Quantum Aspects of Cosmology

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by

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Certificate

This is to certify that this thesis entitled Quantum Aspects of Cosmology submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by K. Nidhi Sudhir at International Center for Theoretical Sciences (ICTS -TIFR), under the supervision of Dr. Suvrat Raju during the academic year 2017-2018.

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Declaration

I hereby declare that the matter embodied in the report entitled Quantum Aspects of Cosmology, are the results of the investigations carried out by me at the Department of Physics, International Center for Theoretical Sciences, under the supervision of Dr. Suvrat Raju and the same has not been submitted elsewhere for any other degree.

Student
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This thesis is dedicated to my beloved parents and school teachers, especially to Gayathri Mam.

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Abstract

It is an interesting task to view the entire universe using quantum mechanics. (Here by the entire universe I mean both the space-time and matter.) And the question of how to do so has been pondered upon by a few minds including those of Wheeler, DeWitt, James Hartle, Stephen Hawking etc. As answers to many questions two formalism viz, the path integral formalism and the canonical formalism were introduced and used to understand the quantum aspects of the universe (or Quantum Gravidynamics). These allow one to construct, for very simple model universes, their wavefunctionals and consequently calculate the expectation values of required observables.

In the following sections we will start by familiarizing ourselves with the concept of wavefunctionals with a couple of simple model examples from QFT. Following this, we look at the Path integral formalism developed for quantum gravidynamics and summarize a paper which points out an inconsistency in two age old proposals. I will keep the canonical formalism for later, as it becomes a crucial part of the second half of the project. Here, I will briefly introduce a few problems faced in the canonical quantization. Thereafter we will understand Geometric Quantization as it gives a natural Hilbert space construction for the respective classical systems. This will allow us to understand better the Hilbert space problem in the canonical theory. As work is still in progress, I conclude this thesis by remarking on how Geometric Quantization comes to the rescue.

Contents

Chapter 1

Wavefunctional in de Sitter space

1.1 Introduction

In this chapter, we familiarize with the concept of wavefunctionals in QFT. Using a few of their simple properties rediscovered and presented below, we proceed to calculate the same for the case of a scalar massless field in the static de Sitter spacetime background.[1]

1.2 Wavefunctionals

Given an action describing the evolution of fields, one could construct a Hamiltonian operator from it using the usual procedures of first quantisation. The resulting operator consists of functional derivatives instead of the usual derivatives with respect to classical observables. The eigenstates of this operator are then represented as functionals on the space of possible field configurations. These, and their superpositions are called wavefunctionals and describe the quantum state of the system. These wavefunctionals can be seen as giving the probability for the occurrence of a particular field on the space-like hypersurface corresponding to time t. To familiarize ourselves with the concept we will begin by examining two simple sample wavefunctionals. Here, I will not construct them from a given action, as the concept will be described in much greater detail in the upcoming chapters.

For now, let's begin with the simple case of a Wavefunctional corresponding to a free scalar field in a static spacetime background. As the action has no interaction terms, the wavefunctional takes the form of a Gaussian as below,

[1]This part of the work was done with Sheryl Mathew and Mrunmay Jagadale, S.N.Bhatt students under Dr. Suvrat Raju

$$\Psi(\phi) = e^{\int\int \frac{G(x,x')}{2}\phi(x)\phi(x')\mathrm{d}^4x\mathrm{d}^4x}$$

Here, if we assume homogeneity and isotropy of the universe, then G(x,x') can only be a function of $|x - x'|$. The two point correlation function for this, given by,

$$< \phi(y), \phi(y') >= \frac{1}{N}\int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x'\Big)\phi(y)\phi(y')D\phi$$

where,

$$N = \int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x'\Big)D\phi$$

is the normalization constant, can be calculated (see Appendix A) to find,

$$< \phi(y), \phi(y') >= G^{-1}(y,y').$$

Here $G^{-1}(y,y')$ is defined such that $\int G^{-1}(y,x)G(x,z)\mathrm{d}x = \delta(y-z)$. Similarly, any higher n-point correlation functions can be calculated (Appendix A). We realize that when n is odd the correlator vanishes and when n is even the correlation functions can be written in terms of the sums of products of the the two point correlators. The above calculation can be done in the momentum space too, that is, with $\phi(k)$. Since $G(x,x')$depends only on $|x-x'|$, in the momentum representation it can only be of the form $G(k)$. The two point correlator in this space is given by,

$$\big\langle\phi(k)\phi(k')\big\rangle = (2\pi)^4\delta(k+k')/G(k)$$

Proceeding to a slightly more realistic example of an interacting scalar field, the wavefunctional looks like e^{-S} where S is given by,

$$S = \int G^{-1}(x_1,x_2)\phi(x_1)\phi(x_2)dx_1dx_2 + \Lambda\int C(x_1,x_2,x_3)\phi(x_1)\phi(x_2)\phi(x_3)dx_1dx_2dx_3$$
$$+ \Lambda^2\int D(x_1,x_2,x_3,x_4)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)dx_1dx_2dx_3dx_4$$

Here, $\Lambda << 1$ and hence the second and third terms can be treated as perturbations. To calculate the three and four point correlation functions we take a series expansion with respect to Λ and obtain results dependent on $G^{-1}(x,x')$, $C(x,x',x'')$ and $D(x,x',x'',x''')$. The entire result and the corresponding Feynman diagrams are given in Appendix A.

Using the above properties of a wavefunctional we can write down its expression in different spacetime backgrounds if we know the correlation functions of the field in the corresponding background. Hence, in the next section we will calculate the correlation functions in de Sitter spacetime (using second quantization). But, before we look at the wavefunctional in de Sitter spacetime, we will also calculate the correlation functions in Anti de Sitter spacetime. Here, we will discover a surprising relation between the correlation functions in dS and Ads which will help us reinterpret the function $G^{-1}(x,x')$.

1.3 Correlation functions in de Sitter spacetime

Since, we are considering the static background case, the Lagrangian for the evolution of a free scalar massless field is,

$$L = \int \frac{1}{2}\sqrt{-g}(\nabla_\mu \phi(x))^2 \mathrm{d}^3x$$

where, the length element is taken to be [2],

$$\mathrm{d}s^2 = -\mathrm{d}x_0^2 + R(x_0)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)$$

In terms of the conformal time η, defined such that, $\mathrm{d}x_0 = R(\eta)\mathrm{d}\eta$, $R(\eta)$ takes the form, $R(\eta) = \frac{1}{H\eta}$. This gives the following form of the length element,

$$\mathrm{d}s^2 = \frac{\alpha^2(-d\eta^2 + dx_1^2 + dx_2^2 + dz^2)}{\eta^2}$$

where $\alpha = \frac{1}{H}$. Given this, the Euler Lagrange E.O.M becomes,

$$0 = [-\partial_\eta^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 + \frac{2}{\eta}\partial_\eta]\phi.$$

with solutions,

$$\begin{aligned}\phi(x,\eta) &= \int (a(\bar{k})e^{\iota\bar{k}\cdot\bar{x}}\eta^{\frac{3}{2}}\mathrm{H}^{(2)}_{\frac{3}{2}}(k\eta)\Lambda_k + a^\dagger(\bar{k})e^{-\iota\bar{k}\cdot\bar{x}}\eta^{\frac{3}{2}}\mathrm{H}^{(1)}_{\frac{3}{2}}(k\eta)\Lambda_k)\frac{\mathrm{d}^3k}{(2\pi)^3} \\ &= \int (a(\bar{k})\eta^{\frac{3}{2}}\mathrm{H}^{(2)}_{\frac{3}{2}}(k\eta)\Lambda_k + a^\dagger(-\bar{k})\eta^{\frac{3}{2}}\mathrm{H}^{(1)}_{\frac{3}{2}}(k\eta)\Lambda_k)e^{\iota\bar{k}\cdot\bar{x}}\frac{\mathrm{d}^3k}{(2\pi)^3}\end{aligned}$$

The corresponding conjugate momentum is,

$$\Pi(x,\eta) = R^2\partial_\eta\phi = R^2\int (a(\bar{k})(\eta^{\frac{3}{2}}\mathrm{H}^{(2)}_{\frac{3}{2}}(k\eta))'\Lambda_k + a^\dagger(-\bar{k})(\eta^{\frac{3}{2}}\mathrm{H}^{(1)}_{\frac{3}{2}}(k\eta))'\Lambda_k)e^{\iota\bar{k}\cdot\bar{x}}\frac{\mathrm{d}^3k}{(2\pi)^3}$$

Now we require these operators to satisfy the following equal time commutation relations,

$$[\phi(x,\eta),\Pi(y,\eta)] = \iota\delta^3(x-y) \quad \big[\phi(x,\eta),\phi(y,\eta)\big] = 0 \quad \big[\Pi(x,\eta),\Pi(y,\eta)\big] = 0$$

and,

$$[a(\bar{k}), a^\dagger(\bar{k'})] = (2\pi)^3\delta^3(k-k').$$

[2]See appendix B

This restricts our normalisation factor Λ_k to take the value, $\Lambda^2 = \pi\frac{H^2}{4}$. This then gives,

$$\langle \phi_{\bar{k}}(\eta)\phi_{\bar{k}'}(\eta)\rangle = \Lambda_k^2\eta^3|\mathrm{H}^{(2)}_{\frac{3}{2}}(k\eta)|^2 \langle 0| \left[a(\bar{k}), a^\dagger(\bar{k'})\right] |0\rangle$$

which in the limit $\eta \to 0$ becomes

$$\lim_{\eta\to 0} \langle \phi_{\bar{k}}(\eta)\phi_{\bar{k}'}(\eta)\rangle = (2\pi)^3\delta^3(\bar{k}-\bar{k}')\frac{H^2}{2k^3}$$

1.4 Correlation functions in Anti de Sitter space

Now let us do the same calculation above for Anti de Sitter spacetime. Here, the length element is given by,

$$ds^2 = \frac{\alpha^2(-dt^2 + dx_1^2 + dx_2^2 + dz^2)}{z^2}{}^3$$

Given this, the E.O.M for the scalar field becomes,

$$\frac{\partial_t^2\phi - \partial_x^2\phi - \partial_y^2\phi - \partial_z^2\phi}{z^2} + \frac{2\partial_z\phi}{z^3} = 0$$

with solutions,

$$\phi(x) = z^{3/2}\mathrm{J}_{3/2}(kz)e^{-\iota(\bar{k}.\bar{x}-\omega t)};\ z^{3/2}\mathrm{J}_{-3/2}(kz)e^{-\iota(\bar{k}.\bar{x}-\omega t)}.$$

Here, $k^2 = \omega^2 - (\bar{k})^2$. Since, the Bessel functions of negative order blow up near $z = 0$, normalisable solutions to our E.O.M correspond to the first kind only. Given this, we may write our field $\phi(x)$ as,

$$\phi(x) = C\int \frac{\mathrm{d}^2k\mathrm{d}\omega}{(2\pi)^3}(z^{3/2}a_{\bar{k},\omega}e^{-\iota(\bar{k}.\bar{x}-\omega t)}\mathrm{J}_{3/2}(kz) + z^{3/2}a^\dagger_{\bar{k},\omega}e^{-\iota(\bar{k}.\bar{x}-\omega t)}\mathrm{J}_{3/2}(kz))$$

and the corresponding conjugate momentum as,

$$\Pi(x) = \frac{\dot{\phi}}{z^2} \quad = \frac{-\iota}{z^2}C\int \frac{\mathrm{d}^2k\mathrm{d}\omega}{\omega}({}^{3/2}a_{\bar{k},\omega}e^{-\iota(\bar{k}.\bar{x}-\omega t)}\mathrm{J}_{3/2}(kz) - z^{3/2}a^\dagger_{\bar{k},\omega}e^{-\iota(\bar{k}.\bar{x}-\omega t)}\mathrm{J}_{3/2}(kz))$$

Now, these operators, should satisfy the following equal time commutation relations,

$$\left[\phi(\bar{x},z,t),\phi(\bar{x'},z',t)\right] = 0 \quad \left[\Pi(\bar{x},z,t),\Pi(\bar{x'},z',t)\right] = 0 \quad \left[\phi(\bar{x},z,t),\Pi(\bar{x'},z',t)\right] = \iota\delta^2(x-x')\delta(z-z)$$

and,

$$\left[a_{\bar{k},\omega}, a^\dagger_{\bar{k}',\omega'}\right] = (2\pi)^3\delta^2(\bar{k}-\bar{k'})\delta(\omega-\omega').$$

[3] see Appendix B

The above conditions require our normalization constant to take the value, $C = \sqrt{\frac{\pi}{2}}$. Hence, in the Fourier space the field takes the form,

$$\phi(\bar{k},\omega,z) = \sqrt{\frac{\pi}{2}}(a_{\bar{k},\omega} + a^{\dagger}_{-\bar{k},-\omega})z^{3/2}\mathrm{J}_{3/2}(kz))$$

The corresponding two point correlation function is,

$$\langle\phi(\bar{k},\omega,z)\phi(\bar{k}',\omega',z')\rangle = (2\pi)^3\delta^2(\bar{k}+\bar{k}')\delta(\omega+\omega')\frac{\pi}{2}z^{3/2}\mathrm{J}_{3/2}(kz)z'^{3/2}\mathrm{J}_{3/2}(kz')$$

When $z = z'$ in the limit $z \to 0$ this becomes,

$$\langle\phi(\bar{k},\omega,z)\phi(\bar{k}',\omega',z)\rangle \sim (2\pi)^3\delta^2(\bar{k}+\bar{k}')\delta(\omega+\omega')z^6k^3$$

Remark: We observe from the above calculations that the two point correlators in dS and AdS spaces are inversely related. Also, from section 1.2, we see that the two point function for a Gaussian Wavefunction is

$$< \phi(y), \phi(y') >= G^{-1}(y,y')$$

. Therefore, at least at this order, we may conclude that the wavefunctional corresponding to de Sitter space may be written as,

$$\Psi[\phi] = Exp[\int \frac{1}{2}\langle\mathscr{O}_1\mathscr{O}_2\rangle\phi_1\phi_2\mathrm{d}x_1\mathrm{d}x_2]$$

where $\langle\mathscr{O}_1\mathscr{O}_2\rangle$ is the AdS two point propagator at the boundary. [4]

In the following section we go on to calculate the three point correlator in AdS space.

