Supersymmetric Yang-Mills theories without anti-commuting variables

विद्या वाचस्पति की

उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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Certificate

Certified that the work incorporated in the thesis entitled **"Supersymmetric Yang-Mills theories without anti-commuting variables"** submitted by **Saurabh Pant** was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

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Declaration

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I declare that this written submission represents my idea in my own words and where others' ideas have been included; I have adequately cited and referenced the original sources. I declare that I have acknowledged collaborative work and discussions wherever such work has been included. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

The work reported in this thesis is the original work done by me under the guidance of Prof. Sudarshan Ananth

Saurabh

Date: July 30, 2024 Saurabh Pant

I dedicate this thesis to my mother for all her love and constant support.

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Abstract

Supersymmetric field theories can be studied via an alternate approach using purely bosonic variables. In this method, a transformation (Nicolai map) of the bosonic fields exists for supersymmetric gauge theories such that the Jacobian of the map is same as the product of fermion and ghost determinants. This thesis investigates the development of supersymmetric Yang-Mills theories without anti-commuting variables, presenting them as entirely bosonic theories.

We derived the second order map (perturbatively in the coupling constant) in the Landau gauge for all pure supersymmetric Yang-Mills theories. This approach yields the well-known old relation that supersymmetric Yang-Mills theories can exist only in $D = 3, 4, 6, 10$ space-time dimensions. We investigated this formalism to the third order in the coupling constant using the rigorous $\mathcal{R}$ prescription. While working on the order g^3 map, we discovered a simpler map through trial and error, also to the third order that works only in space-time dimension six. The existence of two maps at order g^3 in six dimensions highlights the uniqueness of the map and the formalism.

In this approach, correlation functions and scattering amplitudes can be calculated using the inverse map. The light-cone gauge is useful for studying scattering amplitudes as the spinor helicity variables appear naturally in this gauge. We studied the Nicolai map approach in the light-cone gauge for supersymmetric Yang-Mills theory and computed the map perturbatively to order g^2. With the physical helicity fields, we obtained two maps at the second order in coupling and discussed the problems related to the uniqueness of these maps.

List of Publications

1. * Nipun Bhave and **Saurabh Pant**, *Nicolai maps and uniqueness in the light-cone gauge*, Journal of High Energy Physics (JHEP) 09 (2024) 121 , arXiv: 2406.18238 [hep-th].

2. Sudarshan Ananth, Nipun Bhave, Chetan Pandey and **Saurabh Pant**, *Deriving interaction vertices in higher derivative theories*, Phys. Lett. B **853** (2024), 138704, arXiv: 2306.05074 [hep-th].

3. Sudarshan Ananth, Chetan Pandey and **Saurabh Pant**, *Soft factors and interaction vertices from light-cone actions*, Fortschr. Phys. 2024, 2300182, arXiv:2212.13382 [hep-th].

4. * Sudarshan Ananth, Hannes Malcha, Chetan Pandey and **Saurabh Pant**, *Supersymmetric Yang-Mills theory in $D = 6$ without anti-commuting variables*, Physical Review D 103 (2021) 2, 025010, arXiv: 2006.02457 [hep-th] .

5. * Sudarshan Ananth, Olaf Lechtenfeld, Hannes Malcha, Hermann Nicolai, Chetan Pandey and **Saurabh Pant**, *Perturbative linearization of supersymmetric Yang-Mills theory*, Journal of High Energy Physics (JHEP) 10 (2020) 199, arXiv: 2005.12324 [hep-th] .

6. * Sudarshan Ananth, Chetan Pandey and **Saurabh Pant**, *Higher spins, quadratic forms, and amplitudes*, Journal of High Energy Physics (JHEP) 07 (2020) 07, 100, arXiv: 2005.10376 [hep-th] .

7. * Sudarshan Ananth, Hermann Nicolai, Chetan Pandey and **Saurabh Pant**, *Supersymmetric Yang-Mills theories: not quite the usual perspective*, Journal of Physics A: Mathematical and Theoretical 53 (2020) 17, arXiv: 2001.02768 [hep-th], (Invited Contribution).

This thesis is based on the publications that are marked with the *.

Contents

Chapter 1

Introduction

Quantum field theory is a highly successful and experimentally well-tested framework for describing elementary particles and three of the four fundamental forces of nature. It combines special relativity with quantum mechanics. The former explains the dynamics of relativistic particles, and the latter serves as a framework for investigating particles at the quantum scale. The study of quantum field theory and its physical and mathematical properties is particularly exciting in theoretical physics because it enhances our understanding of nature.

In quantum field theory, the primary objects of interest are scattering amplitudes. These are the central building blocks for constructing scattering cross-sections, which describe the probabilities of elementary particles interacting with each other at particle colliders. The study of scattering amplitudes improves the understanding of the mathematical structure of quantum field theories and offers the necessary tools for interpreting modern experiments. However, these amplitudes are often plagued by ultraviolet divergences at high energies. Enhancing the symmetry in the theory might improve the ultraviolet behavior.

Supersymmetry, a space-time symmetry that relates bosons and fermions, is of great interest in this context as it dramatically improves the ultraviolet properties. The vanishing of vacuum energy for supersymmetric theories is one such example. For bosonic and fermionic theories, the vacuum energy is divergent separately and contributes with opposite signs. However, in supersymmetric theories, the number of bosons and fermions are equal, and due to this, they cancel out these divergences to all orders in perturbation theory. The proof of the vanishing of the vacuum energy for interacting supersymmetric theories was first shown by Zumino in [1]. The vanishing of vacuum energy plays an important role in constructing supersymmetric theories (without anti-commuting variables), which is the central theme of this thesis.

Supersymmetry transformations map bosonic particles (fields) into fermionic particles (fields) and vice versa. These transformations are fermionic in nature and involve anti-commutation relations, forming a supersymmetry algebra. In a usual quantum field theory, the maximum allowed symmetry (under certain assumptions) is Poincaré and internal symmetries [2]. Supersymmetric field theories are invariant under the additional supersymmetric transformations, and they circumvent the no-go theorem (Coleman Mandula theorem) [2], allowing the extension of the maximum allowed symmetry for field theories [3, 4].

The first supersymmetric quantum field theory model, known as the Wess-Zumino model, was developed in four dimensions [5]. It included a complex scalar field and a fermionic field, with the theory demonstrating well-behaved ultraviolet properties. Following this, supersymmetric theories involving gauge fields were also constructed, showing similar finiteness. Supersymmetry has since been applied to particle physics and has successfully explained questions like naturalness and the hierarchy problem, unification of standard model interactions, and dark matter contents [6]. While many arguments support the existence of supersymmetry in nature, it has yet to be experimentally observed, possibly due to the spontaneous breaking of supersymmetry at high energy.

Nevertheless, supersymmetry played a huge role in the development of quantum field theory and mathematical physics. Supersymmetric theories have acted as a toy model, and advances in them have improved our understanding of quantum field theory. They have also provided a testing ground for computation tools and techniques that can be applied to quantum field theory. The $\mathcal{N} = 4$ super Yang-Mills theory, in the planar limit, is, from many perspectives, the best example [7, 8]. It has a number of simplifying features; its scattering amplitudes are ultraviolet finite [9, 10], exhibit novel symmetries like conformal invariance [11] and dual conformal invariance [12], surprising duality between scattering amplitudes and Wilson loops [13], and integrable properties [14]. The similarity of $\mathcal{N} = 4$ theory with QCD also helps us to study these theories at strong coupling. It also offers new techniques for the computation of Feynman integrals that are relevant to LHC physics [8].

Despite significant progress in our understanding of the $\mathcal{N} = 4$ super Yang-Mills theory and its scattering amplitude, certain relationships, such as dual conformal invariance and the surprising duality between the Wilson loop and amplitudes, have yet to be understood from the Lagrangian perspective. This theory also lacks well-defined asymptotic states, preventing the S-matrix from being defined in the usual sense. The $\mathcal{N} = 4$ super Yang-Mills theory has been shown to be ultraviolet finite to all orders only in the light-cone gauge, in terms of physical degrees of freedom [9, 10]. A non-perturbative proof of finiteness still does not exist. Given all these open questions, it appears worthwhile to understand all possible aspects of supersymmetric Yang-Mills theories and their properties.

In this thesis, we aim to develop a different framework of supersymmetric theories to answer some of the above questions eventually. Supersymmetric theories can be constructed without anti-commuting variables and thus expressed in terms of purely bosonic variables. This is the central theme of this thesis, and it was first proposed in [15–17] by Hermann Nicolai and further studied by Dietz, Flume, and Lechtenfeld [18–20]. The focus of this thesis is on supersymmetric gauge theories, and they can be devised in a different way provided there exists a non-linear and non-local transformation of the bosonic fields, called the Nicolai map $\mathcal{T}_g$ that linearizes the Yang-Mills action in such a way that the Jacobi determinant of the transformation $\mathcal{T}_g$ exactly equals the product of the Matthews–Salam–Seiler (MSS) [21, 22] and Faddeev–Popov (FP) [23] determinants. This interesting approach gives a fresh insight into the physics of supersymmetric gauge theories.

Some key questions that arise are:
(a) Is the mapped bosonic theory free, and where is the information about the interaction?
(b) Is the information about supersymmetry still intact, and how do we compute the objects of interest, like correlation functions or scattering amplitudes, in this mapped theory?

These questions were partly answered in the 1980s by the work of Dietz and Lechtenfeld [17], where they showed that the correlation functions of fully interacting supersymmetric theory can be computed using the inverse Nicolai map in terms of a free correlator. This allows us to quantize a supersymmetry theory without the use of fermions, as all the information of interaction is in the map. Despite all of this development during the 1980s on Nicolai map [20], there were no results for the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, which is considered one of the most interesting theory from many perspectives [8]. Additionally, for supersymmetric gauge theories, the map was known only up to order g^2 in four dimensions in the Landau gauge. To compute correlators at higher loops, one needs to know the map to higher orders in the coupling constant.

1.1 Results of work

To answer all of the above questions, we embark on a comprehensive exploration of supersymmetric theory of Yang-Mills without any fermions. For the first time, we construct an order g^2 map that works in space-time dimensions $D = 3, 4, 6, 10$ [24]. Since $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in ten dimensions is the parent theory for $\mathcal{N} = 4$ theory [25, 26], one can derive the map for this theory by dimensional reduction of the ten-dimensional map. Subsequently, the correlators for $\mathcal{N} = 4$ theory was computed utilizing our novel map by Nicolai and Plefka [27]. Building upon the old ideas of Dietz and Lechtenfeld [18, 19], we systematically construct the map up to order g^3 and give the direct proof for all statements of the main theorem using the $\mathcal{R}$ operator [28].

Furthermore, we find a simpler map up to the third order that works only in space-time dimension six [29]. This transformation is derived through guess work, and we address the issue of uniqueness of the map. We show that the two order g^3 maps can be connected at the Jacobian level. To better understand the connection between scattering amplitude and the map, we initiate the study of this formalism in the light-cone gauge. We calculate the Nicolai map for super Yang-Mills theory in the light-cone gauge to the second order in the coupling constant [30].

1.2 Thesis outline

We present below an outline of the work and the results presented in the thesis.

In chapter 2, we given an overview of supersymmetric field theories and their properties required for the subsequent chapters. We also discuss the formulation of field theories in the light-cone gauge.

In chapter 3, we first review the construction of the Nicolai map in supersymmetric quantum mechanics as an example. We then discuss the construction of the map up to order g^2 for supersymmetric Yang-Mills theories. We show that the determinant matching condition yields the old result [26] that super Yang-Mills theories can exist only in $D = 3, 4, 6, 10$ space-time dimensions (also known as critical dimensions).

In chapter 4, we present the systematic construction of this formalism. We discuss the general construction of $\mathcal{R}$ operator in the Landau gauge for all on-shell supersymmetric Yang-Mills theories in space-time dimensions $D = 3, 4, 6, 10$. We require using the Landau gauge to prove the distributive property of the $\mathcal{R}$ operator. We outline the relation between the $\mathcal{R}$ operator and the map. We then derive the new result, the map $\mathcal{T}_g$ up to order g^3 using the $\mathcal{R}$ operator. We perform checks for the derived map and find that the map exists in the critical dimensions.

In chapter 5, we present a novel map also to order g^3 that works only in six dimensions. We arrive at the map by trial and error and discover that the guessed map is simpler than one derived using $\mathcal{R}$ prescription (4.2). The existence of two maps at order g^3 in six dimensions raises questions about the uniqueness of this approach. We discuss the uniqueness of the maps and show that one can establish the connection between them at the level of the Jacobi determinant. We also outline an algorithmic approach to determine the map directly.

In chapter 6, we focus on the construction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in the light-cone gauge. We present the formulation of the Nicolai map in four dimensions, in terms of physical degrees of freedom, to second order. We also generalize the map to arbitrary D and establish the connection with the already found map in general gauges [31]. We find that in this gauge, there are two maps in four dimensions that satisfy all the conditions of the main theorem. Using the simple four-dimensional map, we compute the scattering amplitudes. We also discuss the probable connection between Nicolai map and quadratic form structures found in the light-cone Hamiltonians of Yang-Mills theory.

Chapter 2

Aspects of quantum field theory

Here, we briefly introduce the Poincaré symmetry and extend it to super Poincaré by adding a new spacetime symmetry called supersymmetry. We discuss the representations of supersymmetry algebra and review some basic aspects of supersymmetric Yang-Mills theories. In the end, we outline the formulation of quantum field theories in the light-cone gauge.

2.1 Symmetries in quantum field theory

Quantum field theory provides a theoretical framework to understand the dynamics of elementary particles. The set of transformations under which physical laws are invariant (covariant) are called symmetries. Mathematically, these symmetries can be associated with groups, and continuous symmetries have an underlying algebra.

2.1.1 Poincaré symmetry

The set of transformations that leave all relativistic quantum field theories invariant in four-dimensional Minkowski spacetime with a metric signature $(+1, -1, -1, -1)$ are of the form

$$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu}\,, \tag{2.1}$$

which preserves the form of metric and where Λ is known as the Lorentz transformation matrix and obeys $\Lambda^T \eta \Lambda = \eta$. The Lorentz transformations with $\det \Lambda = 1$ and $\Lambda^0_0 = 1$ form the Lorentz group $SO(1,3)$ and obeys the algebra

$$[M_{\mu\nu}\ ,\ M_{\rho\sigma}] = i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho} - i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\rho}M_{\mu\sigma}\,, \tag{2.2}$$

where $M_{\mu\nu}$ are six 4×4 anti-symmetric matrices that generate the different Lorentz transformations (three rotations and three boosts).

One can introduce new generators

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}\,, \qquad K_i \equiv M_{0i}\,, \tag{2.3}$$

where J_i are rotation generators and are hermitian in nature, the boost generators K_i are anti-hermitian, and this makes the Lorentz group non-compact. This non-compactness implies that the Lorentz group does not have a finite dimensional unitary representation.

To study the representations of the Lorentz algebra, we introduce the new linear combinations of the generators J_i and K_i

$$N_i = \frac{1}{2}(J_i + iK_i)\,, \qquad N_i^\dagger = \frac{1}{2}(J_i - iK_i)\,, \tag{2.4}$$

where N_i and $N_i^\dagger$ are hermitian and the algebra is

$$[N_i\,,\, N_j^\dagger] = 0\,, \qquad [N_i\,,\, N_j] = i\epsilon_{ijk}N_k\,, \qquad [N_i^\dagger\,,\, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger\,. \tag{2.5}$$

This means that these new generators obey the Lie algebra of $SU(2)$, and which proves the fact that the Lorentz algebra is isomorphic to two complexified $su(2)$ algebras, and therefore, the representations of these $su(2)$ algebras can be used to construct the finite-dimensional representations of the Lorentz algebra. For example $(\frac{1}{2}, 0)$ has spin half and denotes a left-handed spinor, and $(0, \frac{1}{2})$ describes a right-handed spinor. These spinor representations labeled in terms of $su(2)$ representations are realized by two-component complex spinors. For more details on spinors, refer to Appendix A.

The Lorentz group can be extended by including space-time translations generated by the generator P_μ. In terms of $P_\mu, M_{\mu\nu}$ the Poincaré algebra reads

$$\begin{aligned}
&[P_\mu\,,\; P_\nu] = 0\,,\\
&[M_{\mu\nu}\,,\; M_{\rho\sigma}] = i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho} - i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\rho}M_{\mu\sigma}\,,\\
&[P_\rho\,,\; M_{\mu\nu}] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu\,.
\end{aligned} \tag{2.6}$$

2.1.2 Supersymmetry

The relativistic quantum field theories with additional symmetry that relates bosons (denoting force carriers) and fermions (describing matter particles) are called supersymmetric field theories. The supersymmetry generator Q^I_α (where α is the spinor index and $I = 1, ...\mathcal{N}$ labels the different supercharges) act as

$$Q|\text{boson}\rangle = |\text{fermion}\rangle\,, \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle\,, \tag{2.7}$$

and they change the spin of a particle and, hence, its space-time properties. These generators, Q^I_α, are anti-commuting fermionic generators and transform under the spinor representation of the Lorentz group and satisfy the following algebra

$$\begin{aligned} \{Q^I_\alpha\,,\, Q^J_\beta\} \;&=\; \{\bar{Q}^I_{\dot\alpha}\,,\, \bar{Q}^J_{\dot\beta}\} \;=\; 0\,, \\ \{Q^I_\alpha\,,\, Q^J_{\dot\alpha}\} &= 2\,\delta^{IJ}\sigma^\mu_{\alpha\dot\alpha}P_\mu\,, \end{aligned} \tag{2.8}$$

where $\bar{Q}$ is related to Q by complex conjugation. The commutators involving supersymmetry and Poincaré generators are

$$\begin{aligned} &[P_\mu\,,\, Q^I_\alpha] = [P_\mu\,,\, \bar{Q}^I_{\dot\alpha}] = 0\,, \\ &[M_{\mu\nu}\,,\, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha^{\ \beta}\, Q^I_\beta\,, \\ &[M_{\mu\nu}\,,\, Q^{\dot\alpha\, I}] = i(\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\, \bar{Q}^{\dot\beta\, I}\,. \end{aligned} \tag{2.9}$$

From the above commutation relations, one can find that the particles that belong to the same supermultiplet have the same mass but different spins, and the fermions and bosons in a supermultiplet are equal in number. Also, for supersymmetric theories, the vacuum state is annihilated by all the supersymmetric generators, and the energy P_0 is nonnegative.

Supersymmetric field theories are formulated by constructing supersymmetric representations in terms of fields. Consider the action of infinitesimal supersymmetry transformation on a multiplet of fields $(\phi, \psi, ...)$

$$\delta_\epsilon\phi = (\epsilon Q + \bar\epsilon\bar{Q})\phi\,, \qquad \delta_\epsilon\psi = (\epsilon Q + \bar\epsilon\bar{Q})\psi\,, \tag{2.10}$$

where ϵ^α is an anti-commuting parameter such that $\{[\epsilon^\alpha\,,.]\} = 0$. To check the algebraic structure of the infinitesimal supersymmetry transformation, consider the commutation relation between two successive infinitesimal transformations of the scalar field

$$[\delta_1, \delta_2]\phi = -2i(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1)\partial_\mu\phi\,. \tag{2.11}$$

From the above relation, one can obtain the action of supersymmetry transformation on the other fields and can check that supersymmetry algebra closes on the multiplet. Theories, where one needs the equation of motion to close the algebra, are called on-shell supersymmetry field theories, and if the algebra follows without using the equation of motion, such theories are called off-shell supersymmetric.

Under the action of supersymmetry transformations, the constructed Lagrangian in supersymmetric theories must (at most) transform as a total derivative.

2.2 Supersymmetric Yang-Mills theory

Let us examine a theory with a non-abelian gauge field $A^a_\mu(x)$ and a Majorana fermion $\lambda^a(x)$. As supersymmetry commutes with the gauge symmetry, both fermions and bosons transform under the adjoint representation of the gauge group $SU(N)$ or $U(N)$ labeled by the index a. The gauge transformation of the super multiplet fields are

$$\begin{aligned} A^a_\mu &\to A^a_\mu + \partial_\mu\Lambda^a + gf^{abc}A^b_\mu\Lambda^c\,, \\ \lambda^a &\to \lambda^a + gf^{abc}\lambda^b\Lambda^c\,, \end{aligned} \tag{2.12}$$

where Λ^a denotes an infinitesimal gauge transformation, g is the coupling, and f^{abc} corresponds to the antisymmetric structure constants that satisfy

$$[T^a_r\,, T^b_r] = if^{abc}T^c_r\,, \tag{2.13}$$

where T^a belongs to the generators of the gauge group for any representation r. In addition to this, the structure constants obey the Jacobi identity

$$f^{abc}f^{ade} + f^{acd}f^{abe} + f^{adb}f^{ace} = 0\,. \tag{2.14}$$

For the adjoint representation $(T^b_G)_{ac} = if^{abc}$, and its dimensions for $SU(N)$ gauge group is $N^2 - 1$ and the structure constants obey

$$f^{abc}f^{abd} = N\delta^{cd}\,. \tag{2.15}$$

For the description of spinors, we need to introduce a set of γ matrices that are defined through the Clifford algebra relation

$$\{\gamma_\mu\,,\,\gamma_\nu\} = 2\,\eta_{\mu\nu}\,\mathbb{I}\,, \tag{2.16}$$

where $\mathbb{I}$ denotes the identity matrix in spinor space, and we have suppressed the spinor indices $\alpha, \beta = 1, 2, ...r$ (r denotes the number of independent fermion components). The trace of (2.16) gives

$$\mathrm{tr}\,\mathbb{I} = \mathrm{r}\,, \tag{2.17}$$

where trace over spinor indices are denoted by tr. The trace of an odd number of gamma matrix vanishes, and the traces over the product of gamma matrices are given by

$$\mathrm{tr}(\gamma^{\mu_1}, ..., \gamma^{\mu_\mathrm{n}}) = \sum_{\mathrm{i}=2}^{\mathrm{n}} (-1)^{\mathrm{i}}\,\eta^{\mu_1\mu_\mathrm{i}}\,\mathrm{tr}(\gamma^{\mu_2}....\hat{\gamma}^{\mu_\mathrm{i}}....\gamma^{\mu_\mathrm{n}})\,, \tag{2.18}$$

where the hat on the gamma matrix indicates the excluded matrix from the trace.

Gamma matrices can be constructed by tensoring the product of sigma matrices. For even dimensions, the representation of gamma matrix has $2^{\frac{D}{2}}$ complex degrees of freedom, whereas for odd D, the representation is $2^{\frac{D-1}{2}}$ dimensional. In any even dimension, the degree of freedom of the spinor can be reduced by a factor of two by using the Weyl condition

$$\lambda = \frac{1}{2}(1 \pm \gamma^{D+1})\lambda \,. \tag{2.19}$$

Another condition that one can apply on spinors is the Majorana condition, which is

$$\bar{\lambda} = (\lambda^T C)\,, \tag{2.20}$$

where C describes the charge conjugation matrix. This condition halves the degree of freedom of the spinor and works only in specific dimensions (for more details, see Appendix B). For spinors in specific dimensions, one can apply both the Majorana and Weyl condition that decreases the degrees of freedom by a factor of four.

The $\mathcal{N} = 1$ supersymmetric Yang-Mills action with gauge invariant part is

$$S_{\text{inv}} = \int d^D x \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \frac{i}{2} \bar{\lambda}^a \gamma^\mu (D_\mu \lambda)^a \right] \,, \tag{2.21}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu\, A^c_\nu \,, \qquad D_\mu \lambda^a = \partial_\mu \lambda^a + g\, f^{abc} A^b_\mu\, \lambda^c \,.$$

Supersymmetry requires that the number of bosonic and fermionic degrees of freedom are equal. A gauge field in D dimension has $D-2$ on-shell real degrees of freedom, and the fermion has $2^{[\frac{D}{2}]}$ on-shell, real degrees of freedom. Naively, for any dimension these two numbers are not equal; we must impose additional conditions on spinors for matching. One can impose different conditions like Majorana or Weyl on the fermion to decrease its number of independent components. Below, we discuss the different scenarios when the bosonic and fermionic degrees of freedom can be matched

	A_μ	λ_M	λ_W	λ_{MW}
D=3	1	1	-	-
D= 4	2	2	2	-
D=6	4	-	4	-
D=10	8	16	16	8

where λ_M is Majorana spinor, λ_W is Weyl spinor, and λ_{MW} is Majorana-Weyl spinor. For $D > 10$, the degrees of freedom cannot be matched, so for supersymmetric Yang-Mills theories, the only options are

$$D = 3, 4, 6, 10 \qquad \Longleftrightarrow \qquad r = 2, 4, 8, 16\,, \tag{2.22}$$

and this gives the relation

$$r = 2(D-2)\,, \tag{2.23}$$

which we will later rederive within the framework of the Nicolai map.

Under the following supersymmetry transformations

$$\delta\, A^a_\mu = i\bar{\epsilon}\,\gamma_\mu \lambda^a, \qquad \delta\,\lambda^a = -\frac{1}{2}\epsilon\,\gamma^{\mu\nu} F^a_{\mu\nu}\,,$$

the on-shell action (2.21) is invariant up to a total derivative. One needs to use the Fierz identity to show this invariance, and this condition only holds in $D = 3, 4, 6, 10$ dimensions. It was first proved in [26] that supersymmetric Yang-Mills theories without any matter only exist in these dimensions.

Due to the manifest Lorentz invariance, the gauge theories have an additional redundancy that leads to the propagation of unphysical degrees of freedom. These redundancy in the path integral can be fixed using the gauge fixing procedure [23] which gives additional contribution to the Lagrangian

$$S_{\text{gf}} = \int d^D x\, \left(\frac{1}{2\xi}\,(G^a[A_\mu])^2 + \bar{C}^a \frac{\partial G^a[A_\mu]}{\partial A^b_\mu} (D_\mu C)^a \right)\,, \tag{2.24}$$

which ensures that the gauge field propagator and path integral are properly defined.

The objects of interest, correlation functions, for this theory are computed by taking the full action $S = S_{\text{inv}} + S_{\text{gf}}$. For the operators $\mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)$ of super Yang-Mills theory, the correlation function is defined as

$$\langle\langle \mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\rangle\rangle = \int DAD\lambda\, DC\, D\bar{C}\; e^{i\, S_{\rm inv}\,[g,A,\lambda]+i\, S_{\rm gf}[g,A,C,\bar{C}]}\;\; \mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\,. \tag{2.25}$$

Note that the correlation functions are properly normalized for all g. This is due to fact that the vacuum energy vanishes in supersymmetric theories [1] as the normalization term denotes the sum of vacuum energy.

In this thesis, we will mostly work with the Landau gauge

$$G^a[A_\mu] = \partial^\mu A^a_\mu\,. \tag{2.26}$$

It can be obtained by setting $\xi \rightarrow 0$ in the part of the action that contains gauge fixing term (2.24). The limit corresponds to inserting the delta function that imposes the gauge condition. The ghost propagator

$$G^{ab}(x) \equiv C^a(x)\bar{C}^b(0)\,, \tag{2.27}$$

obeys

$$-\partial^\mu (D_\mu G)^{ab}(x) = \delta^{ab}\delta(x)\,. \tag{2.28}$$

One can study supersymmetric theories in another popular gauge known as the axial gauge; it is defined as

$$G^a(A) = n^\mu A^a_\mu \tag{2.29}$$

where $n^\mu n_\mu = 1$. For the case when $n^2 = 0$ it is known as the light-cone gauge or light-front. In the axial-type gauges, there are no ghost fields, as they can be dropped out from the path integral. This specific light cone gauge is called LC_4, and the field theories using this approach were studied by Leibbrandt and Lee [32, 33]. In this thesis, specifically in the Chapter 6, we adopt the LC_2 light-cone gauge approach, where only the physical degrees of freedom of the theory propagates. We outline below this approach in some detail.

2.3 Field theories in the light-cone gauge

The laws of nature are independent of the choice of frames and coordinates. Similarly, in quantum field theory, the physical observables do not depend on the gauge choice. In certain coordinate systems, some properties become more evident than in others. The light-cone gauge is particularly interesting in this regard because it offers a unique perspective on scattering amplitudes, and hidden symmetries become more apparent in this formalism.

The light-cone coordinates were first proposed by Dirac for relativistic theory in [34], demonstrating that within the light-cone framework, any of the null coordinates can be chosen as time. In this formulation, seven out of ten generators of Poincaré symmetry are independent of time derivatives, and the Hamiltonian eigenvalue equation does not contain square roots.

In four dimensions, with Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, the light cone coordinates are defined as

$$\begin{aligned} x^+ &= \frac{1}{\sqrt{2}}(x^0 + x^3)\,, \qquad & x^- &= \frac{1}{\sqrt{2}}(x^0 - x^3)\,, \\ x &= \frac{1}{\sqrt{2}}(x^1 + i\,x^2)\,, \qquad & \bar{x} &= \frac{1}{\sqrt{2}}(x^1 - i\,x^2)\,, \end{aligned} \tag{2.30}$$

and the corresponding derivatives are

$$\partial^{\pm} = \frac{1}{\sqrt{2}}(\partial^0 \pm \partial^3)\,, \qquad \partial = \frac{1}{\sqrt{2}}(\partial^1 + i\,\partial^2)\,, \qquad \bar{\partial} = \partial^*\,, \tag{2.31}$$

where $\partial^{\pm} = \frac{\partial}{\partial x_{\pm}}$ and $\partial^- x^+ = \partial^+ x^- = -1$. The coordinate x^+ is chosen to be time and the equivalent momentum $p_+ = -p^-$ is the light-cone Hamiltonian.

Light-cone field theories can be developed in two ways. In the first approach, one can start with the covariant Lagrangian, and by using the appropriate gauge condition and constraint equations, the relevant Lagrangian can be expressed in terms of the helicity fields. This method is outlined in detail for $\mathcal{N} = 1$ supersymmetric Yang-Mills in Chapter 6.

