sheaf of GH sections of an SGH principal G-bundle may not have any elements when G is an arbitrary real (connected) Lie group with a GCS.

Even in the case of a trivial SGH principal G-bundle, where G is a real Lie group with a GCS, constant sections are not always GH maps. Consequently, transition maps may not be GH maps. This can be demonstrated as follows:

Let M denote a GC manifold. Consider G, a real Lie group endowed with a regular GCS of type zero. Let $M \times G$ represent the trivial SGH principal G-bundle over M. Then, every smooth section of $M \times G$ is given by a smooth map $f : U \longrightarrow G$, where $U \subseteq M$ is an open set. If a section is a GH map, then the corresponding smooth map f is also a GH map, and by Lemma 2.2.1, df respects the linear Poisson structures at each point. However, in the case of constant sections, f becomes a constant map, implying that df does not respect the linear Poisson structures at any point, as $df = 0$. Thus, constant sections on $M \times G$ are not GH maps in this case.

However, when working with a smooth generalized connection, we only need to focus on smooth sections, which are always available. Hence, we need not concern ourselves with the aforementioned possibility.

4.2.2 Local coordinate description of the Atiyah class

In this subsection, we compute the Atiyah class $a(P)$ in local coordinates following Atiyah [7]. Let G be a (connected) complex Lie group with complex Lie algebra $\mathfrak{g}$. Let P be an SGH principal G-bundle over a regular GC manifold M with local trivializations $\{U_\alpha , \phi_\alpha\}$ and transition maps $\phi_{\alpha\beta}$ (see (4.1.2)).

Let $M_{\mathfrak{g}} := M \times \mathfrak{g}$ denote the trivial SGH vector bundle over M where $\mathfrak{g}$ is the complex Lie algebra of G. Since ϕ_α is a GH homeomorphism and it commutes with the action of G, it induces an SGH vector bundle isomorphism

$$\widehat{\phi_\alpha} : At(P)|_{U_\alpha} \longrightarrow \mathcal{G}M|_{U_\alpha} \oplus M_{\mathfrak{g}}|_{U_\alpha} . \tag{4.2.9}$$

Define the SGH vector bundle homomorphism

$$a_\alpha : \mathcal{G}M|_{U_\alpha} \longrightarrow At(P)|_{U_\alpha} \tag{4.2.10}$$

by $a_\alpha(X) = (\widehat{\phi_\alpha})^{-1}(X \oplus 0)$ for all $X \in \mathcal{G}M|_{U_\alpha}$. Then the map $a_{\alpha\beta} : \mathcal{G}M|_{U_{\alpha\beta}} \longrightarrow At(P)|_{U_{\alpha\beta}}$, defined as

$$a_{\alpha\beta} = a_\beta - a_\alpha \,,$$

gives a representative 1-cocycle for $a(P)$ in $H^1(M, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}\,,\, \mathbf{Ad}(\mathbf{P}))$.

Denote $G_{\mathfrak{g}} := G \times \mathfrak{g}$. Note that $\mathcal{G}G := T^{1,0}G$. Both right and left multiplication maps on G are biholomorphic. Using them we have SGH bundle isomorphisms

$$\xi : \mathcal{G}G \longrightarrow G_{\mathfrak{g}} \quad \text{and} \quad \eta : \mathcal{G}G \longrightarrow G_{\mathfrak{g}} \,,$$

respectively. Thus,

$$\xi \,,\, \eta \in H^0(G, \mathrm{Hom}_{\mathcal{O}_G}(\mathbf{T^{1,0}G}, \mathbf{G}_{\mathfrak{g}}))\,.$$

Now, $\phi_{\alpha\beta}$ is a GH map due to Proposition 4.1.2, thereby it induces elements

$$\xi_{\alpha\beta} \,,\, \eta_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}, \mathbf{M}_{\mathfrak{g}}))\,.$$

Then, for each $X \in \mathcal{G}M|_{U_{\alpha\beta}}$,

$$\begin{aligned}
\widehat{\phi_\alpha}(a_{\alpha\beta}(X)) &= \widehat{\phi_\alpha}((\widehat{\phi_\beta})^{-1}(X \oplus 0) - (\widehat{\phi_\alpha})^{-1}(X \oplus 0)) \\
&= \widehat{\phi_\alpha}((\widehat{\phi_\beta})^{-1}(X \oplus 0)) - (X \oplus 0) \\
&= (X \oplus \xi_{\alpha\beta}(X)) - (X \oplus 0) \\
&= (0 \oplus \xi_{\alpha\beta}(X))\,.
\end{aligned}$$

By the short exact sequence in (4.2.8), we can identify $Ad(P)|_{U_\alpha}$ as an SGH subbundle of $At(P)|_{U_\alpha}$. Then, the SGH vector bundle isomorphism between $Ad(P)|_{U_\alpha}$ and $M_{\mathfrak{g}}|_{U_\alpha}$ is identified with the restriction map

$$\widehat{\phi_\alpha}|_{Ad(P)|_{U_\alpha}} : Ad(P)|_{U_\alpha} \longrightarrow M_{\mathfrak{g}}|_{U_\alpha}\,.$$

Therefore, we get

$$a_{\alpha\beta} = (\widehat{\phi_\alpha})^{-1} \circ \xi_{\alpha\beta}\,, \tag{4.2.11}$$

and since $\xi_{\alpha\beta} = \mathrm{Ad}(\phi_{\alpha\beta}) \cdot \eta_{\alpha\beta}$, we can replace (4.2.11) by

$$a_{\alpha\beta} = (\widehat{\phi_\beta})^{-1} \circ \eta_{\alpha\beta}\,. \tag{4.2.12}$$

Now if $a(P) = 0$, then the coboundary equation is

$$a_{\alpha\beta} = \gamma_\beta - \gamma_\alpha$$

where $\gamma_i \in \Gamma(U_i, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}\,,\ \mathbf{Ad}(\mathbf{P})))$ for $i \in \{\alpha, \beta\}$. For each $i \in \{\alpha, \beta\}$, if we denote

$$\Theta_i := \widehat{\phi_i} \circ \gamma_i\,,$$

then $\Theta_i \in \Gamma(U_i, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}\,,\ \mathbf{M}_{\mathfrak{g}}))$. Thus the coboundary equation becomes

$$\xi_{\alpha\beta} = \mathrm{Ad}(\phi_{\alpha\beta}) \cdot \Theta_\beta - \Theta_\alpha\,, \tag{4.2.13}$$

or

$$\eta_{\alpha\beta} = \Theta_\beta - \mathrm{Ad}(\phi_{\beta\alpha}) \cdot \Theta_\alpha\,. \tag{4.2.14}$$

Remark 4.2.4. *Note that, in case of smooth generalized connection, since we have*

$$H^1(M, \mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M)\,,\ C^\infty(Ad(P)))) = 0\,,$$

the co boundary equation is

$$a_{\alpha\beta} = \gamma'_\beta - \gamma'_\alpha$$

where $\gamma'_i \in C^\infty(U_i, \mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M)\,,\ C^\infty(Ad(P))))$ *for* $i \in \{\alpha, \beta\}$*. Then for each* i *in* $\{\alpha, \beta\}$*, if we again denote*

$$\Theta_i := \widehat{\phi_i} \circ \gamma'_i\,,$$

we get that $\Theta_i \in C^\infty(U_i, \mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M)\,,\ C^\infty(M_{\mathfrak{g}})))$*. Thus the co-boundary equation becomes*

$$\xi_{\alpha\beta} = \mathrm{Ad}(\phi_{\alpha\beta}) \cdot \Theta_\beta - \Theta_\alpha\,, \tag{4.2.15}$$

or

$$\eta_{\alpha\beta} = \Theta_\beta - \mathrm{Ad}(\phi_{\beta\alpha}) \cdot \Theta_\alpha\,. \tag{4.2.16}$$

This completes the primary groundwork for GH connection and smooth generalized connection. In the subsequent section, we will utilize this description to establish the curvature.

4.3 Atiyah class of an SGH vector bundle

In this section, we address the question regarding the relationship between the Atiyah class of an SGH vector bundle and the Atiyah class of its associated SGH principal bundle by establishing that these classes only differ by sign, akin to the classical holomorphic case.

Let E be an SGH vector bundle over a regular GC manifold M with local trivializations $\{U_\alpha\,,\,\phi_\alpha\}$. Let $J^1(E)$ be the first jet bundle of E over M as defined in [103, Section 3.2]. Then by [103, Theorem 3.17], $J_1(E)$ is an SGH vector bundle over M and it fits into the following exact sequence, denoted by $\mathcal{B}(E)$,

$$0 \longrightarrow \mathcal{G}^*M \otimes E \xrightarrow{J} J_1(E) \xrightarrow{\pi_1} E \longrightarrow 0 \tag{4.3.1}$$

of SGH vector bundles over M.

Let $\widehat{\pi_1} : \mathbf{J_1}(\mathbf{E}) \longrightarrow \mathbf{E}$ be the morphism of $\mathcal{O}_M$-sheaves, induced by π_1. Then, we can see that there exists a canonical $\mathbb{C}$-module map of sheaves $\psi : \mathbf{E} \longrightarrow \mathbf{J_1 E}$ such that $\widehat{\pi_1} \circ \psi = \mathrm{Id}_{\mathbf{E}}$. Thus, as a sheaf of $\mathbb{C}$-modules, we have

$$\mathbf{J_1}(\mathbf{E}) = \mathbf{E} \oplus_{\mathbb{C}} (\mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathbf{E})\,.$$

Recall that, for each $m \in M$, $f \in \mathcal{O}_{M,m}$ if and only if $(df)_m \in (\mathcal{G}^*M)_m$. So, we can define the map

$$\phi_m : \mathcal{O}_{M,m} \times \mathbf{J_1}(\mathbf{E})_m \longrightarrow \mathbf{J_1}(\mathbf{E})_m$$

by

$$\phi_m(f, s+\delta) = fs \oplus (f\delta + df \otimes s)$$

where $s \in E_m$, $\delta \in ((\mathcal{G}^*M)_m \otimes_{\mathcal{O}_{M,m}} E_m)$, and $f \in \mathcal{O}_{M,m}$. This defines an action of $\mathcal{O}_M$ on $\mathbf{J_1}(\mathbf{E})$ making it a sheaf of $\mathcal{O}_M$-modules. We obtain the following short exact sequence of $\mathcal{O}_M$-modules

$$0 \longrightarrow \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathbf{E} \xrightarrow{\widehat{J}} \mathbf{J_1}(\mathbf{E}) \xrightarrow{\widehat{\pi_1}} \mathbf{E} \longrightarrow 0 \tag{4.3.2}$$

where $\widehat{J}(\delta) = 0 + \delta$ and $\widehat{\pi_1}(s+\delta) = s$ are the morphisms of $\mathcal{O}_M$-modules induced by the maps J and π_1 in (4.3.1), respectively.

Since $\mathrm{Hom}_{\mathcal{O}_M}(\mathbf{E}, \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathbf{E}) \cong \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathrm{End}(\mathbf{E})$, by [7, Proposition 2] and using (4.3.2), the extension $\mathcal{B}(E)$ defines an element

$$b(E) \in H^1(M, \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathrm{End}(\mathbf{E}))\,.$$

Definition 4.3.1. *([103, Definition 4.4]) $b(E)$ is called the Atiyah class of the SGH vector bundle E over M.*

The following result is standard in the holomorphic case (see [7, Proposition 9]) and follows similarly in the SGH setting. Nonetheless, we provide a proof for the sake of completeness

Proposition 4.3.1. *Let E be an SGH vector bundle of real rank $2l$ over M. Let P_E be the corresponding SGH principal $GL_l(\mathbb{C})$-bundle as in* (4.1.3)*. Then we have*

$$\mathrm{End}(E) \cong Ad(P_E)$$

as SGH vector bundles where $Ad(P_E)$ as in (4.2.7)*.*

Proof. For $Ad(P_E)$, there exist local trivializations $\{U_\alpha, \phi_\alpha\}$ of $Ad(P_E)$ over M such that

1. $\phi_\alpha : Ad(P_E)|_{U_\alpha} \longrightarrow U_\alpha \times \mathfrak{gl}_l(\mathbb{C})$ is a GH homeomorphism where $\mathfrak{gl}_l(\mathbb{C})$ is the complex Lie algebra of $GL_l(\mathbb{C})$;

2. getting $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathfrak{gl}_l(\mathbb{C}) \longrightarrow U_{\alpha\beta} \times \mathfrak{gl}_l(\mathbb{C})$$

defined as

$$\psi_{\alpha\beta}(m\,, X) = (m\,, \mathrm{Ad}(\phi_{\alpha\beta}(m)^{-1})(X)) \;\; \forall \;\; m \in U_{\alpha\beta} \;\text{ and }\; X \in \mathfrak{gl}_l(\mathbb{C})$$

is GH homeomorphism where $\phi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_l(\mathbb{C})$ is the transition map of P_E which is also a GH map by Proposition 4.1.2.

Similarly for $\mathrm{End}(E)$, there exist local trivializations $\{W_\alpha, \widetilde{\phi}_\alpha\}$ of $\mathrm{End}(E)$ over M such that

1. $\widetilde{\phi}_\alpha : \mathrm{End}(E)|_{U_\alpha} \longrightarrow U_\alpha \times \mathrm{End}(\mathbb{C}^l)$ is a GH homeomorphism;

2. again getting $W_{\alpha\beta} = W_\alpha \cap W_\beta \neq \emptyset$, the map

$$\widetilde{\psi}_{\alpha\beta} = \widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1} : W_{\alpha\beta} \times \mathrm{End}(\mathbb{C}^l) \longrightarrow W_{\alpha\beta} \times \mathrm{End}(\mathbb{C}^l)$$

defined as

$$\widetilde{\psi}_{\alpha\beta}(m\,, A) = (m\,, ((g^*_{\alpha\beta})^{-1} \otimes g_{\alpha\beta})(p)(A)) \;\; \forall \;\; m \in W_{\alpha\beta} \;\text{ and }\; A \in \mathrm{End}(\mathbb{C}^l)$$

is a GH homeomorphism where $g_{\alpha\beta} : W_{\alpha\beta} \longrightarrow GL_l(\mathbb{C})$ is the transition map of E which is also a GH map by [103, Proposition 3.2].

Without loss of generality, we can assume that $Ad(P_E)$ and $\mathrm{End}(E)$ both have the same local trivializations over M.

Now given a $\mathbb{C}$-vector space V of complex dimension l and $G = \mathrm{Aut}(V)$, we can have a canonical isomorphism $\mathfrak{gl}_l(\mathbb{C}) \cong \mathrm{End}(V)$. So then we can identify $\mathrm{End}(\mathbb{C}^l) = \mathfrak{gl}_l(\mathbb{C}) = M_l(\mathbb{C})$ where $M_l(\mathbb{C})$ is the set of all $l \times l$ complex matrices. Then we can see that for all $m \in U_{\alpha\beta}$ and $A \in M_l(\mathbb{C})$,

$$\mathrm{Ad}(\phi_{\alpha\beta}(m)^{-1})(A) = \phi_{\alpha\beta}(m)^{-1} A \phi_{\alpha\beta}(m) = ((\phi_{\alpha\beta}(m)^{-1})^* \otimes_{\mathbb{C}} \phi_{\alpha\beta}(m))(A).$$

In other words, $Ad(P_E)$ and $\mathrm{End}(E)$ both has the same transition maps with local trivializations $\{U_\alpha, \phi_\alpha\}$ over M. Hence $Ad(P_E)$ and $\mathrm{End}(E)$ are canonically isomorphic as SGH vector bundles, that is,

$$Ad(P_E) \cong \mathrm{End}(E)\,.$$

□

Corollary 4.3.1. $H^1(M, \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathrm{End}(\mathbf{E})) \cong H^1(M, \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathbf{Ad}(\mathbf{P_E}))$.

Theorem 4.3.1. *Let E be an SGH vector bundle over a regular GC manifold M. Let P be the associated SGH principal $GL_l(\mathbb{C})$-bundle over M, as in* (4.1.4)*, where l is the complex rank of E. Let $b(E)$ and $a(P)$ be the obstruction elements defined by $\mathcal{B}(E)$ and $\mathcal{A}(P)$, as in the equations* (4.3.1) *and* (4.2.8)*, respectively. Then*

$$a(P) = -b(E)\,.$$

Proof. Let E be an SGH vector bundle with local trivializations $\{U_\alpha, \phi_\alpha\}$ where

$$\phi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}^l\,, \tag{4.3.3}$$

are local GH homeomorphisms (cf. (4.1.1)). Then P is defined by the transition functions (cf. (4.1.2)),

$$\phi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_l(\mathbb{C})\,, \tag{4.3.4}$$

where

$$\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times GL_l(C) \longrightarrow U_{\alpha\beta} \times GL_l(\mathbb{C}) \tag{4.3.5}$$

is given by $\psi_{\alpha\beta}(m\,, g) = (m\,, \phi_{\alpha\beta}(m)g)$.

Let $W = \mathbb{C}^l$ so that $E \cong P \times_{GL_l(\mathbb{C})} W$. Let $M_W = M \times W$, a trivial SGH vector bundle over M. The GH homeomorphism ϕ_α induces a sheaf isomorphism of $\mathcal{O}_M|_{U_\alpha}$-modules

$$\widetilde{\phi}_\alpha : \mathbf{E}|_{U_\alpha} \longrightarrow \mathbf{M_W}|_{U_\alpha}\,. \tag{4.3.6}$$

This also induces another canonical $\mathcal{O}_M|_{U_\alpha}$-module isomorphism, again denoted by $\widetilde{\phi}_\alpha$,

$$\widetilde{\phi}_\alpha = \widetilde{\phi}_\alpha \otimes_{\mathcal{O}_M} \mathrm{Id} : \mathbf{E}|_{U_\alpha} \otimes_{\mathcal{O}_M|_{U_\alpha}} \mathcal{G}^*\mathbf{M}|_{U_\alpha} \longrightarrow \mathbf{M_W}|_{U_\alpha} \otimes_{\mathcal{O}_M|_{U_\alpha}} \mathcal{G}^*\mathbf{M}|_{U_\alpha} . \tag{4.3.7}$$

Note that $f \in \mathcal{O}_M$ if and only if $df \in \mathcal{G}^*\mathbf{M}$. Thus the exterior derivative map $d : \mathcal{O}_M \longrightarrow \mathcal{G}^*\mathbf{M}$ is well-defined. Since M_W is a trivial bundle,

$$\mathbf{M_W}|_{U_\alpha} \cong_{\mathcal{O}_M|_{U_\alpha}} \bigoplus_r \mathcal{O}_M|_{U_\alpha} \tag{4.3.8}$$

for some $r \in \mathbb{N}$. Thus we can extend d to a $\mathbb{C}$-linear sheaf homomorphism,

$$d : \mathbf{M_W}|_{U_\alpha} \longrightarrow \mathbf{M_W}|_{U_\alpha} \otimes_{\mathcal{O}_M|_{U_\alpha}} \mathcal{G}^*\mathbf{M}|_{U_\alpha} . \tag{4.3.9}$$

Define a $\mathbb{C}$-homomorphism of sheaves over U_α,

$$D_\alpha : \mathbf{E}|_{U_\alpha} \longrightarrow \mathbf{E}|_{U_\alpha} \otimes_{\mathcal{O}_M|_{U_\alpha}} \mathcal{G}^*\mathbf{M}|_{U_\alpha} \tag{4.3.10}$$

by

$$D_\alpha(s) = (\widetilde{\phi}_\alpha)^{-1} d\widetilde{\phi}_\alpha(s)$$

where the first $\widetilde{\phi}_\alpha$, d and the second $\widetilde{\phi}_\alpha$ are as in the equations (4.3.7), (4.3.9) and (4.3.6), respectively. Now consider the sheaf homomorphism

$$b_\alpha : \mathbf{E}|_{U_\alpha} \longrightarrow \mathbf{J_1}(\mathbf{E})|_{U_\alpha} \tag{4.3.11}$$

defined by

$$b_\alpha(s) = s + D_\alpha(s) \quad \text{for all } s \in \mathbf{E}|_{U_\alpha} .$$

Then for any $f \in \mathcal{O}_M|_{U_\alpha}$ and $s \in \mathbf{E}|_{U_\alpha}$, we have

$$\begin{aligned} b_\alpha(fs) &= fs + (\widetilde{\phi}_\alpha)^{-1} d\widetilde{\phi}_\alpha(fs) \\ &= fs \oplus (s \otimes df + f(\widetilde{\phi}_\alpha)^{-1} d\widetilde{\phi}_\alpha(s)) \\ &= f \cdot (s + D_\alpha(s)) . \end{aligned}$$

Hence, b_α is an $\mathcal{O}_M$-module homomorphism. Consider the sheaf homomorphism

$$b_{\alpha\beta} : \mathbf{E}|_{U_{\alpha\beta}} \longrightarrow \mathbf{J_1}(\mathbf{E})|_{U_{\alpha\beta}} \tag{4.3.12}$$

defined by $b_{\alpha\beta} := b_\beta - b_\alpha$. Note that $b_{\alpha\beta}(s) = D_\beta(s) - D_\alpha(s)$. So

$$b_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathrm{Hom}_{\mathcal{O}_M}(\mathbf{E}, \mathbf{E} \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M})) .$$

This shows that $\{b_{\alpha\beta}\}$ is a representative 1-cocycle for $b(E)$ in $H^1(M, \mathrm{End}(\mathbf{E}) \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M})$.

Consider the following two sheaf homomorphisms over U_α,

$$\widetilde{\phi}_{\alpha\beta} = \widetilde{\phi}_\alpha \circ (\widetilde{\phi}_\beta)^{-1} : \mathbf{M_W}|_{U_{\alpha\beta}} \longrightarrow \mathbf{M_W}|_{U_{\alpha\beta}}\,, \tag{4.3.13}$$

and the second one, also denoted by $\widetilde{\phi}_{\alpha\beta}$,

$$\widetilde{\phi}_{\alpha\beta} : \mathbf{M_W}|_{U_{\alpha\beta}} \otimes_{\mathcal{O}_M|_{U_{\alpha\beta}}} \mathcal{G}^*\mathbf{M}|_{U_{\alpha\beta}} \longrightarrow \mathbf{M_W}|_{U_{\alpha\beta}} \otimes_{\mathcal{O}_M|_{U_{\alpha\beta}}} \mathcal{G}^*\mathbf{M}|_{U_{\alpha\beta}}\,. \tag{4.3.14}$$

By (4.3.8), we can see that $\widetilde{\phi}_{\alpha\beta}$ can be thought of as a $\mathcal{O}_M|_{U_{\alpha\beta}}$-valued matrix, again denoted by

$$\widetilde{\phi}_{\alpha\beta} : \bigoplus_r \mathcal{O}_M|_{U_{\alpha\beta}} \longrightarrow \bigoplus_r \mathcal{O}_M|_{U_{\alpha\beta}}.$$

So $d(\widetilde{\phi}_{\alpha\beta})$ is well understood. Then for any $s \in \mathbf{M_W}|_{U_{\alpha\beta}}$, we get

$$\begin{aligned}
\widetilde{\phi}_\alpha b_{\alpha\beta}(\widetilde{\phi}_\alpha)^{-1}(s) &= \widetilde{\phi}_\alpha(D_\beta((\widetilde{\phi}_\alpha)^{-1}(s)) - D_\alpha((\widetilde{\phi}_\alpha)^{-1}(s))) \\
&= \widetilde{\phi}_\alpha((\widetilde{\phi}_\beta)^{-1} d(\widetilde{\phi}_\beta(\widetilde{\phi}_\alpha)^{-1}(s)) - (\widetilde{\phi}_\alpha)^{-1}(ds)) \\
&= \widetilde{\phi}_{\alpha\beta}(d(\widetilde{\phi}^{-1}_{\alpha\beta}(s))) - ds \\
&= \widetilde{\phi}_{\alpha\beta}\, d(\widetilde{\phi}^{-1}_{\alpha\beta}) \cdot s \\
&= -d(\widetilde{\phi}_{\alpha\beta})\, \widetilde{\phi}^{-1}_{\alpha\beta} \cdot s\,.
\end{aligned} \tag{4.3.15}$$

But, in the notation of Subsection 4.2.2, $d(\widetilde{\phi}_{\alpha\beta})\, \widetilde{\phi}^{-1}_{\alpha\beta} = \xi_{\alpha\beta}$. Here, using $\mathfrak{g} = \mathfrak{gl}_l(\mathbb{C})$, we identify the three sheaves $\mathrm{Hom}_{\mathcal{O}_M}(\mathbf{M_W}, \mathbf{M_W} \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M})$, $\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}, \mathbf{M}_\mathfrak{g})$, and $\mathbf{M}_\mathfrak{g} \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M}$ via their respective canonical $\mathcal{O}_M$-module isomorphisms where $\mathcal{G}\mathbf{M}$, $\mathbf{M}_\mathfrak{g}$ are as in the equations (3.2.3) and (4.2.9) respectively.

Now, $\mathrm{Hom}_{\mathcal{O}_M}(\mathbf{E}, \mathbf{E} \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M})$ is isomorphic to $\mathbf{Ad}(\mathbf{P}) \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M}$ by Proposition 4.3.1. Therefore, upon identifying $\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{G}\mathbf{M}, \mathbf{Ad}(\mathbf{P}))$, $\mathbf{Ad}(\mathbf{P}) \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M}$, and also $\mathrm{Hom}_{\mathcal{O}_M}(\mathbf{E}, \mathbf{E} \otimes_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M})$ via canonical $\mathcal{O}_M$-module isomorphisms, we have, via equations (4.2.11) and (4.3.15), that

$$b_{\alpha\beta} = -a_{\alpha\beta}\,.$$

It follows that $a(P) = -b(E)$. □

4.4 Dolbeault cohomology of SGH vector bundles

In the preceding section, we introduced the concepts of GH connection and smooth generalized connection. Now, every connection theory naturally leads to the theory of its curvature. Considering the classical holomorphic case, we know that another significant aspect of curvature theory is the computation of the Atiyah class via curvature; see [7, Proposition 4]. The key ingredient for such a description is the classical Dolbeault cohomology for holomorphic vector bundles.

