The Einstein-Cartan-Dirac (ECD) theory

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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An ode to reality

The solidified isle is the realm of physical reality,
Whose edges alone are probed by the restless waves of Thought and Reason,
Aided in the foreground by the floral sense of beauty,
Whilst the All-Knowing Sun of Intuition shines brightly above
Illuminating all realms, even those recondite noumenal recesses
Unknown and Unknowable to Thought and Emotion,
Where you reign supreme, Oh Reality! [1]

कस्मिन्नु भगवो विज्ञाते सर्वमिदम् विज्ञातं भवतीति|

[What is it, (lord), which being known, everything else (in a context) becomes known]

---MUNDAKA UPANISHAD (1.1.3)|

Yet, nature is made better by no mean
But nature makes that mean: So, over that art,
Which you say adds to the nature, is an art
That nature makes. [2]

[1] Composed by Prof. R. Srikanth. For the source, click here
[2] An excerpt from 'A winter's tale' by William Shakespeare

Certificate

This is to certify that this dissertation entitled **'The Einstein-Cartan-Dirac (ECD) theory'** submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research (IISER) Pune represents original research carried out by 'Swanand M Khanapurkar' at 'Tata Institute of Fundamental Research (TIFR)', under the supervision of 'Prof. Tejinder P. Singh, Department of Astronomy and Astrophysics (DAA), TIFR' during the academic year 2017-2018.

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Declaration

I hereby declare that the matter embodied in the report entitled **"The Einstein-Cartan-Dirac (ECD) theory"** are the results of the investigations carried out by me at the 'Department of Astronomy and Astrophysics (DAA), Tata Institute of Fundamental Research (TIFR)', under the supervision of 'Prof. Tejinder P. Singh' and the same has not been submitted elsewhere for any other degree.

Student
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Supervisor
Prof. Tejinder P. Singh

Dedicated to

To my parents
For giving me an immense support during the whole of my journey through life

To Prof. Tejinder P. Singh
For the patience and the faith which he kept in me during the thesis work.

To my friends and mentors at IISER-Pune, TIFR and NIRMAN.
In the presence of whom, I happen to get (momentarily) relieved from the curse of Sisyphus

To the magnificent SUNSET at the shores of TIFR, Mumbai


[3].

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[3]Photo credits:- Pranab Das's collection. For the source of this photo, click here

Abstract

There are various generalizations of Einstein's theory of gravity (GR); one of which is the Einstein- Cartan (EC) theory. It modifies the geometrical structure of manifold and relaxes the notion of affine connection being symmetric. The theory is also called U_4 theory of gravitation; where the underlying manifold is not Riemannian. The non-Riemannian part of the space-time is sourced by the spin density of matter. Here mass and spin both play the dynamical role. We consider the minimal coupling of Dirac field with EC theory; thereby calling the full theory as Einstein-Cartan-Dirac (ECD) theory. In the recent works by T.P Singh titled "A new length scale in quantum gravity [4]", the idea of new unified mass dependent length scale L_{cs} has been proposed. We discuss this idea and formulate ECD theory in both - standard as well as this new length scale. We found the non-relativistic limit of ECD theory using WKB-like expansion in $\sqrt{\hbar}/c$ of the ECD field equations with both the length scales. At leading order, ECD equations with standard length scales give Schrödinger-Newton equation. With L_{cs}, in the low mass limit, it gives source-free Poisson equation, suggesting that small masses don't contribute to gravity at leading order. For higher mass limit, it reduces to Poisson equation with delta function source. Next, we formulate ECD theory with both the length scales (especially the Dirac equation which is also called hehl-Datta equation and Contorsion spin coefficients) in Newman-Penrose (NP) formalism. The idea of L_{cs} suggests a symmetry between small and large masses. Formulating ECD theory with L_{cs} in NP formalism is desirable because NP formalism happens to be the common vocabulary for the description of low masses (Dirac theory) and high masses (gravity theories). We propose a conjecture to establish this duality between small and large masses which is claimed to source the torsion and curvature of space-time respectively. We therefore call it "Curvature-Torsion" duality conjecture. In the context of this conjecture, Solutions to HD equations on Minkowski space with torsion have been found and their implications for the conjecture are discussed. Three new works which we have done in this thesis [Non-relativistic limit of ECD theory, formulating ECD theory in NP formalism and attempts to find the solution to non-linear Dirac equation on U_4] are valid for standard theory and also the theory with L_{cs}. The conjecture to establish the Curvature-torsion duality is formulated in the context of idea of L_{cs}.

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Chapter 1

Introduction

1.1 Introducing four broad ideas to establish the grounds for this thesis

1.1.1 Einstein-Cartan theory

Einstein's theory of gravity, more commonly called as "General theory of relativity" (GR), published in 1915 is one of the most important works of 20^{th} century. It has been described as the most beautiful of all the existing physical theories [1]. In GR, Gravity is described as a geometric property of space-time continuum; thereby generalizing special relativity and Newton's law of universal gravitation. In GR, background space-time is Riemann manifold (denoted as V_4) which is torsion less. Affine connection coincides uniquely with levi-civita connection and geodesics coincides with the path of shortest distance.

There are few possible modifications of Einstein's theory of gravity (GR) [consistent with the principle of equivalence]; one of which is the Einstein- Cartan (EC) theory. It modifies the geometrical structure of manifold and relaxes the notion of affine connection being symmetric. The theory is also called U_4 theory of gravitation; where the underlying manifold is not Riemannian. The non-Riemannian part of the space-time is sourced by the spin density of matter. Mass and spin both play the dynamical role. Torsion, as an antisymmetric part of the affine connection was introduced by Elie-Cartan (1922) [8]. In May 1929, He wrote a letter to Albert Einstein suggesting that his studies on torsion might be of physical relevance in General Relativity. The local Minkowskian structure of space-time (which is the essential constraint on manifold if it has to describe physically plausible space-time) is not violated in the presence of torsion. So a manifold with torsion and curvature [with an essential constraint that non-metricity = 0 [2]] can very well describe physical space-time. It is called Riemann-Cartan (U_4) manifold. Since the works of E.Cartan, many people like D. Scima, Kibble, F.Hehl, Trautman etc. have studied the theories of gravity on a Riemann-Cartan space time U_4 over last century. The basic framework of EC theory was laid down by D.Scima (1962, 1964) [9], [11] and Kibble (1961) [10]. Hence the theory is called Einstein-Cartan-Scima-Kibble (ECSK) theory. Modern review on the subject of ECSK theory is by F.Hehl et.al (1976) [2]. It is titled "General relativity with spin and torsion: Foundations and prospects". In a recent work of

Trautman [53], he suggests, "It is possible that the Einstein–Cartan theory will prove to be a better classical limit of a future quantum theory of gravitation than the theory without torsion". It is worth asking the question that why don't we observe torsion in the universe around us. We note that torsion becomes comparable to curvature only at length scales smaller than the Einstein-Cartan radius $r_c = (\lambda_c L_{pl}^2)^{1/3}$ and at densities higher than m/r_c^3 where λ_c and L_{pl} are Compton wavelength and plank length respectively. For nucleons, the Einstein-Cartan radius is about 10^{-27} cms, and the density above which torsion becomes important is about 10^54 gms/cc [52]. These scales are beyond current technology, and since GR is in excellent agreement with observations, it is said that torsion can be safely neglected in today's universe. Literature on Einstein-Cartan theory in the context of cosmology and early universe can be refereed in [50], [51] and the references therein. [36]

When we minimally couple Dirac field on U_4, we get Einstein-Cartan-Dirac (ECD) theory. There are 2 independent geometric fields (metric, torsion) in this theory and one matter field ψ. We get 3 equations of motion. Dirac equation on U_4 becomes non-linear and is then called Hehl-Datta equation [3]. Einstein-Cartan theory and its coupling with Dirac field has been discussed in details in chapter (2). U_4 theory has also been discussed in details in book by Gasperini [29]. We have used some results from this book.

1.1.2 The Schrödinger-Newton equation

The Schrödinger equation describes the evolution of the wave-function over time. Born's probability rule gives a connection between the wave-function and the physical world. However the process of wave-function collapse is one of instantaneous nature and its mechanism is not explained via any acceptable theory. Broadly, this is often called "Quantum-measurement problem". A brief review of various interpretations which revolve around this problem can be looked up in section I.B of [15] and the references therein.

The Schrödinger-Newton equation came first into the discussions within the scientific literature due to Ruffini and Bonazzola in their work [25]. Diosi et.al in their works [21] proposed this equation as a model of wave function collapse; more specifically as a model of gravitational localization of macro objects. Roger Penrose developed this idea further and proposed that Schrödinger-Newton equation describes the basis states for the scheme of gravitationally induced wavefunction collapse. This can be looked up in his works [37], [22]. In deriving Schrödinger-Newton equation, we primarily observe the self-gravity of a quantum mechanical object; that is we observe the modification of Schrödinger's equation due to the gravity of the particle for which the equation is being written. Here, matter is taken to be of quantum nature while gravity is still treated classically. Here we assume the fact that, to leading order, the particle produces a classical potential satisfying the Poisson equation, whose source is a density proportional to the quantum probability density.

$$\nabla^2 \phi = 4\pi G m |\psi|^2 \tag{1.1}$$

The Schrödinger equation is then modified to include this potential and we get the Schrödinger-

Newton equation,

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + m\phi\psi \tag{1.2}$$

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) - Gm^2\int\frac{|\psi(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3r'\psi(\mathbf{r},t) \tag{1.3}$$

Equations (A.12), (A.13) and (A.14) together is called "Schrödinger-Newton" system of equations. By many people, this system of equations was taken as hypothesis to be put to test by experiments, whether there are any observational consequences (Ex. in molecular interferometry etc.) Work by Giulini et.al [20] analyzed the quantitative behavior of Gaussian wave packets moving according to Schrödinger-Newton equation and proved that wave packets disperse due to their own gravitational field significantly at mass scales around 10^{10}u (for a width of 500nm.) This is just 10^3 orders of magnitude more than masses which are envisaged in the future molecular interferometry experiments. Some works [38], [23] propose that this equation sheds some light on the question of necessity of quantum gravity.

Main paper of our interest in this thesis is [19]. Its a recent study by Guilini and Grossardt aimed at knowing whether this equation can be understood as a consequence of known principles and equations. They found that Schrödinger-Newton equation is the non-relativistic limit of self -gravitating Klein-Gordon and Dirac fields. Here the gravity is the classical gravity described by GR (on V_4 manifold).

1.1.3 Tetrad formalism, Spinor formalism, Newman-Penrose (NP) formalism

1) Tetrad formalism in GR

The usual method in approaching the solution to the problems in General Relativity was to use a **local coordinate basis** $\hat{e}^\mu$ such that $\hat{e}^\mu = \partial_\mu$. This coordinate basis field is covariant under General coordinate transformation. However, it has been found useful to employ non-coordinate basis techniques in problems involving Spinors. This is the tetrad formalism which consists of setting up four linearly independent basis vectors called a 'tetrad basis' at each point of a region of spacetime; which are covariant under local Lorentz transformations. [One of the reason of using tetrad formalism for spinors is essentially this fact that transformation properties of spinors can be easily defined in flat space-time]. Tetrads are basically basis vectors on local Minkowski space. Detail account of tetrad formalism in GR can be found in Appendix [B.1].

2) $SL(2,\mathbb{C})$ Spinor formalism

4-vector on a Minkowski space can be represented by a hermitian matrix by some transformation law. Unimodular transformations on complex 2-Dim space induces a Lorentz transformation in Minkowski space. Unimodular matrices form a group under multiplication and is denoted by$SL(2,\mathbb{C})$ - special linear group of 2 x 2 matrices over complex numbers. By a simple counting argument, it has six free real parameters corresponding to those of the Lorentz group. The levi-civita symbol $\epsilon_{AB'}$ acts as metric tensor in $\mathbb{C}^2$, which preserves the scalar product under Unimodular transformations. Spinor P^A of rank 1 is defined as vector in complex

2-Dim space subject to transformations $\in SL(2,\mathbb{C})$. Similarly higher rank spinor are defined. Analogous to a tetrad in Minkowski space, here we have a spin dyad (a pair of 2 spinors $\zeta_{(0)A}$ and $\zeta_{(1)A}$) such that $\zeta_{(0)A}\zeta^{A}_{(1)} = 1$.

3) Newman-Penrose (NP) formalism

NP formalism was formulated by Neuman and Penrose in their work [35]. It is a special case of tetrad formalism; where we choose our tetrad as a set of four null vectors viz.

$$e^{\mu}_{(0)} = l^{\mu}, \quad e^{\mu}_{(1)} = n^{\mu}, \quad e^{\mu}_{(2)} = m^{\mu}, \quad e^{\mu}_{(3)} = \bar{m}^{\mu} \tag{1.4}$$

l^{μ}, n^{μ} are real and $m^{\mu}, \bar{m}^{\mu}$ are complex. The tetrad indices are raised and lowered by flat space-time metric

$$\eta_{(i)(j)} = \eta^{(i)(j)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.5}$$

and the tetrad vectors satisfy the equation $g_{\mu\nu} = e^{(i)}_{\mu} e^{(j)}_{\nu} \eta_{(i)(j)}$. In the formalism, we replace tensors by their tetrad components and represent these components with distinctive symbols. These symbols are quite standard and used everywhere in literature. A brief review of NP formalism can be found in chapter (5).

Now, it can be shown that there is a natural connection between spin dyads and null tetrads [6], [31]. A null tetrad can be associated with a spin dyad by certain identification. This connection is explained in details in Appendix [B.2]. Equations of motion involving spinorial fields (Ex. Dirac field) can be expressed in NP formalism. Dirac equation on V_4 has been studied extensively in [6]. Many systems in gravitational physics are also studied in NP formalism [6]. NP formalism happens to be the common vocabulary between physics of quantum mechanical spinor field systems and classical gravitational field systems.

1.1.4 Unified length scale in quantum gravity L_{cs} and curvature-torsion duality

In the recent works of Tejinder P. Singh [4], [5], it has been argued, why and how Compton wavelength $(\lambda/\hbar c)$ and Schwarzschild radius $(2GM/c^2)$ for a point particle of mass 'm' should be combined into one single new length scale, which is called Compton-Schwarzschild length (L_{CS}). The idea of L_{cs} is more coherent in the framework of U_4. Action principle has been proposed with this new length scale and Dirac equation and Einstein GR equations are shown to be mutually dual limiting cases of this underlying modified action. More details can be looked up in chapter (3). It has been proposed that for $m \ll m_{pl}$, the spin density is more important than mass density. Mass density can be neglected and spin density sources the torsion (coupling is through $\hbar$). Whereas, $m \gg m_{pl}$, mass density dominates spin density. spin density can be neglected and as usual, mass density sources the gravity (coupling is through G). In this manner there exists a symmetry between small mass and large mass in the sense that small mass is the source for torsion and large mass is the source for gravity. [5]. Since both small masses and large masses give same L_{cs} (which is the only free parameter in the theory), there is a sort of duality between solutions to small masse and that of large mass. We call such a duality "Curvature-torsion" duality. We will explain this duality more in chapter (6) and [18]

1.2 Goals and objectives of the Thesis

1.2.1 Finding Non-relativistic limit of ECD theory

As discussed in the section (1.1.2), recent work by Giulini and Großardt [19] derived the non-relativistic limit of self-gravitating Klein-Gordan and Dirac fields. They used WKB-like expansion of Dirac Spinor and metric in (1/c) (as discussed in [26]) and found that, at leading order, the non-relativistic limit gives Schrödinger-Newton equation. This work considers:
Spherically symmetric gravitational fields
Background space-time is Riemannian (V_4)

As a sequel to this study and to the study by TP Singh [5](where ECD equations are modified with the unified length scale L_{cs}), we aim for the following:

Consider the generic metric (with no assumptions of symmetry) and find the non-relativistic limit of Einstein-Dirac system. This would generalize their work. We hypothesize that It will also be possible to find the underlying role of symmetry in the metric (in the context of non-relativisic limit).
If we consider gravitational theories with torsion; especially Einstein-Cartan-Dirac (ECD) theory discussed in section (1.1.1), it is worthwhile seeing whether the effects of torsion (viz. non-linearity in Dirac equation and correction to the gravity equation by spin-density) modify the Schrödinger-Newton equation in its non-relativistic limit. If it doesn't, next question we can ask is - At what order in 1/c, does effects due to torsion start getting manifested in the non-relativistic limit. This is important from the point of view of experimental studies in the detection of torsion and also to study the implications of the ECD theory at low energies.
Find the non-relativistic limit of ECD equations with modified length scale L_{cs}. We wish to analyze the underlying limit at leading order for limiting cases of large mass and small mass.

1.2.2 Formulating ECD theory in NP formalism

Dirac equation has been studied extensively in NP formalism on V_4. It's detail account can be seen in this celebrated book "The mathematical theory of black holes" By S. Chandrasekhar [6]. We wish to formulate ECD theory in NP formalism. More specifically;
We know that Contorsion tensor is completely expressible in terms of components of Dirac spinor. We want to find an explicit expression for Contorsion spin coefficients (in Newman-Penrose) in terms of Dirac spinor components. We will express this in both length scales - standard and unified length scale L_{cs}
Dirac equation on V_4 is presented in equation (108) of [6]. We aim to modify these equations

on U_4. We will express this in both length scales - standard and unified length scale L_{cs}

There are 2 independent reasons for doing this:

1) Many gravitational systems in the literature (especially having some specific symmetries explained in details in chapter (5) are formulated in NP formalism. But the space-time background in all those cases don't have torsion (V_4). It is worthwhile seeing the change in equations when we have torsion in the picture. Most of the important and physically relevant geometrical objects/ identities (Ex. Riemann curvature tensor, Weyl tensor, Bianchi identities, Ricci identities etc.) on U_4 have been formulated in NP formalism in the work [34]. In the context of ECD theory, however, there are 2 important aspects which are not yet accounted viz. Dirac equation on U_4 (Hehl-Datta equation), canonical EM tensor etc. Some works [47], [41], [40] attempt to do that but have not provided explicit corrections to standard NP variables due to torsion. Also, there are notational and sign errors in them. We wish to modify the equations/ physical objects as a sequel to Chandra's work in [6] which is on V_4. In the case of vanishing torsion, our equations/ formulations should boil down to standard equations on V_4 as given in [6]. With this objective, we formulate the equations of ECD theory (which has 3 primary equations on U_4 - Dirac equation, Gravitational equation relating Einstein's tensor and canonical EM tensor, Algebraic equation relating torsion and spin) with standard length scale. **Especially we would like to analyze the Contorsion spin coefficients and thereby use Chandrasekhar's approach to modify Dirac equation.**

2) As explained in section (1.1.4), the idea of L_{cs} in the context of U_4 theory provides a symmetry between small and large mass. There is a duality in the solution to large and small mass (we attempt ton establish it through a conjecture explained in next section). Dirac theory dominates for small masses and gravity dominates for large masses. In order to establish such a duality, its desirable to have a common mathematical language (provided by NP formalism) for dealing with both the domains [4]. To this aim, we formulate the ECD theory in NP formalism with unified length scale L_{cs}.

1.2.3 Testing Curvature-torsion duality conjecture

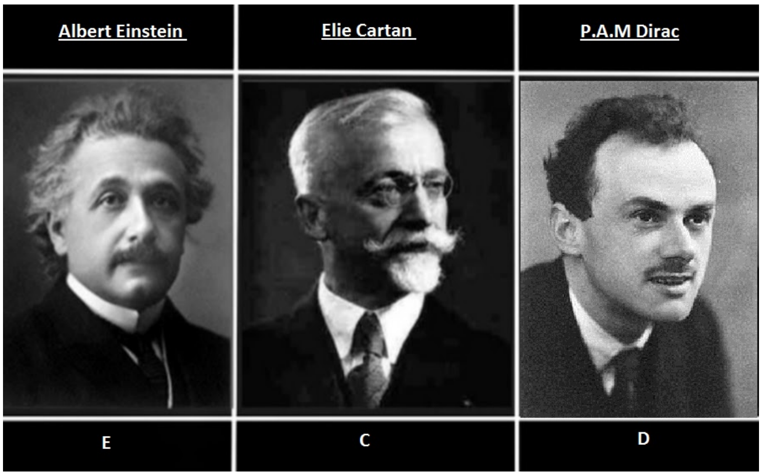
As discussed in section (1.1.4), the idea of L_{cs} proposed in [4] hints at a symmetry between small and large masses. Solution to small mass is dual to the solution to large mass in the sense that both have same L_{cs} which is the only free parameter in the theory. The motivation for such a "curvature-torsion" duality has been discussed in [5]. However, we need to make this duality, both qualitatively and mathematically, more evident. To this aim, we propose a conjecture called "Curvature-torsion duality conjecture" in chapter (6). Further, this chapter discusses the ways in which such a conjecture can be put to a test. After going through arguments presented in this chapter, we find that if a solution to ECD equations on Minkowski space with torsion exists, which make a tensor "T-S" (defined in 6) vanish, existence of such a solution supports the conjecture. So, the last few sections of this chapter are devoted at finding solutions to Hehl-Datta equations on Minkowski space with torsion and test the duality conjecture. A more detailed account of curvature-torsion duality as an idea can be looked up in [18].

1.3 Brief outline of the Thesis

Chapter 2 and 3 are theory chapters. In chapter 2, we have explained Einstein-Cartan-Dirac theory in details starting from first principles. Chapter 3 discusses the idea of unified length scale called Compton-Schwarzschild length scale (L_{cs}) in the theories which attempt to unify quantum mechanics with gravity. This chapter is mainly based on [4] and [5]. Chapter 4 is dedicated at finding Non-relativistic limit of ECD theory with standard as well as unified length scale. One can directly go to summary section 4.5 of this chapter to know some new results. In Chapter 5, we have formulated the ECD theory in NP formalism with standard as well as unified length scale. One can find its summary in section 5.3. In chapter 6, we attempt to establish a duality between curvature-torsion via a conjecture and solve ECD equations on Minkowski space (metric flat) with torsion. Chapter 7 is reserved fro presenting conclusions, outlook and future plans. All the important calculations relating to Non-relativistic limit of ECD equations can be looked up in Appendix A. ECD equations in NP formalism in Appendix B.

Chapter 2

Einstein-Cartan-Dirac (ECD) theory



2.1 Brief Review of classical theories of gravity

Huge strides were made in the European world of 13/14/15 and 16th century about nature of motion seen in the physical world. It took ingenious arguments and efforts of Aristotle, Keplar, Ibn Sina, T. Brahe, Copernicus, Galileo, Leibniz etc. to come up with a coherent, highly falsifiable, internally consistent (that is requiring no additional assumption beyond physical observables), highly predictable and reproducible model/ ontology of the nature of motion. the idea of "conservation of momentum" was an important paradigm shift in our thinking about the ontology of motion. The law also gave a mathematically characterizable notion to the inertia. It stated that the product of "That property of matter which characterizes inertia" (called inertial mass M_i) and "velocity" remains conserved and such a hypothesis (extensively supported by empirical evidences) is sufficient for any type of motion to take place as such; abandoning the idea of "unmoved mover" of Aristotle. Issac Newton in 16^{th} gave (the then universal) law of gravitation. The masses which appear in this law is the attribute of "gravitational mass M_g". This formalism triggered the huge developments in classical physics. Surprisingly, M_i and M_g happened to be numerically exactly the same. It suggested that "acceleration imparted to a body by a gravitational field is independent of the nature of the body". This motivated Einstein to generalize his special theory of relativity to include general coordinate transformations and non-inertial observers. He found that equivalence between inertia and gravity naturally leads to his theory of gravity called as general theory of relativity (GR). It is a classical theory of gravity. In GR, space-time is curved and the amount of curvature is determined by the Energy distribution on space-time. GR can be summed up in the following equation:

$$G_{\mu\nu} = KT_{\mu\nu} \tag{2.1}$$

Where $G_{\mu\nu}$ is Einstein's tensor which characterizes the curvature of space-time manifold and $T_{\mu\nu}$ characterizes the energy distribution on space-time. Gravity is described as a geometric property of space-time continuum. In GR, background space-time is Riemann manifold (denoted as V_4) which is torsion less. Affine connection coincides uniquely with levi-civita connection and geodesics coincides with the path of shortest distance. It is also called V_4 theory of gravitation. Max Born describes GR as (in his own words) "GR seemed and still seems to me at present to be the greatest accomplishment of human thought about nature; it is a most remarkable combination of philosophical depth, physical intuition and mathematical ingenuity. I admire it as a work of art." GR has survived 100 years of challenges, both by experimental tests and by alternative theories. It is the basis for the Standard Model of physical cosmology. The review of GR and cosmology w.r.t its unsolved problems and future directions can be looked up in [54]

2.2 Field theory for first quantized Dirac-field

Under the coordinate transformations, $x \to x' = \Lambda x$, the field ϕ can transform actively or passively as $\phi \to \phi'$. Active transformation of a generic field is governed by the equation: $\phi'(x) = L_\Lambda \phi(\Lambda^{-1}x)$ where L_Λ are the elements of representation of a group of rotations [e.g. if ϕ real scalar field, then $L_\Lambda = \mathbb{I}$, if ϕ is real vector field on 3D space, then $L_\Lambda = R$ where R represents a 3x3 orthogonal matrix. If ϕ is vector field on 4D space-time, then $L_\Lambda = \Lambda$ where Λ represents a 4x4 matrix of Lorentz transformation. ϕ is spinor field on 4D space-time, then $L_\Lambda = S[\Lambda]$ where $S[\Lambda]$ is a spinor representation of Lorentz group]. We denote real tensor fields by ϕ and spinor fields by ψ. We define 2 types of variations - functional variation and total variation and adopt following notation henceforth [7]
*Functional variation in ϕ: $\delta\phi = \phi'(x^\mu) - \phi(x^\mu)$ and
*Total variation in ϕ : $\Delta\phi = \phi'(x'^\mu) - \phi(x^\mu) = \delta\phi + (\partial_\mu\phi)\delta x^\mu$.

2.2.1 Generalized Noether theorem and conserved currents

Let $\phi(x^\mu)$ traces out 4-D region R in a 5-D space (ϕ,x,y,z,t). Initial and final space-like hyperspace; sliced at times t = t_1 and t = t_2 forms a boundary ∂R of region R. Under the condition that the variation of ϕ and x^μ vanish on the boundary ∂R we get, the *Euler-Lagrange* equation of motion for this field ϕ as follows:

$$\frac{\partial L}{\partial \phi} = \partial_\mu \pi^\mu; \qquad \pi^\mu = \left(\frac{\partial L}{\partial(\partial_\mu \phi)}\right) \tag{2.2}$$

Now we vary action S on a classical trajectory and state Noether's theorem as follows: Suppose action is invariant under a group of transformations on x^μ and ϕ [whose infinitesimal version is given by $\Delta x^\mu = X^\mu_\nu \delta\omega^\nu$ and $\Delta\phi = \Phi_\nu \delta\omega^\nu$ and which are characterized by infinitesimal parameter $\delta\omega^\mu$], then there exist one or more conserved quantities which remain invariant under the transformations. For Lagrangian, the condition is that it should either remain invariant or at the most change by total derivative. We will exploit this freedom om Lagrangian later, We will now establish this theorem mathematically. Variation of action over classical trajectory yields:

$$\delta S = \int_{\delta R} \Big[\pi^\mu \Phi_\nu - \Theta^\mu_k X^k_\nu\Big] \delta\omega^\nu \, d\sigma_\mu; \qquad \Theta^\mu_\nu = (\pi^\mu \partial_\nu \phi - L\delta^\mu_\nu) \tag{2.3}$$

Now, if the transformations make $\delta S = 0$ and since $\delta\omega^\nu$ is arbitary, we can write equation 2.3 as follows:

$$\int_{\delta R} J^\mu_\nu d\sigma_\mu = 0; \qquad J^\mu_\nu = \Big[\pi^\mu \Phi_\nu - \Theta^\mu_k X^k_\nu\Big] \tag{2.4}$$

Using Gauss's theorem;

$$\int_{\delta R} J^\mu_\nu d\sigma_\mu = 0 \implies \int_R \partial_\mu J^\mu_\nu d^4x = 0 \implies \partial_\mu J^\mu_\nu = 0 \tag{2.5}$$

We therefore have a conserved and divergence-less current J^μ_ν whose existence follows from the invariance of action under the given (generic) set of transformations. Integrating above equation over $t = const$ hyperspace and by using *Gauss's theorem* we get

$$\frac{\partial Q_\nu}{\partial t} = 0 \qquad \left(Q_\nu = \int_V J^0_\nu d^3x\right) \tag{2.6}$$

where Q_ν is *Noether's Charge.*

2.2.2 Noether's theorem applied to Real Tensor and spinor fields

Ex.1: Translational invariance for real tensor fields

Under the requirement that the laws of physics are to be translationally invariant i.e., using $\Phi_\mu = 0$ and $X^\mu_\nu = \delta^\mu_\nu$ we get $J^\mu_\nu = -\Theta^\mu_\nu$; which, using 2.6 gives conserved four-momentum of the field

$$Q_\nu = \int \Theta^0_\nu d^3x; \quad Q_0 = \int_V \Big(\frac{\partial L}{\partial \dot\phi}\dot\phi - L\Big) d^3x = \int_V \mathcal{H} d^3x = \mathbb{H} = P_0; \quad Q_i = \int_V \Big(\frac{\partial L}{\partial \dot\phi}\partial_i \phi\Big) d^3x = P_i \tag{2.7}$$

Where, $\mathcal{H}$ is the Hamiltonian density and $\mathbb{H}$ is the Total Hamiltonian of the system. Also, $Q_\mu = P_\mu$ and the fact that $\partial_t(Q_\mu) = 0$ suggests that invariance of translations conserve the 4-Momentum P_μ. Here the conservation law is $\partial_\mu \Theta^\mu{}_\nu = 0$. We observe that the Noether theorem's claim (Action remaining invariant) doesn't specify $\Theta^\mu{}_\nu$ uniquely. The conservation law specifies Θ^μ_ν upto addition of divergence of an antisymmetric tensor field 'f' as follows:

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}; \qquad \left(f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}\right); \qquad \partial_\mu T^{\mu\nu} = 0 \tag{2.8}$$

Owing to Gauss's divergence theorem, such an addition of 'f' doesn't change the physical observables viz. Energy and Momentum.