In the second approach, field theories can be constructed from the symmetry principles. For light-cone field theories, Poincaré invariance is not manifest, and must be explicitly checked under the Poincaré group. The non-manifest nature of Poincaré invariance can be used to construct interaction vertices in light-cone field theories. The idea is that Hamiltonian is also an element of the Poincaré algebra and can be fixed using the closure of Poincaré algebra. This method was first used in [35] to derive interaction vertices for arbitrary spins up to order g. The algebra-closure approach allows for the consistent derivation of interaction vertices perturbatively. Recently, this framework [35–37] was expanded to include higher derivative interaction vertices for arbitrary spin theories [38]. There has been considerable work on constructing consistent higher-spin interaction vertices in the light front approach using various methods [39–42]. While these ideas are exciting, they are not covered in this thesis.

Chapter 3

Nicolai map at order g^2 for supersymmetric Yang-Mills theory

The material presented in this chapter is mostly based on author's publication [24].

This chapter focuses on the construction of the Nicolai map up to second order in the coupling constant for pure supersymmetric Yang-Mills theories in arbitrary dimensions. The transformation is derived through trial and error, starting with an educated guess that maps the free bosonic theory to a fully interacting Yang-Mills theory. The determinant matching condition yields an old result that pure supersymmetric Yang-Mills theories can exist only in space-time dimensions $D = 3, 4, 6, 10$.

3.1 Notations and conventions

Supersymmetric theory with at most quadratic fermionic terms in the Lagrangian, can be defined by a non-linear and non-local transformation of the bosonic fields ("Nicolai map") that linearizes the bosonic action in such a way that the Jacobian of the bosonic field transformation equals the fermion determinant obtained upon integrating out all anticommuting fields.

We begin by presenting the construction of the Nicolai map using supersymmetric quantum mechanics as an example, based on the seminal work by Ezawa and Klauder [43]. Consider a system with two degrees of freedom $q(t)$ (bosonic) and $\psi(t)$ (fermionic). The Euclidean action is given by

$$S = \int dt \left[\frac{1}{2}\dot{q}^2(t) + \dot{q}(t)V\left[q(t)\right] + \frac{1}{2}V^2\left[q(t)\right] + \bar{\psi}(t)\left(\frac{d}{dt} + V'\left[q(t)\right]\right)\psi(t)\right] , \tag{3.1}$$

where the form of superpotential $V(q)$ is not required and $V' = \frac{dV}{dq}$.

The above Lagrangian is invariant (up to a total derivative) under infinitesimal supersymmetric transformations

$$\delta q(t) = \bar{\epsilon}\,\psi(t) + \bar{\psi}(t)\,\epsilon, \qquad \delta\,\psi(t) = \epsilon\,(\dot{q}(t) - V[q(t)])\,, \qquad \delta\,\bar{\psi}(t) = \bar{\epsilon}\,(-\dot{q}(t) - V[q(t)])\ , \tag{3.2}$$

where $\epsilon, \bar{\epsilon}, \psi$, and $\bar{\psi}$ are Grassmann variables; all of them anti-commute with each other.

The ansatz for the transformation $\mathcal{T}_g\, q(t)$ is

$$\mathcal{T}_g\, q(t) \equiv q'(t) = q(t) + \int dt'\,\theta(t-t')\,V[q(t')]\,, \tag{3.3}$$

where g is the coupling of $V(q)$ and $\theta(t-t')$ is the step function.

We can show that $\mathcal{T}_g\, q(t)$ maps the free action to the full bosonic action

$$\int dt \left(\frac{1}{2}\frac{d\,q'(t)}{dt}\right)^2 = \int dt \left(\frac{1}{2}\dot{q}^2(t) + \dot{q}(t)V\left[q(t)\right] + \frac{1}{2}V^2\left[q(t)\right]\right)\ . \tag{3.4}$$

The Jacobi determinant of the transformation $\mathcal{T}_g\, q(t)$ is

$$\det\left(\frac{\delta q'(t)}{\delta q(t')}\right) = \det\left(\delta(t-t') + \theta(t-t')V'\left[q(t')\right]\right)\ . \tag{3.5}$$

As the action (3.1) is quadratic in fermions, one can integrate out the fermions using the standard Gaussian integration method (see Appendix C), and we get

$$\begin{aligned}
\Delta_{FD}(q) &= \det\left[\left(\frac{d}{dt} + V'\left[q(t)\right]\right)\,\delta(t-t')\right]\ , \\
&= \det\left[\frac{d}{dt}\,\delta(t-t')\right]\,\det\left(\delta(t-t') + \theta(t-t')V'\left[q(t')\right]\right)\ .
\end{aligned} \tag{3.6}$$

From the above equation, we can see that the Jacobian determinant (3.5) equals the fermion determinant (3.6), up to a constant. Therefore, supersymmetric theories can be formulated without using anticommuting variables. This approach to supersymmetric quantum mechanics is simple and special because the map (3.3) can be easily inferred from the action (3.1) and has a closed form. However, in general, constructing such a map requires tedious derivation and often results in an infinite series, making the verification of the transformation more complex.

Our focus here is on supersymmetric Yang-Mills theories. In this chapter, we will outline the construction of Nicolai maps for supersymmetric Yang-Mills theories. We will explain this result (which for $\mathcal{N} = 1$ super-Yang-Mills theory in four dimensions was obtained and proved long ago in [16]) in simple terms by explicitly rederiving the map up to $\mathcal{O}(g^2)$, and extending it to all pure supersymmetric Yang-Mills theories. As a new result, using this approach, we will recover the well-known relation of [26] that interacting pure supersymmetric Yang-Mills theories can exist only in space-time dimensions $D = 3, 4, 6, 10$.

We work here in Euclidean space, rendering upper and lower indices equivalent. However, the Euclidean metric is not crucial to our discussion, as all of these results can be derived in spacetime with a Lorentzian signature as in [18, 19]. Consider the pure $\mathcal{N} = 1$ super Yang-Mills in D dimensions that contains one spin field $A^a_\mu(x)$ and a spinor field $\lambda^a(x)$ (spinor index suppressed). The invariant part of D dimensional $\mathcal{N} = 1$ super Yang-Mills is

$$S_{\text{inv}} = \int d^D x \, \left(\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \frac{1}{2} \bar{\lambda}^a \gamma^\mu D_\mu \lambda^a \right) .$$

The free scalar propagator is (with the Laplacian $\Box \equiv \partial^\mu \partial_\mu$)

$$C(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ikx}}{k^2} \quad \Rightarrow \quad \Box C(x) = -\delta(x) \,. \tag{3.7}$$

where $\delta(x) \equiv \delta^{(D)}(x)$ is the D-dimensional δ-function. For arbitrary D we have

$$C(x) = \frac{1}{(D-2) D \pi^{D/2}} \Gamma \left(\frac{D}{2} + 1 \right) (x^2)^{1-\frac{D}{2}} \;; \tag{3.8}$$

in particular, for $D = 4$

$$C(x) = \frac{1}{4\pi^2 \, x^2} \,. \tag{3.9}$$

When writing $\partial_\lambda C(x-y) \equiv (\partial/\partial x^\lambda) C(x-y) \equiv \partial^x_\lambda C(x-y)$, the derivative by convention *always acts on the first argument*. Careful track needs to be kept of the sign flips $\partial^x_\lambda C(x-y) = -\partial^y_\lambda C(x-y) = +\partial^x_\lambda C(y-x) = -\partial^y_\lambda C(y-x)$, The free fermionic propagator is

$$\gamma^\mu \partial_\mu S_0(x) = \delta(x) \quad \Rightarrow \quad S_0(x) = -\gamma^\mu \partial_\mu C(x)\,, \tag{3.10}$$

where the spinor indices are suppressed. This implies $S_0(x-y) = -S_0(y-x)$. As discussed in chapter 2, that the independent fermionic degrees of freedom (spinor components) will be designated by r, and it depends on D including extra factors of $\frac{1}{2}$ for Majorana or Weyl spinors, and $\frac{1}{4}$ for Majorana-Weyl spinors, respectively. For pure supersymmetric Yang-Mills theories, the degrees of freedom matches for these case

$$D \;=\; 3,4,6,10 \qquad \Longleftrightarrow \qquad r \;=\; 2,4,8,16\,. \tag{3.11}$$

With Minkowskian signature, for $D = 4$ space-time, this corresponds to a Majorana spinor, for $D = 6$ to a Weyl spinor, while for $D = 10$ we have one more factor of $\frac{1}{2}$ because of the Majorana-Weyl condition. For free theories, this equality follows trivially (see Appendix C)

$$\begin{aligned}
&\int DA\; e^{\frac{1}{2}A\Box A} \;\sim\; [\det(-\Box)]^{-\frac{D}{2}}\,,\\
&\int DC\, D\bar{C}\; e^{\frac{1}{2}\bar{C}\Box C} \;\sim\; [\det(-\Box)]\,,\\
&\int D\chi\; e^{\frac{1}{2}\bar{\chi}\not{\partial}\chi} \;\sim\; [\det(-\Box)]^{\frac{r}{4}}\,,
\end{aligned} \tag{3.12}$$

by demanding the cancellation of the free determinants. It is just a consequence of the equality of bosonic and fermionic degrees of freedom on-shell for free theories.

We shall rederive this constraint in section 3 *without* any use of anti-commuting objects.

3.2 Main theorem

Supersymmetric gauge theories can be formulated purely in terms of bosonic variables if the gauge fields admit a non-linear and non-local transformation $\mathcal{T}_g$ of the Yang-Mills fields

$$\mathcal{T}_g \;:\; A^a_\mu(x) \;\mapsto\; A'^a_\mu(x, g; A)\,,$$

that can be inverted perturbatively such that

1. The transformation $\mathcal{T}_g(A)$ when substituted in the free bosonic Lagrangian (Maxwell theory) yields the full interacting bosonic Lagrangian (Yang-Mills theory).

$$S_0[A'(A)] \; = \; S_g[A] \equiv \frac{1}{4}\int d^Dx \; F^a_{\mu\nu} F^{a\,\mu\nu} \, .$$

2. The map $\mathcal{T}_g$ preserves the gauge condition

$$\mathcal{T}_g[G^a(A)] = G^a(A) \, .$$

3. The Jacobi determinant of $\mathcal{T}_g$ is equal to the product of the Matthews-Salam-Seiler (MSS) determinant (or Paffian) [21, 22] obtained by integrating out the fermions and the Faddeev-Popov (FP) determinant [23] (obtained by integrating out the ghost fields)

$$\det\left(\frac{\delta A'^{\,a}_\mu(x,g;A)}{\delta A^b_\nu(y)}\right) \; = \; \Delta_{MSS}[A]\,\Delta_{FP}[A] \, ,$$

at least order by order in perturbation theory.

3.3 The map at order g^2

The ansatz for the transformation $\mathcal{T}_g\, A$ is

$$\begin{aligned} \mathcal{T}_g\, A \equiv A'^{\,a}_\mu(x) \; &= \; A^a_\mu(x) \, + \, g\, f^{abc}\int du\; \partial_\lambda C(x-u) A^b_\mu(u) A^c_\lambda(u) \\ &+ \frac{1}{2}\, g^2 f^{abc} f^{bde}\int du\, dv\; \partial_\rho C(x-u)\; A^c_\lambda(u) \Big\{ \partial_\mu C(u-v)\; A^d_\lambda(v) A^e_\rho(v) \\ &\quad \partial_\lambda C(u-v)\; A^d_\rho(v) A^e_\mu(v) \, + \partial_\rho C(u-v)\; A^d_\mu(v) A^e_\lambda(v) \Big\} \, + \, \mathcal{O}(g^3) \, , \end{aligned} \tag{3.13}$$

where the dimensional dependence in the measure is suppressed for convenience. The inverse map can be found by inverting the transformation (3.13) perturbatively.

3.4 Consistency tests of the map

We now perform the three checks of the main theorem (3.2) for our ansatz $\mathcal{T}_g(A)$ up to $\mathcal{O}(g^2)$.

3.4.1 Gauge condition

In this chapter, we are working with the gauge $G^a[A_\mu] = \partial_\mu A^a_\mu$. We need to show that

$$\partial_\mu A'^a_\mu = \partial_\mu A^a_\mu + \mathcal{O}(g^3)\,. \tag{3.14}$$

Applying ∂_μ to the terms of the map (3.13), we obtain

$$\begin{aligned}
\partial_\mu A'^a_\mu \;\; = \;\; & \partial_\mu A^a_\mu + g\, f^{abc} \int du\, \partial_\mu\, \partial_\lambda C(x-u) A^b_\mu(u) A^c_\lambda(u) \\
& + \frac{1}{2}\, g^2 f^{abc} f^{bde} \int du\, dv\; \partial_\mu\, \partial_\rho C(x-u)\, A^c_\lambda(u) \Big\{ \partial_\mu C(u-v)\, A^d_\lambda(v) A^e_\rho(v) \\
& \quad \partial_\lambda C(u-v)\, A^d_\rho(v) A^e_\mu(v) \; + \partial_\rho C(u-v)\, A^d_\mu(v) A^e_\lambda(v) \Big\} \; + \; \mathcal{O}(g^3)\,.
\end{aligned}$$

The derivatives in order g term are symmetric under the exchange of μ and λ, but the gauge fields are anti-symmetric due to the color index. Hence, the order g contribution vanishes. Similarly, the $\mathcal{O}(g^2)$ also vanishes due to the manifest anti-symmetry under the exchange of two space-time indices.

3.4.2 Free action condition

Coming to the first statement of the main theorem, we show that the transformation satisfy

$$\frac{1}{2} \int dx\; A'^a_\mu(x)\, (-\Box\, \delta_{\mu\nu} + \partial_\mu \partial_\nu)\, A'^a_\nu(x) = \frac{1}{4} \int dx\; F^a_{\mu\nu} F^a_{\mu\nu}\,. \tag{3.15}$$

The second term on both the left and right side in the above equation can be ignored due to the gauge condition. We plug the map (3.13) in the left-hand side of (3.15) and simplify order by order. At the leading order, we obtain the kinetic term. For order g, we get

$$\int dx\; A'^a_\mu(x)|_{\mathcal{O}(g^0)}\; (-\Box\, \delta_{\mu\nu})\, A'^a_\nu(x)|_{\mathcal{O}(g)}\,,$$

$$= -g\, f^{abc} \int dx\, du\, A^a_\mu(x)\, \Box_x\, \partial_\lambda C(x-u) A^b_\mu(u) A^c_\lambda(u)\,,$$

$$= -g\, f^{abc} \int dx\, \partial_\lambda A^a_\mu(x) A^b_\mu(x) A^c_\lambda(x) = \frac{1}{4} \int dx\, F^a_{\mu\nu} F^a_{\mu\nu}|_{\mathcal{O}(g)}\,. \tag{3.16}$$

In the second line, we partially integrated ∂_λ on the left gauge field and used the relation $\Box_x C(x-u) = -\delta(x-u)$. At order g^2, we have two contribution

$$-\frac{1}{2} \int dx\, A'^a_\mu(x)|_{\mathcal{O}(g)}\, \Box\, A'^a_\mu(x)|_{\mathcal{O}(g)} - \int dx\, A'^a_\mu(x)|_{\mathcal{O}(g^0)}\, \Box\, A'^a_\mu(x)|_{\mathcal{O}(g^2)}\,. \tag{3.17}$$

We first solve for the left part in the above equation

$$= -\frac{g^2}{2} f^{abc} f^{ade} \int dx\, du\, dv\, \partial_\lambda C(x-u) A^b_\mu(u) A^c_\lambda(u) \Box_x\, \partial_\rho C(x-v) A^d_\mu(v) A^e_\rho(v)\,. \tag{3.18}$$

After partial integration and using the relation $\Box_x C(x-u) = -\delta(x-u)$, we obtain

$$= -\frac{g^2}{2} f^{abc} f^{ade} \int dx\, du\, A^b_\mu(x) A^c_\lambda(x)\, \partial_\lambda \partial_\rho C(x-u) A^d_\mu(u) A^e_\rho(u)\,. \tag{3.19}$$

We now simplify the second part of (3.17)

$$\begin{aligned}
&= -\frac{1}{2} g^2 f^{abc} f^{bde} \int dx\, du\, dv\, A^a_\mu(x)\, \Box_x\, \partial_\rho C(x-u)\, A^c_\lambda(u) \Big\{ \partial_\mu C(u-v)\, A^d_\lambda(v) A^e_\rho(v) \\
&\qquad \partial_\lambda C(u-v)\, A^d_\rho(v) A^e_\mu(v)\, + \partial_\rho C(u-v)\, A^d_\mu(v) A^e_\lambda(v) \Big\}\,, \\
&= -\frac{1}{2} g^2 f^{abc} f^{bde} \int dx\, du\, \partial_\rho A^a_\mu(x)\, A^c_\lambda(x) \Big\{ \partial_\mu C(x-u)\, A^d_\lambda(v) A^e_\rho(v) \\
&\qquad \partial_\lambda C(x-u)\, A^d_\rho(v) A^e_\mu(v)\, + \partial_\rho C(x-u)\, A^d_\mu(v) A^e_\lambda(v) \Big\}\,.
\end{aligned}$$

Using the fact that the equation is symmetric under exchange of $a \leftrightarrow c$ and $\mu \leftrightarrow \lambda$. We can rewrite the above equation as

$$\begin{aligned}
&= -\frac{1}{4} g^2 f^{abc} f^{bde} \int dx\, du\, \partial_\rho (A^a_\mu(x)\, A^c_\lambda(x)) \Big\{ \partial_\mu C(x-u)\, A^d_\lambda(v) A^e_\rho(v) \\
&\qquad \partial_\lambda C(x-u)\, A^d_\rho(v) A^e_\mu(v)\, + \partial_\rho C(x-u)\, A^d_\mu(v) A^e_\lambda(v) \Big\}\,, \\
&= \frac{1}{4} g^2 f^{abc} f^{ade} \int dx\;\; A^b_\mu(x)\, A^c_\lambda(x)\;\; A^d_\mu(x) A^e_\lambda(x) \\
&\quad + \frac{1}{2} g^2 f^{abc} f^{ade} \int dx\, du\, A^b_\mu(x)\, A^c_\lambda(x)\, \partial_\rho \partial_\lambda C(x-u)\, A^d_\mu(u) A^e_\rho(u)
\end{aligned}$$

The blue color term from the above equation cancels against the (3.19). We, therefore, proved the first condition of the main theorem that the transformation (3.13) maps the free bosonic theory to a fully interacting Yang-Mills theory. We also found out that the fulfillment of action and the gauge condition *does not* use the special value of D, and therefore work in all dimensions.

The most non-trivial requirement for the map is the matching of determinants. We now verify the third statement of the main theorem for our ansatz (3.13).

3.4.3 Matching of determinants

For the (perturbative) computation of the relevant functional determinants (or rather their logarithms), we use the standard formula

$$\log\det\big(1-X\big) = \mathrm{Tr}\,\log\big(1-X\big) = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\,X^n\,. \tag{3.20}$$

The Jacobi determinant corresponding to the transformation (3.13) is

$$\begin{aligned}
\det\left(\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\right) \;=\;& \delta_{\mu\nu}\delta^{ab}\delta(x-y) + g\,f^{abc}\left\{\partial_\lambda C(x-y)A^c_\lambda(y)\,\delta_{\mu\nu} - \partial_\nu C(x-y)A^c_\mu(y)\right\} \\
&+\frac{1}{2}g^2 f^{adb}f^{dgh}\int dv\;\partial_\rho C(x-y)\Big\{\partial_\mu C(y-v)\,A^g_\nu(v)A^h_\rho(v) \\
&\quad\partial_\nu C(y-v)\,A^g_\rho(v)A^h_\mu(v) + \partial_\rho C(y-v)\,A^g_\mu(v)A^h_\nu(v)\Big\} + \\
&\quad+\frac{1}{2}g^2 f^{adc}f^{dgh}\int du\,dv\;\partial_\rho C(x-u)\,A^c_\lambda(u)\,\delta(y-v) \\
&\qquad\Big\{\partial_\mu C(u-v)\,\big(\delta_{\lambda\nu}\delta^{gb}A^h_\rho(v) + \delta_{\rho\nu}\delta^{hb}A^g_\lambda(v)\big) \\
&\qquad+\partial_\lambda C(u-v)\,\big(\delta_{\rho\nu}\delta^{gb}A^h_\mu(v) + \delta_{\mu\nu}\delta^{hb}A^g_\rho(v)\big) \\
&\qquad+\partial_\rho C(u-v)\,\big(\delta_{\mu\nu}\delta^{gb}A^h_\lambda(v) + \delta_{\lambda\nu}\delta^{hb}A^g_\mu(v)\big)\Big\} + \mathcal{O}(g^3)\,.
\end{aligned}$$

Using the identity (3.20), and the fact that $\partial_\rho C(0) = 0$ one can show that the contribution to the Jacobian determinant at order g vanishes. At order g^2, we have two contributions

$$\log\det\left(\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\right)\Bigg|_{\mathcal{O}(g^2)} = \mathrm{Tr}\left[\frac{\delta \mathrm{A}'}{\delta \mathrm{A}}\bigg|_{\mathcal{O}(\mathrm{g}^2)}\right] - \left(2\cdot\frac{1}{2}\right)\mathrm{Tr}\left[\frac{\delta \mathrm{A}'}{\delta \mathrm{A}}\bigg|_{\mathcal{O}(\mathrm{g})}\frac{\delta \mathrm{A}'}{\delta \mathrm{A}}\bigg|_{\mathcal{O}(\mathrm{g})}\right]\,. \tag{3.21}$$

The trace (Tr) is computed by putting $\mu = \nu, b = a, y = x$ and integrating over x. Solving the above equation and using the $SU(n)$ identity $f^{abc} f^{abd} = n\delta^{cd}$, we obtain

$$\log\det\left(\frac{\delta A'^{a}_{\mu}(x)}{\delta A^{b}_{\nu}(y)}\right) = ng^2 \int dx\, dy \left\{\frac{2D-3}{2}\,\partial_\mu C(x-y)\, A^a_\mu(y)\, \partial_\nu C(y-x)\, A^a_\nu(x) - \frac{D-2}{2}\,\partial_\mu C(x-y) A^a_\nu(y) \partial_\mu C(y-x) A^a_\nu(x)\right\}. \tag{3.22}$$

The path integral for pure $\mathcal{N}=1$ super Yang-Mills reads

$$Z = \int DAD\lambda DCD\bar{C}\; e^{-S_{\text{inv}}(A,\lambda) - S_{\text{gf}}(A,C,\bar{C})}. \tag{3.23}$$

We integrate out the fermionic variables using the standard Grassmann integral (see Appendix C for details) and obtain the Matthews-Salam determinant

$$\det\left[\gamma^\mu\left(\delta^{ab}\partial_\mu - g f^{abm} A^m_\mu\right)\right]^{\frac{1}{2}} = \det\left(\gamma^\mu\partial_\mu\right)^{\frac{1}{2}} \cdot \det(1-Y)^{\frac{1}{2}}, \tag{3.24}$$

where the relevant function matrix is,

$$Y^{ab}(x,y;A) = g\, f^{abm}\gamma^\mu\gamma^\nu \partial_\mu C(x-y)\, A^m_\nu(y). \tag{3.25}$$

The factor of $\frac{1}{2}$ in the determinant accounts for the correct counting of physical fermionic degrees of freedom (technically due to the Majorana or Weyl condition when performing the Berezin integration). Using the identity (3.20), we get

$$\frac{1}{2}\log\det\left(1-Y\right) = \frac{1}{4} n g^2\, \mathrm{tr}\left(\gamma_\alpha\gamma_\lambda\gamma_\beta\gamma_\rho\right) \int dx\, dy\; \partial_\alpha C(x-y) A^m_\lambda(y)\, \partial_\beta C(y-x) A^m_\rho(x). \tag{3.26}$$

Here again, the contribution at order g trivially vanishes, and the trace (tr) is associated with the trace over spinor indices. The trace of four γ-matrices can be obtain from (2.18) and reads

$$\mathrm{tr}\left(\gamma_\alpha\gamma_\lambda\gamma_\beta\gamma_\rho\right) = -r\left(\delta_{\alpha\lambda}\delta_{\beta\rho} - \delta_{\alpha\beta}\delta_{\lambda\rho} + \delta_{\alpha\rho}\delta_{\lambda\beta}\right). \tag{3.27}$$

Plugging the trace formula in the (3.26), we get

$$\log\left(\Delta_{MSS}[A]\right) = ng^2\int dx\,dy\,\Big\{\frac{r}{2}\,\partial_\mu C(x-y)A^a_\mu(y)\,\partial_\nu C(y-x)A^a_\nu(x) - \frac{r}{4}\,\partial_\mu C(x-y)A^a_\nu(y)\,\partial_\mu C(y-x)A^a_\nu(x)\Big\}\,, \tag{3.28}$$

where r counts the number of off-shell fermionic degrees of freedom.

We also have contributions from Ghost determinants. The ghost fields are anti-commuting scalar fields. We integrate them out similar to the fermion case and obtain the following determinant

$$\det\left(\partial^\mu D_\mu\right) = \det\left(\left[\delta^{ab}\partial_\mu - gf^{abm}A^m_\mu\right]\partial^\mu\right) = \det(\Box)\cdot\det(1-Z)\,, \tag{3.29}$$

where $Z^{ab}(x,y;A) = gf^{abm}\partial_\mu C(x-y)A^m_\mu(y)$. The non-trivial contribution from ghost determinant is

$$\log\det\left(1-Z\right) = \frac{1}{2}ng^2\int dx\,dy\,\partial_\mu C(x-y)A^m_\mu(y)\partial_\nu C(y-x)A^m_\nu(x) + \mathcal{O}(g^3)\,, \tag{3.30}$$

Our aim was to show that the ansatz (3.13) satisfies the third statement of the main theorem.

$$\log\det\left(\frac{\delta A'^a_\mu(x,g;A)}{\delta A^b_\nu(y)}\right) = \log\left(\,\Delta_{MSS}[A]\,\Delta_{FP}[A]\,\right)\,.$$

We take the product of fermion (MSS) (3.28) and ghost (FP) (3.30) determinants and equate them to the Jacobi determinant of the transformation (3.22) that gives us the following two relations

$$\frac{2D-3}{2} = \frac{1+r}{2}\,,\quad\text{and}\quad \frac{D-2}{2} = \frac{r}{4}\,, \tag{3.31}$$

which gives

$$\boxed{r = 2(D-2)}\,, \tag{3.32}$$

and are thus satisfied for

$$D = 3,4,6,10 \iff r = 2,4,8,16\,. \tag{3.33}$$

We have discovered that the matching of determinants depends on the dimension of our field theory, which imposes a constraint on the allowed values of space-time dimensions. We showed that the supersymmetric Yang-Mills theories can only exist in $D = 3, 4, 6, 10$ dimensions. This result was first obtained in [26] using the closure of supersymmetry transformation and the specific Fierz identity. We were able to recover this classic old result without requiring the closure of supersymmetry algebra. This means that one can formulate supersymmetric Yang-Mills theories entirely in $D = 3, 4, 6, 10$ space-time dimensions without using anti-commuting variables.

We have confirmed that our ansatz (3.13) satisfies all three conditions of the main theorem. Additionally, we found that the map at order g^2 does not require any use of gauge condition. This map can be used to quantize the super Yang-Mills theory in terms of bosonic variables. A natural question that arises is whether this derived transformation (3.13) can be used to compute objects of interest like scattering amplitudes or correlation functions. The correlation function for $\mathcal{N} = 4$ super Yang-Mills was computed in [27] using our map (3.13). Higher-loop computations require a higher-order map in the coupling constant. In the next chapter, we will discuss the systematic construction of a map at order g^3 in the coupling constant.

Chapter 4

Systematic construction of the map and its extension to g^3

The material presented in this chapter is based on author's publication [28].

In this chapter, we present the formal construction of the Nicolai map up to the third order in the coupling constant for pure supersymmetric Yang-Mills theories in arbitrary dimensions. We derive the infinitesimal generator $\mathcal{R}$, a non-local and non-linear functional differential operator, of the *inverse* Nicolai map. This operator serves as a starting point for constructing Nicolai maps. We perform the checks for the map derived using $\mathcal{R}$ prescription and find that the existence of map in space-time dimensions $D = 3, 4, 6, 10$ holds at the third order.

4.1 Introduction

Supersymmetric Yang-Mills theories can be developed in terms of bosonic variables using the Nicolai map, which was initially constructed and studied through trial and error. For lower order in perturbation theory, the map was simpler and easier to guess, as shown in (3). However, at higher orders in the coupling constant, the form of the map gets complicated due to the rapid increase in a number of terms (4.51). The work of Dietz, Flume, and Lechtenfeld [18–20] introduced a new and systematic way of constructing the map.

In this chapter, we will show how to construct the $\mathcal{R}$ operator for any pure super Yang-Mills theory. We will present the general construction of this operator valid for all on-shell supersymmetric Yang-Mills theories in space-time dimensions $D = 3, 4, 6, 10$. The proof is largely based on existing work [44], but we have generalized it for critical dimensions in the Landau gauge. Our main goal in this chapter is to work out the infinitesimal generator $\mathcal{R}$ and use it to derive the map at order g^3. We will explain in detail how all the necessary conditions are satisfied.