1.5 3-point correlation functions in AdS

1.5.1 Propagator in AdS

The Greens function satisfies the equation,

$$[-\frac{2}{z^3}\partial_z + \frac{1}{z^2}(-\partial_t^2 + \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_z^2)]G(x,x',z,z') = \delta(x-x')\delta(z-z')$$

Therefore we have,

$$[\frac{1}{z^2}(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_z^2 - \partial_t^2) - \frac{2}{z^3}\partial_z]G(x,x',z,z') = \delta(x-x')\delta(z-z')$$
$$[\frac{1}{z^2}(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_z^2 - \partial_t^2) - \frac{2}{z^3}\partial_z]\int_{-\infty}^{\infty} G(p,z,z')e^{-ip(x-x')}\frac{\mathrm{d}^3p}{(2\pi)^3} = \delta(z-z')\int_{-\infty}^{\infty} e^{-ip(x-x')}\frac{\mathrm{d}^3p}{(2\pi)^3}$$
$$\int_{-\infty}^{\infty}[\frac{1}{z^2}(\omega^2 - k_1^2 - k_2^2 + \partial_z^2) - \frac{2}{z^3}\partial_z]G(p,z,z')e^{-ip(x-x')}\frac{\mathrm{d}^3p}{(2\pi)^3} = \delta(z-z')\int_{-\infty}^{\infty} e^{-ip(x-x')}\frac{\mathrm{d}^3p}{(2\pi)^3}$$

[4] Now it seems that this is true at all orders. For further details take a look at [1]

Since Bessel functions of the form $J_\alpha[kz]$ for all real positive values of k, form a basis in the space of functions, exactly like e^{ikx}, we could write the two point correlator as,

$$G(p,z,z') = \int_0^\infty G(p,k,z') z^{3/2} \mathrm{J}_{3/2}(kz)\mathrm{d}k$$

Apart from this, Bessel functions also satisfy the closure relation,

$$\int_0^\infty x\mathrm{J}_\alpha(ux)\mathrm{J}_\alpha(vx)\mathrm{d}x = \frac{\delta(u-v)}{u}$$

Given the above two properties, one can write the Green's Function as,

$$G(x,z;x',z') = \int_{-\infty}^\infty \int_0^\infty \frac{2z^{3/2}{z'}^{3/2}k\mathrm{J}_{3/2}(kz)\mathrm{J}_{3/2}(kz')e^{-\iota\bar{p}.(\bar{x}-\bar{x}')}}{(p^2-k^2)\pi}\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{d}k$$

1.5.2 3-point correlator

Here, we will calculate, the three point correlator corresponding to the following Lagrangian with cubic interaction term.

$$L = \int_{-\infty}^\infty \sqrt{-g}(\nabla_\mu\phi(x))^2 + \sqrt{-g}\frac{\lambda}{3!}\phi^3(x)\mathrm{d}^3x$$

As will be described in the next chapter, Feynman Path Integral approach gives us a wavefunctional corresponding to the above Lagrangian as e^{iS} (where S is the action). This corresponds to,

$$\exp\left[i\int_{-\infty}^\infty \sqrt{-g}(\nabla_\mu\phi(x))^2 + \sqrt{-g}\frac{\lambda}{3!}\phi^3(x)\mathrm{d}^4x\right]$$

From the above we find the form of the three point correlators and find their Fourier transform with respect to the translationally invariant coordinates. This, leaves us with,

$$\delta^3(\bar{k}_1+\bar{k}_2+\bar{k}_3)\int\frac{G(\bar{k}_1,z,z')G(\bar{k}_2,z,z')G(\bar{k}_3,z,z')}{z'^4}\mathrm{d}z'$$

Now, from the expression above for G(p,z,z'), its form as $z\to 0$ can be obtained as,

$$\lim_{z\to 0} G(p,z,z') = -z^3e^{-\iota pz'}(1+\iota z'p) = z^3(pz')^{3/2}\mathrm{H}^{(1)}_{3/2}(pz')$$

Therefore,

$$
\begin{aligned}
\int_0^\infty & G(k_1,z,z')G(k_2,z,z')G(k_3,z,z')\frac{\mathrm{d}z'}{z'^4} \\
&= -z^9\int_0^\infty \Big[e^{-\iota(k_1+k_2+k_3)z'}(1+\iota z')(k_1+k_2+k_3)- \\
&z'^2\big(k_1k_2+k_2k_3+k_1k_3-\iota z'^3(k_1k_2k_3)\big)\Big]\frac{\mathrm{d}z'}{z'^4} \\
&= -z^9\lim_{\epsilon\to 0}\int_\epsilon^\infty \Big[e^{-\iota(k_1+k_2+k_3)z'}(1+\iota z')(k_1+k_2+k_3)- \\
&z'^2\big(k_1k_2+k_2k_3+k_1k_3-\iota z'^3(k_1k_2k_3)\big)\Big]\frac{\mathrm{d}z'}{z'^4}
\end{aligned}
\tag{1.1}
$$

Evaluating this integral from ϵ to ∞ we observe that the terms diverging as $\frac{1}{\epsilon}$ and $\frac{1}{\epsilon^2}$ are contact terms. These are hence removed to give,

$$
\begin{aligned}
\langle\phi(\bar{k_1},\omega,z)\phi(\bar{k_2},\omega',z)\phi(\bar{k_3},\omega',z)\rangle =&(2\pi)^3\delta^3(\bar{k}_1+\bar{k}_2+\bar{k}_3)(1/18)\Big[2(k_1+k_2+k_3)^3 \\
&+6(k_1^3+k_2^3+k_3^3-3k_1k_2k_3) \\
&-3\Big(k_1^3+k_2^3+k_3^3\Big)\Big(1.154+2\iota\pi \\
&+2\log\big(\iota\epsilon(k_1+k_2+k_3)\big)\Big)\Big]
\end{aligned}
\tag{1.2}
$$

Similarly we observe that any term analytic in two or more momentum variables should also correspond to delta functions or their derivatives when Fourier transformed. Consider a term of the form $f(\bar{k}_1,\bar{k}_2,\bar{k}_3)\delta^3(\bar{k}_1+\bar{k}_2+\bar{k}_3)$. Fourier transforming this to the position space, we get,

$$
\int f(\bar{k}_1,\bar{k}_2,-\bar{k}_1-\bar{k}_2)e^{-\iota\bar{k}_1\cdot(\bar{x}_1-\bar{x}_3)-\iota\bar{k}_2\cdot(\bar{x}_2-\bar{x}_3)}\mathrm{d}k_1\mathrm{d}k_2
$$

The analytic terms in our expression are of the form, $f(\bar{k}_1,\bar{k}_2,\bar{k}_3)=k_1^{2l}k_2^{2m}k_3^{2n}$, where l,m,n are non-negative integers. Now from the above equations we see that if the three point correlation function is analytic in at least two momenta $\bar{k}_i$ and $\bar{k}_j$, then their Fourier Transform corresponds to terms of the form $\delta^3(\bar{x}_i-\bar{x}_j)$ or their derivatives. Hence, these too correspond to contact terms and are to be removed. Consequently, the three point correlator becomes,

$$
\begin{aligned}
\langle\phi(\bar{k_1},\omega,z)\phi(\bar{k_2},\omega',z)\phi(\bar{k_3},\omega',z)\rangle =&\frac{(2\pi)^3}{3}\Big((k_1^3+k_2^3+k_3^3)\log(k_1+k_2+k_3) \\
&+k_1k_2k_3\Big)\delta^3(\bar{k}_1+\bar{k}_2+\bar{k}_3)
\end{aligned}
\tag{1.3}
$$

Chapter 2

Path Integral Formalism

2.1 Introduction

In the previous chapter the wavefunctionals that we considered were on a static space-time background. Now, let's proceed to understand the evolution of both the space-time and the matter fields quantum mechanically. Quantum gravidynamics is the name given to the quantum theory of the entire universe (space-time and matter fields). As, in the previous cases of QFT and QM, the wavefunctionals describing the states of systems in quantum gravidynamics can be obtained by both the Feynman Path Integral formalism and the Canonical theory. In this chapter we will try to understand the former.

2.2 Formalism

Let's begin by reviewing the case of quantum field theories on a static space-time background. The quantum mechanical amplitude of a history or a field configuration in this case is given by,

$$e^{\iota S(\phi(x,t))}$$

where $S(\phi(x,t))$ is the action describing the dynamics of the field $\phi(x,t)$. Given this amplitude, one can construct the amplitude for other restricted observations from the superpositions of all allowed histories. For example, since time is a well defined quantity here, one could look for the amplitude corresponding to the fields $\phi_1(x, t = t_{in})$ and $\phi_2(x, t = t_f)$ on the hypersurfaces corresponding to the times $t = t_{in}$ and $t = t_f$, respectively. In this case, we will have to sum the amplitudes of all field configurations which satisfy the boundary conditions $\phi_1(x, t_{in})$ and $\phi_2(x, t_f)$. This, then gives us the transition amplitude from $\phi_1(x, t_{in})$ to $\phi_2(x, t_f)$,i.e.,

$$\big\langle \phi_1(x, t_{in}), \phi_2(x, t_f) \big\rangle = \int_{\phi_1(x,t_{in})}^{\phi_2(x,t_f)} e^{\iota S(\phi(x,t))} \mathscr{D}\phi.$$

The wavefunctional may then be obtained as the transition amplitude from a fixed initial condition to any possible value of the function $\phi(x, t = t_f)$ at the final time t_f,i.e.,

$$\Psi(\Phi) = \int_{\phi_1(x,t_{in})}^{\Phi(x,t_f)} e^{\iota S(\phi(x,t))} \mathscr{D}\phi$$

The initial conditions specified, determine the state.