Ex.2: Rotational invariance for real tensor fields We characterize infinitesimal Lorentz transformations by an antisymmetric tensor $\epsilon^{\mu\nu}$ such that $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Under a requirement that the action should be invariant under Lorentz group i.e. under the infinitesimal transformations $\Delta\phi = 0$ and $\Delta x^\mu = \delta x^\mu$; which is given by following equation:

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu = X^\mu_{\rho\sigma}\epsilon^{\rho\sigma}; \qquad X^\mu_{\rho\sigma} = \frac{\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho}{2} \tag{2.9}$$

using equation (2.3) to find *Noether's current*; we obtain a 3 component Noether's current $J^{\mu\nu\sigma}$ as follows:

$$J^{\mu\nu\sigma} = \frac{-1}{2}\left(\Theta^{\mu\nu}x^\sigma - \Theta^{\mu\sigma}x^\nu\right); \qquad \partial_\mu J^{\mu\nu\sigma} = \frac{-1}{2}\left(\Theta^{\sigma\nu} - \Theta^{\nu\sigma}\right) \tag{2.10}$$

$\Theta^\mu{}_\nu$ is the EM tensor representing 4-Momentum density. Hence RHS in the above expression represents density of angular momentum. Indeed, as we expect from the analogy with classical mechanics, invariance under Lorentz's rotation conserve the angular momentum of the system. The question now is: Does it remain conserved for any $\Theta^\mu{}_\nu$? As we see in the second equation of equation (2.10), only for symmetric $\Theta^\mu{}_\nu$, conservation law seems to hold. We will investigate it in the next section.

Ex.3: Rotational invariance for Spinor fields (this is of our interest) We know that a Spinor field transforms as

$$\psi^\alpha(x) \longrightarrow \psi'^\alpha(x) = S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x); \qquad S[\Lambda] = 1 + \frac{\omega_{\mu\nu}S^{\mu\nu}}{2} \tag{2.11}$$

Corresponding functional and total variation in ψ is then given by

$$\delta[\psi^\alpha(x)] = \left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^\alpha{}_\beta \psi^\beta(x) - \partial_\mu\psi^\alpha(x)\omega^\mu{}_\nu x^\nu; \qquad \Delta\psi = \frac{1}{2}\omega^{\mu\nu}S_{\mu\nu}\psi = \Psi_{\mu\nu}\omega^{\mu\nu}; \qquad \Psi_{\mu\nu} = \frac{1}{2}S_{\mu\nu}\psi \tag{2.12}$$

And the total variation in x^μ is as given in eqn (2.9). Then, by Noether's theorem, the conserved current is:

$$J^\mu_{\nu\sigma} = \pi^\mu\Psi_{\nu\sigma} - \Theta^\mu{}_\alpha X^\alpha_{\nu\sigma} \tag{2.13}$$

$$= \frac{1}{2}\frac{\partial L}{\partial_\mu\psi}S_{\nu\sigma}\psi - \frac{1}{2}\left(\Theta^\mu{}_\nu x_\sigma - \Theta^\mu{}_\sigma x_\nu\right) \tag{2.14}$$

$\Theta^\mu{}_\nu$ is the EM tensor representing 4-Momentum density. Hence the second term in above expression represents density of orbital angular momentum. Therefore $J^\mu_{\nu\sigma}$ can be recognized as the total angular momentum density of the matter provided the first term represents the intrinsic spin density of matter field. We take ν, σ up, define spin density of the matter by a 3 component tensor $S^{\mu\nu\sigma}$ and rewrite the above equation as follows:

$$J^{\mu\nu\sigma} = S^{\mu\nu\sigma} - \frac{1}{2}\left(\Theta^{\mu\nu}x^\sigma - \Theta^{\mu\sigma}x^\nu\right); \qquad S^{\mu\nu\sigma} = \frac{1}{2}\frac{\partial L}{\partial_\mu\psi}S^{\nu\sigma}\psi \tag{2.15}$$

2.2.3 Symmetrization of EM tensor by Belinfante-Rosenfeld transformation

We find that, unless $\Theta^{\sigma\nu}$ is symmetric (which need not be the case always), we don't have a truly conserved angular momentum density current. But we know that Noether conserved currents are arbitary upto addition of divergence-less fields (refer equation 2.8). We can exploit this possibility to modify $\Theta^{\mu\nu}$ to $T^{\mu\nu}$ such that it is a symmetric tensor. The antisymmetric tensor field $f^{\lambda\mu\nu}$' which makes $T^{\mu\nu}$ symmetric is called Belnfinte tensor $B^{\lambda\mu\nu}$. It respects the fact that $\partial_\mu T^{\mu\nu} = 0$ and the fact that new symmetric tensor $T_{\mu\nu}$ defines the **same physical observable** (namely, energy-momentum) of the field.

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu} \qquad \left(B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}\right) \tag{2.16}$$

Is the existence of such a Balinfante tensor (which makes $T^{\mu\nu}$ symmetric) guaranteed? Following theorem proved by Belinfante in [14] gives necessary and sufficient conditions on the existence of $B^{\lambda\mu\nu}$. [We state the converse of the original theorem statement here]
Theorem A [33]: $\exists$ a symmetric stress-energy tensor [equivalently $\exists$ Belinfante tensor $B^{\lambda\mu\nu}$] iff the anti-symmetric part of the conserved canonical EM tensor is a total divergence.
Theorem B [33]: Given a tensor $H^{\lambda\mu\nu}$ such that $\Theta^{[\mu\nu]} = -\frac{1}{2}\partial_\lambda H^{\lambda\mu\nu}$, one can explicitly construct a Belinfante tensor $B^{\lambda\mu\nu}$ such that $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu}$ is symmetric. The explicit construction is as follows:

$$B^{\lambda\mu\nu} = \frac{1}{2}\Big(H^{\lambda\mu\nu} + H^{\mu\nu\lambda} - H^{\nu\lambda\mu}\Big) \tag{2.17}$$

Such a transformation of $\Theta^{\mu\nu}$ to $T^{\mu\nu}$ is called **"Belinfante-Rosenfeld transformation"**.

Einstein's general theory of relativity requires EM tensor in its field equations to be symmetric. $G^{\mu\nu} = kT^{\mu\nu}$ Here $T^{\mu\nu}$ is called **'Dynamic EM tensor'** and is constructed as $T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}$. It is symmetric by construction. We now state an important theorem.
Theorem C[33]: The symmetric EM tensor obtained by Belinfante-Rosenfeld transformation using Belinfante's tensor on matter field **is the same as** dynamic EM tensor which appears on the RHS of field equations of general theory of relativity.

2.2.4 Applying above machinery to Dirac Lagrangian

Lagrangian density pf Dirac field in given by

$$\mathcal{L}_m = \frac{i\hbar c}{2}(\overline{\psi}\gamma^a\partial_a\psi - \partial_a\overline{\psi}\gamma^a\psi) - mc^2\overline{\psi}\psi \tag{2.18}$$

The EM tensor and its antisymmetric part is given by

$$\Theta_{ij} = \frac{i\hbar c}{2}[\overline{\psi}\gamma_i\partial_j\psi - \partial_j\overline{\psi}\gamma_i\psi] \qquad \Theta_{[ij]} = \partial_k S^{ijk} \tag{2.19}$$

Belinfante tensor is $B^{\lambda\mu\nu} = -S^{\lambda\mu\nu} + -S^{\mu\nu\lambda} + S^{\nu\lambda\mu}$. Hence, according to B-R transformations,

$$\Theta^{\mu\nu} \longrightarrow T^{\mu\nu} = \Theta^{\mu\nu} - \partial_\lambda[S^{\lambda\mu\nu} + S^{\mu\nu\lambda} - S^{\nu\lambda\mu}] \tag{2.20}$$

And with the Lagrangian density defined in 2.18, the explicit expression for $S^{\lambda\mu\nu}$ is given by:

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{2.21}$$

[Note = Up till now, we have used Latin symbols and Greek symbols interchangeably. We will define an unambiguous convention for their usage later]

2.3 Einstein-Cartan (EC) theory: Modifying Einstein's GR to include torsion

First we ask the question - Why consider a modified theory of gravity when General theory of relativity works out beautifully well and has stood **all** the experimental tests within the limits of the domain of validity of the theory. To understand this, we must realize that GR was formulated to describe gravitational interactions between macroscopic bodies. It is a classical theory of gravity. It is strongly suspected that at very high energies where the gravitational interaction becomes comparable to other quantum interactions and at very small length scales, the current formulation of gravity would not hold. There were (and still under investigation) many attempts to reconcile gravity with other fundamental interactions. One of the approach to do this is to expand the domains of validity of ordinary GR (validity in terms of micro/macro extent of matter) and to modify it so as to accomodate the new physical principles/ new experiments offered by the expanded domain of validity.

The Einstein-Cartan theory (EC) or also known as Einstein-Cartan-Sciama-Kibble (ECSK) theory [First published in [9], [11] and extensively reviewed in [2]] is one such attempt which **"extends" the geometrical principles and concepts of GR to the certain aspects of micro-physical world**. In ordinary GR, matter is represented by Energy-Momentum tensor, which essentially provides the description of mass density distribution on space-time. However, when we delve into the microscopical scale we see that particles obey the laws of quantum mechanics and special relativity. At such length scales, the 'spin' (along with mass) of the particle has to be taken into account. Just like mass (which is characterized by EM tensor), it is a fundamental **and independent** property of matter . In macro physical limits, mass adds up because of its monopole character, whereas spin, being of dipole character, usually averages out in absence of external forces; hence matter in its macro physical regime can be dynamically characterized only by the energy-momentum tensor. If we wish to extend GR to include micro physics, we must take into account, therefore, that matter is dynamically described by mass and spin density, and since mass is related to curvature via EM tensor in framework of GR, spin should be related, through spin density tensor, to some other geometrical property of space time in the spirit of geometric theory of gravity. This requirement is satisfied by EC or ECSK theory.

EC theory removes the restriction for the affine connection to be symmetric which was considered in GR. The antisymmetric part of the affine connection commonly known as 'torsion'($Q_{\alpha\beta}{}^{\mu}$), transforms like a third rank tensor and is known as Cartan's torsion tensor. It is seen that torsion couples to the intrinsic spin angular momentum of particles [2] just as the symmetric part of the connection (which gets expressed completely in terms of metric and its derivatives) couples to the mass. Since torsion is a geometrical quantity, spin modifies space-time and the resultant space-time is known as 'Riemann-Cartan' space-time (U_4) The field equations that follow are known as Einstein-Cartan field equation. The (U_4) manifold is also metric compatible (See section explained below) and hence can describe physical space-time in agreement with equivalence principle.

Physically, torsion is related to the translation of vector like curvature is related to rotation, when a vector is displaced along infnitesimal path on U_4 manifold. Hence torsion allows for translations to be included and converts the local lorentz symmetry group of GR to the Poincare' group [2]; which is essential because, in microscopic regime, elementary particles are the irreducible representations of Poincare' group, labeled by mass and spin. A detailed account of this motivation to include torsion can be looked in [2]. Another motivation is that in the absence of external forces, the correct conservation law of total angular momentum arises only if torsion, whose origin is spin density, is included into gravitation [**?**],

First we define a connection $\Gamma_{\alpha\beta}{}^{\mu}$ on a general affine manifold (A_4) to allow for the parallel transport of tensorial objects. We define a torsion tensor out of this connection and it is given by,

$$Q_{\alpha\beta}{}^{\mu} = \Gamma_{[\alpha\beta]}{}^{\mu} = \frac{1}{2}(\Gamma_{\alpha\beta}{}^{\mu} - \Gamma_{\beta\alpha}{}^{\mu}) \tag{2.22}$$

It is a third rank tensor that is antisymmetric in its first two indices and has 24 independent components. It can be shown that the general connection $\Gamma_{\alpha\beta}{}^{\mu}$ on (A_4) can be expressed in terms of metric, torsion tensor, and tensor of non metricity ($N_{\alpha\beta\mu} = \nabla_{\mu} g_{\alpha\beta}$)

$$\Gamma_{\alpha\beta}{}^{\mu} = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} - K_{\alpha\beta}{}^{\mu} - V_{\alpha\beta}{}^{\mu} \tag{2.23}$$

where $\left\{ {}^{\mu}_{\alpha\beta} \right\}$ is the Christoffel symbol, $K_{\alpha\beta}{}^{\mu} = -Q_{\alpha\beta}{}^{\mu} - Q^{\mu}{}_{\alpha\beta} + Q_{\beta}{}^{\mu}{}_{\alpha}$ is the contorsion tensor and $V_{\alpha\beta}{}^{\mu} = \frac{1}{2}[N_{\alpha\beta}{}^{\mu} - N^{\mu}{}_{\alpha\beta} - N_{\beta}{}^{\mu}{}_{\alpha}]$ is the definition of V .

Einstein-Cartan manifold (U_4) is a particular case of a general affine manifold in which the metric tensor is covariantly constant.

$$N_{\alpha\beta\mu} = \nabla_{\mu} g_{\alpha\beta} = 0 \tag{2.24}$$

This condition, which preserves scalar products (and then the invariance of lengths and angles) under parallel displacement is called metricity postulate. It secures the local Minkowski structure of space-time in agreement with principle of equivalence. The connection satisfying the condition of eqn (2.24) is called metric compatible connection. The connection of Riemann Cartan manifold (U_4) is then written as:

$$\Gamma_{\alpha\beta}{}^{\mu} = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} - K_{\alpha\beta}{}^{\mu} \tag{2.25}$$

Other quantities such as covariant derivative, Riemann tensor, Ricci tensor and Einstein tensor are defined in a similar fashion as in GR, the only difference being that the Christoffel symbols are replaced by the total connection as defined in equation (2.25),

$$A^{\mu}{}_{;\beta} = \partial_\beta A^\mu + \Gamma_{\beta\alpha}{}^{\mu} A^\alpha \tag{2.26}$$

$$R_{\alpha\beta\mu}{}^{\nu} = \partial_\alpha \Gamma_{\beta\mu}{}^{\nu} - \partial_\beta \Gamma_{\alpha\mu}{}^{\nu} + \Gamma_{\alpha\lambda}{}^{\nu}\Gamma_{\beta\mu}{}^{\lambda} - \Gamma_{\beta\lambda}{}^{\nu}\Gamma_{\alpha\mu}{}^{\lambda} \tag{2.27}$$

$$R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha} \tag{2.28}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.29}$$

However it must be noted that $R_{\mu\nu}$ and $G_{\mu\nu}$ are no longer symmetric. Riemann Tensor has 36 independent components. The Bianchi identities can be defined in a similar way; following the usual definitions. It is worth investigating the anti-symmetric part of $G_{\mu\nu}$. We can show that

$$G_{[\mu\nu]} = R_{[\mu\nu]} = \nabla_\alpha T_{\mu\nu}{}^{\alpha} + 2Q_\alpha T_{\mu\nu}{}^{\alpha} = \widetilde{\nabla}_\alpha T_{\mu\nu}{}^{\alpha} \qquad where \qquad \widetilde{\nabla}_\alpha = \nabla_\alpha + 2Q_\alpha \tag{2.30}$$

where $T_{\mu\nu}{}^{\alpha} = Q_{\mu\nu}{}^{\alpha} + \delta^\alpha_\mu Q_\nu - \delta^\alpha_\nu Q_\mu$ is called as the modified torsion tensor (This is a very important quantity which, as we will see appears in filed equations of EC theory) and the quantity Q_ν is the trace of torsion, given by $Q_\nu = Q_{\nu\alpha}{}^{\alpha}$. $G^{\mu\nu}$ can also be expressed as [2],

$$\begin{aligned} G^{\mu\nu}(\Gamma) =& G^{\mu\nu}(\{\}) + \widetilde{\nabla}_\alpha [T^{\mu\nu\alpha} + T^{\alpha\mu\nu} - T^{\nu\alpha\mu}] \\ &+ \Big[4T^{\mu\alpha}{}_{[\beta}T^{\nu\beta}{}_{\alpha]} + 2T^{\mu\alpha\beta}T^{\nu}{}_{\alpha\beta} - T^{\alpha\beta\mu}T_{\alpha\beta}{}^{\nu} - \frac{1}{2}g^{\mu\nu}(4T^{\beta}_{\alpha}{}_{[\gamma}T^{\alpha\gamma}{}_{\beta]} + T^{\alpha\beta\gamma}T_{\alpha\beta\gamma})\Big] \end{aligned} \tag{2.31}$$

We adopt an important convention henceforth:

- The symbol ∇ is used to indicate total covariant derivative. The symbol $\{\}$ is used to indicate christofell connection. So, $\nabla^{\{\}}$ would mean covariant derivative w.r.t christofell connection.

- Whenever there is a bracket like $(\{\})$ this in front of any object, it indicates the value of object calculated using Christoffel connection. We would also call it "Riemann part of the object" often.

Hence $G^{\mu\nu}(\{\})$ is the Riemann part of Einstein's tensor (the one occurring in GR). By definition, it is symmetric. However it doesn't capture the full symmetric part of total $G^{\mu\nu}$. Hence all the additional part to $G^{\mu\nu}(\{\})$ is asymmetric.

2.4 Lagrangian and the corresponding Field equations of EC theory

The field equations for the Einstein-Cartan theory may be obtained by the usual procedure where the action is constructed and then varied w.r.t. the geometric and matter fields in the Action. Lagrangian of EC theory will have matter lagrangian and a kinetic term for the gravitational field. We apply minimal coupling procedure, where Minkowski metric $\eta_{\mu\nu}$ is replaced by world metric $g_{\mu\nu}$ and partial with covariant derivatives of EC theory (defined later in next section). We keep $\mathcal{L}_g = R$ as in normal GR. We justify this by knowing the fact that in the limit of vanishing torsion, the original field equations of GR are obtained. The action is given by:

$$S = \int d^4x\sqrt{-g}\Big[\mathcal{L}_m(\psi, \nabla\psi, g) - \frac{1}{2\chi}R(g, \partial g, Q)\Big] \tag{2.32}$$

Here $\chi = \frac{8\pi G}{c^4}$ and $\mathcal{L}_m$ denotes the matter Lagrangian density and describes the distribution of matter field. The second term on the RHS represents the Lagrangian density due to the gravitational field. There are 3 fields in this Lagrangian viz. ψ (matter field), $g_{\mu\nu}$ (metric field) and $K_{\alpha\beta\mu}$ (Contorsion field)

varying w.r.t the matter field ψ

$$\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi} = 0 - - - - - - - E.O.M\ for\ matter\ field. \tag{2.33}$$

Varying w.r.t. the metric field,

$$\frac{1}{\sqrt{-g}\chi}\frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} = T^{\mu\nu} \tag{2.34}$$

Varying w.r.t. Contorsion field,

$$\frac{1}{\sqrt{-g}\chi}\frac{\delta(\sqrt{-g}R)}{\delta K_{\alpha\beta\mu}} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta K_{\alpha\beta\mu}} = S^{\mu\beta\alpha} \tag{2.35}$$

These are the generic field equations of Einstein-Cartan theory. $T_{\mu\nu}$ on the RHS of eqn (2.34) dynamical Energy-Momentum Tensor. Similarly, $S^{\mu\beta\alpha}$ on the RHS of eqn (2.35) is the dynamical spin density tensor defined in equation (2.21)

Therefore we notice that, just as mass/energy density of the matter is coupled to the Riemann curvature of space-time via $T_{\mu\nu}$, the spin of matter is coupled to torsion of the space time via $S_{\mu\beta\alpha}$. Using the definition of the curvature tensor and torsion tensor defined in the earlier section, we obtain:

$$\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}} = G^{\mu\nu} - \widetilde{\nabla}_\alpha[T^{\mu\nu\alpha} + T^{\alpha\mu\nu} - T^{\nu\alpha\mu}] = \chi T^{\mu\nu} \tag{2.36}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}R)}{\delta K_{\alpha\beta\mu}} = T^{\mu\beta\alpha} = \chi S^{\mu\beta\alpha} \tag{2.37}$$

Equation (2.36) and (2.37) can be together written as,

$$G^{\mu\nu} = \chi T^{\mu\nu} + \widetilde{\nabla}_\alpha[T^{\mu\nu\alpha} + T^{\alpha\mu\nu} - T^{\nu\alpha\mu}] \tag{2.38}$$

$$= \chi T^{\mu\nu} + \chi\widetilde{\nabla}_\alpha[S^{\mu\nu\alpha} + S^{\alpha\mu\nu} - S^{\nu\alpha\mu}] \tag{2.39}$$

$$G^{\mu\nu} = \chi\Sigma^{\mu\nu} \tag{2.40}$$

Where, $\Sigma^{\mu\nu}$ is the canonical energy momentum tensor. Field equations of EC theory can be summarized below [2], [3], [29].

$$G^{\mu\nu} = \chi\Sigma^{\mu\nu} \tag{2.41}$$

$$T^{\mu\nu\alpha} = \chi S^{\mu\nu\alpha} \tag{2.42}$$

$$\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\bar{\psi}} = 0 \tag{2.43}$$

We now find th explicit expression for $G^{\mu\nu}(\{\})$ using equations (2.31), (2.41), (2.42).

$$G^{\mu\nu}(\{\}) = \chi T^{\mu\nu} + \chi^2\tau^{\mu\nu} = \chi\widetilde{\sigma}^{\mu\nu}; \qquad \widetilde{\sigma}^{\mu\nu} = T^{\mu\nu} + \chi\tau^{\mu\nu} \tag{2.44}$$

where

$$\tau^{\mu\nu} = c^2\Big(4S^{\mu\alpha}{}_{[\beta}S^{\nu\beta}{}_{\alpha]} - 2s^{\mu\alpha\beta}S^{\nu}{}_{\alpha\beta} + S^{\alpha\beta\mu}S_{\alpha\beta}{}^{\nu} + \frac{1}{2}g^{\mu\nu}(4S^{\beta}_{\alpha}{}_{[\gamma}S^{\alpha\gamma}{}_{\beta]} + S^{\alpha\beta\gamma}S_{\alpha\beta\gamma})\Big) \tag{2.45}$$

We again note an important point here though $\widetilde{\sigma}^{\mu\nu}$ defined above is symmetric by definition, it doesn't capture the full symmetry of $\Sigma^{\mu\nu}$.

This term on RHS of equation (2.44) is completely dependent on the spin of the particle. Some important observations can be made from above field equations. eqn (2.42) is an algebric equation; suggesting that torsion can't propogate outside matter field in the EC theory. It is confined to the region of matter fields. However, Spin of the matter fields modifies the Energy momentum tensor as given by eqn (2.44), which in turn modifies the metric, which propogates upto infinity. **The spin content of the matter can influence the geometry outside the matter, though indirectly (through metric) and very weakly.**

2.5 Coupling of EC theory with Dirac field: Einstein-Cartan-Dirac (ECD) theory

We will consider particles with spin-1/2, described by the Dirac field. The matter Lagrangian density for Dirac field is given by ,

$$\mathcal{L}_m = \frac{i\hbar c}{2}(\overline{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\overline{\psi}\gamma^\mu\psi) - mc^2\overline{\psi}\psi \tag{2.46}$$

Here ψ is a spinor. Transformation properties of Spinors are defined in a flat Minkowski space; locally tangent to the U_4 manifold. We know that, at each point, we have a coordinate basis vector field $\hat{e}^\mu = \partial_\mu$. This coordinate basis field is covariant under General coordinate transformation. However, a spinor (as defined on flat Minkowski space-time) is associated with the basis vectors which are covariant under local Lorentz transformations. To this aim, we define at each point of our manifold, a set of 4 orthonormal basis field (called tetrad field), Given by $\hat{e}^i(x)$. These are 4 vectors (one for each μ) et every point. This tetrad field is governed by a relation $\hat{e}^i(x) = e^i_\mu(x)\hat{e}^\mu$ where trasformation matrix e^i_μ is such that,

$$e^{(i)}_{\mu} e^{(k)}_{\nu} \eta_{(i)(k)} = g_{\mu\nu} \tag{2.47}$$

The detail account of Tetrad formalism is given in Appendix [B.1]. Here we will use some results and definitions from this section. Trasformation matrix $e^{(i)}_{\mu}$ allows us to convert the components of any world tensor (tensor which transforms according to general coordinate transformation) to the corresponding components in local Minkowskian space (These latter components being covariant under local Lorentz transformation). Greek indices are raised or lowered using the metric $g_{\mu\nu}$, while the Latin indices are raised or lowered using $\eta_{(i)(k)}$. parenthesis around indices is just a matter of convention.
We adopt an important conventions for the remainder of paper

- Greek indices e.g. α, ζ, δ refer to world components (which transform according to **general coordinate transformation**).
- Latin indices with parenthesis e.g. (a) or (i) refer to tetrad index. (which transform according to **local Lorentz transformation** in flat tangent space).
- Latin index without parenthesis e.g. i,j,b,c would just mean objects in Minkowski space (which transform according to **global Lorentz transformation**).
- 0,1,2,3 indicate world index and (0),(1),(2),(3) indicate tetrad index.
- The symbol ∇ is used to indicate total covariant derivative. The symbol $\{\}$ is used to indicate christofell connection. So, $\nabla^{\{\}}$ would mean covariant derivative w.r.t christofell connection.
- The symbol comma (,) is used to indicate partial derivatives and the symbol semicolon (;) is used to indicate Riemannian covariant derivative. So for tensors, (;) and $\nabla^{\{\}}$ are same, but for spinors (;) would have partial derivatives and riemannian part of spin connection (γ) as described below.

Just as we define affine connection Γ to facilitate parallel transport of geometrical objects with world (greek) indices, we define Spin connection ω for anholonomic objects (those having latin index). As affine connection Γ has 2 parts- riemannian ($\{\}$) part coming from christofell connection and torsional part (made up of contorsion tensor K), similarly, spin connection ω also has 2 parts - Riemannian (denoted by γ) and torsional part (again made up of contorsion tensor K). γ, γ^{o} and K are related by following equation. These symbols and notations ae taken from [34]. All the mathematics is explained in Appendix [B.1].

$$\gamma_{\mu}{}^{(i)(k)} = \gamma^{o}_{\mu}{}^{(i)(k)} - K_{\mu}{}^{(k)(i)} \tag{2.48}$$

Here, $\gamma^{o}_{\mu}{}^{(i)(k)}$ is riemannian part and $K_{\mu}{}^{(k)(i)}$ is the contorsion (torsional part)

The relation between spin conection and affine connection is as follows

$$\begin{aligned}\gamma_\mu{}^{(i)(k)} &= e^{(i)}_\alpha e^{\nu(k)}\Gamma_{\mu\nu}{}^{\alpha} - e^{\nu(k)}\partial_\mu e^{(i)}_\nu \\ &= e^{(i)}_\alpha e^{\nu(k)}\left\{ \begin{matrix}\alpha \\ \mu\nu\end{matrix} \right\} - K_\mu{}^{(k)(i)} - e^{\nu(k)}\partial_\mu e^{(i)}_\nu\end{aligned} \tag{2.49}$$

From above two equations, one can obtain the following crucial equation for Riemannian part of spin connection, entirely in terms of Christoffel symbols and tetrads.[29]

$$\left\{ \begin{matrix}\alpha \\ \mu\nu\end{matrix} \right\} = e^\alpha_{(i)} e_{\nu(k)} \gamma^o_\mu{}^{(k)(i)} + e^\alpha_{(i)}\partial_\mu e^{(i)}_\nu \tag{2.50}$$

Using all the results mentioned above, we define covariant derivative (CD) for Spinors on V_4 and U_4

$$\psi_{;\mu} = \partial_\mu\psi + \frac{1}{4}\gamma^o_{\mu(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi - - - - - - - - - - - - - - - - - - CD\ on\ [V_4] \tag{2.51}$$

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{4}\gamma^o_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{1}{4}K_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi - - - - - - - - CD\ on\ [U_4] \tag{2.52}$$

Substituting this into eqn (2.46), we obtain the explicit form of Lagrangian density; which we vary w.r.t. $\overline{\psi}$ as in eqn (2.43) to obtain Dirac equation on V_4 and U_4.

$$i\gamma^\mu\psi_{;\mu} - \frac{mc}{\hbar}\psi = 0 - - - - - - Dirac\ Eqn\ on\ [V_4] \tag{2.53}$$

$$i\gamma^\mu\psi_{;\mu} + \frac{i}{4}K_{(a)(b)(c)}\gamma^{[(a)}\gamma^{(b)}\gamma^{(c)]}\psi - \frac{mc}{\hbar}\psi = 0 - - - - - - Dirac\ Eqn\ on\ [U_4] \tag{2.54}$$

We next obtain gravitational field equations on both V_4 and U_4 using eqn (2.41) and Lagangian density defined in eqn (2.46).

$$G^{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T^{\mu\nu} - - - - - - - - - - - - - - - - - Gravitation\ Eqn\ on\ [V_4] \tag{2.55}$$

$$G^{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T^{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} - - - - Gravitation\ Eqn\ on\ [U_4] \tag{2.56}$$

Here, $T^{\mu\nu}$ is the dynamical EM tensor which is symmetric and defined below:

$$T_{\mu\nu} = \Sigma_{(\mu\nu)}(\{\}) = \frac{i\hbar c}{4}\left[\bar{\psi}\gamma_\mu\psi_{;\nu} + \bar{\psi}\gamma_\nu\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_\nu\psi - \bar{\psi}_{;\nu}\gamma_\mu\psi\right] \tag{2.57}$$

Equations [2.53 and 2.55] together form a system of equations of Einstein-Dirac theory.

We now aim to establish the field equations of Einstein-Cartan-Dirac theory. First let's define Spin density tensor using Lagrangian density defined in eqn (2.46)

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}\psi \tag{2.58}$$

Using equations (2.58) and (2.42), eqn (2.54) can be simplified to give us the Hehl-Datta equation [2], [3] ('L_{Pl}' being the Planck length). This, Along with equation (2.56) and the equation which couples modified torsion tensor and spin density tensor together define the field equations of:

Einstein-Cartan-Dirac (ECD) theory; as summarized below

$$i\gamma^{\mu}\psi_{;\mu} = +\frac{3}{8}L_{Pl}^{2}\overline{\psi}\gamma^{5}\gamma_{(a)}\psi\gamma^{5}\gamma^{(a)}\psi + \frac{mc}{\hbar}\psi \tag{2.59}$$

$$G^{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T^{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{2.60}$$

$$T^{\mu\nu\alpha} = -K^{\mu\nu\alpha} = \frac{8\pi G}{c^4}S^{\mu\nu\alpha} \tag{2.61}$$

Chapter 3

Introducing unified length scale L_{CS} in quantum gravity

3.1 Brief review of quantum theories of gravity

This section is mainly based upon the review article titled "Conceptual Problems in Quantum Gravity and Quantum Cosmology" by Claus Kiefer [45]. According to our present knowledge; strong, weak, electromagnetic and gravitational interactions are the four fundamental interactions in nature. The first three are described by standard model of particle physics (whose framework is of Quantum field theory) and fourth one is described by GR (whose framework is classical). Though no empirical evidence goes against GR; from purely theoretical point of view, the situation is not satisfactory. The main field equation of GR (2.1) would no longer have the same form if we consider the quantum nature of fields in $T_{\mu\nu}$. The 'semi-classical Einstein equations' with $T_{\mu\nu}$ replaced by its expectation value '$< \psi|\mathbb{T}_{\mu\nu}|\psi >$' leads to problems [23]. In the 1957 Chapel Hill Conference, Richard Feynman gave the argument suggesting that 'It is the superposition principle of QM which strongly points towards the need for quantizing gravity [55]'. Apart from this, 'unavoidable presence of singularities in GR [57]' and 'problem of time in QM [58],[45] ' forms the motivation for quantizing gravity amongst few other motivations. On a side note, it should be noted that the idea of 'emergent gravity' by Padmanabhan [59] is an alternative to the approach of direct quantization of gravitational fields. In brief, we can divide the approaches to quantum gravity in 4 broad groups [61],[60]: 1) Quantize general relativity [2 methods are used in this approach -covariant and canonical quantum gravity.] 2) 'General-relativise' quantum theory [trying to adapt standard quantum theory to the needs of classical general relativity]. 3) General relativity is the low-energy limit of a quantum theory of something quite different [The most notable example of this type is the theory of closed superstrings]. 4) Start ab initio with a radical new theory. [Both classical general relativity and standard quantum theory 'emerge' from a deeper theory that involves a fundamental revision of the concepts of space, time and matter.] We will now introduce the idea of unified length scale (L_{cs}) in quantum gravity.