We here work in Euclidean space and the on-shell action of $\mathcal{N} = 1$ super Yang-Mills in D

dimensions (2.21) that consists of a gauge-invariant part

$$S_{\text{inv}} \;=\; \tfrac{1}{4}\int \mathrm{d}x\; F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) \;+\; \tfrac{1}{2}\int \mathrm{d}x\; \bar{\lambda}^a(x)\,\gamma^\mu (D_\mu\lambda)^a(x)\,, \tag{4.1}$$

and a gauge-fixing part

$$S_{\text{gf}} \;=\; \frac{1}{2\xi}\int \mathrm{d}x\; G^a(A)\,G^a(A) \;+\; \int \mathrm{d}x\; \bar{C}^a(x)\,\frac{\partial G^a(A)}{\partial A^b_\mu(x)}(D_\mu C)^b(x)\,. \tag{4.2}$$

The full action $S_{\text{inv}} + S_{\text{gf}}$ is invariant under the BRST (or Slavnov) variations

$$sA^a_\mu = (D_\mu C)^a\,,\quad s\lambda^a = -gf^{abc}\lambda^b C^c\,,\quad sC^a = -\tfrac{g}{2}f^{abc}C^bC^c\,,\quad s\bar{C}^a = -\tfrac{1}{\xi}\,G^a(A)\,, \tag{4.3}$$

for all positive ξ and an arbitrary gauge-fixing function $G^a(A)$ (which for simplicity we assume not to depend on g). In the remainder, we will specialize in the Landau gauge function $G^a = \partial^\mu A^a_\mu$ which can be obtained for $\xi \to 0$. For the ghost kinetic term, we obtain the standard form

$$\int \mathrm{d}x\; \bar{C}^a(x)\,\frac{\partial G^a(A)}{\partial A^b_\mu(x)}(D_\mu C)^b(x) \;=\; \int \mathrm{d}x\; \bar{C}^a(x)\,\partial^\mu (D_\mu C)^a(x)\,. \tag{4.4}$$

The n-point correlation function of a set of operators $\mathcal{O}_j(x_j)$ in super Yang-Mills theory is

$$\langle\!\langle \mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\rangle\!\rangle = \int DAD\lambda\, DC\, D\bar{C}\; e^{-S_{\text{inv}}[g,A,\lambda]-S_{\text{gf}}[g,A,C,\bar{C}]}\;\mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\,. \tag{4.5}$$

The operators $\mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)$ are bosonic in nature, and we integrate out the fermions and ghosts, and obtain purely bosonic expectation value

$$\langle \mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\rangle = \int DA\, e^{-S_g[A]}\;\Delta_{MSS}[A]\,\Delta_{FP}[A]\;\mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\,. \tag{4.6}$$

In the next section (4.2.2), we will show how one can express correlation functions of fully interacting theory in terms of free bosonic correlators.

We state below the revised version of the main theorem [15].

4.1.1 Main theorem

Supersymmetric gauge theories are characterized by the existence of a non-linear and non-local transformation $\mathcal{T}_g$ of the Yang-Mills fields

$$\mathcal{T}_g \,:\, A^a_\mu(x) \;\mapsto\; A'^a_\mu(x,g;A)\,,$$

which is invertible, at least in the sense of a formal power series such that

1. The Yang-Mills action without gauge-fixing terms is mapped to the bosonic abelian action,

$$S_0[A'] \;=\; S_g[A]\,, \tag{4.7}$$

   where $S_g[A] = \frac{1}{4}\int \mathrm{d}x\, F^a_{\mu\nu}F^a_{\mu\nu}$ is the Yang-Mills action with gauge coupling g.

2. On the gauge surface $G^a[A] \equiv \partial^\mu A^a_\mu = 0$, the Jacobi determinant of $\mathcal{T}_g$ is equal to the product of the MSS [1] and FP determinants

$$\det\left(\frac{\delta A'^a_\mu(x,g;A)}{\delta A^b_\nu(y)}\right) \;=\; \Delta_{MSS}[A]\,\Delta_{FP}[A]\,, \tag{4.8}$$

   at least order by order in perturbation theory.

3. The gauge fixing function

$$G^a[A] \text{ is a fixed point of } \mathcal{T}_g. \tag{4.9}$$

In the next section, we will show the derivation of the $\mathcal{R}$ operator and how to obtain the map $\mathcal{T}_g$ from it. In this chapter, we won't outline the proof of the main theorem using the properties of the $\mathcal{R}$ operator. Our focus here is on the construction of the $\mathcal{R}$ operator and on the properties of the map. For more details on the $\mathcal{R}$ prescription and the main theorem proof, please refer to [28, 45].

[1] With the understanding that Δ_{MSS} is really a Pfaffian for Majorana fermions.

4.2 Introduction to $\mathcal{R}$ prescription

The construction of the $\mathcal{R}$ operator is based on the concept that the derivative of action with respect to coupling constant g can be expressed as a super variation of some fermionic functional. This idea was first given by Zumino in proof of the vanishing of vacuum energy of supersymmetric theories [1].

Consider X be a functional of the bosonic fields A^a_μ, the linear response of its vacuum expectation value to a change in the coupling constant is given by

$$\frac{\mathrm{d}}{\mathrm{d}g}\langle X\rangle \;=\; \frac{\mathrm{d}}{\mathrm{d}g}\langle\!\langle X\rangle\!\rangle \;=\; \left\langle\!\!\left\langle \frac{\mathrm{d}X}{\mathrm{d}g}\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{\mathrm{d}(S_{\text{inv}}+S_{\text{gf}})}{\mathrm{d}g}\, X\right\rangle\!\!\right\rangle \;=:\; \langle \mathcal{R}\, X\rangle\,. \tag{4.10}$$

Here, the vacuum expectation values $\langle\!\langle \cdots \rangle\!\rangle$ and $\langle \cdots \rangle$ were defined in (4.5) and (4.6), and we have dropped subscripts g or ξ for simplicity of notation.

The $\mathcal{R}$ operator is a functional differential operator defined as the generator of infinitesimal shifts of the coupling constant g of the theory. We use the supersymmetry to rewrite the right-hand side in terms of a derivational operator $\mathcal{R}$. We introduce

$$\Delta_\alpha \;=\; -\frac{1}{2r} f^{abc}\int \mathrm{d}x\;\left(\gamma^{\rho\lambda}\lambda^a(x)\right)_\alpha A^b_\rho(x)A^c_\lambda(x)\,, \tag{4.11}$$

and use the standard supersymmetry variations (with supersymmetry parameter stripped off)

$$\delta_\alpha\lambda^a_\beta \;=\; \tfrac{1}{2}(\gamma^{\mu\nu})_{\beta\alpha}F^a_{\mu\nu}\,, \qquad \text{and} \qquad \delta_\alpha A^a_\mu \;=\; -(\bar\lambda^a\gamma_\mu)_\alpha\,, \tag{4.12}$$

to compute

$$\delta_\alpha\Delta_\alpha \;=\; \tfrac{1}{2}f^{abc}\int \mathrm{d}x\; F^a_{\mu\nu}(x)A^b_\mu(x)A^c_\nu(c) \;+\; \tfrac{D-1}{r}f^{abc}\int \mathrm{d}x\;\left(\gamma^\mu\lambda^a(x)\bar\lambda^b(x)\right)_{\alpha\alpha}A^c_\mu(x)\,, \tag{4.13}$$

so that

$$\frac{\mathrm{d}S_{\text{inv}}}{\mathrm{d}g} \;=\; \delta_\alpha\Delta_\alpha \;+\; \left(\tfrac{1}{2}-\tfrac{D-1}{r}\right)f^{abc}\int \mathrm{d}x\;\bar\lambda^a(x)\gamma^\mu\lambda^b(x)A^c_\mu(x)\,. \tag{4.14}$$

Notice that δ_α anticommutes with other anticommuting operators. With

$$\frac{\mathrm{d}S_{\text{gf}}}{\mathrm{d}g} \;=\; f^{abc} \int \mathrm{d}x\; \bar{C}^a(x)\, \partial^\mu \big(A^b_\mu(x)\, C^c(x)\big)\,, \tag{4.15}$$

we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}g}\,\langle X\rangle \;=\; & \left\langle\!\!\left\langle \frac{\mathrm{d}X}{\mathrm{d}g}\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle (\delta_\alpha \Delta_\alpha)\, X\right\rangle\!\!\right\rangle \\ & + \left\langle\!\!\left\langle \left(\tfrac{D-1}{r} - \tfrac{1}{2}\right) f^{abc} \int \mathrm{d}x\; \bar{\lambda}^a(x)\gamma^\mu A^c_\mu(x)\lambda^b(x)\; X \right\rangle\!\!\right\rangle \\ & - \left\langle\!\!\left\langle f^{abc} \int \mathrm{d}x\; \bar{C}^a(x)\, \partial^\mu \big(A^b_\mu(x)\, C^c(x)\big)\; X \right\rangle\!\!\right\rangle .\end{aligned} \tag{4.16}$$

We want to rewrite

$$\langle\!\langle (\delta_\alpha \Delta_\alpha) X\rangle\!\rangle = \langle\!\langle \Delta_\alpha \delta_\alpha X\rangle\!\rangle + \langle\!\langle \delta_\alpha(\Delta_\alpha X)\rangle\!\rangle\,. \tag{4.17}$$

We use the supersymmetry ward identity [45]

$$\langle\!\langle \delta_\alpha Y\rangle\!\rangle = \langle\!\langle (\delta_\alpha S_{\text{gf}})\, Y\rangle\!\rangle\,. \tag{4.18}$$

Employing the BRST transformations (4.3) we find that

$$\delta_\alpha S_{\text{gf}} \;=\; -s \int \mathrm{d}x\; \bar{C}^a(x)\, \delta_\alpha\big(\partial^\mu A^a_\mu(x)\big)\,. \tag{4.19}$$

Thus, the Ward identity becomes

$$\langle\!\langle \delta_\alpha Y\rangle\!\rangle = \left\langle\!\!\left\langle \int \mathrm{d}x\; \bar{C}^a(x)\, \delta_\alpha\big(\partial^\mu A^a_\mu(x)\big)\; s(Y)\right\rangle\!\!\right\rangle . \tag{4.20}$$

In (4.17) we can see that $Y = \Delta_\alpha X$, and from $s(\Delta_\alpha X) = s(\Delta_\alpha)X - \Delta_\alpha s(X)$ we also require the BRST transformation of Δ_α. Using the Jacobi identity, we get

$$s\,(\Delta_\alpha) \;=\; \frac{1}{r} f^{abc} \int \mathrm{d}x\; \big(\gamma^{\rho\lambda}\lambda^a(x)\big)_\alpha \partial_\rho C^b(x) A^c_\lambda(x)\,. \tag{4.21}$$

Subsequently, we can put everything back together,

$$\frac{\mathrm{d}}{\mathrm{d}g}\langle X\rangle \;=\; \left\langle \frac{\mathrm{d}X}{\mathrm{d}g}\right\rangle - \langle\!\langle \Delta_\alpha \delta_\alpha X\rangle\!\rangle + \left\langle\!\!\left\langle \int \mathrm{d}x\, \bar{C}^a(x)\, \delta_\alpha\big(\partial^\mu A^a_\mu(x)\big)\Delta_\alpha\, s(X)\right\rangle\!\!\right\rangle + \langle\!\langle Z\; X\rangle\!\rangle\,, \tag{4.22}$$

with

$$\begin{aligned} Z \;=\; &-\int \mathrm{d}y\, \bar{C}^a(y)\, \delta_\alpha\big(\partial^\mu A^a_\mu(y)\big)\, \frac{1}{r} f^{bcd} \int \mathrm{d}x\, \big(\gamma^{\rho\lambda}\lambda^b(x)\big)_\alpha A^c_\rho(x)\, \partial_\lambda C^d(x) \\ &+\; \left(\tfrac{D-1}{r} - \tfrac{1}{2}\right) f^{abc} \int \mathrm{d}x\, \bar{\lambda}^a(x)\gamma^\mu A^c_\mu(x)\lambda^b(x) \;-\; f^{abc}\int \mathrm{d}x\, \bar{C}^a(x)\, \partial^\mu\big(A^b_\mu(x)\, C^c(x)\big)\,. \end{aligned} \tag{4.23}$$

We want the $\mathcal{R}$ operator to be manifestly distributive. From (4.22), we can see that the first three terms act like derivative operators. The fourth term $\langle\!\langle Z\; X\rangle\!\rangle$ does not behave like a derivative, so to keep the operator distributive, we need to show that the fourth term vanishes.

As it stands, and up to this point, the above derivation is valid *for all values of the gauge parameter* ξ. We can, therefore, take the limit $\xi \to 0$, for which all contributions containing $\partial^\mu A^a_\mu$ simply vanish (recall that physical quantities anyway cannot depend on ξ). We will show in the next subsection (4.2.1) that under these conditions the multiplicative contribution disappears,

$$\lim_{\xi\to 0}\, \langle\!\langle Z\; X\rangle\!\rangle_\xi = 0 \qquad \text{for} \quad \frac{D-1}{r} - \frac{1}{2} = \frac{1}{r}\,, \tag{4.24}$$

and thus only in the critical dimensions $D = 3, 4, 6$ and 10, where $r = 2(D-2)$ indeed.

Note that for off-shell supersymmetric theories with the rescaled fields, the operator $\mathcal{R}$ is manifestly distributive. The construction holds for all general gauges in four dimensions [31].

The generator $\mathcal{R}$ then can be obtained from (4.22) by integrating out the fermions

$$\mathcal{R}\, X \;=\; \frac{\mathrm{d}X}{\mathrm{d}g} \;+\; \underbrace{\delta_\alpha X \cdot \Delta_\alpha}_{} \;+\; \int \mathrm{d}x\, \underbrace{\bar{C}^a(x)\; \underbrace{\delta_\alpha\big(\partial^\mu A^a_\mu(x)\big)\,\Delta_\alpha}_{}\; s}_{}(X)\,, \tag{4.25}$$

where the contractions signify fermionic (gaugino or ghost) propagators in the gauge-field background. For $\mathcal{N} = 1$ super Yang–Mills theory this result was first derived in [18], see also [17].

A key role in the $\mathcal{R}$ operator is played by the fermionic propagator $S^{ab}(x,y;A)$ in a gauge-field dependent background characterized by $A^a_\mu(x)$, with

$$\gamma^\mu \left(D_\mu S\right)^{ab}(x,y;A) \;\equiv\; \gamma^\mu \left[\delta^{ac}\partial_\mu \;+\; g f^{adc} A^d_\mu(x)\right] S^{cb}(x,y;A) \;=\; \delta^{ab}\delta(x-y)\,. \tag{4.26}$$

The limit $g = 0$ gives us the free fermionic propagator $S^{ab}_0(x)$. We also require the implementation of the ghost propagator $G^{ab}(x,y;A)$, obeying

$$\partial^\mu (D_\mu G)^{ab}(x,y;A) \;\equiv\; \left[\delta^{ac}\Box + g f^{adc}\frac{\partial}{\partial x^\mu} A^d_\mu(x)\right] G^{cb}(x,y;A) \;=\; \delta^{ab}\delta(x-y)\,, \tag{4.27}$$

where the differential operator acts on everything to its right.The free ghost propagator satisfies

$$\Box G^{ab}_0(x) \;=\; \delta^{ab}\delta(x) \qquad \Rightarrow \qquad G^{ab}_0(x) \;=\; -\delta^{ab} C(x)\,, \tag{4.28}$$

and the full ghost propagator expands as

$$G^{ab}(x,y) \;=\; G^{ab}_0(x,y) \;-\; g\int \mathrm{d}z\, G^{ac}_0(x,z) f^{cde}\partial^\mu_z\left(A^d_\mu(z) G^{eb}_0(z,y)\right) \;+\; \cdots\,. \tag{4.29}$$

It is important to note that not only $G^{ab}(x,y;A)$ depends on g and the background field $A^a_\mu(x)$ but that $(D_\mu G)^{ab}(x,y;A)$ does so as well

$$-(D_\mu G)^{ab}(x,y;A) \;=\; \delta^{ab}\partial_\mu C(x-y) \;+\; g f^{acb}\int \mathrm{d}z\; \Pi_{\mu\nu}(x-z) A^c_\nu(z) C(z-y) \;+\; \mathcal{O}(g^2)\,, \tag{4.30}$$

with the abelian transversal projector

$$\Pi_{\mu\nu}(x-z) \;\equiv\; \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\delta(x-z) \;\simeq\; \delta_{\mu\nu}\delta(x-z) + \partial_\mu C(x-z)\,\partial^z_\nu\,, \tag{4.31}$$

where $\simeq$ means equality in the sense of a distribution. We will later see that the terms of $\mathcal{O}(g)$ in (4.30) become relevant for the map $\mathcal{T}_g$ from order g^3 onwards.

We use the background-field dependent propagators defined in (4.26) and (4.27).

$$\underbrace{\lambda^a(x)\,\bar\lambda^b}(y) \;\equiv\; S^{ab}(x,y;A)\,, \qquad \text{and} \qquad \underbrace{C^a(x)\,\bar C^b}(y) \;\equiv\; G^{ab}(x,y;A)\,. \tag{4.32}$$

to rewrite (4.25). Here $\lambda^a(x)$ are the gaugino fields (prior to their elimination via the MSS determinant), and $C^a(x)$ and $\bar C^a(x)$ denote the ghost and anti-ghost fields.[2] For the Landau gauge function (2.26) the $\mathcal{R}$ operator is then represented by the functional differential operator

$$\begin{aligned}\mathcal{R} \;=\; &\frac{\mathrm{d}}{\mathrm{d}g} - \frac{1}{2r}\int \mathrm{d}x\,\mathrm{d}y\;\mathrm{Tr}\left(\gamma_\mu S^{ab}(x,y;A)\,\gamma^{\rho\lambda}\right)\; f^{bcd}A^c_\rho(y)A^d_\lambda(y)\,\frac{\delta}{\delta A^a_\mu(x)}\\ &-\frac{1}{2r}\int \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\;(D_\mu G)^{ae}(x,z;A)\,\partial_\nu\,\mathrm{Tr}\left(\gamma^\nu S^{eb}(z,y;A)\,\gamma^{\rho\lambda}\right)\; f^{bcd}A^c_\rho(y)A^d_\lambda(y)\,\frac{\delta}{\delta A^a_\mu(x)}\,.\end{aligned} \tag{4.33}$$

Notice that the first part of the $\mathcal{R}$ operator (first line on the r.h.s. of (4.33)) is *gauge independent*, whereas the second line does depend on the choice of the gauge-fixing function via the ghost propagator. The gauge field $A^a_\mu(x)$ does not depend on g, and the first action of $\mathcal{R}$ to $A^a_\mu(x)$ is straightforward. For all higher orders, we also need

$$\frac{\mathrm{d}S^{ab}(x,y)}{\mathrm{d}g} \;=\; -\int \mathrm{d}z\; S^{ac}(x,z)\,f^{cmd}A^m_\mu(z)\,\gamma^\mu S^{db}(z,y)\,, \tag{4.34}$$

and

$$\frac{\delta S^{ab}(z,y)}{\delta A^m_\mu(x)} \;=\; -g\,S^{ac}(z,x)\,f^{cmd}\gamma^\mu S^{db}(x,y)\,, \tag{4.35}$$

as well as

$$\frac{\mathrm{d}G^{ab}(x,y)}{\mathrm{d}g} \;=\; \int \mathrm{d}z\; G^{ac}(x,z)\,f^{cmd}\overleftarrow{\partial}^\mu_z\,A^m_\mu(z)\,G^{db}(z,y)\,, \tag{4.36}$$

and

$$\frac{\delta G^{ab}(z,y)}{\delta A^m_\mu(x)} \;=\; g\,G^{ac}(z,x)\,f^{cmd}\overleftarrow{\partial}^\mu_x\,G^{db}(x,y)\,. \tag{4.37}$$

These equations are obtained from (4.26) and (4.27). After iteratively computing $\mathcal{R}^n$ for any

[2] Not to be confused with the propagator $C(x)$, which carries no indices.

desired n, we set $g = 0$, which in particular maps $S^{ab}(x,y)$ to the free propagator $S_0^{ab}(x-y)$ and $G^{ab}(x,y)$ to the free propagator $G_0^{ab}(x-y)$. This operator then serves as starting point for construction of $(\mathcal{T}_g^{-1}A)^a_\mu(x)$ at $\mathcal{O}(g^n)$.

In section (4.2.2), we will prove that the $\mathcal{R}^n$ operator is the infinitesimal generator of the inverse Nicolai map. Once the $\mathcal{R}$ operator is constructed, one can systematically derive the Nicolai map. Before jumping into the map, we present below the proof of distributivity of the $\mathcal{R}$ operation.

4.2.1 Distributivity of the $\mathcal{R}$ operation

In this subsection, we generalize the argument from [18] in order to prove that (4.24) holds for any bosonic functional X. Integrating out the gauginos and ghosts yields

$$\begin{aligned} \underset{\sqcup}{Z} \;=\; & \frac{1}{r}\int \mathrm{d}y\, \big(\bar{C}^a(y)\big)\big(\partial^y_\rho\bar{\lambda}^a(y)\gamma^\rho\big)_\alpha\, f^{bcd}\int \mathrm{d}x\,\big(\gamma^{\mu\nu}\lambda^b(x)\big)_\alpha A^c_\mu(x)\,\partial^x_\nu C^d(x) \\ & + \left(\tfrac{D-1}{r}-\tfrac{1}{2}\right) f^{abc}\int \mathrm{d}x\;\bar{\lambda}^a_\alpha(x)\gamma^\mu_{\alpha\beta}A^b_\mu(x)\lambda^c_\beta(x) \;-\; f^{abc}\int \mathrm{d}x\;\bar{C}^a(x)A^b_\mu(x)\partial^\mu_x C^c(x)\,. \end{aligned} \tag{4.38}$$

We use the identity $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = -\gamma^\nu\gamma^\mu + \delta^{\mu\nu}$ and reorder the contracted terms so as to identify any contraction with a fermion or ghost propagator (in the presence of the gauge-field background) to get

$$\begin{aligned} \underset{\sqcup}{Z} \;=\; & -\frac{1}{r}f^{bcd}\int \mathrm{d}x\,\mathrm{d}y\;\mathrm{Tr}\big(\partial^x_\nu G^{da}(x,y)\gamma^\rho\gamma^\nu\gamma^\mu\partial^y_\rho S^{ba}(x,y)\big)\,A^c_\mu(x) \\ & + \frac{1}{r}f^{bcd}\int \mathrm{d}x\,\mathrm{d}y\;\mathrm{Tr}\big(\partial^x_\nu G^{da}(x,y)\gamma^\rho\delta^{\mu\nu}\partial^y_\rho S^{ba}(x,y)\big)\,A^c_\mu(x) \\ & - \left(\tfrac{D-1}{r}-\tfrac{1}{2}\right) f^{abc}\int \mathrm{d}x\;\mathrm{Tr}\left(S^{ca}(x,x)\gamma^\mu\right)A^b_\mu(x) \;+\; f^{abc}\int \mathrm{d}x\;\partial^\mu_x G^{ca}(x,x)A^b_\mu(x)\,. \end{aligned} \tag{4.39}$$

We use the following Schwinger–Dyson identities,

$$\begin{aligned} S^{ba}(x,y) \;&=\; S_0^{ba}(x-y) \;+\; g f^{emn}\int \mathrm{d}z\; S_0^{be}(x-z)A^n_\nu(z)\gamma^\nu S^{ma}(z,y)\,, \\ \gamma^\nu\partial^x_\nu G^{da}(x,y) \;&=\; S_0^{da}(x-y) \;+\; g f^{emn}\int \mathrm{d}z\; S_0^{de}(x-z)A^n_\nu(z)\partial^\nu_z G^{ma}(z,y)\,, \end{aligned} \tag{4.40}$$

and the relation $\gamma^\nu \partial_\nu^x G_0^{da}(x-y) = S_0^{da}(x-y)$. Integrating by parts and using $\gamma^\rho \partial_\rho^y S_0^{da}(x-y) = -\delta^{da}\delta(x-y)$ together with $\mathrm{Tr}\,\mathbf{1} = r$, this gives

$$\begin{aligned}
\underset{\sqcup}{Z} \;=\; & -\frac{1}{r} f^{bca} \int \mathrm{d}x \; \mathrm{Tr}\left(\gamma^\mu S^{ba}(x,x)\right) A_\mu^c(x) \\
& - \frac{g}{r} f^{bcd} f^{emn} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \; \mathrm{Tr}\left(S_0^{de}(x-z) A_\nu^n(z) \partial_z^\nu G^{ma}(z,y) \gamma^\mu \partial_\rho^y S^{ba}(x,y)\gamma^\rho\right) A_\mu^c(x) \\
& - f^{acd} \int \mathrm{d}x \; \partial_x^\mu G^{da}(x,x) A_\mu^c(x) \\
& + \frac{g}{r} f^{bcd} f^{emn} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \; \mathrm{Tr}\left(\partial_x^\mu G^{da}(x,y) S_0^{be}(x-z) \gamma^\nu A_\nu^n(z) \partial_\rho^y S^{ma}(z,y)\gamma^\rho\right) A_\mu^c(x) \\
& - \left(\tfrac{D-1}{r} - \tfrac{1}{2}\right) f^{abc} \int \mathrm{d}x \; \mathrm{Tr}\left(S^{ca}(x,x)\gamma^\mu\right) A_\mu^b(x) \;+\; f^{abc} \int \mathrm{d}x \; \partial_x^\mu G^{ca}(x,x) A_\mu^b(x)\,.
\end{aligned}$$

The pure fermion loops (first and fifth term) cancel, provided (3.32) holds with $D = 3, 4, 6$ or 10, as advertised. The pure ghost loops (third and sixth term) cancel independently of dimension. Finally, we use $S_0^{be}(x-z) = -S_0^{be}(z-x)$ to cancel the two remaining terms,

$$\begin{aligned}
\underset{\sqcup}{Z} \;=\; & -\frac{g}{r} f^{bcd} f^{emn} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \; \mathrm{Tr}\left(S_0^{de}(x-z) A_\nu^n(z) \partial_z^\nu G^{ma}(z,y) \gamma^\mu \partial_\rho^y S^{ba}(x,y)\gamma^\rho\right) A_\mu^c(x) \\
& + \frac{g}{r} f^{bcd} f^{emn} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \; \mathrm{Tr}\left(\partial_x^\mu G^{da}(x,y) S_0^{be}(x-z) \gamma^\nu A_\nu^n(z) \partial_\rho^y S^{ma}(z,y)\gamma^\rho\right) A_\mu^c(x) \\
=\; & 0\,.
\end{aligned}$$

Therefore, we proved that the $\mathcal{R}$ operator (4.25) is distributive

$$\mathcal{R}(X_1 X_2) = \mathcal{R}X_1\; X_2 + X_1 \mathcal{R} X_2\,. \tag{4.41}$$

and the proof only holds true for *Landau gauge*.

4.2.2 Relation between the map and $\mathcal{R}$ operator

In this section, we show that the $\mathcal{R}$ operator plays a key role in our construction of Nicolai maps. For some bosonic operator $X[A]$, the coupling flow equation is

$$\frac{\mathrm{d}}{\mathrm{d}g}\langle X\rangle \;=\; \langle \mathcal{R}\, X\rangle\,, \tag{4.42}$$

where the action of $\mathcal{R}X[A]$ is given in (4.25). Physically, the $\mathcal{R}$ operator can be interpreted as the generator of flow in the physical coupling constant. The above equation can be viewed as a diffusion equation with coupling constant acting as a time parameter [19].

To get a finite shift in the coupling constant, we take higher derivatives of the vacuum expectation value of $X[A]$ and obtain the full expectation value as

$$\begin{aligned} \langle X[A,g]\rangle &= \langle X[A']\rangle_{g'=0} + \sum_{k=1}^{\infty} \frac{g^k}{k!} \frac{d^k}{dg'^k} \langle X[A']\rangle\big|_{g'=0} = \sum_{k=0}^{\infty} \frac{g^k}{k!} \langle R^k(g') X[A']\rangle\big|_{g'=0}, \\ &= \langle \exp(gR)\, X[A']\rangle_0 = \int DA'\, e^{-\frac{1}{2}(\partial_\mu A')^2} \sum_{k=0}^{\infty} \frac{g^k}{k!} R^k X[A']\big|_{g'=0}, \end{aligned} \tag{4.43}$$

where $\langle ....\rangle_0$ denotes the average with free measure.

To relate the above equation to the inverse of the Nicolai map, we start with the correlation function of gauge field A (suppressing color and space-time index) and using (4.5), we get

$$\langle\langle A_1(x_1)....A_n(x_n)\rangle\rangle = \int DAD\lambda\, DC\, D\bar{C}\; e^{-S[A,\lambda,C,\bar{C}]}\; A_1(x_1)....A_n(x_n)\,. \tag{4.44}$$

Integrating out the fermions and ghost field yields

$$\langle A_1(x_1)...A_n(x_n)\rangle = \int DA\, e^{-S_g[A]} \Delta_{MSS}[A]\, \Delta_{FP}[A]\; A_1(x_1)....A_n(x_n)\,. \tag{4.45}$$

Consider the transformation $\mathcal{T}_g$ of the bosonic field that obeys all three conditions of the main theorem (4.1.1). We perform a change of variables in the above equation using the inverse of the transformation $\mathcal{T}_g$ and obtain

$$\langle T_g{}^{-1}[A'_1](x_1)...T_g{}^{-1}[A'_n](x_n)\rangle_0 = \int D_0 A'\, e^{-S_0[A']}\; T_g{}^{-1}[A'_1](x_1)....T_g{}^{-1}[A'_n](x_n)\,, \tag{4.46}$$

where the Jacobian of the inverse transformation cancels the product of fermion and ghost determinants, and we obtain the following relation

$$\langle\langle A_1(x_1)....A_n(x_n)\rangle\rangle = \langle T_g{}^{-1}[A'_1](x_1)...T_g{}^{-1}[A'_n](x_n)\rangle_0\,. \tag{4.47}$$

Therefore, any bosonic correlation function of the fully supersymmetric theories can be computed in terms of the free bosonic correlation function using the inverse map. This statement also holds for fermionic correlators [27, 45].