In this section, we establish the curvature theory for smooth generalized connections and GH connections in Subsection 4.4.2 by utilizing the local description presented in Subsection 4.2.2. Furthermore, we delve into developing a de Rham cohomology $H^{\bullet}_{D}(M)$ for regular GC manifold M in Proposition 4.4.3, and a Dolbeault cohomology $H^{\bullet,\star}_{d_L}(M,E)$ for an SGH vector bundle E in Corollary 4.4.2. These elements provide a crucial relationship between the curvature and the Atiyah class in Theorem 4.4.2. To achieve this description, we rely heavily on the associated Lie algebroid cohomology detailed in Subsection 2.4.2.

4.4.1 Cohomolgy Theory

Let $(M, \mathcal{J}_M)$ be a regular GC manifold with i-eigen bundle L_M. Then, $(TM \oplus T^*M) \otimes \mathbb{C} = L_M \oplus \overline{L_M}$. Note that

$$\overline{\mathcal{G}^*M} = \overline{L_M} \cap (T^*M \otimes \mathbb{C}) \quad \text{and} \quad \overline{\mathcal{G}M} = (\overline{\mathcal{G}^*M})^* \tag{4.4.1}$$

are also smooth vector bundles over M (cf. (3.2.1), (3.2.3)). Let k be the type of $\mathcal{J}_M$. So, on a coordinate neighborhood U (cf. (2.3.8), Corollary 2.3.1),

$$C^\infty(\overline{\mathcal{G}^*M}|_U) = \mathrm{Span}_{C^\infty(U)}\{d\overline{z_1}, \dots, d\overline{z_k}\} \quad \text{and} \quad C^\infty(\overline{\mathcal{G}M}|_U) = \mathrm{Span}_{C^\infty(U)}\{\frac{\partial}{\partial \overline{z_1}}, \dots, \frac{\partial}{\partial \overline{z_k}}\}\,.$$

Let $\mathscr{S}$ denote the induced regular transversely holomorphic, symplectic foliation of complex codimension k corresponding to $\mathcal{J}_M$. Let $d_{\mathscr{S}}$ denote the exterior derivative along the leaves. Define

$$F_M := \ker(d_{\mathscr{S}} : C^\infty_M \longrightarrow C^\infty(T^*\mathscr{S} \otimes \mathbb{C}))$$

as the sheaf of smooth $\mathbb{C}$-valued functions over M which are constant along the leaves. Note that $\mathcal{O}_M \leq F_M \leq C^\infty_M$. For any vector bundle E over M whose transition maps are leaf-wise constant, we denote the sheaf of smooth leaf-wise constant sections of E by $F_M(E)$.

The transition functions of $\mathcal{G}^*M$ and $\overline{\mathcal{G}^*M}$ are constant along the leaves of $\mathscr{S}$. On a coordinate neighborhood U (cf. (2.3.8)),

$$F_M(\overline{\mathcal{G}^*M}|_U) = \mathrm{Span}_{F_M(U)}\{d\overline{z_1}, \dots, d\overline{z_k}\}\,,$$

and

$$F_M(\overline{\mathcal{G}M}|_U) = \mathrm{Span}_{F_M(U)}\{\frac{\partial}{\partial \overline{z_1}}, \dots, \frac{\partial}{\partial \overline{z_k}}\}\,.$$

For any $p, q \geq 0$, define

$$\begin{aligned} \tilde{A}^{p,q} &:= C^\infty(\wedge^p \mathcal{G}^*M \otimes \wedge^q \overline{\mathcal{G}^*M})\,, \\ A^{p,q} &:= F_M(\wedge^p \mathcal{G}^*M \otimes \wedge^q \overline{\mathcal{G}^*M})\,. \end{aligned} \tag{4.4.2}$$

More specifically, given any open set $U \subseteq M$,

$$\begin{aligned} \tilde{A}^{p,q}(U) &= C^\infty(U, \wedge^p \mathcal{G}^*M) \otimes_{C^\infty(U)} C^\infty(U, \wedge^q \overline{\mathcal{G}^*M})\,, \\ A^{p,q}(U) &= F_M(\wedge^p \mathcal{G}^*M)(U) \otimes_{F_M(U)} F_M(\wedge^q \overline{\mathcal{G}^*M})(U)\,. \end{aligned} \tag{4.4.3}$$

Note that $A^{p,q} \leq \tilde{A}^{p,q}$ and $\tilde{A}^{p,q} = A^{p,q} \otimes_{F_M} C^\infty_M$. For any $l \in \{0, \dots, 2k\}$, denote $A^l = \bigoplus_{p+q=l} A^{p,q}$ and $\tilde{A}^l = \bigoplus_{p+q=l} \tilde{A}^{p,q}$. Thus, we get two bigraded sheaves, namely,

$$A := \bigoplus_{p,q} A^{p,q} \quad , \quad \tilde{A} := \bigoplus_{p,q} \tilde{A}^{p,q}\,. \tag{4.4.4}$$

To summarize, $\tilde{A}$ and A are the bigraded sheaves of germs of sections of $\bigoplus_{p,q}(\wedge^p \mathcal{G}^*M \otimes \wedge^q \overline{\mathcal{G}^*M})$, which are smooth and constant along the leaves, respectively.

Let $d : C^\infty(\wedge^\bullet T^*M \otimes \mathbb{C}) \longrightarrow C^\infty(\wedge^{\bullet+1} T^*M \otimes \mathbb{C})$ be the exterior derivative. By Proposition 3.4.1, we can see that $\mathcal{G}M$ and $\overline{\mathcal{G}M}$ both are integrable smooth sub-bundle of $TM \otimes \mathbb{C}$. Thus we can restrict d to $\tilde{A}^\bullet$, $A^\bullet$. We denote these restrictions by $\tilde{D}$ and D, respectively, that is,

$$\tilde{D} := d|_{\tilde{A}^\bullet} \quad , \quad D := d|_{A^\bullet}\,. \tag{4.4.5}$$

In particular, any $\omega \in A^{p,q}$ (respectively, $\tilde{A}^{p,q}$), is locally (cf. (2.3.8)) of the form

$$\omega = \sum_{I,J} f_{IJ}\, dz_I \wedge d\overline{z_J}\,,$$

where $f_{IJ} \in F_M(U)$ (respectively, $C^\infty(U)$), I, J are ordered subsets of $\{1, \dots, k\}$, and $dz_I = \bigwedge_{i \in I} dz_i$, $d\overline{z_J} = \bigwedge_{j \in J} d\overline{z_j}$. Then,

$$D\omega \,(\text{respectively, } \tilde{D}\omega) = \sum_{I,J} \partial f_{IJ}\; dz_I \wedge d\overline{z_J} + \sum_{I,J} \overline{\partial} f_{IJ}\; dz_I \wedge d\overline{z_J}\,, \tag{4.4.6}$$

where ∂f_{IJ} and $\overline{\partial} f_{IJ}$ are defined by

$$\partial f_{IJ} := \sum_{i=1}^{k} \frac{\partial f_{IJ}}{\partial z_i}\, dz_i\ , \qquad \overline{\partial} f_{IJ} := \sum_{i=1}^{k} \frac{\overline{\partial} f_{IJ}}{\partial \overline{z_i}}\, d\overline{z_i}\,. \tag{4.4.7}$$

We identify L_M^* with $\overline{L_M}$ via the symmetric bilinear form defined in (2.3.1), and consider the restrictions of d_L to $C^\infty(\wedge^\bullet \overline{\mathcal{G}^* M})$ and $d_{\overline{L}}$ to $C^\infty(\wedge^\bullet \mathcal{G}^* M)$ where d_L and $d_{\overline{L}}$ defined in the Subsection 2.4.2 by the equations (2.4.11) and (2.4.12), respectively. We denote these by $\tilde{d}_L$ and $\tilde{d}_{\overline{L}}$, respectively. In particular, for any $\omega \in C^\infty(\wedge^p \mathcal{G}^* M)$, locally we can write

$$\omega = \sum_I f_I\, dz_I\,.$$

Then,

$$\tilde{d}_{\overline{L}} \omega = \sum_I d_{\overline{L}} f_I|_{C^\infty(\mathcal{G}^* M)}\ dz_I\,.$$

We know that, for any $f \in C^\infty(U)$, $d_{\overline{L}} f \in L$ and $d_L f \in \overline{L}$. Therefore, if we restrict them to $C^\infty(\mathcal{G}^* M)$ and $C^\infty(\overline{\mathcal{G}^* M})$, respectively, we get that

$$d_{\overline{L}}|_{C^\infty(\mathcal{G}^* M)} f = \partial f\ , \quad d_L|_{C^\infty(\overline{\mathcal{G}^* M})} f = \overline{\partial} f\,,$$

where ∂f and $\overline{\partial} f$ are defined as in (4.4.7). We can further restrict d_L and $d_{\overline{L}}$ to $F_M(\wedge^\bullet \overline{\mathcal{G}^* M})$ and $F_M(\wedge^\bullet \mathcal{G}^* M)$ which we again denote by d_L and $d_{\overline{L}}$, respectively. Thus we can consider the following morphisms of sheaves

$$\begin{aligned} d_L &: F_M(\wedge^\bullet \overline{\mathcal{G}^* M}) \longrightarrow F_M(\wedge^{\bullet+1} \overline{\mathcal{G}^* M})\,, \\ d_{\overline{L}} &: F_M(\wedge^\bullet \mathcal{G}^* M) \longrightarrow F_M(\wedge^{\bullet+1} \mathcal{G}^* M)\,. \end{aligned} \tag{4.4.8}$$

Note that $d_L = \tilde{d}_L|_{F_M(\wedge^\bullet \overline{\mathcal{G}^* M})}$ and $d_{\overline{L}} = \tilde{d}_{\overline{L}}|_{F_M(\wedge^\bullet \mathcal{G}^* M)}$. They induce two differential complexes, namely $(F_M(\wedge^\bullet \overline{\mathcal{G}^* M}), d_L)$ and $(F_M(\wedge^\bullet \mathcal{G}^* M), d_{\overline{L}})$. Subsequently, we can naturally extend d_L and $d_{\overline{L}}$ to $A^{\bullet,\bullet}$, again denoted by d_L and $d_{\overline{L}}$ respectively, and get the following morphisms of sheaves

$$\begin{aligned} d_L &: A^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet+1}\,; \\ d_{\overline{L}} &: A^{\bullet,\bullet} \longrightarrow A^{\bullet+1,\bullet}\,. \end{aligned} \tag{4.4.9}$$

In particular, for any $\omega \in A^{p,q}$, locally

$$\omega = \sum_{I,J} f_{IJ}\, dz_I \wedge d\overline{z_J}\,.$$

Then,

$$d_L \omega = \sum_J d_L f_{IJ}|_{F_M(\overline{\mathcal{G}^* M})} \wedge dz_I \wedge d\overline{z_J}\,, \quad d_{\overline{L}} \omega = \sum_I d_{\overline{L}} f_{IJ}|_{F_M(\mathcal{G}^* M)} \wedge dz_I \wedge d\overline{z_J}\,. \tag{4.4.10}$$

By the equations (4.4.6) and (4.4.10), on $A^{\bullet,\bullet}$, we have

$$D = d_{\overline{L}} + d_L \quad \text{and} \quad D(A^{\bullet,\bullet}) \subseteq A^{\bullet+1,\bullet} \oplus A^{\bullet,\bullet+1}\,.$$

Similarly, one can see that $\tilde{D} = \tilde{d}_{\overline{L}} + \tilde{d}_L$ where $\tilde{d}_{\overline{L}}$ and $\tilde{d}_L$ are considered as a morphism of sheaves between $\tilde{A}^{\bullet,\bullet}$ to $\tilde{A}^{\bullet+1,\bullet}$ and $\tilde{A}^{\bullet,\bullet+1}$, respectively.

Definition 4.4.1. *Any element $\omega \in \tilde{A}^l$ is called a generalized form of order l and any element in $\tilde{A}^{p,q}$ is called a generalized form of type (p,q). Here $\tilde{A}^{p,q}, \tilde{A}^l$ are as in* (4.4.4).

Definition 4.4.2. *Any element $\omega \in A^l$ is called a transverse generalized form of degree l and any element in $A^{p,q}$ is called a transverse generalized form of type (p,q). Here $A^{p,q}, A^l$ are as in* (4.4.4).

Let $Z^\bullet = \ker(D : A^\bullet \longrightarrow A^{\bullet+1})$, i.e., the set of D-closed transverse generalized forms of degree l. Let $B^\bullet(M) = \operatorname{img}(D : A^{\bullet-1}(M) \longrightarrow A^\bullet(M))$, i.e., the set of D-exact transverse generalized forms of degree l. Then, the homology of the cochain complex $\{A^\bullet(M), D\}$ is called the D-cohomology of M, and it is denoted by

$$H_D^\bullet(M) := \frac{Z^\bullet(M)}{B^\bullet(M)} = \frac{\ker(D : A^\bullet(M) \longrightarrow A^{\bullet+1})(M)}{\operatorname{img}(D : A^{\bullet-1}(M) \longrightarrow A^\bullet(M))}\,. \tag{4.4.11}$$

Definition 4.4.3. *Let $(\mathcal{G}^*\mathbf{M})^p := \bigwedge^p_{\mathcal{O}_M} \mathcal{G}^*\mathbf{M}$ for $p \in \mathbb{N}$ and $(\mathcal{G}^*\mathbf{M})^0 := \mathcal{O}_M$. Note that $(\mathcal{G}^*\mathbf{M})^\bullet < F_M(\wedge^\bullet \mathcal{G}^* M)$. We say that a transverse generalized form ω of type $(p,0)$ is a GH p-form if $d_L\omega = 0$, that is, $\omega \in (\mathcal{G}^*\mathbf{M})^p$.*

Let N be another regular GC manifold and let $f : M \longrightarrow N$ be a GH map. Then it follows that

1. $f^*(A_N^{\bullet,\bullet}) \subset A_M^{\bullet,\bullet}$,
2. $f^* \circ d_{L_N} = d_{L_M} \circ f^*$.

Corollary 4.4.1. *Let M be a GC manifold. Given an open set $U \subseteq M$, a smooth map $\psi : (U, \mathcal{J}_U) \longrightarrow \mathbb{C}$ is a GH function, that is, $f \in \mathcal{O}_M(U)$, if and only if $d_L f = 0$ where d_L as defined in* (2.4.11).

Proof. Follows from Lemma 2.3.1. □

Let $Z^{\bullet,\bullet} = \ker(d_L : A^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet+1})$ and let $B^{\bullet,\bullet}(M) = \mathrm{img}(d_L : A^{\bullet,\bullet-1}(M) \longrightarrow A^{\bullet,\bullet}(M))$. Then the homology of the cochain complex $\{A^{\bullet,\bullet}(M), d_L\}$ is called d_L-*cohomology of* M and it is denoted by

$$H^{\bullet,\bullet}_{d_L}(M) := \frac{Z^{\bullet,\bullet}(M)}{B^{\bullet,\bullet}(M)} = \frac{\ker(d_L : A^{\bullet,\bullet}(M) \longrightarrow A^{\bullet,\bullet+1})(M)}{\mathrm{img}(d_L : A^{\bullet,\bullet-1}(M) \longrightarrow A^{\bullet,\bullet}(M))}\,. \tag{4.4.12}$$

One can also consider the homology of the cochain complex $\{\tilde{A}^{\bullet}(M), \tilde{D}\}$ which is called the $\tilde{D}$-cohomology of M, and is denoted by

$$H^{\bullet}_{\tilde{D}}(M) := \frac{\ker(\tilde{D} : \tilde{A}^{\bullet}(M) \longrightarrow \tilde{A}^{\bullet+1})(M)}{\mathrm{img}(\tilde{D} : \tilde{A}^{\bullet-1}(M) \longrightarrow \tilde{A}^{\bullet}(M))}\,.$$

Similarly, the homology of the cochain complex $\{\tilde{A}^{\bullet,\bullet}(M), \tilde{d}_L\}$ which will be called $\tilde{d}_L$-*cohomology of* M, and denoted by

$$H^{\bullet,\bullet}_{\tilde{d}_L}(M) := \frac{\ker(\tilde{d}_L : \tilde{A}^{\bullet,\bullet}(M) \longrightarrow \tilde{A}^{\bullet,\bullet+1})(M)}{\mathrm{img}(\tilde{d}_L : \tilde{A}^{\bullet,\bullet-1}(M) \longrightarrow \tilde{A}^{\bullet,\bullet}(M))}\,.$$

We know that locally (cf. (2.3.8)),

$$C^{\infty}(\overline{\mathcal{G}^*M}|_U) = \mathrm{Span}_{C^{\infty}(U)}\{d\overline{z_1}, \dots, d\overline{z_k}\}\,,$$

and

$$C^{\infty}(\mathcal{G}^*M|_U) = \mathrm{Span}_{C^{\infty}(U)}\{dz_1, \dots, dz_k\}\,,$$

where k is the type of M. Then, by following [68, P-25 , P-42], one immediately obtains the result below.

Proposition 4.4.1. *Let M be a regular GC manifold of type k. Then for any $q > 0$,*

1. *$\tilde{d}_L$-Poincaré Lemma: For sufficiently small open set $U \subset M$, $H^{\bullet,q}_{\tilde{d}_L}(U) = 0$.*

2. *$H^q(M, \tilde{A}^{\bullet,\bullet}) = 0$.*

3. *$\tilde{D}$-Poincaré Lemma: For a sufficiently small open set $U \subset M$, $H^q_{\tilde{D}}(U) = 0$.*

4. *$H^q(M, \tilde{A}^{\bullet}) = 0$.*

Definition 4.4.4. *An open cover $\mathcal{U} = \{U_\alpha\}$ of M is called a transverse good cover if $\mathcal{U}$ is a locally finite transverse open cover (cf. Definition 3.4.1) and any finite intersection $\bigcap_{i=0}^{l} U_{\alpha_i}$ is diffeomorphic to a tubular neighborhood as in Theorem 3.4.1.*

Proposition 4.4.2. *Let M be a regular GC manifold of type k. Assume $M/\mathscr{S}$ has a smooth orbifold structure. Let $\mathfrak{U} = \{U_\alpha\}$ be a sufficiently fine transverse good cover of M. Then for any $q > 0$,*

1. *d_L-Poincaré Lemma: For a sufficiently small transverse open set $U \subset M$,*
$$H^{\bullet,q}_{d_L}(U) = 0\,.$$
2. *D-Poincaré Lemma: For a sufficiently small transverse open set $U \subset M$,*
$$H^q_D(U) = 0\,.$$
3. *$H^q(\mathfrak{U}, A^{\bullet,\bullet}) = 0\,.$*
4. *$H^q(\mathfrak{U}, A^{\bullet}) = 0\,.$*

Proof. By Theorem 3.4.1, there exists a transverse open set (tubular neighborhood) U around a leaf S which is diffeomorphic to $\tilde{S} \times_{\mathrm{Hol}(S)} \mathbb{C}^k$ where $\tilde{S}$ is the universal cover of S and $\mathrm{Hol}(S)$ is the holonomy group of S. Since $\mathrm{Hol}(S)$ is finite, it acts linearly. Recall that $\mathcal{N}^* \otimes \mathbb{C} = \mathcal{G}^*M \oplus \overline{\mathcal{G}^*M}$ where $\mathcal{N}$ is the normal bundle of $\mathscr{S}$. Taking U to be sufficiently small, we have

$$F_M(\overline{\mathcal{G}^*M}|_U) = \mathrm{Span}_{F_M(U)}\{d\overline{z_1}, \ldots, d\overline{z_k}\}\,,$$

and

$$F_M(\mathcal{G}^*M|_U) = \mathrm{Span}_{F_M(U)}\{dz_1, \ldots, dz_k\}\,.$$

Then following the proof in [68, P-25 , P-42], we can prove (1) and (2).

To prove (3) and (4), it is enough to show that there is a partition of unity subordinate to $\mathfrak{U}$ such that they are constant along the leaves. This is obtained easily by pulling back a partition of unity for $M/\mathscr{S}$ subordinate to $\hat{\mathfrak{U}} = (\tilde{\pi}(U_\alpha))$ with respect to the quotient map $\tilde{\pi} : M \longrightarrow M/\mathscr{S}$. □

Proposition 4.4.3. *(de Rham cohomology for regular GC manifold) Let M be a regular GC manifold with induced symplectic foliation $\mathscr{S}$. Assume the leaf space $M/\mathscr{S}$ admits a smooth orbifold structure. Then for $q \geq 0$,*

$$H^q(\mathfrak{U}, \{\mathbb{C}\}) \cong H^q_D(M)\,,$$

where $\{\mathbb{C}\}$ is the sheaf of locally constant $\mathbb{C}$-valued functions and $\mathfrak{U}$ is a sufficeiently fine transverse good cover of M.

Proof. By D-Poincaré Lemma, we have the following exact sequence of sheaves

$$0 \longrightarrow \{\mathbb{C}\} \hookrightarrow A^0 \xrightarrow{D} A^1 \xrightarrow{D} \cdots \tag{4.4.13}$$

on M. This gives the following exact sequence,

$$0 \longrightarrow Z^{\bullet} \hookrightarrow A^{\bullet} \xrightarrow{D} Z^{\bullet+1} \longrightarrow 0\,. \tag{4.4.14}$$

In particular, the sequence

$$0 \longrightarrow \{\mathbb{C}\} \hookrightarrow A^0 \xrightarrow{D} Z^1 \longrightarrow 0 \tag{4.4.15}$$

is exact. By (4) in Proposition 4.4.2, $H^q(\mathfrak{U}, A^{\bullet}) = 0$ for all $q > 0$. Thus, considering the associated long exact sequences in cohomology for these exact sequences of sheaves, as in [68, pp. 40-41, 44], we obtain that for all $q \geq 0$

$$\begin{aligned}
H^q(\mathfrak{U}, \{\mathbb{C}\}) &\cong H^{q-1}(\mathfrak{U}, Z^1) \quad (\text{by } (4.4.15)) \\
&\cong H^{q-2}(\mathfrak{U}, Z^2) \quad (\text{by } (4.4.14)) \\
&\vdots \\
&\cong H^1(\mathfrak{U}, Z^{q-1}) \quad (\text{by } (4.4.14)) \\
&\cong \frac{H^0(\mathfrak{U}, Z^q)}{D(H^0(\mathfrak{U}, Z^{q-1}))} \\
&= \frac{Z^q(M)}{B^q(M)} = H^q_D(M)\,.
\end{aligned}$$

□

Theorem 4.4.1. *(Dolbeault cohomology for regular GC manifold) Let M be a regular GC manifold with induced symplectic foliation $\mathscr{S}$. Assume that the leaf space $M/\mathscr{S}$ admits an orbifold structure. Then for any $p, q \geq 0$,*

$$H^q(M, (\mathcal{G}^*\mathbf{M})^p) \cong H^{p,q}_{d_L}(M)\,.$$

Proof. By d_L-Poincaré Lemma, the following sequences of sheaves,

$$0 \longrightarrow (\mathcal{G}^*\mathbf{M})^{\bullet} \hookrightarrow A^{\bullet,0} \xrightarrow{d_L} Z^{\bullet,1} \longrightarrow 0\,, \tag{4.4.16}$$

$$0 \longrightarrow Z^{\bullet,\bullet} \hookrightarrow A^{\bullet,\bullet} \xrightarrow{d_L} Z^{\bullet,\bullet+1} \longrightarrow 0 \tag{4.4.17}$$

are exact. Let $\mathcal{U}$ be a sufficiently fine transverse good cover of M. By (3) in Proposition 4.4.2, $H^r(\mathcal{U}, A^{\bullet,\bullet}) = 0$ for all $r > 0$. Hence, using the long exact sequences in cohomology associated with (4.4.16) and (4.4.17) (cf. [68, pp. 40-41]), we have,

$$\begin{aligned}
H^q(\mathcal{U}, (\mathcal{G}^*\mathbf{M})^p) &\cong H^{q-1}(\mathcal{U}, Z^{p,1}) \quad (\text{by } (4.4.16)) \\
&\cong H^{q-2}(\mathcal{U}, Z^{p,2}) \quad (\text{by } (4.4.17)) \\
&\cong H^{q-3}(\mathcal{U}, Z^{p,3}) \quad (\text{by } (4.4.17)) \\
&\vdots \\
&\cong H^{1}(\mathcal{U}, Z^{p,q-1}) \quad (\text{by } (4.4.17)) \\
&\cong \frac{H^0(\mathcal{U}, Z^{p,q})}{d_L(H^0(\mathcal{U}, Z^{p,q-1}))} \\
&= \frac{Z^{p,q}(M)}{B^{p,q}(M)} = H^{p,q}_{d_L}(M)\,.
\end{aligned}$$

We can choose $\mathcal{U} = \{U_\alpha\}$ such that any finite intersection $V = \bigcap_{i=0}^{l} U_{\alpha_i}$ is diffeomorphic a tubular neighborhood as in Theorem 3.4.1. Fix such a V. Then $\mathcal{V} := \{V \cap U_\alpha\}$ is a transverse good cover of V. Note that $H^q(V, A^{\bullet,\bullet}) = 0$. Then, as above,

$$H^q(\mathcal{V}, (\mathcal{G}^*\mathbf{M})^p|_V) = H^{p,q}_{d_L}(V)\,.$$

Since $H^{p,q}_{d_L}(W) = 0$ for any finite intersection W of elements in $\mathcal{V}$ by the d_L-Poincaré Lemma, using Leray's theorem we have,

$$H^q(V, (\mathcal{G}^*\mathbf{M})^p|_V) = H^q(\mathcal{V}, (\mathcal{G}^*\mathbf{M})^p|_V) = H^{p,q}_{d_L}(V)\,.$$

Again, by the d_L-Poincaré Lemma, $H^{p,q}_{d_L}(V) = 0$. Thus, by Leray's Theorem, for all $p, q \geq 0$,

$$H^q(\mathcal{U}, (\mathcal{G}^*\mathbf{M})^p) \cong H^q(M, (\mathcal{G}^*\mathbf{M})^p)\,.$$

□

Let E be an SGH vector bundle over M. We put

$$A_E := A \otimes_{\mathcal{O}_M} \mathbf{E}\,, \tag{4.4.18}$$

where A is as in (4.4.4). Since A is an F_M-module, it is also $\mathcal{O}_M$-module, and thus, A_E is well-defined. We can naturally extend d_L from A to A_E, denoted by d_L', as follows: For

$f \in F_M$, $\alpha \in A$ and $\beta \in \mathbf{E}$, we can define $d_L^{'}(f\alpha \wedge \beta) = f d_L(\alpha) \wedge \beta$. This definition is well-defined because if $f \in \mathcal{O}_M$, we have $f\alpha \wedge \beta = \alpha \wedge f\beta$. Then

$$\begin{aligned} d_L^{'}(f\alpha \wedge \beta) &= f d_L \alpha \wedge \beta \quad (\text{as } d_L f = 0) \\ &= d_L \alpha \wedge (f\beta) \\ &= d_L^{'}(\alpha \wedge f\beta)\,. \end{aligned}$$

For notational convenience, we again denote by d_L or by $d_{L,E}$, the natural extension d'_L of the operator d_L. We denote the component of type (p, q) of the cohomology of the complex $(H^0(M, A_E), d_L)$ by $H^{p,q}_{d_L}(M, E)$. Note that tensoring the short exact sequences (4.4.16) and (4.4.17) with the locally free sheaf $\mathbf{E}$ again yields short exact sequences. Then, following the proof of Theorem 4.4.1, we get the following.