3.2 The idea of L_{cs}

Einstein's theory of gravity (GR) and relativistic Quantum mechanics (Ex: Dirac theory for spin-1/2 particles) are the 2 most successful theories of the description of Universe at micro and macro level (in terms of mass 'm' which is being described). Given a relativistic particle of mass 'm', we can associate 2 length scales to it- characterizing its quantum and relativistic behavior. Quantum nature of the particle is associated with its Compton wavelength; given by $\lambda_C = (\hbar/mc)$ and the relativistic nature is associated to the Schwarzschild radius given by $R_S = (2GM/c^2)$. It is through these length scales, that the mass 'm' enters the equation of description of their motion. Example, mass enters Dirac equation through λ_C and it enters GR equations through R_S. Also, It is important to note that neither λ_C (having $\hbar$ and c as fundamental constants) nor R_S (having G and c as fundamental constants) could be used individually to define mass (or units of mass).

Both Dirac theory and general relativity claim to hold for all values of m and it is only through experiments that we find that Dirac equation holds if $m \ll m_p$ or $\lambda_C \gg l_p$ while Einstein equations hold if $m \gg m_p$ or $R_S \gg l_p$. **"From the theoretical viewpoint, it is unsatisfactory that the two theories should have to depend on the experiment to establish their domain of validity"** [4]. If we assert the fact that plank length is the smallest physically meaningful length, then it makes no sense to talk of $R_S < L_{pl}$ when $m < m_{pl}$ and to talk of $\lambda_C < L_{pl}$ when $m > m_{pl}$. Instead it is more reasonable to think of universal length scale which remains above L_{pl} for all masses and whose limiting cases give λ_C for small mass and R_S for large mass. One such way to define a universal length scale is given in [4] as follows

$$\frac{L_{CS}}{2l_p} := \frac{1}{2}\left(\frac{2m}{m_p} + \frac{m_p}{2m}\right) := \cosh z \tag{3.1}$$

. where $z = \ln \frac{2m}{m_p}$. These ideas are discussed in details in the recent works of Tejinder P. Singh [4], [5]. The dynamical process for mass 'm' now involves L_{cs} (mass enters the dynamics through L_{cs}). An action principle has been proposed with this new length scale and Dirac equation and Einstein GR equations are shown to be mutually dual limiting cases of this underlying modified action. The proposed action for this underlying gravitation theory, which gives the required limits is as follows

$$\frac{L_{pl}^2}{\hbar} S = \int d^4x \sqrt{-g} [R - (1/2) L_{CS} \bar{\psi}\psi + L_{CS}^2 \bar{\psi} i \gamma^\mu \partial_\mu \psi] \tag{3.2}$$

Generalizing this on a curved space-time, the action is:

$$\frac{L_{pl}^2}{\hbar} S = \int d^4x \sqrt{-g} \left[R - \frac{1}{2} L_{CS} \bar{\psi}\psi + \pi\hbar i L_{CS}^2 \left(\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma^\mu \psi\right)\right] \tag{3.3}$$

If ∇ and 'R' are taken on V_4, the system is called 'Einstein-Dirac' system. In such a system , for small mass limit, couping to EM tensor in Einstein's equation is through $\hbar$ and not G. Hence we expect gravity to vanish. This creates an unpleasant situation for Einstein's equations. Because vanishing of gravity makes LHS 0; but RHS is non-zero (it is EM tensor coupled

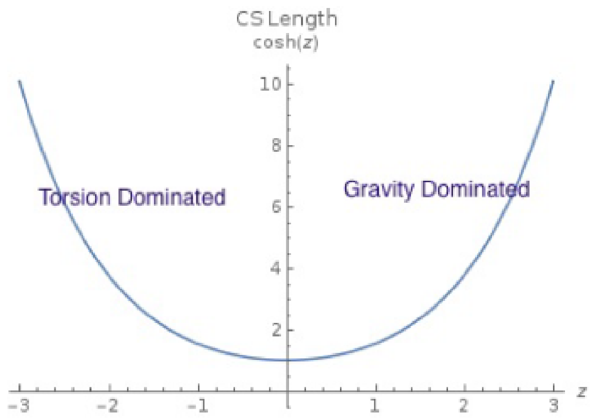


Figure 2 – Plot of the scaled Compton-Schwarzschild Length $\frac{L_{CS}}{2l_p} = cosh(z)$ as a function of the logarithmic mass $z = ln(2m/m_p)$. The CS length L_{CS} attains the minimum value $2l_p$ at $z = 0$, i.e. at $m = m_p/2$. For $z > 0$, i.e. $m > m_p/2$, L_{CS} increases with increasing mass: this is the gravity dominated regime. For $z < 0$, i.e. $m < m_p/2$, L_{CS} increases with decreasing mass: this is the torsion dominated regime. For any given value of L_{CS}, two values of m, say m_q and m_c, are possible, and they are related as $m_q m_c = m_p^2/4$. This figure has been taken from [1]

Figure 3.1: L_{cs} Vs. mass behavior and its description [4]

through $\hbar$). This compels us to introduce torsion in the theory. Because it would now add torsion field in the LHS and then it couples to EM tensor via $\hbar$. Further arguments can be looked up in [5]. So the idea of L_{cs} is more coherent with the framework of Einstein-Cartan manifold (U_4 manifold). For $m \ll m_{pl}$, the spin density is more important than mass density. Mass density can be neglected and spin density sources the torsion (coupling is through $\hbar$). Whereas, $m \gg m_{pl}$, mass density dominates spin density. spin density can be neglected and as usual, mass density sources the gravity (coupling is through G). **Spin density and torsion are significant in micro-regime; whereas gravity and mass density are important in macro-regime.** In this manner there exists a symmetry between small mass and large mass in the sense that small mass is the source for torsion and large mass is the source for gravity. The solution for small mass is dual to the 'wave-function collapsed' solution for large mass in the sense that both the solutions have same value for L_{cs} which is the only free parameter in the theory.[5]

3.3 ECD equations with L_{cs}

The set of ECD field equations with the L_{CS} incorporated in them are obtained by varying the Action (3.3) w.r.t all the 3 fields (Here we have also given gravity equation with riemannian part of Einstein tensor.)[5]

$$G_{\mu\nu} = \frac{8\pi L_{CS}^2}{\hbar c}\Sigma_{\mu\nu} \tag{3.4}$$

$$G_{\mu\nu}(\{\}) = \frac{8\pi L_{CS}^2}{\hbar c}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi L_{CS}^2}{\hbar c}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{3.5}$$

$$T^{\mu\nu\alpha} = \frac{8\pi L_{CS}^2}{\hbar c}S^{\mu\nu\alpha} \tag{3.6}$$

$$i\gamma^{\mu}\psi_{;\mu} = \frac{3}{8}L_{CS}^2\bar{\psi}\gamma^5\gamma_{\nu}\psi\gamma^5\gamma^{\nu}\psi + \frac{1}{2L_{CS}}\psi = 0 \tag{3.7}$$

A note on length scales: We use l to denote a length scale in the theory. For standard ECD theory, the typical scales that can be considered are the Planck length $l_1 = L_{Pl} = \sqrt{\frac{G\hbar}{c^3}}$, half the Compton wavelength $l_2 = \frac{\lambda_C}{2} = \frac{\hbar}{2mc}$. For the modified ECD theory, we take $l_1 = l_2 = L_{CS}$, in terms of the new unified length scale.

Chapter 4

Non-relativistic limit of ECD field equations

4.1 Theoretical background and notations/ representations used in this chapter

4.1.1 Notations, conventions, representations

- Greek indices e.g. α, ζ, δ refer to world components (which transform according to **general coordinate transformation**).
- Latin indices with parenthesis e.g. (a) or (i) refer to tetrad index. (which transform according to **local Lorentz transformation** in flat tangent space).
- Latin index without parenthesis e.g. i,j,b,c would just mean objects in Minkowski space (which transform according to **global Lorentz transformation**).
- 0,1,2,3 indicate world index and (0),(1),(2),(3) indicate tetrad index.
- The Lorentz Signature used in this report is Diag(+, -, -, -).
- We use Dirac basis to represent the gamma matrices. These are basically matrix representation of clifford algebra $Cl_{1,3}[\mathbb{R}]$

$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \frac{i}{4!}\epsilon_{ijkl}\gamma^i\gamma^j\gamma^k\gamma^l = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \alpha^i = \beta\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{4.1}$$

4.2 Non-relativistic limit of the Einstein-Dirac equations

4.2.1 Ansatz for the spinor and the metric

Ansatz for Dirac spinor: We need to choose an appropriate expansion ansatz for the spinor so as to obtain the non-relativistic limit. We expand $\psi(x,t)$ as $\psi(x,t) = e^{iS(x,t)\hbar}$ (which can be done for any complex function of x and t). We can either expand S as a perturbative power series in the parameter $\sqrt{\hbar}$ or $(1/c)$ and obtain the semi-classical or non-relativistic limit respectively, at various orders. The scheme for non-relativistic limit has been employed by Kiefer and Singh [26]. Giulini and Grossardt in their work [19], combine both these schemes and construct a new ansatz using the parameter $\sqrt{\hbar}/c$ as follows:

$$\psi(\mathbf{r},t) = e^{\frac{ic^2}{\hbar}S(\mathbf{r},t)} \sum_{n=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n a_n(\mathbf{r},t) \tag{4.2}$$

where $S(\mathbf{r},t)$ is a scalar function and $a_n(\mathbf{r},t)$ is a spinor field. We use this ansatz in our calculations, and by taking the limit $c \to \infty$ arrive at the non-relativistic limit.

Ansatz for metric: We first express the generic form of the metric in a power series with same parameter as that used to expand the spinor viz. $\sqrt{\hbar}/c$

$$g_{\mu\nu}(\mathbf{r},t) = \eta_{\mu\nu} + \sum_{n=1}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n g^{[n]}_{\mu\nu}(\mathbf{r},t) \tag{4.3}$$

where $g^{[n]}_{\mu\nu}(x)$ are infinitely many metric functions indexed by n. In the non-relativistic scheme, gravitational potentials cannot produce velocities comparable to c - they are weak potentials. Therefore we assume that the leading function $g^{[0]}_{\mu\nu}(x) = \eta_{\mu\nu}$. With this, we get the following generic power series for tetrads and spin coefficients and Einstein tensor

$$e^{\mu}_{(i)} = \delta^{\mu}_{(i)} + \sum_{n=1}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\mu[n]}_{(i)} \qquad \gamma_{(a)(b)(c)} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{[n]}_{(a)(b)(c)} \tag{4.4}$$

$$e^{(i)}_{\mu} = \delta^{(i)}_{\mu} + \sum_{n=1}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n e^{(i)[n]}_{\mu} \qquad G_{\mu\nu} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n G^{[n]}_{\mu\nu} \tag{4.5}$$

where $e^{\mu(n)}_{(i)}[g^{[n]}_{\mu\nu}]$, $e^{(i)[n]}_{\mu}[g^{[n]}_{\mu\nu}]$, $\gamma^{[n]}_{(a)(b)(c)}[g^{[n]}_{\mu\nu}]$ and $G^{[n]}_{\mu\nu}$ are infinitely many tetrad, spin coefficient and Einstein tensor functions indexed by n. They are functions of metric functions $g^{[n]}_{\mu\nu}$ and their various derivatives.

4.2.2 Analyzing Dirac equation with above ansatz

We will now expand the Dirac equation on V_4 as given in eqn (2.53) with the above ansatz. We also note that $\gamma^{(a)}\psi_{;(a)} = e^{(a)}_{\mu} e^{\nu}_{(a)} \gamma^{\mu}\psi_{;\nu} = \delta^{\mu}_{\nu}\gamma^{\mu}\psi_{;\nu} = \gamma^{\mu}\psi_{;\mu}$.

$$i\gamma^{\mu}\psi_{;\mu} - \frac{mc}{\hbar}\psi = 0 \tag{4.6}$$

$$\Rightarrow \; i\gamma^0\partial_0\psi + \frac{i}{4}\gamma^{(0)}\gamma^{o}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi + i\gamma^{\alpha}\partial_{\alpha}\psi + \frac{i}{4}\gamma^{(j)}\gamma^{o}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{mc}{\hbar}\psi = 0 \tag{4.7}$$

We separate spatial and temporal parts. Substituting appropriate expansions from (4.4), (4.5) into above equations and multiplying by $\gamma^{(0)}c$ on both sides yields:

$$\Rightarrow \left[1 + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right] i\partial_t\psi + \frac{ic}{4}\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{o[n]}_{(0)(b)(c)}\right]\gamma^{[(b)}\gamma^{(c)]}\psi +$$
$$\left[1 + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\alpha[n]}_{(a)}\right] ic\alpha.\nabla\psi + \frac{ic}{4}\alpha^{(j)}\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{o[n]}_{(j)(b)(c)}\right]\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{\beta mc^2}{\hbar}\psi = 0 \tag{4.8}$$

Dividing both sides by $\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right]$, we obtain

$$i\partial_t\psi = -\frac{ic}{4}\frac{\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{o[n]}_{(0)(b)(c)}\right]}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right]}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\alpha[n]}_{(a)}\right]}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right]} ic\alpha.\nabla\psi - \frac{ic}{4}\alpha^{(j)}\frac{\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{o[n]}_{(j)(b)(c)}\right]}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right]}\gamma^{[(b)}\gamma^{(c)]}\psi + \frac{1}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right]}\frac{\beta mc^2}{\hbar}\psi \tag{4.9}$$

We consider the terms of order c^2, c, 1 and neglect the terms having order of $O\left(\frac{1}{c^n}\right)$; $n \geq 1$. This is sufficient to get the behaviour of the functions in the spinor ansatz. It will turn out later that this is also sufficient to get the equation which is followed by leading order spinor term a_0. We obtain the following equations:

$$i\partial_t\psi + \frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi + \ ic\alpha.\nabla\psi + \frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi - \beta\frac{mc^2}{\hbar}\psi + \beta\frac{mc}{\sqrt{\hbar}}e^{0[1]}_{(0)}\psi - \beta m\left[\left(e^{0[1]}_{(0)}\right)^2 - e^{0[2]}_{(0)}\right]\psi = 0 \tag{4.10}$$

Substituting the spinor ansatz i.e. eqn (4.2) in equation (4.10), the various terms are evaluated as follows:

Term 1

$$\begin{aligned} i\partial_t\psi &= i\partial_t\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \\ &= ie^{\frac{ic^2S}{\hbar}}\frac{c^2}{\hbar}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[\dot{a}_{n-2} + i\dot{S}a_n\right] \\ &= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[-\dot{S}a_{n-1} + i\dot{a}_{n-3}\right] \end{aligned} \tag{4.11}$$

Term 2

$$+\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi = +\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \tag{4.12}$$

$$= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[i\sqrt{\hbar}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}a_{n-3}\right] \tag{4.13}$$

Term 3

$$\begin{aligned} ic\alpha^j\partial_j\psi &= ic\vec{\alpha}\cdot\vec{\nabla}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \\ &= ic\vec{\alpha}\cdot\left[e^{\frac{ic^2S}{\hbar}}\frac{c^2}{\hbar}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left(i\vec{\nabla}Sa_n + \vec{\nabla}a_{n-2}\right)\right] \\ &= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[-\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}Sa_n + i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}a_{n-2}\right] \end{aligned} \tag{4.14}$$

Term 4

$$+\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi = +\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \tag{4.15}$$

$$= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \left[i\sqrt{\hbar}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}a_{n-3}\right] \tag{4.16}$$

Term 5

$$-\beta\frac{mc^2}{\hbar}\psi = -\beta\frac{mc^2}{\hbar}e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n$$

$$= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n (-\beta m a_{n-1}) \tag{4.17}$$

Term 6

$$+\beta\frac{mc}{\sqrt{\hbar}}e^{0[1]}_{(0)}\psi = +\beta\frac{mc}{\sqrt{\hbar}}e^{0[1]}_{(0)}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \tag{4.18}$$

$$= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \left[\beta m\, e^{0[1]}_{(0)}a_{n-2}\right] \tag{4.19}$$

Term 7

$$-\beta m\left[\left(e^{0[1]}_{(0)}\right)^2 - e^{0[2]}_{(0)}\right]\psi = -\beta m\left[\left(e^{0[1]}_{(0)}\right)^2 - e^{0[2]}_{(0)}\right]\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \tag{4.20}$$

$$= -e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \left[\beta m\left((e^{0[1]}_{(0)})^2 - e^{0[2]}_{(0)}\right)\right]a_{n-3} \tag{4.21}$$

After substituting equations (4.11), (4.12), (4.14), (4.15), (4.17), (4.18) and (4.20) into (4.10) and sorting by powers of n we get,

$$e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\Bigg[\Big(-\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}S\Big)a_n - \Big(\dot{S}+\beta m\Big)a_{n-1} + \Big(i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla} + \beta m\, e^{0[1]}_{(0)}\Big)a_{n-2}$$
$$+i\dot{a}_{n-3} + \left(i\sqrt{\hbar}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]} + i\sqrt{\hbar}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]} - \beta m\Big((e^{0[1]}_{(0)})^2 - e^{0[2]}_{(0)}\Big)\right)a_{n-3}\Bigg] = 0 \tag{4.22}$$

At order $n = 0$ the equation reduces to,

$$\nabla S = 0 \tag{4.23}$$

which implies the scalar 'S' is a function of time only i.e., $S = S(t)$. Dirac spinor is a 4-component spinor $a_n = (a_{n,1}, a_{n,2}, a_{n,3}, a_{n,4})$. We split it into two two-component spinors $a_n^{>} = (a_{n,1}, a_{n,2})$ and $a_n^{<} = (a_{n,3}, a_{n,4})$. For order $n = 1$, the equation is $\Big(\dot{S}+\beta m\Big) = 0$; which can be written as following two equations:

$$(m+\dot{S})a_0^{>} = 0 \tag{4.24a}$$

$$(m-\dot{S})a_0^{<} = 0 \tag{4.24b}$$

This implies that either $S = -mt$ and $a_0^< = 0$ or $S = +mt$ and $a_0^> = 0$. The wave function at this order is $\psi = e^{\frac{\pm imc^2 t}{\hbar}}$. It represents the particles of positive energy (lower sign) and negative energy (upper sign) at rest. We will restrict to the former case i.e. $S = -mt$ and $a_0^< = 0$, which represents positive energy (lower sign) solutions. It has been implicitly assumed that 2 cases (of positive and negative energies) can be treated separately. We digress at this point and analyze the metric energy-momentum tensor now with the results obtained in equation (4.23) and the fact that $a_0^< = 0$.

4.2.3 Analyzing the Energy momentum tensor $T_{\mu\nu}$ with above ansatz

The dynamical Energy momentum tensor given in equation (2.57). Lets consider the "kT_{00}" component.

Analyzing kT_{00} (after raising the index on gamma matrices):

$$kT_{00} = \frac{4i\pi G\hbar}{c^4}\left[\bar{\psi}\gamma^0\Big(\partial_t\psi + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\psi\Big) - \Big(\partial_t\bar{\psi} + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\bar{\psi}\Big)\gamma^0\psi\right] \tag{4.25}$$

$$\Rightarrow kT_{00} = \frac{4i\pi G\hbar}{c^4}\Big(1 + \sum_{n=1}^{\infty}(\frac{\sqrt{\hbar}}{c})^n e^{0[n]}_{(0)}\Big)\Big[\bar{\psi}\gamma^{(0)}\Big(\partial_t\psi + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\psi\Big) \\ - \Big(\partial_t\bar{\psi} + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\bar{\psi}\Big)\gamma^{(0)}\psi\Big] \tag{4.26}$$

After putting spinor anstaz eqn (4.2) in eqn (4.25), we obtain following power series for kT_{00}. We have given expression for the leading order only.

$$kT_{00} = \frac{4i\pi G}{c^2}\Bigg\{\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n a_n^\dagger\Bigg)\Bigg(\sum_{m=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m\Big[i\dot{S}a_m + \dot{a}_{m-2}\Big]\Bigg) \\ +\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n\Big[i\dot{S}a_n^\dagger - \dot{a}_{n-2}^\dagger\Big]\Bigg)\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m a_m\Bigg)\Bigg\} + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{4.27}$$

Explicit expression for leading order is obtained by considering $(n + m = 0)$ as follows:

$$kT_{00} = \frac{4\pi Gi}{c^2}\Big\{i(-m)a_0^{>\dagger}a_0^> + i(-m)a_0^{>\dagger}a_0^>\Big\} + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{4.28}$$

$$kT_{00} = \frac{8\pi Gm\,|a_0^>|^2}{c^2} + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{4.29}$$

Analyzing $kT_{0\mu}$:

$$kT_{0\mu} = \frac{2i\pi G\hbar}{c^4}\Big[c\bar{\psi}\gamma_0\Big(\partial_\mu\psi + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big) + c\bar{\psi}\gamma_\mu\Big(\partial_0\psi + \frac{1}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big) \\ - c\Big(\partial_\mu\bar{\psi} + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_0\psi - c\Big(\partial_0\bar{\psi} + \frac{1}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_\mu\psi\Big] \tag{4.30}$$

We will first find the coefficient of the term of order $\frac{1}{c^2}$ which is the leading order of T_{00}. Now, all the terms containing spin coefficients $\gamma_{\mu(i)(j)}$ have leading order of $\frac{1}{c^3}$. So it will not contribute at the order $\frac{1}{c^2}$. So what we get is (here, index on gamma matrices is raised):

$$kT_{0\mu} = \frac{2i\pi G\hbar}{c^4}\Bigg[c\bar{\psi}\gamma^0\partial_\mu\psi - c\bar{\psi}\gamma^\mu\partial_0\psi - c\partial_\mu\bar{\psi}\gamma^0\psi + c\partial_0\bar{\psi}\gamma^\mu\psi\Bigg] \tag{4.31}$$

$$= \frac{-2i\pi G\hbar}{c^3}\Big(1+\sum_{n=1}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n e^{0[n]}_{(0)}\Big)\Bigg[\bar{\psi}\gamma^{(0)}\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^{(0)}\psi\Bigg] \tag{4.32}$$

$$+\frac{2i\pi G\hbar}{c^4}\Big(1+\sum_{n=1}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n e^{\mu[n]}_{(a)}\Big)\Bigg[\partial_t\bar{\psi}\gamma^{(a)}\psi - \bar{\psi}\gamma^{(a)}\partial_t\psi\Bigg]$$

There are two types of terms in equation above. One having coefficient $\frac{2i\pi G\hbar}{c^3}$ and other with coefficient $\frac{2i\pi G\hbar}{c^4}$. We call them term 1 and 2 respectively. We analyze both of them independently. Term 1 gives

$$(term\ 1) = \frac{2i\pi G\hbar}{c^3}\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n\Big(a^\dagger_{n_1}\partial_\mu a_{n_2} - \partial_\mu a^\dagger_{n_1}a_{n_2}\Big); \qquad n = n_1+n_2$$

$$= \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{4.33}$$

$$(term\ 2) = \frac{2i\pi G}{c^2}\Bigg\{\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n a^\dagger_n\Bigg)\alpha^{(a)}\Bigg(\sum_{m=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m\Big[i\dot{S}a_m + \dot{a}_{m-2}\Big]\Bigg)$$

$$+\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n\Big[i\dot{S}a^\dagger_n - \dot{a}^\dagger_{n-2}\Big]\Bigg)\alpha^{(a)}\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m a_m\Bigg)\Bigg\} + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big)$$

$$= \frac{4\pi Gm}{c^2}(a^\dagger_0\alpha^{(a)}a_0) + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big)$$

$$= \frac{4\pi Gm}{c^2}\Bigg[\begin{pmatrix} a_0^{>} & 0\end{pmatrix}^\dagger\begin{pmatrix}0 & \sigma^{(a)}\\ \sigma^{(a)} & 0\end{pmatrix}\begin{pmatrix}a_0^{>}\\ 0\end{pmatrix}\Bigg] + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big)$$

$$= \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{4.34}$$

So we find, in both term 1 and 2, terms of the $O\Big(\frac{1}{c^2}\Big)$ are zero. Hence

$$kT_{0\mu} = \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{4.35}$$

Analyzing $kT_{\mu\nu}$

$$kT_{\mu\nu} = \frac{2i\pi G\hbar}{c^3}\Bigg[+\bar{\psi}\gamma_\mu\Big(\partial_\nu\psi + \frac{1}{4}[\gamma^o_{\nu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big) + \bar{\psi}\gamma_\nu\Big(\partial_\mu\psi + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big)$$
$$- \Big(\partial_\nu\bar{\psi} + \frac{1}{4}[\gamma^o_{\nu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_\mu\psi - \Big(\partial_\mu\bar{\psi} + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_\nu\psi\Bigg] \tag{4.36}$$

Here also, we will first find the coefficient of the term of order $\frac{1}{c^2}$ which is the leading order of kT_{00}. All the terms containing spin coefficients $\gamma_{\mu(i)(j)}$ have leading order of $\frac{1}{c^3}$. So it will not contribute at the order $\frac{1}{c^2}$. So what we get is (here, index on gamma matrices is raised):

$$
\begin{aligned}
kT_{\mu\nu} &= \frac{2i\pi G\hbar}{c^3}\Bigg[-\bar{\psi}\gamma^{\mu}\partial_{\nu}\psi - \bar{\psi}\gamma^{\nu}\partial_{\mu}\psi + \partial_{\nu}\bar{\psi}\gamma^{\mu}\psi + \partial_{\mu}\bar{\psi}\gamma^{\nu}\psi\Bigg] \\
&= \frac{2i\pi G\hbar}{c^3}\left(1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\mu[n]}_{(a)}\right)\left[\psi^{\dagger}\alpha^{(a)}\partial_{\nu}\psi - \partial_{\nu}\psi^{\dagger}\alpha^{(a)}\psi\right] \\
&+ \frac{2i\pi G\hbar}{c^3}\left(1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\nu[n]}_{(b)}\right)\left[\partial_{\mu}\psi^{\dagger}\alpha^{(b)}\psi - \psi^{\dagger}\alpha^{(b)}\partial_{\mu}\psi\right] \\
&= \frac{2i\pi G\hbar}{c^3}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left(e^{\mu}_{(a)}a^{\dagger}_{n_1}\alpha^{(a)}\partial_{\nu}a_{n_2} - e^{\mu}_{(a)}\partial_{\nu}a^{\dagger}_{n_1}\alpha^{(a)} + e^{\nu}_{(b)}a^{\dagger}_{n_1}\alpha^{(b)}\partial_{\mu}a_{n_2} - e^{\nu}_{(b)}\partial_{\mu}a^{\dagger}_{n_1}\alpha^{(b)}a_{n_2}\right)
\end{aligned}
\tag{4.37}
$$

$$
kT_{\mu\nu} = \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{4.38}
$$

From order analysis of components of the metric energy-momentum tensor, summarized in equations (4.29) , (4.35) and (4.38), we have proved a crucial result viz.

$$
\frac{|T_{00}|}{|T_{0i}|} \ll 1, \qquad \frac{|T_{00}|}{|T_{ij}|} \ll 1, \qquad k|T_{00}| \sim O\left(\frac{1}{c^2}\right) \qquad ; i,j \in (1,2,3) \tag{4.39}
$$

Owing to Einstein's equations, the same relation then exists amongst the components of Einstein tensor as well viz.

$$
\frac{|G_{00}|}{|G_{0i}|} \ll 1, \qquad \frac{|G_{00}|}{|G_{ij}|} \ll 1, \qquad |G_{00}| \sim O\left(\frac{1}{c^2}\right) \qquad ; i,j \in (1,2,3) \tag{4.40}
$$

4.2.4 Constraints imposed on metric as an implication of above analysis

We proved an important fact in the previous two sections viz. $|G_{00}| \sim O\left(\frac{1}{c^2}\right)$ and all other components of G are of higher order. For a generic metric ansatz, $G_{\mu\nu}$ has been explicitly calculated in appendix [A.1]. At this point, we make an important assumption – the metric field is asymptotically flat. This fact suggests the following important constraints on metric components [proved in appendix (A.2)]

1) $G^{[1]}_{\mu\nu} = 0$ $(\forall \mu,\nu)$ and non-allowance of solutions which don't respect asymptotic flatness of metric gives following result for metric and other quantities :

$$
g^{[1]}_{\mu\nu} = 0, \qquad e^{\mu[1]}_{(i)} = 0, \qquad e^{(i)[1]}_{\mu} = 0, \qquad \gamma^{[1]}_{(i)(j)(k)} = 0 \qquad \forall\ \ ij,k,\mu,\nu \in (0,1,2,3) \tag{4.41}
$$

This is proved in appendix (A.2.1)

2) We also have $G^{[2]}_{\mu\nu} = 0$ (except for $\mu = 0$ and $\nu = 0$). This imposes different kind of restrictions on $g^{[2]}_{\mu\nu}$. We see that the form which $g^{[2]}_{\mu\nu}$ can take is $g^{[2]}_{\mu\nu} = F(\mathbf{r},t)\delta_{\mu\nu}$ for some field $F(\mathbf{r},t)$. This is proved in appendix (A.2.2). The full metric is then given by:

$$g_{\mu\nu}(\mathbf{r},t)=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}+\left(\frac{\hbar}{c^2}\right)\begin{bmatrix}F&0&0&0\\0&F&0&0\\0&0&F&0\\0&0&0&F\end{bmatrix}(\mathbf{r},t)+\sum_{n=3}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\begin{bmatrix}g_{00}^{[n]}&g_{01}^{[n]}&g_{02}^{[n]}&g_{03}^{[n]}\\g_{10}^{[n]}&g_{11}^{[n]}&g_{12}^{[n]}&g_{13}^{[n]}\\g_{20}^{[n]}&g_{21}^{[n]}&g_{22}^{[n]}&g_{23}^{[n]}\\g_{30}^{[n]}&g_{31}^{[n]}&g_{32}^{[n]}&g_{33}^{[n]}\end{bmatrix}(\mathbf{r},t) \tag{4.42}$$

where $g_{00}^{[2]}=g_{11}^{[2]}=g_{22}^{[2]}=g_{33}^{[2]}=F(\mathbf{r},t)$

With this form of metric, all the other objects (tetrads, spin coefficients etc.) have been calculated in Appendix sections [A.3], [A.5], [A.4] and [A.6]. We have used these results in the next section.