Take equation (4.43) and replace the product of bosonic operator $X[A]$ with $A_1(x_1)....A_n(x_n)$. Comparing (4.43) and (4.45), we get

$$A(x,g,A') = T_g^{-1}\,A' \equiv \sum_{k=0}^{\infty} \frac{g^k}{k!} R^k A'\big|_{g'=0}\,, \tag{4.48}$$

the inverse Nicolai map that relates the interacting theory to a free theory at a finite value of the coupling constant. The actual map $\mathcal{T}_g$ can then be obtained order by order in g by inverting the above power series. Let us assume the following form for the map

$$\mathcal{T}_g A \;=\; \sum_{n=0}^{\infty} \frac{g^n}{n!}\, T_n A\,. \tag{4.49}$$

Expanding $\mathcal{T}_g^{-1}\mathcal{T}_g = \mathbf{1}$ in powers of g and matching coefficients we readily obtain

$$\begin{aligned} T_0 A \;&=\; A\,, \\ T_1 A \;&=\; -\mathcal{R}\, T_0 A\big|_{g=0}\,, \\ T_2 A \;&=\; -\mathcal{R}^2 T_0 A\big|_{g=0} - 2\mathcal{R}\, T_1 A\big|_{g=0}\,, \\ T_3 A \;&=\; -\mathcal{R}^3 T_0 A\big|_{g=0} - 3\mathcal{R}^2 T_1 A\big|_{g=0} - 3\mathcal{R}\, T_2 A\big|_{g=0}\,. \end{aligned} \tag{4.50}$$

One can now systematically derive the $\mathcal{T}_g$ map. We present below the explicit expression for $(\mathcal{T}_g\, A)^a_\mu(x)$.

Note that the $\mathcal{T}_g$ map can be constructed using an alternate but equivalent construction devised by Lechtenfeld and Rupprecht in [46, 47]. It follows from an old work of Lechtenfeld [44]. The central idea of this construction is to derive directly the $\mathcal{T}_g$ map from the coupling flow equation (4.42). In this approach, instead of working with the functional differential operator, one works with the path-ordered integral equation (eq (43) in [46]) to construct the Nicolai map directly without computing the inverse map first.

4.3 Result and discussion

We now present the main new result, which is the explicit formula for $\mathcal{T}_g$ to cubic order $\mathcal{O}(g^3)$ [3]

$$
\begin{aligned}
(\mathcal{T}_g A)^a_\mu(x) \;=\; & A^a_\mu(x) \;+\; g\, f^{abc} \int \mathrm{d}y\, \partial_\rho C(x-y) A^b_\mu(y) A^c_\rho(y) \\
& + \frac{3g^2}{2} f^{abc} f^{bde} \int \mathrm{d}y\, \mathrm{d}z\, \partial_\rho C(x-y) A^c_\lambda(y) \partial_{[\rho} C(y-z) A^d_\mu(z) A^e_{\lambda]}(z) \\
& + \frac{g^3}{2} f^{abc} f^{bde} f^{cmn} \int \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \partial_\rho C(x-y) \\
& \quad \times \partial_\lambda C(y-z) A^d_\lambda(z) A^e_\sigma(z) \partial_{[\rho} C(y-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
& + g^3 f^{abc} f^{bde} f^{dmn} \int \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \partial_\rho C(x-y) A^c_\lambda(y) \Big\{ \\
& \qquad + 2\, \partial_{[\rho} C(y-z) A^e_{\sigma]}(z) \partial_{[\lambda} C(z-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
& \qquad - 2\, \partial_{[\lambda} C(y-z) A^e_{\sigma]}(z) \partial_{[\rho} C(z-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
& \qquad - \partial_\sigma C(y-z) A^e_\sigma(z) \partial_{[\rho} C(z-w) A^m_\mu(w) A^n_{\lambda]}(w) \\
& \qquad - 2\, \partial_{[\sigma} C(y-z) A^e_{\mu]}(z) \partial_{[\rho} C(z-w) A^m_\lambda(w) A^n_{\sigma]}(w) \\
& \qquad + \partial_{[\rho} C(y-z) A^e_\mu(z) \partial_{|\sigma|} C(z-w) A^m_{\lambda]}(w) A^n_\sigma(w) \Big\} \\
& + \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \Big\{ \\
& \qquad + 2\, \partial_\rho C(x-y) A^c_{[\rho}(y) \partial_{\mu]} C(y-z) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \\
& \qquad - \partial_\mu C(x-y)\, \partial_\rho \left(A^c_\rho(y) C(y-z) \right) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \Big\} \\
& - \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}y\, \mathrm{d}z\, A^c_\mu(x) C(x-y) A^e_\rho(y) \partial_\lambda C(y-z) A^m_\rho(z) A^n_\lambda(z) \\
& + \mathcal{O}(g^4)\,.
\end{aligned}
\tag{4.51}
$$

The first two lines above correspond to the result obtained in (3). The last two lines are the new terms arising from the g-dependence of $(D_\mu G)^{ab}$ in (4.27); they are crucial for the fulfillment of the conditions in the main theorem. While the result up to $\mathcal{O}(g^2)$ was originally obtained by trial and error in [16, 24], this becomes tricky at higher orders because the number of terms is

[3] The terms in the map are written in the compact notation: *e.g.* $[ab] = \frac{1}{2}(ab - ba)$.

significantly larger at $\mathcal{O}(g^3)$ than below. In addition, from the last term we see that new structures appear which could be difficult to guess. In the following section we will verify that this result indeed satisfies all three statements of the main theorem (subsection 4.1.1) simultaneously, providing a highly non-trivial test.

4.4 Checks of the map

A general all order proof of the statements in the main theorem is given in [28, 45]. The checks of the main theorem that we performed up to $\mathcal{O}(g^2)$ can be found in (3). Thus, we only present here the calculations at third order in g.

4.4.1 Gauge condition

We start with the third statement of the main theorem (4.1.1) and verify that $\partial_\mu A'^a_\mu(x) = \partial_\mu A^a_\mu(x) + \mathcal{O}(g^4)$. The action of ∂_μ to the terms of order g^3 in (4.51) and removing all terms that are manifestly anti-symmetric under the exchange of indices μ and ρ yields

$$
\begin{aligned}
\partial_\mu A'^a_\mu(x)\big|_{\mathcal{O}(g^3)} = g^3 f^{abc} f^{bde} f^{dmn} \int \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \partial_\mu \partial_\rho C(x-y) A^c_\lambda(y) \Big\{ \\
+ 2\, \partial_{[\rho} C(y-z) A^e_{\sigma]}(z) \partial_{[\lambda} C(z-w) A^m_{\mu}(w) A^n_{\sigma]}(w) \\
- 2\, \partial_{[\sigma} C(y-z) A^e_{\mu]}(z) \partial_{[\rho} C(z-w) A^m_{\lambda}(w) A^n_{\sigma]}(w) \Big\} \\
- \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w \\
\times \Box C(x-y)\, \partial_\rho \left(A^c_\rho(y) C(y-z) \right) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \\
- \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}y\, \mathrm{d}z \\
\times \partial_\mu \left(A^c_\mu(x) C(x-y) \right) A^e_\rho(y) \partial_\lambda C(y-z) A^m_\rho(z) A^n_\lambda(z)\,.
\end{aligned}
$$

The first two terms cancel each other. In the third term we use $\Box\, C(x-y) = -\delta(x-y)$. It is then easy to see that

$$
\partial_\mu A'^a_\mu(x)\big|_{\mathcal{O}(g^3)} = 0\,. \tag{4.52}
$$

4.4.2 Free action

The first statement in the main theorem states that the transformed gauge field must satisfy

$$\frac{1}{2}\int \mathrm{d}x\; A'^a_\mu(x)\,(-\Box\,\delta_{\mu\nu}+\partial_\mu\partial_\nu)\,A'^a_\nu(x) \;=\; \frac{1}{4}\int \mathrm{d}x\; F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) \;+\; \mathcal{O}(g^4)\,. \tag{4.53}$$

Note that, the fulfillment of this condition does not require any special value of dimension D. We can ignore the second term on the l.h.s. and the corresponding term on the r.h.s. of this equation due to the Landau gauge condition. The order g^2 calculation, is given in detail in (3). At third order, (4.53) has two contributions which must cancel each other:

$$0 \overset{!}{=} \int \mathrm{d}x\,\left(A'^a_\mu(x)\big|_{\mathcal{O}(g^3)}\,\Box\,A'^a_\mu(x)\big|_{\mathcal{O}(g^0)} + A'^a_\mu(x)\big|_{\mathcal{O}(g^2)}\,\Box\,A'^a_\mu(x)\big|_{\mathcal{O}(g^1)}\right). \tag{4.54}$$

To check this we collect all the terms to obtain

$$\begin{aligned}
&\int \mathrm{d}x\,\left(A'^a_\mu(x)\big|_{\mathcal{O}(g^3)}\,\Box\,A'^a_\mu(x)\big|_{\mathcal{O}(g^0)} + A'^a_\mu(x)\big|_{\mathcal{O}(g^2)}\,\Box\,A'^a_\mu(x)\big|_{\mathcal{O}(g^1)}\right)\\
&\quad= -\frac{g^3}{2} f^{abc} f^{bde} f^{cmn} \int \mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z\;\mathrm{d}w\;\partial_\rho C(x-y)\\
&\qquad\qquad\qquad \times\,\partial_\lambda C(y-z) A^d_\lambda(z) A^e_\sigma(z)\partial_{[\rho} C(y-w) A^m_\mu(w) A^n_{\sigma]}(w)\,\Box\,A^a_\mu(x)\\
&\qquad + g^3 f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z\;\mathrm{d}w\;\partial_\rho C(x-y) A^c_\lambda(y)\,\Big\{\\
&\qquad\qquad + 2\,\partial_{[\rho} C(y-z) A^e_{\sigma]}(z)\partial_{[\lambda} C(z-w) A^m_\mu(w) A^n_{\sigma]}(w)\,\Box\,A^a_\mu(x)\\
&\qquad\qquad - 2\,\partial_{[\lambda} C(y-z) A^e_{\sigma]}(z)\partial_{[\rho} C(z-w) A^m_\mu(w) A^n_{\sigma]}(w)\,\Box\,A^a_\mu(x)\\
&\qquad\qquad - \partial_\sigma C(y-z) A^e_\sigma(z)\partial_{[\rho} C(z-w) A^m_\mu(w) A^n_{\lambda]}(w)\,\Box\,A^a_\mu(x)\\
&\qquad\qquad - 2\,\partial_{[\sigma} C(y-z) A^e_{\mu]}(z)\partial_{[\rho} C(z-w) A^m_\lambda(w) A^n_{\sigma]}(w)\,\Box\,A^a_\mu(x)\\
&\qquad\qquad + \partial_{[\rho} C(y-z) A^e_\mu(z)\partial_{|\sigma|} C(z-w) A^m_{\lambda]}(w) A^n_\sigma(w)\,\Box\,A^a_\mu(x)\Big\}\\
&\qquad + \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z\;\mathrm{d}w\;\Big\{\\
&\qquad\qquad + 2\,\partial_\rho C(x-y) A^c_{[\rho}(y)\partial_{\mu]} C(y-z) A^e_\lambda(z)\partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w)\,\Box\,A^a_\mu(x)\\
&\qquad\qquad - \partial_\mu C(x-y)\,\partial_\rho\left(A^c_\rho(y) C(y-z)\right) A^e_\lambda(z)\partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w)\,\Box\,A^a_\mu(x)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&\qquad\qquad \times A^c_\mu(x) C(x-y) A^e_\rho(y) \partial_\lambda C(y-z) A^m_\rho(z) A^n_\lambda(z) \, \Box \, A^a_\mu(x) \\
&+ \frac{3g^3}{2} f^{abc} f^{bde} \int \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}w \; \partial_\rho C(x-y) A^c_\lambda(y) \partial_{[\mu} C(y-z) A^d_\lambda(z) A^e_{\rho]}(z) \\
&\qquad\qquad \times \Box \left(f^{amn} \partial_\sigma C(x-w) A^m_\mu(w) A^n_\sigma(w) \right) .
\end{aligned}
\tag{4.55}
$$

It is easy to simplfiy the above expression. We first perform integration by parts on each term such that the Laplacian acts on the $C(x-y)$, which gives a δ-function and we obtain

$$
\begin{aligned}
&\int \mathrm{d}x \left(A'^a_\mu(x)\big|_{\mathcal{O}(g^3)} \Box \, A'^a_\mu(x)\big|_{\mathcal{O}(g^0)} + A'^a_\mu(x)\big|_{\mathcal{O}(g^2)} \Box \, A'^a_\mu(x)\big|_{\mathcal{O}(g^1)} \right) \\
&= -\frac{g^3}{2} f^{abc} f^{bde} f^{cmn} \int \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}w \; \partial_\rho A^a_\mu(x) \\
&\qquad\qquad \times \partial_\lambda C(x-z) A^d_\lambda(z) A^e_\sigma(z) \partial_{[\rho} C(x-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
&\quad + g^3 f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}w \; \partial_\rho A^a_\mu(x) A^c_\lambda(x) \Big\{ \\
&\qquad\qquad + 2 \, \partial_{[\rho} C(x-z) A^e_{\sigma]}(z) \partial_{[\lambda} C(z-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
&\qquad\qquad - 2 \, \partial_{[\lambda} C(x-z) A^e_{\sigma]}(z) \partial_{[\rho} C(z-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
&\qquad\qquad - \partial_\sigma C(x-z) A^e_\sigma(z) \partial_{[\rho} C(z-w) A^m_\mu(w) A^n_{\lambda]}(w) \\
&\qquad\qquad - 2 \, \partial_{[\sigma} C(x-z) A^e_{\mu]}(z) \partial_{[\rho} C(z-w) A^m_\lambda(w) A^n_{\sigma]}(w) \\
&\qquad\qquad + \partial_{[\rho} C(x-z) A^e_\mu(z) \partial_{|\sigma|} C(z-w) A^m_{\lambda]}(w) A^n_\sigma(w) \Big\} \\
&\quad + \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}w \Big\{ \\
&\qquad\qquad + 2 \, \partial_\rho A^a_\mu(x) A^c_{[\rho}(x) \partial_{\mu]} C(x-z) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \\
&\qquad\qquad - \partial_\mu A^a_\mu(x) \, \partial_\rho \left(A^c_\rho(x) C(x-z) \right) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \Big\} \\
&\quad - \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&\qquad\qquad \times \Box \, A^a_\mu(x) A^c_\mu(x) C(x-y) A^e_\rho(y) \partial_\lambda C(y-z) A^m_\rho(z) A^n_\lambda(z) \\
&\quad + \frac{3g^3}{2} f^{abc} f^{bde} f^{amn} \int \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}w \\
&\qquad\qquad \times A^c_\lambda(x) \partial_{[\mu} C(x-z) A^d_\lambda(z) A^e_{\rho]}(z) \partial_\rho \partial_\sigma C(x-w) A^m_\mu(w) A^n_\sigma(w) \, .
\end{aligned}
\tag{4.56}
$$

We notice that we can replace any $\partial_\rho A^a_\mu(x) A^c_\lambda(x)$ by $\frac{1}{2}\partial_\rho\left(A^a_\mu(x)A^c_\lambda(x)\right)$ if the full expression is symmetric under simultaneous exchange $a \leftrightarrow c$ and $\mu \leftrightarrow \lambda$. This allows us to integrate the respective terms by parts again and most terms cancel. Subsequently, we obtain

$$
\begin{aligned}
&\int \mathrm{d}x \left(A'^a_\mu(x)\big|_{\mathcal{O}(g^3)} \Box A'^a_\mu(x)\big|_{\mathcal{O}(g^0)} + A'^a_\mu(x)\big|_{\mathcal{O}(g^2)} \Box A'^a_\mu(x)\big|_{\mathcal{O}(g^1)} \right) \\
&\quad = \frac{g^3}{2} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x\, \mathrm{d}w\, \partial_\rho A^a_\mu(x) A^c_\lambda(x) A^e_\sigma(x) \partial_{[\lambda} C(x-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
&\qquad + \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}w\, \Big\{ \\
&\qquad\qquad + \partial_\rho A^a_\mu(x) A^c_\rho(x) \partial_\mu C(x-z) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \\
&\qquad\qquad - \partial_\rho A^a_\mu(x) A^c_\mu(x) \partial_\rho C(x-z) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \\
&\qquad\qquad - \partial_\mu A^a_\mu(x)\, \partial_\rho \left(A^c_\rho(x) C(x-z) \right) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \Big\} \\
&\qquad - \frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \Box A^a_\mu(x) A^c_\mu(x) C(x-y) A^e_\rho(y) \partial_\lambda C(y-z) A^m_\rho(z) A^n_\lambda(z)\,.
\end{aligned}
$$

The first term vanishes by the Jacobi identity, *i.e.*

$$
\begin{aligned}
&\frac{g^3}{2} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x\, \mathrm{d}w\, \partial_\rho A^a_\mu(x) A^c_\lambda(x) A^e_\sigma(x) \partial_{[\lambda} C(x-w) A^m_\mu(w) A^n_{\sigma]}(w) \\
&\quad = \frac{g^3}{6} \left(f^{abc} f^{bde} + f^{eba} f^{bdc} + f^{cbe} f^{bda} \right) f^{dmn} \int \mathrm{d}x\, \mathrm{d}w \\
&\qquad \times \partial_\rho A^a_\mu(x) A^c_\lambda(x) A^e_\sigma(x) \partial_{[\lambda} C(x-w) A^m_\mu(w) A^n_{\sigma]}(w) \;=\; 0\,.
\end{aligned} \tag{4.57}
$$

The second, third and fourth term in (4.57) can be integrated by parts and after removing the terms that are anti-symmetric under the exchange of two indices, we get

$$
\begin{aligned}
&\int \mathrm{d}x \left(A'^a_\mu(x)\big|_{\mathcal{O}(g^3)} \Box A'^a_\mu(x)\big|_{\mathcal{O}(g^0)} + A'^a_\mu(x)\big|_{\mathcal{O}(g^2)} \Box A'^a_\mu(x)\big|_{\mathcal{O}(g^1)} \right) \\
&\quad = -\frac{g^3}{3} f^{abc} f^{bde} f^{dmn} \int \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}w\, \Big\{ \\
&\qquad\qquad + \partial_\mu \partial_\rho A^a_\mu(x) A^c_\rho(x) C(x-z) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \\
&\qquad\qquad - \Box A^a_\mu(x) A^c_\mu(x) C(x-z) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \\
&\qquad\qquad - \partial_\mu \partial_\rho A^a_\mu(x) A^c_\rho(x) C(x-z) A^e_\lambda(z) \partial_\sigma C(z-w) A^m_\lambda(w) A^n_\sigma(w) \Big\}\,.
\end{aligned} \tag{4.58}
$$

Thus, the condition (4.53) holds up $\mathcal{O}(g^3)$. Notice that the very existence of a non-local field transformation mapping one local action to another local action is a remarkable fact in itself, independently of supersymmetry but in the absence of supersymmetry, locality would be spoilt by the Jacobian.

4.4.3 Determinants consistency test

Finally, we need to perturbatively show that on the gauge surface the Jacobian determinant is equal to the product of the MSS and FP determinants. This is done order by order in g by considering the logarithms of the determinants rather than the determinants themselves; since the relevant checks up to $\mathcal{O}(g^2)$ were already performed in Chapter 3, we can here concentrate on the third order, *viz.*

$$\log\det\left(\frac{\delta A'^{a}_{\mu}(x)}{\delta A^{b}_{\nu}(y)}\right)\bigg|_{\mathcal{O}(g^3)} \stackrel{!}{=} \log\Big(\Delta_{MSS}[A]\,\Delta_{FP}[A]\Big)\bigg|_{\mathcal{O}(g^3)} . \tag{4.59}$$

Of the three statements in subsection 4.1.1 this is the most complicated condition to verify. Moreover, it is the only condition that depends on the dimension of our field theory and will impose the constraint (3.32) on the latter.

The ghost determinant is computed from the functional matrix (3.29)

$$\mathbf{X}^{ab}(x,y;A) = g\, f^{abc} C(x-y) A^c_\mu(y)\partial^y_\mu\,, \tag{4.60}$$

using the well-known equation

$$\log\det\left(1-\mathbf{X}\right) = \mathrm{Tr}\,\log\left(1-\mathbf{X}\right)\,. \tag{4.61}$$

Up to $\mathcal{O}(g^3)$ this yields

$$\begin{aligned}\log\det\left(1-\mathbf{X}\right) = \frac{1}{2}\,n\,g^2 &\int \mathrm{d}x\,\mathrm{d}y\;\partial_\mu C(x-y)A^a_\nu(y)\partial_\nu C(y-x)A^a_\mu(x)\\ &+\frac{1}{3}\,g^3\,f^{adm}f^{bem}f^{cde}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad\times\partial_\mu C(x-y)A^b_\nu(y)\partial_\nu C(y-z)A^c_\rho(z)\partial_\rho C(z-x)A^a_\mu(x)\,.\end{aligned} \tag{4.62}$$

For the MSS determinant (3.24) we have

$$\mathbf{Y}^{ab}_{\alpha\beta}(x,y;A) = g\, f^{abc}\partial_\rho C(x-y)\left(\gamma^\rho\gamma^\lambda\right)_{\alpha\beta}A^c_\lambda(y)\,. \tag{4.63}$$

We include an additional factor of $\frac{1}{2}$ to correctly account for the Majorana (or Weyl) condition and we get

$$\begin{aligned}\frac{1}{2}\log\det(1-\mathbf{Y}) = &\frac{1}{4}\, n\, g^2\, \mathrm{Tr}(\gamma^\rho\gamma^\lambda\gamma^\sigma\gamma^\alpha)\int \mathrm{d}x\, \mathrm{d}y\; \partial_\rho C(x-y) A^a_\lambda(y)\partial_\sigma C(y-x) A^a_\alpha(x)\\ &+\frac{1}{6}\, g^3\, f^{adm} f^{bem} f^{cde}\, \mathrm{Tr}(\gamma^\rho\gamma^\lambda\gamma^\sigma\gamma^\alpha\gamma^\beta\gamma^\tau)\int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\\ &\times \partial_\rho C(x-y) A^b_\lambda(y)\partial_\sigma C(y-z) A^c_\alpha(z)\partial_\beta C(z-x) A^a_\tau(x)\,.\end{aligned} \tag{4.64}$$

The fermion and ghost determinant have no contribution at $\mathcal{O}(g)$ and the results at $\mathcal{O}(g^2)$ are presented in (3.4.3) . Evaluating the trace in (4.64) and multiplying the two determinants subsequently yields the right hand side of (4.59)

$$\begin{aligned}\log\left(\Delta_{MSS}[A]\,\Delta_{FP}[A]\right)\big|_{\mathcal{O}(g^3)} = f^{adm} f^{bem} f^{cde}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\Big\{&\\ -r\,\partial_\mu C(x-y) A^b_\mu(y)\partial_\rho C(y-z) A^c_\lambda(z)\partial_\rho C(z-x) A^a_\lambda(x)&\\ +\frac{r+1}{3}\,\partial_\mu C(x-y) A^b_\rho(y)\partial_\rho C(y-z) A^c_\lambda(z)\partial_\lambda C(z-x) A^a_\mu(x)&\\ +\frac{r}{2}\,\partial_\mu C(x-y) A^b_\rho(y)\partial_\rho C(y-z) A^c_\mu(z)\partial_\lambda C(z-x) A^a_\lambda(x)&\\ -\frac{r}{6}\,\partial_\mu C(x-y) A^b_\rho(y)\partial_\lambda C(y-z) A^c_\mu(z)\partial_\rho C(z-x) A^a_\lambda(x)&\\ +\frac{r}{2}\,\partial_\mu C(x-y) A^b_\rho(y)\partial_\lambda C(y-z) A^c_\mu(z)\partial_\lambda C(z-x) A^a_\rho(x)\Big\}&\,.\end{aligned} \tag{4.65}$$

We thus end up with a total of five independent structures; we use color coding to help us identify the corresponding terms in the Jacobian determinant.

At $\mathcal{O}(g^3)$ the logarithm of the Jacobian determinant schematically consists of three terms

$$\begin{aligned}\log\det\left(\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\right)\bigg|_{\mathcal{O}(g^3)} = &\;\mathrm{Tr}\left[\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^3)}\right] - \left(2\cdot\frac{1}{2}\right)\mathrm{Tr}\left[\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^2)}\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\right]\\ &+\frac{1}{3}\,\mathrm{Tr}\left[\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\right]\,.\end{aligned} \tag{4.66}$$

and the final trace is done by setting $\mu=\nu, a=b, x=y$ and integrating over x.

The computation is straightforward and we find

$$
\begin{aligned}
&\frac{1}{3}\,\mathrm{Tr}\left[\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\bigg|_{\mathcal{O}(g^1)}\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\bigg|_{\mathcal{O}(g^1)}\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\bigg|_{\mathcal{O}(g^1)}\right] \\
&\quad= f^{adm} f^{bem} f^{cde} \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\bigg\{ \\
&\qquad\quad -\frac{3-D}{3}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\rho C(y-z)A^c_\lambda(z)\partial_\lambda C(z-x)A^a_\mu(x) \\
&\qquad\quad +\partial_\mu C(x-y)A^b_\rho(y)\partial_\rho C(y-z)A^c_\mu(z)\partial_\lambda C(z-x)A^a_\lambda(x) \\
&\qquad\quad -\frac{1}{3}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\lambda C(y-z)A^c_\mu(z)\partial_\rho C(z-x)A^a_\lambda(x)\bigg\}\,.
\end{aligned}
\tag{4.67}
$$

The second term gives

$$
\begin{aligned}
&-\left(2\cdot\frac{1}{2}\right)\mathrm{Tr}\left[\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\bigg|_{\mathcal{O}(g^2)}\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\bigg|_{\mathcal{O}(g^1)}\right] \\
&\quad= f^{adm} f^{bem} f^{cde} \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\bigg\{ \\
&\qquad\quad +\frac{1-D}{2}\,\partial_\mu C(x-y)A^b_\mu(y)\partial_\rho C(y-z)A^c_\lambda(z)\partial_\rho C(z-x)A^a_\lambda(x) \\
&\qquad\quad +\frac{1}{2}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\rho C(y-z)A^c_\lambda(z)\partial_\lambda C(z-x)A^a_\mu(x) \\
&\qquad\quad -\frac{3-D}{2}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\rho C(y-z)A^c_\mu(z)\partial_\lambda C(z-x)A^a_\lambda(x) \\
&\qquad\quad +\frac{1}{2}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\lambda C(y-z)A^c_\mu(z)\partial_\lambda C(z-x)A^a_\rho(x)\bigg\}\,.
\end{aligned}
\tag{4.68}
$$

Finally, the first term gives

$$
\begin{aligned}
&\mathrm{Tr}\left[\frac{\delta A'^a_\mu(x)}{\delta A^b_\nu(y)}\bigg|_{\mathcal{O}(g^3)}\right] \\
&\quad= f^{adm} f^{bem} f^{cde} \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\bigg\{ \\
&\qquad\quad +\frac{7-3D}{2}\,\partial_\mu C(x-y)A^b_\mu(y)\partial_\rho C(y-z)A^c_\lambda(z)\partial_\rho C(z-x)A^a_\lambda(x) \\
&\qquad\quad -\frac{3-2D}{6}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\rho C(y-z)A^c_\lambda(z)\partial_\lambda C(z-x)A^a_\mu(x) \\
&\qquad\quad -\frac{3-D}{2}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\rho C(y-z)A^c_\mu(z)\partial_\lambda C(z-x)A^a_\lambda(x) \\
&\qquad\quad +\frac{3-D}{3}\,\partial_\mu C(x-y)A^b_\rho(y)\partial_\lambda C(y-z)A^c_\mu(z)\partial_\rho C(z-x)A^a_\lambda(x)
\end{aligned}
\tag{4.69}
$$

$$
\begin{aligned}
&- \frac{5-2D}{2}\, \partial_\mu C(x-y) A^b_\rho(y) \partial_\lambda C(y-z) A^c_\mu(z) \partial_\lambda C(z-x) A^a_\rho(x) \Big\} \\
&- \frac{2}{3}\, f^{aem} f^{bde} f^{cdm} \int \mathrm{d}x\ \mathrm{d}y\ A^b_\mu(x) A^c_\rho(x) C(x-y) \partial_\rho C(x-y) A^a_\mu(y) \\
&+ \frac{1}{3}\, f^{adm} f^{bce} f^{dem} \int \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\ A^a_\mu(x) \left(\partial_\rho C(x-y)\right)^2 \partial_\lambda C(y-z) A^b_\lambda(z) A^c_\mu(z) \\
&- \frac{1}{3}\, f^{adm} f^{bce} f^{dem} \int \mathrm{d}x\ \mathrm{d}y\ C(0) A^a_\mu(x) \partial_\rho C(x-y) A^b_\rho(y) A^c_\mu(y)\,.
\end{aligned}
\tag{4.70}
$$

There are two special features about this part of the Jacobian determinant. First, we have to use the *gauge condition* $G^a[A] \equiv \partial^\mu A^a_\mu = 0$ to eliminate two terms. Secondly, we find terms that do not match any of the five structures from the fermion and ghost determinants and, hence, must cancel among themselves. However, before addressing those terms, let us first analyze the color-coded terms. Imposing the equality (4.59) yields the following conditions

$$
\begin{aligned}
-r &= \frac{1-D}{2} + \frac{7-3D}{2} = 4-2D \\
\frac{r+1}{3} &= -\frac{3-D}{3} + \frac{1}{2} - \frac{3-2D}{6} = \frac{2D-3}{3} \\
\frac{r}{2} &= 1 - \frac{3-D}{2} - \frac{3-D}{2} = D-2 \\
-\frac{r}{6} &= -\frac{1}{3} + \frac{3-D}{3} = \frac{2-D}{3} \\
\frac{r}{2} &= \frac{1}{2} - \frac{5-2D}{2} = D-2\,.
\end{aligned}
\tag{4.71}
$$

Happily, all five equations are satisfied with $r = 2(D-2)$, so we recover the result (5.16)

$$
D = 3, 4, 6, 10 \qquad \Longleftrightarrow \qquad r = 2, 4, 8, 16\,, \tag{4.72}
$$

thus extending the result of (3) to cubic order. It remains to be shown that the remaining (black) terms from (4.70) vanish. Using the Jacobi identity in the first term and $f^{abc} f^{abd} = n\, \delta^{cd}$ in the latter two yields

$$
\begin{aligned}
&- \frac{n}{3} f^{abc} \int \mathrm{d}x\ \mathrm{d}y\ A^b_\mu(x) A^c_\rho(x) C(x-y) \partial_\rho C(x-y) A^a_\mu(y) \\
&+ \frac{n}{3} f^{abc} \int \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\ A^a_\mu(x) \left(\partial_\rho C(x-y)\right)^2 \partial_\lambda C(y-z) A^b_\lambda(z) A^c_\mu(z) \\
&- \frac{n}{3} f^{abc} \int \mathrm{d}x\ \mathrm{d}y\ C(0) A^a_\mu(x) \partial_\rho C(x-y) A^b_\rho(y) A^c_\mu(y)\,.
\end{aligned}
\tag{4.73}
$$

The second term is rewritten using the identity

$$\Box\left(C^2(x-y)\right) \;=\; -2\,C(0)\delta(x-y) \;+\; 2\,\partial_\rho C(x-y)\partial_\rho C(x-y)\,, \tag{4.74}$$

with a formally divergent piece $C(0)$ which can be appropriately regulated. This simplifies the expression above to

$$\begin{aligned}
&-\frac{n}{3}\,f^{abc}\int \mathrm{d}x\;\mathrm{d}y\;A^b_\mu(x)A^c_\rho(x)C(x-y)\partial_\rho C(x-y)A^a_\mu(y)\\
&+\frac{n}{3}\,f^{abc}\int \mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z\;A^a_\mu(x)C(0)\delta(x-y)\partial_\rho C(y-z)A^b_\rho(z)A^c_\mu(z)\\
&+\frac{n}{6}\,f^{abc}\int \mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z\;A^a_\mu(x)\,\Box\,\left(C^2(x-y)\right)\partial_\rho C(y-z)A^b_\rho(z)A^c_\mu(z)\\
&-\frac{n}{3}\,f^{abc}\int \mathrm{d}x\;\mathrm{d}y\;C(0)A^a_\mu(x)\partial_\rho C(x-y)A^b_\rho(y)A^c_\mu(y)\,.
\end{aligned} \tag{4.75}$$

Integrating by parts the above terms and they cancels in pairs.