Corollary 4.4.2. $H^{p,q}_{d_L}(M, E) \cong H^q(M, (\mathcal{G}^*\mathbf{M})^p \otimes_{\mathcal{O}_M} \mathbf{E})$.

Suppose, the leaf space $\mathscr{M} := M/\mathscr{S}$ is a smooth manifold. Then, Theorem 3.4.5 shows that, for $l, p, q \geq 0$, A^l (respectively, $A^{p,q}$), as defined in (4.4.2), is isomorphic to $\tilde{\pi}^{-1}(\Omega^l_{\mathscr{M}})$ (respectively, $\tilde{\pi}^{-1}(\Omega^{p,q}_{\mathscr{M}})$) where $\Omega^l_{\mathscr{M}}$ is the sheaf of $\mathbb{C}$-valued smooth l-forms on $\mathscr{M}$. In particular, The map $\tilde{\tilde{\pi}}^{\#}$, in Subsection 3.4.3, induces the pullback map $\tilde{\pi}^*$ from $\Omega^l_{\mathscr{M}}(\mathscr{M})$ (respectively, $\Omega^{p,q}_{\mathscr{M}}(\mathscr{M})$) to $A^l(M)$ (respectively, $A^{p,q}(M)$) which is an isomorphism of $\mathbb{C}$-vector spaces. By the definitions of D and d_L (see (4.4.5) and (4.4.9)), we have the following commutative diagrams.

$$\begin{array}{ccc} \Omega^l_{\mathscr{M}}(\mathscr{M}) & \xrightarrow{\tilde{\pi}^*(\cong)} & A^l(M) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle D} \\ \Omega^{l+1}_{\mathscr{M}}(\mathscr{M}) & \xrightarrow[\tilde{\pi}^*(\cong)]{} & A^l(M) \end{array} \qquad \begin{array}{ccc} \Omega^{p,q}_{\mathscr{M}}(\mathscr{M}) & \xrightarrow{\tilde{\pi}^*(\cong)} & A^{p,q}(M) \\ {\scriptstyle \overline{\partial}}\downarrow & & \downarrow{\scriptstyle d_L} \\ \Omega^{p,q+1}_{\mathscr{M}}(\mathscr{M}) & \xrightarrow[\tilde{\pi}^*(\cong)]{} & A^{p,q+1}(M) \end{array}$$

This shows that we have surjective homomorphisms at the level of de Rham cohomology and Dolbeault cohomology, respectively:

$$\tilde{\pi}^* : H^l_{dR}(\mathscr{M}, \mathbb{C}) \longrightarrow H^l_D(M) \quad \text{and} \quad \tilde{\pi}^* : H^{p,q}_{\overline{\partial}}(\mathscr{M}) \longrightarrow H^{p,q}_{d_L}(M)\,.$$

Since $\tilde{\pi}$ is a submersion, $\tilde{\pi}^*$ is one-to-one. Consequently, $\tilde{\pi}^*$ is an isomorphism of $\mathbb{C}$-vector spaces. Thus, we can conclude the following.

Corollary 4.4.3. *Let M be a regular GC manifold such that the leaf space $\mathscr{M}$ of the induced foliation is a smooth manifold. Then,*

$$H^{\bullet}_{dR}(\mathscr{M}, \mathbb{C}) \cong H^{\bullet}_D(M) \quad \textit{and} \quad H^{\bullet,\bullet}_{\overline{\partial}}(\mathscr{M}) \cong H^{\bullet,\bullet}_{d_L}(M)\,.$$

Let V be a holomorphic vector bundle over $\mathscr{M}$. Note that, by Lemma 3.1.3, any holomorphic vector bundle is an SGH vector bundle over $\mathscr{M}$ and vice versa. Let $\mathbf{V}$ be the sheaf of holomorphic sections on $\mathscr{M}$. For, $p, q \geq 0$, consider

$$\Omega^{p,q}_{\mathscr{M}}(V) := \Omega^{p,q}_{\mathscr{M}} \otimes_{\mathcal{O}_{\mathscr{M}}} \mathbf{V}\,,$$

the sheaf of V-valued differential forms of type (p, q) , with the natural extension of $\overline{\partial}$-operator on $H^0(\mathscr{M}, \Omega^{p,q}_{\mathscr{M}}(V))$, again denoted by $\overline{\partial}$. Let $H^{\bullet,\bullet}_{\overline{\partial}}(\mathscr{M}, V)$ denote the V-valued Dolbeault cohomology of $\mathscr{M}$. Consider the SGH vector bundle $E := \tilde{\pi}^*V$ over M. Then, preceding discussions and Theorem 3.4.5 show that, for $p, q \geq 0$, $A^{p,q}_E$, as defined in (4.4.18), is isomorphic to $\tilde{\pi}^{-1}(\Omega^{p,q}_{\mathscr{M}}(V)))$. In particular, the pullback map $\tilde{\pi}^*$, induced by the map $\tilde{\tilde{\pi}}^{\#}$ (cf. Subsection 3.4.3) provides the following isomorphism of $\mathbb{C}$-valued vector spaces,

$$\tilde{\pi}^* : H^0(\mathscr{M}, \Omega^{p,q}_{\mathscr{M}}(V)) \longrightarrow A^{p,q}_E(M)\,,$$

where A_E as defined in 4.4.18. Consequently, we have the following commutative diagram.

$$\begin{array}{ccc} H^0(\mathscr{M}, \Omega^{p,q}_{\mathscr{M}}(V)) & \xrightarrow{\tilde{\pi}^*(\cong)} & A^{p,q}_E(M) \\ \Big\downarrow \overline{\partial} & & \Big\downarrow d_L \\ H^0(\mathscr{M}, \Omega^{p,q+1}_{\mathscr{M}}(V)) & \xrightarrow[\tilde{\pi}^*(\cong)]{} & A^{p,q+1}_E(M) \end{array}$$

This shows that $\tilde{\pi}^*$ is surjective at the level of Dolbeault cohomology. Since $\tilde{\pi}^*$: $H^{\bullet,\bullet}_{\overline{\partial}}(\mathscr{M}, V) \longrightarrow H^{\bullet,\bullet}_{d_L}(M, E)$ is one-to-one, $\tilde{\pi}^*$ is an isomorphism. Thus we get the following extension of Corollary 4.4.3.

Corollary 4.4.4. *Let M be a regular GC manifold such that the leaf space $\mathscr{M}$ of the induced foliation is a smooth manifold. Let $\tilde{\pi} : M \longrightarrow \mathscr{M}$ be the quotient map. Let V be a holomorphic vector bundle on $\mathscr{M}$. Then,*

$$H^{\bullet,\bullet}_{\overline{\partial}}(\mathscr{M}, V) \cong H^{\bullet,\bullet}_{d_L}(M, E) \;\; (\textit{via } \tilde{\pi}^*) \;\; \textit{where} \;\; E = \tilde{\pi}^*V\,.$$

Remark 4.4.1. *Corollary 4.4.4 is useful for studying* $\dim_{\mathbb{C}} H^{\bullet,\bullet}_{d_L}(M, E)$. *This can be illustrated as follows:*

Let $\mathcal{O}_{\mathbb{CP}^n}(1)$ denote the hyperplane bundle over $\mathbb{CP}^n$, and let $\mathcal{O}_{\mathbb{CP}^n}(-1) := \mathcal{O}_{\mathbb{CP}^n}(1)^*$ be the tautological line bundle over $\mathbb{CP}^n$. For, $m \in \mathbb{Z}$, set

$$\mathcal{O}_{\mathbb{CP}^n}(m) = \begin{cases} \mathcal{O}_{\mathbb{CP}^n}(-1)^{\otimes m} & \textit{for } m \leq 0, \\ \mathcal{O}_{\mathbb{CP}^n}(1)^{\otimes m} & \textit{for } m \geq 0. \end{cases}$$

Let M be a regular GC manifold with the leaf space $\mathbb{CP}^n$. Consider the following SGH vector bundle over M,

$$\mathcal{O}_M(m) := \tilde{\pi}^*\mathcal{O}_{\mathbb{CP}^n}(m) \text{ where } m \in \mathbb{Z} \text{ and } \tilde{\pi} : M \longrightarrow \mathbb{CP}^n \text{ is the quotient map}.$$

By Corollary 4.4.2 and Corollary 4.4.4, for $m \in \mathbb{Z}$, we have

$$\dim_{\mathbb{C}} H^\bullet(M, (\mathcal{G}^*\mathbf{M})^\bullet \otimes_{\mathcal{O}_M} \mathcal{O}_M(m)) = \dim_{\mathbb{C}} H^\bullet(\mathbb{CP}^n, \Omega^\bullet_{\mathbb{CP}^n} \otimes_{\mathcal{O}_{\mathbb{CP}^n}} \mathcal{O}_{\mathbb{CP}^n}(m)),$$

where $\Omega^\bullet_{\mathbb{CP}^n}$ denotes the sheaf of holomorphic $\bullet$-forms on $\mathbb{CP}^n$. Then, using Bott formula (cf. [28] and [124, Chapter 1]), for $p, q \geq 0$, we get,

$$\dim_{\mathbb{C}} H^q(M, (\mathcal{G}^*\mathbf{M})^p \otimes_{\mathcal{O}_M} \mathcal{O}_M(m)) = \begin{cases} \binom{m+n-p}{m}\binom{m-1}{p} & \text{for } q = 0, 0 \leq p \leq n, m > p\,; \\ \binom{-m+p}{-m}\binom{-m-1}{n-p} & \text{for } q = n, 0 \leq p \leq n, m < p-n\,; \\ 1 & \text{for } m = 0, 0 \leq p = q \leq n\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

In particular, for $p = 0$, we have

$$\dim_{\mathbb{C}} H^q(M, \mathcal{O}_M(m)) = \begin{cases} \binom{m+n}{m} & \text{for } q = 0, m \geq 0\,; \\ \binom{-m-1}{-m-1-n} & \text{for } q = n, m \leq -n-1\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

4.4.2 Cohomology class of the curvature

Let G be a complex Lie group with complex Lie algebra $\mathfrak{g}$. Let $G \hookrightarrow P \longrightarrow M$ be an SGH principal bundle. Then, by applying Corollary 4.4.2 to $\mathbf{Ad}(\mathbf{P})$ where $Ad(P)$ is as in (4.2.7), we have the following.

Corollary 4.4.5. *$H^q(M, (\mathcal{G}^*\mathbf{M})^p \otimes_{\mathcal{O}_M} \mathbf{Ad}(\mathbf{P})) \cong H^{p,q}_{d_L}(M, Ad(P))$.*

Let $\Theta = \{\Theta_\alpha\} \in \tilde{A}^{1,0} \otimes_{\mathcal{O}_M} \mathbf{Ad}(\mathbf{P})$ be a smooth generalized connection on P (see Definition 4.2.3 and Section 4.2.2), where

$$\Theta_\alpha \in C^\infty(U_\alpha, \mathrm{Hom}_{C^\infty_M}(C^\infty(\mathcal{G}M), C^\infty(M_{\mathfrak{g}})))$$

on a trivializing neighborhood U_α of P. Then, the curvature of this smooth generalized connection is defined on U_α by

$$\Omega_\alpha := \tilde{D}\Theta_\alpha + \frac{1}{2}[\Theta_\alpha, \Theta_\alpha] \tag{4.4.19}$$

where Θ_α is considered as $\mathfrak{g}$ valued function. Then, by using either of the equations (4.2.15) or (4.2.16), we get

$$\begin{gathered} \Omega_\alpha = \mathrm{Ad}(\phi_{\alpha\beta})\,\Omega_\beta \quad \text{on } U_\alpha \cap U_\beta\,, \\ \text{or} \\ \Omega_\beta = \mathrm{Ad}(\phi_{\beta\alpha})\,\Omega_\alpha \quad \text{on } U_\alpha \cap U_\beta\,. \end{gathered} \tag{4.4.20}$$

So, after patching, we get an element $\Omega \in C^\infty(M, \wedge^2(\mathcal{G}M \oplus \overline{\mathcal{G}M})^*) \otimes Ad(P))$. Ω is called the curvature of the smooth generalized connection Θ.

Let $\Theta \in A^{1,0}_{Ad(P)}$ be such that $\Theta_\alpha \in A^{1,0}_{M_\mathfrak{g}}$ for every α. This type of connection always exists due to the existence of partition of unity on the orbifold leaf space $M/\mathscr{S}$. We can then reformulate equation (4.4.19) as

$$\Omega_\alpha = D\Theta_\alpha + \frac{1}{2}[\Theta_\alpha, \Theta_\alpha] \quad \text{on } U_\alpha\,. \tag{4.4.21}$$

This shows that the $(1,1)$ component of Ω_α, denoted by $\Omega^{1,1}_\alpha$, is given by

$$\Omega^{1,1}_\alpha = d_L\Theta_\alpha \quad \text{on } U_\alpha\,. \tag{4.4.22}$$

By (4.4.20), we have

$$\begin{gathered} \Omega^{1,1}_\alpha = \mathrm{Ad}(\phi_{\alpha\beta})\Omega^{1,1}_\beta \quad \text{on } U_\alpha \cap U_\beta\,, \\ \text{or} \\ \Omega^{1,1}_\beta = \mathrm{Ad}(\phi_{\beta\alpha})\Omega^{1,1}_\alpha \quad \text{on } U_\alpha \cap U_\beta\,. \end{gathered} \tag{4.4.23}$$

After patching, we get a global element $\Omega \in \Big(F_M(\wedge^2(\mathcal{G}^*M \oplus \overline{\mathcal{G}^*M})) \otimes_{\mathcal{O}_M} \mathbf{Ad(P)}\Big)(M)$ whose $(1,1)$-component is $\Omega^{1,1}$. Then from the equations (4.2.15), (4.2.16), (4.4.22) and (4.4.23), we can see that the d_L-cohomology class $[\Omega^{1,1}]$ of $\Omega^{1,1}$ is independent of the choice of a smooth generalized connection of type $(1,0)$. Note that $[\Omega^{1,1}]$ maps to $a(P)$, as defined in Theorem 4.2.2, via the isomorphism in Corollary 4.4.5. We summarise our results as follows.

Theorem 4.4.2. *Let P be an SGH principal G-bundle over a regular GC manifold M where G is a complex Lie group. Assume that the leaf space of the induced symplectic foliation on M admits a smooth orbifold structure. Let Θ be a smooth generalized* *connection of type* $(1,0)$ *on* P*, which is constant along the leaves. Let* $\Omega^{1,1}$ *denote the corresponding* $(1,1)$ *component of the curvature. Let* $[\Omega^{1,1}]$ *be the* d_L*-cohomology class in* $H^{1,1}_{d_L}(M, Ad(P))$*. Then* $[\Omega^{1,1}]$ *corresponds to* $a(P) \in H^1(M, \mathcal{G}^*\mathbf{M} \otimes_{\mathcal{O}_M} \mathbf{Ad}(\mathbf{P}))$ *via the isomorphism in Corollary 4.4.5.*

4.5 Generalized Chern-Weil Theory and characteristic classes

An application of the theory of curvature can be seen in the classical Chern-Weil theory, which provides the characteristic classes. In general, for a detailed study on Chern-Weil theory for principal bundles, and characteristic classes, we refer to [23, 47, 67, 97, 115, 116, 119, 147]. In this section, we present a related Chern-Weil theory for SGH principal G-bundles, under the conditions outlined in Theorem 4.4.2, thereby introducing a new type of characteristic classes for SGH principal G-bundles.

Let G be a complex Lie group with $\mathfrak{g}$ denoting its complex Lie algebra. Let $\mathrm{Sym}^k(\mathfrak{g}^*)$ denotes the set of all symmetric k-linear mappings $\mathfrak{g} \times \mathfrak{g} \times \cdots \times \mathfrak{g} \longrightarrow \mathbb{C}$ on the Lie algebra $\mathfrak{g}$. Define the right adjoint action of the Lie group G on $\mathrm{Sym}^k(\mathfrak{g}^*)$ by

$$\begin{aligned} &(f, g) \mapsto \mathrm{Ad}(g^{-1})f \\ \text{where} \quad &(\mathrm{Ad}(g^{-1})f)(x_1, \dots, x_k) = f(\mathrm{Ad}(g^{-1})x_1, \dots, \mathrm{Ad}(g^{-1})x_k) \end{aligned} \tag{4.5.1}$$

for any $f \in \mathrm{Sym}^k(\mathfrak{g}^*)$ and $g \in G$. Denote the space of $\mathrm{Ad}(G)$-invariant forms by

$$\mathrm{Sym}^k(\mathfrak{g}^*)^G := \{f \in \mathrm{Sym}^k(\mathfrak{g}^*) \mid \mathrm{Ad}(g^{-1})f = f \ \forall\ g \in G\} \tag{4.5.2}$$

Now, given any $f \in \mathrm{Sym}^k(\mathfrak{g}^*)^G$, we define a $2k$-form in A^{2k}, of type (k,k), by

$$f(\Omega^{1,1})(X_1, X_2 \dots, X_{2k}) := \frac{1}{(2k)!}\sum_{\sigma} \epsilon_\sigma f(\Omega^{1,1}(X_{\sigma(1)}, X_{\sigma(2)}), \dots, \Omega^{1,1}(X_{\sigma(2k-1)}, X_{\sigma(2k)})) \tag{4.5.3}$$

for $X_1, \dots, X_{2k} \in F_M(\mathcal{G}M \oplus \overline{\mathcal{G}M})$, where σ is an element of the symmetric group S_{2k}, ϵ_σ denotes the sign of the permutation $\sigma \in S_{2k}$, and $\Omega^{1,1}$ is defined as in Theorem 4.4.2.

Let $\mathbb{C}[\mathfrak{g}]$ be the algebra of $\mathbb{C}$-valued polynomials on $\mathfrak{g}$. Consider the same adjoint action of G on $\mathbb{C}[\mathfrak{g}]$ as in (4.5.1), and let $\mathbb{C}[\mathfrak{g}]^G$ denote the subalgebra of fixed points under this action. Then any $f \in \mathrm{Sym}^k(\mathfrak{g}^*)^G$ can be viewed as a homogeneous polynomial function of degree k in $\mathbb{C}[\mathfrak{g}]^G$ that is,

$$\mathrm{Sym}^k(\mathfrak{g}^*)^G < \mathbb{C}[\mathfrak{g}]^G \quad \text{for any } k \geq 0\,.$$

We set

$$\mathrm{Sym}(\mathfrak{g}^*)^G := \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(\mathfrak{g}^*)^G\,.$$

Then, $\mathrm{Sym}(\mathfrak{g}^*)^G$ can be viewed as a sub-algebra of $\mathbb{C}[\mathfrak{g}]^G$.

Since $d_L\Omega^{1,1} = 0$, we can see that $d_L f(\Omega^{1,1}) = 0$ for any $f \in \mathrm{Sym}^k(\mathfrak{g}^*)^G$. Thus $f(\Omega^{1,1}) \in H^{k,k}_{d_L}(M)$. We define a map

$$\begin{aligned} \Phi_k : \mathrm{Sym}^k(\mathfrak{g}^*)^G &\longrightarrow H^{k,k}_{d_L}(M)\,, \\ f &\mapsto [f(\Omega^{1,1})]\,. \end{aligned} \tag{4.5.4}$$

Using the algebra structure of $\mathrm{Sym}(\mathfrak{g}^*)^G$, we extend the map in (4.5.4) to an algebra homomorphism

$$\begin{aligned} \Phi : \mathrm{Sym}(\mathfrak{g}^*)^G &\longrightarrow H^{*}_{d_L}(M)\,, \\ f &\mapsto [f(\Omega^{1,1})]\,, \end{aligned} \tag{4.5.5}$$

where $H^*_{d_L}(M) := \bigoplus_{k,l} H^{k,l}_{d_L}(M)$.

Note that $\mathrm{img}\,\Phi \subseteq \bigoplus_{k\geq 0} H^{k,k}_{d_L}(M)$. We show that Φ is independent of the choice of a smooth generalized connection of type $(1,0)$ which is constant along the leaves. For that, consider two smooth generalized connections Θ, Θ' of type $(1,0)$ on the SGH principal bundle P over M which are constant along the leaves. Define

$$\begin{aligned} \omega &= \Theta - \Theta'\,; \\ \omega_t &= \Theta' + t\omega\,, \quad \text{for } t \in [0,1]\,. \end{aligned}$$

From the equations (4.2.15) and (4.2.16), one can see that ω_t is a 1-parameter family of smooth generalized connections of type $(1,0)$ constant along the leaves. Let Ω_t be the curvature of ω_t and let $\Omega_t^{1,1}$ be the $(1,1)$ component of Ω_t. By (4.4.22), we have

$$\begin{aligned} \Omega_t^{1,1} &= d_L\omega_t\,, \\ &= d_L\Theta' + t\, d_L\omega\,, \\ \implies \quad \frac{d\Omega_t^{1,1}}{dt} &= d_L\omega\,. \end{aligned} \tag{4.5.6}$$

Consider the transverse generalized $(2k-1)$-form of type $(k, k-1)$, defined by

$$\varphi = k\int_0^1 f(\omega, \Omega_t^{1,1}, \ldots, \Omega_t^{1,1})\; dt\,. \tag{4.5.7}$$

Note that

$$\begin{aligned} k\, d_L f(\omega, \Omega_t^{1,1}, \ldots, \Omega_t^{1,1}) &= k f(d_L\omega, \Omega_t^{1,1}, \ldots, \Omega_t^{1,1})\,, \\ &= k f(\frac{d\Omega_t^{1,1}}{dt}, \Omega_t^{1,1}, \ldots, \Omega_t^{1,1}) \;\; (\text{by } \; (4.5.6))\,, \\ &= \frac{d}{dt}(f(\Omega_t^{1,1}, \ldots, \Omega_t^{1,1}))\,. \end{aligned} \tag{4.5.8}$$

Hence,

$$d_L\varphi = \int_0^1 \frac{d}{dt}(f(\Omega_t^{1,1}, \ldots, \Omega_t^{1,1}))\; dt \;=\; f(\Omega_1^{1,1}, \ldots, \Omega_1^{1,1}) - f(\Omega_0^{1,1}, \ldots, \Omega_0^{1,1})\,.$$

This shows that the algebra homomorphism Φ, in (4.5.5), is independent of the choice of smooth generalized connections of type $(1,0)$ which are constant along the leaves.

Definition 4.5.1. *The algebra homomorphism* Φ*, defined in* (4.5.5)*, is called the generalized Chern-Weil homomorphism.*

4.5.1 Generalized Chern classes

Let $P \longrightarrow M$ be an SGH principal G-bundle over a regular GC manifold M. Let G be a complex Lie group with a canonical faithful representation such as a classical complex Lie group. Then the complex Lie algebra $\mathfrak{g}$ is identified with a complex subalgebra of $M_l(\mathbb{C})$ where l is the dimension of the representation. For any $A \in \mathfrak{g}$, consider the following characteristic polynomial

$$\det\left(I + t\frac{A}{2\pi\, i}\right) = \sum_{k=0}^{l} f_k(A)\, t^k\,, \tag{4.5.9}$$

where $f_k \in \mathbb{C}[\mathfrak{g}]$ is an elementary symmetric polynomial of degree k and I is the identity matrix. Since the right hand side of (4.5.9) is invariant under $\mathrm{Ad}(G)$-action, we have

$$f_k \in \mathrm{Sym}^k(\mathfrak{g}^*)^G\,.$$

Following [147, Example 32.3], we define an analogue of Chern classes for an SGH principal G-bundles where G is a complex Lie group with a holomorphic faithful representation.

Definition 4.5.2. *The* k*-th generalized Chern class of* P*, denoted by* $\mathbf{g}c_k(P)$*, is defined as the image of* f_k *under the generalized Chern-Weil homomorphism, that is,*

$$\mathbf{g}c_k(P) := \Phi(f_k)\,,$$

where Φ *as defined in* (4.5.5).

Proposition 4.5.1. $\mathbf{g}c_1(P) = \left[\left(\frac{1}{2\pi\, i}\right)\mathrm{Trace}(\Omega^{1,1})\right]$ *, where* $\Omega^{1,1}$ *as defined in* (4.4.22).

Proof. Consider the usual determinant map $\det : M_l(\mathbb{C}) \longrightarrow \mathbb{C}$, and the smooth map $(\det \circ \psi)(z) = \sum_{k=0}^{l} f_k(A)\, z^k$, where $\psi(z) = I + z\frac{A}{2\pi\, i}$ for all $z \in \mathbb{C}$, $A \in \mathfrak{g}$. Here, $\mathfrak{g}$ is identified with a complex subalgebra of $M_l(\mathbb{C})$. After differentiating both sides with respect to z, at $z = 0$, we get $f_1(A) = \left(\frac{\mathrm{Trace}(A)}{2\pi\, i}\right)$ which concludes the proof. □

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Chapter 5

Connections on Strong Generalized Holomorphic Vector Bundles and Hodge theory

In this chapter, we extend the concept of GH connections to SGH vector bundles, which are also GH vector bundles in the sense described by Lang et al. ([103, Definition 4.1]) (see Remark 5.1.7). We achieve this by introducing two new types of connections: smooth generalized connections and transverse generalized connections. Furthermore, we delve into related topics such as Hodge theory and explore related dualities and vanishing theorems. For an extensive exploration of the theory of connection on vector bundles, we refer to [51, 95, 96, 144]. In general, for a detailed study of Hodge theory, as well as related dualities and vanishing theorems, we refer to [20, 54, 68, 86, 120, 136, 150, 160] and the references therein. To delve into elliptic operators on manifolds, we refer to [133, 134] and the references therein.