4.2.5 Non-Relativistic (NR) limit of Einstein-Dirac equations

Dirac equation: Equation (4.22) becomes the following

$$e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[m\ a_{n-1}+i\dot{a}_{n-3}+i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}a_{n-2}-\beta m a_{n-1}-\beta\frac{mF(\mathbf{r},t)}{2}a_{n-3}\right]=0 \tag{4.43}$$

We have already used the results from analysis of this equation for $n=0$ and $n=1$. We now analyze it for $n=2$ and $n=3$. At order $n=2$ the equation (4.22) results in

$$\begin{pmatrix}\dot{S}+\mathrm{m}&0\\0&\dot{S}\text{ - m}\end{pmatrix}\begin{pmatrix}a_1^>\\a_1^<\end{pmatrix}-i\sqrt{\hbar}\begin{pmatrix}0&\vec{\sigma}\cdot\vec{\nabla}\\\vec{\sigma}\cdot\vec{\nabla}&0\end{pmatrix}\begin{pmatrix}a_0^>\\a_0^<\end{pmatrix}=0 \tag{4.44}$$

The first of these is trivially satisfied. The second one yields an expression for $a_1^<$ in terms of $a_0^>$

$$a_1^<=\frac{-i\sqrt{\hbar}\vec{\sigma}\cdot\vec{\nabla}}{2m}a_0^> \tag{4.45}$$

At order $n=3$,

$$\begin{pmatrix}\dot{S}+\mathrm{m}&0\\0&\dot{S}\text{ - m}\end{pmatrix}\begin{pmatrix}a_2^>\\a_2^<\end{pmatrix}-i\sqrt{\hbar}\begin{pmatrix}0&\vec{\sigma}\cdot\vec{\nabla}\\\vec{\sigma}\cdot\vec{\nabla}&0\end{pmatrix}\begin{pmatrix}a_1^>\\a_1^<\end{pmatrix}-\begin{pmatrix}i\partial_t-\frac{mF(\mathbf{r},t)}{2}&0\\0&i\partial_t+\frac{mF(\mathbf{r},t)}{2}\end{pmatrix}\begin{pmatrix}a_0^>\\a_0^<\end{pmatrix}=0 \tag{4.46}$$

Upon using equation (4.45), the first branch of (4.46) yields,

$$i\hbar\frac{\partial a_0^>}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2a_0^>+\frac{m\hbar F(\mathbf{r},t)}{2}a_0^> \tag{4.47}$$

Einstein's equations: Next, we go to Einstein's equations. G_{00} is evaluated in Appendix [A.6]. We equate it with kT_{00} and obtain:

$$\frac{\hbar\nabla^2F(\mathbf{r},t)}{c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)=\frac{8\pi Gm\,|a_0^>|^2}{c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right) \tag{4.48}$$

Equating the functions at order $\frac{1}{c^2}$, we obtain:

$$\nabla^2 F(\mathbf{r},t) = \frac{8\pi G m\, |a_0^{>}|^2}{\hbar} \tag{4.49}$$

If we recognize the quantity $\frac{\hbar F(\mathbf{r},t)}{2}$ as the Newtonian potential ϕ, then we get Schrödinger-Newton system of equations with $m\phi$ as the gravitational potential energy and $m\,|a_0^{>}|^2$ as mass density $\rho(\mathbf{r},t)$. The physical picture, which this system of equations suggests, has been given in the introduction.

$$i\hbar\frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} + m\phi(\mathbf{r},t)a_0^{>} \tag{4.50}$$

$$\nabla^2\phi(\mathbf{r},t) = 4\pi G m\,|a_0^{>}|^2 = 4\pi G\rho(\mathbf{r},t) \tag{4.51}$$

$$i\hbar\frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} - Gm^2\int\frac{|a_0^{>}(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' a_0^{>} \tag{4.52}$$

This completes the derivation of the Schrödinger-Newton equation from the Einstein-Dirac equations, in the non-relativistic limit.

4.3 Non-relativistic limit of Einstein-Cartan-Dirac equations

Dirac equation on U_4 (which is also known as the Hehl-Datta equation) is given by equation (2.59)

$$i\gamma^\mu\psi_{;\mu} - \frac{3}{8}L_{Pl}^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi - \frac{mc}{\hbar}\psi = 0 \tag{4.53}$$

We have already evaluated first and the last term after putting ansatz for spinor (4.2) and metric (4.42). The second term (arising because of torsion) induces non-linearity into the Dirac equation. We now evaluate this term by following similar procedure as we did for the other two terms. First we multiply the mid-term by $\gamma^0 c$ as done while getting equation (4.8) from (4.7) and get the following:

$$\gamma^{(0)}\frac{3c}{8}L_{Pl}^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi = -\frac{3c}{8}l_{Pl}^2 e^{\frac{ic^2 S}{\hbar}}\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n^\dagger\right)\gamma_{(a)}\left(\sum_{l=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_l\right)\gamma_5\gamma^{(a)}\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_m\right) \tag{4.54}$$

Next, we divide it by $\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e_{(0)}^{0[n]}\right]$ as done while getting equation (4.9) from (4.8). This is equivalent to dividing by $\left[1-\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]$ or equivalently multiplying by $\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]$ as given in (A.4). We get the following:

The non-linear term:

$$e^{\frac{ic^2 S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]\frac{3G}{8}\left(\sum_{n_1,n_2,n_3=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_{n_1-i}^\dagger\gamma_a a_{n_2-j}\gamma^5\gamma^a a_{n_3-k}\right) \tag{4.55}$$

where $n = n_1 + n_2 + n_3$. This term modifies Equation (4.43) as follows

$$e^{\frac{ic^2 S}{\hbar}} \frac{c^3}{\hbar^{3/2}} \sum_{n=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n \Bigg[m\, a_{n-1} + i\dot{a}_{n-3} + i\sqrt{\hbar}\overrightarrow{\alpha} \cdot \overrightarrow{\nabla} a_{n-2} - \beta m a_{n-1} - \beta \frac{mF(\mathbf{r},t)}{2} a_{n-3} + \frac{3G}{8}\Big(\sum_{n_1,n_2,n_3=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n a^{\dagger}_{n_1-i}\gamma_a a_{n_2-j}\gamma^5\gamma^a a_{n_3-k}\Big)\Bigg] = 0 \tag{4.56}$$

where $n = n_1 + n_2 + n_3, i + j + k = 5$ and, whatever value of i, j, k, n_1, n_2, n_3 is chosen from (0,1,2,3,4,5) the fact that $i \leq n_1$, $j \leq n_2$ and $k \leq n_3$ is to be respected. We find from the above expression that the non-linear term with starts contributing finitely from n = 5 onwards. So, the analysis for $n = 0, 1, 2, 3$ as given in Appendix remains as it is and we obtain Schrödinger equation for $a_0^{>}$ viz. $i\hbar\frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} + \frac{m\hbar F(\mathbf{r},t)}{2} a_0^{>}$.

Next, we go to Einstein's equations (gravitation equation of ECD theory). The equations of interest here are as given by eqn (2.60) as $G_{\mu\nu}(\{\}) = \chi T_{\mu\nu} - \frac{1}{2}\chi^2 g_{\mu\nu} S^{\alpha\beta\lambda} S_{\alpha\beta\lambda}$. The tensors $G_{\mu\nu}$ and $T_{\mu\nu}$ are already analyzed in above section. We will analyze the second term on the right hand side, which is $(-\frac{1}{2}\chi^2 g_{\mu\nu} S^{\alpha\beta\lambda} S_{\alpha\beta\lambda})$. It contains the products of spin density tensor which is given by eqn (2.58). We consider only first term in the expansion of metric because other terms combined with the coupling constant are already higher orders.

$$\frac{-1}{2}\chi^2 g_{00} S^{\alpha\beta\lambda} S_{\alpha\beta\lambda} = -g_{00}\frac{2\pi^2 G^2 \hbar^2}{c^6} \sum_{N=0}^{\infty} \Big(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a^{\dagger}_k \gamma^0 \gamma^{[c}\gamma^a\gamma^{b]} \Big)\Big(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a^{\dagger}_m \gamma^0 \gamma_{[c}\gamma_a\gamma_{b]} n_m \Big) = \sum_{n=6}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{4.57}$$

We find that this addition doesn't contribute at the order $1/c^2$ on the RHS of equation (2.60). Hence we get back Poisson equation. Recognizing the quantity $\frac{\hbar F(\mathbf{r},t)}{2}$ as the potential ϕ, at leading order, we find that ECD theory also yields Schrödinger-Newton equation. Torsion does not contribute at leading non-relativistic order.

4.4 Non-relativistic limit of ECD field equations with new length scale L_{cs}

The motivation for introducing a new length scale [?] in the ECD theory is as follows. Given a relativistic particle of mass m, it could satisfy either the flat space-time Dirac equation, or the Einstein equations for a point mass, or the ECD equations which couple the Dirac field to its self-gravity and torsion. How are we to know which of these three equations does the dynamics satisfy? There is no mass scale in the equations to determine this. To resolve this problem, we introduced a new length scale L_{CS} in the ECD equations, with the following properties: for $m \gg m_{Pl}, L_{CS} = 2Gm/c^2$; for $m \ll m_{Pl} = \hbar/2mc$; for $m = m_{Pl}/2, L_{CS} = 2L_{Pl}$. In other words, for large masses the length scale in the problem is Schwarzschild radius, and for small masses the length scale is half of the Compton wavelength [4, 5, 18]. An example of a function which can satisfy these properties is

$$\frac{L_{CS}}{2L_{p}l} := \frac{1}{2}\left(\frac{2m}{m_P l} + \frac{m_P l}{2m}\right) := \cosh z \tag{4.58}$$

where $z = \ln 2m/m_{Pl}$. We desire that the field equations with this L_{CS} should reduce to Einstein equations for large masses, and to the Hehl-Datta equation for small masses. An action which yields such equations is

$$\frac{L_{Pl}^2}{\hbar}S = \int d^4x\,\sqrt{-g}\left[\frac{1}{8\pi}R\; - \frac{1}{2}\; L_{CS}\,\overline{\psi}\psi \;+\; L_{CS}^2\;\left\{\frac{i}{2}\overline{\psi}\gamma^{\mu}\;\nabla_{\mu}\psi - \frac{i}{2}(\nabla_{\mu}\overline{\psi})\gamma^{\mu}\psi\right\}\;\right] \tag{4.59}$$

The ECD field equations following from this action, with L_{cs} incorporated in them, are the following:

$$\gamma^{\mu}\psi_{;\mu} = +\frac{3}{8}L_{cs}^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{1}{2L_{CS}}\psi \tag{4.60}$$

$$G^{\mu\nu}(\{\}) = \frac{8\pi L_{CS}^2}{\hbar c}T^{\mu\nu} - \frac{1}{2}\left(\frac{8\pi L_{CS}^2}{\hbar c}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{4.61}$$

$$T^{\mu\nu\alpha} = -K^{\mu\nu\alpha} = \frac{8\pi L_{CS}^2}{\hbar c}S^{\mu\nu\alpha} \tag{4.62}$$

Here we analyze the non-relativistic limit of these equations.

4.4.1 Analysis for lower mass limit of L_{cs}

Lower mass limit of L_{cs} is $\frac{\lambda_C}{2} = \frac{\hbar}{2mc}$. The Dirac equation in the Riemann-Cartan spacetime with new length scale L_{CS} in its lower mass limit is given by (4.60):

$$i\gamma^{\mu}\psi_{;\mu} = \frac{3\hbar^2}{32m^2c^2}\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{1}{2L_{CS}}\psi \tag{4.63}$$

We have already evaluated first and the last term after putting ansatz for spinor (4.2) and metric (4.42). The second term (arising because of torsion) induces non-linearity into the Dirac equation. We now evaluate this term by following similar procedure as we did for the other two terms. First we multiply the middle term by $\gamma^0 c$ as done while getting equation (4.8) from (4.7) and get the following:

$$\gamma^{(0)}\frac{3c}{32}\lambda_C^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi = \frac{3c}{32}\lambda_C^2 e^{\frac{ic^2S}{\hbar}}\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n^{\dagger}\right)\gamma_{(a)}\left(\sum_{l=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_l\right)\gamma_5\gamma^{(a)}\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_m\right) \tag{4.64}$$

Next, we divide it by $\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e_{(0)}^{0[n]}\right]$ as done while getting equation (4.9) from (4.8). This is equivalent to dividing by $1-\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)$ as given in (A.4) or multiplying by $1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)$. We get:

$$e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]\frac{3\hbar^{3/2}}{32m^2}\left(\sum_{n_1,n_2,n_3=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_{n_1-i}^{\dagger}\gamma_a a_{n_2-j}\gamma^5\gamma^a a_{n_3-k}\right) \tag{4.65}$$

where $n = n_1+n_2+n_3, i+j+k = 4$ and, whatever value of i,j,k,n_1,n_2,n_3 is chosen from (0,1,2,3,4) the fact that $i \le n_1$, $j \le n_2$ and $k \le n_3$ is to be respected. We find from the above expression that the non-linear term with L_{CS} starts contributing finitely from n = 4 onwards. So, the analysis for $n = 0,1,2,3$ as given iearlier remains as it is and we obtain Schrödinger equation for $a_0^{>}$ viz. $i\hbar\frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} + \frac{m\hbar F(\mathbf{r},t)}{2}a_0^{>}$.

Now, the gravitational equation of ECD with L_{cs} in its lower mas limit is given by (4.61). We will consider terms only up till second order in $(1/c)$. So we stick to equation for 00 component. We neglect the 2nd term om the right hand side of (4.61) because it is of higher order. The equation for 00 component is:

$$G_{00} = \left(\frac{2\pi\hbar}{m^2c^3}\right)\left(\frac{i\hbar c}{4}\right)\left[2\bar{\psi}\gamma_0\psi_{;0} - 2\bar{\psi}_{;0}\gamma_0\psi\right] \tag{4.66}$$

$$G_{00} = e^0_{(0)}\left(\frac{i\pi\hbar^2}{m^2c^3}\right)\left[\psi^\dagger(\partial_t\psi) - (\partial_t\psi^\dagger)\psi\right] \tag{4.67}$$

After substituting spinor ansatz (4.2), we obtain following equation for the right hand side:

$$G_{00} = \left(\frac{i\pi\hbar}{m^2c}\right)\left[\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m a_m^\dagger\right)\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n [\dot{a}_{n-2} + i\dot{S}a_n]\right)\right.$$
$$\left. - \left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m [\dot{a}_{m-2}^\dagger - i\dot{S}a_m^\dagger]\right)\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right)\right] \tag{4.68}$$

This implies that

$$\frac{\hbar\nabla^2 F}{c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) = \frac{1}{c}\left(\frac{2\pi\hbar}{m}|a_0^>|^2\right) + \frac{1}{c^2}\left(\frac{2\pi\hbar^{3/2}}{m}\left[a_1^{>\dagger}a_0^> + a_0^{>\dagger}a_1^>\right]\right) + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{4.69}$$

This leads us to conclude that $a_0^> = 0$ and hence

$$\nabla^2 F = 0 \implies \nabla^2\phi = 0 \tag{4.70}$$

With this new length scale, there is no contribution to gravity in the small mass limit, at the leading order. This makes the theory different from the standard ECD theory. Another possible interpretation of the modified Poisson equation (4.69) might be to write it at order $1/c^2$ as

$$\nabla^2\phi = \frac{4\pi Gm}{\alpha}\left(|a_0^>|^2 + \hbar^{1/2}\left[a_1^{>\dagger}a_0^> + a_0^{>\dagger}a_1^>\right]\right) \tag{4.71}$$

where $\alpha \equiv 4Gm^2/\hbar c$ is the dimensionless gravitational fine structure parameter. Further implications of this equation are at present under investigation.

4.4.2 Analysis for higher mass limit of L_{cs}

The high mass limit of L_{cs} is $2Gm/c^2$. We have shown elsewhere that in the large mass limit these equations reduce to Einstein equations for a point mass. The non-relativistic limit will then inevitably be the Poisson equation for a point mass.

This can also be seen as follows. The Einstein equation in the Riemann-Cartan spacetime with new length scale L_{CS} is given by (4.61). We neglect terms higher order in L_{CS} because it is easy to deduce from the fact that L_{CS}^2 in higher mass limit is already fourth order in $(1/c)$. So only first term of right hand side is significant. We consider the "00" component of the above equation

$$G_{00} = \frac{8\pi L_{CS}^2}{\hbar c}T_{00} \tag{4.72}$$

The stress tensor is given by (2.57). Its "00" component is given by [we neglect orders greater than $1/c^2$].

$$T_{00} = \frac{i\hbar c}{4}\Big[2\bar{\psi}\gamma^0\psi_{;0} - 2\bar{\psi}_{;0}\gamma^0\psi\Big] \tag{4.73}$$

The Dirac equation with L_{cs} in its higher mass limit is given by (4.60). Now, for large masses ($m \gg m_{Pl}$), amplitude of state ψ is negligible (except in a very narrow region where mass m gets localized). This is possible if we assume the localization process. In such a case, the kinetic energy term can be neglected and we obtain the following equations

$$\begin{aligned} \psi_{;0} &= -\frac{3}{8}i\gamma^0 L_{CS}^2\bar{\psi}\gamma^5\gamma_a\psi\gamma^5\gamma^a\psi - \frac{i\gamma^0}{2L_{CS}}\psi \\ \psi^\dagger_{;0} &= \frac{3}{8}iL_{CS}^2(\gamma^0\bar{\psi}\gamma^5\gamma_a\psi\gamma^5\gamma^a\psi)^\dagger + \frac{i}{2L_{CS}}\psi^\dagger\gamma^0 \end{aligned} \tag{4.74}$$

Substituting above equation (4.74) in eqn (4.73) and neglecting higher order terms in L_{CS} we get,

$$\frac{8\pi L_{CS}^2}{\hbar c}T_{00} = 4\pi L_{CS}(\psi^\dagger\gamma^0\psi) \tag{4.75}$$

Substituting for L_{CS} in the large mass limit in eqn (4.75) ,

$$\frac{8\pi L_{CS}^2 T_{00}}{\hbar c} = 4\pi L_{CS}(\psi^\dagger\gamma^0\psi) = \frac{8\pi Gm\bar{\psi}\psi}{c^2} \tag{4.76}$$

In the localization process we replace $\bar{\psi}\psi$ with a spatial Dirac delta function [5]. Substituting equation (4.76) and G_{00} from Appendix [A.44] in equation (4.72) and equating at order $\frac{1}{c^2}$, we get the Poisson equation as the non-relativistic weak field limit of the modified Einstein equation in the large mass limit,

$$\nabla^2 F(\mathbf{r},t) = \frac{8\pi Gm}{\hbar}\delta(\mathbf{r}) \tag{4.77}$$

As earlier, we recognize $\frac{\hbar F}{2}$ as Newtonian potential ϕ and hence, we get

$$\nabla^2\phi = 4\pi Gm\delta(\mathbf{r}) \tag{4.78}$$

The large mass non-relativistic limit with this new length scale is not the Schrödinger-Newton equation, but the Poisson equation for a classical point mass.

4.4.3 Some comments on analysis for intermediate mass

For an intermediate mass, L_{cs} is given by equation (4.58). With this, the ECD equations become:

$$i\gamma^\mu\psi_{;\mu} = \frac{3}{8}\Big(\frac{2Gm}{c^2} + \frac{\hbar}{2mc}\Big)^2\bar{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{1}{\Big(\frac{4Gm}{c^2} + \frac{\hbar}{mc}\Big)}\psi \tag{4.79}$$

$$G_{\mu\nu}(\{\}) = \frac{8\pi}{\hbar c}\Big(\frac{2Gm}{c^2} + \frac{\hbar}{2mc}\Big)^2 T_{\mu\nu} - \frac{32\pi^2}{\hbar^2c^2}\Big(\frac{2Gm}{c^2} + \frac{\hbar}{2mc}\Big)^4 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{4.80}$$

First we will analyze HD equation. The three non-linear terms appear in this equation with coefficients $\frac{3G^2m^2}{2c^4}$, $\frac{3L_{pl}^2}{4}$ and $\frac{3\hbar^2}{32m^2c^2}$. We have already done the order analysis of all these terms and shown to be higher order; not contributing to the equation at leading order. So we neglect them. What we get is:

$$i\gamma^\mu\psi_{;\mu} = \frac{1}{\left(\frac{4Gm}{c^2} + \frac{\hbar}{mc}\right)}\psi = \frac{mc\psi}{\hbar}\left(\frac{1}{1+\frac{4m^2}{m_{pl}^2}}\right) \tag{4.81}$$

$$\implies \left[1 + \frac{4m^2}{m_{pl}^2}\right] i\gamma^\mu\psi_{;\mu} = \frac{mc\psi}{\hbar} \tag{4.82}$$

This is a very interesting equation. If mass m is too small compared to m_{pl}, we can neglect the second term on left hand side and this basically gives Schrödinger's equation. On the other hand, if mass is too large, we neglect the first term on the left hand side, and then the equation becomes such that we can safely assume the localization process. [basically it justifies eq. (4.74)]. We plan to investigate the intermediate mass case more rigorously in the future.

4.5 Summary of important results

- At leading order, non-relativistic limit of self-gravitating Dirac field on V_4 (commonly called as Einstein-Dirac system) is Schrödinger-Newton equation with no assumption of symmetry on metric.
- Non-relativistic limit of self-gravitating Dirac field on U_4 (commonly called as Einstein-Cartan-Dirac system) is also Schrödinger-Newton equation at leading order.
- Non-relativistic limit of ECD theory with L_{cs} in its low mass limit produces a source-free Poisson equation. This will be interpreted in chapter (7).
- Non-relativistic limit of ECD theory with L_{cs} in its higher mass limit produces Poisson equation with delta function source. This will be interpreted in chapter (7).

The work in this chapter is based on the paper titled "The non-relativistic limit of the Einstein-Cartan-Dirac equations" which is `"under preparation"` [16]

Chapter 5

Brief review of Newmann-Penrose (NP) formalism and formulation of ECD equations in NP formalism

There has been a variety of different (physically and mathematically equivalent) ways of writing the field equations of General theory of relativity. Initially, it was formulated in standard coordinate-basis version using the metric tensor components as the basic variable and the Christoffel symbols as connection. Later various methods like that of differential forms developed by Cartan (Lovelock and Rund, 1975), the space-time (orthonormal) tetrad version of Ricci (Levy, 1925) and the spin coefficient version of Newman and Penrose (Newman and Penrose, 1962; Geroch et al, 1973; Penrose, 1968; Penrose and Rindler, 1984; Penrose and Rindler, 1986; Newman and Tod, 1980; Newman and Unti, 1962) are developed. All references in parenthesis are taken from Scholarpedia article titled "Spin-Coefficient formalism".

Dirac equation on V_4 has been studied extensively in NP formalism. It's detail account can be seen in [6]. From this chapter onwards, we follows the notations/ representations/ conventions and symbols of this celebrated book "The mathematical theory of black holes" By S. Chandrasekhar [6]. Our aim in this chapter is as follows

- We know that Contorsion tensor is completely expressible in terms of components of Dirac spinor. We want to find an explicit expression for Contorsion spin coefficients (in Newman-Penrose) in terms of Dirac spinor components.
- Dirac equation on V_4 is presented in equation (108) of [6]. We aim to modify these equations on U_4.

We will first present a brief review of NP formalism and then formulate ECD equations in NP formalism.

5.1 Newman-Penrose formalism

NP formalism was formulated by Neuman and Penrose in their work [35]. It is a special case of tetrad formalism (introduced in Appendix [B.1]); where we choose our tetrad as a set of four null vectors viz.

$$e^{\mu}_{(0)} = l^{\mu}, \quad e^{\mu}_{(1)} = n^{\mu}, \quad e^{\mu}_{(2)} = m^{\mu}, \quad e^{\mu}_{(3)} = \bar{m}^{\mu} \tag{5.1}$$

l^{μ}, n^{μ} are real and $m^{\mu}, \bar{m}^{\mu}$ are complex. The tetrad indices are raised and lowered by flat space-time metric

$$\eta_{(i)(j)} = \eta^{(i)(j)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{5.2}$$

and the tetrad vectors satisfy the equation $g_{\mu\nu} = e^{(i)}_{\mu} e^{(j)}_{\nu} \eta_{(i)(j)}$. In the formalism, we replace tensors by their tetrad components and represent these components with distinctive symbols. These symbols are quite standard and used everywhere in literature. It was Hermann Bondi who first suggested the use of null-tetrads for the analysis of electromagnetic and gravitational radiation since they propogate along these null directions. Some important features of NP formalism are can be jot down as follows: (these are partially also the reasons why we adopted this formalism to represent our equations)

- With NP formalism, equations can be partially grouped together into sets of linear equations (Newman and Unti, 1962)
- All are complex equations; thereby reducing the total number of equations by half
- It allows one to concentrate on individual 'scalar' equations with particular physical or geometric significance.
- It allows one to search for solutions with specific special features, such as the presence of one or two null directions that might be singled out by physical or geometric considerations. Ex. it turns out to be a very useful tool in solving problems involving massless fields etc.
- Newman and Penrose also showed that their formalism is completely equivalent to the SL(2,$\mathbb{C}$) spinor approach. [We are gonna follow SL(2,$\mathbb{C}$) spinor approach]
- In NP formalism, equations are written out explicitly without the use of the index and summation conventions.
- While dealing with Spinors on curved space-times, it becomes very easy to establish the knowledge of physical/ geometric properties of complicated space-times (e.g. space-time around Kerr black hole etc.) and the knowledge of various properties of Spinors simultaneously in a common vocabulary of NP formalism. Various commonly occurring space-times have been formulated in NP formalism in [6]. This point is the main reason why we adopt this formalism.

The orthonormality condition on null tetrads imply $l.m = l.\bar{m} = n.m = n.\bar{m} = 0$, $l.l = n.n = m.m = \bar{m}.\bar{m} = 0$ and $l.n = 1$ and $m.\bar{m} = -1$. The Ricci rotation coefficients (defined in appendix [B.1]) for null tetrads are called spin coefficients and are defined as follows

$$\gamma_{(l)(m)(n)} = e^{\nu}_{(n)} e^{\mu}_{(m)} \nabla_{\nu} e_{(l)\mu} \tag{5.3}$$

The covariant derivative defined in the above equation can be taken w.r.t both V_4 **and** U_4 **manifold.** We are here interested in U_4. Spin coefficients are denoted by following symbols

$$\begin{aligned}
\kappa &= \gamma_{(2)(0)(0)} & \rho &= \gamma_{(2)(0)(3)} & \epsilon &= \frac{1}{2}(\gamma_{(1)(0)(0)} + \gamma_{(2)(3)(0)}) \\
\sigma &= \gamma_{(2)(0)(2)} & \mu &= \gamma_{(1)(3)(2)} & \gamma &= \frac{1}{2}(\gamma_{(1)(0)(1)} + \gamma_{(2)(3)(1)}) \\
\lambda &= \gamma_{(1)(3)(3)} & \tau &= \gamma_{(2)(0)(1)} & \alpha &= \frac{1}{2}(\gamma_{(1)(0)(3)} + \gamma_{(2)(3)(3)}) \\
\nu &= \gamma_{(1)(3)(1)} & \pi &= \gamma_{(1)(3)(0)} & \beta &= \frac{1}{2}(\gamma_{(1)(0)(2)} + \gamma_{(2)(3)(2)})
\end{aligned} \tag{5.4}$$

These are 12 complex spin coefficients, corresponding to 24 real components of γ. We separate the Riemann part and the torsional part from the covariant derivative of equation (5.3). The result is

$$\begin{aligned}
\gamma_{(l)(m)(n)} &= e^{\nu}_{(n)} e^{\mu}_{(m)} \nabla_{\nu} e_{(l)\mu} \\
&= e^{\nu}_{(n)} e^{\mu}_{(m)} \left[\delta^{\alpha}_{\mu} \partial_{\nu} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + K_{\nu\mu}{}^{\alpha} \right] e_{(l)\alpha} \\
&= \gamma^{o}_{(l)(m)(n)} + K_{(n)(m)(l)}
\end{aligned} \tag{5.5}$$

In terms of the symbols (defined in equation (5.4)), we adopt notation of [34] where $\kappa = \kappa^o + \kappa_1$ and so on for all the 12 spin coefficients. κ^o denotes Riemann part and and κ_1 denote torsional part. The torsional part of spin coefficients (which distinguishes it from V_4) is called Contorsion spin coefficients. The spin coefficients and contorsion spin coefficients are given in the figure (5.1).