Thus, (4.59) is satisfied. Note that we had to make use of the Landau gauge condition (2.26) to achieve this equality. The map at order g^2 did not require the use of the gauge condition for matching determinants (2.26). This feature, which arises only from $\mathcal{O}(g^3)$ onwards, is entirely due to the g-dependence of the ghost propagator in (4.27). The dependence of the map and the $\mathcal{R}$ operator on the gauge condition is an artifact of working with on-shell supersymmetry. It was shown in [31] that while working with off-shell supersymmetry with rescaled fields, the map and the prescription can be derived for arbitrary gauges.

In this chapter, we extended the map and the framework to the third order in the coupling constant. We found that the number of terms in the map increases exponentially for higher order in the coupling constant. It was shown in [27] that the correlation functions can be obtained in this formalism in terms of free bosonic expectation value using the inverse Nicolai map. A natural and interesting direction in this context would be to establish a direct connection between the map and amplitudes. While working on the order g^3 map, we discovered a different map that satisfies all conditions of the main theorem in six dimensions. We will discuss the construction of this map and its properties in the next chapter.

Chapter 5

An alternate six dimensional map and uniqueness

The material presented in this chapter is mostly based on the author's publication [29].

We focus here on a stand-alone result—a new map, also to third order in the coupling constant, but works only in six dimensions. The map was derived by trial and error by starting with an educated guess. This map is simpler than the one in [28] and highlights a potential ambiguity in the $\mathcal{R}$ operator formalism and was also discussed in [48, 49]. We also examine the uniqueness of this approach and comment on how the maps can be related to each other. At the end, we outline an algorithmic approach to determine the map recursively.

5.1 Introduction

We present here a new and *simple* expression for the map at order g^3. We discovered this simpler map while working on the calculations of the third-order map presented in (4). We started with an educated guess and arrived at the correct map by demanding that the ansatz satisfies all three statements of the main theorem *only* in $D = 6$.

We work here in Euclidean space using the Landau gauge[1].

$$G^a[A_\mu] = \partial^\mu A^a_\mu \,. \tag{5.1}$$

The results presented below may be adapted to other gauges (the light-cone gauge being of particular interest given potential links to [50–52]). The free scalar propagator is ($\Box \equiv \partial^\mu \partial_\mu$)

$$C(x) = \int \frac{d^Dk}{(2\pi)^D} \frac{e^{ikx}}{k^2} \quad \Rightarrow \quad -\Box C(x) = \delta(x) \,. \tag{5.2}$$

[1]The gauge surface restriction will prove unnecessary for this particular map

The free fermion propagator is (spinor indices suppressed)

$$\gamma^\mu \partial_\mu S_0(x) = \delta(x) \quad \Rightarrow \quad S_0(x) = -\gamma^\mu \partial_\mu C(x)\,, \tag{5.3}$$

$S_0(x-y) = -S_0(y-x)$. In a gauge-field dependent background

$$\gamma^\mu (D_\mu S)^{ab}(x) \equiv \gamma^\mu \Big[\delta^{ac}\partial_\mu - g f^{acd} A^d_\mu(x)\Big] S^{cb}(x) = \delta^{ab}\delta(x)\,. \tag{5.4}$$

To avoid redundancy, we do not state the main theorem again in this chapter. We will refer to the statements of the main theorem (4.1.1) from the last chapter.

5.2 A simple six-dimensional map

The new result in this chapter is the following explicit expression for $\mathcal{T}_g$ to $\mathcal{O}(g^3)$.

$$\begin{aligned}
(\mathcal{T}_g A)^a_\mu(x) \;=\; & A^a_\mu(x) + g\, f^{abc} \int dy\, \partial_\lambda\, C(x-y)\, A^b_\mu(y)\, A^c_\lambda(y) \\
& + \frac{3}{2} g^2 f^{abc} f^{bde} \int dy\, dz\, \partial_\rho\, C(x-y)\, A^c_\sigma(y)\, \partial_{[\rho}\, C(y-z)\, A^d_\mu(z)\, A^e_{\sigma]}(z) \\
& + \frac{3}{2} g^3 f^{abc} f^{bde} f^{dmn} \int dy\, dz\, dw\, \partial_\rho\, C(x-y)\, A^c_\lambda(y) \Big\{ \\
& \qquad + \partial_\lambda\, C(y-z)\, A^e_\sigma(z)\, \partial_{[\mu}\, C(z-w)\, A^m_\rho(w)\, A^n_{\sigma]}(w) \\
& \qquad + \partial_\mu\, C(y-z)\, A^e_\sigma(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w) \\
& \qquad + \partial_\rho\, C(y-z)\, A^e_\sigma(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w) \Big\} \\
& - g^3 f^{abc} f^{bde} f^{dmn} \int dy\, dz\, dw\, \partial_\rho\, C(x-y)\, A^c_\lambda(y) \Big\{ \\
& \qquad + \partial_\sigma\, C(y-z)\, A^e_\sigma(z)\, \partial_{[\mu}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\rho]}(w) \\
& \qquad + \partial_\sigma\, C(y-z)\, A^e_\rho(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w) \\
& \qquad + \partial_\sigma\, C(y-z)\, A^e_\mu(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w) \\
& \qquad + \partial_\sigma\, C(y-z)\, A^e_\lambda(z)\, \partial_{[\mu}\, C(z-w)\, A^m_\rho(w)\, A^n_{\sigma]}(w) \Big\}\,,
\end{aligned} \tag{5.5}$$

where $[\mu\nu\rho] = \frac{1}{6}[\mu\nu\rho - \mu\rho\nu + \nu\rho\mu - \nu\mu\rho + \rho\mu\nu - \rho\nu\mu]$.

It is important to note that this result differs from the six-dimensional map (4.51) derived using rigorous $\mathcal{R}$ prescription. The maps (4.51) and (5.5) agrees up to order g^2. All terms above have the base structure $\partial CA\ \partial CA\ \partial CAA$ at $\mathcal{O}(g^3)$, while the map in (4.51) also includes the structures $\partial C\ \partial CAA\ \partial CAA$, $A\ CA\ \partial CAA$ and $\partial C\ \partial(AC)\ A\,\partial CAA$.

Further, terms that overlap with those in (4.51), appear here with different coefficients. As a consequence, the expression above is *not a subset* of the result in (4).

Finally, while the result in Chapter 4 was valid in all the critical dimensions, we will see that the result (5.5) constitutes a map **only in six dimensions**.

5.3 Consistency checks of the map

In this section, we prove that expression in (5.5) satisfies all three requirements, (4.7), (4.8) and (4.9), necessary for it to be a map. The calculations up to $\mathcal{O}(g^2)$ are identical to those presented in Chapter 3, so the focus here will be on $\mathcal{O}(g^3)$.

5.3.1 Gauge condition

We begin with the third requirement, listed in (4.9). We need to show that $\partial_\mu A'^a_\mu(x) = \partial_\mu A^a_\mu(x) + \mathcal{O}(g^4)$. We apply ∂_μ to the terms of order g^3 in (5.5). This gives us a symmetric $\partial_\mu\,\partial_\rho$ at the beginning of the expression, so we eliminate all terms that are anti-symmetric under the exchange $\mu \leftrightarrow \rho$ and find

$$
\begin{aligned}
\partial_\mu A'^a_\mu(x)\big|_{\mathcal{O}(g^3)} = \frac{3}{2}\, g^3\, f^{abc}\, f^{bde}\, f^{dmn} \int dy\, dz\, dw\, \partial_\mu\partial_\rho\, C(x-y)\, A^c_\lambda(y) \Big\{ \\
+\, \partial_\mu\, C(y-z)\, A^e_\sigma(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w) \\
+\, \partial_\rho\, C(y-z)\, A^e_\sigma(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w) \Big\} \\
-\, g^3\, f^{abc}\, f^{bde}\, f^{dmn} \int dy\, dz\, dw\, \partial_\mu\partial_\rho\, C(x-y)\, A^c_\lambda(y) \Big\{ \\
+\, \partial_\sigma\, C(y-z)\, A^e_\rho(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w) \\
+\, \partial_\sigma\, C(y-z)\, A^e_\mu(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w) \Big\}\,.
\end{aligned} \tag{5.6}
$$

The first two terms cancel each other under the exchange of μ and ρ. Similarly, the other two terms also cancel out confirming that $\partial_\mu A'^a_\mu(x) = \partial_\mu A^a_\mu(x) + \mathcal{O}(g^4)\,.$

5.3.2 Free Action to full action

We now move to the first requirement in (4.7) which states that the transformed gauge field must satisfy

$$\frac{1}{2}\int dx\, A'^{a}_{\mu}(x)\,(-\Box\,\delta_{\mu\nu}+\partial_\mu\partial_\nu)\,A'^{a}_{\nu}(x) \;=\; \frac{1}{4}\int dx\, F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) \;+\; \mathcal{O}(g^4)\,. \tag{5.7}$$

Because of the invariance of the gauge function, we ignore the second term on the l.h.s. and the corresponding term on the r.h.s. of this equation [28]. At third order, (5.7) has two contributions

$$0 \overset{!}{=} \int dx\Big(A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^3)}\,\Box\, A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^0)} + A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^2)}\,\Box\, A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^1)}\Big)\,. \tag{5.8}$$

This expression reads

$$\begin{aligned}
&\int dx\,\Big(A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^3)}\,\Box\, A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^0)} + A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^2)}\,\Box\, A'^{a}_{\mu}(x)\big|_{\mathcal{O}(g^1)}\Big) \\[2ex]
&\quad = \frac{3}{2}\,g^3\, f^{abc}\, f^{bde}\, f^{dmn} \int dx\,dy\,dz\,dw\,\partial_\rho\, C(x-y)\, A^c_\lambda(y)\,\Big\{ \\
&\qquad\qquad + \partial_\lambda\, C(y-z)\, A^e_\sigma(z)\,\partial_{[\mu}\, C(z-w)\, A^m_\rho(w)\, A^n_{\sigma]}(w)\Box\, A^a_\mu(x) \\
&\qquad\qquad + \partial_\mu\, C(y-z)\, A^e_\sigma(z)\,\partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w)\Box\, A^a_\mu(x) \\
&\qquad\qquad + \partial_\rho\, C(y-z)\, A^e_\sigma(z)\,\partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w)\Box\, A^a_\mu(x)\Big\} \\
&\qquad - g^3\, f^{abc}\, f^{bde}\, f^{dmn} \int dx\,dy\,dz\,dw\,\partial_\rho\, C(x-y)\, A^c_\lambda(y)\,\Big\{ \\
&\qquad\qquad + \partial_\sigma\, C(y-z)\, A^e_\sigma(z)\,\partial_{[\mu}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\rho]}(w)\Box\, A^a_\mu(x) \\
&\qquad\qquad + \partial_\sigma\, C(y-z)\, A^e_\rho(z)\,\partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w)\Box\, A^a_\mu(x) \\
&\qquad\qquad + \partial_\sigma\, C(y-z)\, A^e_\mu(z)\,\partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w)\Box\, A^a_\mu(x) \\
&\qquad\qquad + \partial_\sigma\, C(y-z)\, A^e_\lambda(z)\,\partial_{[\mu}\, C(z-w)\, A^m_\rho(w)\, A^n_{\sigma]}(w)\Box\, A^a_\mu(x)\Big\} \\
&\qquad + \frac{3}{2}\,g^3\, f^{abc}\, f^{bde} \int dx\,dy\,dz\,dw\;\partial_\rho\, C(x-y)\, A^c_\lambda(y)\;\partial_{[\rho}\, C(y-z)\, A^d_\mu(z)\, A^e_{\lambda]}(z) \\
&\qquad\qquad\quad \times\,\Box\,\left(f^{amn}\partial_\sigma C(x-w)A^m_\mu(w)A^n_\sigma(w)\right)\,.
\end{aligned}$$

We simplify the r.h.s. to obtain

$$
\begin{aligned}
&= \frac{3}{2}\, g^3\, f^{abc}\, f^{bde}\, f^{dmn} \int dx\, dz\, dw\, \partial_\rho\, A^a_\mu(x)\, A^c_\lambda(x) \Big\{ \\
&\qquad + \partial_\lambda\, C(x-z)\, A^e_\sigma(z)\, \partial_{[\mu}\, C(z-w)\, A^m_\rho(w)\, A^n_{\sigma]}(w) \\
&\qquad + \partial_\mu\, C(x-z)\, A^e_\sigma(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w) \\
&\qquad + \partial_\rho\, C(x-z)\, A^e_\sigma(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w) \Big\} \\
&\quad - g^3\, f^{abc}\, f^{bde}\, f^{dmn} \int dx\, dz\, dw\, \partial_\rho A^a_\mu(x)\, A^c_\lambda(x) \Big\{ \\
&\qquad + \partial_\sigma\, C(x-z)\, A^e_\sigma(z)\, \partial_{[\mu}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\rho]}(w) \\
&\qquad + \partial_\sigma\, C(x-z)\, A^e_\rho(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\lambda(w)\, A^n_{\mu]}(w) \\
&\qquad + \partial_\sigma\, C(x-z)\, A^e_\mu(z)\, \partial_{[\sigma}\, C(z-w)\, A^m_\rho(w)\, A^n_{\lambda]}(w) \\
&\qquad + \partial_\sigma\, C(x-z)\, A^e_\lambda(z)\, \partial_{[\mu}\, C(z-w)\, A^m_\rho(w)\, A^n_{\sigma]}(w) \Big\} \\
&\quad + \frac{3}{2}\, g^3\, f^{abc}\, f^{bde} f^{amn} \int dx\, dz\, dw \\
&\qquad A^c_\lambda(x)\ \partial_{[\rho}\, C(x-z)\, A^d_\mu(z)\, A^e_{\lambda]}(z) \partial_\rho \partial_\sigma C(x-w) A^m_\mu(w) A^n_\sigma(w)\,.
\end{aligned}
$$

This can be simplified with some re-writing [for example, $\partial_\rho A^a_\mu(x) A^c_\lambda(x) \to \frac{1}{2}\partial_\rho \left(A^a_\mu(x) A^c_\lambda(x)\right)$ based on the symmetries $a \leftrightarrow c$ and $\mu \leftrightarrow \lambda$]. The r.h.s. simplifies to

$$
= \frac{3}{4}\, g^3 f^{abc} f^{bde} f^{dmn} A^a_\mu(x) A^c_\lambda(x) A^e_\sigma(x) \partial_{[\,\sigma} C(x-w) A^m_\lambda(w) A^n_{\mu\,]}(w)\,. \tag{5.9}
$$

There is a symmetry to these terms: the ∂CAA blocks are invariant under a cyclic permutation of the Lorentz indices. This motivates re-writing the term as

$$
\begin{aligned}
&\tfrac{1}{4}\, g^3 f^{abc} f^{bde} f^{dmn} \left[A^a_\mu(x) A^c_\lambda(x) A^e_\sigma(x) + A^a_\sigma(x) A^c_\mu(x) A^e_\lambda(x) + A^a_\lambda(x) A^c_\sigma(x) A^e_\mu(x) \right] \\
&\qquad \times \partial_{[\,\sigma} C(x-w) A^m_\lambda(w) A^n_{\mu\,]}(w) \\
&= \tfrac{1}{4}\, g^3 \left[f^{abc} f^{bde} + f^{eba} f^{bdc} + f^{cbe} f^{bda} \right] f^{dmn} A^a_\mu(x) A^c_\lambda(x) A^e_\sigma(x) \\
&\qquad \times \partial_{[\sigma} C(x-w) A^m_\lambda(w) A^n_{\mu]}(w)\,.
\end{aligned} \tag{5.10}
$$

The above term vanishes by invoking the Jacobi identity. Note here that for the first time in this computation, we required Jacobi identity for some simplification.

$$f^{abc} f^{bde} + f^{eba} f^{bdc} + f^{cbe} f^{bda} = 0\,. \tag{5.11}$$

Thus (5.7) holds up to $\mathcal{O}(g^3)$.

5.3.3 Jacobians, fermion and ghost determinants

We now turn to (4.8), the second condition of the main theorem. As discussed before, this is the most constraining of the three requirements, demanding that the bosonic Jacobian determinant equal the product of the MSS and FP determinants. Again, this check up to $\mathcal{O}(g^2)$ was performed in Chapter 3 allowing us to concentrate here on $\mathcal{O}(g^3)$.

$$\log\det\left(\frac{\delta A'^{a}_{\mu}(x)}{\delta A^{b}_{\nu}(y)}\right)\Bigg|_{\mathcal{O}(g^3)} \stackrel{!}{=} \log\left(\Delta_{MSS}[A]\,\Delta_{FP}[A]\right)\Bigg|_{\mathcal{O}(g^3)}. \tag{5.12}$$

It is this non-trivial requirement which results in a dimensional dependence. We prove that the map in (5.5) satisfies (5.12) only for $D = 6$.

Fermion determinant

The fermion determinant was computed in detail in (4). The result obtained in (4) holds true for our case. We therefore just state the result (4.64) here

$$\begin{aligned} g^3\, f^{abm}\, f^{bcn}\, f^{cap} \int dx\, dy\, dz\, \Big\{ \\ &- r\, \partial_\rho C(x-y)\, A^m_\rho(y)\, \partial_\lambda C(y-z)\, A^n_\sigma(z)\, \partial_\lambda C(z-x)\, A^p_\sigma(x) \\ &+ \frac{r}{3}\, \partial_\rho C(x-y)\, A^m_\lambda(y)\, \partial_\lambda C(y-z)\, A^n_\sigma(z)\, \partial_\sigma C(z-x)\, A^p_\rho(x) \\ &+ \frac{r}{2}\, \partial_\rho C(x-y)\, A^m_\lambda(y)\, \partial_\lambda C(y-z)\, A^n_\rho(z)\, \partial_\sigma C(z-x)\, A^p_\sigma(x) \\ &- \frac{r}{6}\, \partial_\rho C(x-y)\, A^m_\lambda(y)\, \partial_\sigma C(y-z)\, A^n_\rho(z)\, \partial_\lambda C(z-x)\, A^p_\sigma(x) \\ &+ \frac{r}{2}\, \partial_\rho C(x-y)\, A^m_\lambda(y)\, \partial_\sigma C(y-z)\, A^n_\rho(z)\, \partial_\sigma C(z-x)\, A^p_\lambda(x) \Big\}, \end{aligned} \tag{5.13}$$

where r represents the number of spinor components.

Ghost determinant

The same holds true for the ghost determinant computation also. The deteminant reads (4.62)

$$
\begin{aligned}
&+\frac{1}{3}\, g^3\, f^{abm}\, f^{bcn}\, f^{cap} \int dx\, dy\, dz \\
&\qquad \partial_\rho\, C(x-y)\, A^m_\rho(y)\, \partial_\lambda\, C(y-z)\, A^n_\lambda(z)\, \partial_\sigma\, C(z-x)\, A^p_\sigma(x)\,. \qquad (5.14)
\end{aligned}
$$

Bosonic Jacobian

The map (5.5) differs from the six-dimensional map (4.51). We, therefore, explicitly compute the Jacobian determinant.

At $\mathcal{O}(g^3)$, the logarithm of the Jacobian determinant schematically consists of three terms

$$
\begin{aligned}
\log\det\left(\frac{\delta A'^{a}_{\mu}(x)}{\delta A^b_\nu(y)}\right)\Bigg|_{\mathcal{O}(g^3)} = \operatorname{Tr}\left[\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^3)}\right] - \left(2\cdot\frac{1}{2}\right)\operatorname{Tr}\left[\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^2)}\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\right] \\
+\frac{1}{3}\operatorname{Tr}\left[\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\frac{\delta A'}{\delta A}\bigg|_{\mathcal{O}(g^1)}\right], \qquad (5.15)
\end{aligned}
$$

and the final trace involves setting $\mu=\nu, a=b, x=y$ and integrating over x.

All terms at $\mathcal{O}(g^3)$ are of the form $\partial CA\,\partial CA\,\partial CAA$. The functional derivative on the very first field, in this structure, vanishes trivially (3.4.3). The functional differentiation of the field in the middle block produces the structure $\partial CA\,\partial C\,\partial CAA$ not seen elsewhere. These terms also vanish by applying integration by parts and exploiting the structural properties.

Therefore, the only non-trivial contribution to the Jacobian comes from functional differentiation of either field from the last block, and it is of the same structure as those from the fermion and ghost contributions. The table below offers a summary of the various contributions to the Jacobian from (5.15).

Jacobian table

In the table, columns $2-5$ capture bosonic contributions, summed up in column 6. Column 7 contains the sums of the fermion and ghost contributions. The detailed breakdown for the bosonic contributions is as follows: Column 2 contains the contributions from $\mathcal{O}(g)$ terms when "cubed". Column 3 lists contributions from $\mathcal{O}(g) \times \mathcal{O}(g^2)$. Column 4 has contributions from the 9 terms in the bosonic result (first three lines of $\mathcal{O}(g^3)$ from (5.5)). In column 5, we present contributions from the next four lines of (5.5) (12 terms).

Group	$(g)^3$	$(g) \times (g^2)$	9 Terms	12 Terms	Boson	MSS+FP
1	0	$\frac{1-D}{2}$	$\frac{5-2D}{2}$	$\frac{2}{3}(3-D)$	$\frac{30-13D}{6}$	$-r$
2	$\frac{D-3}{3}$	$\frac{1}{2}$	$\frac{D-3}{2}$	0	$\frac{5D-12}{6}$	$\frac{r+1}{3}$
3	1	$\frac{D-3}{2}$	$\frac{1}{2}$	$\frac{D-3}{3}$	$\frac{5D-6}{6}$	$\frac{r}{2}$
4	$-\frac{1}{3}$	0	0	$\frac{3-D}{3}$	$\frac{2-D}{3}$	$-\frac{r}{6}$
5	0	$\frac{1}{2}$	$\frac{D-3}{2}$	$\frac{2D-6}{3}$	$\frac{7D-18}{6}$	$\frac{r}{2}$

In column 7, we now set (3.32)

$$r = 2(D-2)\,. \tag{5.16}$$

The *main result* is that Columns 6 and 7 are equal *only for* $D = 6$.

This completes our proof of (4.7), (4.8) and (4.9). It is curious that we have not had to invoke the gauge condition, which was needed in Chapter 4, in this proof.

5.4 Uniqueness of the map

In this chapter, we discovered a different six-dimensional map (5.5) which is simpler than one obtained in Chapter 4. The simpler map has 21 terms as compared to 34 terms in (4.51). These two maps cannot be related at the level of the transformation by performing any partial integration. We notice that the product of the fermion and ghost determinant for both the maps is the same, and they both satisfy the determinant matching condition. Therefore, the extra contribution comes only from the Jacobian side.

The form of the guessed map usually depends on how the Lagrangian is written. We know that we can always rewrite the Lagrangian with some partial integrations, so the guess for the map will change accordingly. The statement (4.9) of the main theorem is the non-trivial requirement for any field transformations to be the Nicolai map. This condition pertains to the derivative of the map, not the map itself, which is why non-uniqueness arises at the level of the map.

We found out that if we add terms proportional to $D-6$ to the Jacobian determinant (columns 4 and 5) of the map (5.5), we obtain the Jacobian determinant (4.70) (colored terms) of the map (4.51). Therefore, the two maps (5.5) and (4.51) can be related to each other at the level of the Jacobian determinant. We discovered this equality while working on the light-cone Nicolai map presented in Chapter 6.

5.5 A potential algorithm to generate the map to third order and beyond

In this section, we outline an algorithmic approach to determining the map T_g. This involves perturbatively generating higher-order expressions in a manner reminiscent of that in [19]. However, the approach presented here comes with the potential advantage of leading to the map directly instead of generating the inverse map T_g^{-1}.

As mentioned already below equation (5.5), the structure of the map presented in this paper is simpler than that in (4). The entire map in (5.5), at order g^3, involves a single structure. We present below an algorithm that generates exactly this structure suggesting a simple all-order generalization of our results.

We start by noting that the “base" structure - the order g result - has the form: ∂CAA. Our claim is that there exists a realization of the map T_g, to all orders, generated entirely by linking a series of ∂CA factors to this base structure.

We illustrate this first at order g^2. The map, at this order, would necessarily involve one ∂CA block in addition to the base structure.

$$\mathcal{T}(g^2) \sim g^2 \ \partial CA \ \partial CAA \, .$$

We are now guided by the following algorithm.

- Sprinkle Lorentz indices on the base ∂CAA block, such that the indices are all distinct. A set of three terms having the same “external" structure but with the three indices on the base-block permuted cyclically constitute a “triplet".
- Choose the two Lorentz indices on the first “block" to be different, for example we can choose them to be ρ and λ respectively.
- Discard all terms with μ on the ∂ of the first block (Note that $A_\mu \Box$ acting on such terms, from the left, would trivially vanish).

Focussing on order g^2

$$\mathcal{T}(g^2) \sim g^2 \ \partial CA \ \partial CAA \, , \tag{5.17}$$

We assign Lorentz indices to the ∂CAA block. At this level, we have three sets of indices: ρ and λ (summed over) and μ (the free index). There is a single "triplet" configuration at this level, with μ, ρ, and λ all distributed on the final block. Therefore, the algorithm yields three terms in the map T_g:

$$T_g : g^2 \ \partial_\rho C A_\lambda \ \partial_{[\rho} C A_\lambda A_{\mu]} \, . \tag{5.18}$$

Moving to order g^3, our procedure asks that we add two ∂CA structures to the base structure.

So we have

$$O(g^3) = g^3 \;\; \partial C A \;\; \partial C A \;\; \partial C A A \, . \tag{5.19}$$

We again distribute Lorentz indices on the ∂CAA block. At this order, we have four sets of indices to work with: ρ, σ, λ all summed over and μ which is free. There are 4 ways of selecting 3 different indices (triplets) from the available set. Without loss of generality, we choose the Lorentz indices on the first block to be ρ and λ respectively. This leaves us with two indices and two slots, which is two arrangements for each triplet, except for one, where we have the same index (σ in this convention), and hence only one arrangement. This gives us seven triplets, or 21 terms at order g^3, and the map

$$\begin{aligned} T_g : g^3 \quad & \partial_\rho C A_\lambda \partial_\lambda C A_\sigma \partial_{[\mu} C A_\rho A_{\sigma]} \\ & \partial_\rho C A_\lambda \partial_\sigma C A_\lambda \partial_{[\mu} C A_\rho A_{\sigma]} \\ & \partial_\rho C A_\lambda \partial_\mu C A_\sigma \partial_{[\sigma} C A_\rho A_{\lambda]} \\ & \partial_\rho C A_\lambda \partial_\sigma C A_\mu \partial_{[\sigma} C A_\rho A_{\lambda]} \\ & \partial_\rho C A_\lambda \partial_\rho C A_\sigma \partial_{[\sigma} C A_\lambda A_{\mu]} \\ & \partial_\rho C A_\lambda \partial_\sigma C A_\rho \partial_{[\sigma} C A_\lambda A_{\mu]} \\ & \partial_\rho C A_\lambda \partial_\sigma C A_\sigma \partial_{[\mu} C A_\lambda A_{\rho]} \, , \end{aligned} \tag{5.20}$$

exactly matching the structures that appear in (5.5). We note that while this algorithm does not determine the overall constants, it does generate the terms in sets that conveniently satisfy the gauge constraint. It is fairly straightforward to write down the structures expected at order g^4 although performing the relevant checks (particularly of the determinants) is technically more involved.

* * *

We conclude that (5.5) represents an alternate Nicolai map [28] in six dimensions, up to $\mathcal{O}(g^3)$, distinct from the map in (4). This raises the possibility that a dimension-dependent map exists that differs for each critical dimension. However, we note that the checks to this order for this particular map do not guarantee its validity at higher orders[2]. The result in (4) is different because it is derived using the $\mathcal{R}$-prescription and is limited to $\mathcal{O}(g^3)$ only because the procedure becomes technically involved at higher orders. However, this prescription can be used to derive maps for all orders, which in the usual sense will not have any non-uniqueness. Ambiguities in constructing Nicolai maps have been previously flagged [28] and studied more recently in [48, 49].