We begin by introducing a smooth generalized connection for SGH vector bundles, which is a smooth version of a GH connection. Next, we provide a local description in 5.1.2 and delineate this connection's characteristics in Proposition 5.1.1-5.1.3. We then delve into the theory of its associated curvature. However, a smooth generalized connection proves insufficient in capturing the transverse geometry of a GC manifold. Hence, we introduce the transverse generalized connection along with its corresponding curvature. Essentially, they are smooth generalized connections but remain constant along the leaf direction of the induced foliation. Assuming the leaf space to be an orbifold, we describe the theory of the transverse generalized connection and its curvature. We also provide generalized Chern classes for SGH vector bundles, similar to those discussed in Subsection 4.5.1. Within this framework, we establish both generalized Poincaré duality and generalized Serre duality, along with providing a Hodge decomposition in Theorem 5.2.2. Furthermore, we extend our analysis to establish the generalized Hodge decomposition and generalized Serre duality for SGH vector bundles in Theorem 5.2.3-5.2.4. Assuming the leaf space to be a Kähler orbifold, we present the generalized Kodaira vanishing theorem and generalized Serre theorem in Theorem 5.2.5. This chapter is based on [127, Sections 10 and 11] and splits into two sections:

1. Connections on SGH vector bundles (Section 5.1).

2. Dualities and vanishing theorems for SGH vector bundles (Section 5.2).

5.1 Connections on SGH vector bundles

Let M be a regular GC manifold and E be an SGH vector bundle over M. Set

$$\tilde{A}_E := \tilde{A} \otimes_{C^\infty_M} C^\infty(E) = \tilde{A} \otimes_{\mathcal{O}_M} \mathbf{E}\,, \tag{5.1.1}$$

where $\tilde{A}$ is defined as in (4.4.4).

Definition 5.1.1. *A smooth generalized connection on an SGH vector bundle E, is a $\mathbb{C}$-linear sheaf homomorphism*

$$\nabla : \tilde{A}^0_E \longrightarrow \tilde{A}^1_E$$

which satisfies the Leibniz rule

$$\nabla(fs) = \tilde{D}f \otimes s + f\nabla(s)$$

for any local function on M and any local section s of E.

Let $\{U_\alpha, \phi_\alpha\}$ be a system of local trivializations of E. Then, on U_α, we may write

$$\nabla|_{U_\alpha} = \phi_\alpha^{-1} \circ (\tilde{D} + \theta_\alpha) \circ \phi_\alpha$$

where θ_α is a matrix valued generalized 1-form. On $U_\alpha \cap U_\beta$, we have

$$\begin{aligned}
&\phi_\beta^{-1} \circ (\tilde{D} + \theta_\beta) \circ \phi_\beta = \phi_\alpha^{-1} \circ (\tilde{D} + \theta_\alpha) \circ \phi_\alpha \,, \\
\Longrightarrow\ &\phi_\beta^{-1} \circ \theta_\beta \circ \phi_\beta - \phi_\alpha^{-1} \circ \theta_\alpha \circ \phi_\alpha = \phi_\alpha^{-1} \circ \tilde{D} \circ \phi_\alpha - \phi_\beta^{-1} \circ \tilde{D} \circ \phi_\beta \,, \\
\Longrightarrow\ &\begin{cases} \phi_\beta^{-1} \circ \theta_\beta \circ \phi_\beta - \phi_\alpha^{-1} \circ \theta_\alpha \circ \phi_\alpha = \phi_\beta^{-1} \circ (\phi_{\alpha\beta}^{-1} \circ \tilde{D} \circ \phi_{\alpha\beta} - \tilde{D}) \circ \phi_\beta \,, \\ \quad \text{or} \\ \phi_\beta^{-1} \circ \theta_\beta \circ \phi_\beta - \phi_\alpha^{-1} \circ \theta_\alpha \circ \phi_\alpha = \phi_\alpha^{-1} \circ (\tilde{D} - \phi_{\alpha\beta} \circ \tilde{D} \circ \phi_{\alpha\beta}^{-1}) \circ \phi_\alpha \,. \end{cases}
\end{aligned}$$

Thus, we get the following co-boundary equation for ∇.

$$\begin{cases} \theta_\beta - \mathrm{Ad}(\phi_{\beta\alpha}) \cdot \theta_\alpha = \phi_{\alpha\beta}^{-1} \tilde{D}(\phi_{\alpha\beta}) \\ \quad \text{or} \\ \mathrm{Ad}(\phi_{\alpha\beta}) \cdot \theta_\beta - \theta_\alpha = -\phi_{\alpha\beta} \tilde{D}(\phi_{\alpha\beta}^{-1}) \end{cases} \quad \text{on } U_\alpha \cap U_\beta \,. \tag{5.1.2}$$

A straightforward modification of the proof of [86, Proposition 4.2.3] yields the following.

Proposition 5.1.1. *Let E be an SGH vector bundle over M. Then*

1. *For any two smooth generalized connections ∇, ∇' on an SGH vector bundle E, $\nabla - \nabla'$ is $\tilde{A}^0$-linear.*

2. *For any $\theta \in \tilde{A}^1_{\mathrm{End}(E)}(M)$, $\nabla + \theta$ is also a smooth generalized connection of E.*

3. *The set of all smooth generalized connections on E, is an affine space over the (infinite-dimensional) $\mathbb{C}$-vector space $\tilde{A}^1_{\mathrm{End}(E)}(M)$.*

Any SGH vector bundle E is also a complex vector bundle and it admits a hermitian metric h. The pair (E, h) is then known as a hermitian vector bundle.

Definition 5.1.2. *Given a hermitian vector bundle (E, h), a smooth generalized connection ∇ is called a generalized hermitian connection with respect to h if for any two local sections s, s', one has*

$$\tilde{D}(h(s, s')) = h(\nabla s, s') + h(s, \nabla s') \,. \tag{5.1.3}$$

Let θ be a element in $\tilde{A}^1_{\mathrm{End}(E)}(M)$ and ∇ be a generalized hermitian connection. Then, by Proposition 5.1.1, $\nabla + \theta$ is also a smooth generalized connection. Now, one can see that $\nabla + \theta$ satisfies (5.1.3) if and only if $h(\theta s, s') + h(s, \theta s') = 0$ for all smooth local sections s, s'. Consider the subsheaf

$$\mathrm{End}(E,h) := \{\theta \in C^\infty(\mathrm{End}(E)) \,|\, h(\theta s, s') + h(s, \theta s') = 0 \ \forall \text{ local sections } s, s'\}$$

of $C^\infty(\mathrm{End}(E))$. Note that $\mathrm{End}(E,h)$ has the structure of a real vector bundle.

Proposition 5.1.2. *The set of all generalized hermitian connections on (E,h) is an affine space over the (infinite-dimensional) $\mathbb{R}$-vector space $\tilde{A}^1_{\mathrm{End}(E,h)}(M)$ where $\tilde{A}^1_{\mathrm{End}(E,h)} = \tilde{A}^1 \otimes_{C^\infty_{M,\mathbb{R}}} C^\infty(\mathrm{End}(E,h))$, and $C^\infty_{M,\mathbb{R}}$ is the sheaf of $\mathbb{R}$-valued smooth functions.*

Proof. Follows from Proposition 5.1.1 after considering E as a real vector bundle. □

Remark 5.1.1. *(cf. [86, Section 4.2])* $\mathrm{End}(E,h)$ *is not always an SGH vector bundle. It is not even always a complex vector bundle. For example, if $E = M \times \mathbb{C}$ is the trivial SGH vector bundle. Then* $\mathrm{End}\,E$ *is again $M \times \mathbb{C}$ but* $\mathrm{End}(E,h)$ *is just the $M \times i\mathbb{R}$.*

Now $\tilde{A}^1_E = \tilde{A}^{1,0}_E + \tilde{A}^{0,1}_E$, as in (4.4.2). So, we can decompose any smooth generalized connection ∇ into two components, $\nabla^{1,0}$ and $\nabla^{0,1}$ such that $\nabla = \nabla^{1,0} + \nabla^{0,1}$ where

$$\nabla^{1,0} : \tilde{A}^0_E \longrightarrow \tilde{A}^{1,0}_E \quad ; \quad \nabla^{0,1} : \tilde{A}^0_E \longrightarrow \tilde{A}^{0,1}_E\,.$$

Note that for any local function f on M and local section s of E,

$$\nabla^{0,1}(fs) = \tilde{d}_L f \otimes s + f\nabla^{0,1}(s)\,.$$

Definition 5.1.3. *A smooth generalized connection ∇ on E is compatible with the GCS if $\nabla^{0,1} = \tilde{d}_L$.*

After some straightforward modifications of the proofs in [86, Corollary 4.2.13, Proposition 4.2.14], one obtains the following.

Proposition 5.1.3. *Let E be an SGH vector bundle over M with a hermitian structure h.*

1. *The space of smooth generalized connections on E, compatible with the GCS, forms an affine space over the $\mathbb{C}$-vector space $\tilde{A}^{1,0}_{\mathrm{End}(E)}(M)$.*

2. *There exists a unique generalized hermitian connection ∇ on E with respect to h which is also compatible with the GCS. This smooth generalized connection is called generalized Chern connection.*

Definition 5.1.4. *The curvature of a smooth generalized connection* ∇, *denoted by* Ω_∇ *and referred to as a smooth generalized curvature, is the composition*

$$\Omega_\nabla := \nabla \circ \nabla : \tilde{A}^0_E \longrightarrow \tilde{A}^1_E \longrightarrow \tilde{A}^2_E \,.$$

Example 5.1.1. *Let* $E = M \times \mathbb{C}^l$ *be the trivial SGH vector bundle. By Proposition 5.1.1, any smooth generalized connection is of the form* $\nabla = \tilde{D} + \theta$ *where* $\theta \in \tilde{A}^1_{\mathrm{End}(E)}(M)$. *Note that* $\tilde{D} : \tilde{A}^p \to \tilde{A}^{p+1}$ *extends naturally to* $\tilde{D} : \tilde{A}^p_E \to \tilde{A}^{p+1}_E$ *by the Leibniz rule. So, for any local section* $s \in \tilde{A}^0_E$, *we have*

$$\begin{aligned}\Omega_\nabla(s) &= (\tilde{D} + \theta)(\tilde{D}s + \theta s)\\ &= \tilde{D}(\tilde{D}s) + (\tilde{D}(\theta s) + \theta \wedge \tilde{D}s) + \theta \wedge \theta(s)\\ &= (\theta \wedge \theta + \tilde{D}(\theta))(s)\,.\end{aligned}$$

For any smooth generalized connection ∇ on an SGH vector bundle E with local trivialization $\{U_\alpha, \phi_\alpha\}$, we know that $\nabla = \phi_\alpha^{-1} \circ (\tilde{D} + \theta_\alpha) \circ \phi_\alpha$. By (5.1.2), on $U_{\alpha\beta}$,

$$\theta_\beta = \mathrm{Ad}(\phi_{\beta\alpha}) \cdot \theta_\alpha + \phi^{-1}_{\alpha\beta} \tilde{D}(\phi_{\alpha\beta})\,.$$

This implies

$$\begin{aligned}\tilde{D}(\theta_\beta) = \tilde{D}(\phi_{\beta\alpha}) \wedge \tilde{D}(\phi_{\alpha\beta}) + \tilde{D}(\phi_{\beta\alpha}) \wedge \theta_\alpha \wedge \phi_{\alpha\beta} + \mathrm{Ad}(\phi_{\beta\alpha}) \cdot \tilde{D}(\theta_\alpha)\\ - \phi_{\beta\alpha} \wedge \theta_\alpha \wedge \tilde{D}(\phi_{\alpha\beta})\,,\end{aligned} \tag{5.1.4}$$

and also,

$$\begin{aligned}\theta_\beta \wedge \theta_\beta = \mathrm{Ad}(\phi_{\beta\alpha})(\theta_\alpha \wedge \theta_\alpha) + \phi_{\beta\alpha}\tilde{D}(\phi_{\alpha\beta}) \wedge \phi_{\beta\alpha}\tilde{D}(\phi_{\alpha\beta})\\ + \phi_{\beta\alpha}\tilde{D}(\phi_{\alpha\beta}) \wedge \mathrm{Ad}(\phi_{\beta\alpha}) \cdot \theta_\alpha + \mathrm{Ad}(\phi_{\beta\alpha}) \cdot \theta_\alpha \wedge \phi_{\beta\alpha}\tilde{D}(\phi_{\alpha\beta})\,.\end{aligned}$$

Note that $\tilde{D}(\phi_{\beta\alpha})\phi_{\alpha\beta} = -\phi_{\beta\alpha}\tilde{D}(\phi_{\alpha\beta})$ since $\tilde{D}(\phi_{\beta\alpha}\,\phi_{\alpha\beta}) = 0$. Thus, we get

$$\begin{aligned}\theta_\beta \wedge \theta_\beta = \mathrm{Ad}(\phi_{\beta\alpha})(\theta_\alpha \wedge \theta_\alpha) - \tilde{D}(\phi_{\beta\alpha}) \wedge \tilde{D}(\phi_{\alpha\beta}) - \tilde{D}(\phi_{\beta\alpha}) \wedge \theta_\alpha \wedge \phi_{\alpha\beta}\\ + \phi_{\beta\alpha} \wedge \theta_\alpha \wedge \tilde{D}(\phi_{\alpha\beta})\end{aligned} \tag{5.1.5}$$

Hence, by combining (5.1.4) and (5.1.5) on $U_{\alpha\beta}$, we have

$$\begin{aligned}\phi_\beta \circ \Omega_\nabla \circ \phi_\beta^{-1} &= (\theta_\beta \wedge \theta_\beta + \tilde{D}(\theta_\beta))\\ &= \mathrm{Ad}(\phi_{\beta\alpha})(\theta_\alpha \wedge \theta_\alpha + \tilde{D}(\theta_\alpha))\\ &= \mathrm{Ad}(\phi_{\beta\alpha})(\phi_\alpha \circ \Omega_\nabla \circ \phi_\alpha^{-1})\,.\end{aligned}$$

This implies that $\Omega_\nabla \in \tilde{A}^2_{\mathrm{End}(E)}(M)$. Now, assume that E admits a hermitian structure such that ∇ is a generalized hermitian connection with respect to h. Without loss of generality, we can assume that $(E,h)|_{U_\alpha}$ is isomorphic to $U_\alpha \times \mathbb{C}^l$ with constant hermitian structure. Then we can easily see that, on U_α, $\overline{\theta_\alpha}^t = -\theta_\alpha$ and so, by Example 5.1.1, $\overline{\Omega_\nabla}^t = -\Omega_\nabla$. Note that, using (5.1.3), we have, for any local $s_i \in \tilde{A}^{k_i}_E$ $(i = 1, 2)$,

$$\tilde{D}h(s_1, s_2) = h(\nabla s_1, s_2) + (-1)^{k_1} h(s_1, \nabla s_2)\,.$$

This implies that for $s_i \in \tilde{A}^0_E$,

$$\begin{aligned} 0 &= \tilde{D}(\tilde{D}h(s_1, s_2)) \\ &= \tilde{D}(h(\nabla s_1, s_2) + h(s_1, \nabla s_2)) \\ &= h(\Omega_\nabla s_1, s_2) + h(s_1, \Omega_\nabla s_2)\,. \end{aligned} \tag{5.1.6}$$

If we further assume that ∇ is compatible with the GCS, we get,

$$\Omega_\nabla = \nabla^2 = (\nabla^{1,0})^2 + \nabla^{1,0} \circ \tilde{d_L} + \tilde{d_L} \circ \nabla^{1,0}\,.$$

Thus, $h(\Omega_\nabla s_1, s_2)$ and $h(s_1, \Omega_\nabla s_2)$ are of type $(2,0)+(1,1)$ and $(1,1)+(0,2)$, respectively. So, by (5.1.6), $(\nabla^{1,0})^2 = 0$. We have proved the following.

Proposition 5.1.4. *Let E be an SGH vector bundle over M with a hermitian structure h. Let ∇ be a smooth generalized connection with curvature Ω_∇. Then*

1. *If ∇ is a generalized hermitian connection with respect to h, Ω_∇ satisfies*

$$h(\Omega_\nabla s_1, s_2) + h(s_1, \Omega_\nabla s_2) = 0 \quad \textit{for any sections } s_1, s_2\,.$$

2. *If ∇ is compatible with the GCS, then Ω_∇ has no $(0,2)$-part, that is,*

$$\Omega_\nabla \in (\tilde{A}^{2,0}_{\mathrm{End}(E)} \oplus \tilde{A}^{1,1}_{\mathrm{End}(E)})(M)\,.$$

3. *If ∇ is a generalized Chern connection on (E,h), Ω_∇ is of type $(1,1)$, skew-hermitian and real.*

Recall the transversely holomorphic symplectic foliation $\mathscr{S}$ of M and the corresponding leaf space $M/\mathscr{S}$. We have seen that a smooth generalized connection on an SGH vector bundle E over M with trivializations $\{U_\alpha, \phi_\alpha\}$ is equivalent to a family $\{\theta_\alpha \in \tilde{A}^1_{\mathrm{End}(E)|_{U_\alpha}}(U_\alpha)\}$ satisfying (5.1.2). If each θ_α is constant along the leaves of $\mathscr{S}$, that is, if we replace $\tilde{A}^\bullet$ by $A^\bullet$, we get the following notion.

Definition 5.1.5. *Let E be an SGH vector bundle on M.*

1. *A transverse generalized connection on E, is a $\mathbb{C}$-linear sheaf homomorphism*

$$\nabla : A_E^0 \longrightarrow A_E^1$$

which satisfies the Leibniz rule

$$\nabla(fs) = Df \otimes s + f\nabla(s)$$

for any local function $f \in F_M$ and any local section s of $F_M(E)$.

2. *A transverse generalized curvature is the curvature of a transverse generalized connection ∇, denoted by Ω_∇. Note that, $\Omega_\nabla \in A^2_{\mathrm{End}(E)}(M)$.*

Remark 5.1.2. *A transverse generalized connection is also a smooth generalized connection in the sense that given a transverse generalized connection ∇, we can consider a $\mathbb{C}$-linear sheaf homomorphism*

$$\tilde{\nabla} : \tilde{A}_E^0 = C_M^\infty \otimes_{F_M} A_E^0 \longrightarrow \tilde{A}_E^1 = C_M^\infty \otimes_{F_M} A_E^1 \,,$$

defined by

$$\tilde{\nabla}(fs) = \tilde{D}f \otimes s + f\nabla s$$

for any local smooth function f and any local section $s \in A_E^0$. One can check that $\tilde{\nabla}$ is a smooth generalized connection.

Remark 5.1.3. *A smooth generalized connection always exists. A transverse generalized connection exists locally. For it to exist globally we need a smooth partition of unity, which is constant along the leaves. If we assume $M/\mathscr{S}$ is a smooth orbifold, such a partition of unity exists. Henceforth, in this section, we always assume that $M/\mathscr{S}$ is a smooth orbifold.*

We can replicate all the definitions and results for smooth generalized connections in this section, except those concerning hermitian structure, to transverse generalized connections by making the following substitutions.

	Replaced by
C^∞_M	F_M
$\tilde{A}^\bullet$	$A^\bullet$
$\tilde{A}^{\bullet,\bullet}$	$A^{\bullet,\bullet}$
$\tilde{D}$	D
$\tilde{d}_L$	d_L

Table 5.1: Replacement table

For the results concerning the generalized hermitian connection, some extra care is needed. Consider the trivial SGH vector bundle $E = M \times \mathbb{C}^r$ with a hermitian structure h and let ∇ be a smooth generalized connection which satisfies (5.1.3). Any hermitian metric h given on E is given by a function, again denoted by h, on M that associates to any $x \in M$, a positive-definite hermitian matrix $h(x) = (h_{ij}(x))$. So we can think of h as a smooth global section of $E^* \otimes E^*$, that is,

$$h \in C^\infty(M, E^* \otimes E^*)\,.$$

Now ∇ is of the form $\nabla = \tilde{D} + \theta$ for some $\theta = (\theta_{ij}) \in \tilde{A}^1_{\mathrm{End}(E)}(M)$. Let e_i be the constant i-th unit vector considered as a section of E. The assumption on ∇ will yield

$$\tilde{D}h(e_i, e_j) = h(\sum_k \theta_{ki} e_k, e_j) + h(e_i, \sum_l \theta_{lj} e_j)\,,$$

or equivalently $\tilde{D}h = \theta^t \cdot h + h \cdot \overline{\theta}$. Furthermore, if we assume that ∇ is compatible with the GCS, then θ is of type $(1, 0)$. This implies

$$\begin{aligned} &\tilde{d}_L h = h \cdot \overline{\theta} \\ \Longrightarrow\ &\theta = \overline{h}^{-1} \tilde{d}_L h\,. \end{aligned}$$

This shows that hermitian structure uniquely determines the smooth generalized connection. Thus, for a transverse generalized connection, we would like to have a hermitian metric which is constant along the leaves.

Definition 5.1.6. *A hermitian metric h on an SGH vector bundle E is called a transverse hermitian metric if $h \in F_M(M, E^* \otimes E^*)$, that is, h is constant along the leaves of $\mathscr{S}$.*

Remark 5.1.4. *Our assumption that $M/\mathscr{S}$ is a smooth orbifold ensures that a transverse hermitian metric always exists.*

With this notion of transverse hermitian metric and using the substitutions in Table 5.1, we can replicate all the relevant definitions and extend the results in Proposition 5.1.3 and Proposition 5.1.4 to the transverse generalized connections. In particular, we have the following.

Theorem 5.1.1. *Let E be an SGH vector bundle over M such that $M/\mathscr{S}$ is a smooth orbifold. Let h be a transverse hermitian metic on E.*

1. *There exists a unique transverse generalized hermitian connection ∇ on E with respect to h which is also compatible with the GCS. This transverse generalized connection is called transverse generalized Chern connection.*

2. *The transverse generalized curvature of ∇, Ω_∇ is of type $(1,1)$, skew-hermitian and real.*

3. *The space of transverse generalized connections on E, compatible with the GCS, forms an affine space over the $\mathbb{C}$-vector space $A^{1,0}_{\mathrm{End}(E)}(M)$.*

5.1.1 Generalized Chern classes for SGH vector bundles

Let $E \longrightarrow M$ be an SGH vector bundle over M of complex rank l. Then, following Subsection 4.5.1, consider the following characteristic polynomial

$$\det\left(I - t\frac{A}{2\pi\, i}\right) = \sum_{j=0}^{l} g_j(A)\, t^j\,,$$

where $g_j \in \mathbb{C}[M_l(\mathbb{C})]$ is the elementary symmetric polynomial of degree j and I is the identity matrix. Then, we can define an analogue of Chern classes similar to the classical case, as follows.

Definition 5.1.7. *Let E be an SGH vector bundle over M. The j-th generalized Chern class of E, denoted by $\mathbf{g}c_j(E)$, is defined as the image of g_j under the generalized Chern-Weil homomorphism, that is,*

$$\mathbf{g}c_j(E) := \Phi(g_j)\,,$$

where Φ as defined in (4.5.5).

Example 5.1.2. *Let E be an SGH vector bundle over M where the leaf space $M/\mathscr{S}$ admits an orbifold structure. Let ∇ be the transverse generalized Chern connection and Ω_∇ be its curvature. Then $\mathbf{g}c_1(E) = -\frac{1}{2\pi i}[\mathrm{Trace}(\Omega_\nabla)]$.*

Remark 5.1.5. *Note that, if the leaf space $M/\mathscr{S}$ is a smooth manifold, and if we have an SGH vector bundle E over M which is the pullback of a holomorphic vector bundle V over $M/\mathscr{S}$, then, by using Corollary 4.4.4, we can conclude that the generalized Chern classes of E are the pullback of the Chern classes of V, that is*

$$\mathbf{g}c_j(E) = \tilde{\pi}^*(c_j(V)) \ \ \textit{for}\ \ 0 \leq j \leq l\,,$$

where $c_j(V)$ is the j-th Chern class of V, $\tilde{\pi} : M \longrightarrow M/\mathscr{S}$ is the quotient map, and l is the complex rank of E.

Remark 5.1.6. *Given an SGH line bundle over M, the image of its isomorphism class in $H^1(M, \mathcal{O}^*_M)$, under the connecting homomorphism in the long exact sequence of sheaf cohomologies derived from the short exact sequence in Theorem 3.3.1, may not give the first generalized Chern class of the bundle. This is because the first generalized Chern class lies in $H^{1,1}_{d_L}(M)$ which under suitable conditions lies in $H^2_D(M)$. But the latter basically describes the cohomology of the leaf space of the GCS and may not be the same as the de Rham cohomology of M. If the bundle is the pullback of a holomorphic line bundle on the leaf space of the symplectic foliation, then there is no such discrepancy.*

Remark 5.1.7. *It is important to note that if we substitute F_M with $\mathcal{O}_M$ in Definition 5.1.5 and refer to Remark 3.1.1, we get a GH connection on an SGH vector bundle as defined by Lang et al [103, Definition 4.1]. In this framework, the subsequent result has been established concerning the existence of a GH connection on an SGH vector bundle.*

Theorem 5.1.2. *([103, Sections 4.1–4.2]) Let E be an SGH vector bundle over a regular GC manifold. Then, the following are equivalent:*

1. *E admits a GH connection.*

2. *The short exact sequence, as defined in* (4.3.2)*, splits.*

3. $b(E) = 0$ *where* $b(E)$ *is the Atiyah class of the SGH vector bundle* E *as defined in Definition 4.3.1.*

Theorem 5.1.3. *Consider* E *as an SGH vector bundle over a regular GC manifold* M*. Let* P *denote the corresponding SGH principal bundle, as in* (4.1.4)*. Then,* E *admits a GH connection if and only if* P *admits a GH connection.*

Proof. Follows from Theorem 4.2.2, Theorem 4.3.1, and Theorem 5.1.2. □

With this, we conclude the groundwork necessary for establishing an analogue of Hodge theory and related dualities and vanishing theorems in the following section.