Now, the square of the wavefunctionals in QFT give us the probability of occurrence of a field on a hypersurface at time t. In quantum gravidynamics the wavefunctionals are similarly expected to give the probability of occurrence of a particular space-time geometry and field. A generalization of the path integral formalism in QFT can then be constructed for this case. But, now the concept of time is not well defined. What then is analogous to the above case, where we described the boundary conditions as corresponding to a particular time? In quantum gravidynamics, boundary conditions are specified on spacelike hypersurfaces. That is, if the amplitude corresponding to a particular field configuration (or history) and four geometry is given by,

$$e^{\iota S(\phi(x,t),g_{\mu\nu}(x,t))}$$

then the transition amplitude is defined with respect to the values of the field and the four geometries on an initial and final spacelike hypersurface.This is then obtained by integrating over all possible field configurations and four geometries with the specified boundary conditions on the specified boundary hypersurfaces. Now, it is easy to specify the fields on a hypersurface, but what about the four geometry? Given the set of all four geometries in which a particular spacelike hypersurface occurs (but the geometry is allowed to vary off the surface), we can choose to represent all of them in a fixed gauge near the surface. Having done so the four geometry is now described by just the spatial metric (h_{ij}) on the hypersurface. Then, specifying the four geometry at the boundary corresponds to specifying the spatial metric on the spacelike hypersurface. That is, the transition amplitude is written as,

$$\langle h_{ij}, \phi | h'_{ij}, \phi' \rangle = \int_{h_{ij},\phi}^{h'_{ij},\phi'} e^{\iota S(\phi(x,t),g_{\mu\nu}(x,t))} \mathscr{D}\phi \mathscr{D}g$$

Given this, the wavefunction corresponding to a particular state (specified by its initial boundary condition) is given by,

$$\Psi(\phi, h_{ij}) = \int_{h'_{ij},\phi'}^{\phi,h_{ij}} e^{\iota S(\phi(x,t),g_{\mu\nu}(x,t))} \mathscr{D}\phi \mathscr{D}g$$

Now, an interesting state to study would be the ground state. But given that in closed universes the concept of energy is not well defined, exactly as is the case with time, what does the so called ground state mean? Hartle proposes that the ground state could be the one corresponding to the state of maximum symmetry. To obtain such a wavefunction from the above path integral Hartle and Hawking propose to extend the conclusion of the following argument in the case of usual Quantum Mechanics to Quantum Gravidynamics. In QM, [1] consider the following transition amplitude,

$$\langle x, 0|0, t' \rangle = \sum_n \psi_n^*(x)\psi_n(0)e^{-\iota E_n t} = \int e^{\iota S(x(t))}\delta x(t)$$

Wick rotating time to $t \to -\iota\tau$ and taking the limit $\tau \to -\infty$ gives, having normalized the groundstate energy to zero,

$$\psi_0^*(x)\psi_0(0) = \int e^{-I(x(\tau))}\delta x(\tau)$$

That is, the path integral corresponding to the Euclidean action gives the ground state in the limit $\tau \to \infty$. Hence, Hartle and Hawking propose that the ground state in quantum gravidynamics, should similarly be defined by the path integral corresponding to the Euclidean action i.e.,

$$\Psi_{grnd}(\phi, h_{ij}) = \int_{\phi_{in}, h_{ij,in}}^{\phi, h_{ij}} e^{-I(\phi', h'_{ij})}\mathscr{D}\phi'\mathscr{D}h'_{ij}$$

The advantage of the Euclidean path integral (atleast in the case of usual QM and QFT [2]), is that, it is convergent. On the contrary the oscillatory integral in a Lorentzian path integral is not trivial to compute and need not converge. This is where Turok et al brought Picard Lefschetz Theory to rescue the Lorentzian Path Integral. Before we look in detail the proposals of Hartle & Hawking, and Vilenkin, let's understand Picard Lefschetz Theory in some detail.

[1] derivation from the Hartle Hawking paper cited as [2]

[2] We will see in Chapter 4, that in the case of quantum gravidynamics, the Euclidean path integral does not converge.

Chapter 3

Picard Lefschetz Theory

[1]

Picard Lefschetz theory prescribes a way of writing oscillatory integrals of the form $\int e^{\iota S(x_i)} \mathrm{d}^n x$, where $x_i \in \mathbb{R}$, i ranges as $0 \leq i \leq n$ and $S(x_i)$ is a polynomial in x_i, as a sum of convergent integrals. Given that the function $S(x_i)$ lives on $\mathbb{R}^n$, we begin by complexifying the domain to $\mathbb{C}^n$, and defining the holomorphic function $S(z_i)$.[2].On the complexified domain manifold, we introduce the Kähler metric,

$$\mathrm{d}s^2 = \frac{1}{2}(\mathrm{d}z_i \otimes \mathrm{d}\bar{z}_i + \mathrm{d}\bar{z}_i \otimes \mathrm{d}z_i)$$

and the Kähler form $\Omega = \frac{1}{2}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ (here a sum over the i's is understood). Now, lets define the following for $\iota S(z) = I(z)$, [3]

$$h = Re(I) = \frac{I + \bar{I}}{2}, \qquad H = Im(I) = \frac{I - \bar{I}}{2}$$

Proceeding to find the critical points of I(z) we realize the following.

Given a holomorphic function of z_i such as I(z), the Cauchy Riemann conditions imply that $\bar{\partial}_{\bar{i}} I(z) = 0$. It then follows that if z_p is a critical point of of I(z), i.e., if $\partial_i I(z) = 0$ for all i's, then, $\partial_i h(z_p) = 0 = \bar{\partial}_{\bar{i}} h(z_p)$. This implies that the critical points of I(z) are critical points of h(z) too. The converse of this statement can also be easily proved, allowing us to say that the set of critical points of I(z) and h(z) are equivalent. Lets name this set α and its elements z_α.

Now that we know that we do not miss any critical points of $I(z)$ while working with h(z), lets go on to define the ascent and descent curves (parametrised

[1]This chapter is based on the appendix of [3]. Hence most of the notation will be similar to this reference.

[2]Here we are concerned only with Euclidean domains for the function $S(x_i)$. The theory is in general valid for affine varieties $\mathscr{Y}$ for which a complexification $\mathscr{X}$ exists such that $\bar{y} = y$ for all $y \in \mathscr{Y}$ - Reference- [3]

[3]Hereafter, I denote by z or x the n tuples of z_i and x_i respectively.

by t) from each of α_i. The later is defined as,

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = -2\frac{\partial h}{\partial \bar{z}_i} = -\bar{\partial}_{\bar{i}}\bar{I}, \qquad \frac{\mathrm{d}\bar{z}_i}{\mathrm{d}t} = -2\frac{\partial h}{\partial z_i} = -\partial_i I$$

These are so called because the value of h decreases monotonically along these curves.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -2\partial_i h \bar{\partial}_{\bar{i}} h \leq 0$$

Similarly one may define the ascent curves given by,

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = 2\frac{\partial h}{\partial \bar{z}_i} = \bar{\partial}_{\bar{i}}\bar{I}, \qquad \frac{\mathrm{d}\bar{z}_i}{\mathrm{d}t} = 2\frac{\partial h}{\partial z_i} = \partial_i I$$

along which the value of h monotonically increases. But, along these curves the value of H is conserved since,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}z_i}{\mathrm{d}t}\frac{\partial H}{\partial z_i} + \frac{\mathrm{d}\bar{z}_i}{\mathrm{d}t}\frac{\partial H}{\partial \bar{z}_i} = \pm\frac{1}{2\iota}(\partial_i I \bar{\partial}_{\bar{i}}\bar{I} - \partial_i I \bar{\partial}_{\bar{i}}\bar{I}) = 0$$

where the '+' corresponds to the ascend curves and '-' corresponds to the descent curves.

Given the Kähler symplectic form Ω, one can understand the ascend and descend curves as solutions to Hamiltonian flow equations, where the role of the Hamiltonian is played by H. This becomes evident when we rewrite the ascend and descend equations as the following,

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = \{H, z_i\}, \qquad \frac{\mathrm{d}\bar{z}_i}{\mathrm{d}t} = \{H, \bar{z}_i\}$$

where the Poisson bracket defined by the Kähler two form is[4],

$$\{f, g\} = -2\iota(\partial_i f \bar{\partial}_{\bar{i}} g - \bar{\partial}_{\bar{i}} f \partial_i g)$$

What we have in our hands right now is the collection of all ascend and descend curves originating from all critical points of the holomorophic function S(z). Of these, the ascend ones are such that the integrand $e^{\iota S(z)}$ diverges as $z \to \infty$, while the along descend ones it converges to zero.

Intuitive picture of relative Homologies: Relative homologies on $\mathbb{C}^n$ can be intuitively thought of as the set of all paths connecting two distinct regions of the space. It is represented as $H(\mathscr{X}, \mathscr{Y}; \mathbb{Z})$ which is to be understood as the set of paths connecting regions $\mathscr{X}$ and $\mathscr{Y}$ in $\mathbb{C}^n$.

Now consider the following definitions,

1.$H(\mathscr{X}, \mathscr{X}_{-T}; \mathbb{Z})$**:** Let $T >> 1$, and $\mathscr{X}_{-T} = \{x \in \mathscr{X} | h(x) < -T\}$. Then $H(\mathscr{X}, \mathscr{X}_{-T}; \mathbb{Z})$ represents the set of paths on which the integrand $e^{\iota S(z)}$ converges.

[4]Geometric quantization is dealt with in detail in Chapter-6

Similarly if $\mathscr{X}^T = \{x \in \mathscr{X} | h(x) > T\}$, then $H(\mathscr{X}, \mathscr{X}^T; \mathbb{Z})$ represents paths on which the integrand diverges.

2. **Lefschetz Thimbles corresponding to the critical point α:** A descend (ascend) Lefschetz Thimble corresponding to the critical point α is defined as the moduli space of the end points of a solution (c(t)) of the descend (ascend) equations such that $c(-\infty) = \alpha$. That is,

$$\{c(0) \in \mathscr{X} | \dot{c}(t) = \pm\{ImI, c\}, c(-\infty) = \alpha\}$$

Here, '-' corresponds to the ascend thimble and '+' corresponds to the descend thimble.

Now, a very powerfull fact which underlies Picard Lefschetz theory is that the set of all descend (ascend) thimbles corresponding to all critical points of S(z) is the exhaustive set of generators of $H(\mathscr{X}, \mathscr{X}_{-T}; \mathbb{Z})$ $(H(\mathscr{X}, \mathscr{X}^T; \mathbb{Z}))$. This allows one to evaluate the integral of $e^{\iota S(z)}$ along some integration contour as a sum of convergent integrals. Lets see how.

An incredible advantage of the structure constructed above is that $Im(\iota S(z))$ is constant, i.e., along the thimbles one does not have to deal with oscillatory integrals. Now, to ensure the same for the decomposition of the original integration contour into thimbles, we define the following pairing operation. If $\mathscr{J}_\alpha$ and $\mathscr{K}_\alpha$ are respectively the descend and ascend thimbles corresponding to the critical point α, then,

$$\langle \mathscr{J}_\alpha, \mathscr{K}_\beta \rangle = \delta_{\alpha\beta}.$$

Given this, if $\mathscr{L}$ denotes the initial integration contour, define,

$$n_\alpha = \langle \mathscr{L}, \mathscr{K}_\alpha \rangle.$$

Then,

$$\mathscr{L} = \sum_\alpha n_\alpha \mathscr{J}_\alpha$$

Hence, we have found a decomposition of the intial integration contour convergent integrals.

A Thing to notice about the above decomposition:
Not all the critical points of S(z) are included in the new contour.For example, if a critical point has $h > 0$, then the ascend lines will not intersect the initial integration contour corresponding to $\mathscr{L}$ as on it $h = Re[\iota S(z)] = 0$. Hence, we realize that a semi-classical approximation following the above decomposition picks only certain critical points.

Let's consider an example.

3.1 Example

Let $S(x) = x^3 + x$ defined on $\mathbb{R}$. Complexifying the domain we end up with $\mathbb{C}$. The holomorphic extension of S(x) is $S(z) = z^3 + z$, where $z = x + \iota y$. It follows that,

$$h = Re[\iota S] = -(3yx^2 - y^3 + y), \qquad H = Im[\iota S] = x^3 - 3y^2x + x$$

The critical points corresponding to this are, $\alpha_1 = (0, \frac{1}{\sqrt{3}})$ and $\alpha_1 = (0, \frac{-1}{\sqrt{3}})$. Now the value of h at each of these critical points viz., $\frac{-2}{3\sqrt{3}}$ and $\frac{2}{3\sqrt{3}}$, respectively says that the deformed contour passes through the critical point α_1 and not α_2. This can be seen in the plot below showing the ascend and descend curves corresponding to both the critical points.