The non-uniqueness of the map, its dependence on the gauge condition, and the complicated structure at higher points naturally raise the question of whether there are better variables that can help us avoid these issues. We believe that light-cone variables might provide a better understanding of these issues. In the next chapter, we will study the characterization of supersymmetric theories without fermions in the light-cone gauge.

[2]If this map survives to higher orders, the gauge condition may become necessary, in keeping with Chapter 4.

Chapter 6

Nicolai maps and quadratic forms in the light-cone gauge

The material presented in this chapter is mostly based on author's publications [30, 52].

This chapter discusses the formulation of supersymmetric Yang-Mills theories without fermions in the light-cone gauge. We compute the map, in terms of the physical degrees of freedom, to second order in the coupling constant and then extend it to all critical dimensions. We demonstrate the existence of two maps in four dimensions and address the uniqueness of these maps in this approach. Using the simpler map, we compute the scattering amplitudes in terms of the free bosonic correlator. The Hamiltonians for the pure and maximally supersymmetric theories in the light-cone gauge in four dimensions can be written as quadratic forms. We briefly review the construction of quadratic forms for arbitrary spin theories and comment on the possible connections between the Nicolai map and quadratic form structures found in the light-cone Hamiltonians of Yang-Mills theory.

The chapter is organized as follows: In section (6.1), we review the formulation of the $\mathcal{N} = 1$ super Yang-Mills theory in the light cone gauge. In section (6.2), we construct the map in $D = 4$, in the physical degrees of freedom, to second order in the coupling constant. In section (6.3), we generalize the map to arbitrary D and establish its connection with the already found map in general gauges [31]. In section (6.4), we comment on the uniqueness of the light-cone Nicolai map in $D = 4$. In section (6.5), we compute three-point and four-point tree-level gluon scattering amplitude (correlation function) using this approach. In the last section (6.6), we show the construction of quadratic forms for arbitrary spin theories and discuss the possible connection between the Nicolai map in the light-cone gauge and mathematical structures like quadratic form that one finds in the light-cone Hamiltonian.

Note: This chapter contains some overlapping results with [53].

6.1 Notations and conventions

Take a gauge theory with bosonic field (A_μ) and fermionic field (ψ) in the adjoint representation. Here, we work in the light-cone gauge in D dimensional space-time. The Nicolai map formalism, in particular the matching of fermion and Jacobian, will fix the allowed dimensions where the supersymmetric Yang-Mills theories can exist. These dimensions are also known as the 'critical dimensions' [28].

The light-cone coordinates were introduced in chapter 2. In this chapter, we will stay in Minkowski space-time and the light cone coordinates are

$$x^{\pm} = \frac{(x^0 \, \pm \, x^{D-1})}{\sqrt{2}} \,. \tag{6.1}$$

The transverse coordinates are given by x_i where $i = 1, ...D-2$. The derivatives with respect to the light-cone coordinates are denoted by $\partial_{\pm}$ $(-\partial^{\mp})$ while those with respect to the transverse coordinates are denoted by ∂_i.

Gamma matrices satisfy $\{\gamma^\mu \, , \, \gamma^\nu\} = -2\eta^{\mu\nu}$ where $\eta_{\mu\nu}$ is the light-cone metric. The $\gamma^{\pm}$ are defined as

$$\gamma^{\pm} = \frac{1}{\sqrt{2}} \left(\gamma^0 \pm \gamma^{D-1}\right) \,. \tag{6.2}$$

They satisfy

$$\gamma^{\pm\,2} = 0 \,, \qquad \gamma^{+\,\dagger} = \gamma^- \,, \qquad \left\{\gamma^{\pm} \, , \, \gamma^i\right\} = 0 \,, \qquad \left\{\gamma^+ , \gamma^-\right\} = 2 \,. \tag{6.3}$$

The gamma matrices satisfy the following trace identity

$$\begin{aligned} \mathrm{Tr}\left(\gamma^\mu \, \gamma^\nu\right) &= -r \, \eta^{\mu\nu} \,, \\ \mathrm{Tr}\left(\gamma^\mu \, \gamma^\nu \, \gamma^\rho \, \gamma^\sigma\right) &= -r(\eta^{\mu\nu} \, \eta^{\rho\sigma} - \eta^{\mu\rho} \, \eta^{\nu\sigma} + \eta^{\mu\sigma} \, \eta^{\rho\nu}) \,, \end{aligned} \tag{6.4}$$

where $r = 2^{\frac{[D]}{2}}$ and it counts the number of off-shell fermionic degrees of freedom.

We introduce two hermitian projection operators

$$P_+ = \frac{1}{2}\gamma^-\gamma^+ \,, \qquad P_- = \frac{1}{2}\gamma^+\gamma^- \quad \text{which satisfy} \;\; P_\pm^2 = P_\pm \;\;,\;\; P_+P_- = P_-P_+ = 0. \tag{6.5}$$

We start with the action (2.21)

$$S = \int d^Dx \left(-\frac{1}{4} F^{\,a}_{\mu\nu} F^{\mu\nu\,a} + \frac{i}{2}\bar{\psi}^a\,\gamma^\mu\,(D_\mu\psi)^a \right)\,, \tag{6.6}$$

where $F^{\,a}_{\mu\nu} = \partial_\mu A^{\,a}_\nu - \partial_\nu A^{\,a}_\mu + g f^{abc} A^{\,b}_\mu A^{\,c}_\nu$, $\bar{\psi} = \psi^\dagger\gamma^0$ and $D_\mu = \partial_\mu\delta^{ac} + g f^{abc} A^b_\mu$. The f^{abc} are the structure constants of the gauge group $SU(n)$.

Here, we will not explicitly distinguish between Majorana, Weyl, and Majorana-Weyl spinors to keep notations simple. Note that this is justified because our calculations require only basic Clifford algebra and the trace relations (6.4).

The equation of motion corresponding to the gauge field is

$$D_\mu F^{\mu\nu\,a} - \frac{i}{2} g f^{abc}\bar{\psi}^b\,\gamma^\nu\psi^c = 0\,. \tag{6.7}$$

We make the gauge choice $A^{\,a}_- = 0$ which renders $\nu = +$ in (6.7) as a constraint equation and we get

$$A^{\,a}_+ = -\frac{1}{\partial^+}(\partial_i A^{i\,a}) - g f^{abc}\frac{1}{\partial^{+\,2}}(A^b_i\partial^+ A^{i\,c}) - \frac{i}{2} g f^{abc}\frac{1}{\partial^+}(\bar{\psi}^a_+\gamma^+\psi^a_+)\,, \tag{6.8}$$

The operator $\frac{1}{\partial^+}$ is an artifact of the light-cone gauge. It is formally defined as

$$\frac{1}{\partial^+} f(x^-) = -\int dy^-\;\theta(x^- - y^-)\,f(y^-)\,, \tag{6.9}$$

where $\theta(x^- - y^-)$ is the step function. The operator acts like an integral operator, not a differential operator. In momentum space, the operator has a well-defined pole prescription [9].

The equation of motion for the fermion field is given by $\gamma^\mu D^{ac}_\mu\psi^c = 0$. The fermion field ψ can be decomposed into $\psi_\pm$ using the projection operators

$$\psi_\pm = P_\pm \psi \quad , \quad \bar{\psi}_\pm = \bar{\psi} P_\mp \ \text{ and } \ \psi = \psi_+ + \psi_- \quad , \quad \bar{\psi} = \bar{\psi}_+ + \bar{\psi}_- \, . \tag{6.10}$$

Acting P_+ and P_- on the equation of motion, we get the following two equations

$$D_-^{ac}\, \psi_-^c = -\frac{1}{2}\gamma^+\gamma^i\, D_i^{ac}\, \psi_+^c \qquad\qquad i = 1, ...D-2\,, \tag{6.11}$$

$$D_+^{ac}\, \psi_+^c = -\frac{1}{2}\gamma^-\gamma^i\, D_i^{ac}\, \psi_-^c \qquad\qquad i = 1, ...D-2\,. \tag{6.12}$$

Since (6.11) is a constraint so we solve for ψ_-^a and we obtain

$$\psi_-^a = \frac{1}{2}\gamma^+\gamma^i \frac{1}{\partial^+} D_i^{ac}\psi_+^c\,. \tag{6.13}$$

Expanding the fermion term in terms of $\psi_\pm$ (and its conjugate) and substituting the constraint equations (6.8),(6.11) in (6.6), we obtain the Lagrangian in the light-cone gauge

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{2}\, A_i^a \Box A_i^a - g f^{abc}\left(\frac{\partial_i}{\partial^+} A_i^a\, \partial^+ A_j^b A_j^c + \partial_i A_j^a A_i^b A_j^c\right) \\
& - g^2 f^{abc} f^{ade}\left(\frac{1}{4} A_i^b A_j^c A_i^d A_j^e + \frac{1}{2}\frac{1}{\partial^+}\left(\partial^+ A_i^b A_i^c\right)\frac{1}{\partial^+}\left(\partial^+ A_j^d A_j^e\right)\right) \\
& + \frac{i}{2}\, \bar{\psi}_+^a\, \gamma^+\left(D_+^{ad} - \frac{1}{2}\,\gamma^i D_i^{ac}\frac{1}{\partial^+}\gamma^j D_j^{cd}\right)\psi_+^d \\
& - \frac{1}{8}\, g^2 f^{abc} f^{ade}\frac{1}{\partial^+}\left(\bar{\psi}_+^b \gamma^+ \psi_+^c\right)\frac{1}{\partial^+}\left(\bar{\psi}_+^d \gamma^+ \psi_+^e\right)\,.
\end{aligned} \tag{6.14}$$

The last term of (6.14) is a four fermion interaction term that was absent in the original action. This is a feature which distinguishes the light-cone formulation of the theory from other gauges.

We now restrict to four dimensions and introduce the transverse coordinates

$$x = \frac{(x^1 + i x^2)}{\sqrt{2}}\,, \qquad \bar{x} = x^*\,, \tag{6.15}$$

and their derivatives $\bar{\partial}\,,\, \partial$ respectively.

We also introduce the helicity field

$$A^a = \frac{A_1^a + iA_2^a}{\sqrt{2}}\,, \tag{6.16}$$

and its conjugate $\bar{A}^a$.

The fermion fields in $D = 4$ satisfy the Majorana condition $\psi = C\bar{\psi}^T$ where the charge conjugation matrix C is

$$C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}\,. \tag{6.17}$$

The fermion field $\psi^a(x)$ takes the form

$$\psi = \begin{pmatrix} \frac{\bar{\partial}}{\partial^+}\bar{\chi} \\ -\bar{\chi} \\ \chi \\ \frac{\partial}{\partial^+}\chi \end{pmatrix}\,. \tag{6.18}$$

The Lagrangian (6.14) can now be simplified and written purely in terms of the physical fields $(A^a, \bar{A}^a, \chi^a, \bar{\chi}^a)$ as

$$\begin{aligned}
\mathcal{L} \;=\;& \bar{A}^a \Box A^a - 2g f^{abc}\left(\frac{\bar{\partial}}{\partial^+}A^a\partial^+\bar{A}^b A^c + \frac{\partial}{\partial^+}\bar{A}^a\partial^+ A^b\bar{A}^c\right) \\
& -2g^2 f^{abc} f^{ade}\frac{1}{\partial^+}\left(\partial^+ A^b\bar{A}^c\right)\frac{1}{\partial^+}\left(\partial^+\bar{A}^d A^e\right) \\
& + \frac{i}{\sqrt{2}}\bar{\chi}^a\left(\frac{\Box}{\partial^+}\delta^{ac} - 2g f^{abc}\frac{1}{\partial^+}(\partial\bar{A}^b + \bar{\partial}A^b) + 2g f^{abc}\frac{\bar{\partial}}{\partial^+}(A^b + 2g f^{abc}\bar{A}^b\frac{\partial}{\partial^+}\right)\chi^c \\
& + i\sqrt{2}\,g^2 f^{abc} f^{bde}\,\bar{\chi}^a\frac{1}{\partial^{+2}}(A^d\partial^+\bar{A}^e + \bar{A}^d\partial^+ A^e)\chi^c - i\sqrt{2}\,g^2 f^{abd} f^{bec}\bar{\chi}^a\bar{A}^d\frac{1}{\partial^+}(A^e\chi^c) \\
& + g^2 f^{abc} f^{ade}\frac{1}{\partial^+}(\bar{\chi}^b\chi^c)\frac{1}{\partial^+}(\bar{\chi}^d\chi^e)\,.
\end{aligned} \tag{6.19}$$

where $\Box = (-2\partial^+\partial^- + 2\partial\bar{\partial})$ and the f^{abc} are the structure constants of the gauge group.

6.2 Light-cone Nicolai map in four dimensions

The statement of the Nicolai map in our context is the following: there exists a non-linear and non-local transformation $\mathcal{T}_g(A)$ which satisfies the following three properties:

1. The transformation $\mathcal{T}_g(A)$ when substituted in the free bosonic Lagrangian (Maxwell theory) yields the full interacting bosonic Lagrangian (Yang-Mills theory).

2. The Jacobian of the transformation is equal to the fermion determinant (in the light-cone gauge there are no ghosts, they decouple from the path integral).

3. The transformation preserves the gauge choice.

Essentially, this means that one works with a free bosonic theory to compute correlators in a supersymmetric gauge theory - through the inverse transformations $\mathcal{T}_g^{-1}(A')$ [27].

6.2.1 The transformation

Consider a field transformation for the physical fields A^a and $\bar{A}^a$, obtained through trial and error such that the Yang-Mills Lagrangian can be mapped to a free bosonic theory: $\bar{A}'^{\,a}\Box A'^{\,a}$. We introduce a Green's function through $\Box C(x-y) = -\delta^{(4)}(x-y)$ to write such an ansatz up to $\mathcal{O}(g^2)$

$$\begin{aligned} A'^{\,a}(x,g;A,\bar{A}) \;=\;& A^a(x) + 2gf^{abc}\int dy\; \partial^+ C(x-y)\frac{\bar{\partial}}{\partial^+}\,A^b(y)A^c(y) \\ & -g^2 f^{abc}f^{bde}\int dy\; \partial^+ C(x-y)A^c(y)\frac{1}{\partial^{+\,2}}\left(\partial^+ A^d(y)\bar{A}^e(y)\right) \\ & -2g^2 f^{abc}f^{bde}\int dy\,dz\;\left(\partial\, C(x-y)\bar{A}^c(y) - \partial^+ C(x-y)\frac{\partial}{\partial^+}\bar{A}^c(y)\right) \\ & \qquad\qquad \times\, \partial^+\, C(y-z)\frac{\bar{\partial}}{\partial^+}A^d(z)A^e(z)\;. \end{aligned} \tag{6.20}$$

Here $dy\,,\; dz$ denote the four dimensional space-time measure. In this section, all measures and delta functions will be assumed to be four-dimensional, and the dimension will be suppressed henceforth. The transformation for $\bar{A}'^{\,a}$ is just the complex conjugate of the above.

The map in Landau gauge at order g^2 (3.13) contains three terms of the form $\partial C\, A\, \partial C\, A\, A$ (with space-time indices and color indices suppressed), where all of them contribute to the Jacobian (as shown in Chapter 3). In the light-cone Nicolai map at order g^2, we find that there are two kinds of terms: one is similar to the Landau gauge case, and the other has a single Green's function (line 2 of eq. (6.20)). The latter term produces the pure Yang-Mills quartic vertex (line 2 of eq. (6.19)) but does not contribute to the Jacobi determinant at order g^2, as we show below.

The functional variation of the fields are

$$\frac{\delta A^a(x)}{\delta A^b(w)} = \frac{\delta \bar{A}^a(x)}{\delta \bar{A}^b(w)} = \delta^{ab}\delta(x-w)\,, \qquad \frac{\delta A^a(x)}{\delta \bar{A}^b(w)} = \frac{\delta \bar{A}^a(x)}{\delta A^b(w)} = 0\,. \tag{6.21}$$

The Jacobi determinant of the map can be computed using the relation

$$\log\det(1+\mathbf{X}) = \mathrm{Tr}\log(1+\mathbf{X}) = \mathrm{Tr}\,\mathbf{X} - \frac{1}{2}\mathrm{Tr}\,\mathbf{X}^2 \pm .. \tag{6.22}$$

The Jacobi matrix[1] of the above transformation (6.20) is

$$\begin{aligned}
\frac{\delta A'^{\,a}(x)}{\delta A^m(w)} \;=\;& \delta^{am}\delta(x-w) + 2g f^{abc}\int dy\, \Big\{ \partial^+ C(x-y)\frac{\bar\partial}{\partial^+}\,\delta^{bm}\delta(y-w)A^c(y) \\
&\qquad + \partial^+ C(x-y)\frac{\bar\partial}{\partial^+}\,A^b(y)\,\delta^{cm}\delta(y-w)\Big\} \\
&-2g^2 f^{abc} f^{bde}\int dy\, dz\, \Big(\partial\, C(x-y)\bar{A}^c(y) - \partial^+ C(x-y)\frac{\partial}{\partial^+}\bar{A}^c(y)\Big) \\
&\times \Big\{ \partial^+\, C(y-z)\frac{\bar\partial}{\partial^+}\delta^{dm}\delta(z-w)A^e(z) + \partial^+\, C(y-z)\frac{\bar\partial}{\partial^+}A^d(z)\,\delta^{em}\delta(z-w)\Big\}\,,
\end{aligned} \tag{6.23}$$

where we have dropped all terms that vanish after taking the trace as they are proportional to $\partial_\mu C(0)$. We use (6.22) to compute the determinant order by order in the coupling constant.

[1]The matrix elements $\dfrac{\delta A'^{\,a}(x)}{\delta \bar{A}^m(w)}$ and $\dfrac{\delta \bar{A}'^{\,a}(x)}{\delta A^m(w)}$ do not contribute to the trace at order g^2.

After partial integrations, taking the trace (by setting $a = m$ and $x = w$), using $f^{abc} f^{abd} = n\delta^{cd}$ and integrating over x, the Jacobi determinant up to $O(g^2)$ reads

$$
\begin{aligned}
\log\det\left(\frac{\delta A_i'^a(x)}{\delta A_j^m(w)}\right) \; = \; 2ng^2 \int dx\, dy \, \Big\{ & \bar\partial\, C(x-y)\, A^b(y)\, \bar\partial\, C(y-x)\, A^b(x) \\
& + \partial^+ C(x-y) \frac{\bar\partial}{\partial^+} A^b(y)\, \partial^+ C(y-x) \frac{\bar\partial}{\partial^+} A^b(x) \\
& - 2\, \bar\partial\, C(x-y)\, A^b(y) \partial^+ C(y-x) \frac{\bar\partial}{\partial^+} A^b(x) \\
& + \frac{\partial\bar\partial}{\partial^+} C(x-y) \bar A^b(y)\, \partial^+ C(y-x) A^c(x) \\
& - \partial\, C(x-y)\, \bar A^b(y)\, \partial^+ C(y-x) \frac{\bar\partial}{\partial^+} A^b(x) \\
& - \bar\partial\, C(x-y)\, A^b(y)\, \partial^+ C(y-x) \frac{\partial}{\partial^+} \bar A^b(x) \\
& + \partial^+ C(x-y) \frac{\partial}{\partial^+} \bar A^b(x)\, \partial^+ C(y-x) \frac{\bar\partial}{\partial^+} A^b(x) \Big\} + c.c. \,.
\end{aligned}
\tag{6.24}
$$

Here i, j run over the transverse variables $x, \bar x$.

6.2.2 The fermion determinant

The construction of the Nicolai map in the light-cone gauge is complicated due to the presence of a four-fermion interaction term in (6.19). In [54], it was first shown how to compute the Nicolai maps for theories with four fermion interaction terms. We explain here why this term does not contribute to the fermion determinant at order g^2.

As a simple example, let us start with $D = 4$. However, this analysis holds for any dimension D. The path integral takes the form

$$
Z = \int DA^a D\bar A^a D\chi^a D\bar\chi^a \; \exp\left[i \int d^4x\, \mathcal{L}_1 + B \right], \tag{6.25}
$$

where $\mathcal{L}_1$ contains all terms in (6.19) except the four fermion interaction term and B is the four fermionic interaction. We now expand the exponent of B to the linear order (since we are working to order g^2 and B is exactly of that order)

$$Z = \int DA^a D\bar{A}^a D\chi^a D\bar{\chi}^a \ (1 + i\int d^4x\, B)\, \text{exp}\left[i\int d^4x\, \mathcal{L}_1\right] , \tag{6.26}$$

$$Z = Z_0 + Z_1\,. \tag{6.27}$$

The fermion determinant can be evaluated in the term Z_0. Let us denote this determinant by $\Delta_F(A, \bar{A}, g)$. The form of Z_0 is then

$$Z_0 = \int DA^a D\bar{A}^a \ \Delta_F(A, \bar{A}, g) \ \text{exp}\left[i\int d^4x\, \mathcal{L}_{YM}\right] , \tag{6.28}$$

where $\mathcal{L}_{YM}$ denotes the Yang-Mills Lagrangian. The term Z_1 is the path integral Z_0 with the four fermion interaction term as an insertion. The term Z_1 may be computed using the standard technique of introducing sources by considering

$$\begin{aligned} Z_0[J] \;&=\; \int DA^a D\bar{A}^a D\chi^a D\bar{\chi}^a \\ &\text{exp}\left[i\int d^4x\, \mathcal{L}_{YM} + \frac{i}{\sqrt{2}}\left((\bar{\chi}^a + \bar{J}^b Q^{-1\,ba})Q^{ac}(\chi^c + Q^{-1\,cd}J^d) - \bar{J}^a Q^{-1\,ac}J^c\right)\right] . \end{aligned} \tag{6.29}$$

Here Q^{ac} denotes the quadratic operator in the fermionic part of the Lagrangian

$$\begin{aligned} Q^{ac}(x;A) \;&=\; \frac{\Box}{\partial^+}\delta^{ac} - 2gf^{abc}\frac{1}{\partial^+}(\partial\bar{A}^b + \bar{\partial}A^b) + 2gf^{abc}\frac{\bar{\partial}}{\partial^+}(A^b + 2gf^{abc}\bar{A}^b\frac{\partial}{\partial^+} \\ &+ 2\,g^2 f^{abc}f^{bde}\,\frac{1}{\partial^{+2}}(A^d\partial^+\bar{A}^e + \bar{A}^d\partial^+A^e) - 2\,g^2 f^{abd}f^{bec}\bar{A}^d\frac{1}{\partial^+}(A^e\,, \end{aligned} \tag{6.30}$$

while $Q^{-1\,ac}$ denotes the fermion propagator in the presence of gauge field. A change of variables in the path integral from χ^a to $\chi^{'\,a} = \chi^a + Q^{-1\,ac}J^c$ and similarly for its complex conjugate allows us to integrate the fermion fields and get a factor of $\Delta_F(A, \bar{A}, g)$. By differentiating $Z_0[J]$ with respect to the sources and putting them to zero, we find the form of Z_1 to be

$$Z_1 = \int DA^a D\bar{A}^a \ \Delta_F(A, \bar{A}, g) G_4(A, \bar{A}, g) \ \text{exp}\left[i\int d^4x\, \mathcal{L}_{YM}\right] . \tag{6.31}$$

Thus,

$$Z = \int DA^a D\bar{A}^a \; \Delta_F(A,\bar{A},g)[1+G_4(A,\bar{A},g)] \;\exp\left[i\int d^4x\,\mathcal{L}_{YM}\right] . \tag{6.32}$$

We can define $\Delta_F'(A,\bar{A},g) = \Delta_F(A,\bar{A},g)[1+G_4(A,\bar{A},g)]$ as the effective fermion determinant. Note that G_4 is a product of two fermion propagators. The fermion propagator in the presence of a gauge field is an infinite series in the coupling g. Since Z_1 is itself at order g^2, we must only consider the series's leading term, which is nothing but the free fermion propagator given by $\partial^+ C(x-y)$. But since all the four fermion fields are at the same space-time point, G_4 vanishes as $\partial^+ C(0) = 0$. Thus, four fermion term starts contributing to the fermion determinant from order g^4.

We work here at order g^2, so we receive contributions to the fermion determinant only from the quadratic operator (6.30) contributions. The quadratic operator Q^{ac} relevant to order g^2 (again dropping terms which vanish after 'trace-ing'), is

$$Q^{ac}(x;A) \;=\; \frac{\Box}{\partial^+}\delta^{ac} - 2gf^{abc}\frac{1}{\partial^+}(\partial\bar{A}^b + \bar{\partial}A^b) + 2gf^{abc}\frac{\bar{\partial}}{\partial^+}(A^b + 2gf^{abc}\bar{A}^b\frac{\partial}{\partial^+} \, , \tag{6.33}$$

which may be written as

$$\begin{aligned} Q^{ac}(x,y;A) \;=\; & \frac{\Box}{\partial^+}\Bigg(\delta^{ac} + 2gf^{abc}\int dy\,\partial^+ C(x-y)\frac{1}{\partial^+}(\partial\bar{A}^b(y) + \bar{\partial}A^b(y)) \\ & -2gf^{abc}\int dy\,\bar{\partial}C(x-y)A^b(y) - 2gf^{abc}\int dy\,\partial^+ C(x-y)\bar{A}^b(y)\frac{\partial^{(y)}}{\partial^+}\Bigg) . \end{aligned} \tag{6.34}$$

As $\det(Q) = \det(\Box/\partial^+)\times\det(1+\mathbf{Y})$, we use (6.22) to compute the fermion determinant order by order in g upto an overall constant $\det(\Box/\partial^+)$. The fermion determinant to order g^2 is

$$\begin{aligned} \log\det(1+\mathbf{Y}) \;=\; 2ng^2\int dx\,dy\,\Big\{ & \bar{\partial}\,C(x-y)\,A^b(y)\,\bar{\partial}\,C(y-x)\,A^b(x) \\ & +\partial^+ C(x-y)\frac{\bar{\partial}}{\partial^+}A^b(y)\,\partial^+ C(y-x)\frac{\bar{\partial}}{\partial^+}A^b(x) \\ & -2\,\bar{\partial}\,C(x-y)\,A^b(y)\partial^+ C(y-x)\frac{\bar{\partial}}{\partial^+}A^b(x) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial\bar{\partial}}{\partial^+}C(x-y)\bar{A}^b(y)\,\partial^+C(y-x)A^c(x)\\
&-\partial\,C(x-y)\,\bar{A}^b(y)\,\partial^+C(y-x)\frac{\bar{\partial}}{\partial^+}A^b(x)\\
&-\bar{\partial}\,C(x-y)\,A^b(y)\,\partial^+C(y-x)\frac{\partial}{\partial^+}\bar{A}^b(x)\\
&+\partial^+C(x-y)\frac{\partial}{\partial^+}\bar{A}^b(x)\,\partial^+C(y-x)\frac{\bar{\partial}}{\partial^+}A^b(x)\Bigg\}+c.c.\,.
\end{aligned}
\tag{6.35}
$$

Thus, we find that the Jacobi determinant of the bosonic transformation (6.24) exactly matches the fermion determinant (6.35) upto $O(g^2)$. We, therefore, have shown that the transformation (6.20) satisfies all the three statements of the main theorem.

6.3 Extension of the light-cone map to all critical dimensions

We now generalize our construction to the critical dimensions (6.36). The first step in this process is to start with an appropriate Lagrangian in a non-helicity basis and then guess the form of the Nicolai map. Note that the Lagrangian in a helicity basis, in four dimensions, is a special case and is related to the little group $SO(2)$.

6.3.1 The move away from a helicity basis

Consider the Lagrangian (6.14), gauge fixed and expressed in light-cone coordinates in non-helicity basis. Note that the number of bosons and fermion for supersymmetric gauge theory match only in $d = 3, 4, 6, 10$. We prove that for theories with interactions, the fermion and the Jacobi determinant match only in the critical dimensions, establishing the old relation that supersymmetric Yang-Mills theories exist in these critical dimensions.