5.2 Dualities and vanishing theorems for SGH vector bundles

In this section, we extend some classical results like Serre duality, Poincaré duality, Hodge decomposition and vanishing theorems to the cohomology theory of Section 4.4.1 following the approach of [6] and [86].

5.2.1 Generalized Serre duality and Hodge decomposition

Let M^{2n} be a compact regular GC manifold of type k. Then the leaf space $M/\mathscr{S}$, as defined in (3.4.2), is a compact space. Let us assume $M/\mathscr{S}$ is a smooth orbifold. Then, by the integrability condition of the GCS, $M/\mathscr{S}$ is a complex orbifold, and hence, orientable. Thus the cohomology $H^{2k}(M/\mathscr{S})$ is nontrivial. Therefore, there exists a $(2n-2k)$-form χ on M (see [6, Section 2.8]) which restricts to a volume form on each leaf such that for any $X_1, \ldots, X_{2n-2k} \in C^\infty(T\mathscr{S})$ and $Y \in C^\infty(TM)$,

$$d\chi(X_1, \ldots, X_{2n-2k}, Y) = 0\,. \tag{5.2.1}$$

Fix a Riemannian metric on the leaf space. This induces a transverse Riemmaninan metric on M. We can complete the transverse metric by a Riemannian metric along the leaves to obtain a Riemannian metric on M for which the leaves are minimal. In fact, χ is associated to this metric.

Now, define a Hodge-star operator on $A^\bullet$,

$$\star : A^\bullet \longrightarrow A^{2k-\bullet}\,, \tag{5.2.2}$$

as follows: Let U be an open set in M on which the GCS is equivalent to a product GCS (see Theorem 2.3.1). This implies that the symplectic foliation on U is trivial. Let $e_1, \ldots, e_{2k}$ be transverse generalized 1-forms such that $\{e_1, \ldots, e_{2k}\}$ is an orthonormal frame of $A^1(U)$. Then, for any $r > 0$, $\star : A^r(U) \longrightarrow A^{2k-r}(U)$ is defined by,

$$\star(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}) = \mathrm{Sign}(i_1, \ldots, i_r, j_1, \ldots, j_{2k-r})\, e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{2k-r}} \tag{5.2.3}$$

where $\{j_1, \ldots, j_{2k-r}\}$ is the increasing complementary sequence of $\{i_1, \ldots, i_r\}$ in the set $\{1, 2, \ldots, 2k\}$ and $\mathrm{Sign}(i_1, \ldots, i_r, j_1, \ldots, j_{2k-r})$ denotes the sign of the permutation $\{i_1, \ldots, i_r, j_1, \ldots, j_{2k-r}\}$. A simple calculation will show that

$$\star\star = (-1)^{r(2k-r)}\,. \tag{5.2.4}$$

Define a hermitian product on $A^r(M)$, by

$$h(\alpha, \beta) := \int_M \alpha \wedge \overline{\star\beta} \wedge \chi\,. \tag{5.2.5}$$

Define another operator $D^* : A^r \longrightarrow A^{r-1}$ by

$$D^* := (-1)^{2k(r-1)-1} \star D \star\,. \tag{5.2.6}$$

For any $\alpha \in A^{r-1}(M)$ and $\beta \in A^r(M)$,

$$d(\alpha \wedge \star\beta \wedge \chi) = D\alpha \wedge \star\beta \wedge \chi - \alpha \wedge \star D^*\beta \wedge \chi + (-1)^{2k-1}\alpha \wedge \star\beta \wedge d\chi\,.$$

Using (5.2.1) and integrating both sides, we get $h(D\alpha, \beta) = h(\alpha, D^*\beta)$. The operator D^* is called the formal adjoint of D.

Since $M/\mathscr{S}$ is a complex orbifold, $\mathscr{S}$ is hermitian as well. The operator $\star$ induces a (vector space) isomorphism between $A^{p,q}(M)$ and $A^{k-q\,,\,k-p}(M)$, that is,

$$A^{p,q}(M) \cong A^{k-q\,,\,k-p}(M)\,, \quad (\text{as } \mathbb{C}\text{-vector spaces})\,,$$

where $A^{p,q}$ as defined in (4.4.3). Moreover, $D = d_L + d_{\overline{L}}$ on $A^{p,q}$ where $d_L, d_{\overline{L}}$ are defined as in (4.4.9). Then the operator D^*, restricted to $A^{p,q}$, decomposes into the sum of two operators

$$d_L^* := - \star\, d_{\overline{L}} \star\,;$$
$$d_{\overline{L}}^* := - \star\, d_L \star\,,$$

respectively of type $(-1, 0)$ and $(0, -1)$. One can see that d_L^* and $d_{\overline{L}}^*$ are the formal adjoints of $d_{\overline{L}}$ and d_L, respectively. Define the following operators

$$\Delta_D := D^*D + DD^*\,;$$

$$\Delta_{d_{\overline{L}}} := d_{\overline{L}} d_L^* + d_L^* d_{\overline{L}}\,; \tag{5.2.7}$$

$$\Delta_{d_L} := d_{\overline{L}}^* d_L + d_L d_{\overline{L}}^*\,.$$

Note that, similar to the classical case, Δ_D, $\Delta_{d_{\overline{L}}}$, and Δ_{d_L} are self-adjoint operators.

For any $p, q, r \geq 0$, define

$$\mathcal{H}_D^r := \ker(\Delta_D) = \{\alpha \in A^r(M) \,|\, D\alpha = D^*\alpha = 0\}\,;$$

$$\mathcal{H}_{d_{\overline{L}}}^{p,q} := \ker(\Delta_{d_{\overline{L}}}) = \{\alpha \in A^{p,q}(M) \,|\, d_{\overline{L}}\alpha = d_L^*\alpha = 0\}\,; \tag{5.2.8}$$

$$\mathcal{H}_{d_L}^{p,q} := \ker(\Delta_{d_L}) = \{\alpha \in A^{p,q}(M) \,|\, d_L\alpha = d_{\overline{L}}^*\alpha = 0\}\,.$$

Definition 5.2.1. *A form $\alpha \in \mathcal{H}_D^r$ is called a transverse GH harmonic form of order r and if $\alpha \in \mathcal{H}_{d_L}^{p,q}$, it's called transverse GH form of type (p, q).*

Theorem 5.2.1. *Let M be a compact regular GC manifold of type k. Let $\mathscr{S}$ be the induced transversely holomorphic foliation. Assume that $M/\mathscr{S}$ is a smooth orbifold. Then we have the following.*

1. *$\mathcal{H}_D^\bullet$ and $\mathcal{H}_{d_L}^{\bullet,\bullet}$ both are finite dimensional.*

2. *There are orthogonal decompositions*

   (a) $A^\bullet(M) = \mathcal{H}_D^\bullet \oplus \mathrm{img}(\Delta_D) = \mathcal{H}_D^\bullet \oplus \mathrm{img}(D) \oplus \mathrm{img}(D^*)\,,$

(b) $A^{\bullet,\bullet} = \mathcal{H}_{d_L}^{\bullet,\bullet} \oplus \operatorname{img}(\Delta_{d_L}) = \mathcal{H}_{d_L}^{\bullet,\bullet} \oplus \operatorname{img}(d_L) \oplus \operatorname{img}(d_{\overline{L}}^*)$.

Proof. Since $M/\mathscr{S}$ is a compact complex orbifold, $A^\bullet$ and $A^{\bullet,\bullet}$ both are hermitian vector bundles over M. Now, a simple local coordinate calculation shows that both Δ_D and Δ_{d_L} are strongly elliptic operators. This implies that the complexes $\{A^\bullet, D\}$ and $\{A^{\bullet,\bullet}, d_L\}$ both are transversely elliptic. Thus by [6, Theorem 2.7.3], we are done. □

Corollary 5.2.1. *Let $H_D^\bullet(M)$ and $H_{d_L}^{\bullet,\bullet}(M)$ are defined as in* (4.4.11), (4.4.12)*, respectively. Then $H_D^\bullet(M)$ and $H_{d_L}^{\bullet,\bullet}(M)$ are finite dimensional and isomorphic to $\mathcal{H}_D^\bullet$ and $\mathcal{H}_{d_L}^{\bullet,\bullet}$, respectively.*

Proof. Follows from Theorem 5.2.1. □

The operator $\star$ induces a $\mathbb{C}$-linear isomorphism

$$\star : \mathcal{H}_{d_L}^{\bullet,*}(M) \cong \mathcal{H}_{d_{\overline{L}}}^{k-*,\,k-\bullet}(M)\,.$$

On the other hand, consider the following hermitian map

$$\tilde{h} : A^\bullet(M) \times A^{2k-\bullet}(M) \longrightarrow \mathbb{C}$$

defined by $\tilde{h}(\alpha,\beta) = \int_M \alpha \wedge \beta \wedge \chi$. It induces a non-degenerate pairing

$$\Phi : H_D^\bullet(M) \times H_D^{2k-\bullet}(M) \longrightarrow \mathbb{C}\,.$$

Theorem 5.2.2. *Let M be a compact regular GC manifold of type k. Let $\mathscr{S}$ be the induced transversely holomorphic foliation. Assume that $M/\mathscr{S}$ is a smooth orbifold. Then*

1. *$H_D^\bullet(M)$ satisfies the generalized Poincaré duality, that is,*

$$H_D^\bullet(M) \cong (H_D^{2k-\bullet}(M))^*\,.$$

2. *$H_{d_L}^{\bullet,\bullet}(M)$ satisfies the generalized Serre duality, that is,*

$$H_{d_L}^{\bullet,\bullet}(M) \cong (H_{d_L}^{k-\bullet,k-\bullet}(M))^*\,.$$

3. *Moreover, if $\mathscr{S}$ is also transversely Kählerian, we have a generalized Hodge decomposition,*

$$H_D^\bullet(M) = \bigoplus_{p+q=\bullet} H_{d_L}^{p,q}(M)\,.$$

Proof. (1) and (2) follows from the preceding discussion.

(3) Since $\mathscr{S}$ is transversely Kählerian, we can prove, analogous to the classical Kähler case, that $\Delta_D = 2\Delta_{d_L}$. Since $A^\bullet(M) = \bigoplus_{p+q=\bullet} A^{p,q}(M)$, every α is a transverse GH harmonic form of order $\bullet$ if and only if its each component is a transverse GH harmonic form of type (p,q) where $p+q=\bullet$. Then using Theorem 5.2.1, we have the direct decomposition

$$H^\bullet_D(M) = \bigoplus_{p+q=\bullet} H^{p,q}_{d_L}(M)\,.$$

□

As the complex conjugation operator induces an isomorphism (of real vector spaces) $H^{\bullet,*}_{d_L}(M) \cong H^{*,\bullet}_{d_L}(M)$, the operator $\star$ also induces a unitary isomorphism

$$\overline{\star} : H^{\bullet,*}_{d_L}(M) \longrightarrow H^{k-\bullet,\,k-*}_{d_L}(M)$$

defined by $\overline{\star}(\alpha) = \star(\overline{\alpha}) = \overline{\star(\alpha)}$.

Let E be an SGH vector bundle on M with a transverse hermitian structure H. Then H can be considered as a $\mathbb{C}$-antilinear isomorphism $H : E \cong E^*$. Consider the following operators

1. $$\overline{\star}_E : A^{\bullet,\bullet}_E \longrightarrow A^{k-\bullet,k-\bullet}_{E^*} \tag{5.2.9}$$

   defined by $\overline{\star}_E(\phi\otimes s) = \overline{\star}(\phi)\otimes H(s)$ for any local sections $\phi \in A^{\bullet,\bullet}$ and $s \in F_M(E)$ where $\overline{\star}(\phi) = \star(\overline{\phi})$.

2. $$d^*_{L,E} : A^{\bullet,\bullet}_E \longrightarrow A^{\bullet,\bullet-1}_E \tag{5.2.10}$$

   defined by $d^*_{L,E} = -\overline{\star}_{E^*}\circ d_{L,E^*}\circ\overline{\star}_E$ where $d_{L,E^*} : A^{\bullet,\bullet}_{E^*} \longrightarrow A^{\bullet,\bullet+1}_{E^*}$ is the natural extension of d_L, as defined in (4.4.9), described in Subsection 4.4.1.

3. $\Delta_{d_{L,E}} := d^*_{L,E}\, d_{L,E} + d_{L,E}\, d^*_{L,E}\,.$

Let M be a compact GC manifold. Then, we can define a natural hermitian scalar product on $A^{\bullet,\bullet}_E(M)$, similarly as in (5.2.5),

$$h_E(\alpha,\beta) := \int_M \alpha\wedge\overline{\star}_E(\beta)\wedge\chi\,. \tag{5.2.11}$$

for any local section $\alpha, \beta \in A_E^{\bullet,\bullet}(M)$ where $\wedge$ is the exterior product on $A^{\bullet,\bullet}$ and the evaluation map $E \otimes E^* \longrightarrow \mathbb{C}$ in bundle part. Then, similarly, we can prove that

Lemma 5.2.1. *$d_{L,E}^*$ is the formal adjoint of $d_{L,E}$ and $\Delta_{d_{L,E}}$ is self-adjoint.*

Set $\mathcal{H}_{d_{L,E}}^{\bullet,\bullet} := \ker(\Delta_{d_{L,E}}) = \{\alpha \in A_E^{\bullet,\bullet}(M) \,|\, d_{L,E}\,\alpha = d_{L,E}^*\,\alpha = 0\}$.

Theorem 5.2.3. *(Generalized Hodge decomposition for SGH vector bundle) Let (E, H) be an SGH vector bundle with a transverse hermitian structure H, over a compact GC manifold M. Assume $M/\mathscr{S}$ is a smooth orbifold. Then*

1. *$\mathcal{H}_{d_{L,E}}^{\bullet,\bullet}$ is finite dimensional.*

2. *$A_E^{\bullet,\bullet}(M) = \mathcal{H}_{d_{L,E}}^{\bullet,\bullet} \oplus \mathrm{img}(\Delta_{d_{L,E}}) = \mathcal{H}_{d_{L,E}}^{\bullet,\bullet} \oplus \mathrm{img}(d_{L,E}) \oplus \mathrm{img}(d_{L,E}^*)$*

Proof. Follows from Theorem 5.2.1 by replacing $\{A^{\bullet,\bullet}, d_L\}$ and Δ_{d_L} with $\{A_E^{\bullet,\bullet}, d_{L,E}\}$ and $\Delta_{d_{L,E}}$, respectively. □

Consider the natural pairing

$$\tilde{h}_E : A_E^{\bullet,\bullet}(M) \times A_{E^*}^{k-\bullet,k-\bullet}(M) \longrightarrow \mathbb{C}$$

defined by $\tilde{h}_E(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \chi$ where $\wedge$ is the exterior product on $A^{\bullet,\bullet}$ and the evaluation map $E \otimes E^* \longrightarrow \mathbb{C}$ in bundle part.

Theorem 5.2.4. *(Generalized Serre duality for SGH vector bundle) Let E be an SGH vector bundle with the same assumption as in Theorem 5.2.3. Then there exists a natural $\mathbb{C}$-linear isomorphism between $H_{d_L}^{\bullet,\bullet}(M, E)$ and $(H_{d_L}^{k-\bullet,k-\bullet}(M, E^*))^*$, that is,*

$$H_{d_L}^{\bullet,\bullet}(M, E) \cong (H_{d_L}^{k-\bullet,k-\bullet}(M, E^*))^* \quad (\textit{as } \mathbb{C}\textit{-vector spaces}),$$

where $k = \mathrm{Type}(M)$.

Proof. Consider the natural pairing $\tilde{h}_E$. It induces a pairing

$$\Phi_E : H_{d_L}^{\bullet,\bullet}(M, E) \times H_{d_L}^{k-\bullet,k-\bullet}(M, E^*) \longrightarrow \mathbb{C}.$$

defined as $\Phi_E([\alpha], [\beta]) = \tilde{h}_E(\alpha, \beta)$ where $[\alpha], [\beta]$ denote the classes of α, β, respectively. One can easily check that this is well-defined. To show that Φ_E is non-degenerate, by Theorem 5.2.3, it is enough to show that for any $0 \neq \alpha \in \mathcal{H}_{d_{L,E}}^{\bullet,\bullet}$, there exist a $\beta \in \mathcal{H}_{d_{L,E^*}}^{\bullet,\bullet}$ such that $\int_M \alpha \wedge \beta \wedge \chi \neq 0$. Note that, $\bar{\star}_E$ induces a $\mathbb{C}$-antilinear isomorphism $\bar{\star}_E : \mathcal{H}_{d_{L,E}}^{\bullet,\bullet} \longrightarrow \mathcal{H}_{d_{L,E^*}}^{k-\bullet,k-\bullet}$. This implies there exist β s.t $\bar{\star}_E(\alpha) = \beta$. Thus $\Phi_E([\alpha], [\beta]) = h_E(\alpha, \alpha) \neq 0$ and this proves the theorem. □

5.2.2 Generalized Vanishing Theorems

Let g be a transversely hermitian metric and I be the transverse complex structure corresponding to the GCS on M where M and $M/\mathscr{S}$ satisfy the same conditions as before with one exception, namely, M need not be compact. Define a transverse generalized form of type $(1,1)$ by

$$\omega := g(I(\cdot),\cdot) \in A^{1,1}(M)\,.$$

This form is called the transverse generalized fundamental form. We define four operators, in particular, an analogue $\mathcal{L}$ of the Lefschetz operator, and a corresponding dual Lefschetz operator Λ, as follows.

$$(1)\ \mathcal{L} : A^{\bullet} \longrightarrow A^{\bullet+2}\,; \quad \alpha \mapsto \alpha \wedge \omega\,,$$

$$(2)\ \Lambda := \star^{-1} \circ \mathcal{L} \circ \star : A^{\bullet} \longrightarrow A^{\bullet-2}\,,$$

$$(3)\ d_L^* := -\star\, d_{\overline{L}} \star\,,$$

$$(4)\ d_{\overline{L}}^* := -\star\, d_L \star\,,$$

where $\star$ is defined in (5.2.2). Note that d_L^* and $d_{\overline{L}}^*$ are well defined even if M is not compact. But if M is compact, they are formal adjoints with respect to the hermitian inner product h (see (5.2.5)).

Now, assume $D\omega = 0$. This implies that $\mathscr{S}$ is transversely Kählerian with transversely Kähler metric g. Thus, $M/\mathscr{S}$ is a Kähler orbifold. Trivial modification of the proofs of [86, Proposition 1.2.26, Proposition 3.1.12] yields the following identities analogous to the Kähler identities in the classical case.

Proposition 5.2.1. *Let M be a regular GCS such that the leaf space $M/\mathscr{S}$ is a Kähler orbifold. Then*

1. $[\Lambda, \mathcal{L}] = (k-(p+q))\,\mathrm{Id}_{A^{p,q}}$
2. $[d_L, \mathcal{L}] = [d_{\overline{L}}, \mathcal{L}] = 0$ *and* $[d_L^*, \Lambda] = [d_{\overline{L}}^*, \Lambda] = 0\,.$
3. $[d_L^*, \mathcal{L}] = i d_{\overline{L}}\,,\ [d_{\overline{L}}, \mathcal{L}] = -i d_L$ *and* $[\Lambda, d_L] = -i d_{\overline{L}}^*\,,$ *and* $[\Lambda, d_{\overline{L}}] = i d_L^*\,.$

For the rest of this section, assume that M is compact and $M/\mathscr{S}$ is a Kähler orbifold. Let E be an SGH vector bundle over M. Consider the natural extension of $\mathcal{L}$, Λ on $A_E^{\bullet,\bullet}$, which will be again denoted by $\mathcal{L}$, Λ, respectively. Fix a transverse hermitian structure on E (see Definition 5.1.6). Let ∇_E be the transverse generalized Chern connection and Ω_∇ be its curvature. Let $\{U_\alpha, \phi_\alpha\}$ be an orthonormal trivialization of E. Then, on $U_\alpha \times \mathbb{C}^r$, with respect to such trivialization,

1. $\bar{\star}_E$ can be identified with the complex conjugate $\bar{\star}$ of the operator $\star$ defined in (5.2.3).

2. $\nabla_E = D + \theta_\alpha$, $\nabla_E^{1,0} = d_{\overline{L}} + \theta_\alpha^{1,0}$ and $\theta_\alpha^* = -\theta_\alpha$.

3. $$\begin{aligned}(\nabla_E^{1,0})^* &= -\bar{\star} \circ \nabla_{E^*}^{1,0} \circ \bar{\star} \quad \text{(by (5.2.10))}\\ &= -\bar{\star} \circ (d_{\overline{L}} - \theta_\alpha^{1,0}) \circ \bar{\star} \quad \text{(by (5.2.10))}\\ &= d_{\overline{L}}^* - (\theta_\alpha^{1,0})^* .\end{aligned}$$

$$\begin{aligned}[\Lambda, \nabla_E^{0,1}] + i(\nabla_E^{1,0})^* &= [\Lambda, d_L] + [\Lambda, \theta_\alpha^{0,1}] + i d_{\overline{L}}^* - i(\theta_\alpha^{1,0})^*\\ &= [\Lambda, \theta_\alpha^{0,1}] - i(\theta_\alpha^{1,0})^* \quad \text{(by (3) Proposition 5.2.1)} .\end{aligned}$$

So, the global operator $[\Lambda, \nabla_E^{0,1}] + i(\nabla_E^{1,0})^*$ is linear. For any point $x \in M$, we can always choose an orthonormal trivialization $\{U_\alpha, \phi_\alpha\}$ such that $x \in U_\alpha$ and $\theta_\alpha(x) = 0$. Since M is compact, $(\nabla_E^{1,0})^* = -\bar{\star}_{E^*} \circ \nabla_{E^*}^{1,0} \circ \bar{\star}_E$ is the formal adjoint of $\nabla_E^{1,0}$.

Lemma 5.2.2. *Let ∇_E be the transverse generalized Chern connection on E and Ω_∇ be its curvature. Then, we have*

1. $[\Lambda, \nabla_E^{0,1}] = -i((\nabla_E^{1,0})^*) = i(\bar{\star}_{E^*} \circ \nabla_{E^*}^{1,0} \circ \bar{\star}_E)$.

2. *For an arbitrary $\alpha \in \mathcal{H}_{d_{L,E}}^{\bullet,\bullet}$,*
$$\frac{i}{2\pi} h_E(\Omega_\nabla \Lambda\alpha, \alpha) \leq 0; \quad \textit{and} \quad \frac{i}{2\pi} h_E(\Lambda\Omega_\nabla \alpha, \alpha) \geq 0 ,$$
where h_E is the natural hermitian product, defined in (5.2.11).

Proof. (1). follows from the preceding discussion.

(2). By Proposition 5.1.1, $\Omega_\nabla = \nabla_E^{1,0} \circ d_{L,E} + d_{L,E} \circ \nabla_E^{1,0}$. Let α be an element in $\mathcal{H}^{p,q}_{d_{L,E}}$. Since $\Lambda\alpha \in A_E^{p-1,q-1}(M)$, $\Omega_\nabla \Lambda\alpha \in A_E^{p,q}(M)$. So, we can compute

$$\begin{aligned}
h_E(i\Omega_\nabla \Lambda\alpha, \alpha) &= ih_E(\nabla_E^{1,0} d_{L,E}\Lambda\alpha, \alpha) + ih_E(d_{L,E}\nabla_E^{1,0}\Lambda\alpha, \alpha) \\
&= ih_E(d_{L,E}\Lambda\alpha, (\nabla_E^{1,0})^*\alpha) + ih_E(\nabla_E^{1,0}\Lambda\alpha, d_{L,E}^*\alpha) \\
&= h_E(d_{L,E}\Lambda\alpha, -i(\nabla_E^{1,0})^*\alpha) + 0 \quad (\text{as } \alpha \in \mathcal{H}^{p,q}_{d_{L,E}}) \\
&= h_E(d_{L,E}\Lambda\alpha, [\Lambda, \nabla_E^{0,1}]\alpha) \quad (\text{by (1)}) \\
&= h_E(d_{L,E}\Lambda\alpha, \Lambda\nabla_E^{0,1}\alpha) - h_E(d_{L,E}\Lambda\alpha, \nabla_E^{0,1}\Lambda\alpha) \\
&= -h_E(d_{L,E}\Lambda\alpha, d_{L,E}\Lambda\alpha) \quad (\text{as } \nabla_E^{0,1} = d_{L,E}) \\
&\leq 0\,.
\end{aligned}$$

Similarly, we can show $h_E(i\Lambda\Omega_\nabla\alpha, \alpha) \geq 0$. □

Definition 5.2.2.

1. *A real $(1,1)$-transverse generalized form α (that is, $\alpha = \overline{\alpha}$) is called (semi-) positive if for all GH tangent vectors $0 \neq v \in \mathcal{G}M$, one has*

$$-i\alpha(v, \overline{v}) > 0\,(\geq 0)\,.$$

2. *Let ∇ be a transverse generalized hermitian connection with respect to a transverse hermitian structure H on E such that $\Omega_\nabla \in A^{1,1}_{\mathrm{End}(E)}(M)$. The transverse generalized curvature Ω_∇ is (Griffiths-) positive if, for any local section $0 \neq s \in F_M(E)$, one has*

$$H(\Omega_\nabla(s), s)(v, \overline{v}) > 0$$

for all $0 \neq v \in \mathcal{G}M$.

Definition 5.2.3. *An SGH line bundle E over M is positive if its first generalized Chern class $\mathbf{g}c_1(E) \in H^2_D(M)$ (by Theorem 5.2.2) can be represented by a closed positive $(1,1)$-transverse generalized form where $\mathbf{g}c_1(E)$ is defined in Example 5.1.2.*

Theorem 5.2.5. *Let M be a compact regular GC manifold of type k. Let the leaf space $M/\mathscr{S}$ of the induced foliation be a Kähler orbifold. Let E be a positive SGH line bundle on M. Then, we have the following*

1. *(Generalized Kodaira vanishing theorem)*

$$H^q(M, (\mathcal{G}^*\mathbf{M})^p \otimes_{\mathcal{O}_M} \mathbf{E}) = 0 \quad \textit{for } p+q > k\,.$$

2. *(Generalized Serre's theorem) For any SGH vector bundle $E^{'}$ on M, there exists a constant m_0 such that*

$$H^q(M, \mathbf{E}^{'} \otimes_{\mathcal{O}_M} \mathbf{E}^m) = 0 \quad \textit{for } m \geq m_0 \textit{ and } q > 0\,.$$

Proof. Choose a transverse hermitian structure on E such that the curvature of the transverse generalized Chern connection ∇_E is positive, that is, $\frac{i}{2\pi}\Omega_{\nabla_E}$ is a transverse Kähler form (that is, D-closed transverse generalized fundamental form) on M. We endow M with this corresponding transverse Kähler structure.