The directional derivatives w.r.t these null tetrads are given by

$$D = l^{\mu} \frac{\partial}{\partial x^{\mu}} = e_0 \qquad \Delta = n^{\mu} \frac{\partial}{\partial x^{\mu}} = e_1 \qquad \delta = m^{\mu} \frac{\partial}{\partial x^{\mu}} = e_2 \qquad \delta^* = \bar{m}^{\mu} \frac{\partial}{\partial x^{\mu}} = e_3 \tag{5.6}$$

5.2 ECD equations in NP formalism

5.2.1 Notations/ representations and spinor analysis

- The Lorentz Signature used in this chapter is Diag (+ - - -)
- The 4 component Dirac-spinor is

$$\psi = \begin{bmatrix} P^A \\ \bar{Q}_{B'} \end{bmatrix} \tag{5.7}$$

  where P^A and $\bar{Q}_{B'}$ are 2-dim complex vectors in $\mathbb{C}^2$ (also called spinors) Please see section for details. We use following notations for Dirac spinor components (consistent with the notations of Chandra's book [6]) $P^0 = F_1$, $P^1 = F_2$, $\bar{Q}^{1'} = G_1$ and $\bar{Q}^{0'} = -G_2$.

$$
\begin{array}{ll}
\kappa = l^\nu m^\mu \nabla_\nu l_\mu & \kappa_1 = K_{\lambda\mu\nu} l^\lambda m^\mu l^\nu \\
\rho = \bar{m}^\nu m^\mu \nabla_\nu l_\mu & \rho_1 = K_{\lambda\mu\nu} \bar{m}^\lambda m^\mu l^\nu \\
\sigma = m^\nu m^\mu \nabla_\nu l_\mu & \sigma_1 = K_{\lambda\mu\nu} m^\lambda m^\mu l^\nu \\
\tau = n^\nu m^\mu \nabla_\nu l_\mu & \tau_1 = K_{\lambda\mu\nu} n^\lambda m^\mu l^\nu \\
\epsilon = \frac{1}{2} l^\nu (n^\mu \nabla_\nu l_\mu - \bar{m}^\mu \nabla_\nu m_\mu) & \epsilon_1 = \frac{1}{2} K_{\lambda\mu\nu} l^\lambda (n^\mu l^\nu - \bar{m}^\mu m^\nu) \\
\alpha = \frac{1}{2} \bar{m}^\nu (n^\mu \nabla_\nu l_\mu - \bar{m}^\mu \nabla_\nu m_\mu) & \alpha_1 = \frac{1}{2} K_{\lambda\mu\nu} \bar{m}^\lambda (n^\mu l^\nu - \bar{m}^\mu m^\nu) \\
\beta = \frac{1}{2} m^\nu (n^\mu \nabla_\nu l_\mu - \bar{m}^\mu \nabla_\nu m_\mu) & \beta_1 = \frac{1}{2} K_{\lambda\mu\nu} m^\lambda (n^\mu l^\nu - \bar{m}^\mu m^\nu) \\
\gamma = \frac{1}{2} n^\nu (n^\mu \nabla_\nu l_\mu - \bar{m}^\mu \nabla_\nu m_\mu) & \gamma_1 = \frac{1}{2} K_{\lambda\mu\nu} n^\lambda (n^\mu l^\nu - \bar{m}^\mu m^\nu) \\
\pi = - l^\nu \bar{m}^\mu \nabla_\nu n_\mu & \pi_1 = - K_{\lambda\mu\nu} l^\lambda \bar{m}^\mu n^\nu \\
\lambda = - \bar{m}^\nu \bar{m}^\mu \nabla_\nu n_\mu & \lambda_1 = - K_{\lambda\mu\nu} \bar{m}^\lambda \bar{m}^\mu n^\nu \\
\mu = - m^\nu \bar{m}^\mu \nabla_\nu n_\mu & \mu_1 = - K_{\lambda\mu\nu} m^\lambda \bar{m}^\mu n^\nu \\
\nu = - n^\nu \bar{m}^\mu \nabla_\nu n_\mu & \nu_1 = - K_{\lambda\mu\nu} n^\lambda \bar{m}^\mu n^\nu
\end{array}
$$

Figure 5.1: 12 complex spin coefficients their 'torsional' parts

- We define 4 null vectors (and their corresponding co-vectors) on Minkowski space

$$l^a = \frac{1}{\sqrt{2}}(1,0,0,1),\ m^a = \frac{1}{\sqrt{2}}(0,1,-i,0),\ \bar{m}^a = \frac{1}{\sqrt{2}}(0,1,i,0),\ n^a = \frac{1}{\sqrt{2}}(1,0,0,-1) \tag{5.8}$$

$$l_a = \frac{1}{\sqrt{2}}(1,0,0,-1),\ m_a = \frac{1}{\sqrt{2}}(0,-1,i,0),\ \bar{m}_a = \frac{1}{\sqrt{2}}(0,-1,-i,0),\ n_a = \frac{1}{\sqrt{2}}(1,0,0,1) \tag{5.9}$$

  We also define, what is called as **Van der Waarden** symbols as follows:

$$\sigma^a = \sqrt{2}\begin{bmatrix} l^a & m^a \\ \bar{m}^a & n^a \end{bmatrix} \qquad \tilde{\sigma}^a = \sqrt{2}\begin{bmatrix} n^a & -m^a \\ -\bar{m}^a & l^a \end{bmatrix} \tag{5.10}$$

- We use following representation of gamma matrices [its the **complex version of Weyl or chiral representation**]

$$\gamma^a = \begin{bmatrix} 0 & (\tilde{\sigma}^a)^* \\ (\sigma^a)^* & 0 \end{bmatrix} (a = 0,1,2,3) \quad where \quad \gamma^0 = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix},\ \gamma^i = \begin{bmatrix} 0 & (-\sigma^i)^* \\ (\sigma^i)^* & 0 \end{bmatrix} \tag{5.11}$$

The reason for choosing complex Weyl representation is the fact that the spinor and gamma matrix defined in equation (5.7) and (5.11) gives us equation (97) and (98) of section (103) given in Chandra's book [6]. **We want to keep everything in accordance with [6] as a standard reference.** (Equation (99) is the complex version of what we will get). For representing equations or physical objects having spinors and gamma matrices on a curved space time, we adopt Tetrad formalism. Using tetrads, we follow the prescription described briefly in [31]. We summarize and comment on it as follows:-

Given a curved manifold $\mathcal{M}$ with all conditions necessary for the existence of spin structure. Let U be a chart on $\mathcal{M}$ with coordinate functions (x^α), then the prescription for representing spinorial objects (objects with spinors and gamma matrices) is as follows:-
1) choose an Orthonormal tetrad field $e^\mu_{(a)}(x^\alpha)$ on U
2) Define the Van der Waarden symbols (the $\sigma^{(a)}$ and $\tilde{\sigma}^{(a)}$) in this tetrad basis exactly as defined on Minkowski space in equation (5.10). Choose a representation of gamma matrix (we will stick to the one chosen above in equation (5.11))
3) The σ's in local coordinate frame are obtained through following equation:-

$$\sigma^\mu(x^\alpha) = e^\mu_{(a)}(x^\alpha)\sigma^{(a)} = \sqrt{2}\begin{bmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{bmatrix} \qquad \tilde{\sigma}^\mu = e^\mu_{(a)}\tilde{\sigma}^{(a)}\sqrt{2}\begin{bmatrix} n^\mu & -m^\mu \\ -\bar{m}^\mu & l^\mu \end{bmatrix} \tag{5.12}$$

and similar transformation for and gamma matrix. **So components of any world object which is indexed by the components of gamma matrices or Spinors is now a function of chosen orthonormal tetrad**. It is defined a-priori in a local tetrad basis [whose components are exactly the same as defined on a flat Minkowski space] and then carried to curved space via tetrad. (This is unlike a normal world objects which are first defined naturally at a point on a manifold and then carried to local tangent space via tetrad).

Dirac equation on V_4 has been studied extensively in NP formalism. It's detail account can be seen in [6](Dirac equation on V_4 is presented in equation (108)). We aim to modify these equations on U_4. To this aim, we want to modify section 102(d) of Chandra's book [6] **to include torsion** in the theory and modify Dirac equation accordingly on U_4. For calculating covariant derivative of spinor, we require the spinor affine connection coefficients. They are defined through the requirement that ϵ_{AB} and σ's are covariantly constant. The whole analysis remains as it is up till eqn (91) of Chandra's book except, everywhere, the covariant derivative would now be evaluated on U_4. The covariant derivatives are defined as:

$$\nabla_\mu P^A = \partial_\mu P^A + \Gamma^A_{\mu B} P^B \tag{5.13}$$
$$\nabla_\mu \bar{Q}^{A'} = \partial_\mu \bar{Q}^{A'} + \bar{\Gamma}^{A'}_{\mu B'} \bar{Q}^{B'} \tag{5.14}$$

Here Γ terms are the terms that add to the partial derivative while calculating the full derivative of spinorial objects on U_4. Their values can be determined completely in terms of Spin coefficients and we now evaluate its tetrad components. **Using Friedman's lemma** (proved on page 542 of Chandra's book [6]), we can express various spin coefficients $\Gamma_{(a)(b)(c)(d')}$ in terms of covariant derivative of basis null vectors (which we had defined earlier viz. l,n,m,$\bar{m}$). The covariant derivative here is exactly the same as defined in equation equation 3.3 (and explicitly written in eqn 3.5) of [34]. We have also defined in (5.5). Using this covariant derivative, it can be easily seen how equations (95) and (96) will get modified. For instance, Chandra's equations (95) and (96) gets modified as $\Gamma_{0000'} = \kappa^o + \kappa_1$ and $\Gamma_{1101'} = \mu^o + \mu_1$. Here the subscript 0 in κ^o and μ^o is just used to indicate the original κ and μ defined on V_4 as in, those original equations of Chandra's book. Likewise, 12 independent spin coefficients are calculated in terms of covariant derivatives of null vectors and defined in tabular equation (5.15).

$$\Gamma_{(a)(b)(c)(d')} = \begin{array}{|c||c|c|c|} \hline (c)(d') \backslash (a)(b) & 00 & 01 \text{ or } 10 & 11 \\ \hline\hline 00' & \kappa^o+\kappa_1 & \epsilon^o+\epsilon_1 & \pi^o+\pi_1 \\ \hline 10' & \rho^o+\rho_1 & \alpha^o+\alpha_1 & \lambda^o+\lambda_1 \\ \hline 01' & \sigma^o+\sigma_1 & \beta^o+\beta_1 & \mu^o+\mu_1 \\ \hline 11' & \tau^o+\tau_1 & \gamma^o+\gamma_1 & \nu^o+\nu_1 \\ \hline \end{array} \tag{5.15}$$

We note that, for generic case, all the 12 terms will have Contorsion spin coefficients.

5.2.2 Contorsion spin coefficients in terms of Dirac spinor components

The spin density tensor of matter ($S^{\mu\nu\lambda}$) can be written as a world tensor in U_4 made up of the Dirac spinor, its adjoint, and gamma matrices:

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{5.16}$$

The ECD field equations show that $T^{\mu\nu\alpha} = kS^{\mu\nu\alpha}$ where $T^{\mu\nu\alpha}$ is the modified torsion tensor defined in Eq. 2.3 of [2]. It can be shown that, for Dirac field, $T^{\mu\nu\alpha} = -K^{\mu\nu\alpha} = kS^{\mu\nu\alpha}$ as in Eq. 5.6 of [3]. Here, k is a gravitational coupling constant containing the length scale l_1, i.e., $\frac{8\pi l_1^2}{\hbar c}$. For the standard theory, $l_1 = L_{Pl}$. Substituting (5.16) in the field equations, we obtain following:

$$K^{\mu\nu\alpha} = -kS^{\mu\nu\alpha} = 2i\pi l_1^2\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{5.17}$$

where the γ^{μ}'s are those defined in (**??**), generalised with world indices using orthonormal tetrads. We subsequently rewrite $K^{\mu\nu\alpha}$ (of which only four independent components are excited by the Dirac field) in the NP formalism; i.e., in the null tetrad basis, as follows:

$$K_{(i)(j)(k)} = e_{(i)\mu}e_{(j)\nu}e_{(k)\alpha}K^{\mu\nu\alpha} \tag{5.18}$$

where $e_{(i)\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ for $i = 0, 1, 2, 3$ To calculate the contorsion spin coefficients, we need to evaluate the contorsion tensor with world indices as defined in (5.17). Consider the product $\gamma^\alpha\gamma^\beta\gamma^\mu$, which is defined as:

$$\gamma^\alpha\gamma^\beta\gamma^\mu = \begin{pmatrix} 0 & (\tilde{\sigma}^\alpha)^*(\sigma^\beta)^*(\tilde{\sigma}^\mu)^* \\ (\sigma^\alpha)^*(\tilde{\sigma}^\beta)^*(\sigma^\mu)^* & 0 \end{pmatrix} \tag{5.19}$$

The explicit form of this matrix is fairly expansive, and a full treatment is given in Appendix A. Eventually, we substitute in for the Dirac bispinor (as defined in [6]), and obtain the expressions for the contorsion spin coefficients in terms of the spinor components. We have, for example, for ρ –

$$\rho = -K_{(0)(2)(3)} = -2\sqrt{2}i\pi l_1^2[F_2\bar{F}_2 - G_1\bar{G}_1] \tag{5.20}$$

All the contorsion spin coefficients can be found in a similar fashion. After evaluating those, the eight non-zero spin coefficients excited by the Dirac spinor given in (**??**) – of which four are independent – are as follows:

$$\tau_1 = -2\beta_1 = K_{012} = 2\sqrt{2}i\pi l_1^2(F_2\bar{F}_1 + G_2\bar{G}_1) \tag{5.21}$$

$$\pi_1 = -2\alpha_1 = K_{013} = 2\sqrt{2}i\pi l_1^2(-F_1\bar{F}_2 - G_1\bar{G}_2) \tag{5.22}$$

$$\mu_1 = -2\gamma_1 = -K_{123} = 2\sqrt{2}i\pi l_1^2(F_1\bar{F}_1 - G_2\bar{G}_2) \tag{5.23}$$

$$\rho_1 = -2\epsilon_1 = -K_{023} = 2\sqrt{2}i\pi l_1^2(G_1\bar{G}_1 - F_2\bar{F}_2) \tag{5.24}$$

From the above relations, we have:

$$\mu_1 = -\mu_1^* \tag{5.25}$$

$$\rho_1 = -\rho_1^* \tag{5.26}$$

$$\pi_1 = +\tau_1^* \tag{5.27}$$

The table (5.15) is modified as follows:

$\Gamma_{(a)(b)(c)(d')} =$ (5.28)

(c)(d') \ (a)(b)	00	01 or 10	11
00'	κ_0	$\epsilon_0 - \rho_1/2$	$\pi_0 + \pi_1$
10'	$\rho_0 + \rho_1$	$\alpha_0 - \pi_1/2$	λ_0
01'	σ_0	$\beta_0 - \tau_1/2$	$\mu_0 + \mu_1$
11'	$\tau_0 + \tau_1$	$\gamma_0 - \mu_1/2$	ν_0

Next, we formulate ECD theory in the NP formalism. There are three equations in this theory - the Dirac equation on U_4 (known as the Hehl-Datta equation), the gravitation field equation on U_4, and an algebraic equation relating torsion and spin. The algebraic equation is given in 5.17. In the next two sections, we formulate the Dirac equation and the gravitation field equations explicitly on U_4 respectively.

5.2.3 The Dirac equation with torsion in the NP formalism

The Dirac equation on U_4 (also known as the Hehl-Datta equation) is:

$$i\gamma^\mu \nabla_\mu \psi = \frac{mc}{\hbar}\psi = \frac{\psi}{2l_2} \tag{5.29}$$

where ∇ here denotes covariant derivative on U_4. $l_2 = \frac{\lambda_c}{2}$ for standard theory. It can be written in the following matrix form:

$$i\begin{pmatrix} 0 & (\tilde{\sigma}^\mu)^* \\ (\sigma^\mu)^* & 0 \end{pmatrix} \nabla_\mu \begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} = \frac{1}{2\sqrt{2}l_2}\begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} \tag{5.30}$$

This can be written as a pair of matrix equations:

$$\begin{pmatrix} \sigma^\mu_{00'} & \sigma^\mu_{10'} \\ \sigma^\mu_{01'} & \sigma^\mu_{11'} \end{pmatrix} \nabla_\mu \begin{pmatrix} P^0 \\ P^1 \end{pmatrix} + \frac{i}{2\sqrt{2}l_2}\begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} = 0 \tag{5.31}$$

$$\begin{pmatrix} \sigma^\mu_{11'} & -\sigma^\mu_{10'} \\ -\sigma^\mu_{01'} & \sigma^\mu_{00'} \end{pmatrix} \nabla_\mu \begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} + \frac{i}{2\sqrt{2}l_2}\begin{pmatrix} P^0 \\ P^1 \end{pmatrix} = 0 \tag{5.32}$$

Working out explicitly, the first equation is:

$$\begin{aligned}\frac{i}{2\sqrt{2}l_2}\bar{Q}^{1'} &= \sigma^{\mu}_{00'}\nabla_\mu P^0 + \sigma^{\mu}_{10'}\nabla_\mu P^1 = (\partial_{00'}P^0 + \Gamma^0{}_{i00'}P^i) + (\partial_{10'}P^1 + \Gamma^1{}_{i10'}P^i) \\ &= (D + \Gamma^0{}_{000'}P^0 + \Gamma^0{}_{100'}P^1) + (\delta^* + \Gamma^1{}_{010'}P^0 + \Gamma^1{}_{110'}P^1) \qquad (5.33) \\ \Rightarrow \frac{i}{2\sqrt{2}l_2}G_1 &= (D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1)\end{aligned}$$

where we have used the gamma matrices as defined in (??), computed the covariant derivatives using (5.13), (5.14) and the spin connections in terms of contorsion spin coefficients as given in (5.28). Using this procedure (a full treatment given in Appendix B), the four Dirac equations are obtained as:

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1) = ib(l_2)G_1 \qquad (5.34)$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 + \frac{3}{2}(\mu_1 F_2 - \tau_1 F_1) = ib(l_2)G_2 \qquad (5.35)$$

$$(D + \epsilon_0^* - \rho_0^*)G_2 - (\delta + \pi_0^* - \alpha_0^*)G_1 - \frac{3}{2}(\tau_1 G_1 - \rho_1 G_2) = ib(l_2)F_2 \qquad (5.36)$$

$$(\Delta + \mu_0^* - \gamma_0^*)G_1 - (\delta^* + \beta_0^* - \tau_0^*)G_2 - \frac{3}{2}(\mu_1 G_1 - \pi_1 G_2) = ib(l_2)F_1 \qquad (5.37)$$

Substituting in the spinorial form of the contorsion spin coefficients in (5.21) - (5.24), we obtain:

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + ia(l_1)[(-F_1\bar{F}_2 - G_1\bar{G}_2)F_2 + (F_2\bar{F}_2 - G_1\bar{G}_1)F_1] = ib(l_2)G_1 \qquad (5.38)$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 + ia(l_1)[(F_1\bar{F}_1 - G_2\bar{G}_2)F_2 - (F_2\bar{F}_1 + G_2\bar{G}_1)F_1] = ib(l_2)G_2 \qquad (5.39)$$

$$(D + \epsilon_0^* - \rho_0^*)G_2 - (\delta + \pi_0^* - \alpha_0^*)G_1 - ia(l_1)[(F_2\bar{F}_2 - G_1\bar{G}_1)G_2 + (F_2\bar{F}_1 + G_2\bar{G}_1)G_1] = ib(l_2)F_2 \qquad (5.40)$$

$$(\Delta + \mu_0^* - \gamma_0^*)G_1 - (\delta^* + \beta_0^* - \tau_0^*)G_2 - ia(l_1)[(F_1\bar{F}_1 - G_2\bar{G}_2)G_1 - (-F_1\bar{F}_2 - G_1\bar{G}_2)G_2] = ib(l_2)F_1 \qquad (5.41)$$

where $a(l_1) = 3\sqrt{2}\pi l_1^2$ and $b(l_2) = \frac{1}{2\sqrt{2}l}$.

These equations can be condensed into the following form:

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 = i[b(l_2) + a(l_1)\xi]G_1 \qquad (5.42)$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 = i[b(l_2) + a(l_1)\xi]G_2 \qquad (5.43)$$

$$(D + \epsilon_0^* - \rho_0^*)G_2 - (\delta + \pi_0^* - \alpha_0^*)G_1 = i[b(l_2) + a(l_1)\xi^*]F_2 \qquad (5.44)$$

$$(\Delta + \mu_0^* - \gamma_0^*)G_1 - (\delta^* + \beta_0^* - \tau_0^*)G_2 = i[b(l_2) + a(l_1)\xi^*]F_1 \qquad (5.45)$$

where $\xi = F_1\bar{G}_1 + F_2\bar{G}_2$ and $\xi^* = \bar{F}_1 G_1 + \bar{F}_2 G_2$.

5.2.4 The gravitation equation on U_4 in NP formalsim

The equation of interest here is (??), reproduced here:

$$G_{\mu\nu}(\{\}) = \frac{8\pi l_1^2}{\hbar c}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi l_1^2}{\hbar c}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \qquad (5.46)$$

On the left hand side, we have $G_{\mu\nu}(\{\})$, which has been completely evaluated in the NP formalism in [6]. There are two terms on right hand side – the first of these is the metric energy-momentum tensor ($T_{\mu\nu}$) formulated on U_4 and is given by equation 2.57. In what follows, we will give a prescription to compute the various components of $T_{\mu\nu}$, under the definition:

$$T_{\mu\nu} = \frac{i\hbar c}{4}\Big[\bar{\psi}\gamma_\mu \nabla^{\{\}}_\nu \psi + \bar{\psi}\gamma_\nu \nabla^{\{\}}_\mu \psi - \nabla^{\{\}}_\mu \bar{\psi}\gamma_\nu \psi - \nabla^{\{\}}_\nu \bar{\psi}\gamma_\mu \psi\Big] \tag{5.47}$$

First, we choose a tetrad basis and construct Van der Waarden symbols as defined in (??). using these, we construct Dirac gamma matrices in the complex Weyl representation as defined in (??). Now, the expression for the covariant derivatives of spinors – see (5.13),(5.14),(??) – can be expressed in terms of the gamma matrices, yielding:

$$\begin{aligned} T_{\mu\nu} = \frac{i\hbar c}{4}\Big[&\bar{\psi}\gamma_\mu\partial_\nu\psi + \frac{1}{4}\bar{\psi}(\gamma_\mu\gamma^\alpha\nabla^{\{\}}_\nu\gamma_\alpha)\psi + \bar{\psi}\gamma_\nu\partial_\mu\psi + \frac{1}{4}\bar{\psi}(\gamma_\nu\gamma^\alpha\nabla^{\{\}}_\mu\gamma_\alpha)\psi \\ &- \partial_\mu\bar{\psi}\gamma_\nu\psi - \frac{1}{4}(\bar{\gamma}^\alpha\bar{\nabla}^{\{\}}_\mu\bar{\gamma}_\alpha)\bar{\psi}\gamma_\nu\psi - \partial_\nu\bar{\psi}\gamma_\mu\psi - \frac{1}{4}(\bar{\gamma}^\alpha\bar{\nabla}^{\{\}}_\nu\bar{\gamma}_\alpha)\bar{\psi}\gamma_\mu\psi\Big] \end{aligned} \tag{5.48}$$

Here, the gamma matrices and other variables are expressed in the basis of null vectors l, n, m and $\bar{m}$. For the generic metric energy-momentum tensor $T_{\mu\nu}$, no further simplification is possible. The expression for $T_{\mu\nu}$ in the NP formalism will however simplify under certain symmetries or specific conditions that the system in question is subjected to. For example, if the background metric is $\eta_{\mu\nu}$, then (for illustration purposes) the T_{12} component of metric EM tensor is given by:

$$\begin{aligned} T^{(\mathrm{NP})}_{12} = \frac{i\hbar c}{4\sqrt{2}}\Big(&i\bar{F}_2(\delta+\delta^*)F_1 - i\bar{F}_1(\delta+\delta^*)F_2 - i\bar{G}_2(\delta+\delta^*)G_1 + i\bar{G}_1(\delta+\delta^*)G_2 \\ &- i\bar{F}_2(\delta-\delta^*)F_1 - i\bar{F}_1(\delta-\delta^*)F_2 + i\bar{G}_2(\delta-\delta^*)G_1 + i\bar{G}_1(\delta-\delta^*)G_2\Big) \\ &- i(\delta+\delta^*)\bar{F}_2F_1 + (\delta+\delta^*)i\bar{F}_1F_2 + (\delta+\delta^*)i\bar{G}_2G_1 - (\delta+\delta^*)i\bar{G}_1G_2 \\ &+ (\delta-\delta^*)i\bar{F}_2F_1 + (\delta-\delta^*)i\bar{F}_1F_2 - (\delta-\delta^*)i\bar{G}_2G_1 - (\delta-\delta^*)i\bar{G}_1G_2\Big) \end{aligned} \tag{5.49}$$

With this prescription, we are able to evaluate all the components of $T_{\mu\nu}$, achieving a particularly simple form in the case of a Minkowskian background metric.

In (??), we also have an additional term in terms of the spin density tensor, given as $\frac{4\pi l_1^2}{\hbar c} g_{\mu\nu} S^{\alpha\beta\lambda} S_{\alpha\beta\lambda}$. Using our expression for the spin density, we can evaluate this term:

$$\frac{4\pi l_1^2}{\hbar c} g_{\mu\nu} S^{\alpha\beta\lambda} S_{\alpha\beta\lambda} = \frac{-\pi l_1^2 \hbar c}{4}\left(\bar{\psi}\gamma^{[\alpha}\gamma^\beta\gamma^{\lambda]}\psi\right)\left(\bar{\psi}\gamma_{[\alpha}\gamma_\beta\gamma_{\lambda]}\psi\right) \tag{5.50}$$

$$= \frac{-\pi l_1^2 \hbar c}{4}\left(\bar{\psi}\gamma^{[(i)}\gamma^{(j)}\gamma^{(k)]}\psi\right)\left(\bar{\psi}\gamma_{[(i)}\gamma_{(j)}\gamma_{(k)]}\psi\right)) \tag{5.51}$$

$$= 6\pi\hbar c l_1^2 g_{\mu\nu}(F_1\bar{G}_1 + F_2\bar{G}_2)(\bar{F}_1G_1 + \bar{F}_2G_2) \tag{5.52}$$

$$= 6\pi\hbar c l_1^2 g_{\mu\nu}\xi\xi^* \tag{5.53}$$

$$= 12\pi\hbar c l_1^2 (l_{(\mu}n_{\nu)} - m_{(\mu}\bar{m}_{\nu)})\xi\xi^* \tag{5.54}$$

i.e., we find it turns out to be proportional to the ξ parameter introduced.

5.3 Summary of important results

- Dirac equation has been modified on U_4 [5.42 - 5.45]
- Contorsion spin coefficients are expressed completely in terms of Dirac Spinor in section (5.2.2).
- Prescription for formulating dynamic EM tensor and Spin density tensor in NP formalism has been presented.

This work is based on the paper titled "The non-relativistic limit of the Einstein-Cartan-Dirac equations" which is `"under preparation"` [17]

Chapter 6

Conjecture: Curvature-Torsion Duality

6.1 Curvature-Torsion duality

In chapter (3), the idea of L_{cs} is introduced. It asserts a symmetry between small mass (m) and large mass (M), which give the same value of L_{cs}. Both the masses enter the ECD equations through the same L_{cs}. The solution to the large mass M (for which mass density and correspondingly the 'curvature' is dominant) is dual to the solution of small mass m (for which spin density and correspondingly the 'torsion' is dominant). Both the solutions are labeled by L_{cs}; since it is the only coupling constant in the theory. Qualitatively, we call this the **'Curvature-Torsion' duality**. We want to establish this duality in the context of ECD system of equations with L_{cs} and make this duality, mathematically more evident.

$$G_{\mu\nu}(\{\}) = \frac{8\pi L_{CS}^2}{\hbar c} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(\frac{8\pi L_{CS}^2}{\hbar c}\right)^2 S_{\alpha\beta\epsilon} S^{\alpha\beta\epsilon} \tag{6.1}$$

$$i\gamma^\mu \psi_{;\mu} = +\frac{3}{8} L_{CS}^2 \bar{\psi}\gamma^5\gamma_\nu \psi \gamma^5 \gamma^\nu \psi + \frac{1}{2L_{CS}}\psi = 0 \tag{6.2}$$

This is the system of equations which we have to understand in details, find possible solutions, put bounds etc. By 'a solution', we mean 3 quantities - (ψ, g, K) where g and K are metric tensor and Contorsion tensor respectively. These quantities are the 3 independent fields in our theory.

We know that affine connection is made up of Christoffel symbols and Contorsion tensor. With this affine connection, we construct The total curvature tensor 'R'. It is composed of two terms R^0 and Q. This notation, we adopt from [32]. It can be written as $R = R^0 + Q$. R^0 is the usual Riemann curvature tensor expressible completely in terms of Christoffel symbols and their derivatives and Q is expressible completely in terms of Contorsion tensor K. The full equation is:

$$R^\alpha{}_{\beta\mu\nu}(\Gamma) = R^\alpha{}_{\beta\mu\nu}(\{\}) + \nabla_\mu^{\{\}} K^\alpha{}_{\nu\beta} - \nabla_\nu^{\{\}} K^\alpha{}_{\mu\beta} + K^\alpha{}_{\mu\rho} K^\rho{}_{\nu\beta} - K^\alpha{}_{\nu\rho} K^\rho{}_{\mu\beta} \tag{6.3}$$

$$R = R^0 + Q - - - - - - - - - - - - - - - \; Symbolic\ equation \tag{6.4}$$

Note that, in the symbolic equation, we have dropped the Indices here. The symbols shouldn't be confused with curvature scalar. Also, Curvature and torsion should be thought of as independent here. Q has information about 'torsion' and R has information about 'curvature'. In a completely torsion dominated theory (e.g. teleparallel gravity), R = 0; $R^0 = -Q$ and in curvature dominated theory (e.g. Einstein's GR), Q=0; $R = R^0$. We know from chapter (3), large masses contribute to gravity which is described by the curvature; determined by levi-civita connection (that is R^0). Torsion is negligible for large masses. Whereas, for small masses, the total curvature is zero.

6.2 Establishing this duality through a conjecture

We know that, for a given L_{cs}, a solution (if it exists) is valid for both LM (large mass M) and its dual SM (small mass, m_q). This leads to an apparent contradiction because 'one solution' which fixes (ψ,g,K) can't physically describe both, SM and LM. It will be physically valid either for LM or SM as we expect the large mass solution to be gravity dominated, and the small mass solution to be torsion dominated. This is possible only if for a given L_{cs}, there are two solutions, one that is curvature dominated, and another that is torsion dominated. To account this, we propose the following conjecture: Assuming that a solution exists for a given L_{cs}, we call it solution (1) [S1]; characterized by three curvature parameters [R(1),R^0(1),Q(1)]. It is governed by equation $R_{(1)} = R^0_{(1)} + Q_{(1)}$. Without loss of generality, we assume it to be curvature dominated. Conjecture is that, given a solution(1), there exists a solution(2) [S2] by construction; characterized by curvature parameters [R(2),R^0(2),Q(2)] and governed by $R_{(2)} = R^0_{(2)} + Q_{(2)}$; such that

$$R_{(2)} - Q_{(2)} = -[R_{(1)} - Q_{(1)}] \Rightarrow R^0_{(2)} = -R^0_{(1)} \tag{6.5}$$

This conjecture forces solution(2) to be torsion dominated. The properties of solution(1) and solution(2) are summarized in the table below. In the large mass limit, Q(1) is zero and we have the pure curvature solution R(1)= R^0(1) (This is general relativity). In the small mass limit, R(2) is zero, and we have the solution Q(2) = - R^0(2) (This is teleparallel gravity). Duality map implies that R(1) = Q(2). These ideas are discussed in details in [18].