The initial contour is shown in black line in 3.1a and similarly the deformed contour in 3.1b. Now since the new contour contains a critical point, one can carry out a saddle point approximation around it. This gives us the integrand,

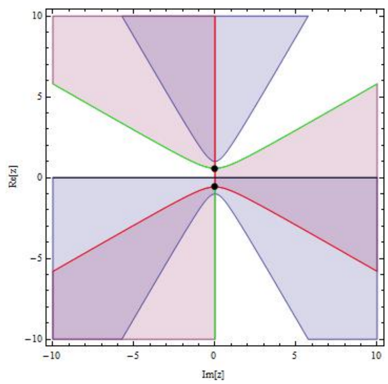
$$e^{\frac{I(z)}{\hbar}} = e^{\frac{1}{\hbar}(\frac{-2}{3\sqrt{3}} - 2\sqrt{3}(z-\alpha_1)^2)}$$

to be integrated along the deformed contour. Now, if the value of $\hbar << 1$ and if the tangent to the contour at the critical point is parallel to the real axis, then one can approximate the above integral as,

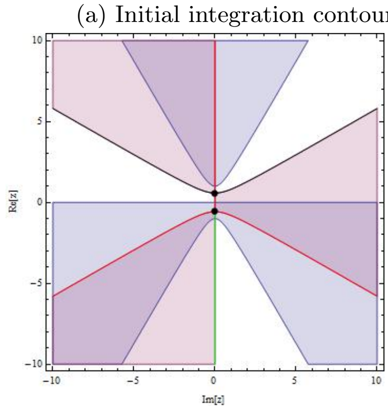
$$\int_{\mathscr{J}_\alpha} e^{\frac{I(z)}{\hbar}} \mathrm{d}z = \int_{\mathscr{J}_\alpha} e^{\frac{1}{\hbar}(\frac{-2}{3\sqrt{3}} - 2\sqrt{3}(z-\alpha_1)^2)} \mathrm{d}z \sim \int_{x+\iota Im(\alpha)} e^{\frac{1}{\hbar}(\frac{-2}{3\sqrt{3}} - 2\sqrt{3}(z-\alpha_1)^2)} \mathrm{d}z$$

where the second integral is taken along the line $x + \iota Im(\alpha)$. This then is the usual Gaussian integral, over the real line i.e.,

$$\int e^{\frac{1}{\hbar}(\frac{-2}{3\sqrt{3}} - 2\sqrt{3}(x-Re[\alpha_1])^2)} \mathrm{d}x = exp(\frac{-2}{3\sqrt{3}\hbar}) \sqrt{\frac{\pi\hbar}{2\sqrt{3}}}$$



(a) Initial integration contour



(b) Deformed contour

Figure 3.1: Deformation of $\mathscr{L}$ on to relevant Lefschetz Thimbles

The blue and purple shaded regions correspond to $h \geq 0$ and $H \geq 0$ respectively. The ascend (in red) and descend (in green) from both the critical are shown.The initial integration contour is shown in black in figure 3.1a. Since the only ascend curve which intersects $\mathscr{L}$ originates from the critical point above, the contour is deformed as in 3.1b

Chapter 4

How did the universe begin?

In this chapter we will take a look at two proposals for the beginning of the universe: 1) "No boundary proposal" by Hartle and Hawking; 2) "Universe from nothing" by Vilenkin. We will also summarize the conclusions of a recent paper by Turok et al, which based on the sound mathematical prescription of Picard Lefschetz Theory suggest that the above two proposals cannot be right.

4.1 No Boundary Proposal

This proposal consists of two parts, the first of which I already described in the chapter on path integrals, is that the ground state may be described by the Euclidean path integral,

$$\Psi_{grnd}(\phi, h_{ij}) = \int_{\phi_{in}, h_{ij,in}}^{\phi, h_{ij}} e^{-I(\phi', h'_{ij})} \mathscr{D}\phi' \mathscr{D}h'_{ij}.$$

The second part of this proposal tells us what class of geometries to sum over. Now, according to Hartle and Hawking this class should include only all possible compact Euclidean geometries. One might think that since we are trying to find the ground state of the wavefunction, we should sum over geometries which tend to maximally symmetric spaces at infinity. This in the case of $\Lambda \leq 0$ corresponds to euclidean flat space and Ads spaces. But Hartle and Hawking argue that, these are valid expectations for particle scattering experiments and are not necessary for cosmological problems where one is concerned only with the bulk. Also, even if the entire class of non compact geometries with maximally symmetric asymptotes is taken, the major contribution to the path integral will come from non compact spaces which are a disjoint union of a compact and a non compact geometry. The former will have a boundary whereas the latter will have no interior. Given this, we are back on the proposal that we need to sum over only compact geometries.

Even so, for the general case, one will have to sum over all the dynamical degrees of freedom. These degrees of freedom are described by the functions corresponding to the spatial part of the metric and all regular field configurations in the universe. Together these form the 'Superspace'. Calculating a path integral in the full Superspace is not tractable. Hence, as has been explained again in the chapter on canonical formalism, we consider a subspace called the Mini-Superspace. This corresponds to a homogeneous and isotropic universe with a single scalar field or a cosmological constant. Homogeneity and isotropy imply that the only degree of freedom for the gravity part is the scaling factor. Also, both the scaling factor and the scalar field are independent of the spatial coordinates.

Let's see what the Hartle-Hawking proposal implies for the case of gravity with a positive cosmological constant in Minisuperspace. The line element in this case is [1],

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a^2(t)\mathrm{d}\Omega^3$$

where $\mathrm{d}\Omega^2$ corresponds to the metric of a three sphere. The action then becomes,

$$S = \int (\sqrt{g}\mathrm{R} + \sqrt{g}\Lambda)\mathrm{d}^4x \tag{4.1}$$

$$= 2\pi^2 \int_0^1 \left[-3a\frac{\dot{a}^2}{N} + N(3a - a^3\Lambda)\right]\mathrm{d}t \tag{4.2}$$

Here the range of t is taken to be 0 to 1 (The infinite integration range of t is substituted by an integral over N in the path integral from 0^+ to ∞). Now, taking $t \to -\iota\tau$ the amplitude corresponding to the Euclidean action becomes,

$$e^{(2\pi^2)\int_0^1 \left[3a\frac{\dot{a}^2}{N} + N(3a - a^3\Lambda)\right]\mathrm{d}t}$$

Now for the Euclidean path integral, the boundary condition on the final three hypersurface corresponds to a particular value for $a_0 = a(t=1)$. Given this the sum over all compact Euclidean four geometries will imply an integration over a in the range 0 to ∞ apart from an integral over N (lapse function). The former can be understood as the universe starting from zero volume with a(t=0)=0. Then the Euclidean wavefunction corresponds to the following path integral,

$$\Psi(a_0) = \int \int_{0^+}^{\infty} e^{(2\pi^2)\int_0^1 \left[3a\frac{\dot{a}^2}{N} + N(3a - a^3\Lambda)\right]\mathrm{d}t} \mathscr{D}a \,\mathrm{d}N$$

Now, the above is not convergent for integration over real values of the scaling factor[2]. Hartle and Hawking try to resolve this issue by picking a complex contour of integration for a. This then allows them to obtain a saddle point

[1] notice that while taking the above line element we have chosen a particular gauge. This becomes necessary when we calculate the path integral over all possible geometries lest we over count them.

[2] The divergence of the path integral as a result of wick rotated time is an important issue. We will discuss about the same in section 4.3

approximation given by, [3]

$$\Psi(a) \propto e^{\frac{12\pi^2}{\Lambda}} cos(4\pi^2\sqrt{\frac{\Lambda}{3}}(a_0^2 - \frac{3}{\Lambda})^{\frac{3}{2}})$$

In the following, we will consider Vilenkin's proposal and compare the form of wavefunction obtained by him with the above.

4.2 Universe from Nothing

Vilenkin proposes a model in which some of the unanswered questions about the initial state[4] of the universe are naturally avoided. He suggests that the universe could have tunneled out of "nothing", with finite size and then continued with its classical evolution. Here, by "nothing" he means "a state of no classical spacetime".The idea of "nothing" will become clearer as we move through this section.

Let's motivate this idea for the de sitter space. The scaling factor for this four geometry goes as, $a(t) = H^{-1}cosh(Ht)$ and hence for $t \leq 0$ the universe contracts while for $t \geq 0$ it expands.[5] Now, if a(t) was the coordinate of a particle, this would correspond to a particle bouncing off a potential. What Vilenkin proposes is that, the universe could have tunneled out of this potential with a finite size given by $a(t=0) = H^{-1}$, and then continued expanding for $t > 0$.

Let's make this analogy concrete and see what wavefunction does it predict for the universe.The action for a homogeneous and isotropic universe with positive cosmological constant is as given by equation 4.1 where R is evaluated for the dS metric. Now, the tunneling probability for the wavefunction is described by the Euclidean solution to the above action. This can be obtained by Wick rotating the time in the classical solution $a(t) = H^{-1}cosh(Ht)$, and gives, $a(t) = H^{-1}cos(Ht)$. This Euclidean solution (which happens to be compact) is a called the de Sitter instanton, and corresponds to the "nothing" state for this system. Given this, the tunneling of the universe may be pictured as in the diagram below [6].

The Schroedinger equation corresponding to the above action is given by,

$$\big(a^{-p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \big(\frac{3\pi}{2G}\big)^2 a^2(1 - a^2H^2)\big)\Psi(a) = 0$$

Here, unlike other quantum mechanical systems, the eigen value for a system including gravity can only be zero because of the diffeomorphism invariance of the

[3]For details refer to "Wavefunction of the universe" by Hartle and Hawking [2]

[4]Like the origin of the thermal state of the universe, what exactly is the Big Bang? etc

[5]H is the Hubble's constant and as per Vilenkin's convention $H^2 = \frac{8\pi G}{3}\Lambda$. Hartle on the other hand uses a different notation and hence for this section, take $H^2 = k\Lambda$ for some constant k.

[6]The diagram is from Vilenkin's Birth of Inflationary Universes paper[4]

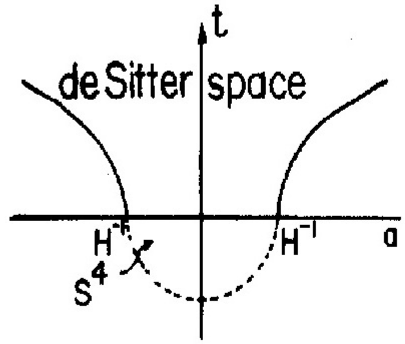


Figure 4.1: Vilenkin's Tunneling Proposal

theory. The first term is a generalized operator form for a few possible operator orderings. Now, given this Hamiltonian constraint, $U(a) = \left(\frac{3\pi}{2G}\right)^2 a^2(1 - a^2H^2)$ acts as the potential. For values of $a < H^{-1}$ the solution to the above has an exponential form, whereas for $a > H^{-1}$ the solution is oscillatory. Their WKB approximations are as follows[7],
1) $0 < a < H^{-1}$ (under Barrier solution)

$$\Psi_{\pm}^{(1)}(a) \propto exp^{\left(\pm \int_a^{H^{-1}} |p(a')| \mathrm{d}a'\right)}$$

2) $a > H^{-1}$

$$\Psi_{\pm}^{(2)}(a) \propto exp^{\left(\pm \iota \int_{H^{-1}}^{a} |p(a')| \mathrm{d}a' \mp \frac{\iota\pi}{4}\right)}$$

where $p(a) = \left(-2U(a)\right)^{\frac{1}{2}}$. Now, in the second set of solutions the plus sign corresponds to outgoing wave whereas the minus to incoming waves. Since, the tunneling solution should correspond to the outgoing wave, in the region $a > H^{-1}$ the plus signed solution is to be picked. Now, corresponding to this, the under barrier solution will have to be of the form,

$$\Psi(a < H^{-1}) \sim \Psi_{+}^{(2)}(a) + \frac{\iota}{2}\Psi_{-}^{(2)}(a)$$

Where, the contribution from the second term is negligible. Now, one can solve the Hamiltonian constraint for $0 < a < H^{-1}$ near the potential barrier such that it tends to $\Psi_{+}^{(2)}(a)$ as a gets smaller (away from the barrier) . This solution can then be analytically continued to obtain the solution above the barrier. Doing so we end up with the following form of the wavefunction,

$$\Psi(a > H^{-1}) \sim e^{-\frac{\pi}{2GH^2}} e^{-\iota\frac{\pi}{2GH^2}(a^2H^2-1)^{\frac{3}{2}}} \tag{4.3}$$

[7]The following equations are from Vilenkin's Creation of Universes paper [5]

As can be seen from equation (4.3), the wavefunction corresponding to Vilenkin's proposal has a suppressed exponential unlike that derived by Hartle and Hawking. Also, it has been rightly observed by Vilenkin that the Hartle-Hawking solution corresponds to a superposition of a contracting and an expanding universe. And that this is to be expected because the path integral that they started out with was real and hence will give a superposition of a wavefunction and its complex conjugate. Hence, Vilenkin proposes that the ground state wavefunction should be obtained from a Lorentzian path integral instead of a Euclidean one, and summed over compact Lorentzian geometries.