We guess the field transformation for the physical fields A_i such that the pure Yang-Mills theory (first two lines of equation (6.14)) can be expressed as $\frac{1}{2}{A_i}'^a\Box A_i'^a$. In this section, all measures and delta functions will be d-dimensional (dimensions will get fixed using the determinant matching). Again, we introduce $\Box C(x-y) = -\delta^{(d)}(x-y)$ to write the map to order g^2

$$
\begin{aligned}
A_i'^{\,a}(x) = \;& A_i^a(x) + g f^{abc} \int dy \, \left(\partial^+ C(x-y) \frac{\partial_j}{\partial^+} A_j^b(y) A_i^c(y) - \, \partial_j C(x-y) A_j^b(y) A_i^c(y) \right) \\
& - \frac{g^2}{2} f^{abc} f^{bde} \int dy \; \partial^+ C(x-y) A_i^c(y) \frac{1}{\partial^{+\,2}} \left(\partial^+ A_j^d(y) A_j^e(y) \right) \\
& + \frac{g^2}{2} f^{abc} f^{bde} \int dy \, dz \, \Big\{ \partial_j C(x-y) A_k^c(y) \\
& \qquad \times \left(\partial_i C(y-z) A_k^d(z) A_j^e(z) + \partial_k C(y-z) A_j^d(z) A_i^e(z) \right) \\
& \qquad - \partial_i C(x-y) A_j^c(y) \, \partial^+ C(y-z) \frac{\partial_k}{\partial^+} A_k^d(z) A_j^e(z) \\
& \qquad + \partial^+ C(x-y) \frac{\partial_j}{\partial^+} A_j^c(y) \, \partial^+ C(y-z) \frac{\partial_k}{\partial^+} A_k^d(z) A_i^e(z) \\
& \qquad + 2 \, \partial_i C(x-y) A_j^c(y) \partial_k C(y-z) A_k^d(z) A_j^e(z) \\
& \qquad - 2 \, \partial^+ C(x-y) \frac{\partial_j}{\partial^+} A_j^c(y) \, \partial_k C(y-z) A_k^d(z) A_i^e(z) \\
& \qquad + \partial^- C(x-y) A_k^c(y) \partial^+ C(y-z) A_k^d(z) A_i^e(z) \\
& \qquad + \partial^+ C(x-y) A_k^c(y) \partial^- C(y-z) A_k^d(z) A_i^e(z) \\
& \qquad - \partial_j C(x-y) A_k^c(y) \partial_j C(y-z) A_k^d(z) A_i^e(z) \Big\} \, .
\end{aligned}
\tag{6.36}
$$

6.3.2 Jacobian

We calculate below the Jacobian of the transformation (6.36).

$$
\begin{aligned}
\frac{\delta A_i'^{\,a}(x)}{\delta A_m^p(w)} = \;& \delta_i^m \delta^{ap} \delta(x-w) + g f^{abc} \int dy \, \Big\{ \partial^+ C(x-y) \delta(y-w) \Big(\frac{\partial_j}{\partial^+} \delta_j^m \delta^{bp} A_i^c(y) \\
& + \frac{\partial_j}{\partial^+} A_j^b(y) \delta_i^m \delta^{cp} \Big) - \partial_j C(x-y) \delta(y-w) \left(\delta_j^m \delta^{bp} A_i^c(y) + A_j^b(y) \delta_i^m \delta^{cp} \right) \Big\} \\
& + \frac{g^2}{2} f^{abc} f^{bde} \int dy \, dz \, \Big\{ \partial_j C(x-y) A_k^c(y) \, \delta(z-w) \Big\{ \partial_i C(y-z) \delta_k^m \delta^{dp} A_j^e(z) \\
& + \partial_i C(y-z) A_k^d(z) \delta_j^m \delta^{ep} + \partial_k C(y-z) \delta_j^m \delta^{dp} A_i^e(z) \\
& + \partial_k C(y-z) A_j^d(z) \delta_i^m \delta^{ep} \Big\} - \partial_i C(x-y) A_j^c(y) \delta(z-w) \\
& \times \left(\partial^+ C(y-z) \frac{\partial_k}{\partial^+} \delta_k^m \delta^{dp} A_j^e(z) + \partial^+ C(y-z) \frac{\partial_k}{\partial^+} A_k^d(z) \delta_j^m \delta^{ep} \right) \\
& + \partial^+ C(x-y) \frac{\partial_j}{\partial^+} A_j^c(y) \, \delta(z-w) \, \partial^+ C(y-z)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\frac{\partial_k}{\partial^+} \delta_k^m \delta^{dp} A_i^e(z) + \frac{\partial_k}{\partial^+} A_k^d(z) \delta_i^m \delta^{ep} \right) + 2\, \partial_i C(x-y) A_j^c(y)\, \delta(z-w) \\
&\times \left(\partial_k C(y-z) \delta_k^m \delta^{dp} A_j^e(z) + \partial_k C(y-z) A_k^d(z) \delta_j^m \delta^{ep} \right) \\
&- 2\, \partial^+ C(x-y) \frac{\partial_j}{\partial^+} A_j^c(y)\, \delta(z-w) \\
&\times \left(\partial_k C(y-z) \delta_k^m \delta^{dp} A_i^e(z) + \partial_k C(y-z) A_k^d(z) \delta_i^m \delta^{ep} \right) \\
&+ \partial^- C(x-y) A_k^c(y)\, \delta(z-w) \\
&\times \left(\partial^+ C(y-z) A_k^d(z) \delta_i^m \delta^{ep} + \partial^+ C(y-z) A_i^e(z) \delta_k^m \delta^{dp} \right) \\
&+ \partial^+ C(x-y) A_k^c(y)\, \delta(z-w) \\
&\times \left(\partial^- C(y-z) A_k^d(z) \delta_i^m \delta^{ep} + \partial^- C(y-z) A_i^e(z) \delta_k^m \delta^{dp} \right) \\
&- \partial_j C(x-y) A_k^c(y)\, \delta(z-w) \\
&\times \left(\partial_j C(y-z) \delta_k^m \delta^{dp} A_i^e(z) + \partial_j C(y-z) A_k^d(z) \delta_i^m \delta^{ep} \right) \Bigg\} ,
\end{aligned}
\tag{6.37}
$$

where we have only written the non-trivial terms relevant till order g^2. We know

$$
\log \det(1 + \mathbf{X}) = \mathrm{Tr} \log(1 + \mathbf{X}) = \mathrm{Tr}\, \mathbf{X} - \frac{1}{2} \mathrm{Tr}\, \mathbf{X}^2 \pm .. \tag{6.38}
$$

We take the trace by setting $a = m$, $w = x$ and integrating over x. We also use the $SU(n)$ identity $f^{abc} f^{abd} = n \delta^{cd}$ to obtain the Jacobi determinant up to $O(g^2)$

$$
\begin{aligned}
\log \det \left(\frac{\delta A_i'^a(x)}{\delta A_j^m(w)} \right) \;=\; ng^2 \int dx\, dy\, (D-2) \Bigg\{ & \partial_i\, C(x-y)\, A_i^b(y)\, \partial_j\, C(y-x)\, A_j^b(x) \\
& + \partial^+ C(x-y) \frac{\partial_j}{\partial^+} A_j^b(y)\, \partial^+ C(y-x) \frac{\partial_i}{\partial^+} A_i^b(x) \\
& - 2\, \partial_i\, C(x-y)\, A_i^b(y) \partial^+ C(y-x) \frac{\partial_j}{\partial^+} A_j^b(x) \\
& - \frac{1}{2} \partial_i\, C(x-y)\, A_j^b(y)\, \partial_i\, C(y-x)\, A_j^b(x) \\
& + \frac{1}{2} \frac{\partial_i^2}{\partial^+} C(x-y) A_j^b(y)\, \partial^+ C(y-x) A_j^b(y) \Bigg\} ,
\end{aligned}
\tag{6.39}
$$

where we have used the relation $2\, \partial^- C(x-y) = \frac{\partial_i^2}{\partial^+} C(x-y) + \frac{1}{\partial^+} \delta(x-y)$.

6.3.3 Fermion Determinant

Similar to the (6.2.2) case, the presence of the four fermion interaction terms in (6.14) makes the fermion determinant computation complicated. We follow the steps shown in subsection (6.2.2) to get the fermion determinant, and its contribution at order g^2 is trivial.

We simplify the quadratic operator in (6.14) by expanding the covariant derivatives and using the constraint equation (6.8). We get

$$\Delta = \det\left\{\frac{1}{2}\frac{\Box}{\partial^+}\delta^{ac} - \frac{1}{2}g f^{abc}\gamma^i\gamma^j\frac{\partial_i}{\partial^+}(A_j^b - \frac{1}{2}g f^{abc}\gamma^i\gamma^j A_i^b\frac{\partial_j}{\partial^+} - g f^{abc}\frac{\partial_i}{\partial^+}A_i^b \right.$$
$$\left. - g^2 f^{abc} f^{bde}\frac{1}{\partial^{+2}}\left(A_i^d\partial^+ A_i^e\right) - \frac{1}{2}g^2 f^{ade} f^{bcd}\gamma^i\gamma^j A_i^e\frac{1}{\partial^+}A_j^b\right\}. \tag{6.40}$$

The non-trivial part of the quadratic operator relevant to order g^2 is

$$\Delta = \det\left(\frac{1}{2}\frac{\Box}{\partial^+}\right)\cdot\det\left\{\delta^{ac} + g f^{abc}\gamma^i\gamma^j\int dy\,\partial^+ C(x-y)A_i^b(y)\,\frac{\partial_j}{\partial^+}\right.$$
$$\left. + g f^{abc}\gamma^i\gamma^j\int dy\,\partial_i C(x-y)A_j^b(y) + 2\,g f^{abc}\int dy\,\partial^+ C(x-y)\frac{\partial_i}{\partial^+}A_i^b(y)\right\}. \tag{6.41}$$

We now compute the fermion determinant perturbatively using (6.22)

$$\begin{aligned}
\log\det(1+\mathbf{Y}) \;=\; n g^2\int dx\,dy\,\frac{r}{4}\Big\{ & 2\,\partial_i\,C(x-y)\,A_i^b(y)\,\partial_j\,C(y-x)\,A_j^b(x) \\
& + 2\,\partial^+ C(x-y)\frac{\partial_j}{\partial^+}A_j^b(y)\,\partial^+ C(y-x)\frac{\partial_i}{\partial^+}A_i^b(x) \\
& - 4\,\partial_i\,C(x-y)\,A_i^b(y)\partial^+ C(y-x)\frac{\partial_j}{\partial^+}A_j^b(x) \\
& - \partial_i\,C(x-y)\,A_j^b(y)\,\partial_i\,C(y-x)\,A_j^b(x) \\
& + \frac{\partial_i^2}{\partial^+}C(x-y)A_j^b(y)\,\partial^+ C(y-x)A_j^b(y)\Big\},
\end{aligned} \tag{6.42}$$

where $r = \mathrm{Tr}\,\mathbf{1}$ and it counts the number of off-shell fermionic degrees of freedom.

6.3.4 Existence of map in critical dimensions

We see now that the Jacobian determinant (6.39) matches against the fermion determinant (6.42) if and only if

$$\frac{r}{2} = D - 2\,,$$
$$\frac{r}{2} = D - 2\,,$$
$$-r = -2(D-2)\,,$$
$$-\frac{r}{4} = -\frac{D-2}{2}\,,$$
$$+\frac{r}{4} = \frac{D-2}{2}\,,$$

all implying that $r = 2(D-2)$ which happens for $D = 3, 4, 6$ and 10. The map (6.36) satisfies all three conditions of the main theorem listed in the section (6.2), hence confirming that a Nicolai map exists for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in the light-cone gauge for all critical dimensions.

The construction of the Nicolai map was studied in general gauges ($n^\mu A_\mu = 0$) in [31]. In particular, the map up to order g^2 was constructed in the axial ($n^2 = 1$) and light-cone gauge ($n^2 = 0$) using the $\mathcal{R}$ operator. We can recover our map (6.36) (to order g^2) from the one derived in [31]. The following conditions need to be employed in equation (4.1) of ref. [31] : n^μ is chosen to be null. In the light cone coordinates, this means setting the components $n^+ = n^i = 0$ and $n^- = 1$. The ghost propagator (in the light-cone gauge) is $\frac{1}{\partial^+}$. Using the above conditions and the constraint equation (6.8) omitting the fermion term, we obtain our map (6.36) to order g^2. We also obtain extra pieces at order g^2, which implies that our map is a subset of their map.

Note that we cannot recover our light-cone map (6.36) from the maps derived in Chapter 3 and Chapter 4. Although, the map (3.13) up to order g^2 does not require the use of the gauge condition to prove the main theorem, but the maps are structurally intrinsic to the Landau gauge and do not have pieces that are proportional to $\partial_\mu A_\mu$, which might be trivial in the Landau gauge but will contribute in other gauges.

6.4 On the issue of uniqueness

The map obtained in the preceding section (6.36) can be written in four dimensions in terms of the helicity variables and fields. The map up to order g reads

$$\begin{aligned} A'^{a}(x,g;A,\bar{A}) \;=\; & A^a(x) + 2g f^{abc} \int dy \,\Big\{ \partial^+ C(x-y) \frac{\bar\partial}{\partial^+} A^b(y) A^c(y) \\ & + \partial^+ C(x-y) \frac{\partial}{\partial^+} \bar{A}^b(y) A^c(y) - \partial\, C(x-y) \bar{A}^b(y)\, A^c(y) + \mathcal{O}(g^2) \Big\}\,. \end{aligned} \tag{6.43}$$

This map is distinct from the one (6.20) obtained in section (6.2). This difference can be understood at the level of Lagrangian. The order g terms involving the gauge fields in (6.19) and (6.14) are related by partial integrations in $D = 4$. However, the maps (6.20) and (6.43) to the cubic order are not related by any partial integrations.

As discussed in the last chapter 5, we can, in principle, write distinct maps by writing the Lagrangian in different ways by performing partial integrations. Out of all three conditions of the main theorem, the most complicated one is the determinant matching condition. In our case, we have two maps at order g^2 that satisfy all three conditions of the Main theorem (listed in section 2). The matching of the determinant is about the equality of the derivative (Jacobian) of the map with the fermion determinant and not about the map itself, hence the non-uniqueness. The issue of uniqueness can be fixed if one computes the map to higher orders. Notice that for any finite order in a perturbation theory, one can always find the simplest map relevant to that order, which will simplify the computation of correlation functions (scattering amplitudes). Therefore, the non-uniqueness of the map can be helpful for computations of physical objects at a specific order in the coupling (as shown below).

We showed in Chapter 5 that the Nicolai maps can be related at the level of the Jacobi determinant. The existence of more than one map at a particular order in the coupling constant is due to the freedom in writing the Lagrangian. In this chapter, we found two distinct four-dimensional maps (6.20) and (6.43). We can relate the map (6.36) that works in all critical dimensions to the simple four-dimensional map (6.20) at the level of the Jacobi determinant by adding and subtracting a term of the form $4ng^2\partial C(x-y)\,\bar{A}^a\bar{\partial}C(y-x)\,A^a$ in (6.24). This implies that when the Jacobian (6.39) is written in $D=4$ in the helicity basis, there is a simplification as some terms get canceled. The existence of a simpler map (6.20) in four dimensions is a direct consequence of this simplification.

6.5 Scattering amplitudes

In this section, we present the computation of the correlation function of full supersymmetric Yang-Mills theory using the Nicolai map. It was shown in [20, 27] that the correlation function of interacting super Yang-Mills theory can be computed using the inverse Nicolai map. The idea is that the information of supersymmetry is contained in the bosonic map, and the computation of correlators using this approach precisely agrees with the standard quantum field theory method. Also, the amount of labor required to determine the correlators is comparable. The n-point Yang-Mills correlators can be computed as

$$\left\langle\!\left\langle A_1(x_1)....A_n(x_n)\right\rangle\!\right\rangle = \left\langle T_g{}^{-1}[A'_1](x_1)...T_g{}^{-1}[A'_n](x_n)\right\rangle_0 . \tag{6.44}$$

The inverse transformation corresponding to (6.20) is

$$\begin{aligned} T_g{}^{-1}A'^a \;\equiv\;& A^a(x,g;A,\bar{A}) = A'^a(x) - 2gf^{abc}\int dy\;\partial^+C(x-y)\frac{\bar{\partial}}{\partial^+}A'^b(y)A'^c(y) \\ &+g^2f^{abc}f^{bde}\int dy\;\partial^+C(x-y)A'^c(y)\frac{1}{\partial^{+2}}\left(\partial^+A'^d(y)\bar{A}'^e(y)\right) \\ &+2g^2f^{abc}f^{bde}\int dy\,dz\;\left(\partial C(x-y)\bar{A}'^c(y)-\partial^+C(x-y)\frac{\partial}{\partial^+}\bar{A}'^c(y)\right) \\ &\qquad\times\,\partial^+\,C(y-z)\frac{\bar{\partial}}{\partial^+}A'^d(z)A'^e(z)\,. \end{aligned} \tag{6.45}$$

The three-point correlation function can be computed using (6.44)

$$\begin{aligned} \left\langle\!\left\langle A^{a_1}(x_1)\,\bar{A}^{a_2}(x_2)\,\bar{A}^{a_3}(x_3)\right\rangle\!\right\rangle \;=\;& \int DA'\,e^{i\bar{A}'\Box A'}\,A^{a_1}(x_1)\,\bar{A}^{a_2}(x_2)\,\bar{A}^{a_3}(x_3) \\ =\;& -2gf^{a_1bc}\int dy\,\partial^+C(x_1-y)\left\langle\frac{\bar{\partial}}{\partial^+}A'^b(y)\,A'^c(y)\,\bar{A}'^{a_2}(x_2)\,\bar{A}'^{a_3}(x_3)\right\rangle_0 , \end{aligned}$$

where $\langle ....\rangle_0$ is the expectation value for free correlator.

The two-point correlator for gauge fields is

$$\left\langle \bar{A}'^{a_1}(x_1)\, A'^{a_2}(x_2)\right\rangle_0 = \delta^{a_1 a_2}\, C(x_1 - x_2)\,. \tag{6.46}$$

Computing all possible free Wick contraction in (6.46) and then using (6.46), we get

$$\begin{aligned}\left\langle\!\left\langle A^{a_1}(x_1)\, \bar{A}^{a_2}(x_2)\, \bar{A}^{a_3}(x_3)\right\rangle\!\right\rangle \;=\; & -g\, f^{a_1 a_2 a_3} \int dy \left\{ \frac{\bar{\partial}}{\partial^+} C(y - x_2)\, C(x_3 - y)\, \partial^+\, C(x_1 - y) \right. \\ & \left. - \frac{\bar{\partial}}{\partial^+} C(y - x_3)\, C(x_2 - y)\, \partial^+\, C(x_1 - y) \right\}.\end{aligned} \tag{6.47}$$

We obtain the three-point correlation function of gauge fields in the light-cone gauge. To extract the three-point amplitude, we take all external legs on-shell and perform the Fourier transform. We get

$$M_3(1^+, 2^-, 3^-) = -g\, f^{a_1 a_2 a_3}\, p^+ \left(\frac{\bar{k}}{k^+} - \frac{\bar{l}}{l^+} \right) \delta^4(l + p - k)\,. \tag{6.48}$$

Using the spinor helicity variables [55, 56], we get (suppressing color and coupling constant)

$$M_3(1^+, 2^-, 3^-) = \frac{\langle 23\rangle^4}{\langle 12\rangle\, \langle 23\rangle\, \langle 31\rangle}\,, \tag{6.49}$$

which matches with the standard three-point amplitude [57]. For the 4-point correlator, we get contributions from both order g and order g^2 term of the inverse map (6.45)

$$\begin{aligned}
\langle\!\langle A^{a_1}(x_1)\, A^{a_2}(x_2)\, \bar{A}^{a_3}(x_3)\, \bar{A}^{a_4}(x_4) \rangle\!\rangle \;=\; & \int DA'\, e^{i\bar{A}'\Box A'}\, A^{a_1}(x_1)\, A^{a_2}(x_2)\, \bar{A}^{a_3}(x_3)\, \bar{A}^{a_4}(x_4) \\
&= 4g^2 f^{a_1bc}\, f^{a_3de} \int dy\, dz\, \left(\partial^+ C(x_1-y)\, \partial^+ C(x_3-z)\right) \\
&\quad \times \left\langle \frac{\bar\partial}{\partial^+}\, A'^b(y)\, A'^c(y)\, A'^{a_2}(x_2)\, \frac{\partial}{\partial^+}\, \bar{A}'^d(y)\, \bar{A}'^e(y) \bar{A}'^{a_4}(x_4) \right\rangle_0 \\
&\quad + 2g^2 f^{a_1bc}\, f^{bde} \int dy\, dz\, \left(\partial\, C(x_1-y) - \partial^+ C(x_1-y) \frac{\partial}{\partial^+}\right) \\
&\quad \times \partial^+ C(y-z) \left\langle \bar{A}'^c(y) \frac{\bar\partial}{\partial^+} A'^d(z) A'^e(z)\, A'^{a_2}(x_2)\, \bar{A}'^{a_3}(x_3)\, \bar{A}'^{a_4}(x_4) \right\rangle_0
\end{aligned}$$

$$\begin{aligned}
&+ g^2 f^{a_1bc} f^{bde} \int dy\; \partial^+ C(x_1-y) \\
&\left\langle A'^c(y) \frac{1}{\partial^{+\,2}} \left(\partial^+ A'^d(y) \bar{A}'^e(y)\right)\, A'^{a_2}(x_2)\, \bar{A}'^{a_3}(x_3)\, \bar{A}'^{a_4}(x_4) \right\rangle_0 \\
&+ \text{all possible contractions}\,.
\end{aligned} \tag{6.50}$$

Taking all possible Wick contractions, we get the four-point correlator. To obtain the amplitude, we rewrite it in momentum space and take all external legs on-shell. We get

$$M_4(1^+, 2^+, 3^-, 4^-) = \frac{\langle 34\rangle^4}{\langle 12\rangle\, \langle 23\rangle\, \langle 34\rangle\, \langle 41\rangle}\,. \tag{6.51}$$

One can similarly compute higher-point tree and loop level amplitudes by deriving the maps to higher order in the coupling. We can also compute fermionic correlators using this approach.

6.6 Quadratic forms

The light-cone Hamiltonians for the pure and the maximally supersymmetric theories in four dimensions can be expressed as quadratic forms [50–52]. In this section, we briefly show the construction of quadratic forms for theories of arbitrary spin without supersymmetry. We then discuss quadratic form structures for pure Yang-Mills theory. We then comment on the possible connection between the Nicolai map in the light-cone gauge and the quadratic form.

The light-cone Hamiltonian for arbitrary spin theories can be constructed by the closure of Poincaré algebra [35, 36]. The idea is that Hamiltonian itself appears as an element of the Poincaré algebra and can be fixed using the closure of Poincaré algebra.

$$H = \int d^3x \left(\partial\bar{\phi}\bar{\partial}\phi - g \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \bar{\phi}\, \partial^{+\lambda} \left[\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\lambda-n}}\phi \, \frac{\bar{\partial}^{n}}{\partial^{+n}}\phi \right] + \mathcal{O}(g^2) \right) , \tag{6.52}$$

As is well known, for odd λ, non-trivial cubic vertices require the introduction of an antisymmetric structure constant f^{abc}.

In this section, we prove that the Hamiltonians in (6.52) are quadratic forms. Specifically, this means they can be written using "covariant" derivatives as follows

$$H = \int d^3x \; \bar{\mathcal{D}}\phi \, \mathcal{D}\bar{\phi} \, . \tag{6.53}$$

The ansatz for covariant derivatives read

$$\mathcal{D}\bar{\phi} = \partial\bar{\phi} - 2g \sum_{n=0}^{\lambda-1} (-1)^n \binom{\lambda-1}{n} \frac{\bar{\partial}^n}{\partial^{+n+1}} \left[\frac{\bar{\partial}^{\lambda-n-1}}{\partial^{+\lambda-n-1}}\phi \, \partial^{+\lambda}\bar{\phi} \right] , \tag{6.54}$$

$$\bar{\mathcal{D}}\phi = \bar{\partial}\phi - 2g \sum_{n=0}^{\lambda-1} (-1)^n \binom{\lambda-1}{n} \frac{\partial^n}{\partial^{+n+1}} \left[\frac{\partial^{\lambda-n-1}}{\partial^{+\lambda-n-1}}\bar{\phi} \, \partial^{+\lambda}\phi \right] , \tag{6.55}$$

where g is the coupling constant (structure constants, relevant to odd spins, are not shown explicitly). To prove that the Hamiltonian defined as a quadratic form is equivalent to those in (6.52) we need to examine the $O(g)$ contributions from (6.53). These are

$$-2g \sum_{n=0}^{\lambda-1} (-1)^n \binom{\lambda-1}{n} \left[\bar{\partial}\phi \, \frac{\bar{\partial}^n}{\partial^{+n+1}} \left(\frac{\bar{\partial}^{\lambda-n-1}}{\partial^{+\lambda-n-1}}\phi \, \partial^{+\lambda}\bar{\phi} \right) \right] , \tag{6.56}$$

and its complex conjugate. We partially integrate this expression to obtain

$$-2g \sum_{n=0}^{\lambda-1} (-1)^{\lambda+n+1} \binom{\lambda-1}{n} \bar{\phi} \, \partial^{+\lambda} \left[\frac{\bar{\partial}^{n+1}}{\partial^{+n+1}}\phi \frac{\bar{\partial}^{\lambda-n-1}}{\partial^{+\lambda-n-1}}\phi \right] . \tag{6.57}$$

We split (6.57) into two halves P and Q. In P, we shift $n \to \lambda - n - 1$ and invoke the identity

$$\binom{\lambda-1}{n} = \binom{\lambda-1}{\lambda-1-n} . \tag{6.58}$$

This yields

$$P = -g\sum_{n=0}^{\lambda-1}(-1)^n\binom{\lambda-1}{n}\,\bar{\phi}\,\partial^{+\,\lambda}\left[\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\,\lambda-n}}\phi\frac{\bar{\partial}^{n}}{\partial^{+\,n}}\phi\right] . \tag{6.59}$$

In the other half, Q, we shift $n \to n-1$ to obtain

$$Q = -g\sum_{n=1}^{\lambda}(-1)^{\lambda+n}\binom{\lambda-1}{n-1}\,\bar{\phi}\,\partial^{+\,\lambda}\left[\frac{\bar{\partial}^{n}}{\partial^{+\,n}}\phi\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\,\lambda-n}}\phi\right] . \tag{6.60}$$

Thus we have

$$\begin{aligned}\mathcal{H} = P + Q = &\left\{-g\sum_{n=0}^{\lambda-1}(-1)^n\binom{\lambda-1}{n}\,\bar{\phi}\,\partial^{+\,\lambda}\left[\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\,\lambda-n}}\phi\frac{\bar{\partial}^{n}}{\partial^{+\,n}}\phi\right]\right.\\ &\left.-g\sum_{n=1}^{\lambda}(-1)^{\lambda+n}\binom{\lambda-1}{n-1}\,\bar{\phi}\,\partial^{+\,\lambda}\left[\frac{\bar{\partial}^{n}}{\partial^{+\,n}}\phi\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\,\lambda-n}}\phi\right]\right\},\\ &- = g\sum_{n=0}^{\lambda}(-1)^n\left(\left[\binom{\lambda-1}{n}+\binom{\lambda-1}{n-1}\right]\bar{\phi}\,\partial^{+\,\lambda}\left[\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\,\lambda-n}}\phi\frac{\bar{\partial}^{n}}{\partial^{+\,n}}\phi\right]\right) .\end{aligned} \tag{6.61}$$

Using the Pascal triangle property

$$\binom{\lambda-1}{n} + \binom{\lambda-1}{n-1} = \binom{\lambda}{n} , \tag{6.62}$$

this is

$$\mathcal{H} = P + Q = -g\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\,\bar{\phi}\,\partial^{+\,\lambda}\left[\frac{\bar{\partial}^{\lambda-n}}{\partial^{+\,\lambda-n}}\phi\frac{\bar{\partial}^{n}}{\partial^{+\,n}}\phi\right] , \tag{6.63}$$

reproducing the structures in (6.52) and confirming that the higher spin Hamiltonians are indeed quadratic forms.

The focus of our thesis is on Yang-Mills theories and Nicolai map. We first rederive the quadratic form structure form for spin-1 theory using our arbitrary spin result. We plug $\lambda = 1$ in (6.54) and simplify, we get

$$\bar{\mathcal{D}}A^a = \bar{\partial}A^a - g f^{abc}\frac{1}{\partial^+}(\bar{A}^b\partial^+ A^c)\,, \tag{6.64}$$

and $\mathcal{D}\bar{A}^a$ is obtained by complex conjugation. The light-cone Hamiltonian for this theory may be written as

$$\mathcal{H} = 2\int d^3x\, \mathcal{D}\bar{A}^a\bar{\mathcal{D}}A^a\,, \tag{6.65}$$

We now propose an alternative way to express this Hamiltonian. We introduce new variables A'^a , $\bar{A}'^a$ given by

$$A'^a(x) = A^a(x) - 2g f^{abc}\int d^2y\, \partial C^T(x-y)\frac{1}{\partial_-}[\bar{A}^b(x')\partial_- A^c(x')]\,, \tag{6.66}$$

where $x' = (x^+, x^-, y, \bar{y})$ and $2\,\partial\bar{\partial}C^T(x-y) = -\delta^2(x-y)$. In these new variables, the Hamiltonian takes the form

$$\mathcal{H} = -2\int d^3x\, \bar{A}'^a\partial\bar{\partial}A'^a\,. \tag{6.67}$$

Thus, (6.66) is a bosonic transformation that maps the Yang-Mills Hamiltonian to a free Hamiltonian. It is therefore of interest to explore whether this transformation (6.66) has a connection with the Nicolai map (6.20). We believe that the expression (6.66) represents a good place to begin an investigation of possible links between the Nicolai map and quadratic form.

* * *

In this chapter, we discussed the construction of the Nicolai map in terms of physical degrees of freedom up to the second order in the coupling constant. We found two maps in four dimensions that satisfy all three statements of the main theorem. The non-uniqueness of the maps can be helpful in computing scattering amplitudes (correlation functions) to a particular order in the coupling. In the end, we reviewed quadratic form structures in the light-cone Hamiltonian and commented on the possible connection between the Nicolai map and quadratic forms.

Chapter 7

Conclusion and Outlook

This section offers a synopsis of the results presented in this thesis. We highlight our findings and discuss some interesting open problems and research avenues that we hope to work and explore in the future. The thesis focuses on investigating aspects of supersymmetric Yang-Mills theories purely with bosonic variables. We revisited earlier work from 1979, which involved quantizing supersymmetric theories in an alternate way. The basic idea is that for supersymmetric theories involving only quadratic fermions in the Lagrangian, there exists a transformation of the bosonic fields which are non-local and non-linear in nature and it maps the interacting theory to a free bosonic theory. This transformation is such that its Jacobi determinant equals the product of fermion and ghost determinant (for gauge theories).

7.1 Summary of the results

We started with the on-shell supersymmetric Yang-Mills theory in the Landau gauge in chapter 3 and constructed the map up to order g^2 through guess work. We found that our form of the guess map (3.13) satisfies all three conditions of the main theorem (3.2). We discovered that the determinant matching condition is dimensional dependent, and it constraints the allowed value of space-time dimensions where super Yang-Mills theories can exist to be $D = 3, 4, 6, 10$. This result was first obtained in [26]; we recovered this old classic result without requiring any closure of supersymmetry algebra.

To understand the map and its mathematical properties better, we needed the transformation to a higher order in the coupling. We extended the map to the third order in Chapter 4. The form of the map gets complicated at higher orders and the number of terms increases exponentially, so we used an old method developed by Lechtenfeld, Dietz, and Flume [18–20] to construct the map systematically. We re-derived the infinitesimal generator $\mathcal{R}$ of the inverse Nicolai map for all on-shell supersymmetric Yang-Mills theories in space-time dimensions $D = 3, 4, 6, 10$. Using the $\mathcal{R}$ operator (4.22), we obtained the map (4.51) to order g^3 that satisfies the main theorem

(4.1.1). We found that starting from order g^3, we require using the Landau gauge condition for determinant matching.