Solution	**Governing eqn**	**Valid for**	**Dominated by**	**Physical for**
Solution(1) →	$R_{(1)} = R^0_{(1)} + Q_{(1)}$	M and m	1) **Curvature**. 2) $R_{(1)} = R^0_{(1)}$	Large mass (M)
Solution(2) →	$R_{(2)} = R^0_{(2)} + Q_{(2)}$	M and m	1) **Torsion**. 2) $R^0_{(2)} = -Q_{(2)}$	Small mass (m)

This conjecture automatically provides a natural duality between curvature and torsion for Large mass and small mass respectively. In terms of above vocabulary, we summarize the curvature-torsion duality in figure below (6.2).

Here, we have plotted "R-Q" Vs. $z = ln\left[\frac{m}{m_{pl}}\right]$. 'M' and 'm' have same L_{cs}. For M, R^0 (or equivalently 'R-Q') is positive and dominates as mass goes high. It is shown as "solution 1"

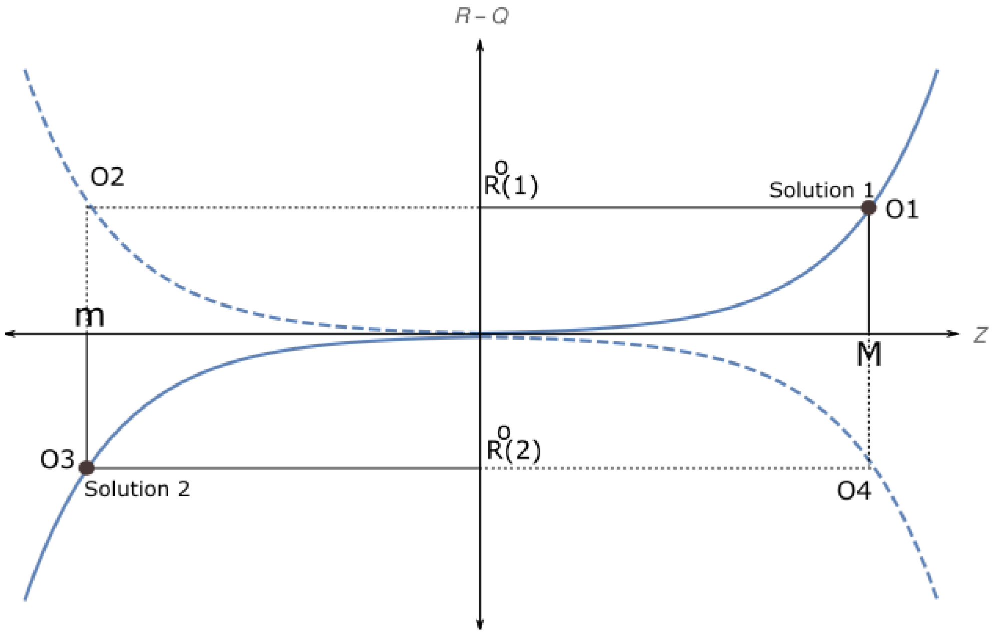


Figure 6.1: Curvature-Torsion duality

[S1] in the first quadrant. In the limit of very high masses, curvature is fully given by Riemann curvature tensor R^0. For small mass m, R^0 (or equivalently 'R-Q') is negative and goes on becoming more negative as as mass goes further low. As mass tends to zero, the total curvature also tends to zero and torsion balances Reimann curvature tensor R^0. Its solution is solution-2 [S2] in third quadrant. At $m = m_{pl}$, we have R-Q $= 0$ or $R^0 = 0$; where the total curvature is sourced only by torsion. There also exists a unphysical "mirror universe" in which torsion is sourced by torsion and curvature by large masses. It is shown by dotted graph which rolls down from second quadrant to fourth quadrant.

6.3 Attempting Solution(s) for this conjecture to support the curvature-torsion duality

We proposed the Curvature-torsion duality conjecture in the previous section. At $m = m_{pl}$, R-Q $= 0$ or $R^0 = 0$. One of the allowed solution to this is on Minkowski space with torsion. So, we next attempt to find the solutions to ECD equations on Minkowski space with torsion and test the duality conjecture. We also propose a 'test' which can make our claims falsifiable. First we establish the ingredients of ECD equations on Minkowski space with torsion in this section

6.3.1 Dirac equation (Hehl-Datta Equation) on Minkowski Space with Torsion

Dirac equation on U_4; called as Hehl datta (HD) equations are written explicitly in equations [5.42 - 5.45]. On Minkowski space with torsion, they are as follows (In NP formalism):

$$DF_1 + \delta^* F_2 = i[b(l_2) + a(l_1)\xi]G_1 \tag{6.6}$$
$$\Delta F_2 + \delta F_1 = i[b(l_2) + a(l_1)\xi]G_2 \tag{6.7}$$
$$DG_2 - \delta G_1 = i[b(l_2) + a(l_1)\xi^*]F_2 \tag{6.8}$$
$$\Delta G_1 - \delta^* G_2 = i[b(l_2) + a(l_1)\xi^*]F_1 \tag{6.9}$$

In a Cartesian coordinate system (ct, x, y, z)[1] we have:

$$(\partial_0 + \partial_3)F_1 + (\partial_1 + i\partial_2)F_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_1 \tag{6.10}$$
$$(\partial_0 - \partial_3)F_2 + (\partial_1 - i\partial_2)F_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_2 \tag{6.11}$$
$$(\partial_0 + \partial_3)G_2 - (\partial_1 - i\partial_2)G_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_2 \tag{6.12}$$
$$(\partial_0 - \partial_3)G_1 - (\partial_1 + i\partial_2)G_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_1 \tag{6.13}$$

In cylindrical polar coordinates (ct, r, ϕ, z), we have:

$$r\partial_t F_1 + e^{i\phi} r\partial_r F_2 + ie^{i\phi}\partial_\phi F_2 + r\partial_z F_1 = ir\sqrt{2}[b(l_2) + a(l_1)\xi]G_1 \tag{6.14}$$
$$r\partial_t F_2 + e^{-i\phi} r\partial_r F_1 - ie^{-i\phi}\partial_\phi F_1 - r\partial_z F_2 = ir\sqrt{2}[b(l_2) + a(l_1)\xi]G_2 \tag{6.15}$$
$$r\partial_t G_2 - e^{-i\phi} r\partial_r G_1 + ie^{-i\phi}\partial_\phi G_1 + cr\partial_z G_2 = ir\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_2 \tag{6.16}$$
$$r\partial_t G_1 - e^{i\phi} r\partial_r G_2 - ie^{i\phi}\partial_\phi G_2 - r\partial_z G_1 = ir\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_1 \tag{6.17}$$

Likewise, in spherical polar coordinates (ct, r, θ, ϕ):

$$\partial_t F_1 + \cos\theta\partial_r F_1 - \frac{\sin\theta}{r}\partial_\theta F_1 + \frac{ie^{i\phi}}{r\sin\theta}\partial_\phi F_2 + e^{i\phi}\sin\theta\partial_r F_2 + \frac{e^{i\phi}\cos\theta}{r}\partial_\theta F_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_1 \tag{6.18}$$

$$\partial_t F_2 - \cos\theta\partial_r F_2 - \frac{\sin\theta}{r}\partial_\theta F_2 + \frac{ie^{-i\phi}}{r\sin\theta}\partial_\phi F_1 + e^{-i\phi}\sin\theta\partial_r F_1 - \frac{e^{-i\phi}\cos\theta}{r}\partial_\theta F_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_2 \tag{6.19}$$

$$\partial_t G_2 + \cos\theta\partial_r G_2 - \frac{\sin\theta}{r}\partial_\theta G_2 - \frac{ie^{-i\phi}}{r\sin\theta}\partial_\phi G_1 - e^{-i\phi}\sin\theta\partial_r G_1 + \frac{e^{-i\phi}\cos\theta}{r}\partial_\theta G_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_2 \tag{6.20}$$

$$\partial_t G_1 - \cos\theta\partial_r G_1 - \frac{\sin\theta}{r}\partial_\theta G_1 - \frac{ie^{i\phi}}{r\sin\theta}\partial_\phi G_2 - e^{i\phi}\sin\theta\partial_r G_2 - \frac{e^{i\phi}\cos\theta}{r}\partial_\theta G_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_1 \tag{6.21}$$

6.3.2 The Dynamical EM tensor ($T_{\mu\nu}$) on Minkowski space with torsion

The dynamical EM tensor given in equation (2.57). On Minkowski space, it assumes the following form:

$$T_{\mu\nu} = \Sigma_{(\mu\nu)}(\{\}) = \frac{i\hbar c}{4}\Big[\bar{\psi}\gamma_\mu\partial_\nu\psi + \bar{\psi}\gamma_\nu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\nu\psi - \partial_\nu\bar{\psi}\gamma_\mu\psi\Big] \tag{6.22}$$

[1]Setting $c = 1$ by convention

Its 10 components are given by following 10 equations:

$$T_{21} = \frac{i\hbar c}{4}\Big(\bar{F}_1\partial_1 F_1 + \bar{F}_2\partial_1 F_2 + \bar{G}_1\partial_1 G_1 + \bar{G}_2\partial_1 G_2 - \bar{F}_2\partial_0 F_1 - \bar{F}_1\partial_0 F_2 + \bar{G}_2\partial_0 G_1 + \bar{G}_1\partial_0 G_2$$
$$- \partial_1\bar{F}_1 F_1 - \partial_1\bar{F}_2 F_2 - \partial_1\bar{G}_1 G_1 - \partial_1\bar{G}_2 G_2 + \partial_0\bar{F}_2 F_1 + \partial_0\bar{F}_1 F_2 - \partial_0\bar{G}_2 G_1 - \partial_0\bar{G}_1 G_2\Big) \tag{6.23}$$

$$T_{31} = \frac{i\hbar c}{4}\Big(\bar{F}_1\partial_2 F_1 + \bar{F}_2\partial_2 F_2 + \bar{G}_1\partial_2 G_1 + \bar{G}_2\partial_2 G_2 + i\bar{F}_2\partial_0 F_1 - i\bar{F}_1\partial_0 F_2 - i\bar{G}_2\partial_0 G_1 + i\bar{G}_1\partial_0 G_2$$
$$- \partial_2\bar{F}_1 F_1 - \partial_2\bar{F}_2 F_2 - G_1\partial_2\bar{G}_1 - \partial_2\bar{G}_2 G_2 - i\partial_0\bar{F}_2 F_1 + i\partial_0\bar{F}_1 F_2 + i\partial_0\bar{G}_2 G_1 - i\partial_0\bar{G}_1 G_2\Big) \tag{6.24}$$

$$T_{41} = \frac{i\hbar c}{4}\Big(\bar{F}_1\partial_3 F_1 + \bar{F}_2\partial_3 F_2 + \bar{G}_1\partial_3 G_1 + \bar{G}_2\partial_3 G_2 - \bar{F}_1\partial_0 F_1 + \bar{F}_2\partial_0 F_2 + \bar{G}_1\partial_0 G_1 - \bar{G}_2\partial_0 G_2$$
$$- \partial_3\bar{F}_1 F_1 - \partial_3\bar{F}_2 F_2 - \partial_3\bar{G}_1 G_1 - \partial_3\bar{G}_2 G_2 + \partial_0\bar{F}_1 F_1 - \partial_0\bar{F}_2 F_2 - \partial_0\bar{G}_1 G_1 + \partial_0\bar{G}_2 G_2\Big) \tag{6.25}$$

$$T_{32} = \frac{i\hbar c}{4}\Big(i\bar{F}_2\partial_1 F_1 - i\bar{F}_1\partial_1 F_2 - i\bar{G}_2\partial_1 G_1 + i\bar{G}_1\partial_1 G_2 - \bar{F}_2\partial_2 F_1 - \bar{F}_1\partial_2 F_2 + \bar{G}_2\partial_2 G_1 + \bar{G}_1\partial_2 G_2$$
$$- i\partial_1\bar{F}_2 F_1 + i\partial_1\bar{F}_1 F_2 + i\partial_1\bar{G}_2 G_1 - i\partial_1\bar{G}_1 G_2 + \partial_2\bar{F}_2 F_1 + \partial_2\bar{F}_1 F_2 - \partial_2\bar{G}_2 G_1 - \partial_2\bar{G}_1 G_2\Big) \tag{6.26}$$

$$T_{42} = \frac{i\hbar c}{4}\Big(- \bar{F}_1\partial_1 F_1 + \bar{F}_2\partial_1 F_2 + \bar{G}_1\partial_1 G_1 - \bar{G}_2\partial_1 G_2 - \bar{F}_2\partial_3 F_1 - \bar{F}_1\partial_3 F_2 + \bar{G}_2\partial_3 G_1 + \bar{G}_1\partial_3 G_2$$
$$+ \partial_1\bar{F}_1 F_1 - \partial_1\bar{F}_2 F_2 - \partial_1\bar{G}_1 G_1 + \partial_1\bar{G}_2 G_2 + \partial_3\bar{F}_2 F_1 + \partial_3\bar{F}_1 F_2 - \partial_3\bar{G}_2 G_1 - \partial_3\bar{G}_1 G_2\Big) \tag{6.27}$$

$$T_{43} = \frac{i\hbar c}{4}\Big(- \bar{F}_1\partial_2 F_1 + \bar{F}_2\partial_2 F_2 + \bar{G}_1\partial_2 G_1 - \bar{G}_2\partial_2 G_2 + i\bar{F}_2\partial_3 F_1 - i\bar{F}_1\partial_3 F_2 - i\bar{G}_2\partial_3 G_1 + i\bar{G}_1\partial_3 G_2$$
$$+ \partial_2\bar{F}_1 F_1 - \partial_2\bar{F}_2 F_2 - \partial_2\bar{G}_1 G_1 + \partial_2\bar{G}_2 G_2 - i\partial_3\bar{F}_2 F_1 + i\partial_3\bar{F}_1 F_2 + i\partial_3\bar{G}_2 G_1 - i\partial_3\bar{G}_1 G_2\Big) \tag{6.28}$$

$$T_{11} = \frac{i\hbar c}{2}\Big(\bar{G}_1\partial_0 G_1 + \bar{G}_2\partial_0 G_2 - \partial_0\bar{G}_1 G_1 - \partial_0\bar{G}_2 G_2 + \bar{F}_1\partial_0 F_1 + \bar{F}_2\partial_0 F_2 - \partial_0\bar{F}_1 F_1 - \partial_0\bar{F}_2 F_2\Big) \tag{6.29}$$

$$T_{22} = \frac{i\hbar c}{2}\Big(- \bar{F}_2\partial_1 F_1 - \bar{F}_1\partial_1 F_2 + \bar{G}_2\partial_1 G_1 + \bar{G}_1\partial_1 G_2 + \partial_1\bar{F}_2 F_1 + \partial_1\bar{F}_1 F_2 - \partial_1\bar{G}_2 G_1 - \partial_1\bar{G}_1 G_2\Big) \tag{6.30}$$

$$T_{33} = \frac{i\hbar c}{2}\Big(i\bar{F}_2\partial_2 F_1 - i\bar{F}_1\partial_2 F_2 - i\bar{G}_2\partial_2 G_1 + i\bar{G}_1\partial_2 G_2 - i\partial_2\bar{F}_2 F_1 + i\partial_2\bar{F}_1 F_2 + i\partial_2\bar{G}_2 G_1 - i\partial_2\bar{G}_1 G_2\Big) \tag{6.31}$$

$$T_{44} = \frac{i\hbar c}{2}\Big(- \bar{F}_1\partial_3 F_1 + \bar{F}_2\partial_3 F_2 + \bar{G}_1\partial_3 G_1 - \bar{G}_2\partial_3 G_2 + \partial_3\bar{F}_1 F_1 - \partial_3\bar{F}_2 F_2 - \partial_3\bar{G}_1 G_1 + \partial_3\bar{G}_2 G_2\Big) \tag{6.32}$$

6.3.3 Calculation of the Spin density part which acts as a correction to T_{ij}

The second term on RHS of equation (A.11) on Minkowski space is given as $\frac{4\pi(L_{cs})^2}{\hbar c}\eta_{ij}S^{abc}S_{abc}$ which can be written as

$$\frac{4\pi l^2}{\hbar c}\eta_{\mu\nu}S^{\alpha\beta\gamma}S_{\alpha\beta\gamma} = 6\pi\hbar c l^2\eta_{\mu\nu}(F_1\bar{G}_1 + F_2\bar{G}_2)(\bar{F}_1 G_1 + \bar{F}_2 G_2) = 6\pi\hbar c l^2 g_{\mu\nu}\xi\xi^* \tag{6.33}$$

6.4 Solutions to HD equation on M_4 with torsion and testing duality conjecture

6.4.1 Attempting a non-static solution by working in 1+1 dimensions

In the following analysis, we will assume an ansatz of the form $F_1 = G_2$ and $F_2 = G_1$, and further assume that the Dirac states are a function of only t and z. The four equations – in Cartesian (6.10) - (6.13) as well as cylindrical polar coordinates (6.14) - (6.17)) – reduce to the following two independent equations[2]

$$\begin{aligned}
&\partial_t\psi_1 + \partial_z\psi_2 - i\sqrt{2}b\psi_1 + \frac{ia}{\sqrt{2}}(|\psi_2|^2 - |\psi_1|^2)\psi_1 = 0\\
&\partial_t\psi_2 + \partial_z\psi_1 + i\sqrt{2}b\psi_2 + \frac{ia}{\sqrt{2}}(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0
\end{aligned} \tag{6.34}$$

where $\psi_1 = F_1 + F_2$ and $\psi_2 = F_1 - F_2$. Writing $\sqrt{2}b = -m$ and $a = 2\sqrt{2}\lambda$, we have:

$$\begin{aligned}
&\partial_t\psi_1 + \partial_z\psi_2 + im\psi_1 + 2i\lambda(|\psi_2|^2 - |\psi_1|^2)\psi_1 = 0\\
&\partial_t\psi_2 + \partial_z\psi_1 - im\psi_2 + 2i\lambda(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0
\end{aligned} \tag{6.35}$$

[2]We note that $\xi = 2Re(F_1\bar{F}_2)$, thus $\xi = \xi^*$. Furthermore, a and b are henceforth shorthand for $a(l)$ and $b(l)$.

These equations are identical to those studied in [48], which investigates the convergence and stability of the difference scheme for the non-linear Dirac equation in 1 + 1 dimensions. Proceeding as in [48], we use the following solitary wave ansatz:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A(z) \\ iB(z) \end{pmatrix} e^{-i\Lambda t} \tag{6.36}$$

where $A(z)$ and $B(z)$ are real functions. Substituting in, we have:

$$\begin{aligned} B' - (\sqrt{2}b + \Lambda)A - \frac{a}{\sqrt{2}}(A^2 - B^2)A = 0 \\ A' - (\sqrt{2}b - \Lambda)B - \frac{a}{\sqrt{2}}(A^2 - B^2)B = 0 \end{aligned} \tag{6.37}$$

which admits the following solutions:

$$A(z) = \frac{-i2^{3/4}(\sqrt{2}b - \Lambda)}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b + \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} \tag{6.38}$$

$$B(z) = \frac{-i2^{3/4}(\sqrt{2}b + \Lambda)}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b - \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} \tag{6.39}$$

It can be seen upon the substitutions $\lambda = 0.5$ (equivalently $a = \sqrt{2}$) and $m = 1$ (equivalently $m_0 = -1$), that this is a generalisation of the equations for $A(z)$ and $B(z)$ in [48](see section III). A similar solution is found in [39], with $a(l_1) = a(L_{pl})$ and $b(l_2) = b(\lambda_c)$.

In terms of the spinor components:

$$F_1 = G_2 = \frac{\sqrt{(2b^2 - \Lambda^2)}}{2}\left[\frac{-i2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b - \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} + \frac{2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b + \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}\right] e^{-i\Lambda t} \tag{6.40}$$

$$F_2 = G_1 = \frac{\sqrt{(2b^2 - \Lambda^2)}}{2}\left[\frac{-i2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b - \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} - \frac{2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b + \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}\right] e^{-i\Lambda t} \tag{6.41}$$

and the parameter ξ characterising torsion takes the form:

$$\xi = \frac{-2\sqrt{2}(2b^2 - \Lambda^2)(\sqrt{2}b - \Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2})}{a[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]^2} \tag{6.42}$$

The probability density is given by the zeroth component of the four-vector fermion current $j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = 2\Big(|F_1|^2 + |F_2|^2\Big) = \Big(|A|^2 + |B|^2\Big)$ For the subsequent analysis, we define the following dimensionless variables:

$$\begin{aligned} p &= \sqrt{2}bz \\ w &= -\frac{\Lambda}{\sqrt{2}b} \\ \tilde{A}(p) &= \frac{\sqrt{a}}{2\sqrt{b}}A(z) \\ \tilde{B}(p) &= \frac{\sqrt{a}}{2\sqrt{b}}B(z) \\ \tilde{j}^0 &= \frac{a}{4b}j^0 = 0 \end{aligned} \tag{6.43}$$

With these definitions, we have $[p] = [w] = [\tilde{A}(p)] = [\tilde{B}(p)] = [\tilde{j}^0] = 0$; i.e., all these quantities are now dimensionless. Scaled thusly, $A(p)$ and $B(p)$ take the form:

$$A(p) = \frac{2i(1+w)}{\sqrt{a}} \frac{\sqrt{b(1-w)}\cosh(p\sqrt{1-w^2})}{(w\cosh(2p\sqrt{1-w^2})+1)} \tag{6.44}$$

$$B(p) = \frac{2i(1-w)}{\sqrt{a}} \frac{\sqrt{b(1+w)}\sinh(p\sqrt{1-w^2})}{(w\cosh(2p\sqrt{1-w^2})+1)} \tag{6.45}$$

There are six unique cases (corresponding to values of w) which give different solutions. In each case, we will consider torsion-less limit (the linear Dirac equation) in order to compare and contrast the behaviour. The equations and plots for the linear case can be found in Appendix D.

Case I: $w \in (-\infty, -1)$
The equations reduce to:

$$\tilde{A}(p) = i(1+w)\frac{\sqrt{(|w|+1)}\cos(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))} \tag{6.46}$$

$$\tilde{B}(p) = i(w-1)\frac{\sqrt{(|w|-1)}\sin(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))} \tag{6.47}$$

$$\tilde{j}^0 = \left[\frac{(w+1)^2(|w|+1)\cos^2(p\sqrt{w^2-1}) + (w-1)^2(|w|-1)\sin^2(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))^2}\right] \tag{6.48}$$

(6.49)

Comments: This solution has an infinite number of singularities placed periodically at non-zero values of p, and is clearly unphysical. An example of this case (with $w = -2$) can be seen in the left column of Fig. 6.3.
Comparison with torsionless case: For $w \in (-\infty, -1)$, the linear Dirac equation gives plane waves solutions, which are physically meaningful, and the probability density fluctuates sinusoidally. It is the addition of torsion that makes this case unphysical. A plot has been made (for $w = -2$) in Fig. C.1.

Case II: $w = \pm 1$ (trivial case)
The equations reduce to:

$$\tilde{A}(p) = 0 \qquad \tilde{B}(p) = 0 \qquad \tilde{j}^0 = 0 \tag{6.50}$$

Case III: $w \in (-1, 0)$
The equations reduce to:

$$\tilde{A}(p) = i(1+w)\frac{\sqrt{(1+|w|)}\cosh(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2}))} \tag{6.51}$$

$$\tilde{B}(p) = i(1-w)\frac{\sqrt{(1-|w|)}\sinh(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2}))} \tag{6.52}$$

$$\tilde{j}^0 = \left[\frac{(w+1)^2(|w|+1)\cosh^2(p\sqrt{1-w^2}) + (1-w)^2(1-|w|)\sinh^2(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2}))^2}\right] \tag{6.53}$$

Comments: This solution has two singularities placed symmetrically around the origin at two finite (non-zero) values of p. In the infinite limit, it decays to zero. However, owing to the presence of singularities, we may still conider it an unphysical solution. An example (with $w = -0.5$) can be seen in the left column of Figure. 6.4
Comparison with torsionless case: For $w \in (-1, 0)$ the linear Dirac equation has unphysical solutions. The solutions grow exponentially to infinity as $p \to \pm\infty$. For $w = -0.5$, this solution is plotted in Fig. C.1. As can be seen, for this case, both the linear (torsionless) and non-linear (with torsion) Dirac equations give unphysical solutions.

<u>Case IV</u>: $w = 0$
The equations reduce to:

$$\tilde{A}(p) = i\cosh(p) \tag{6.54}$$

$$\tilde{B}(p) = i\sinh(p) \tag{6.55}$$

$$\tilde{j}^0 = [\cosh^2(p) + \sinh^2(p)] \tag{6.56}$$

Comments: This solution blows up exponentially as $p \to \pm\infty$. Thus, it is clearly unphysical. This case (with $w = 0$ has been plotted in the right column of Fig. 6.4
Comparison with torsionless case: For $w = 0$, the linear Dirac equation is unphysical. The solutions exponentially increase to infinity as $p \to +\infty$. A plot of the solutions (for $w = 0$) is available in Fig. C.1. Thus, for this case, both the linear and non-linear Dirac equations give unphysical solutions.

<u>Case V</u>: $w \in (0, 1)$
The equations reduce to:

$$\tilde{A}(p) = i(1+w)\frac{\sqrt{(1-w)}\cosh(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))} \tag{6.57}$$

$$\tilde{B}(p) = i(1-w)\frac{\sqrt{(1+w)}\sinh(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))} \tag{6.58}$$

$$\tilde{j}^0 = \left[\frac{(1+w)^2(1-w)\cosh^2(p\sqrt{1-w^2}) + (1-w)^2(1+w)\sinh^2(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))^2}\right] \tag{6.59}$$

Comments: In this case, we have no singularities anywhere. All the functions (including the probability density) asymptotically vanish. Therefore, this case represents a physically viable solution. Depending on the exact nature of solution, we can consider two sub-cases: (a) with $w \in (0, \frac{1}{3})$ and (b) with $w \in [\frac{1}{3}, 1)$.
We see that (a) has a local minimum at the origin and two global maxima symmetric around the origin at non-zero p. A plot is provided in Fig. 6.2 (in blue). On the other hand, (b) has global maximum at the origin and monotonically decays to zero at infinity. Two examples of this can be seen in Fig. 6.2 (in orange and green). The solution for case (b) resembles a 'blob'; further analysis of this can be found in the discussion.
Comparison with torsionless case: For $w \in (0, 1)$ the linear Dirac equation gives unphysical

solutions. The solutions increase exponentially to infinity as $p \to \pm\infty$. A plot of this solution (with $w = 0.5$) can be seen in Fig. C.1. The addition of torsion, as seen, makes the solutions physically meaningful.

Case VI: $w \in (1, \infty)$

The equations reduce to:

$$\tilde{A}(p) = -(1+w)\frac{\sqrt{(w-1)}\cos(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1})} \tag{6.60}$$

$$\tilde{B}(p) = -(1-w)\frac{\sqrt{(w+1)}\sin(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1})} \tag{6.61}$$

$$\tilde{j}^0 = \left[\frac{(1+w)^2(w-1)\cos^2(p\sqrt{w^2-1}) + (1-w)^2(w+1) + \sin^2(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1}))^2}\right] \tag{6.62}$$

Comments: This solution has an infinite number of singularities placed periodically over non-zero values of p, and is thus clearly unphysical. A plot (with $w = 2$) is given in the left column of Fig. 6.3

Comparison with torsionless case: For $w \in (1, \infty)$ the linear Dirac equation gives (physically meaningful) plane waves solutions. The probability density fluctuates sinusoidally. The addition of torsion makes this solution ultimately unphysical. A plot (with $w = 2$) is available in Fig. C.1.

The following table summarises the various cases:

Cases	**Solution(s) of the linear Dirac equation**	**Solution(s) of the Dirac equation with torsion**
Case I	Physical (Plane wave)	Unphysical (infinite singularities)
Case II	Trivial solution	Trivial solution
Case III	Unphysical (blows up exponentially at infinity)	Unphysical, (two singularities)
Case IV	Unphysical (blows up exponentially at infinity)	Unphysical (blows up exponentially at infinity)
Case V	Unphysical (blows up exponentially at infinity)	Physical (No singularity)
Case VI	Physical (Plane wave)	Unphysical (infinite singularities)

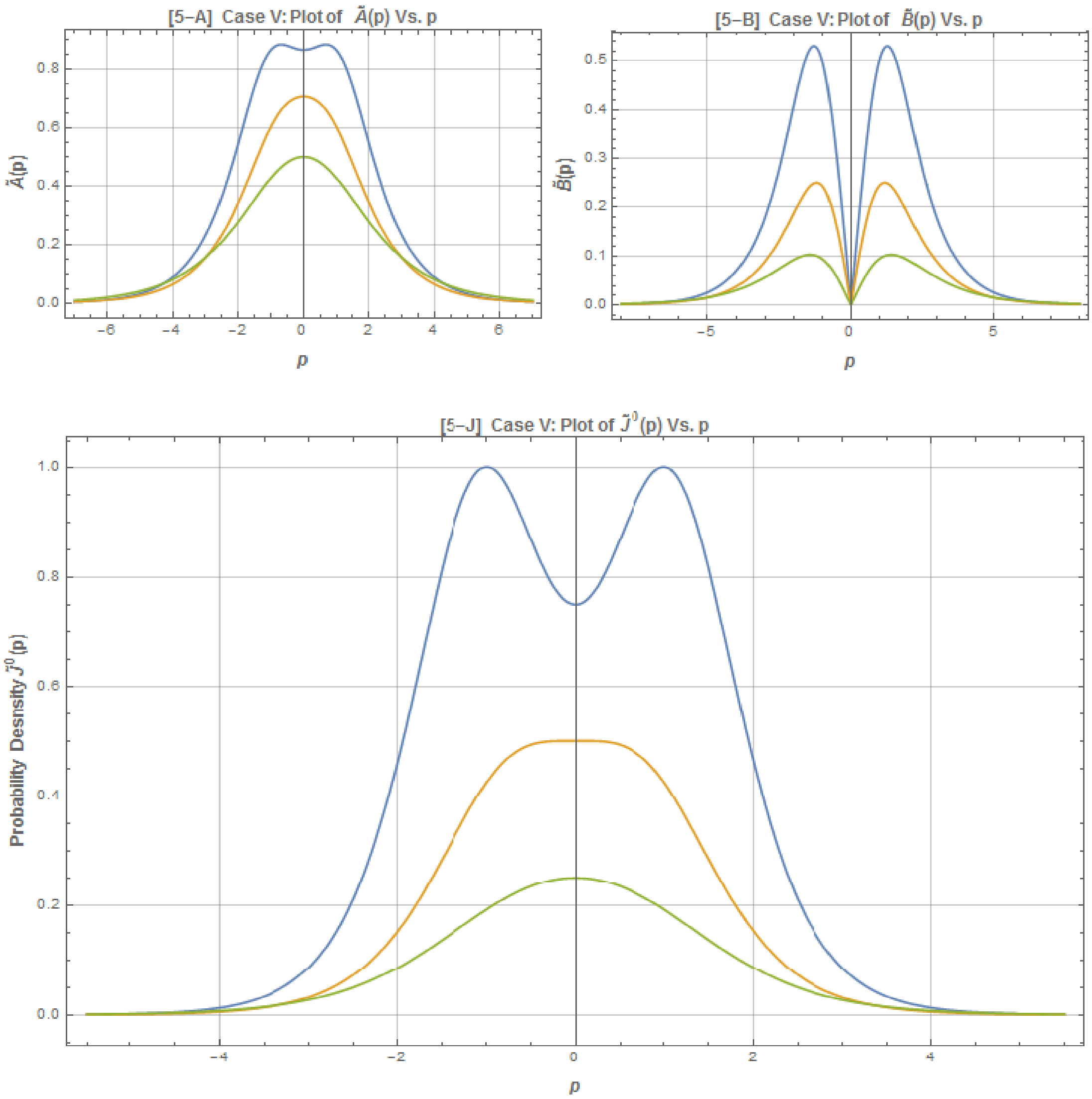


Figure 6.2: **Case V**. In all plots: *Green*: Case V(a) with w=0.75], *Orange*: Case V(a) with $w = 0.5$, *Blue*: Case V(b) with $w = 0.25$. Case V(a) has global maxima at origin. Case V(b) has local minima at origin and two maximas at two symmetrically opposite sides of origin at *non-zero* p. Both cases V(a) and V(b) are asymptotically vanishing.