4.3 Evaluations using Picard Lefschetz Theory

In the section on Hartle-Hawking proposal we saw that Wick rotating the action for gravity led to a divergent integral with respect to a. This is because of the negative kinetic energy terms associated with the scaling factor. That is, unlike on-gravitational systems where the Euclidean action results in an exponentially suppressed path integral, path integrals corresponding to gravitational systems diverge because of the negative kinetic energy contribution. Hence, Turok *et al* believe that instead of Wick rotating, Lorentzian path integrals should be studied as such. This becomes possible with Picard Lefschetz Theory. In this case, the semi-classical approximation corresponding to the case of pure gravity with a positive cosmological constant results in results in the following wavefunction.

$$\Psi(a_0) \propto e^{-\frac{12\pi^2}{\Lambda} - \iota 4\pi^2 \sqrt{\frac{\Lambda}{3}} (a_0^2 - \frac{3}{\Lambda})^{\frac{3}{2}}}$$

This agrees with the result that Vilenkin obtained but not Hartle and Hawking. This may be understood as follows. Since in the semi-classical approximation, only the classical solutions have a major contribution, we can write the metric as (using Friedmann equations),

$$\mathrm{d}s^2 = -\frac{\mathrm{d}q^2}{4q(\frac{1}{3}\Lambda q - 1)} + q\mathrm{d}\Omega_3^2$$

where $q = a^2$ Now, for $q > \frac{3}{\Lambda}$ the metric is Lorentzian, while for $0 \leq q < \frac{3}{\Lambda}$ it is Euclidean. $q_b = \frac{3}{\Lambda}$ is where the Lorentzian classical solution bounces.We then take the branch cut such that it runs from the point q_b leftwards. Now while integrating over the q variable in the path integral in the range 0 to $q_1 = a_0^2$, one has to choose a contour of integration avoiding the branch cut. It seems that Picard Lefschetz Theory corresponds to taking the contour which passes below q_b whereas Wick rotation corresponds to passing above it. This is illustrated in the figure 4.2.

Hence, trusting the mathematical foundations of Picard Lefschetz Theory one is forced to conclude that the semi-classical approximation obtained by Hartle and Hawking has the wrong sign.

But so what if this is true? Turok *et al* go on to calculate the semi-classical approximation for gravity plus gravitational waves (perturbations). The action

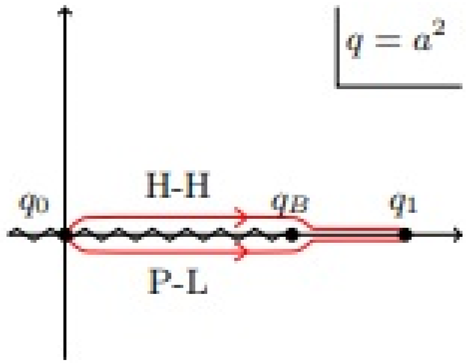


Figure 4.2: The above figure shows the integration contour corresponding to Hartle Hawking calculation (HH) and using Picard Lefschetz Theory (PL). The figure is from "No Smooth Beginning for Space-time"[6].

corresponding to gravitational waves is,

$$S = \frac{1}{2}\int N_s \mathrm{d}t\mathrm{d}^3x\big[q^2\big(\frac{\dot{\phi}}{N_s}\big)^2 - l(l+2)\phi^2\big].$$

In the following semi-classical approximation corresponding to the perturbations, Turok *et al* ignore gravitational back-reactions. They take the classical solutions of the gravitational action as a static background and N_s corresponds to this static value. At this order it seems that the wavefunction factorizes, giving the perturbation part as,

$$\Psi(\phi) \propto e^{\frac{3l(l+1)(l+2)}{2\hbar\Lambda}\phi_1^2} \times phase$$

We observe that the perturbations exponentially increase. Since, this is contrary to experimental observations, it looks like both the proposals for the smooth beginning of universe will have to be wrong.

Conclusions

From the above discussion, it seems to me that the proposal made by Hartle and Hawking, that the ground state wavefunctional should be described by the Euclidean path integral, is not justified. This becomes obvious in the case of Minisuperspace model. Wick rotating the Lorentzian action gives a negative kinetic energy contribution from the scaling factor. Hence, the exponential factor (viz., $-S_e$ where S_e is the Euclidean action) now remains unsuppressed for arbitrarily large values of $\dot{a}(t)$. Then, the trick used in Chapter 2 to obtain the ground state becomes invalid. Hence, we will have to confront the Lorentzian path integral. But, then the calculation done by Turok *et al*, using Picard Lefschetz Theory predict undamped perturbations. This calculation was done keeping in mind the proposals of both Hartle-Hawking and Vilenkin, that only compact geometries (Euclidean and Lorentzian compact geometries respectively) need to be summed over in the path integral for the ground state. The consequent semi-classical approximation of the wavefunction, predicting undamped perturbations suggests that both the above proposals are unphysical.

Chapter 5

Canonical Formalism

In the above sections, we have mostly been looking at the wavefunction(al) of the universe from the path integral point of view. But given an action one can construct an operator constraint to be satisfied by the wavefunctional, analogous to the Schroedinger's equation. In the following few sections we will look at the operator constraint and a few problems that arise when we try to solve them. This includes the definition of an inner product which defines the Hilbert space of wavefunctionals. We will later on try to resolve this problem using geometric quantization and solve a simple system.

5.1 Wheeler de Witt Equation

Given a general geometry of spacetime, the metric corresponding to it can be written in the following form,

$$\begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

Here, one has chosen a particular foliation of spacetime. In terms of these variables, the Einstein Lagrangian density becomes,

$$\mathscr{L} = \sqrt{-g}\mathrm{R} \tag{5.1}$$

$$= \alpha\gamma^{\frac{1}{2}}(K_{ij}K^{ij} - K^2 + {}^{(3)}R) - 2(\gamma^{\frac{1}{2}}K)_{,0} + 2(\gamma^{\frac{1}{2}}K\beta^i - \gamma^{\frac{1}{2}}\gamma^{ij}\alpha_{,j})_{,i} \tag{5.2}$$

where, $g = det(g_{\mu\nu}) = \alpha^2\gamma$, $\gamma = det(\gamma_{ij})$ and,

$$K_{ij} = \frac{1}{2}\alpha^{-1}(\beta_{i,j} + \beta_{j,i} - \gamma_{ij,0}), \qquad K^{ij} = \gamma^{ik}\gamma^{jl}K_{kl}, \quad K = \gamma^{ij}K_{ij} \tag{5.3}$$

K_{ij} is called the second fundamental form or the extrinsic curvature tensor. This tensor gives a measure of the curvature of the spacelike hypersurfaces embedded in spacetime. Unlike K_{ij}, ${}^{(3)}R_{ij}$ gives the intrinsic curvature of these hypersurfaces, and these become identical only in flat spacetimes. Given the above Lagrangian density, the corresponding Hamiltonian may be calculated and takes the form,

$$H = \int (\pi\alpha_{,0} + \pi^i\beta_{i,0} + \alpha\mathscr{H} + \beta_i\chi^i)\mathrm{d}^3x$$

Here, $\mathscr{H} = \gamma^{\frac{1}{2}}(K_{ij}K^{ij} - K^2 -^{(3)} R), \qquad \chi^i = -2\pi^{ij}_{,j} - \gamma^{il}(2\gamma_{jl,k} - \gamma_{jk,l})\pi^{jk}$

and π, π^i, π^{ij} are respectively the conjugate momentums of $\alpha, \beta^i, \gamma^{ij}$ respectively.

Now, take a look at the structure of $\mathscr{H}$. It is of the form, extrinsic curvature $(K_{ij}K^{ij} - K^2)$ plus the negative of the intrinsic curvature $-^{(3)}R$. Interpreting the former as the kinetic energy and the latter as the potential energy, we see that $\mathscr{H}$ has a structure very similar to the usual Hamiltonian i.e, sum of kinetic and potential energies. Let's keep this in mind, as we will use this feature of $\mathscr{H}$ to identify the operator constraint corresponding to it (in the quantum theory) as the one describing the dynamics of space-time.

Returning to the classical theory we see that, since α and β_i are cyclic the following primary constraints follow from the Lagrangian,

$$\pi = 0, \quad \pi^i = 0s \tag{5.4}$$

Also, since these have to be true at all times, their Poisson Brackets with the total Hamiltonian operator gives the following secondary constraints,

$$\mathscr{H} = 0, \quad \chi^i = 0. \tag{5.5}$$

But the above classical constraints translate to the following operator equations in the quantum theory.

$$\pi\Psi = 0, \quad \pi^i\Psi = 0, \quad \mathscr{H}\psi = 0, \quad \chi^i\Psi = 0$$

Here, now $\pi, \pi^i, \mathscr{H}$ and χ^i are the operators corresponding to the classical functions where all the conjugate momenta have been replaced by the corresponding differential operators (i.e., $\pi^{ij} \to -i\hbar\frac{\partial}{\partial\gamma^{ij}}$ etc). Now, as mentioned above, the structure of $\mathscr{H}$ suggests that the dynamics of the system is described by the operator constraint $\mathscr{H}\psi = 0$. This, therefore is the analogue of Schroedinger equation in quantum gravidynamics for a pure gravity Lagrangian and is called the Wheeler DeWitt equation. Explicitly written down, it takes the following form,

$$\big(G_{ijkl}\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}} + \gamma^{\frac{1}{2}}{}^{(3)}\mathrm{R}\big)\Psi(\gamma_{ij}) = 0 \tag{5.6}$$

where, $G_{ijkl} \equiv \frac{1}{2}\gamma^{\frac{-1}{2}}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl})$.

In general, when we include matter fields, the Wheeler DeWitt equation takes the form,

$$(\mathscr{H} + \mathscr{H}_m)\Psi(\gamma_{ij}, \phi) = 0 \tag{5.7}$$

where the second term is the usual Hamiltonian operator corresponding to the matter fields. Given a universe described by some Hamiltonian, the solutions to the above equation give us the wavefunctionals corresponding to it. But it is to be noted that unlike usual quantum mechanics, this operator contains functional derivatives. This translates to saying that Eq.5.6 and Eq.5.7 are not single equations but ∞^3 equations corresponding to each point on the spacelike hypersurface.

A simple model universe which acts as the playground for many calculations is the Minisuperspace model. This, as we have seen before, corresponds to a closed homogeneous and isotropic universe with a scalar field. The assumptions of homogeneity and isotropy reduce the degrees of freedom to two, one corresponding to the scaling factor and the other corresponding to the field.

Returning to equations.5.6 and 5.7, we see that the Hamiltonian constraint can only have a zero eigenvalue. This is to be expected because the theory is diffeomorphism invariant and time is ill defined. How then do we understand an evolving universe? This is called the problem of time. For the case of closed universes, one could pick a few intrinsic parameters to play the role of time, and then look at the evolution of other parameters with respect to these. For a detailed discussion of the same and for the case of open universes refer to [7].

Apart from the above, another issue one faces when we try to solve the Wheeler DeWitt equation is the lack of a good definition for the inner product. In other words we don't have a Hilbert space to which the wavefunctionals belong.

DeWitt proposes the following form of inner product in analogy with the Klein Gordon inner product,

$$(\Psi_b, \Psi_a) = Z \int_\Sigma \Psi_b^*[\gamma_{ij}] \times \prod_x \big(\mathrm{d}\Sigma^{ij} G_{ijkl} \frac{\overrightarrow{\delta}}{i\delta\gamma_{kl}} - \frac{\overleftarrow{\delta}}{i\delta\gamma_{kl}} G_{ijkl} \mathrm{d}\Sigma_{ij}\big)\Psi_a[\gamma_{ij}]$$

But exactly like its analogue, the above inner product also suffers from the "negative probability" issue. This problem is called the Hilbert space problem. But now, a natural inner product and hence a Hilbert space, arises from geometric quantization of any quantizable theory. Hence, in the work in progress, we are trying to use Geometric Quantization to understand the Hilbert space problem. To understand this idea better, let's take a look at Geometric Quantization in the next chapter.

Chapter 6

Geometric Quantization

To construct the quantum theory corresponding to a classical system, one must associate to each classical observable (a continuous function on the phase space of the system) a corresponding operator acting on an appropriate Hilbert space of states. Geometric quantization is a prescription to do the same.[1]

6.1 Phase space as a Symplectic Manifold

Consider a classical system living on a phase space of dimension 2, with any point on it defined by (p, q). Differential geometry provides a natural way of understanding this space (and its generalizations to higher dimensions). The space described by the variable q constitutes a (one dimensional) manifold denoted as $\mathscr{B}$. Consequently, at each point on $\mathscr{B}$, there exists an open neighborhood U equipped with a chart $\phi : U \to \mathbb{R}$. The inverse of this function gives us a parametrization of U in terms of $t \in \mathbb{R}$ and hence, the tangent vector at any point as $\frac{\mathrm{d}q}{\mathrm{d}t} = \dot{q}$. This in turn implies that the points on the tangent bundle of $\mathscr{B}$ is of the form $(\dot{q}, q)$. It is the cotangent bundle corresponding to $\mathscr{B}$ that constitutes the phase space of the system.

Before we proceed further let's consider the following definitions:

1. **Symplectic two form (ω):** a closed non degenerate two form. A manifold equipped with a symplectic two form is called a symplectic manifold. In the above case, one could define the following symplectic form,

$$\omega = \mathrm{d}p \wedge \mathrm{d}q.$$

The cotangent bundle equipped with this two form is now a symplectic manifold. Let's call it $\mathscr{M}$. By virtue of the 'exact'ness of ω one can write it as, $\omega = \mathrm{d}(p\mathrm{d}q)$ where $p\mathrm{d}q$ is called the tautological 1-form.