(1) With respect to this transverse Kähler structure, the operator $\mathcal{L}$ is nothing but the curvature operator $\frac{i}{2\pi}\Omega_{\nabla_E}$. Then, for $\alpha \in \mathcal{H}^{p,q}_{d_{L,E}}$,

$$\begin{aligned} 0 \leq\, & h_E(\frac{i}{2\pi}[\Lambda, \Omega_{\nabla_E}]\alpha, \alpha) \quad \text{(by (2) Lemma 5.2.2)} \\ &= h_E([\Lambda, \mathcal{L}]\alpha, \alpha) \\ &= (k-(p+q))h_E(\alpha,\alpha) \quad \text{(by (1) Proposition 5.2.1)}\,. \end{aligned}$$

By Corollary 4.4.2 and Theorem 5.2.3, we get

$$\mathcal{H}^{p,q}_{d_{L,E}} \cong H^{p,q}_{d_L}(M, E) \cong H^q(M, (\mathcal{G}^*\mathbf{M})^p \otimes_{\mathcal{O}_M} \mathbf{E})\,.$$

Hence $H^q(M, (\mathcal{G}^*\mathbf{M})^p \otimes_{\mathcal{O}_M} \mathbf{E}) = 0$ for $p+q > k$.

(2) Let $m \neq 0$. Choose a transverse hermitian structure on $E^{'}$ and denote its associated transverse generalized Chern connection by $\nabla_{E'}$. Then we have an induced transverse Chern connection on $E^{''} := E^{'} \otimes E^m$, denoted by ∇, corresponding to the induced transverse hermitian structure,

$$\nabla = \nabla_{E'} \otimes 1 + 1 \otimes \nabla_{E^m}\,,$$

where ∇_{E^m} is induced by ∇_E. Its curvature is of the form

$$\begin{aligned}\frac{i}{2\pi}\Omega_{\nabla} &= \frac{i}{2\pi}\Omega_{\nabla_{E'}} \otimes 1 + \frac{i}{2\pi}(1 \otimes \Omega_{\nabla_{E^m}}) \\ &= \frac{i}{2\pi}\Omega_{\nabla_{E'}} \otimes 1 + m(1 \otimes \frac{i}{2\pi}\Omega_{\nabla_E}) \quad (\text{as } \Omega_{\nabla_{E^m}} = m\Omega_{\nabla_E})\end{aligned}$$

By (2) in Lemma 5.2.2, for $\alpha \in \mathcal{H}^{p,q}_{d_{L,E''}}$, we have

$$\begin{aligned}0 \leq \frac{i}{2\pi} & h_{E''}([\Lambda, \Omega_{\nabla}]\alpha, \alpha) \\ &= \frac{i}{2\pi}h_{E''}([\Lambda, \Omega_{E'}]\alpha, \alpha) + m\, h_{E''}([\Lambda, \mathcal{L}]\alpha, \alpha) \\ &= \frac{i}{2\pi}h_{E''}([\Lambda, \Omega_{E'}]\alpha, \alpha) + m(k-(p+q))h_{E''}(\alpha, \alpha) \quad (\text{by (1) Proposition 5.2.1})\,.\end{aligned}$$

Since $h_{E''}$ is a positive-definite hermitian matrix on each fiber of $E^{''}$, we can consider the fiber wise Cauchy-Schwarz inequality

$$|h_{E''}([\Lambda, \Omega_{E'}]\alpha, \alpha)| \leq ||[\Lambda, \Omega_{E'}]|| \cdot h_{E''}(\alpha, \alpha)\,.$$

By compactness of M, we have a global upper bound C for the operator norm $||[\Lambda, \Omega_{E'}]||$, independent of m, and a corresponding global inequality. Thus, we get,

$$\begin{aligned}0 &\leq |\frac{i}{2\pi}h_{E''}([\Lambda, \Omega_{E'}]\alpha, \alpha)| + (m(k-(p+q)))\, h_{E''}(\alpha, \alpha) \\ &= \left(\frac{C}{2\pi} + (m(k-(p+q)))\right)\, h_{E''}(\alpha, \alpha)\,.\end{aligned}$$

Hence, if $C+2\pi\, m(k-(p+q)) < 0$, then $\alpha = 0$. When $p = k$ and $q > 0$, $m > \frac{C}{2\pi} \geq \frac{C}{2\pi\, q}$ ensures $\alpha = 0$. So, if we take $m_0 > \frac{C}{2\pi}$, by Corollary 4.4.2 and Theorem 5.2.3, we get

$$\mathcal{H}^{k,q}_{d_{L,E''}} \cong H^q(M, (\mathcal{G}^*\mathbf{M})^k \otimes_{\mathcal{O}_M} (\mathbf{E}^{'} \otimes_{\mathcal{O}_M} \mathbf{E^m})) = 0 \quad \text{for } m \geq m_0 \text{ and } q > 0\,.$$

Now, we apply these arguments to the SGH bundle $(\mathcal{G}M)^k \otimes E^{'}$ instead of $E^{'}$. The constant m_0 might change in the process but this will prove the assertion. □

———— ○ ————

Chapter 6

Generalized Complex Structure on Torus Principal Bundles

In Chapter 3, we have seen how the total space of SGH bundles admits a GCS, which is locally a product GCS induced from both the base and the fiber. The natural query arising from this observation is why not consider it locally equivalent (via B-transformations and diffeomorphisms) to a product GCS. While this question seems plausible locally, complications arise when considering the GCS on different trivializations; suitable patching for B-transformations or diffeomorphisms may not happen, and transition maps may not preserve the B-transformations. However, if globally, there exists a B-transformation inducing local B-transformations, then patching becomes straightforward.

The objective of this chapter can be approached in two ways. First, it delves into this scenario for certain principal G-bundles over complex manifolds, where G denotes a symplectic manifold. It demonstrates that even if the GCS is locally equivalent (via B-transformations and diffeomorphisms) to a product GCS, Lemma 3.1.1 remains valid. Additionally, when G is a torus, the condition of locally equivalent (via B-transformations and diffeomorphisms) to a product GCS, is equivalent to the flatness of those certain principal torus bundles, as described in Theorems 6.2.1-6.2.3. Second, it provides a class of GC manifolds whose leaf space is a smooth manifold.

An application of Theorem 6.2.1 is that the spectral sequence developed by Angella et al. [4] can be applied to describe the generalized Dolbeault cohomology of the total space of the bundle. This is explained in the more general setting of symplectic fiber bundles with suitable assumptions on the GCS that are slightly more general than the hypotheses of [4] (see Theorems 4.2.1 and 4.2.2). The case of principal torus bundles is stated in Corollary 4.3.1, and a Künneth formula for the generalized Dolbeault cohomology of these bundles is given in Corollary 4.4.1. It is important to note that by using torus action on a GC manifold, one can also study various geometric properties. For example, see [17, 105, 106]. This chapter is based on [126] and is structured into three sections:

1. GC structure on principal bundles (Section 6.1).
2. Tubular neighborhood of the fiber of a torus bundle (Section 6.2).
3. A spectral sequence for the generalized Dolbeault cohomology (Section 6.3).

6.1 GC structure on principal bundles

The following construction of a generalized complex structure on a smooth principal torus bundle over a complex manifold is mentioned as Example 2.16 in the thesis of Cavalcanti [36]. We present a detailed argument for the convenience of the reader.

Proposition 6.1.1. *Let (E, π, M) be a smooth principal $\mathbb{T}^{2l}$-bundle over a complex manifold M with characteristic class of type $(1,1)$. Then, the total space E admits a family of regular GCS with the fibers as leaves of the associated symplectic foliation.*

Proof. Consider a connection $(\theta_1, \ldots, \theta_{2l})$ on the principal bundle E corresponding to a decomposition $\mathbb{T}^{2l} = \prod_{j=1}^{2l} S^1$ of Lie groups. By the hypothesis, we may choose the connection so that its curvature form is of type $(1,1)$. Then, for each j there exists a 2-form χ_j of type $(1,1)$ on M such that

$$d\theta_j = \pi^* \chi_j \,. \tag{6.1.1}$$

Note that

$$\omega := \sum_{j=1}^{l} \theta_{2j-1} \wedge \theta_{2j} \tag{6.1.2}$$

is a $\mathbb{T}^{2l}$-invariant 2-form on E which restricts to an invariant symplectic form on each fiber of E.

Let Ω be a local generator of $\wedge^{(n,0)}(T^*M \otimes \mathbb{C})$ where $n = \dim_{\mathbb{C}}(M)$. More precisely, if $(z_1, \ldots, z_n)$ is a system of local holomorphic coordinates on M, we may take

$$\Omega = dz_1 \wedge \ldots \wedge dz_n \,.$$

In addition, let η be an arbitrary real closed 2-form on E. Define

$$\rho := e^{\eta+i\omega} \wedge \pi^*\Omega\,. \tag{6.1.3}$$

Then, it is clear that

$$\pi^*\Omega \wedge \pi^*\overline{\Omega} \wedge \omega^l \neq 0\,. \tag{6.1.4}$$

Moreover, as $d\Omega = 0$ and $d\eta = 0$, we have

$$d\rho = e^\eta \wedge i\left(e^{i\omega} - \frac{(i\omega)^l}{l!}\right) d\omega \wedge \pi^*\Omega\,. \tag{6.1.5}$$

Using (6.1.2), we have

$$d\omega \wedge \pi^*\Omega = \sum_{j=1}^{l}(\pi^*\chi_{2j-1} \wedge \theta_{2j} - \theta_{2j-1} \wedge \pi^*\chi_{2j}) \wedge \pi^*\Omega\,.$$

Note that

$$\chi_j \wedge \Omega = 0$$

for each j, as χ_j is of type $(1,1)$ and Ω is of type $(n,0)$. Hence,

$$d\omega \wedge \pi^*\Omega = 0 = d\rho\,. \tag{6.1.6}$$

By Definition 2.3.3 (cf. [70, Theorem 3.38 and Theorem 4.8]), (6.1.4) and (6.1.6) imply that E admits a generalized complex structure whose canonical line bundle is locally generated by ρ (see also [38, Section 1]). □

Let K be an even-dimensional compact Lie group and let G denote the complexification of K. Let (E_K, π, M) be a smooth principal K-bundle over a complex manifold M. We say that (E_K, π, M) admits a complexification if it can be obtained by a smooth reduction of structure group from a holomorphic principal G-bundle $(E_G, \widetilde{\pi}, M)$.

Theorem 6.1.1. *Let K be an even dimensional compact Lie group and let G denote the complexification of K. Let (E_K, π, M) be a smooth principal K-bundle over a complex manifold M, which admits a complexification. Then E_K admits a family of generalized complex structures.*

Proof. Let $\mathbb{T}$ be a maximal torus of K. Let B be a Borel subgroup of G containing K. Then, by [130, Section 5] there exists a complex manifold $X = E_G/B$, such that E_K admits the structure of a principal T-bundle, (E_K, π', X) say, over X. Moreover, this principal $\mathbb{T}$-bundle over X admits a complexification. So, by [130, Section 4], (E_K, π', X) admits a $(1,0)$ connection with $(1,1)$ curvature. Now, applying Theorem 6.1.1 to the bundle (E_K, π', X), we conclude that E_K admits a family of generalized complex structures. □

Specific examples of bundles that admit a complexification include the unitary frame bundle associated with a holomorphic vector bundle of even rank over a complex manifold. We refer the reader to [130, Section 3] for more examples. The following example was kindly shared with us by Ajay Singh Thakur.

Example 6.1.1. *Let $E \to M$ be a smooth vector bundle of rank n over a complex manifold M. Let $\pi : \mathbb{P}(E) \to M$ be the associated projective bundle over M with fiber $\mathbb{CP}^{n-1}$, where $\mathbb{P}(E)$ is the space of lines in E. Let $\mathcal{L}$ be the tautological complex line bundle over $\mathbb{P}(E)$. The restriction of $\mathcal{L}$ to each fiber is the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^{n-1}$. Let ω be the first Chern class of the dual bundle $\mathcal{L}^*$. Note that the Fubini-Study metric ω_{FS} on $\mathbb{CP}^{n-1}$ is first Chern class of the line bundle $\mathcal{O}(1)$. Therefore, ω a closed two form on $\mathbb{P}(E)$, whose restriction to each fiber $\mathbb{CP}^{n-1}$ is the symplectic form ω_{FS}. If Ω be a local generator of $\wedge^{(n,0)}(T^*M \otimes \mathbb{C})$, then define*

$$\rho := e^{i\omega} \wedge \pi^*\Omega. \tag{6.1.7}$$

As $d\Omega = 0$ and $d\omega = 0$, we have $d\rho = 0$. Hence, $\mathbb{P}(E)$ admits a generalized complex structure.

6.2 Tubular neighborhood of the fiber of a torus bundle

Let θ be a *Maurer-Cartan* connection 1-form on S^1. Consider a decomposition

$$\mathbb{T}^{2l} = \prod_{j=1}^{2l} S^1 \tag{6.2.1}$$

of Lie groups. Let $P_i : \mathbb{T}^{2l} = \prod_{j=1}^{2l} S^1 \longrightarrow S^1$ be the projection map on i-th coordinate for $i = 1, \ldots, 2l$. Then $(P_1^*\theta, \ldots, P_{2l}^*\theta)$ is the *Maurer-Cartan* connection on $\mathbb{T}^{2l} = \prod_{j=1}^{2l} S^1$. Note that the 2-form

$$\omega_{\mathbb{T}} = \sum_{j=1}^{l} (P_{2j-1}^*\theta \wedge P_{2j}^*\theta) \tag{6.2.2}$$

gives a symplectic form on $\mathbb{T}^{2l}$.

Let $M \times \mathbb{T}^{2l} \xrightarrow{\mathrm{Pr}_2} \mathbb{T}^{2l}$ is the natural projection map. For each $i \in \{1, 2, ..., 2l\}$, define

$$\tilde{\theta}_i = \mathrm{Pr}_2^* \, P_i^* \theta \, . \tag{6.2.3}$$

Then, $(\tilde{\theta}_1, \ldots, \tilde{\theta}_{2l})$ is a $\mathbb{T}^{2l}$-invariant connection of the trivial principal $\mathbb{T}^{2l}$-bundle $M \times \mathbb{T}^{2l} \xrightarrow{\mathrm{Pr}_1} M$.

Now, let $\pi : E \to M$ be a smooth principal $\mathbb{T}^{2l}$-bundle that satisfies the hypothesis of Proposition 6.1.1. Let $\{U_\alpha\}$ be a locally finite open cover of M such that E admits a local trivialization

$$\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{T}^{2l} \tag{6.2.4}$$

over each U_α. Let $i_\alpha : U_\alpha \times \mathbb{T}^{2l} \hookrightarrow M \times \mathbb{T}^{2l}$ be the natural inclusion map. Then, $\phi_\alpha^* i_\alpha^* \tilde{\theta}_i$, $1 \le i \le 2l$, are 1-forms on $\pi^{-1}(U_\alpha)$. Let $\{\psi_\alpha\}$ be a smooth partition of unity on M subordinate to $\{U_\alpha\}$. For each $j \in \{1, \ldots, 2l\}$, define an S^1-invariant 1-form θ_j on E by

$$\theta_j = \sum_\alpha (\psi_\alpha \circ \pi) \ \phi_\alpha^* i_\alpha^* \tilde{\theta}_j \, . \tag{6.2.5}$$

Then, $\Theta := (\theta_1, \ldots, \theta_{2l})$ gives a connection on the principal bundle E corresponding to the decomposition (6.2.1) which is invariant under the $\mathbb{T}^{2l}$ action. Define

$$\omega \ := \sum_{j=1}^{1} \theta_{2j-1} \wedge \theta_{2j} \, . \tag{6.2.6}$$

If the curvature of Θ is of type $(1, 1)$, then by Proposition 6.1.1, we have a family of GCS on E with the canonical line bundle U_E locally generated by

$$\rho \ = \ e^{\eta + i\omega} \wedge \pi^*(\Omega) \tag{6.2.7}$$

where η is a closed real 2-form on E. In general, ρ only gives an *almost GCS* on E, i.e., the integrability condition (2.3.4) may not be satisfied.

Let

$$\rho_0 \ := \ e^{i\omega} \wedge \pi^*(\Omega) \, . \tag{6.2.8}$$

For each $b \in M$, let

$$i_b : \pi^{-1}(b) = E_b \longrightarrow E \tag{6.2.9}$$

be the natural inclusion map. Consider the $\mathbb{T}^{2l}$-invariant symplectic structure induced by ω on E_b,

$$\omega_b := i_b^*\omega\,.$$

Given $b \in U_\alpha$, consider the map (cf. (6.2.4))

$$\phi_{\alpha,b}^{-1} := \phi_\alpha^{-1}(b,\cdot) : \mathbb{T}^{2l} \longrightarrow E_b.$$

Similarly, using the identification of $\{b\} \times \mathbb{T}^{2l}$ with $\mathbb{T}^{2l}$, denote by

$$\phi_{\alpha,b} : E_b \longrightarrow \mathbb{T}^{2l} \tag{6.2.10}$$

the restriction of the map ϕ_α to E_b. Consider the family of symplectic forms $\tilde{\omega}_b$ on $\mathbb{T}^{2l}$ defined by

$$\tilde{\omega}_b = (\phi_{\alpha,b}^{-1})^*\omega_b\,. \tag{6.2.11}$$

Lemma 6.2.1. *For any $b \in U_\alpha \cap U_\beta$,*

$$(\phi_{\alpha,b}^{-1})^*\omega_b = (\phi_{\beta,b}^{-1})^*\omega_b\,.$$

Proof. Consider the composition of the maps $E_b \xrightarrow{\phi_{\alpha,b}} \mathbb{T}^{2l} \xrightarrow{\phi_{\beta,b}^{-1}} E_b$. Note that $\phi_{\beta,b}^{-1} \circ \phi_{\alpha,b} \in \mathbb{T}^{2l}$. Then, we have,

$$(\phi_{\beta,b}^{-1} \circ \phi_{\alpha,b})^*\omega_b = \omega_b \qquad \text{(as } \omega_b \text{ is } \mathbb{T}^{2l}\text{-invariant)}$$

$$\Longrightarrow \phi_{\alpha,b}^* \circ (\phi_{\beta,b}^{-1})^*\omega_b = \omega_b\,,$$

$$\Longrightarrow (\phi_{\beta,b}^{-1})^*\omega_b = (\phi_{\alpha,b}^{-1})^*\omega_b.$$

□

Thus, the symplectic form $\tilde{\omega}_b$ that defined on $\mathbb{T}^{2l}$ by (6.2.11) does not depend on the choice of local trivialization.

Remark 6.2.1. *Lemma 6.2.1 does not depend on the choice of connection i.e, for any connection $(\theta_1, \ldots, \theta_{2l})$ on the principal bundle E corresponding to a decomposition $\mathbb{T}^{2l} = \prod_{j=1}^{2l} S^1$ of Lie groups and the corresponding ω as in Proposition 6.1.1, we can use the same techniques and get the same result.*

Now by a similar argument as in Lemma 6.2.1 and as $\tilde{\theta}_i$ is $\mathbb{T}^{2l}$-invariant,

$$\phi^*_{\alpha,b}(\tilde{\theta}_i\,|_W) = \phi^*_{\beta,b}(\tilde{\theta}_i\,|_W)\,,$$

for any open set $W \subset U_\alpha \cap U_\beta$ and any $b \in W$. Therefore,

$$i^*_b \circ \phi^*_\alpha(\tilde{\theta}_i\,|_W) = i^*_b \circ \phi^*_\beta(\tilde{\theta}_i\,|_W)\,,$$

for any $b \in W \subset U_\alpha \cap U_\beta$.

Then, it follows from the local finiteness of $\{U_\alpha\}$ and (6.2.5) that for any $b \in M$ and $i \in \{1, \ldots, 2l\}$, there exists a suitable open set W containing b such that

$$\begin{aligned} i^*_b\theta_i &= \sum_\beta \psi_\beta(b)\, i^*_b \circ \phi^*_\beta(\tilde{\theta}_i\,|_W) \\ &= \sum_\beta \psi_\beta(b)\, i^*_b \circ \phi^*_\alpha(\tilde{\theta}_i\,|_W) \\ &= i^*_b \circ \phi^*_\alpha(\tilde{\theta}_i\,|_W) \end{aligned}$$

for any α satisfying $b \in U_\alpha$. Hence, we have,

$$\begin{aligned} i^*_b\theta_i &= i^*_b \circ \phi^*_\alpha(\tilde{\theta}_i\,|_W) \\ &= i^*_b \circ \phi^*_\alpha(\tilde{\theta}_i) \\ &= (\phi_\alpha \circ i_b)^*\tilde{\theta}_i \end{aligned} \tag{6.2.12}$$

for any α such that $b \in U_\alpha$.

Let us explicitly calculate $\tilde{\omega}_b$ for every $b \in M$. As in (6.2.6), $\omega = \sum_{j=1}^{1} \theta_{2j-1} \wedge \theta_{2j}$.

We have

$$\begin{aligned}
i_b^*\omega &= \sum_{j=1}^{l} i_b^*\theta_{2j-1} \wedge i_b^*\theta_{2j} \\
&= \sum_{j=1}^{l} (\phi_\alpha \circ i_b)^*\tilde{\theta}_{2j-1} \wedge (\phi_\alpha \circ i_b)^*\tilde{\theta}_{2j} \quad \text{(by (6.2.12))} \\
&= \sum_{j=1}^{l} (\phi_\alpha \circ i_b)^*(\tilde{\theta}_{2j-1} \wedge \tilde{\theta}_{2j}) \\
&= \sum_{j=1}^{l} (\mathrm{Pr}_2 \circ \phi_\alpha \circ i_b)^*(P_{2j-1}^*\theta \wedge P_{2j}^*\theta) \\
&= \sum_{j=1}^{l} (\mathrm{Pr}_2 \circ \tilde{i}_b \circ \phi_{\alpha,b})^*(P_{2j-1}^*\theta \wedge P_{2j}^*\theta) \\
&= \sum_{j=1}^{l} \phi_{\alpha,b}^*(P_{2j-1}^*\theta \wedge P_{2j}^*\theta) \quad (\text{as } \mathrm{Pr}_2 \circ \tilde{i}_b = \mathrm{Id}_{\mathbb{T}^{2l}}, \text{ see (6.2.20)}) \\
&= \phi_{\alpha,b}^*\,\omega_{\mathbb{T}} \quad \text{(see (6.2.2))}
\end{aligned}$$

for any α such that $b \in U_\alpha$. Therefore, for each $b \in M$,

$$\tilde{\omega}_b = (\phi_{\alpha,b}^{-1})^*(i_b^*\omega) = (\phi_{\alpha,b}^{-1})^*(\phi_{\alpha,b}^*\omega_{\mathbb{T}}) = \omega_{\mathbb{T}}\,. \tag{6.2.13}$$

In other words, $\tilde{\omega}_b$ is independent of the choice of $b \in M$.

Lemma 6.2.2. *Let $(\theta_1, \ldots, \theta_{2l})$ and $(\theta_1^{'}, \ldots, \theta_{2l}^{'})$ be any two connections on E corresponding to a decomposition of $\mathbb{T}^{2l} = \prod_{i=1}^{2l} S^1$ of Lie groups. Then, for each $j \in \{1, \cdots, 2l\}$, there exists a 1-form $\beta_j \in \Omega^1(M)$ such that*

$$\theta_j - \theta_j^{'} = \pi^*\beta_j\,.$$

Proof. Denote the connections by $\Theta := (\theta_1, \ldots, \theta_{2l})$ and $\Theta^{'} := (\theta_1^{'}, \ldots, \theta_{2l}^{'})$. For each $x \in M$, define the value of β_j at x, by

$$(\beta_j)_x(v) := (\theta_j - \theta_j^{'})_y(w) \tag{6.2.14}$$

for any $v \in T_xM$, where $\pi(y) = x$ and $d\pi_y(w) = v$. First, we show that the definition is independent of the choice of w for fixed v and y. Let $w, w^{'} \in (d\pi_y)^{-1}(v)$. Then, $d\pi_y(w - w^{'}) = 0$. So $w - w^{'} \in T_y(E_x)$. There exists a vector W in the Lie algebra $\mathfrak{t}$ of $\mathbb{T}^{2l}$ such that the fundamental vector field $W^\#$ of W satisfies

$$W^\#(y) = w - w^{'}\,.$$

It follows that

$$\Theta_y(w - w^{'}) = W = \Theta^{'}_y(w - w^{'}) .$$

Thus,

$$(\theta_j - \theta^{'}_j)_y(w) = (\theta_j - \theta^{'}_j)_y(w^{'}) ,$$

showing that the definition of $(\beta_j)_x(v)$ in (6.2.14) is independent of the choice of w.

Moreover, as the structure group is abelian, the connections Θ and $\Theta^{'}$ on E are $\mathbb{T}^{2l}$-invariant. Given any $y, y^{'} \in \pi^{-1}(x)$, there exists $g \in \mathbb{T}^{2l}$ such that $y = r_g(y') = y^{'} \cdot g$. Let $w^{'} \in T_{y^{'}M}$ such that $(dr_g)_{y^{'}}(w^{'}) = w$ and $d\pi_y(w) = v$. Then,

$$\begin{aligned}(\theta_j - \theta^{'}_j)_y(w) &= (\theta_j - \theta^{'}_j)_{gy^{'}}((dr_g)_{y^{'}}(w^{'})) \\ &= (r_g^*(\theta_j - \theta^{'}_j))_{y^{'}}(w^{'}) \\ &= (\theta_j - \theta^{'}_j)_{y^{'}}(w^{'}) .\end{aligned}$$

This proves that the definition of $(\beta_j)_x$ as in (6.2.14) is independent of choices of both y and w.