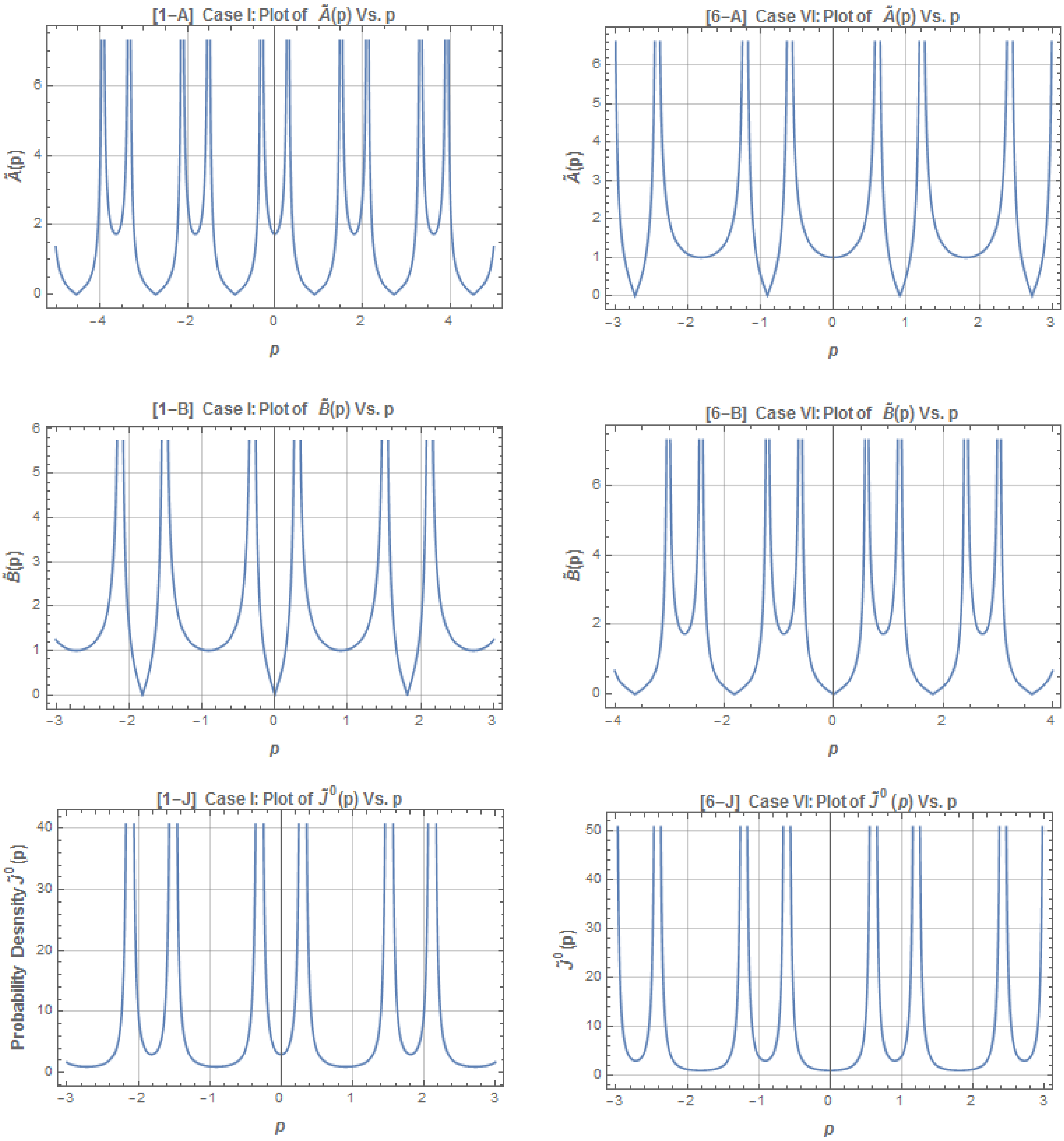


Figure 6.3: **Case I and Case VI**. The left column shows plots for Case 1 with $w = -2$. The right column shows plots for Case 6 with $w = +2$. Both the cases have unphysical solutions.

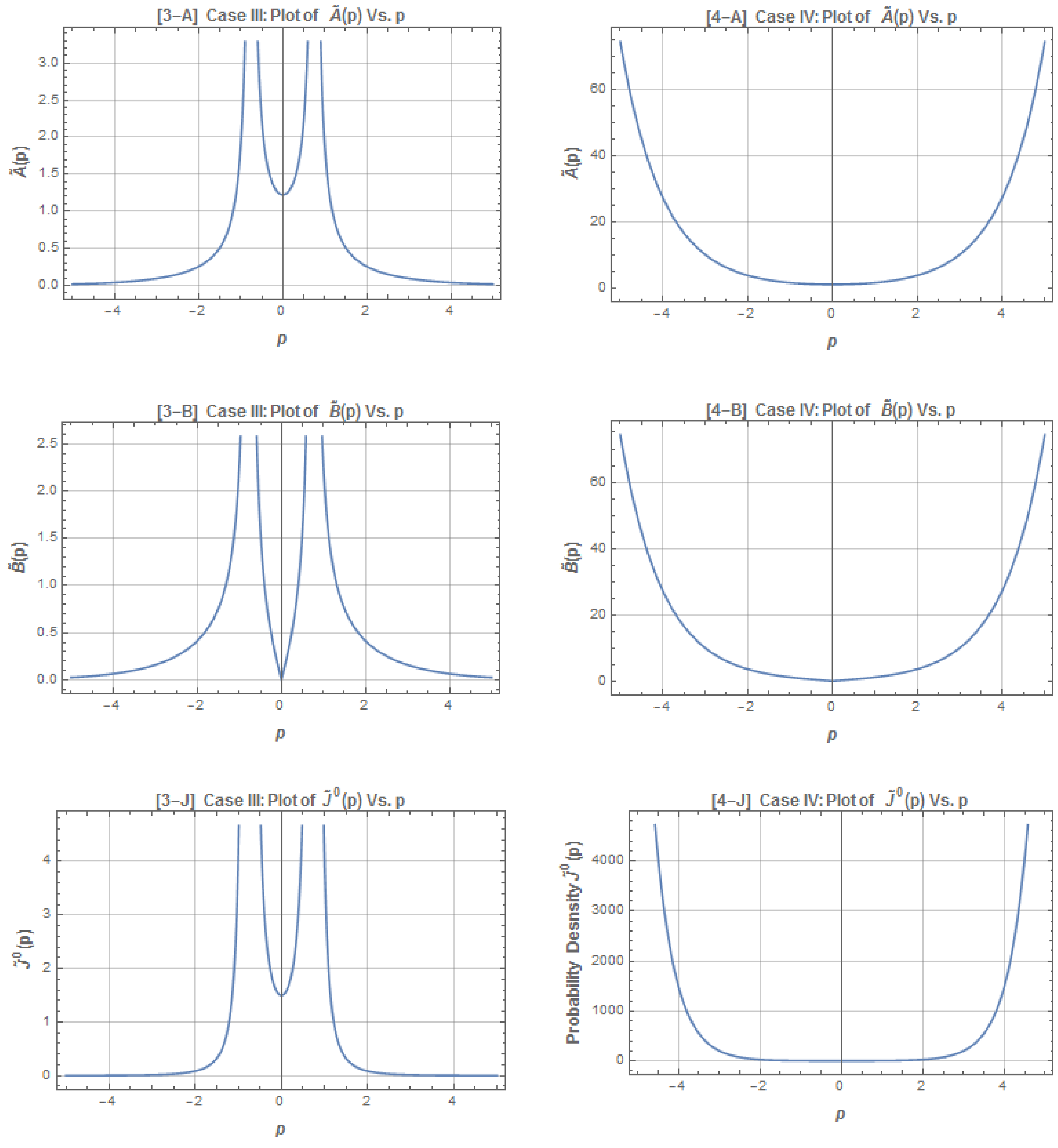


Figure 6.4: **Case III and Case IV**. Case III on the left, with $w = -0.5$. Case IV on the right, with $w = 0$. Both the cases have unphysical solutions.

$(T-S)_{\mu\nu}$ **for non-static solutions in** $1+1$ **dim** (t,z)

$$(T-S)_{\mu\nu} = \hbar c \begin{bmatrix} \left(\Lambda[A^2+B^2] - \frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 & -\Lambda AB & 0 \\ 0 & \left(\frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 & 0 \\ -\Lambda AB & 0 & \left(\frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 \\ 0 & 0 & 0 & \left([AB'-BA'] + \frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) \end{bmatrix} \tag{6.63}$$

Λ is a free parameter in the solution.

6.4.2 Attempting plane wave solutions

We begin by substituting the following plane wave ansatz in (6.10 - 6.13) as follows:

$$\begin{bmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{bmatrix} e^{ik.x} \tag{6.64}$$

With this ansatz, ξ and ξ^* are as follows:

$$\xi = u^A \bar{v}_{A'} \tag{6.65}$$

$$\xi^* = \bar{u}^{A'} v_A \tag{6.66}$$

We assume ξ to be a real constant such that

$$\xi = \xi^* = \frac{\pm 1}{12\pi l_0^3}; \quad \text{For some constant length } l_0 \tag{6.67}$$

Putting the above ansatz in (6.10 - 6.13), we obtain (with $\mu(\xi) = \sqrt{2}[b(l_2) + a(l_1)\xi]$):

$$(k_0+k_3)u^0 + (k_1+ik_2)u^1 - \mu(\xi)\bar{v}_{0'} = 0 \tag{6.68}$$

$$(k_0-k_3)u^1 + (k_1-ik_2)u^0 - \mu(\xi)\bar{v}_{1'} = 0 \tag{6.69}$$

$$(k_0+k_3)\bar{v}_{1'} - (k_1-ik_2)\bar{v}_{0'} - \mu(\xi)u^1 = 0 \tag{6.70}$$

$$(k_0-k_3)\bar{v}_{0'} - (k_1+ik_2)\bar{v}_{1'} - \mu(\xi)u^0 = 0 \tag{6.71}$$

Note that μ here is a function of ξ, which remains a undetermined quantity until a complete solution is obtained. Ensuring ξ is a real constant however restricts us to a small and rather specific class of solutions.

We can write the equations in matrix form:

$$\begin{pmatrix} (k_0+k_3) & (k_1+ik_2) & -\mu(\xi) & 0 \\ (k_1-ik_2) & (k_0-k_3) & 0 & -\mu(\xi) \\ 0 & -\mu(\xi) & -(k_1-ik_2) & (k_0+k_3) \\ -\mu(\xi) & 0 & (k_0-k_3) & -(k_1+ik_2) \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6.72}$$

The first assumption we make is that $k_1 = k_2 = k_3 = 0$; this is equivalent to working in a rest frame. The matrix equation then reduces to:

$$\begin{pmatrix} k_0 & 0 & -\mu(\xi) & 0 \\ 0 & k_0 & 0 & -\mu(\xi) \\ 0 & -\mu(\xi) & 0 & k_0 \\ -\mu(\xi) & 0 & k_0 & 0 \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6.73}$$

For a solution to this system, we need a null determinant. In other words,

$$(k_0^2 - \mu(\xi)^2)^2 = 0 \Rightarrow k_0 = \pm\mu(\xi) \tag{6.74}$$

By considering $Sign(\xi)$ and $Sign(\mu(\xi))$, we have four broad cases:

Case I: $\xi = \frac{+1}{12\pi l_0^3}$ and $k_0 = +\mu(\xi)$

In this case, the most general solution is given by

$$\psi_{(\mathrm{I})} = \frac{1}{\sqrt{12\pi l_0^3}} \begin{pmatrix} \phi \\ \phi \end{pmatrix} e^{+i|\mu(\xi)|x_0} \tag{6.75}$$

where $\phi = \frac{\alpha_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\beta_1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ such that $|\alpha_1|^2 + |\beta_1|^2 = 1$. For all finite l_0, only positive frequency solutions exist. We can explicitly write:

$$k_0 = +\mu(\xi) = \left(\frac{l_0^3 + l_1^2 l_2}{2 l_0^3 l_2} \right) \tag{6.76}$$

Case II: $\xi = \frac{+1}{12\pi l_0^3}$ and $k_0 = -\mu(\xi)$

The general solution is:

$$\psi_{(\mathrm{II})} = \frac{1}{\sqrt{12\pi l_0^3}} \begin{pmatrix} \phi \\ \phi \end{pmatrix} e^{-i|\mu(\xi)|x_0} \tag{6.77}$$

with $\phi = \frac{\alpha_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\beta_1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ such that $|\alpha_1|^2 + |\beta_1|^2 = 1$. For all finite l_0, only negative frequency solutions exist. We can explicitly write:

$$k_0 = -\mu(\xi) = -\left(\frac{l_0^3 + l_1^2 l_2}{2 l_0^3 l_2} \right) \tag{6.78}$$

Case III: $\xi = \frac{-1}{12\pi l_0^3}$ and $k_0 = +\mu(\xi)$

We will first evaluate k_0, given as:

$$k_0 = +\mu(\xi) = \left(\frac{l_0^3 - l_1^2 l_2}{2 l_0^3 l_2} \right) \tag{6.79}$$

For $l_0^3 > l_1^2 l_2$ (case III(a)), we have a positive frequency solution, whereas for $l_0^3 < l_1^2 l_2$ (case III(b)), we have negative frequency solution. The generic wavefunctions can be written as:

$$\psi_{\mathrm{III}(a)} = \frac{1}{\sqrt{12\pi l_0^3}} \begin{pmatrix} \rho \\ -\rho \end{pmatrix} e^{i|\mu(\xi)|x_0} \qquad l_0^3 > l_1^2 l_2 \tag{6.80}$$

$$\psi_{\mathrm{III}(b)} = \frac{1}{\sqrt{12\pi l_0^3}} \begin{pmatrix} \rho \\ -\rho \end{pmatrix} e^{-i|\mu(\xi)|x_0} \qquad l_0^3 < l_1^2 l_2 \tag{6.81}$$

where $\rho = \frac{\alpha_2}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\beta_2}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$ such that $|\alpha_2|^2 + |\beta_2|^2 = 1$.
Case IV: $\xi = \frac{-1}{12\pi l_0^3}$ and $k_0 = -\mu(\xi)$
First evaluating k_0:

$$k_0 = -\mu(\xi) = -\left(\frac{l_0^3 - l_1^2 l_2}{2l_0^3 l_2}\right) \tag{6.82}$$

For $l_0^3 > l_1^2 l_2$ (case IV(a)), we have a negative frequency solution and for $l_0^3 < l_1^2 l_2$ (Case IV(b)), we have a positive frequency solution. The generic wave functions in this case are:

$$\psi_{\text{IV}(A)} = \frac{1}{\sqrt{12\pi l_0^3}}\begin{pmatrix}\rho\\-\rho\end{pmatrix} e^{-i|\mu(\xi)|x_0} \qquad l_0^3 > l_1^2 l_2 \tag{6.83}$$

$$\psi_{\text{IV}(B)} = \frac{1}{\sqrt{12\pi l_0^3}}\begin{pmatrix}\rho\\-\rho\end{pmatrix} e^{i|\mu(\xi)|x_0} \qquad l_0^3 < l_1^2 l_2 \tag{6.84}$$

where $\rho = \frac{\alpha_2}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\beta_2}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$ such that $|\alpha_2|^2 + |\beta_2|^2 = 1$.

Now, in the case of vanishing torsion (i.e., in the limit of $a(l_1) \longrightarrow 0$, we are left with two cases for k_0, viz. $k_0 = \pm\mu(\xi) = \pm\sqrt{2}b(l_2)$, with corresponding positive and negative frequency solutions.

This analysis was done in the rest frame of the plane wave (where the velocity of wave propagation is zero). However, it can be generalized by boosting the 4-momentum vector as follows:

$$\begin{bmatrix}k_0\\0\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}E\\k_1\\k_2\\k_3\end{bmatrix} = k^\mu \tag{6.85}$$

where $k_0, E \geq 0$ and $E^2 - k_1^2 - k_2^2 - k_3^2 = k_0^2$.

Similarly, the generic spinor, represented in the rest frame, transforms as follows in the boosted frame:

$$\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix} e^{\pm ik_0x_0} \mapsto \frac{1}{\sqrt{k_0}}\begin{bmatrix}\sqrt{k.\sigma}\phi_1\\\sqrt{k.\bar{\sigma}}\phi_2\end{bmatrix} e^{\pm ik.x} \tag{6.86}$$

This scheme can be applied to all the four cases to obtain generic results; the general nature and properties of the solution do not change significantly.

Ultimately, we find that the spatial behavior of the Dirac state, probability current and the tensor $(T-S)_{\mu\nu}$ is either sinusoidal or constant over all of space.

6.4.3 Solution by reduction to (2+1) Dim in cylindrical coordinates (t,r,ϕ)

We put z-dependence to zero in the equations [6.14 - 6.17] and get the following equations:

$$r\partial_t F_1 + cr\partial_r F_2 e^{i\phi} + ic\partial_\phi F_2 e^{i\phi} F_1 = icr\sqrt{2}(b + a\xi)G_1 \tag{6.87}$$
$$r\partial_t F_2 + cr\partial_r F_1 e^{-i\phi} - ic\partial_\phi F_1 e^{-i\phi} = icr\sqrt{2}(b + a\xi)G_2 \tag{6.88}$$
$$r\partial_t G_2 - cr\partial_r G_1 e^{-i\phi} + ic\partial_\phi G_1 e^{-i\phi} = icr\sqrt{2}(b + a\xi^*)F_2 \tag{6.89}$$
$$r\partial_t G_1 - cr\partial_r G_2 e^{i\phi} - ic\partial_\phi G_2 e^{i\phi} = icr\sqrt{2}(b + a\xi^*)F_1 \tag{6.90}$$

We take the ansatz, $F_2 = G_2$ and $F_1 = -G_1$

$$r\partial_t F_1 + r\partial_r F_2 e^{i\phi} + i\partial_\phi F_2 e^{i\phi} = -ir\sqrt{2}(b + a\xi)F_1 \tag{6.91}$$
$$r\partial_t F_2 + r\partial_r F_1 e^{-i\phi} - i\partial_\phi F_1 e^{-i\phi} = ir\sqrt{2}(b + a\xi)F_2 \tag{6.92}$$

We choose following ansatz in the above equation

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} iA(r)e^{\frac{i\phi}{2}} \\ B(r)e^{\frac{-i\phi}{2}} \end{bmatrix} e^{-i\omega t} \tag{6.93}$$

Putting this ansatz in above equations, we obtain the 2 differential equations as follows:

$$-rB\omega + r\partial_r A + \frac{A}{2} = r\sqrt{2}[b + a(B^2 - A^2)]B \tag{6.94}$$
$$rA\omega + r\partial_r B + \frac{B}{2} = r\sqrt{2}[b + a(B^2 - A^2)]A \tag{6.95}$$

We add and subtract above 2 equations and put following in it:

$$\psi_1 = B(r) + A(r) \tag{6.96}$$
$$\psi_2 = B(r) - A(r) \tag{6.97}$$

And we obtain:

$$-r\omega\psi_2 + r\psi_1' + \frac{\psi_1}{2} - r\sqrt{2}(b + a\psi_1\psi_2)\psi_1 = 0 \tag{6.98}$$
$$r\omega\psi_1 + r\psi_2' + \frac{\psi_2}{2} + r\sqrt{2}(b + a\psi_1\psi_2)\psi_2 = 0 \tag{6.99}$$

We aim to solve this system of equations. With $\omega = 0$, We get

$$\psi_1 = \left[\frac{c_2 e^{\sqrt{2}br}}{r^{\left(\frac{1-2\sqrt{2}ac_1}{2}\right)}}\right] \qquad \psi_2 = \left[\frac{c_1 e^{-\sqrt{2}br} r^{\left(\frac{-1-2\sqrt{2}ac_1}{2}\right)}}{c_2}\right] \tag{6.100}$$

This is clearly unphysical because ψ_1 blows up $\forall$ non-zero c_2; and making c_2 zero blows up ψ_2. So, we conclude that, **static solution to the above system of equation is unphysical**. So ω can't be zero. Some further attempts to solve it numerically are given in Appendix.

6.4.4 Solution by reduction to (3+1) Dim in spherical coordinates (t,r,θ,ϕ)

We begin by putting following ansatz in HD equations with spherical coordinates:

$$\begin{bmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)e^{+i\phi/2} \\ R_{+\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)e^{-i\phi/2} \\ R_{+\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)e^{+i\phi/2} \\ R_{-\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)e^{-i\phi/2} \end{bmatrix} e^{-i\omega t} \tag{6.101}$$

With this ansatz, equations [6.18 - 6.21] take the following form:

$$\begin{aligned}&\Big(- i\omega R_{-\frac{1}{2}}S_{-\frac{1}{2}} + \cos\theta R'_{-\frac{1}{2}}S_{-\frac{1}{2}} - \frac{\sin\theta}{r}R_{-\frac{1}{2}}S'_{-\frac{1}{2}} + \frac{1}{2r\sin\theta}R_{+\frac{1}{2}}S_{+\frac{1}{2}} + \sin\theta R'_{+\frac{1}{2}}S_{+\frac{1}{2}} + \frac{\cos\theta}{r}R_{+\frac{1}{2}}S'_{+\frac{1}{2}}\Big)\\ &= i\sqrt{2}(b+a\xi)R_{+\frac{1}{2}}S_{-\frac{1}{2}}\end{aligned} \tag{6.102}$$

$$\begin{aligned}&\Big(- i\omega R_{+\frac{1}{2}}S_{+\frac{1}{2}} - \cos\theta R'_{+\frac{1}{2}}S_{+\frac{1}{2}} + \frac{\sin\theta}{r}R_{+\frac{1}{2}}S'_{+\frac{1}{2}} - \frac{1}{2r\sin\theta}R_{-\frac{1}{2}}S_{-\frac{1}{2}} + \sin\theta R'_{-\frac{1}{2}}S_{-\frac{1}{2}} + \frac{\cos\theta}{r}R_{-\frac{1}{2}}S_{-\frac{1}{2}}\Big)'\\ &= i\sqrt{2}(b+a\xi)R_{-\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)\end{aligned} \tag{6.103}$$

$$\begin{aligned}&\Big(- i\omega R_{-\frac{1}{2}}S_{+\frac{1}{2}} + \cos\theta R'_{-\frac{1}{2}}S_{+\frac{1}{2}} - \frac{\sin\theta}{r}R_{-\frac{1}{2}}S'_{+\frac{1}{2}} + \frac{1}{2r\sin\theta}R_{+\frac{1}{2}}S_{-\frac{1}{2}} - \sin\theta R'_{+\frac{1}{2}}S_{-\frac{1}{2}} - \frac{\cos\theta}{r}R_{+\frac{1}{2}}S'_{-\frac{1}{2}}\Big)\\ &= i\sqrt{2}(b+a\xi^*)R_{+\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)\end{aligned} \tag{6.104}$$

$$\begin{aligned}&\Big(- i\omega R_{+\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta) - \cos\theta R'_{+\frac{1}{2}}S_{-\frac{1}{2}} + \frac{\sin\theta}{r}R_{+\frac{1}{2}}S'_{-\frac{1}{2}} - \frac{1}{2r\sin\theta}R_{-\frac{1}{2}}S_{+\frac{1}{2}} - \sin\theta R'_{-\frac{1}{2}}S_{+\frac{1}{2}} - \frac{\cos\theta}{r}R_{-\frac{1}{2}}S'_{+\frac{1}{2}}\Big)\\ &= i\sqrt{2}(b+a\xi^*)R_{-\frac{1}{2}}S_{-\frac{1}{2}}\end{aligned} \tag{6.105}$$

Where

$$\xi = R_{-\frac{1}{2}}S_{-\frac{1}{2}}\bar{R}_{+\frac{1}{2}}\bar{S}_{-\frac{1}{2}} + R_{+\frac{1}{2}}S_{+\frac{1}{2}}\bar{R}_{-\frac{1}{2}}\bar{S}_{-\frac{1}{2}} \tag{6.106}$$

$$\xi^* = \bar{R}_{-\frac{1}{2}}\bar{S}_{-\frac{1}{2}}R_{+\frac{1}{2}}S_{-\frac{1}{2}} + \bar{R}_{+\frac{1}{2}}\bar{S}_{+\frac{1}{2}}R_{-\frac{1}{2}}S_{-\frac{1}{2}} \tag{6.107}$$

6.5 Summary

- Curvature-Torsion duality conjecture presented.

- Formulated the ECD theory on Minkowski space with torsion.
- Solution to Dirac equation on M_4 with torsion by reducing the problem to (1+1)- Dim found. However, it cannot make T-S vanish for any values of free parameters.
- Plane wave solutions to Dirac equation on M_4 with torsion exist. Explicit expression of plane wave solutions with only time dependence found. However, it cannot make T-S vanish for any values of free parameters.
- Solution by reducing the problem to (2+1)-Dim attempted. Equations are presented. However solution is not found yet. More has been discussed in chapter (7)
- Solution by reducing the problem to (3+1)-Dim attempted. Equations are presented. However solution is not found yet. More has been discussed in chapter (7)

Chapter 7

Discussion

7.1 Conclusions and outlook

As discussed in (1.2.1), we found the non-relativistic limit of Einstein-Dirac system [That is self-gravitating Dirac field on V_4] with **generic metric** and found that it indeed reproduces the results of [19] viz. at leading order, the NR limit is Schrödinger-Newton equation. In short, We generalized their work (by considering generic metric). Next, we found the NR limit for Einatein-Cartan-Dirac system [That is self-gravitating Dirac field on U_4]. At leading order, it also turns out to be Schrödinger-Newton equation. This suggests that, at leading order, there is NO effect of torsion in the non-relativistic limit. So in order to experimentally probe the effects of torsion, we will have to go higher orders. Our method of finding NR limit also provides a prescription for finding the correction terms due to torsion. When we compare it with Einstein-Dirac system, we can analyze the orders of the coupled equations which are altered due to torsion through this prescription. This has huge implications for anyone who would like to design experiments to detect torsion in future. This was all w.r.t standard ECD theory (as in, ECD with standard length scales as couplings). We also have some interesting results after we take the NR limit of ECD equations modified with L_{cs}. In high mass limit, we obtain Poisson equation with delta function source. We showed that, this result is valid for all energy levels; not only in Non-relativistic limit. This has interesting implications. We know from [15] that very large masses are highly localized (In terms of their wave-function, it is already in a collapsed state). So it behaves classically. Hence, we obtain Poisson equation with delta function source (localized source for point particles) even for relativistic case. This is consistent with ordinary GR and Newton's law. It proves that, the modification of theory with L_{cs} is consistent with the known theories in large mass limit. In the small mass limit however, since L_{cs} goes as 1/c as opposed to $1/c^2$ (which was the case with large mass limit), we find that Poisson equation is $\nabla^2\phi = 0$. So for $m \ll m_{pl}$, we find that, gravitational field as well as quantum state vanishes at $1/c^2$. This gives a falsifiable test for the idea of L_{cs}. Gravity between very small masses would be weaker than the predictions of GR if one does an experiment to test the inverse square law between the pair of very small masses.

In chapter (5), we formulated ECD theory in NP formalism. Dirac equation is modified on U_4 and presented in NP formalism in equations [5.42 - 5.45]. We also provided the prescription

for finding the expression of EM tensor in NP formalism and also calculated the spin density term which acts as a correction to the dynamic (and symmetrical) EM tensor; together which contribute to the Einstein's tensor made up from Christoffel connection. Contorsion spin coefficients in NP formalism are also expressed in terms of Dirac state.

Chapter (6) discusses the curvature-torsion duality. As we have mentioned in chapter (3), the idea of L_{cs} naturally hints towards a symmetry between higher and lower masses. In this chapter, we have made this duality mathematically more evident through a conjecture. One way to test the conjecture is to find the solutions on Minkowski space with torsion and test the components of tensor "T-S". This tensor doesn't vanish for the 2 solutions which are presented in section (**??**) and section (**??**). So these solutions do not support our conjecture. Solutions by reducing the problem to (2+1)-Dim and (3+1)-Dim are under investigation. The big picture which Curvature-torsion duality presents, has been discussed in details in an essay submitted to Gravity research foundation. It can be looked up in [18].

Solutions to linear Dirac equation on Minkowski space has been studied extensively. In this work, we attempted finding solutions to HD equations on Minkowski space with torsion. We wanted to see whether presence of torsion induce any non-trivial (and physically relevant) modifications to the solutions for linear (non-torsional) case. Solutions after reducing the problem to (1+1) dimension in the variables (t, z) were found. We found a finite parameter range w $\in$ (0,1), where this solution vanishes at infinity in the non-static case and has finite maxima (or finite local minima) at orgin. For w $\in$ (1/3,1), the solution (and the probability density) decreases monotonically from a finite value at center and asymptotically reaches zero at infinity. This is the sought after finite solution - the 'blob'.