[1] In the following sections I discuss material mostly from [8] and [9].Hence the notations will be the same.

6.2 A Few Definitions

In this section, we will consider a few definitions required to understand the mathematical structure behind geometric quantization. As, the problem that I pass on to later is relatively simple, and because I can't afford to be very rigorous in a brief summary as below, I will restrict all my definitions to the simpler cases. Wherever I make such simplifications I will make it a point to bring it to the notice of the reader via footnote.

1. **Vector Bundle**

A vector bundle is a map $f : E \to B$ such that around any point $b \in B$ there exists an open interval U in which $f^{-1}U \equiv U \times F$. Here, F has the structure of a vector space and is called the fiber of the bundle.[2]

A complex line bundle would then correspond to the case where, $F = \mathbb{C}$, whereas a tangent bundle corresponds to the case where the fiber at a point is the entire tangent space at that point. Now given the above structure one may make the following insightful definitions:
a) Wavefunction: sections of the complex line bundle $\pi : L \to \mathscr{M}$.
b) Tangent vector field: sections of the tangent bundle $\pi : T\mathscr{M} \to \mathscr{M}$

2. **Inner product on** $\Gamma(\mathbb{C})$

Given a complex line bundle $\pi : L \to \mathscr{M}$, the fiber at each point $x \in \mathscr{M}$ is a complex vector space of dimension one. Hence, on each fiber one can define a natural inner product $c_1^* c_2$ where $c_1, c_2 \in F_x$. Extending this to the entire manifold $\mathscr{M}$, one can define the inner product between two sections of the complex line bundle s_1 and s_2 as,

$$\langle\langle s_1, s_2 \rangle\rangle := \int \bar{s_1} s_2 \omega^n$$

where 2n is the dimension of the phase space and $\omega^n := \omega \wedge \omega \wedge ...$ Now, the set of L^2 functions defined with respect to the inner product above, on $\mathscr{M}$ is a Hilbert space.

3. **Hamiltonian vector field**

Given the symplectic two form ω, the Hamiltonian vector field corresponding to a function f(p,q) is defined as the tangent vector field (Σ_f) such that at any point, $\mathrm{d}f = -\omega(\Sigma_f, .)$. It can be easily checked that along the integral curve of Σ_f, the function f remains constant.

3. **Poisson Bracket:** given two vector fields X_f and X_g corresponding to the functions f and g respectively, the Poisson Bracket is defined as,

$$\{f, g\} = \omega(X_f, X_g)$$

As may be proved from the exactness of ω, the Poisson Bracket satisfies the Jacobi identity given by,

[2]This definition is highly simplified, but will save our purpose.

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$$

That is, the pair $(\{.,.\}, C^\infty(\mathscr{M}))$ constitute a Lie algebra.

4. **Connection** [3] Lets denote, the space of sections corresponding to a given vector bundle, $\pi : E \to \mathscr{M}$ as $\Gamma(E)$. Given this, a connection ∇ on a complex vector bundle $\Gamma(E) : E \to \mathscr{M}$ is a $\mathbb{C}$ bilinear map,

$$\Gamma(T\mathscr{M}) \times \Gamma(E) \to \Gamma(E), \qquad (X, s) \to \nabla_X s$$

such that
a)$\nabla_{fX} s = f \nabla_X s$, (linear with respect to the first slot)
b)$\nabla_X(fs) = s\nabla_X f + f\nabla_X s$, (a derivation with respect to the second slot)
for all $f \in C^\infty(\mathscr{M}, \mathbb{C})$ and all $X \in \Gamma(T\mathscr{M})$[4]
A connection is then called Hermitian if,

$$X\big\langle\big\langle s, s'\big\rangle\big\rangle = \big\langle \nabla_X s, s'\big\rangle + \big\langle s, \nabla_X s'\big\rangle$$

for all sections $s, s' \in \Gamma(E)$ with respect to the fiber inner product $\big\langle .,. \big\rangle$

6.3 Prequantization

The first step to quantization involves constructing the right Hilbert space. This is done by finding a Hermitian connection ∇ on the complex line bundle $\pi : L \to \mathscr{M}$ such that the curvature corresponding to it, defined as,

$$R^\nabla(X, Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s$$

is such that, $R^\nabla(X, Y)s = \omega(X, Y)s$. Though the definition of such a connection is not unique, the following one does the job,

$$\nabla_X f = Xf - \frac{\iota}{\hbar}\theta(X)f$$

Here, $\theta(X)$ is the tautological one form defined above. Having defined this connection, one then obtains a 'prequantum Hilbert space' [5] with curvature described by ω and an L^2 inner product defined as above.

Now that we have a Hilbert space, let's proceed to define operators corresponding to the classical observables described by f(p,q) in the phase space. This is done with the following prequantization map,

$$Q_f(s) = \iota\hbar\nabla_{\Sigma_f} s + fs \tag{6.1}$$

[3]Definition from Eugene Lerman's crash course on Geometric Quantisation [8]
[4]If the above definition looks too complicated, lets just remind ourselves that the covariant derivative is a connection.
[5]it's so called because this is not yet the right Hilbert space. See the next section

This map is constructed such that it maps the Lie algebra of Poisson Brackets to the Lie algebra of the operators,i.e.,

$$[Q_f, Q_g]s = Q_{f,g}s.$$

The particular map is also such that the operators are skew-Hermitian with respect to the L^2 inner product defined above, implying,

$$\langle\langle Q_f s, s'\rangle\rangle + \langle\langle s, Q_f s'\rangle\rangle = 0$$

6.4 Polarization

But we have not quite reached the end of the quantization procedure. Notice that the Pre Quantum Hilbert space that we constructed depends on too many variables. Whereas in the usual Schroedinger picture our wavefunctions depend either on the position variables or the momentum variables, the wavefunctions in the Pre-Quantum Hilbert space depend on both. Now, in John Baez's words - [6] "It doesn't seem to be true that God created a classical universe on the first day and then quantized it on the second day"

It therefore seems necessary that we reduce the degrees of freedom of our Hilbert space by half, and this is done by choosing a Polarisation. Before we get into what a Polarization actually is, we have to thrive through a couple more of definitions.[7]

1) **Lagrangian subspace**
Given a symplectic vector space (V, ω), its Lagrangian subspace $\mathbb{L}$ is defined, such that,
a) $\omega(v_1, v_2) = 0$ for all $v_1, v_2 \in \mathbb{L}$;
b) $\mathbb{L}$ is maximally isotropic, i.e., if $\mathbb{L}'$ is another Lagrangian subspace of (V, ω) which contains $\mathbb{L}$ then, $\mathbb{L}' = \mathbb{L}$.
It may be shown that the condition of maximal isotropy implies that the number of independent variables in the Lagrangian subspace is exactly half the number in V.

2) **Extension to manifolds**
An immersed submanifold $\mathbb{L}$ of $(\mathscr{M}, \omega)$ is called Lagrangian if, the tangent space of $\mathbb{L}$ at each point on the manifold is a Lagrangian subspace of the tangent space of $\mathscr{M}$ at that point, i.e., $\mathrm{T}_x\mathbb{L} \subset (\mathrm{T}_x\mathscr{M}, \omega_x)$

Now, with the above structure one can define a few polarizations, of which a) Real polarization and b) Complex (or Kähler) Polarization, are two examples. Below we will define the real polarization and will keep the later for the next section as it requires the introduction of some additional structure on the real manifold.

[6] came across the same while reading through Ivan Todorov's Quantization is a mystery
[7] These I borrow from Eugene Lerman's Crash course on geometric quantization

Real Polarization
On a symplectic manifold $(\mathscr{M}, \omega)$ a real polarization is a subbundle $\mathscr{F} \subset \mathrm{T}\mathscr{M}$ such that,
i)$\mathscr{F}$ is Lagrangian
ii)$\mathscr{F}$ is integrable (involutive) i.e., for all $X, Y \in \mathscr{F}_x$, the Lie bracket $[X, Y] \in \mathscr{F}_x$ for all $x \in \mathscr{M}$

Example: on the two dimensional phase space described by $\mathbb{R}^2$,the subspace spanned by vectors proportional to $(\frac{\partial}{\partial p})$ form a polarization.
[8]Having chosen a polarization we now have to look for wavefunctions and operators which comply with it. That is, we now define the following.

1) Sections which are covariantly constant along $\mathscr{F}$: If $s \in \Gamma(\mathrm{L})$, it is covariantly constant along $\mathscr{F}$ if for all $X \in \mathscr{F}$, $\nabla_X s = 0$.

2) Polarization preserving functions on phase space $C^\infty_{\mathscr{F}}(\mathscr{M})$: These are defined as,

$$C^\infty_{\mathscr{F}}(\mathscr{M}) := f \in C^\infty(\mathscr{M}) | [\Sigma_f, X] \in \Gamma(\mathscr{F}) \text{for all} X \in \Gamma(\mathscr{F}).$$

These functions are such that the flow of their Hamiltonian vector preserves $\mathscr{F}$ and are closed under Poisson Bracket i.e., if f,g are two polarization preserving maps so is $\{f, g\}$.

Now, the definition of an operator corresponding to a function on the phase space (mentioned above) is such that when defined for polarization preserving functions, it takes covariantly constant sections to other covariantly constant sections. Also, by the above mentioned closure property they preserve the Poisson Bracket algebra of the functions. Hence, with the above what we have achieved is the construction of a rightly 'sized' Hilbert space, with wavefunctions given by covariantly constant sections and operators by those corresponding to polarization preserving functions.

Let's take the simple example of the 1D Harmonic Oscillator, to illustrate Geometric Quantization.The Hamiltonian for the same is given by,

$$\mathrm{H} = \frac{p^2}{2m} + \frac{q^2}{2m}$$

Now, given the two form $\omega = \mathrm{d}p \wedge \mathrm{d}q$, the Hamiltonian vector field corresponding to q is as per definition, the vector field X such that $\omega(X, .) = -\mathrm{d}q$. This gives, $X = -\frac{\partial}{\partial p}$. Similarly, the vector field corresponding to p is fiven by, $\frac{-\partial}{\partial q}$. Let's pick a polarization such that the operator corresponding to q remains q. Then from 6.1 this requires that,

[8]Apart from the above structure, one assumes that the space of leaves of the polarization satisfies a few other properties like being a Hausdorff manifold. See- [8] for further details

$$\nabla_{-\frac{\partial}{\partial p}} s(p,q) = 0 \tag{6.2}$$

$$\implies -\frac{\partial}{\partial p} s(p,q) + \frac{\iota}{\hbar} p\mathrm{d}q(-\frac{\partial}{\partial p}) s(p,q) = 0 \tag{6.3}$$

$$\implies \frac{\partial}{\partial p} s(p,q) = 0. \tag{6.4}$$

This implies that the sections s (or wavefunctions) in this representation are are independent of the variable p. Well, this didn't tell us much. Also, we see that the inner product as defined above is divergent for this case. Hence, let's go on to take a look at complex polarization.

6.5 Kähler Polarization

To define Kähler or complex polarization on a real 2n dimensional manifold, we need to introduce a complex structure on it. A complex structure J is defined as a (real) map from the $\mathrm{T}\mathscr{M}_x \to \mathrm{T}\mathscr{M}_x$ such that $J^2 = -1$[9]. Given this structure multiplication with a complex number $(a + \iota b)v$ corresponds to $(a + Jb)v$.[10](?)

This structure allows us to shift to the variables $(z, \bar{z})$ instead of (p,q). Here, $z = \frac{p+\iota q}{\sqrt{2}}$ and $\bar{z} = \frac{p-\iota q}{\sqrt{2}}$. In these variables, the following definitions are taken, $\theta = \frac{\iota}{2}(z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z)$ and $\omega = \iota \mathrm{d}z \wedge \mathrm{d}\bar{z}$. With respect to these, a polarization in a complex manifold is defined exactly as for the case of a real manifold- an integral Lagrangian submanifold. Given this, let's take a look at our 1D Harmonic oscillator again. Taking the mass and angular frequency of the Harmonic Oscillator as 1, the Hamiltonian function in the phase space can be written as,

$$\mathrm{H} = \frac{z\bar{z} + \bar{z}z}{2}$$

Now, the Hamiltonian vector field corresponding to z and $\bar{z}$ are respectively, $X_z = \iota\frac{\partial}{\partial \bar{z}}$ and $X_{\bar{z}} = -\iota\frac{\partial}{\partial z}$. Now, let's pick a representation in which the operator corresponding to z is z itself. In this case,

$$\nabla_{X_z}\Psi(z, \bar{z}) = 0$$

Since, $\theta(X_z) = \frac{\iota}{2}(z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z)(\iota\frac{\partial}{\partial \bar{z}}) = -\frac{z}{2}$,this gives

$$(\iota\frac{\partial}{\partial \bar{z}} + \iota\frac{z}{2})\Psi(z, \bar{z}) = 0$$

Solving this, we get, the following form of the wavefunction,

$$\Psi(z, \bar{z}) = \psi(z)e^{\frac{-z\bar{z}}{2\hbar}}$$

[9]example $J = -\iota\sigma_2$, where σ_2 is a Hermitian Pauli matrix, provides a complex structure in 2 dimensions

[10]The complex structure is said to be compatible with the two form ω if, $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in \mathrm{T}\mathscr{M}_x$ and all $x \in \mathscr{M}$. One can also define a metric on the manifold as $g(X, Y) := \omega(JX, Y)$. This along with ω gives $\mathscr{M}$ the structure of a Kähler manifold.