Next, we take advantage of the above independence of choices to show that β_j is a smooth form. Let

$$f := \phi_\alpha^{-1} \circ i$$

where $i : U_\alpha \longrightarrow U_\alpha \times \mathbb{T}^{2l}$ is the inclusion map defined by $i(z) := (z, 1)$. Here, 1 denotes the identity element of $\mathbb{T}^{2l}$. Then, $d\pi_{f(z)}(df_z(v)) = v$ for all $z \in U$ and $v \in T_zU$. It follows that

$$\beta_j = (\theta_j - \theta^{'}_j) \circ df ,$$

completing the proof of the lemma. □

Let $(\theta_1, \ldots, \theta_{2l})$ be the connection defined in (6.2.5), and let Ω denote a local $(n, 0)$ form on M as in Proposition 6.1.1. Let $(\theta^{'}_1, \ldots, \theta^{'}_{2l})$ be any connection on E, corresponding to the same decomposition $\mathbb{T}^{2l} = \prod_{i=1}^{2l} S^1$ of Lie groups, such that $d\rho'_0 = 0$ where

$$\rho^{'}_0 := e^{i\omega^{'}} \wedge \pi^*(\Omega) \quad \text{and} \quad \omega^{'} := \sum_{i=1}^{l} \theta^{'}_{2i-1} \wedge \theta^{'}_{2i} . \tag{6.2.15}$$

Then, following the Proposition 6.1.1, we get a family of GCS on E with the canonical line bundle $U^{'}_E$, locally generated by

$$\rho^{'} = e^{\eta + i\omega^{'}} \wedge \pi^*(\Omega) \tag{6.2.16}$$

where η is a closed real 2-form on E.

Fix α. Let $\mathrm{Pr}_1 : U_\alpha \times \mathbb{T}^{2l} \longrightarrow U_\alpha$ and $\mathrm{Pr}_2 : U_\alpha \times \mathbb{T}^{2l} \longrightarrow \mathbb{T}^{2l}$ be the natural projections. Note that $\mathrm{Pr}_1 \circ \phi_\alpha = \pi$ on $E\,|_{U_\alpha} = \pi^{-1}(U_\alpha)$. On $\pi^{-1}(U_\alpha)$, we have the GCS given by $\rho_0'|_{\pi^{-1}(U_\alpha)} = e^{i\omega'} \wedge \pi^*\Omega$. Hence, we get a GCS on $U_\alpha \times \mathbb{T}^{2l}$ given by

$$\widetilde{\rho}_\alpha := (\phi_\alpha^{-1})^*(\rho_0'|_{\pi^{-1}(U_\alpha)}) = e^{i(\phi_\alpha^{-1})^*\omega'} \wedge (\phi_\alpha^{-1})^*\pi^*\Omega = e^{i(\phi_\alpha^{-1})^*\omega'} \wedge \mathrm{Pr}_1^*\Omega\,. \tag{6.2.17}$$

Consider the following decomposition.

$$\Omega^k_{\mathbb{C}}(U_\alpha \times \mathbb{T}^{2l}) = \sum_{r+p+q=k} \mathrm{Pr}_2^*(\Omega^r_{\mathbb{C}}(\mathbb{T}^{2l})) \otimes_{C^\infty(U_\alpha \times \mathbb{T}^{2l},\mathbb{C})} \Big(\mathrm{Pr}_1^*(\Omega^{p,0}(U_\alpha)) \otimes_{C^\infty(U_\alpha,\mathbb{C})} \mathrm{Pr}_1^*(\Omega^{0,q}(U_\alpha)) \Big) \tag{6.2.18}$$

Accordingly, $i(\phi_\alpha^{-1})^*\omega' \in C^\infty(\wedge^2 T^*_{\mathbb{C}}(U_\alpha \times \mathbb{T}^{2l}))$ decomposes into six components,

$$\begin{array}{lll} A^{200} & & \\ A^{110} & A^{101} & \\ A^{020} & A^{011} & A^{002}\,. \end{array}$$

Here, the first superscript in A^{rpq} corresponds to the $\mathbb{C}$-valued de Rham grading on $\Omega^\bullet_{\mathbb{C}}(\mathbb{T}^{2l})$, and the last two superscripts correspond to the Dolbeault grading on $\Omega^\bullet_{\mathbb{C}}(U_\alpha)$. Furthermore, the exterior derivative decomposes into the sum of three operators

$$d = d_F + \partial + \overline{\partial}\,,$$

each of degree 1 in their respective component of the tri-grading. Note that d_F is the fiber-wise exterior derivative.

Denote the imaginary part of A^{200} by $\widehat{\omega}$. In other words,

$$A^{200} = i\widehat{\omega}\,. \tag{6.2.19}$$

Consider the maps,

$$\mathbb{T}^{2l} \xrightarrow{\quad \tilde{i}_b \quad} U_\alpha \times \mathbb{T}^{2l} \xrightarrow{\quad \phi_\alpha^{-1} \quad} \pi^{-1}(U_\alpha)$$

where

$$\tilde{i}_b(x) := (b, x)\,. \tag{6.2.20}$$

Recall the connection form θ_j in (6.2.5). Then, applying Lemma 6.2.2 and equation (6.2.13), for each $b \in M$, we get

$$
\begin{aligned}
& i_b^*\theta_j^{'} = i_b^*\theta_j \quad \forall\, j \\
\Longrightarrow\ & i_b^*\omega^{'} = i_b^*\omega \quad (\omega \text{ as in } (6.2.6)) \\
\Longrightarrow\ & (\phi_{\alpha,b}^{-1})^* i_b^*\omega^{'} = (\phi_{\alpha,b}^{-1})^* i_b^*\omega \\
\Longrightarrow\ & (\phi_{\alpha,b}^{-1})^* i_b^*\omega^{'} = \tilde{\omega}_b \\
\Longrightarrow\ & (\phi_{\alpha,b}^{-1})^* i_b^*\omega^{'} = \omega_{\mathbb{T}}\,.
\end{aligned}
$$

Then, we have,

$$
\begin{aligned}
\tilde{i}_b^* A^{200} &= i(\tilde{i}_b^*(\phi_\alpha^{-1})^*\omega^{'})) \\
&= i(\phi_\alpha^{-1} \circ \tilde{i}_b)^*\omega^{'} \\
&= i(i_b \circ \phi_{\alpha,b}^{-1})^*\omega^{'} \\
&= i(\phi_{\alpha,b}^{-1})^* i_b^*\omega^{'} \\
&= i\omega_{\mathbb{T}}\,.
\end{aligned}
$$

Hence, by (6.2.19) we get,

$$\tilde{i}_b^*\widehat{\omega} = \omega_{\mathbb{T}}\ \forall\, b \in U_\alpha\,. \tag{6.2.21}$$

Consider the GCS $\widetilde{\rho}_\alpha$ on $U_\alpha \times \mathbb{T}^{2l}$ from (6.2.17),

$$e^{i(\phi_\alpha^{-1})^*\omega^{'}} \wedge \mathrm{Pr}_1^*\,\Omega = e^{\sum A^{rpq}} \wedge \mathrm{Pr}_1^*\,\Omega\,.$$

Note that only the components A^{200} , A^{101} and A^{002} act non-trivially, via the wedge product, on $\mathrm{Pr}_1^*\,\Omega$ in the expression $e^{i(\phi_\alpha^{-1})^*\omega^{'}} \wedge \mathrm{Pr}_1^*\,\Omega$, as Ω is a pure $(n,0)$-type form. Therefore, $\widetilde{\rho}_\alpha$ simplifies to

$$\widetilde{\rho}_\alpha = e^{i\widehat{\omega} + A^{101} + A^{002}} \wedge \mathrm{Pr}_1^*\,\Omega\,. \tag{6.2.22}$$

Then by equation (6.1.6), we get that $d(i(\phi_\alpha^{-1})^*\omega^{'}) \wedge \mathrm{Pr}_1^*\,\Omega = 0$ which implies the following four equations

$$\overline{\partial} A^{002} = 0 \tag{6.2.23}$$

$$\overline{\partial}A^{101} + d_F A^{002} = 0 \tag{6.2.24}$$

$$\overline{\partial}A^{200} + d_F A^{101} = 0 \tag{6.2.25}$$

$$d_F A^{200} = 0\,. \tag{6.2.26}$$

The last equation just states that the pullback of $i(\phi_\alpha^{-1})^*\omega^{'}$ to any fiber is a closed form, as we already know by (6.2.21). Moreover, since $\overline{A^{101}}$ and $\overline{A^{002}}$ are of the type (110) and (020), respectively, their wedge products with $\mathrm{Pr}_1^*\,\Omega$ vanish. Therefore, in general, the exponent of e in (6.2.22) may be modified to

$$i\widehat{\omega} + A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}} + \widehat{A}\,,$$

where $\widehat{A}$ is a real 2-form of type (011). Therefore, we obtain,

$$\widetilde{\rho}_\alpha = e^{\widehat{B}+i\widehat{\omega}} \wedge \mathrm{Pr}_1^*\,\Omega\,, \tag{6.2.27}$$

where $\widehat{B} = A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}} + \widehat{A}\,.$

Lemma 6.2.3. *The form $\widehat{\omega}$ is the pullback of the form $\omega_{\mathbb{T}}$ on $\mathbb{T}^{2l}$ under the projection map* $\mathrm{Pr}_2 : U_\alpha \times \mathbb{T}^{2l} \longrightarrow \mathbb{T}^{2l}$*, i.e.*

$$\widehat{\omega} = \mathrm{Pr}_2^*\,\omega_{\mathbb{T}}\,.$$

Proof. Since $\widehat{\omega}$ is of type (200), it is of the form

$$\widehat{\omega} = \sum_{1\le j\le\, l(2l-1)} a_j\,\mathrm{Pr}_2^*\,\omega_j$$

where $a_j \in C^\infty(U_\alpha \times \mathbb{T}^{2l}, \mathbb{R})$ and $\{\omega_j : 1 \le j \le l(2l-1)\}$ is a global frame of the trivial bundle of smooth 2-forms on $\mathbb{T}^{2l}$. For any $b, b^{'} \in U_\alpha$, by (6.2.21),

$$\widetilde{i}_b^*\widehat{\omega} = \widetilde{i}_{b'}^*\widehat{\omega}$$

$$\Longrightarrow\ \sum_j a_j(b,\cdot)\,\omega_j = \sum_j a_j(b^{'},\cdot)\,\omega_j$$

$$\Longrightarrow\ \sum_j (a_j(b,\cdot) - a_j(b^{'},\cdot))\,\omega_j = 0$$

$$\Longrightarrow a_j(b,\cdot) = a_j(b^{'},\cdot\,).$$

Hence, there exists smooth functions $b_j \in C^\infty(\mathbb{T}^{2l}, \mathbb{R})$ such that

$$a_j = \mathrm{Pr}_2^*\,b_j\,.$$

Then,

$$\widehat{\omega} = \mathrm{Pr}_2^* \bar{\omega} \quad \text{where} \quad \bar{\omega} = \sum_j b_j \omega_j \in \Omega^2(\mathbb{T}^{2l})\,.$$

Finally, using (6.2.13) and the fact that $\mathrm{Pr}_2 \circ \tilde{i}_b = \mathrm{Id}$, we have

$$\omega_{\mathbb{T}} = \tilde{i}_b^* \widehat{\omega} = \tilde{i}_b^* \,\mathrm{Pr}_2^* \bar{\omega} = \bar{\omega}\,.$$

□

Let $\rho_1^{'} = e^{i\,\mathrm{Pr}_2^* \omega_{\mathbb{T}}} \wedge \mathrm{Pr}_1^* \Omega$. Then from (6.2.27) and Lemma 6.2.3, we can see that the generalized complex structure $\widetilde{\rho}_\alpha$, from (6.2.17), is of the form

$$\widetilde{\rho}_\alpha = e^{\widehat{B}} \rho_1^{'} \qquad \text{on } U_\alpha \times \mathbb{T}^{2l}\,, \tag{6.2.28}$$

where $\widehat{B} = A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}} + \widehat{A}$, as defined in (6.2.27). Now $d\widetilde{\rho}_\alpha = 0$ because $d\rho^{'} = 0$, where $\rho^{'}$ as in (6.2.16). This implies

$$\begin{aligned} & e^{\widehat{B}} \wedge d\widehat{B} \wedge \rho_1^{'} = 0 \quad (\text{as } \ d\rho_1^{'} = 0) \\ \Longrightarrow\ & d\widehat{B} \wedge \rho_1^{'} = 0 \\ \Longrightarrow\ & d\widehat{B} \wedge \mathrm{Pr}_1^* \Omega = 0\,. \end{aligned}$$

So, to ensure $d\widehat{B} = 0$, it is enough to show that $(d\widehat{B})^{012}$ and $(d\widehat{B})^{111}$ both are zero. This imposes the following two constraint equations,

$$(d\widehat{B})^{012} = \partial A^{002} + \overline{\partial} \widehat{A} = 0 \tag{6.2.29}$$

$$(d\widehat{B})^{111} = \partial A^{101} + \overline{\partial A^{101}} + d_F \widehat{A} = 0\,. \tag{6.2.30}$$

For each $j \in \{1, 2, \ldots, 2l\}$, set

$$\theta_{j,\alpha}^{''} := (\phi_\alpha^{-1})^* \theta_j^{'}\,.$$

Then $(\theta_{1,\alpha}^{''}, \ldots, \theta_{2l,\alpha}^{''})$ defines a $\mathbb{T}^{2l}$-invariant connection on $U_\alpha \times \mathbb{T}^{2l}$. Consider the connection $(\tilde{\theta}_1, \ldots, \tilde{\theta}_{2l})$ on $U_\alpha \times \mathbb{T}^{2l}$ as defined in (6.2.3). By Lemma 6.2.2, there exist $\beta_{j,\alpha} \in \Omega^1(U_\alpha)$ such that

$$\theta_{j,\alpha}^{''} - \tilde{\theta}_j = \mathrm{Pr}_1^* \beta_{j,\alpha}\,. \tag{6.2.31}$$

Then, we have,

$$
\begin{aligned}
(\phi_\alpha^{-1})^*\omega' &= \sum_{j=1}^{l} \theta''_{2j-1,\alpha} \wedge \theta''_{2j,\alpha}\,, \\
&= \Pr_2^* \omega_{\mathbb{T}} + \sum_{j=1}^{l} (\tilde{\theta}_{2j-1} \wedge \Pr_1^* \beta_{2j,\alpha} + \Pr_1^* \beta_{2j-1,\alpha} \wedge \tilde{\theta}_{2j}) \\
&+ \sum_{j=1}^{l} \Pr_1^* (\beta_{2j-1,\alpha} \wedge \beta_{2j,\alpha})\,.
\end{aligned}
$$

Let $\beta^{pq}_{j,\alpha}$ correspond to the Dolbeault grading of $\beta_{j,\alpha}$, on $\Omega^\bullet_{\mathbb{C}}(U_\alpha)$. One can see that

$$
\begin{aligned}
A^{002} &= \sum_{j=1}^{l} \Pr_1^* (\beta^{01}_{2j-1,\alpha} \wedge \beta^{01}_{2j,\alpha})\,, \\
A^{101} &= \sum_{j=1}^{l} (\tilde{\theta}_{2j-1} \wedge \Pr_1^* \beta^{01}_{2j,\alpha} + \Pr_1^* \beta^{01}_{2j-1,\alpha} \wedge \tilde{\theta}_{2j})\,.
\end{aligned} \tag{6.2.32}
$$

Set $A^{02} := \sum_{j=1}^{l} (\beta^{01}_{2j-1,\alpha} \wedge \beta^{01}_{2j,\alpha})$. Then,

$$A^{002} = \Pr_1^* A^{02}\,. \tag{6.2.33}$$

From (6.2.23), we get $\overline{\partial} A^{02} = 0$. By using local $\overline{\partial}$-Poincaré Lemma on U_α, there exists a smooth form η of type (01) on U_α such that

$$A^{02} = \overline{\partial}\eta \quad \text{on } \; U_\alpha\,. \tag{6.2.34}$$

Let us assume that $\widehat{A} = \Pr_1^* A^{11}$ where A^{11} is a real form of type (11) on U_α . Then, equation (6.2.29) is equivalent to

$$\overline{\partial}(A^{11} - \partial\eta) = 0 \quad \text{on } \; U_\alpha\,. \tag{6.2.35}$$

Again, by using the local $\overline{\partial}$-Poincaré Lemma on U_α, a smooth form η' of type (10) on U_α such that

$$A^{11} - \partial\eta = \overline{\partial}\eta' \quad \text{on } \; U_\alpha\,. \tag{6.2.36}$$

Since A^{11} is real, $A^{11} - \partial\eta - \overline{\partial\eta}$ is both ∂ and $\overline{\partial}$ closed form. Then, by the local $\partial\overline{\partial}$-Lemma on U_α, there exists a smooth function $\chi \in C^\infty(U_\alpha, \mathbb{R})$ such that, on U_α

$$
\begin{aligned}
& A^{11} - \partial\eta - \overline{\partial\eta} = i\partial\overline{\partial}\chi \\
\Longrightarrow\; & A^{11} = \partial\eta + \overline{\partial\eta} + i\partial\overline{\partial}\chi\,.
\end{aligned} \tag{6.2.37}
$$

So, we can see that the general solution of equation (6.2.35) is (6.2.37). Thus, for any choice of such a χ, we get a desirable A^{11} as well as $\widehat{A}$ such that the first condition (6.2.29) is satisfied. By (6.2.32), we observe that

$$\partial A^{101} + \overline{\partial A^{101}} = \sum_{j=1}^{l} \left[\tilde{\theta}_{2j-1} \wedge \mathrm{Pr}_1^*(\partial \beta^{01}_{2j,\alpha} + \overline{\partial \beta^{01}_{2j,\alpha}}) + \mathrm{Pr}_1^*(\partial \beta^{01}_{2j-1,\alpha} + \overline{\partial \beta^{01}_{2j-1,\alpha}}) \wedge \tilde{\theta}_{2j} \right] .$$

Then, the second equation (6.2.30) is equivalent to

$$\partial \beta^{01}_{j,\alpha} + \overline{\partial \beta^{01}_{j,\alpha}} = 0 \quad \text{for all} \;\; j \in \{1, \ldots, 2l\} . \tag{6.2.38}$$

Since $d\tilde{\theta}_j = 0$, by (6.2.31), the curvature of the connection $(\theta''_{1,\alpha}, \ldots, \theta''_{2l,\alpha})$ is

$$(\mathrm{Pr}_1^*\, d\beta_{1,\alpha}, \ldots, \mathrm{Pr}_1^*\, d\beta_{2l,\alpha}) . \tag{6.2.39}$$

Theorem 6.2.1. *Let E be a principal $\mathbb{T}^{2l}$-bundle over an n-dimensional complex manifold M. Let $\Theta' := (\theta'_1, \ldots, \theta'_{2l})$ be any connection on E corresponding to a decomposition $\mathbb{T}^{2l} = \prod_{i=1}^{2l} S^1$ of Lie groups. Let $\omega' := \sum_{i=1}^{l} \theta'_{2i-1} \wedge \theta'_{2i}$. Let Ω be a local generator of $\wedge^{(n,0)}(T^*M \otimes \mathbb{C})$ over the trivializing open set $U_\alpha \subset M$. Set*

$$\rho' := e^{i\omega'} \wedge \pi^*\Omega$$

Then we have the following

1. *The condition $d\rho' = 0$ gives a GCS of type $\dim_{\mathbb{C}}(M)$ if and only if the curvature of the connection Θ' is of type $(1,1)$.*

2. *Let $\{U_\alpha, \phi_\alpha\}$ be a local trivialization. Then, ρ' is equivalent (via B-field transformation and diffeomorphism) to the product GCS*

$$(\phi_\alpha^{-1})^*(\rho'|_{\pi^{-1}(U_\alpha)}) \cong e^{i\,\mathrm{Pr}_2^*\omega_{\mathbb{T}}} \wedge \mathrm{Pr}_1^*\Omega .$$

on every $U_\alpha \times \mathbb{T}^{2l}$ if and only if the curvature of the connection Θ' is trivial.

Proof. 1. The sufficiency direction follows from the proof of Proposition 6.1.1.

For the other direction, let $\{U_\alpha, \phi_\alpha\}$ be a local trivialization. Consider the connection $(\tilde{\theta}_1, \ldots, \tilde{\theta}_{2l})$ on $U_\alpha \times \mathbb{T}^{2l}$ as defined in (6.2.3). Let $d\rho' = 0$.

Then, on $U_\alpha \times \mathbb{T}^{2l}$, we have

$$
\begin{aligned}
& d(\phi_\alpha^{-1})^*(\rho') = 0 \\
\Longrightarrow\ & d(e^{\widehat{B}}\rho_1') = 0\,, \quad \text{where } \rho_1' = e^{i\,\mathrm{Pr}_2^*\omega_{\mathbb{T}}} \wedge \mathrm{Pr}_1^*\Omega \\
\Longrightarrow\ & d\widehat{B} \wedge \mathrm{Pr}_1^*\Omega = 0 \\
\Longrightarrow\ & d(A^{101} + A^{002}) \wedge \mathrm{Pr}_1^*\Omega = 0\,, \quad \text{as } \Omega \text{ is of type } (n,0) \\
\Longrightarrow\ & \overline{\partial} A^{101} = 0\,, \quad \text{as } d_F A^{002} = 0 \\
\Longrightarrow\ & \sum_{j=1}^{l} \Big[\tilde{\theta}_{2j-1} \wedge \mathrm{Pr}_1^*(\overline{\partial}\beta^{01}_{2j,\alpha}) + \mathrm{Pr}_1^*(\overline{\partial}\beta^{01}_{2j-1,\alpha}) \wedge \tilde{\theta}_{2j}\Big] = 0\,, \quad \text{see (6.2.32)} \\
\Longrightarrow\ & \overline{\partial}\beta^{(0,1)}_{j,\alpha} = 0 \quad \text{for all } j\,.
\end{aligned}
$$

This shows that the $(0,2)$ component of the curvature is zero. Since the curvature is real, it follows that the $(2,0)$ component of the curvature is also zero.

2. By part (1), the curvature is assumed to be of type $(1,1)$. Then it follows from (6.2.39), that the curvature is of the form

$$(\mathrm{Pr}_1^*\Omega_{1,\alpha}, \ldots, \mathrm{Pr}_1^*\Omega_{2l,\alpha})$$

where $\Omega_{j,\alpha} = \partial\beta^{01}_{j,\alpha} + \overline{\partial\beta^{01}_{j,\alpha}}$ for all j. Then, by equation (6.2.38), the GCS is a product on local trivializations if and only if the curvature is zero.

□

Theorem 6.2.2. *Let E be a principal G-bundle over an n-dimensional complex manifold M where the structure group is a symplectic manifold (G, ω_G). If there exists a GCS, ρ', of type $\dim_{\mathbb{C}}(M)$ such that, on each trivialization $\{U_\alpha, \phi_\alpha\}$, it is equivalent (via B-field transformation and diffeomorphism) to the product GCS*

$$(\phi_\alpha^{-1})^*(\rho'|_{\pi^{-1}(U_\alpha)}) \cong e^{i\,\mathrm{Pr}_2^*\omega_G} \wedge \mathrm{Pr}_1^*\Omega\,,$$

then E is a flat symplectic G-bundle where Ω is a local generator of $\wedge^{(n,0)}(T^*M \otimes \mathbb{C})$.