While working on the order g^3 map, we discovered a new simpler map that applies specifically in six dimensions (5.5). In Chapter 5, we presented the map arrived at by guess work. We emphasized the uniqueness of this approach and determined that the two six-dimensional maps (4.51) and (5.5) cannot be related to each other by partial integrations but can be connected through the Jacobian determinant. Additionally, we outlined an algorithmic approach for recursively determining the map.

In Chapter 6, we studied supersymmetric Yang-Mills theories without fermion fields in the light-cone gauge. We computed the map, in the physical degrees of freedom, to second order in the coupling constant and generalized it to all critical dimensions. We identified two maps, (6.20) and (6.36), in four dimensions that obeys the main theorem. The simpler four-dimensional map (6.20) cannot be derived using the $\mathcal{R}$ prescription. We computed the tree-level three-point and four-point correlation functions for Yang-Mills theory using the inverse map (6.5). The non-uniqueness of these maps can provide insights into studying correlation functions (scattering amplitudes). To establish the connection of quadratic forms with the Nicolai map, we first derived the quadratic form structures for arbitrary spin theories. Subsequently, we discussed the potential relationship between the Nicolai map and structures like quadratic forms that appear in the light-cone Hamiltonian for pure Yang-Mills theory.

7.2 Future directions

This formalism (Nicolai map) presents several possible future research directions and we discuss a few of them below:

We presented two four-dimensional maps at order g^2 in the light-cone gauge and two maps in six dimensions at g^3 in the Landau gauge. We found out that the non-uniqueness arises because the determinant matching condition from the main theorem comments only about the derivatives of the transformation (map) and not the map itself. To better understand the uniqueness, we need a new condition that can impose more constraints on the transformation. Another step would be to study these maps at higher orders in the coupling. In particular, we plan to construct an order g^3 map in the light-cone gauge by taking the 'simple' four-dimensional map (6.20).

In quantum field theory, scattering amplitudes and correlation functions carry the physical information. In this Nicolai map approach, they can be computed in terms of free bosonic expectation value using the inverse map. Although this formalism does not offer any extra advantage for the computation of correlators, we expect that this method might help us quantize supersymmetric theories that cannot be quantized using the standard perturbative techniques. In this context, supermembrane theories are of interest [58]. We aim to set up an integral path approach in terms of the Nicolai map for calculating quantities of physical interest, which are expected to be vertex operators [59].

The maximally supersymmetric Yang-Mills theory, $\mathcal{N} = 4$ super Yang-Mills, in the planar limit, has a lot of simplifying features: its scattering amplitudes are ultra-violet finite, exhibit novel symmetries, and show surprising duality between scattering amplitudes and Wilson loops. The proof of ultraviolet finiteness for $\mathcal{N} = 4$ super Yang-Mills (by power counting) exists only in the light-cone gauge [9, 10]. We aim to construct the map for $\mathcal{N} = 4$ theory in light-cone superspace and understand its finiteness properties in this mapped formalism. Our first task in this program is to establish a direct connection between the Nicolai map and scattering amplitudes in momentum space that we hope will offer new insights into understanding the surprising relations exhibited by $\mathcal{N} = 4$ amplitudes at the level of Lagrangian.

$\mathcal{N} = 8$ supergravity is a maximally supersymmetric theory of gravity. It shares many properties with $\mathcal{N} = 4$ theory, and at the level of scattering amplitude, they exhibit dualities such as BCJ relations, KLT relations, and double copy. It has been shown to be finite up to five loops using modern amplitude techniques and is hoped to be finite to all orders [60]. Our long-term aim is to formulate supergravity theories without anti-commuting variables and understand the finiteness properties within this approach.

Appendix

A Spinors and the representation of Lorentz group

This section focuses on the spinor representations of the Lorentz group and its various characteristics. The group is associated with symmetry transformations that keep the form of the Minkowski metric intact (2.1.1). We briefly reviewed the representation of the Lorentz group in chapter 2 and found that the representations of complex $su(2)$ algebra can serve to study the finite dimensional representation of the Lorentz group.

The idea is that two mutually commuting $su(2)$ algebra is part of $so(1,3)$ algebra (2.5) and their complex linear combinations are isomorphic to $so(1,3)$ algebra

$$so(1,3) \cong su(2) \times su(2)^*\,, \tag{7.1}$$

so the representations of Lorentz algebra can be labeled by pairs (n,m), which are representations of $SU(2)$, and the spin of the representation can be identified with $n+m$ and has dimension $(2n+1)(2m+1)$. Some examples of the simplest representations are

(a) $(0,0)$ with spin zero is the scalar representation.

(b) $(\frac{1}{2},0)$ denotes a left-handed spinor.

(c) $(0,\frac{1}{2})$ describes a right-handed spinor.

(d) $(\frac{1}{2},\frac{1}{2})$ is for vector represenation.

These spinors $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ are two component objects and are called Weyl spinors. One can use these representations and can generate any other representations by multiplying or adding them together. For example $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ yields a spin 1 representation and is denoted as a four-vector.

An alternate way to understand the spinors is via the relation between the Lorentz group and $SL(2,\mathbb{C})$. The latter is the group of determinant one 2×2 complex matrices. Consider a point

y^μ in Minkowski space as a 2 by 2 Hermitian matrix

$$Y = y_\mu \sigma^\mu = \begin{pmatrix} y_0 + y_3 & y_1 - iy_2 \\ y_1 + iy_2 & y_0 - y_3 \end{pmatrix}, \tag{7.2}$$

where $\sigma^\mu = (1, \sigma^i)$, and σ_i are the Pauli matrices. The matrix Y is Hermitian ($Y = Y^\dagger$) and determinant is equal to a Lorentz scalar $y_\mu y^\mu$. Thus, there exists a one-to-one correspondence between y^μ and 2×2 matrix Y.

Consider now an $SL(2, C)$ transformation S that acts on Y as

$$Y \to Y' = SYS^\dagger\,. \tag{7.3}$$

The transformation maintains the hermiticity of Y: $(Y')^\dagger = Y'$, and also preserves the determinant $\det Y' = \det Y$. This implies that the above transformation (7.3) must be a Lorentz transformation. One simple way to check this is by counting the independent parameters: A general complex 2×2 matrix has four complex entries, and due to the determinant condition, it reduces to three complex or six real parameters, which agrees with the three rotations and three boosts parameters of Lorentz group . Additionally, since the transformation S and $-S$ define Y same way, this means that $SL(2, \mathbb{C})$ is the double cover of $SO(1, 3)$ and

$$SO(1,3) \cong \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}\,. \tag{7.4}$$

The above relation can be used to study the representation of $SO(1, 3)$ in terms of $SL(2, \mathbb{C})$.

Take a two-component, complex entity $\psi_\alpha = (\psi_1, \psi_2)$ as the basic representation of $SL(2, \mathbb{C})$ that transform as

$$\psi_\alpha \to \psi'_\alpha = S_\alpha^\beta \psi_\beta\,, \tag{7.5}$$

where $\alpha, \beta = 1, 2$, the matrix $S \in SL(2, \mathbb{C})$ and from the classification of (7.1), the spinor ψ corresponds to $(\frac{1}{2}, 0)$ known as left-handed Weyl spinor.

The complex conjugate representation $\bar{\psi}$ transform as

$$\bar{\psi}_{\dot{\alpha}} \to (S^*)_{\dot{\alpha}}^{\dot{\beta}}\,\bar{\psi}_{\dot{\beta}} \qquad \dot{\alpha}, \dot{\beta} = 1, 2\,, \tag{7.6}$$

and it corresponds to $(0, \frac{1}{2})$ known as right-handed Weyl spinor. To construct Lorentz scalars out of these spinors, we introduce invariant tensors of $SL(2, \mathbb{C})$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \tag{7.7}$$

that enables the raising and lowering of spinor indices (example $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$), just as we use Minkowski metric to raise and lower vector indices. The only difference is that $\epsilon^{\alpha\beta}$ is anti-symmetric because the spinors are anti-commuting Grassmann variables. An example of an invariant is

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi\,. \tag{7.8}$$

One can construct other representations of the Lorentz group using these spinors. The Dirac spinor, a four-component, is an example that can be built using the two Weyl spinors. It is denoted as

$$\lambda = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix}\,, \tag{7.9}$$

and in four dimensions, it forms a reducible representation of the Lorentz group.

B Construction of spinors in various dimensions

We here examine the Clifford algebra in different spacetime dimensions and explore its irreducible representations and properties, which help us in finding spinors in allowed spacetime dimensions. Spinors are integral to the construction of supersymmetric theories and the supercharges transform as spinors under the Lorentz group. Thus, to study supersymmetry algebra, one needs to understand the properties of spinors in arbitrary dimensions.

To introduce spinors, we start with Clifford algebra relation in D dimensions, involving a set of γ matrices that adhere to the relation

$$\{\gamma_\mu\,,\,\gamma_\nu\} = 2\,\eta_{\mu\nu}\,\mathbb{I}\,, \tag{7.10}$$

where $\mu = 0, 1, 2, ..., D-1$, $\mathbb{I}$ is the identity matrix, and the spinor indices α, β are suppressed.

Gamma matrices in higher dimensions can be constructed by tensoring the product of sigma matrices. For even dimensions, the representation of the gamma matrix has $2^{\frac{D}{2}}$ complex degrees of freedom. For odd-dimensions, the representation is $2^{\frac{D-1}{2}}$ dimensional and includes an additional matrix $\gamma_{D+1} = i^n\,\gamma_0\gamma_1\,....\gamma_{D-1}$ that anti-commutes with all γ_μ's and squares to one. Gamma matrices satisfy the following hermiticity properties

$$\gamma_0^\dagger = -\gamma_0\,, \qquad \gamma_i^\dagger = \gamma_i\,. \tag{7.11}$$

A Clifford algebra in $D+2$ dimension can be constructed using the D-dimensional Clifford algebra. The Pauli matrices are given as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{7.12}$$

and the gamma matrix γ_μ generates the Clifford algebra in D dimensions.

The matrices that forms the $(D+2)$-dimensional Clifford algebra can be constructed as

$$\begin{aligned} \Gamma_\mu &= 1 \otimes \gamma_\mu \,, \\ \Gamma_D &= \sigma_1 \otimes \gamma_{D+1} \,, \\ \Gamma_{D+1} &= \sigma_3 \otimes \gamma_{D+1} \,. \end{aligned} \tag{7.13}$$

Once we have the Clifford algebra, we can define the appropriate spinor. In even dimension, one can use the matrix γ_{D+1} to project the spinor into left and right handed spinor components

$$\lambda_L = \frac{1}{2}(1-\gamma_{D+1})\lambda \,, \qquad \lambda_R = \frac{1}{2}(1+\gamma_{D+1})\lambda \,. \tag{7.14}$$

and this halves the number of independent spinor components. A second constraint that one can apply to spinors is the Majorana condition.

$$\bar{\lambda} = (\lambda^T C) \,, \tag{7.15}$$

by choosing the appropriate representation of the gamma matrix, and it is consistent only in dimensions $D \equiv 1,2,3,4 \bmod 8$ [61, 62]. We can have one more condition where the fermion is both real and respects chirality, and it is called Majorana-Weyl spinor. They exists in $D = 2 \bmod 8$, such as $D = 2, 10$ and so on. We outline below in the table the possible spinors in various dimensions

Spinors/ dimensions	2	3	4	6	8	10
Majorana	△	△	△		△	△
Weyl	△		△	△	△	△
Majorana-Weyl	△					△

The Majorana and Weyl spinors exist separately in four dimensions but not both simultaneously. In supersymmetric theories, we will mostly use Majorana representation because it applies to more dimensions. The six-dimensional case can be separately studied for the Weyl spinor.

For the case of the existence of Majorana spinors in Euclidean spacetime, we use the definition proposed by Nicolai in [63, 64]. This approach keeps all the properties of the Majorana spinor defined above intact, and one can easily rotate the correlation functions between Euclidean and Minkowski spacetime.

C Gaussian integrals

Gaussian integrals are fundamentally important in mathematics and physics, especially within quantum field theory, where they present as infinite-dimensional integrals that are difficult to solve. They come in two forms: those involving ordinary real or complex variables and those involving Grassmann variables. We outline here how to solve both kinds of integrals

C.1 Gaussian integral with ordinary variables

An example of the simple Gaussian integral with an ordinary real variable is

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}ax^2\right) dx = \sqrt{\frac{2\pi}{a}}\,, \tag{7.16}$$

that can be generalized to a higher dimensional integral which frequently appears in the theory of quantum fields. The generalization of the above integral in n dimensions is

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}\right) dx_1\, dx_2 .....dx_n\,, \tag{7.17}$$

where $\mathbf{x}^T$ denotes a row vector, $\mathbf{x}$ refers to a column vector, and $\mathbf{A}$ is a n-dimensional symmetric, non singular matrix. To solve the above integral, consider the following change of variables

$$\mathbf{x} = \mathbf{O}\mathbf{y}\,, \quad d\mathbf{x} = \mathbf{O}d\mathbf{y}\,; \quad dx_1 dx_2 ...... = |\mathbf{O}|\, dy_1 dy_2 .... \tag{7.18}$$

where $\mathbf{O}$ is a special orthogonal matrix with $\det\mathbf{O} = 1$. Plugging the above transformation in (7.17), we get

$$\int \exp\left(-\frac{1}{2}\,\mathbf{y}^T\,\mathbf{O}^{-1}\mathbf{A}\mathbf{O}\,\mathbf{y}\right) dy_1\, dy_2 .....dy_n\,, \tag{7.19}$$

such that $\mathbf{O}^{-1}\mathbf{A}\mathbf{O}$ is a diagonal matrix and the integral gives

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} d_i y_i^2\right) dy_1\, dy_2 .....dy_n = \prod_{i=1}^{n} \sqrt{\frac{2\pi}{d_i}} = \frac{(2\pi)^{n/2}}{(\det\mathbf{A})^{1/2}}\,. \tag{7.20}$$

If $n \to \infty$, the numerator blows up, but in quantum field theory, this term comes as an overall normalization constant so that it can be ignored. The functional determinant for the gauge theory can be computed using the above calculation, and it reads

$$\int DA\, e^{-\frac{i}{2}\int d^Dx\, A^a_\mu M^{\mu\nu} A^a_\nu} = [\det M]^{-D/2}\,. \tag{7.21}$$

where M is the quadratic operator, and the normalization is hidden.

C.2 Grassmann integrals

To understand the fermions better, we need to know the properties of anti-commuting variables known as Grassmann numbers. Given any two Grassmann numbers ψ, χ their product anti-commutes $\psi\chi = -\chi\psi$ and $\psi^2 = \chi^2 = 0$.

Take a function $f(\theta)$ of Grasmann variable θ and expand it using the Taylor-series as $f(\theta) = a + b\theta$. Note that the product of two Grassmann numbers acts like a regular number. To solve the integral involving Grassmann numbers, we require the technique of Berezin integration [22]. The standard Berezin integral over θ is

$$\int d\theta\, \theta = 1\,, \qquad \int d\theta\ = 0. \tag{7.22}$$

and the integral of $f(\theta)$ will be

$$\int d\theta f(\theta) = \int d\theta\,(a + b\theta) = a\int d\theta + b\int d\theta\,\theta = b\,. \tag{7.23}$$

Generalizing this to a simple Gaussian integral involving two Grasmman variables gives

$$\int d\theta_1 d\theta_2\, \exp\,(-\frac{1}{2}a\,\theta_1\theta_2) = \int d\theta_1 d\theta_2\,(1 - \frac{1}{2}a\,\theta_1\theta_2) = \frac{1}{2}a\,. \tag{7.24}$$

Consider now a Gaussian integral involving n variables

$$I_n(B) = \int d^n\theta\, \exp\,(-\frac{1}{2}\boldsymbol{\theta}^T B\,\boldsymbol{\theta})\,, \tag{7.25}$$

where $\boldsymbol{\theta}^T = [\theta_1, \theta_2....\theta_n]$ and B is an anti-symmetric matrix. As an example, we first solve for

$n = 4$, where B is an antisymmetric 4 by 4 matrix

$$\int d^4\theta \, \exp\,(-\frac{1}{2}\boldsymbol{\theta}^T B\,\boldsymbol{\theta}) = \int d\theta_1 \,.....\theta_4 \left[1 - \frac{1}{2}\boldsymbol{\theta}^T B\,\boldsymbol{\theta} + \frac{1}{2!}\,(-\frac{1}{2}\boldsymbol{\theta}^T B\,\boldsymbol{\theta})^2 + ...\right] . \tag{7.26}$$

here only the second term survives after the Grassmann integration. The term $\boldsymbol{\theta}^T B\,\boldsymbol{\theta}$ can be simplified as

$$\begin{bmatrix}\theta_1 & \theta_2 & \theta_3 & \theta_4\end{bmatrix} \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix} \begin{bmatrix}\theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4\end{bmatrix}$$

$$= 8\,\theta_1\theta_2\theta_3\theta_4\,(b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23})\,, \tag{7.27}$$

and this is equal to $|B|^{1/2}$, so

$$\int d^4\theta \, \exp\,(-\frac{1}{2}\boldsymbol{\theta}^T B\,\boldsymbol{\theta}) = (\det B)^{1/2}\,, \tag{7.28}$$

This can be generalized for any n, we get

$$\int d^n\theta \, \exp\,(-\frac{1}{2}\boldsymbol{\theta}^T B\,\boldsymbol{\theta}) = (\det B)^{1/2}\,. \tag{7.29}$$

The above relation can be used to compute the functional determinant of Majorana fermions

$$\int D\lambda \; e^{-\frac{1}{2}\int d^Dx\,\bar{\lambda}^a\,M\lambda^a} = (\det M)^{1/2}\,. \tag{7.30}$$

We can generalize the relation (7.29) for the case of complex Grassmann numbers. First, we outline the variables

$$\eta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2)\,, \qquad \bar{\eta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)\,, \tag{7.31}$$

and the integral

$$I_n(C) = \int d\bar{\eta}_1 d\eta_1 \,......\, d\bar{\eta}_n d\eta_n \, \exp\,(-\sum_{i,j=1}^{n} \bar{\eta}_i\, C_{ij}\, \eta_j)\,, \tag{7.32}$$

can be solved by Taylor expanding the above integral and using the steps similar to (7.27)

$$
\begin{aligned}
I_n(C) &= (-1)^n \int d\bar{\eta}_1 d\eta_1 \,......\, d\bar{\eta}_n d\eta_n \; \bar{\eta}_1 \eta_1 \,.....\bar{\eta}_n \eta_n \det(C) \\
&= \det(C)\,. \qquad (7.33)
\end{aligned}
$$

The above formula can be used to compute the Fadeev-Popov determinant.

Bibliography

[1] B. Zumino, “Supersymmetry and the Vacuum,” Nucl. Phys. B **89** (1975), 535

[2] S. R. Coleman and J. Mandula, “All Possible Symmetries of the S Matrix,” Phys. Rev. **159** (1967), 1251-1256

[3] Y. A. Golfand and E. P. Likhtman, “Extension of the Algebra of Poincare Group Generators and Violation of p Invariance,” JETP Lett. **13** (1971), 323-326

[4] R. Haag, J. T. Lopuszanski and M. Sohnius, “All Possible Generators of Supersymmetries of the s Matrix,” Nucl. Phys. B **88** (1975), 257

[5] J. Wess and B. Zumino, “Supergauge Transformations in Four-Dimensions,” Nucl. Phys. B **70** (1974), 39-50

[6] S. P. Martin, “A Supersymmetry primer,” Adv. Ser. Direct. High Energy Phys. **18** (1998), 1-98 [arXiv:hep-ph/9709356 [hep-ph]].

[7] N. Arkani-Hamed, F. Cachazo and J. Kaplan, “What is the Simplest Quantum Field Theory?,” JHEP **09** (2010), 016 [arXiv:0808.1446 [hep-th]].

[8] J. M. Henn, “What Can We Learn About QCD and Collider Physics from N=4 Super Yang–Mills?,” Ann. Rev. Nucl. Part. Sci. **71** (2021), 87-112 [arXiv:2006.00361 [hep-th]].

[9] S. Mandelstam, “Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model,” Nucl. Phys. B **213** (1983), 149-168

[10] L. Brink, O. Lindgren and B. E. W. Nilsson, “The Ultraviolet Finiteness of the N=4 Yang-Mills Theory,” Phys. Lett. B **123** (1983), 323-328

[11] M. F. Sohnius and P. C. West, “Conformal Invariance in N=4 Supersymmetric Yang-Mills Theory,” Phys. Lett. B **100** (1981), 245

[12] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, “Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang-Mills theory,” Nucl. Phys. B **828** (2010), 317-374 [arXiv:0807.1095 [hep-th]].

[13] L. F. Alday and J. M. Maldacena, JHEP **06** (2007), 064 [arXiv:0705.0303 [hep-th]].

[14] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et al.* "Review of AdS/CFT Integrability: An Overview," Lett. Math. Phys. **99** (2012), 3-32 [arXiv:1012.3982 [hep-th]].

[15] H. Nicolai, "On a New Characterization of Scalar Supersymmetric Theories," Phys. Lett. B **89** (1980), 341

[16] H. Nicolai, "Supersymmetry and Functional Integration Measures," Nucl. Phys. B **176** (1980), 419-428

[17] H. Nicolai, "Supersymmetric Functional Integration Measures," CERN-TH-4010-84.

[18] R. Flume and O. Lechtenfeld, "On the Stochastic Structure of Globally Supersymmetric Field Theories," Phys. Lett. B **135** (1984), 91-95

[19] K. Dietz and O. Lechtenfeld, "Nicolai Maps and Stochastic Observables From a Coupling Constant Flow," Nucl. Phys. B **255** (1985), 149-173

[20] K. Dietz and O. Lechtenfeld, "Ghost free quantization of nonabelian gauge theories via the Nicolai transformation of their supersymmetric extensions," Nucl. Phys. B **259** (1985), 397

[21] P. T. Matthews and A. Salam, "Propagators of quantized field," Nuovo Cim. **2** (1955), 120-134

[22] F. A. Berezin, "The method of second quantization," Pure Appl. Phys. **24** (1966), 1-228

[23] L. D. Faddeev and V. N. Popov, "Feynman Diagrams for the Yang-Mills Field," Phys. Lett. B **25** (1967), 29-30

[24] S. Ananth, H. Nicolai, C. Pandey and S. Pant, "Supersymmetric Yang–Mills theories:not quite the usual perspective," J. Phys. A **53** (2020) no.17, 17 [arXiv:2001.02768 [hep-th]].

[25] F. Gliozzi, J. Scherk and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model," Nucl. Phys. B **122** (1977), 253-290

[26] L. Brink, J. H. Schwarz and J. Scherk, "Supersymmetric Yang-Mills Theories," Nucl. Phys. B **121** (1977), 77-92

[27] H. Nicolai and J. Plefka, “$N = 4$ super-Yang-Mills correlators without anticommuting variables,” Phys. Rev. D **101** (2020) no.12, 125013 [arXiv:2003.14325 [hep-th]].

[28] S. Ananth, O. Lechtenfeld, H. Malcha, H. Nicolai, C. Pandey and S. Pant, “Perturbative linearization of supersymmetric Yang-Mills theory,” JHEP **10** (2020), 199 [arXiv:2005.12324 [hep-th]].

[29] S. Ananth, H. Malcha, C. Pandey and S. Pant, “Supersymmetric Yang-Mills theory in $D = 6$ without anticommuting variables,” Phys. Rev. D **103** (2021) no.2, 025010 [arXiv:2006.02457 [hep-th]].

[30] N. Bhave and S. Pant, “Nicolai maps and uniqueness in the light-cone gauge,” JHEP **09** (2024), 121 [arXiv:2406.18238 [hep-th]].

[31] H. Malcha and H. Nicolai, “Perturbative linearization of super-Yang-Mills theories in general gauges,” JHEP **06** (2021), 001 [arXiv:2104.06017 [hep-th]].

[32] G. Leibbrandt, “Introduction to Noncovariant Gauges,” Rev. Mod. Phys. **59** (1987), 1067

[33] Lee, H. C.. "Yang-Mills Theories in Axial and Light-Cone Gauges, Analytic Regularization and Ward Identities." Chinese Journal of Physics 23.2 (1985): 90-143.

[34] P. A. M. Dirac, “Forms of Relativistic Dynamics,” Rev. Mod. Phys. **21** (1949), 392-399

[35] A. K. H. Bengtsson, I. Bengtsson and L. Brink, “Cubic Interaction Terms for Arbitrary Spin,” Nucl. Phys. B **227** (1983), 31-40

[36] Y. S. Akshay and S. Ananth, “Factorization of cubic vertices involving three different higher spin fields,” Nucl. Phys. B **887** (2014), 168-174 [arXiv:1404.2448 [hep-th]].

[37] S. Ananth, A. Kar, S. Majumdar and N. Shah, “Deriving spin-1 quartic interaction vertices from closure of the Poincaré algebra,” Nucl. Phys. B **926** (2018), 11-19 [arXiv:1707.05871 [hep-th]].

[38] S. Ananth, N. Bhave, C. Pandey and S. Pant, “Deriving interaction vertices in higher derivative theories,” Phys. Lett. B **853** (2024), 138704 [arXiv:2306.05074 [hep-th]].

[39] R. R. Metsaev, “S matrix approach to massless higher spins theory. 2: The Case of internal symmetry,” Mod. Phys. Lett. A **6** (1991), 2411-2421 .

[40] R. R. Metsaev, “Poincare invariant dynamics of massless higher spins: Fourth order analysis on mass shell,” Mod. Phys. Lett. A **6** (1991), 359-367 .

[41] A. K. H. Bengtsson, Non-constructibility of Higher Spin Quartic Amplitudes, JHEP **12** (2016), 134 [arXiv:1607.06659 [hep-th]].

[42] D. Ponomarev,“Basic introduction to higher-spin theories,” [arXiv:2206.15385 [hep-th]].

[43] H. Ezawa and J. R. Klauder, “Fermion without fermions: The Nicolai map revisited,” Prog. Theor. Phys. **74** (1985), 904-915

[44] O. Lechtenfeld, “Construction of the Nicolai mapping in supersymmetric field theories,” BONN-IR-84-42.

[45] H. Malcha, “The Nicolai Map and its Application in Supersymmetric Field Theories,” [arXiv:2305.13798 [hep-th]].

[46] O. Lechtenfeld and M. Rupprecht, “Universal form of the Nicolai map,” Phys. Rev. D **104** (2021) no.2, L021701 [arXiv:2104.00012 [hep-th]].

[47] O. Lechtenfeld, “The Nicolai-map approach to supersymmetry,” J. Phys. Conf. Ser. **2667** (2023) no.1, 012001 [arXiv:2309.00481 [hep-th]].

[48] O. Lechtenfeld and M. Rupprecht, “Is the Nicolai map unique?,” JHEP **09** (2022), 139 [arXiv:2207.09471 [hep-th]].

[49] M. Rupprecht, “The coupling flow of $\mathcal{N} = 4$ super Yang-Mills theory,” JHEP **04** (2022), 004 [arXiv:2111.13223 [hep-th]].

[50] S. Ananth, L. Brink, S. S. Kim and P. Ramond, “Non-linear realization of PSU(2, 2|4) on the Light-Cone”, Nucl. Phys. B **722**, 166-190 (2005) [arXiv:hep-th/0505234 [hep-th]]

[51] S. Ananth, L. Brink and M. Mali, “Yang-Mills theories and quadratic forms,” JHEP **08** (2015), 153 [arXiv:1507.01068 [hep-th]].

[52] S. Ananth, C. Pandey and S. Pant, “Higher spins, quadratic forms and amplitudes,” JHEP **07** (2020) no.07, 100 [arXiv:2005.10376 [hep-th]].

[53] O. Lechtenfeld, "Nicolai maps for super Yang–Mills on the light cone," Phys. Lett. B **856** (2024), 138950 [arXiv:2406.04406 [hep-th]].

[54] L. Casarin, O. Lechtenfeld and M. Rupprecht, "Nicolai maps with four-fermion interactions," JHEP **12** (2023), 132 [arXiv:2310.19946 [hep-th]].

[55] H. Elvang and Y. t. Huang, "Scattering Amplitudes," [arXiv:1308.1697 [hep-th]].

[56] S. Ananth, "Spinor helicity structures in higher spin theories," JHEP **11** (2012), 089 [arXiv:1209.4960 [hep-th]].

[57] S. J. Parke and T. R. Taylor, "An Amplitude for n Gluon Scattering," Phys. Rev. Lett. **56** (1986), 2459

[58] O. Lechtenfeld and H. Nicolai, "A perturbative expansion scheme for supermembrane and matrix theory," JHEP **02** (2022), 114 [arXiv:2109.00346 [hep-th]].

[59] A. Dasgupta, H. Nicolai and J. Plefka, "Vertex operators for the supermembrane," JHEP **05** (2000), 007 [arXiv:hep-th/0003280 [hep-th]].

[60] Z. Bern, J. J. M. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, "Supergravity amplitudes, the double copy and ultraviolet behavior," [arXiv:2304.07392 [hep-th]].

[61] P. C. West, "Supergravity, brane dynamics and string duality," [arXiv:hep-th/9811101 [hep-th]].

[62] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge University Press, 2007, ISBN 978-0-511-25228-0, 978-0-521-63304-8, 978-0-521-67228-3

[63] H. Nicolai, "A Possible constructive approach to (SUPER phi**3) in four-dimensions. 1. Euclidean formulation of the model," Nucl. Phys. B **140** (1978), 294-300

[64] P. van Nieuwenhuizen and A. Waldron, "On Euclidean spinors and Wick rotations," Phys. Lett. B **389** (1996), 29-36 [arXiv:hep-th/9608174 [hep-th]].