Also, in this representation, the action of the operator corresponding to $\bar{z}$ is,

$$(\hbar\frac{\partial}{\partial z}+\frac{\bar{z}}{2})\Psi(z,\bar{z})$$

Hence, the operator corresponding to the Hamiltonian function is,

$$H\Psi(z,\bar{z})=\frac{1}{2}((\hbar\frac{\partial}{\partial z}+\frac{\bar{z}}{2})(z\Psi(z,\bar{z})+z(\hbar\frac{\partial}{\partial z}+\frac{\bar{z}}{2})\Psi(z,\bar{z}))$$

Putting in the above form of the wavefunction into the eigenvalue equation for the Hamiltonian, we get,

$$\mathrm{H}\Psi(z,\bar{z})=E\Psi(z,\bar{z}) \tag{6.5}$$

$$\implies(\hbar+2z\hbar\frac{\partial}{\partial z}-2E)\psi(z)=0 \tag{6.6}$$

Solving this we get,
$\psi(z)=z^{m-\frac{1}{2}}$ where $m=\frac{E}{\hbar}$.
Now, since this should correspond to a single valued wavefunction, $m-\frac{1}{2}=n$ where $n\in$ Z. Therefore, the quantization of energy takes the form, $E=n\hbar+\frac{\hbar}{2}$ and the corresponding wavefunctions are of the form, $\Psi(z,\bar{z})=z^n e^{-\frac{z\bar{z}}{2\hbar}}$.

Given these wavefunctions, and since the manifold is flat, we get the following normalization condition from the natural L^2 inner product,

$$||\Psi(z,\bar{z})||^2=\int\int(z\bar{z})^n e^{-\frac{z\bar{z}}{\hbar}}\mathrm{d}z\mathrm{d}\bar{z}.$$

Including the exponential factor into the definition of the inner product, we may look at $\psi(z)=z^n$ as the wavefunctions.

6.6 Segal Bargmann Transform

[11]In the above sections we have seen the advantage of the complex representation over the real one. But even so after solving for the solutions of the Hamiltonian constraint in the complex representation, one would like to transform it to the usual real representation. One easy way is to integrate out the p variables in the complex solutions obtained. But, this is a one way transformation and will not provide us $\phi(z)$ from $\Psi(q)$. Therefore, we look for a unitary transform between, $\phi(z)$ and $\Psi(q)$, allowing us to go back and forth between them.

Let us suppose that the transform is given as,

$$\Psi(q)=\int_{\mathbb{C}}A(q,z)\phi(z)\mathrm{d}z \tag{6.7}$$

$$\phi(z)=\int_{\mathbb{R}}A(q,z)\Psi(q)\mathrm{d}q \tag{6.8}$$

[11]Refer [10]

where in equation (4.1) we need to choose the right contour of integration. Now, since the action of an operator is independent of the representation, we expect the transform to satisfy the following,

$$Op(z)\phi(z) = \int_{\mathbb{R}} A^{-1}(q,z)(Op(q)\Psi(q))\mathrm{d}q$$

and

$$Op(q)\phi(q) = \int_{\mathbb{C}} A(q,z)(Op(z)\Psi(z))\mathrm{d}z$$

The latter for operators corresponding to the observable, z and $\bar{z}$ is as below.
a) operator corresponding to z.

$$\frac{1}{\sqrt{2}}(-\iota\hbar\frac{\partial}{\partial q} - \iota q)\Psi(q) = \int_{\mathbb{C}} A(q,z)z\phi(z)\mathrm{d}z \tag{6.9}$$

b) operator corresponding to $\bar{z}$

$$\frac{1}{\sqrt{2}}(-\iota\hbar\frac{\partial}{\partial q} + \iota q)\Psi(q) = \int_{\mathbb{C}} A(q,z)\hbar\frac{\partial}{\partial z}\phi(z)\mathrm{d}z \tag{6.10}$$

$$= [A(q,z)\hbar\phi(z)]_{B.I}^{B.F} - \hbar\int_{\mathbb{C}} \frac{\partial A(q,z)}{\partial z}\phi(z)\mathrm{d}z \tag{6.11}$$

Here B.F and B.I correspond to the final and initial boundaries of the integration contour, respectively. Now, assuming that, the first term in the above equations vanishes (implying that A(q,z) will have to decay sufficiently fast at the boundaries), we have,

$$\frac{1}{\sqrt{2}}(-\iota\hbar\frac{\partial}{\partial q} + \iota q)\Psi(q) = -\hbar\int_{\mathscr{C}} \frac{\partial A(q,z)}{\partial z}\phi(z)\mathrm{d}z \tag{6.12}$$

Now, combining equations 6.9 and 6.12 we get the P.D.E.,

$$\frac{\partial^2 A}{\partial q \partial z} - \frac{q}{\hbar}\frac{\partial A}{\partial z} + \frac{z}{\hbar}\frac{\partial A}{\partial q} + \frac{qzA}{\hbar^2} - \iota\frac{\sqrt{2}A}{\hbar} = 0$$

Solving this gives,

$$A(q,z) = e^{\frac{-z^2}{2\hbar} - \iota\frac{\sqrt{2}zq}{\hbar} + \frac{q^2}{2\hbar}}$$

Now, as per the derivation, A(q,z) should go to zero at the boundaries of the contour of integration. Hence, the contour should be picked such that, A(q,z) dies down at the infinities along it. Plotting the region in which $|A(q,z)| \leq 1$ for different values of q, we (see figure Fig. 6.1) see that the integral is convergent (for any finite value of q) along any contour deformable to the real line. Hence, the contour of integration is the real line in the complex plane. Also, given this, the inverse transform is carried out by $A^{-1}(q,z)$,i.e.,

$$\phi(z) = \int_{\mathbb{R}} (e^{\frac{z^2}{2\hbar} + \iota\frac{\sqrt{2}zq}{\hbar} - \frac{q^2}{2\hbar}})\Psi(q)\mathrm{d}q$$

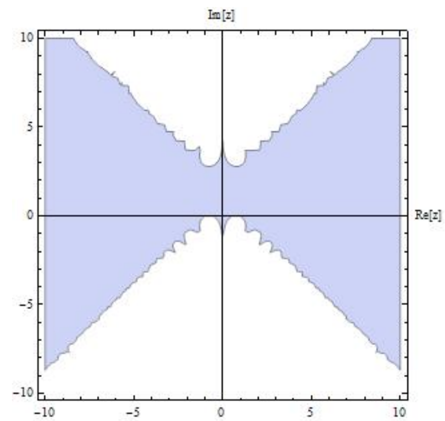


Figure 6.1: Plot of region corresponding to $|A(q, z)| \leq 1$ for q=1. Though the region for different values of q get shifted, the plot characteristics do not change. Hence, the contour along which the integral is convergent, in all cases is equivalent to the real axis.

How is all this relevant to our problem?

We saw in the above that sections that Geometric Quantization gives us a well defined Hilbert space at least in the complex polarization. One can extend the above structure for the case of wavefunctionals if we are interested in the Hilbert space problem in the most general cases. But for the case of Minisuperspace it is much simpler. Having a natural inner product in the complex representation, one could then solve simple systems in it and then transform back to the real representation. We are currently trying to do this for a homogeneous and isotropic universe with a massless scalar field.

Summary and Future Work

In this thesis, I primarily studied about the concepts of quantum gravidynamics. I understood the construction of wavefunctionals from both path integral and canonical formalisms. En route, we followed the discussion of Hartle *et al* and Turok *et al*, and it has helped me form an opinion on the validity of boundary proposals for the universe, suggested by Hartle-Hawking and Vilenkin. I have come to the conclusion that a smooth beginning for the universe as proposed by the above two is not possible. Moving on, I understood a few problems faced in the canonical quantization theory. Moreover, we have been trying to understand the Hilbert space problem using Geometric quantization, and are presently trying out the idea on the Minisuperspace model.

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Appendix A

Correlation Functions

Two Point Function

The two point function is the integral

$$I = \frac{1}{N}\int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x'\Big)\phi(y)\phi(y')D\phi$$

where

$$N = \int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x'\Big)D\phi$$

and $G(x,x')$ is symmetric in x and x'. Note that we have

$$\begin{aligned}
I &= \frac{\partial}{\partial J(y')}\frac{\partial}{\partial J(y)}\frac{1}{N}\int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x' + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi\bigg|_{J=0} \\
&= \frac{\partial}{\partial J(y')}\frac{\partial}{\partial J(y)}Z[J]\bigg|_{J=0}
\end{aligned}$$

where we have defined the partition function as

$$Z[J] = \frac{1}{N}\int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x' + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi$$

Making a change of variables $\chi(x) = \phi(x) - \int \frac{J(z)}{2}G^{-1}(x,z)\mathrm{d}z$, where the function $G^{-1}(x,z)$ is defined such that $\int G^{-1}(x,z)G(z,w)\mathrm{d}z = \delta(x-w)$, and noting that $D\phi = D\chi$, we get

$$
\begin{aligned}
Z[J] &= \frac{1}{N}\int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x' + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi \\
&= \frac{1}{N}\int \exp\Big(-\int \chi(x)\chi(x')G(x,x')\mathrm{d}x\mathrm{d}x' - \int \chi(x)\frac{J(z')}{2}G^{-1}(x',z')G(x,x')\mathrm{d}x\mathrm{d}x'\mathrm{d}z' \\
&- \int \chi(x')\frac{J(z)}{2}G^{-1}(x,z)G(x,x')\mathrm{d}x\mathrm{d}x'\mathrm{d}z - \int \frac{J(z)J(z')}{4}G^{-1}(x,z)G(x,x')G^{-1}(x',z')\mathrm{d}x\mathrm{d}x'\mathrm{d}z\mathrm{d}z' \\
&+ \int J(x)\chi(x)\mathrm{d}x + \int \frac{J(x)J(z)}{2}G^{-1}(x,z)\mathrm{d}x\mathrm{d}z\Big)D\chi \\
&= \frac{1}{N}\int \exp\Big(-\int \chi(x)\chi(x')G(x,x')\mathrm{d}x\mathrm{d}x' - \int \frac{J(x)}{2}\chi(x)\mathrm{d}x - \int \frac{J(x')}{2}\chi(x')\mathrm{d}x' \\
&- \int \frac{J(z)J(z')}{4}G^{-1}(z,z')\mathrm{d}z\mathrm{d}z' + \int J(x)\chi(x)\mathrm{d}x + \int \frac{J(x)J(z)}{2}G^{-1}(x,z)\mathrm{d}x\mathrm{d}z\Big)D\chi \\
&= \frac{1}{N}\int \exp\Big(-\int \chi(x)\chi(x')G(x,x')\mathrm{d}x\mathrm{d}x'\Big)\exp\Big(\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Big)D\chi \\
&= \exp\Big(\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Big)
\end{aligned}
$$

We can use this expression for the partition function to find the integral I.

$$
\begin{aligned}
I &= \frac{\partial}{\partial J(y')}\frac{\partial}{\partial J(y)}Z[J]\Big|_{J=0} \\
&= \frac{\partial}{\partial J(y')}\frac{\partial}{\partial J(y)}\exp\Big(\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Big)\Big|_{J=0} \\
&= \frac{1}{4}\frac{\partial}{\partial J(y')}\Bigg(\Big(\int J(z)G^{-1}(z,y)\mathrm{d}z + \int J(z')G^{-1}(y,z')\mathrm{d}z'\Big)\exp\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Bigg)\Bigg|_{J=0} \\
&= \frac{1}{2}\frac{\partial}{\partial J(y')}\Bigg(\Big(\int J(z)G^{-1}(y,z)\mathrm{d}z\Big)\exp\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Bigg)\Bigg|_{J=0} \\
&= \Bigg(\frac{G^{-1}(y,y')}{2} + \frac{1}{2}\Big(\int J(z)G^{-1}(y,z)\mathrm{d}z\Big)\Big(\int J(z)G^{-1}(z,y')\mathrm{d}z\Big)\Bigg)\exp\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Bigg|_{J=0} \\
I &= \frac{G^{-1}(y,y')}{2}
\end{aligned}
$$