Proof. Consider the map

$$\psi : U_{\alpha\beta} \times G \longrightarrow U_{\alpha\beta} \times G$$

defined by

$$\psi(m,f) = (m, \phi_{\alpha\beta}(m)f) \quad \text{for all } (m,f) \in U_{\alpha\beta} \times G\,,$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \longrightarrow G$ is the transition map. By the assumption on the GCS, there exists a real closed form $B_\alpha \in \Omega^2(U_\alpha \times G)$, such that

$$(\phi_\alpha^{-1})^*(\rho'|_{\pi^{-1}(U_\alpha)}) = e^{B_\alpha + i\,\mathrm{Pr}_2^*\,\omega_G} \wedge \mathrm{Pr}_1^*\,\Omega\,. \tag{6.2.40}$$

On $U_{\alpha\beta} \times G$, we again denote $B_\alpha|_{U_{\alpha\beta}\times G}$ and $B_\beta|_{U_{\alpha\beta}\times G}$ by B_α and B_β , respectively. Let $(B_\alpha + i\omega_G)^{rpq}$ denote the (rpq)-component of $B_\alpha + i\omega_G$ corresponding to the decomposition of $\Omega^2_{\mathbb{C}}(U_{\alpha\beta} \times G)$, as defined in (6.2.18) . Similar meanings are assigned to $(\psi^*(B_\alpha + i\omega_G))^{rpq}$, $(B_\beta + i\omega_G)^{rpq}$, and $(\psi^*(B_\beta + i\omega_G))^{rpq}$. By (6.2.40), on $U_{\alpha\beta} \times G$, we have

$$\begin{aligned}
&\psi^*(e^{B_\alpha + i\,\mathrm{Pr}_2^*\,\omega_G} \wedge \mathrm{Pr}_1^*\,\Omega) = e^{B_\beta + i\,\mathrm{Pr}_2^*\,\omega_G} \wedge \mathrm{Pr}_1^*\,\Omega \\
\Longrightarrow\ & e^{\psi^*(B_\alpha + i\,\mathrm{Pr}_2^*\,\omega_G)} \wedge \mathrm{Pr}_1^*\,\Omega = e^{B_\beta + i\,\mathrm{Pr}_2^*\,\omega_G} \wedge \mathrm{Pr}_1^*\,\Omega \\
\Longrightarrow\ & e^{\sum(\psi^*(B_\alpha + i\,\mathrm{Pr}_2^*\,\omega_G))^{r0q}} \wedge \mathrm{Pr}_1^*\,\Omega = e^{\sum(B_\beta + i\,\mathrm{Pr}_2^*\,\omega_G)^{r0q}} \wedge \mathrm{Pr}_1^*\,\Omega\,, \quad \text{as } \Omega \text{ is of type } (n,0) \\
\Longrightarrow\ & \sum(\psi^*(B_\alpha + i\,\mathrm{Pr}_2^*\,\omega_G))^{r0q} = \sum(B_\beta + i\,\mathrm{Pr}_2^*\,\omega_{\mathbb{T}})^{r0q}\,.
\end{aligned}$$

For $m \in U_{\alpha\beta}$, consider the map $\tilde{i}_m$, as defined in (6.2.20). Then,

$$\begin{aligned}
&\sum \tilde{i}_m^*(\psi^*(B_\alpha + i\,\mathrm{Pr}_2^*\,\omega_G))^{r0q} = \sum \tilde{i}_m^*(B_\beta + i\,\mathrm{Pr}_2^*\,\omega_G)^{r0q} \\
\Longrightarrow\ & \tilde{i}_m^*(\psi^*(B_\alpha + i\,\mathrm{Pr}_2^*\,\omega_G))^{200} = \tilde{i}_m^*(B_\beta + i\,\mathrm{Pr}_2^*\,\omega_G)^{200} \\
\Longrightarrow\ & \phi_{\alpha\beta}(m)^*(\tilde{i}_m^* B_\alpha + i\omega_G) = \tilde{i}_m^* B_\beta + i\omega_G \\
\Longrightarrow\ & \phi_{\alpha\beta}(m)^*\omega_G = \omega_G\,, \quad \text{as } B_\alpha\,,\ B_\beta \text{ and } \phi_{\alpha\beta}(m) \text{ are real}\,.
\end{aligned}$$

This shows that E is a symplectic G-bundle. Due to the property that E is a symplectic bundle, one can see that $(\psi^{-1})^*$ preserves $e^{i\,\mathrm{Pr}_2^*\,\omega_G} \wedge \mathrm{Pr}_1^*\,\Omega$, i.e,

$$(\psi^{-1})^*\rho_0 = \rho_0\,, \quad \text{where } \rho_0 = e^{i\,\mathrm{Pr}_2^*\,\omega_G} \wedge \mathrm{Pr}_1^*\,\Omega\,.$$

Let L be the $+i$-eigenbundle (i.e, null space) of the product GCS , ρ_0 . At a point $(m,f) \in U_{\alpha\beta} \times G$, L can be written in the following form,

$$L_{(m,f)} = \left(T_m^{0,1}U_{\alpha\beta} \oplus (T_m^{1,0}U_{\alpha\beta})^*\right) \oplus \{X - i\omega_{G,f}(X) \,|\, X \in T_fG \otimes \mathbb{C}\}\,. \tag{6.2.41}$$

For any $X+\eta \in L$, $\psi_*(X)+(\psi^{-1})^*\eta$ is again an element of L which is verified from the following:

$$\begin{aligned}(\psi_*(X)+(\psi^{-1})^*\eta)\cdot\rho_0 &= \rho_0(\psi_*(X))+(\psi^{-1})^*\eta\wedge\rho_0\\ &= (\psi^{-1})^*\left(\psi^*\left(i_{\psi_*(X)}(\psi^{-1})^*\rho_0\right)\right)+(\psi^{-1})^*(\eta\wedge\rho_0)\,, \quad \text{as } (\psi^{-1})^*\rho_0=\rho_0\\ &= (\psi^{-1})^*(i_X\rho_0+\eta\wedge\rho_0)\\ &= 0\,, \quad \text{as } (X+\eta)\cdot\rho_0=0\,.\end{aligned}$$

It follows that

$$X+\eta\in L \quad \text{if and only if} \quad \psi_*(X)+(\psi^{-1})^*\eta\in L \tag{6.2.42}$$

Note that for $(m,f)\in U_{\alpha\beta}\times G$,

$$(\psi)_{*(m,f)}=\begin{pmatrix} Id_{U_{\alpha\beta}} & 0\\ (r_f)_*\circ(\phi_{\alpha\beta})_{*m} & (\phi_{\alpha\beta}(m))_{*f}\end{pmatrix}, \tag{6.2.43}$$

$$(\psi^{-1})^*_{(m,f)}=\begin{pmatrix} Id_{U_{\alpha\beta}} & (\phi^{-1}_{\alpha\beta})^*_m\circ(r_{\phi_{\alpha\beta}(m)\cdot f})^*\\ 0 & (\phi^{-1}_{\alpha\beta}(m))^*_f\end{pmatrix},$$

where the map $r_f : G\longrightarrow G$ is the right translation by f.

Let $e\in G$ be the identity element and $Y\in T_eG$. Then, for $(m,e)\in U_{\alpha\beta}\times G$, we have

$$\begin{aligned}\begin{pmatrix}(\psi)_* & 0\\ 0 & (\psi^{-1})^*\end{pmatrix}(Y-i\omega_{G,e}(Y)) &= (\psi)_*(Y)-(\psi^{-1})^*(i\omega_{G,e}(Y))\\ &= \left\{(\phi_{\alpha\beta}(m))_{*e}(Y)-i(\phi^{-1}_{\alpha\beta}(m))^*_e(\omega_{G,e}(Y))\right\} \quad \text{(By (6.2.43))}\\ &\quad -i\left((\phi^{-1}_{\alpha\beta})^*_m\circ(r_{\phi_{\alpha\beta}(m)})^*(\omega_{G,e}(Y))\right),\end{aligned}$$

By (6.2.41) and (6.2.42), $\{Y-i\omega_{G,e}(Y)\}\in L_{(m,e)}$ implies that

$$(\psi)_*(Y)-(\psi^{-1})^*(i\omega_{G,e}(Y))\in L_{(m,\phi_{\alpha\beta}(m))}\,.$$

Then, it follows that

$$\eta := -i\left((\phi^{-1}_{\alpha\beta})^*_m\circ(r_{\phi_{\alpha\beta}(m)})^*(\omega_{G,e}(Y))\right)\in(T^{1,0}_mU_{\alpha\beta})^*\,.$$

Since $(\phi_{\alpha\beta}^{-1})_m^*$ and $(r_{\phi_{\alpha\beta}(m)})^*$ both are real linear operators, and $\omega_{G,e}(Y)$ is a real 1-form, we get $\overline{\eta} = -\eta$. This contradicts the fact that $\eta \in (T_m^{1,0}U_{\alpha\beta})^*$, and so,

$$(\phi_{\alpha\beta}^{-1})_m^* \circ (r_{\phi_{\alpha\beta}(m)})^*(\omega_{G,e}(Y)) = 0\,.$$

Now $(r_{\phi_{\alpha\beta}(m)})^*$ is an isomorphism and $\omega_{G,e}$ is non-degenerate which shows that $(\phi_{\alpha\beta}^{-1})_m^*$ will vanish. As $m \in U_{\alpha\beta}$ is arbitrary, we get

$$(\phi_{\alpha\beta})_m^* = 0\,, \quad \text{for all } m\,.$$

Hence E is a flat symplectic principal G-bundle. □

Theorem 6.2.3. *Let E be a principal $\mathbb{T}^{2l}$-bundle over an n-dimensional complex manifold M. Then, E is a flat symplectic $\mathbb{T}^{2l}$-bundle if and only if there exists a GCS, ρ', of type $\dim_{\mathbb{C}}(M)$ such that, on each trivialization $\{U_\alpha, \phi_\alpha\}$, it is equivalent (via B-field transformation and diffeomorphism) to the product GCS*

$$(\phi_\alpha^{-1})^*(\rho'|_{\pi^{-1}(U_\alpha)}) \cong e^{i\,\mathrm{Pr}_2^*\,\omega_{\mathbb{T}}} \wedge \mathrm{Pr}_1^*\,\Omega\,,$$

where $\omega_{\mathbb{T}}$ is a symplectic form on $\mathbb{T}^{2l}$ and Ω is a local generator of $\wedge^{(n,0)}(T^*M \otimes \mathbb{C})$.

Proof. One way is straightforward as one can construct such a GCS by using Theorem 6.2.1. The converse direction follows from Theorem 6.2.2. □

Remark 6.2.2. *Theorem 6.2.3 and Theorem 6.2.2 do not imply that all GCS of type $\dim_{\mathbb{C}} M$ is the product GCS on a trivializing open set. Even in the simplest case, it may happen that there exists a GCS which cannot be the product GCS in a trivializing open neighborhood. The following example demonstrates this.*

Example 6.2.1. *Let $E = F \times \mathbb{C}$ be the trivial bundle over $\mathbb{C}$ with symplectic fiber F of dimension $2l$. Let ω_F be a symplectic form on F. Set*

$$A := iz\overline{z}\,dz\,.$$

Let σ be a real closed, but not exact, 1-form on F. Consider the following 2-form

$$i\omega = i\omega_F + (A - \overline{A}) \wedge \sigma\,.$$

Now $d(A - \overline{A}) = -i\,(\overline{z} + z)\,dz \wedge d\overline{z}$. This implies $d\omega \neq 0$ but $d(i\omega) \wedge dz = 0$. Note that $\omega^l \wedge dz \wedge d\overline{z} = \omega_F^l \wedge dz \wedge d\overline{z} \neq 0$. Thus, we have a GCS of type 1 given by

$$\rho = e^{i\omega} \wedge dz\,.$$

If possible, let there exists a closed real $B \in \Omega^2(E)$ *such that*

$$\rho = e^{B+i\omega_F} \wedge dz\,.$$

Let C^{rpq} *denote the* (rpq)*-component of* $C := B + i\omega_F$ *in the natural decomposition of* $\Omega^2_{\mathbb{C}}(F \times \mathbb{C})$, *as given in* (6.2.18)*. Then, we have*

$$\begin{aligned} &\rho = e^{B+i\omega_F} \wedge dz \\ \Longrightarrow\ & e^{-\overline{A}\wedge\sigma} \wedge dz = e^{\sum C^{r0q}} \wedge dz \\ \Longrightarrow\ & -\overline{A} \wedge \sigma = C^{101}\,. \end{aligned}$$

Since C^{011} *is real, it is of the form* $C^{011} = if\, dz \wedge d\overline{z}$ *for some* $f \in C^\infty(\mathbb{C} \times F, \mathbb{R})$. *Let* d_F *be the exterior derivative in fiber direction. So,*

$$\begin{aligned} & dC = 0 \\ \Longrightarrow\ & (dC)^{111} = 0 \\ \Longrightarrow\ & (i\, d_F f + i(z+\overline{z})\,\sigma) \wedge dz \wedge d\overline{z} = 0 \\ \Longrightarrow\ & d_F f + (z+\overline{z})\,\sigma = 0\,. \end{aligned}$$

Fixing any $z \in \mathbb{C} - \{i\mathbb{R}\}$, *we have*

$$d_F g = \sigma\,, \quad \text{where} \quad g = -\frac{1}{z+\overline{z}} f(-,z) \in C^\infty(F,\mathbb{R})\,.$$

This contradicts that σ *is not exact. If possible, let there exists a real automorphism* h *on* $F \times \mathbb{C}$ *such that*

$$h^*\rho = e^{B+i\omega_F} \wedge dz\,.$$

Note that $h^*dz = dz$, *and* $i\, h^*\omega \wedge dz = (B + i\omega_F) \wedge dz$. *So, it follows that*

$$\begin{aligned} & ih^*\omega_F + h^*(-\overline{A}\wedge\sigma) = \sum C^{r0q} \\ \Longrightarrow\ & h^*(-\overline{A}\wedge\sigma) = C^{101} \end{aligned}$$

Then, we can continue as before. Thus we conclude that ρ *is not equivalent to the product GCS.*

6.3 A spectral sequence for the generalized Dolbeault cohomology

A *generalized holomorphic bundle* over a GC manifold B consists of a complex vector bundle W with a Lie algebroid connection

$$D: C^\infty(\wedge^i L^* \otimes W) \longrightarrow C^\infty(\wedge^{i+1} L^* \otimes W)$$

satisfying $D \circ D = 0$ (cf. [70, Definition 4.27]). For a generalized holomorphic bundle (W, D), the Lie algebroid cohomology is defined as

$$H^\bullet(L, W) = \frac{\ker(D: C^\infty(\wedge^\bullet L^* \otimes W) \longrightarrow C^\infty(\wedge^{\bullet+1} L^* \otimes W))}{\operatorname{img}(D: C^\infty(\wedge^{\bullet-1} L^* \otimes W) \longrightarrow C^\infty(\wedge^\bullet L^* \otimes W))}. \tag{6.3.1}$$

For any $2n$-dimensional GC manifold B with canonical line bundle U, the corresponding involutive maximal isotropic subbundle L, and the operator $\bar\partial$ as in equation (2.4.3) gives a Lie algebroid connection. Thus $\{U, \bar\partial\}$ is a generalized holomorphic bundle over B and also note that

$$GH^{n-\bullet}_{\bar\partial}(B) = H^\bullet(L, U). \tag{6.3.2}$$

Now coming back to our situation, let L be the null space of the canonical line bundle of E, denoted as U_E, as in Theorem 6.2.1. On a local trivialization $\{U_\alpha\}$, for a local holomorphic coordinate system $(z_1, \cdots, z_n) \in U_\alpha$, assume that, the GCS on $U_\alpha \times \mathbb{T}^{2l}$ is

$$(\phi_\alpha^{-1})^*(\rho_0|_{\pi^{-1}(U_\alpha)}) = e^{i\,\mathrm{Pr}_2^*\omega_{\mathbb{T}}} \wedge \mathrm{Pr}_1^*\Omega \tag{6.3.3}$$

where $\Omega = dz_1 \wedge \cdots \wedge dz_n$ and the null space is

$$L|_{\pi^{-1}(U_\alpha)} \;=\; \mathrm{Pr}_1^*(T^{0,1}U_\alpha \oplus (T^{1,0}U_\alpha)^*) \oplus \mathrm{Pr}_2^*\{X - i\omega_{\mathbb{T}}(X) \,|\, X \in T(\mathbb{T}^{2l}) \otimes \mathbb{C}\}.$$

Further, note that,

$$\mathrm{Pr}_2^*\{X - i\omega_{\mathbb{T}}(X) \,|\, X \in T(T^{2l}) \otimes \mathbb{C}\} = \{X - i\,\mathrm{Pr}_2^*\omega_{\mathbb{T}}(X) \,|\, X \in \mathrm{Pr}_2^* T(\mathbb{T}^{2l}) \otimes \mathbb{C}\}.$$

Consider the Courant involutive subbundle $S < L$ such that on local trivialization

$$S|_{\pi^{-1}(U_\alpha)} \;=\; \{X - i\,\mathrm{Pr}_2^*\omega_{\mathbb{T}}(X) \,|\, X \in \mathrm{Pr}_2^* T(\mathbb{T}^{2l}) \otimes \mathbb{C}\}.$$

Then following [4, Section 2], for any generalized holomorphic bundle V over E, the subspaces

$$F^p\, C^\infty(\wedge^{p+q} L^* \otimes V) = \{\phi \in C^\infty(\wedge^{p+q} L^* \otimes V) \,|\, \phi(X_1, \cdots, X_{p+q}) = 0 \text{ for } X_{n_1}, \cdots, X_{n_{q+1}} \in S\}$$

of $C^\infty(\wedge^\bullet L^*\otimes V)$ give a bounded decreasing filtration of $\{C^\infty(\wedge^\bullet L^*\otimes V), D\}$ such that the corresponding spectral sequence $\{E_r^{\bullet,\bullet}\}_r$ converges to the Lie algebroid cohomology $H^\bullet(L,V)$ described in (6.3.1). By definition,

$$\begin{aligned}E_0^{p,q} &= \frac{F^p\, C^\infty(\wedge^{p+q}L^*\otimes V)}{F^{p+1}\, C^\infty(\wedge^{p+q}L^*\otimes V)}\\ &= \frac{\{\phi\in C^\infty(\wedge^{p+q}L^*\otimes V)\,|\,\phi(X_1,\cdots,X_{p+q})=0 \text{ for } X_{n_1},\cdots,X_{n_{q+1}}\in S\}}{\{\phi\in C^\infty(\wedge^{p+q}L^*\otimes V)\,|\,\phi(X_1,\cdots,X_{p+q})=0 \text{ for } X_{n_1},\cdots,X_{n_q}\in S\}}.\end{aligned}$$

Locally, we have,

$$F^p\, C^\infty(\wedge^{p+q}L^*\otimes V) = \bigoplus_{p\le i\le p+q} C^\infty\left(\wedge^i \operatorname{Pr}_1^*(L_M^*|_{U_\alpha})\otimes\wedge^{p+q-i}S^*|_{\pi^{-1}(U_\alpha)}\right)\otimes_{C^\infty(U_\alpha\times\mathbb{T}^{2l},\mathbb{C})} C^\infty(V).$$

If $V=\pi^*(V')$, for a holomorphic vector bundle V' over M and $L_M = T^{0,1}M\oplus(T^{1,0}M)^*$, then

$$\begin{aligned}E_0^{p,q} &\cong C^\infty(\wedge^p \operatorname{Pr}_1^*(L_M^*|_{U_\alpha})\otimes \operatorname{Pr}_1^*(V'|_{U_\alpha}))\otimes_{C^\infty(U_\alpha\times\mathbb{T}^{2l},\mathbb{C})} C^\infty(\wedge^q S^*|_{\pi^{-1}(U_\alpha)})\\ &\cong C^\infty(\operatorname{Pr}_1^{-1}(\wedge^p(L_M^*|_{U_\alpha})\otimes V'|_{U_\alpha}))\otimes_{C^\infty(U_\alpha,\mathbb{C})} C^\infty(\wedge^q S^*|_{\pi^{-1}(U_\alpha)})\\ &\cong C^\infty(\wedge^p(L_M^*|_{U_\alpha})\otimes V'|_{U_\alpha})\otimes_{C^\infty(U_\alpha,\mathbb{C})} C^\infty(\wedge^q S^*|_{\pi^{-1}(U_\alpha)}).\end{aligned}$$

The differential d_0 on $E_0^{p,q}$ is given by $id\otimes d_S$ where d_S is the differential on the Lie algebroid complex $C^\infty(\wedge^\bullet S^*)$. For $b\in M$ and $E_b=\pi^{-1}(b)$, by [150, Section 9.2], we get a flat holomorphic vector bundle $\mathcal{H}^\bullet=\cup_{b\in M}H^\bullet(E_b,\mathbb{C})$ over M where $H^\bullet(E_b,\mathbb{C})$ denotes the $\mathbb{C}$-valued de Rham cohomology of E_b. Now, consider the Lie algebroid corresponding to the relative tangent bundle $\mathcal{T}$ of the principal bundle E, and the corresponding Lie algebroid cohomology $H^\bullet(\mathcal{T})$. Then, by [80, Chapter I.2.4],

$$H^\bullet(\mathcal{T})\cong C^\infty(M,\mathcal{H}^\bullet)\,. \tag{6.3.4}$$

Since $T(U_\alpha\times\mathbb{T}^{2l}) = \operatorname{Pr}_1^* T(U_\alpha)\oplus \operatorname{Pr}_2^* T(\mathbb{T}^{2l})$, we have $\mathcal{T}|_{\pi^{-1}(U_\alpha)} = \operatorname{Pr}_2^* T(\mathbb{T}^{2l})$. Moreover, as $\omega_{\mathbb{T}}$ is closed, we have a Lie algebroid isomorphism,

$$\mathcal{T}|_{\pi^{-1}(U_\alpha)}\xrightarrow{\cong} S|_{\pi^{-1}(U_\alpha)},\quad X\mapsto X-i\operatorname{Pr}_2^*\omega_{\mathbb{T}}(X)\,.$$

Applying (6.3.4), locally we have,

$$E_1^{p,q}\cong C^\infty(\wedge^p\,(L_M^*|_{U_\alpha})\otimes V'|_{U_\alpha})\otimes_{C^\infty(U_\alpha,\mathbb{C})} C^\infty(U_\alpha,\mathcal{H}^q|_{U_\alpha})$$

with the differential $d_1 = \bar{\partial}_M$, the usual Dolbeault operator on M. Hence, globally we have,

$$E_1^{p,q} \cong C^\infty(\wedge^p L_M^* \otimes V' \otimes \mathcal{H}^q)$$

with the differential d_1 being the Lie algebroid connection for the holomorphic bundle $V' \otimes \mathcal{H}^\bullet$. Hence, we obtain,

$$E_2^{p,q} \cong H^p(L_M, V' \otimes \mathcal{H}^q).$$

Thus, we have a description of the generalized Dolbeault cohomology of the total space which extends the description in [4, Theorem 2.1].

Theorem 6.3.1. *Let $\pi : E \longrightarrow M$ be a fiber bundle over a complex manifold M of complex dimension n with a symplectic fiber (F, ω_F). Assume that there exists $\omega \in \Omega^2(E)$ such that*

1. *it defines a generalized complex structure $\mathcal{J}$ on E which is locally of the form $\rho := e^{i\omega} \wedge \pi^*(\Omega)$,*

2. *on each local trivialization $\{U_\alpha, \phi_\alpha\}$, the GCS is equivalent (via B-field transformation and diffeomorphism) to the product GCS as in (6.3.3), i.e*

$$(\phi_\alpha^{-1})^*(\rho|_{\pi^{-1}(U_\alpha)}) \cong e^{i\,\mathrm{Pr}_2^*\omega_F} \wedge \mathrm{Pr}_1^*\Omega\,.$$

Here, Ω is a local generator of $\wedge^{(n,0)}(T^*M \otimes \mathbb{C})$. Let L be the $+i$-eigenbundle of $\mathcal{J}$. Let V be a complex vector bundle over E such that $V = \pi^* V'$ for a holomorphic vector bundle V' over the complex manifold M. Considering V as a generalized holomorphic bundle, there exists a spectral sequence $\{E_r^{\bullet,\bullet}\}_r$ which converges to $H^\bullet(L, V)$ such that

$$E_2^{p,q} \cong H^p(L_M, V' \otimes \mathcal{H}^q)\,.$$

Proof. Follows from the preceding description of the generalized Dolbeault cohomology of the total space. □

Since $U_E = \pi^* U_M$, where U_M is the canonical line bundle for M, we have the following theorem.

Theorem 6.3.2. *Consider the same setting as in the preceding theorem and* $\dim_{\mathbb{R}} F = 2l$. *Then there exists a spectral sequence $\{E_r^{\bullet,\bullet}\}_r$ which converges to $GH_{\bar{\partial}}^{n+l-\bullet}(E)$ such that*

$$E_2^{p,q} \cong GH_{\bar{\partial}}^{n-p}(M, \mathcal{H}^{l-q}).$$

Corollary 6.3.1. *For a flat $\mathbb{T}^{2l}$-principal bundle E with the family of GCS as defined in Proposition 6.1.1, there exists a spectral sequence $\{E_r^{\bullet,\bullet}\}_r$ which converges to $GH_{\bar{\partial}}^{n+l-\bullet}(E)$ such that*

$$E_2^{p,q} \cong GH_{\bar{\partial}}^{n-p}(M, \mathcal{H}^{l-q}).$$

Example 6.3.1. *(Generalized Dolbeault cohomology of trivial torus bundles) When the torus bundle is trivial $E = M \times \mathbb{T}^{2l}$, the flat holomorphic vector bundle $\mathcal{H}^\bullet$ is also trivial i.e, $\mathcal{H}^\bullet = M \times H^\bullet(\mathbb{T}^{2l}, \mathbb{C})$. So the $E_2^{p,q}$ term of the spectral sequence as in Theorem 6.3.1 is of the form*

$$E_2^{p,q} \cong H^p(L_M, U_M \otimes H^{l-q}(\mathbb{T}^{2l}, \mathbb{C})) = GH_{\bar{\partial}}^{q}(\mathbb{T}^{2l}) \otimes GH_{\bar{\partial}}^{n-p}(M).$$

Now, each element in E_2 term is already a global form on $M \times \mathbb{T}^{2l}$. Hence, d_k vanishes for any $k \geq 2$, and $E_2 = E_\infty$. Therefore, we get the following analogue of the Künneth formula

Corollary 6.3.2. *For the family of generalized complex structures as defined in Proposition 6.1.1, when $E = M \times \mathbb{T}^{2l}$, the generalized Dolbeault cohomology group of E has a decomposition in terms of the generalized Dolbeault cohomology groups of the fiber space and the base manifold, i.e,*

$$GH_{\bar{\partial}}^{n+l-m}(E) \cong \bigoplus_{p+q=m} \Big(GH_{\bar{\partial}}^{q}(\mathbb{T}^{2l}) \otimes GH_{\bar{\partial}}^{n-p}(M) \Big)$$

where $-l \leq q \leq l$, $-n \leq p \leq n$ and $-(n+l) \leq m \leq (n+l)$.

Remark 6.3.1. *In Theorem 6.3.1, if the form ω is closed, one may construct a B-transformation so that the GCS is the product GCS on each trivializing neighborhood. But even if ω is not closed, it may still be possible to construct such a B-transformation. The following example will show such a construction in the simplest case.*

Example 6.3.2. *Let $E = \mathbb{C} \times F$ be the trivial bundle over $\mathbb{C}$ with symplectic fiber F. Let ω_F is a symplectic form on F. Set*

1. $A_1 := (\frac{z^2}{2} + z\bar{z})\, d\bar{z}$.
2. $A_2 := z\, d\bar{z}$.

Let σ be a real closed 1-form on F. Define

$$i\omega_j = i\omega_F + (A_j - \overline{A_j}) \wedge \sigma\,, \quad \textit{for } j = 1, 2\,.$$

Not that $d(A_1 - \overline{A_1}) = 2(z+\overline{z})\, dz \wedge d\overline{z}$ *and* $d(A_2 - \overline{A_2}) = 2dz \wedge d\overline{z}$. *This implies that*

$$d\omega_j \neq 0 \quad \text{and} \quad d(i\omega_j) \wedge dz = 0\,.$$

One can see that $\omega_j^l \wedge dz \wedge d\overline{z} = \omega_F^l \wedge dz \wedge d\overline{z} \neq 0$ *which implies that*

$$\rho_j = e^{i\omega_j} \wedge dz \quad \text{for } j = 1, 2\,,$$

gives a GCS on E. *One may write*

$$\rho_j = e^{B_j + i\omega_F} \wedge dz\,, \quad \text{where } B_j = (A_j + \overline{A_j}) \wedge \sigma \text{ is a real 2-form}\,.$$

Notice that $dA_j = -d\overline{A_j}$. *This shows that* $dB_j = 0$ *for* $j = 1, 2$. *Hence each* ρ_j *is equivalent to the product GCS.*

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