Therefore

$$
\langle\phi(y)\phi(y')\rangle = \frac{1}{N}\int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x'\Big)\phi(y)\phi(y')D\phi = \frac{G^{-1}(y,y')}{2}
$$

Note :

$$\Big\langle \phi(y_1)...\phi(y_n)\Big\rangle_0 = \frac{\partial}{\partial J(y_1)}...\frac{\partial}{\partial J(y_n)}N\int \exp\Big(-\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x' + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi\Bigg|_{J=0}$$

$$= \frac{\partial}{\partial J(y_1)}...\frac{\partial}{\partial J(y_n)}N\int CD\phi\Bigg|_{J=0}$$

$$\Big\langle \phi(y_1)...\phi(y_n)\Big\rangle_0 = \frac{\partial}{\partial J(y_1)}...\frac{\partial}{\partial J(y_n)}\exp\int \frac{J(z)J(z')}{4}G^{-1}(z,z')\mathrm{d}z\mathrm{d}z'\Bigg|_{J=0}$$

Therefore,
if n is odd

$$\Big\langle \phi(y_1)...\phi(y_n)\Big\rangle_0 = 0$$

and if n is even
$\Big\langle \phi(y_1)...\phi(y_n)\Big\rangle_0 = \frac{1}{2}\Big($Sum of all possible* combinations of $G^{-1}(y_1,y_2)...G^{-1}(y_{n-1},y_n)\Big)$
*Without repeats and $G^{-1}(y_i,y_j)$ is considered same as $G^{-1}(y_j,y_i)$.

Three and Four Point Functions

For three point and four point functions, consider the probability distribution function: $p[\phi] = e^{-S}$, where

$$S = \int G^{-1}(x_1,x_2)\phi(x_1)\phi(x_2)dx_1dx_2 + \Lambda\int C(x_1,x_2,x_3)\phi(x_1)\phi(x_2)\phi(x_3)dx_1dx_2dx_3$$
$$+\Lambda^2\int D(x_1,x_2,x_3,x_4)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)dx_1dx_2dx_3dx_4$$

and $C(x_1,x_2,x_3)$ and $D(x_1,x_2,x_3,x_4)$ are fully symmetric. Expanding e^{-S} to first order in Λ gives :

$$e^{-S} = e^{-S_{free}}(1 - \Lambda\int C(x_1,x_2,x_3)\phi(x_1)\phi(x_2)\phi(x_3)dx_1dx_2dx_3)$$

where $S_{free} = \int G^{-1}(x_1,x_2)\phi(x_1)\phi(x_2)dx_1dx_2$, so the partition function in this case is :

$$Z[J] = \frac{1}{N}\int \exp\Big(-S + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi$$
$$= \frac{1}{N}\int \exp\Big(-S_{free} + \int J(x)\phi(x)\mathrm{d}x\Big)(1 - \Lambda\int C(x_1,x_2,x_3)\phi(x_1)\phi(x_2)\phi(x_3)dx_1dx_2dx_3)D\phi$$
$$= \frac{1}{N}\int \exp\Big(-S_{free} + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi$$
$$-\Lambda\frac{1}{N}\int(\exp\Big(-S_{free} + \int J(x)\phi(x)\mathrm{d}x\Big)\int C(x_1,x_2,x_3)\phi(x_1)\phi(x_2)\phi(x_3)dx_1dx_2dx_3)D\phi$$
$$= Z_{free}[J] - \Lambda\int C(x_1,x_2,x_3)\frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)}\frac{\partial}{\partial J(x_3)}Z_{free}[J]$$

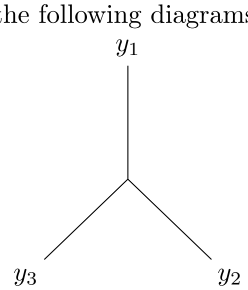
where we have defined

$$Z_{free}[J] = \frac{1}{N}\int \exp\Big(-S_{free} + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi$$
$$= \exp\Big(\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Big)$$

The three point function is then :

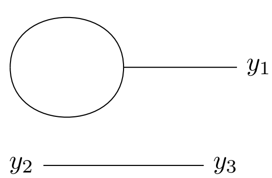
$$\big\langle\phi(y_1)\phi(y_2)\phi(y_3)\big\rangle = \frac{\partial}{\partial J(y_1)}\frac{\partial}{\partial J(y_2)}\frac{\partial}{\partial J(y_3)}Z[J]\bigg|_{J=0}$$
$$= -\Lambda\int C(x_1,x_2,x_3)\frac{\partial}{\partial J(y_1)}\frac{\partial}{\partial J(y_2)}\frac{\partial}{\partial J(y_3)}\frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)}\frac{\partial}{\partial J(x_3)}Z_{free}[J]\bigg|_{J=0}$$
$$= -\Lambda\int C(x_1,x_2,x_3)\frac{\partial}{\partial J(y_1)}\cdots\frac{\partial}{\partial J(x_3)}\exp\Big(\frac{1}{4}\int J(z)G^{-1}(z,z')J(z')\mathrm{d}z\mathrm{d}z'\Big)\bigg|_{J=0}$$

We can use Feynman diagrams to depict the various kinds of terms that occur in the final expression above. The correlator $\big\langle\phi(y_1)\phi(y_2)\phi(y_3)\big\rangle$ is the sum of the following diagrams :



$$= \frac{-3!\Lambda}{2^3}\int C(x_1,x_2,x_3) * G^{-}(x_1,y_1) * G^{-}(x_2,y_2) * G^{-}(x_3,y_3)$$

and



$$= \frac{-3\Lambda}{2^3}G^{-}(x_2,x_3)\int C(x_1,x_2,x_3) * G^{-}(y_1,y_2) * G^{-}(x_1,y_3)$$

plus equivalent terms.

For four point functions, we expand e^{-S} to order Λ^2. We get the partition function :

$$Z[J] = \frac{1}{N}\int \exp\Big(-S_{free} + \int J(x)\phi(x)\mathrm{d}x\Big)(1 - \Lambda C + \frac{\Lambda^2 C^2}{2} - \Lambda^2 D^2) D\phi$$

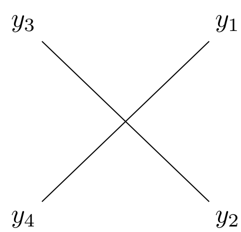
where we have compactly written the integrals as

$$C = \int C(x_1, x_2, x_3)\phi(x_1)\phi(x_2)\phi(x_3)dx_1 dx_2 dx_3$$

and

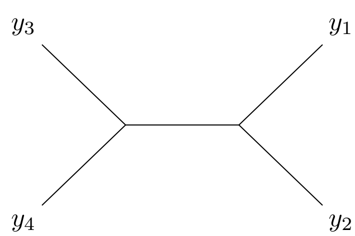
$$D = \int D(x_1, x_2, x_3, x_4)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)dx_1 dx_2 dx_3 dx_4$$

The following connected Feynman diagrams contribute to the four point function at order Λ^2:



$= \frac{-4!\Lambda^2}{2^4}\int D(x_1, x_2, x_3, x_4) * G^-(x_1, y_1) * G^-(x_2, y_2) * G^-(x_3, y_3) * G^-(x_4, y_4)$

and



$= \frac{36\Lambda^2}{2^5}\int C(x_1, x_2, x_3)C(x_4, x_5, x_6)G^-(y_1, x_1)G^-(y_2, x_2)G^-(x_3, x_4)G^-(x_5, y_3)G^-(x_6, y_4)$
plus equivalent terms.

A.0.1 Correlators in momentum space

Working in the momentum space gives us simple answers for the correlators we found previously. Here we derive these simple momentum space correlators. We first restate our assumptions, which are that $G(x, y)$ is symmetric (i.e. $G(x, y) = G(y, x)$) and translation invariant (i.e. only depends on $x - y$). Keeping this in mind, Fourier transforming the equation

$$\int G(x, y)G^-(y, z)dy = \delta(x - z)$$

gives

$$\tilde{G}^{-}(k) = 1/\tilde{G}(k)$$

where $\tilde{G}$ is in the monentum space. Fourier transforming the relation $G(x, y) = G(y, x)$ gives $\tilde{G}(k) = \tilde{G}(-k)$. We can write the partition function in terms of integrals in momentum space as follows :

$$\begin{aligned}
Z[J] &= \frac{1}{N}\int \exp\Big(-\frac{1}{2}\int \phi(x)\phi(x')G(x,x')\mathrm{d}x\mathrm{d}x' + \int J(x)\phi(x)\mathrm{d}x\Big)D\phi \\
&= \frac{1}{N}\int \exp\Big(-\frac{1}{2}\int \phi(x)\phi(x')(\int \tilde{G}(k)e^{\iota k.(x-x')}\frac{\mathrm{d}k}{(2\pi)^4})\mathrm{d}x\mathrm{d}x' + \int J(x)(\int \phi(k)e^{\iota k.x}\frac{\mathrm{d}k}{(2\pi)^4})\mathrm{d}x\Big)D\phi \\
&= \frac{1}{N}\int \exp\Big(-\frac{1}{2}\int \phi(-k)\phi(k)\tilde{G}(k)\frac{\mathrm{d}k}{(2\pi)^4} + \int J(-k)\phi(k)\frac{\mathrm{d}k}{(2\pi)^4}\Big)D\phi \\
&= \exp\Big(\frac{1}{2}\int J(-k)\frac{1}{\tilde{G}(k)}J(k)\frac{\mathrm{d}k}{(2\pi)^4}\Big)
\end{aligned}$$

Given periodic boundary condition, it can be proved that Jacobian is constant:
$D\phi$ (in position space) = $constant * D\phi$ (in momentum space).
We'll assume this result here without proving.

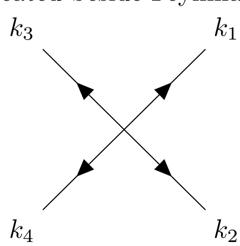
From the equation

$$Z[J] = \frac{1}{N}\int \exp\Big(-\frac{1}{2}\int \phi(-k)\phi(k)\tilde{G}(k)\frac{\mathrm{d}k}{(2\pi)^4} + \int J(-k)\phi(k)\frac{\mathrm{d}k}{(2\pi)^4}\Big)D\phi$$

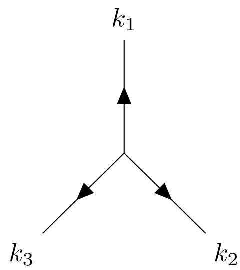
we see that two-point correlator is given by

$$\begin{aligned}
\big\langle\phi(k_1)\phi(k_2)\big\rangle &= \Big((2\pi)^4\frac{\delta}{\delta J(-k_1)}\Big)\Big((2\pi)^4\frac{\delta}{\delta J(-k_2)}\Big)Z[J] \\
&= \Big((2\pi)^4\frac{\delta}{\delta J(-k_1)}\Big)\Big((2\pi)^4\frac{\delta}{\delta J(-k_2)}\Big)\exp\Big(\frac{1}{2}\int J(-k)\frac{1}{\tilde{G}(k)}J(k)\frac{\mathrm{d}k}{(2\pi)^4}\Big) \\
&= (2\pi)^4\delta(k_1+k_2)/G(k_1)
\end{aligned}$$

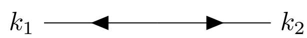
We can calculate three point and four point correlators by taking functional derivative with $J(k)$ the required number of times. The following terms indicated beside Feynman diagrams emerge:



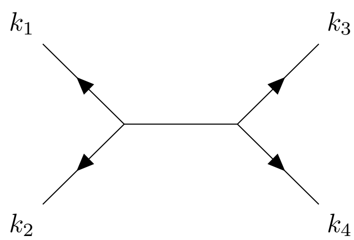
$$= 4!D(k_1,k_2,k_3,k_4)/(G(k_1)*G(k_2)*G(k_3)*G(k_4))$$



$$= 3!C(k_1, k_2, k_3)/(G(k_1) * G(k_2) * G(k_3))$$



$$= (2\pi)^4 \delta(k_1 + k_2)/G(k_1)$$



$$= \frac{1}{G(k_1)G(k_2)G(k_3)G(k_4)} \int \frac{d^4k}{(2\pi)^4} C(k_1, k_2, k)C(k_3, k_4, k)/G(k)$$