7.2 Future plans

- Continue the self-study of gravitational theories with torsion from both theoretical and experimental perspectives.

- To find the non-relativistic limit of ECD equations with new length scale L_{cs} for the masses which are comparable to plank mass. We speculate that it will be something different from Schrödinger-Newton equation.

- To understand the implications of the idea of L_{cs} (in its low mass limit) in the known theories of particle physics. In its low mass limit, Dirac equation has cubic non-linear term with λ_c as coupling constant. It can be tested against known experimental data and also to make quantitative predictions for the new experiments. Another plan is to work on the falsifiable test for the idea of L_{cs} presented in the first paragraph of discussions.

- To find a solution to Hehl-Datta equation on Minkowski space with torsion (either by continuing the study of reducing the equations to 2+1 Dim and 3+1 Dim as mentioned in sections (6.4.3) and (6.4.4) or by some other method) such that the tensor "T-S" becomes zero. The aim is testing the hypothesis of curvature-torsion duality.

Appendix A

Results of Long calculations used in Chapter 4 - Non-relativistic limit of ECD equations

A.1 Form of Einstein's tensor evaluated from the generic metric upto second order

We have used the ansatz for metric [defined in equation (4.3)]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{n=1}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n g^{[n]}_{\mu\nu}(x)$$

The metric and its inverse, up to second order, can be written as following:

$$g_{\mu\nu} = \eta_{\mu\nu} + \Big(\frac{\sqrt{\hbar}}{c}\Big) g^{[1]}_{\mu\nu} + \Big(\frac{\hbar}{c^2}\Big) g^{[2]}_{\mu\nu} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{A.1}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \Big(\frac{\sqrt{\hbar}}{c}\Big) g^{\mu\nu[1]} - \Big(\frac{\hbar}{c^2}\Big) [g^{\mu[1]}_{\ \beta} g^{\beta\nu[1]} + g^{\mu\nu[2]}] + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{A.2}$$

We evaluate Christoffel symbols, Riemann curvature tensor, Ricci tensor and scalar curvature up to second order using above 2 equations and obtain Einstein tensor at the end. Einstein's tensor $G_{\mu\nu}$ is then given by

$$G_{\mu\nu} = \Big(\frac{\sqrt{\hbar}}{c}\Big) G^{[1]}_{\mu\nu} + \Big(\frac{\hbar}{c^2}\Big) G^{[2]}_{\mu\nu} \tag{A.3}$$

Where

$$G^{[1]}_{\mu\nu} = -\frac{1}{2}\Box \bar{g}^{[1]}_{\mu\nu}; \qquad \text{where } \ \bar{g}^{[1]}_{ij} = g^{[1]}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} g^{[1]}; \qquad g^{[1]} = (\eta^{\mu\nu} g^{[1]}_{\mu\nu}) \tag{A.4}$$

$$G^{[2]}_{\mu\nu} = -\frac{1}{2}\Box \bar{g}^{(2)}_{\mu\nu} + f(g^{[1]}_{\mu\nu}) \qquad \text{where } \ \bar{g}^{[2]}_{ij} = g^{[2]}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} g^{[2]}; \qquad g^{[2]} = (\eta^{\mu\nu} g^{[2]}_{\mu\nu}) \tag{A.5}$$

f is a function of $g^{[1]}_{\mu\nu}$ and is given by following equation:

$$f(g^{[1]}_{\mu\nu}) = -\frac{1}{4}\Big[2\partial^{\lambda}g^{[1]}\partial_{\nu}g^{[1]}_{\lambda\mu} - 2\partial^{\lambda}g^{[1]}\partial_{\lambda}g^{[1]}_{\mu\nu} - \partial_{\rho}g_{\nu}^{\lambda[1]}\partial_{\mu}g_{\lambda}^{\rho[1]} - \partial_{\rho}g_{\nu}^{\lambda[1]}\partial_{\lambda}g_{\mu}^{\rho[1]} +$$
$$\partial_{\rho}g_{\nu}^{\lambda[1]}\partial^{\rho}g^{[1]}_{\lambda\mu} + \partial_{\nu}g_{\rho}^{\lambda[1]}\partial_{\mu}g_{\lambda}^{\rho[1]} + \partial_{\nu}g_{\rho}^{\lambda[1]}\partial_{\lambda}g_{\mu}^{\rho[1]} - \partial_{\nu}g_{\rho}^{\lambda[1]}\partial^{\rho}g^{[1]}_{\lambda\mu}\Big]$$
$$-\frac{1}{8}\Big[2\partial^{\lambda}g^{[1]}\partial_{\nu}g^{[1]}_{\lambda\mu} - 2\eta_{\mu\nu}\partial^{\lambda}g^{[1]}\partial_{\lambda}g^{[1]} - \partial_{\rho}g_{\nu}^{\lambda[1]}\partial_{\mu}g_{\lambda}^{\rho[1]} - \partial_{\rho}g_{\mu}^{\lambda[1]}\partial_{\lambda}g_{\nu}^{\rho[1]}$$
$$+\partial_{\rho}g_{\mu}^{\lambda[1]}\partial^{\rho}g^{[1]}_{\lambda\nu} + \partial_{\mu}g_{\rho}^{\lambda[1]}\partial_{\nu}g_{\lambda}^{\rho[1]} + \partial_{\mu}g_{\rho}^{\lambda[1]}\partial_{\lambda}g_{\nu}^{\rho[1]} - \partial_{\nu}g_{\rho}^{\lambda[1]}\partial^{\rho}g_{\lambda\mu[1]}\Big]$$

A.2 Constraints imposed on metric due to asymptotic flatness condition

A.2.1 Constraint on $g^{[1]}_{\mu\nu}$

First we analyze the off-diagonal form of $g^{[1]}_{\mu\nu}$. Off-diagonal components of $G^{[1]}_{\mu\nu}$ is zero. This implies (for off-diagonal components alone), from equation (A.4), $\Box\bar{g}^{[1]}_{\mu\nu} = \Box g^{[1]}_{\mu\nu} = 0$. Non-trivial solution to this equation (which is a gravitational wave solution) doesn't respect asymptotic flatness. So the only solution allowed is trivial solution viz. $g^{[1]}_{\mu\nu} = 0$. Now, for diagonal components, we assume the metric form to be the most generic:

$$g^{[1]}_{\mu\nu} = \begin{pmatrix} f_1^{[1]} & 0 & 0 & 0 \\ 0 & f_2^{[1]} & 0 & 0 \\ 0 & 0 & f_3^{[1]} & 0 \\ 0 & 0 & 0 & f_4^{[1]} \end{pmatrix} \tag{A.6}$$

$$\bar{g}^{[1]}_{00} = \frac{f_1^{[1]} + f_2^{[1]} + f_3^{[1]} + f_4^{[1]}}{2} \tag{A.7}$$
$$\bar{g}^{[1]}_{11} = \frac{f_1^{[1]} + f_2^{[1]} - f_3^{[1]} - f_4^{[1]}}{2} \tag{A.8}$$
$$\bar{g}^{[1]}_{22} = \frac{f_1^{[1]} + f_3^{[1]} - f_2^{[1]} - f_4^{[1]}}{2} \tag{A.9}$$
$$\bar{g}^{[1]}_{33} = \frac{f_1^{[1]} + f_4^{[1]} - f_2^{[1]} - f_3^{[1]}}{2} \tag{A.10}$$

And the fact that Einstein's tensor is zero for all the components implies,

$$\Box\bar{g}^{[1]}_{00} = \Box\frac{f_1^{[1]} + f_2^{[1]} + f_3^{[1]} + f_4^{[1]}}{2} = 0 \Longrightarrow \Box f_1^{[1]} + \Box f_2^{[1]} + \Box f_3^{[1]} + \Box f_4^{[1]} = 0 \tag{A.11}$$
$$\Box\bar{g}^{[1]}_{11} = \Box\frac{f_1^{[1]} + f_2^{[1]} - f_3^{[1]} - f_4^{[1]}}{2} = 0 \Longrightarrow \Box f_1^{[1]} + \Box f_2^{[1]} = \Box f_3^{[1]} + \Box f_4^{[1]} \tag{A.12}$$
$$\Box\bar{g}^{[1]}_{22} = \Box\frac{f_1^{[1]} + f_3^{[1]} - f_2^{[1]} - f_4^{[1]}}{2} = 0 \Longrightarrow \Box f_1^{[1]} + \Box f_3^{[1]} = \Box f_2^{[1]} + \Box f_4^{[1]} \tag{A.13}$$
$$\Box\bar{g}^{[1]}_{33} = \Box\frac{f_1^{[1]} + f_4^{[1]} - f_2^{[1]} - f_3^{[1]}}{2} = 0 \Longrightarrow \Box f_1^{[1]} + \Box f_4^{[1]} = \Box f_2^{[1]} + \Box f_3^{[1]} \tag{A.14}$$

One should note that individually, $\Box f_i^{[1]} = 0$ only implies $f_i^{[1]} = 0$ (no wave solution allowed) Even $f_i^{[1]} =$ constant is NOT allowed as constant solution also contradicts asymptotic flatness. Equations (A.12), (A.13) and (A.14) imply that

$$\Box f_2^{[1]} = \Box f_1^{[1]} \implies f_2^{[1]} = f_1^{[1]} + c_1 \tag{A.15}$$

$$\Box f_3^{[1]} = \Box f_1^{[1]} \implies f_3^{[1]} = f_1^{[1]} + c_2 \tag{A.16}$$

$$\Box f_4^{[1]} = \Box f_1^{[1]} \implies f_4^{[1]} = f_1^{[1]} + c_3 \tag{A.17}$$

However, all the constants c_1, c_2, c_3 should be zero [As constant + asymptotic flat function can't give overall asymptotic flat function]. Now, equation (A.11) implies, $4\Box f_1^{[1]} = 0 \implies f_1^{[1]} = 0$. Hence all the functions $f_i^{[1]} = 0 \; \forall \; i$. HENCE

$$g_{\mu\nu}^{[1]} = 0 \; \forall \; \mu, \nu \tag{A.18}$$

A.2.2 Constraint on $g_{\mu\nu}^{[2]}$

Here also, first we analyze the off-diagonal form of $g_{\mu\nu}^{[2]}$. Off-diagonal components of $G_{\mu\nu}^{[2]}$ is zero. This implies, from equation (A.5), $\Box \bar{g}_{\mu\nu}^{[2]} = \Box g_{\mu\nu}^{[2]} = 0$. Non-trivial solution to this equation (which is a gravitational wave solution) doesn't respect asymptotic flatness. So the only solution allowed is trivial solution viz. $g_{\mu\nu}^{[2]} = 0$. Now, for diagonal components, we again assume the metric form to be the most generic:

$$g_{\mu\nu}^{[2]} = \begin{pmatrix} f_1^{[2]} & 0 & 0 & 0 \\ 0 & f_2^{[2]} & 0 & 0 \\ 0 & 0 & f_3^{[2]} & 0 \\ 0 & 0 & 0 & f_4^{[2]} \end{pmatrix} \tag{A.19}$$

$$\bar{g}_{00}^{[2]} = \frac{f_1^{[2]} + f_2^{[2]} + f_3^{[2]} + f_4^{[2]}}{2} \tag{A.20}$$

$$\bar{g}_{11}^{[2]} = \frac{f_1^{[2]} + f_2^{[2]} - f_3^{[2]} - f_4^{[2]}}{2} \tag{A.21}$$

$$\bar{g}_{22}^{[2]} = \frac{f_1^{[2]} + f_3^{[2]} - f_2^{[2]} - f_4^{[2]}}{2} \tag{A.22}$$

$$\bar{g}_{33}^{[2]} = \frac{f_1^{[2]} + f_4^{[2]} - f_2^{[2]} - f_3^{[2]}}{2} \tag{A.23}$$

And the fact that Einstein's tensor is zero for all the components except '00' component implies,

$$\Box \bar{g}_{00}^{[2]} = \Box \frac{f_1^{[2]} + f_2^{[2]} + f_3^{[2]} + f_4^{[2]}}{2} \implies \Box f_1^{[2]} + \Box f_2^{[2]} + \Box f_3^{[2]} + \Box f_4^{[2]} \neq 0 \tag{A.24}$$

$$\Box \bar{g}_{11}^{[2]} = \Box \frac{f_1^{[2]} + f_2^{[2]} - f_3^{[2]} - f_4^{[2]}}{2} \implies \Box f_1^{[2]} + \Box f_2^{[2]} = \Box f_3^{[2]} + \Box f_4^{[2]} \tag{A.25}$$

$$\Box \bar{g}_{22}^{[2]} = \Box \frac{f_1^{[2]} + f_3^{[2]} - f_2^{[2]} - f_4^{[2]}}{2} \implies \Box f_1^{[2]} + \Box f_3^{[2]} = \Box f_2^{[2]} + \Box f_4^{[2]} \tag{A.26}$$

$$\Box \bar{g}_{33}^{[2]} = \Box \frac{f_1^{[2]} + f_4^{[2]} - f_2^{[2]} - f_3^{[2]}}{2} \implies \Box f_1^{[2]} + \Box f_4^{[2]} = \Box f_2^{[2]} + \Box f_3^{[2]} \tag{A.27}$$

Equations (A.25), (A.26) and (A.27) imply that

$$\Box f_2^{[2]} = \Box f_1^{[2]} \implies f_2^{[2]} = f_1^{[2]} \tag{A.28}$$

$$\Box f_3^{[2]} = \Box f_1^{[2]} \implies f_3^{[2]} = f_1^{[2]} \tag{A.29}$$

$$\Box f_4^{[2]} = \Box f_1^{[2]} \implies f_4^{[2]} = f_1^{[2]} \tag{A.30}$$

[we have already seen why addition of constant to above solution contradicts our claim of asymptotic flatness.] With equations (A.28), (A.29) and (A.30), we find that $f_1^{[2]} = f_2^{[2]} = f_3^{[2]} = f_4^{[2]} = F(\mathbf{r}, t)$.

$$g_{\mu\nu}^{[2]} = \begin{pmatrix} F(\mathbf{r},t) & 0 & 0 & 0 \\ 0 & F(\mathbf{r},t) & 0 & 0 \\ 0 & 0 & F(\mathbf{r},t) & 0 \\ 0 & 0 & 0 & F(\mathbf{r},t) \end{pmatrix} \tag{A.31}$$

A.3 Metric and Christoffel symbol components

The form of metric defined in equation (4.42) is as follows:

$$g_{\mu\nu} = \begin{pmatrix} 1 + \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{A.32}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 - \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{A.33}$$

Christoffel Connection:

The non-zero Christoffel connection components (the first term; which is second order in 1/c) corresponding to metric $g_{\mu\nu}$ defined above are as follows:

$$\begin{aligned} \Gamma^0_{0\mu} &= \frac{\hbar \partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \\ \Gamma^\mu_{00} &= \frac{\hbar \partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \\ \Gamma^\mu_{\mu\mu} &= \frac{-\hbar \partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \end{aligned} \tag{A.34}$$

[Here $\mu = 1, 2, 3$ i.e., it refers to the spatial coordinates.]

Other non zero Christoffel connection components have all orders of terms from order 3 viz. $\sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)$

A.4 Tetrad components

Tetrads are introduced in the section "Preliminaries: Einstein-Cartan-Dirac equations". The metric and corresponding tetrad field on the whole manifold is defined below:

$$dS^2 = \left[1 + \frac{\hbar F(\mathbf{r},t)}{c^2}\right] c^2 dt^2 - \left[1 - \frac{\hbar F(\mathbf{r},t)}{c^2}\right] d\mathbf{r}^2 \tag{A.35}$$

$$\hat{e}_{(0)} = \frac{1}{c}\left(1 + \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_t, \quad \hat{e}_{(1)} = \left(1 - \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_x, \quad \hat{e}_{(2)} = \left(1 - \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_y, \quad \hat{e}_{(3)} = \left(1 - \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_z \tag{A.36}$$

With this, the transformation matrix which relates the world components with anholonomic components (defined in equation 2.47)

$$e^{(i)}_{\mu} = \begin{pmatrix} 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{A.37}$$

$$e^{\mu}_{(i)} = \begin{pmatrix} 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{A.38}$$

$$e_{\nu(k)} = \begin{pmatrix} 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{A.39}$$

$$e^{\nu(k)} = \begin{pmatrix} 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{A.40}$$

A.5 Components of the Riemann part of Spin Connection $\gamma^o_{(a)(b)(c)}$

The form of spin connections are defined in equations (2.48), (2.49). We use the relation between Christoffel connection and tetrad transformation matrix (defined in Eqn. (2.50)) to calculate $\gamma^o_{(a)(b)(c)}$ as follows:

$$
\begin{aligned}
\gamma^o_{(0)(0)(0)} &= \frac{-\hbar\partial_0 F}{2c^2}\frac{\left(1+\frac{\hbar F}{2c^2}\right)}{\left(1-\frac{\hbar F}{2c^2}\right)} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) &\qquad \gamma^o_{(i)(0)(0)} &= \left(\frac{-\hbar\partial_i F}{2c^2}\right)\frac{\hbar F/2c^2}{\left(1+\frac{\hbar F}{2c^2}\right)} + \sum_{n=5}^{\infty} O\left(\frac{1}{c^n}\right)\\
\gamma^o_{(0)(i)(0)} &= \frac{-\hbar\partial_i F}{2c^2}\frac{\left(1+\frac{\hbar F}{2c^2}\right)}{\left(1-\frac{\hbar F}{2c^2}\right)} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) &\qquad \gamma^o_{(0)(0)(i)} &= \frac{\hbar\partial_i F}{2c^2}\frac{1}{\left(1+\frac{\hbar F}{2c^2}\right)}\\
\gamma^o_{(i)(i)(i)} &= \frac{\hbar\partial_i F}{2c^2}\frac{\hbar F/2c^2}{\left(1+\frac{\hbar F}{2c^2}\right)} + \sum_{n=5}^{\infty} O\left(\frac{1}{c^n}\right) &\qquad \gamma^o_{(i)(i)(0)} &= \gamma^o_{(i)(0)(i)} = +\sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)\\
\gamma^o_{(0)(i)(i)} &= \frac{-\hbar\partial_0 F}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) &\qquad \gamma^o_{(0)(i)(j)} &= \gamma^o_{i0j} = \gamma^o_{ij0} = +\sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)\\
\gamma^o_{(i)(j)(j)} &= \frac{-\hbar\partial_0 F}{2c^2}\frac{\left(1-\frac{\hbar F}{2c^2}\right)}{\left(1+\frac{\hbar F}{2c^2}\right)} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) &\qquad \gamma^o_{(i)(j)(k)} &= \gamma^o_{(i)(j)(i)} = \gamma^o_{(j)(j)(i)} = +\sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)
\end{aligned}
\tag{A.41}
$$

The contorsion spin coefficients (which when gets added to Riemann spin coefficient, gives total spin connection) gets manifested as a non-linear term in Hehl-Datta equation. It is completely expressible in terms of Dirac spinor. So it can be calculated with spinor ansatz. We have done this while calculating the Non-relativistic limit of ECD system of equations.

A.6 Components for Einstein's tensor

In this appendix, we aim to calculate the components of Einstein's tensor. $G^{[1]}_{\mu\nu}$ has been proved to be zero. $G^{[2]}_{\mu\nu}$ has been defined in Eqn. (A.5). We found the form of $g^{[2]}_{\mu\nu}$ in appendix section (A.2.2). Since $g^{[1]}_{\mu\nu}$ is zero, f$[g^{[1]}_{\mu\nu}]$ defined in Eqn. (A.5) is also zero. With this, we compute $G^{[2]}_{\mu\nu}$:

$$G^{[2]}_{\mu\nu} = -\frac{1}{2}\Box\bar{g}^{[2]}_{\mu\nu}; where \qquad \bar{g}^{[2]}_{\mu\nu} = g^{[2]}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}h_{\alpha\beta}) \tag{A.42}$$

$$\eta^{\mu\nu}h_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} = \frac{-2\hbar F(\mathbf{r},t)}{c^2} \tag{A.43}$$

It can easily be seen that $G_{\mu\nu}$ for $\mu \neq \nu$ is equal to 0.

We now calculate the diagonal components,

$$G_{00} = -\frac{1}{2}\Box\bar{g}^{[2]}_{00} = -\frac{\hbar}{c^2}\Box F(\mathbf{r},t) = \left[-\frac{\hbar\partial_t^2 F(\mathbf{r},t)}{c^4} + \frac{\hbar\nabla^2 F(\mathbf{r},t)}{c^2}\right] \tag{A.44}$$

$$G_{\alpha\alpha} = 0; \quad because \quad \bar{g}^{[2]}_{\alpha\alpha} = 0; \qquad \alpha \in (1,2,3) \tag{A.45}$$

Thus,

$$G_{\mu\nu} = \frac{\hbar}{c^2}\begin{pmatrix} \nabla^2 F(\mathbf{r},t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{A.46}$$

A.7 Generic components of $T_{\mu\nu}$

$T_{\mu\nu}$ has been defined in equation Eqn. (2.57). With the spin coefficients in above sections, we get the following metric energy-momentum tensor, whose components are given on the next page.

$$
T_{\mu\nu} = \frac{i\hbar c}{4}
\begin{pmatrix}
\begin{array}{r}
2\bar{\psi}\gamma_0(\partial_0\psi \\
+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha0}\gamma^\alpha\gamma^0]\psi) \\
-(\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\
+\gamma_{0\alpha0}\gamma^\alpha\gamma^0]\bar{\psi})2\gamma_0\psi
\end{array}
&
\begin{array}{r}
\bar{\psi}\gamma_0\partial_1\psi + \bar{\psi}\gamma_1(\partial_0\psi \\
+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha0}\gamma^\alpha\gamma^0]\psi) \\
-\partial_1\bar{\psi}\gamma_0\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\
+\gamma_{0\alpha0}\gamma^\alpha\gamma^0]\bar{\psi})\gamma_1\psi)
\end{array}
&
\begin{array}{r}
\bar{\psi}\gamma_0\partial_2\psi + \bar{\psi}\gamma_2(\partial_0\psi \\
+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha0}\gamma^\alpha\gamma^0]\psi) \\
-\partial_2\bar{\psi}\gamma_0\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\
+\gamma_{0\alpha0}\gamma^\alpha\gamma^0]\bar{\psi})\gamma_2\psi)
\end{array}
&
\begin{array}{r}
\bar{\psi}\gamma_0\partial_3\psi + \bar{\psi}\gamma_3(\partial_0\psi \\
+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha0}\gamma^\alpha\gamma^0]\psi) \\
-\partial_3\bar{\psi}\gamma_0\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\
+\gamma_{0\alpha0}\gamma^\alpha\gamma^0]\bar{\psi})\gamma_3\psi)
\end{array}
\\
\begin{array}{r}
\bar{\psi}\gamma_1(\partial_0\psi \\
+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha0}\gamma^\alpha\gamma^0]\psi) \\
+\bar{\psi}\gamma_0\partial_1\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\
+\gamma_{0\alpha0}\gamma^i\gamma^0]\bar{\psi})\gamma_1\psi - \partial_1\bar{\psi}\gamma_0\psi
\end{array}
&
2(\bar{\psi}\gamma_1\partial_1\psi - \partial_1\psi\bar{\gamma_1}\psi)
&
\begin{array}{r}
\bar{\psi}\gamma_1\partial_2\psi + \bar{\psi}\gamma_2\partial_1\psi \\
-\partial_2\bar{\psi}\gamma_1\psi - \partial_1\bar{\psi}\gamma_2\psi
\end{array}
&
\begin{array}{r}
\bar{\psi}\gamma_1\partial_3\psi + \bar{\psi}\gamma_3\partial_1\psi \\
-\partial_3\bar{\psi}\gamma_1\psi - \partial_1\bar{\psi}\gamma_3\psi
\end{array}
\\
\begin{array}{r}
\bar{\psi}\gamma_2(\partial_0\psi \\
+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha0}\gamma^\alpha\gamma^0]\psi) \\
+\bar{\psi}\gamma_0\partial_2\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\
+\gamma_{0\alpha0}\gamma^i\gamma^0]\bar{\psi})\gamma_2\psi - \partial_2\bar{\psi}\gamma_0\psi
\end{array}
&
\begin{array}{r}
\bar{\psi}\gamma_2\partial_1\psi + \bar{\psi}\gamma_1\partial_2\psi \\
-\partial_1\bar{\psi}\gamma_2\psi - \partial_2\bar{\psi}\gamma_1\psi
\end{array}
&
2(\bar{\psi}\gamma_2\partial_2\psi - \partial_2\psi\bar{\gamma_2}\psi)
&
\begin{array}{r}
\bar{\psi}\gamma_2\partial_3\psi + \bar{\psi}\gamma_3\partial_2\psi \\
-\partial_3\bar{\psi}\gamma_2\psi - \partial_2\bar{\psi}\gamma_3\psi
\end{array}
\\
\begin{array}{r}
\bar{\psi}\gamma_3(\partial_0\psi \\
+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha0}\gamma^\alpha\gamma^0]\psi) \\
+\bar{\psi}\gamma_0\partial_3\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\
+\gamma_{0\alpha0}\gamma^i\gamma^0]\bar{\psi})\gamma_3\psi - \partial_3\bar{\psi}\gamma_0\psi
\end{array}
&
\begin{array}{r}
\bar{\psi}\gamma_3\partial_1\psi + \bar{\psi}\gamma_1\partial_3\psi \\
-\partial_1\bar{\psi}\gamma_3\psi - \partial_3\bar{\psi}\gamma_1\psi
\end{array}
&
\begin{array}{r}
\bar{\psi}\gamma_3\partial_2\psi + \bar{\psi}\gamma_2\partial_3\psi \\
-\partial_2\bar{\psi}\gamma_3\psi - \partial_3\bar{\psi}\gamma_2\psi
\end{array}
&
2(\bar{\psi}\gamma_3\partial_3\psi - \partial_3\psi\bar{\gamma_3}\psi)
\end{pmatrix}
\tag{A.47}
$$

Appendix B

Tetrad formalism/ NP formalism and formulating ECD equations in NP formalism

B.1 Tetrad formalism and formulating Covariant derivative for Spinors

The usual method in approaching the solution to the problems in General Relativity was to use a **local coordinate basis** $\hat{e}^{\mu}$ such that $\hat{e}^{\mu} = \partial_{\mu}$. This coordinate basis field is covariant under General coordinate transformation. However, it has been found useful to employ non-coordinate basis techniques in problems involving Spinors. This is the tetrad formalism which consists of setting up four linearly independent basis vectors called a 'tetrad basis' at each point of a region of spacetime; which are covariant under local Lorentz transformations. [One of the reason of using tetrad formalism for spinors is essentially this fact that transformation properties of spinors can be easily defined in flat space-time]. The tetrad basis is given by $\hat{e}^{(i)}(x)$. These are 4 vectors (one for each μ) et every point. This tetrad field is governed by a relation $\hat{e}^{i}(x) = e^{i}_{\mu}(x)\hat{e}^{\mu}$ where trasformation matrix e^{i}_{μ} is such that,

$$e^{(i)}_{\mu} e^{(k)}_{\nu} \eta_{(i)(k)} = g_{\mu\nu}; \tag{B.1}$$

Any 'object' now can be expressed in coordinate or tetrad basis as follows:

$$V = V^{(a)}\hat{e}_{(a)} - - - - - - - - Tetrad\ \ basis \tag{B.2}$$

$$V = V^{\mu}\partial_{\mu} - - - - - - - - -Coordinate\ \ basis \tag{B.3}$$

Trasformation matrix $e^{(i)}_{\mu}$ allows us to convert the components of any world tensor (tensor which transforms according to general coordinate transformation) to the corresponding components in local Minkowskian space (These latter components being covariant under local

Lorentz transformation). [Ex. $T_{\mu\nu} = e_\mu^{(i)} e_\nu^{(k)} T_{(i)(k)}$]. Greek indices are raised or lowered using the metric $g_{\mu\nu}$, while the Latin indices are raised or lowered using $\eta_{(i)(k)}$. parenthesis around indices is just a matter of convention already defined. A

$$A_{(a),(b)} = e^\mu_{(b)} \frac{\partial}{\partial x^\mu} A_{(a)} = e^\mu_{(b)} \frac{\partial}{\partial x^\mu} [e^\nu_{(a)} A_\nu] \tag{B.4}$$

$$= e^\mu_{(b)} [A_\nu \partial_\mu e^\nu_{(a)} + e^\nu_{(a)} \partial_\mu A_\nu] \tag{B.5}$$

$$= e^\mu_{(b)} [A^\rho \nabla_\mu e_{(a)\rho} + e^\nu_{(a)} \nabla_\mu A_\nu - \Gamma^\nu_{\mu\rho} \not{e}^\rho_{(a)} A_\nu + \Gamma^\nu_{\mu\rho} \not{e}^\rho_{(a)} A_\nu] \tag{B.6}$$

$$\tag{B.7}$$

From this, we get the expression for Covariant derivative of object with tetrad index

$$\nabla_{(b)} A_{(a)} = \partial_{(b)} A_{(a)} - e^\rho_{(c)} \nabla_\mu e_{(a)\rho} e^\mu_{(b)} A^{(c)} \tag{B.8}$$

$$= \partial_{(b)} A_{(a)} - \gamma_{(c)(a)(b)} A^{(c)} \tag{B.9}$$

where $\gamma_{(c)(a)(b)}$ are called **Ricci rotation coefficients** which are anti-symmetric in first pair of indices and are defined as

$$\gamma_{(c)(a)(b)} = e^\mu_{(c)} \nabla_\nu e_{(a)\mu} e^\nu_{(b)} \tag{B.10}$$

$$= e^\nu_{(b)} e^\mu_{(c)} \left[\delta^\alpha_\mu \partial_\nu - \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} + K_{\nu\mu}{}^\alpha \right] e_{(a)\alpha} \tag{B.11}$$

$$= \gamma^o_{(c)(a)(b)} - K_{(b)(a)(c)} \tag{B.12}$$

B.2 Natural connection between $SL(2,\mathbb{C})$ Spinor formalism and NP formalism

4-vector on a Minkowski space can be represented by a hermitian matrix by some transformation law. Unimodular transformations on complex 2-Dim space induces a Lorentz transformation in Minkowski space. Unimodular matrices form a group under multiplication and is denoted by$SL(2,\mathbb{C})$ - special linear group of 2 x 2 matrices over complex numbers. By a simple counting argument, it has six free real parameters corresponding to those of the Lorentz group. For a Lorentz transformation acting on Minkowski space, there are strictly speaking two transformations $\pm L \in SL(2,\mathbb{C})$. But this sign ambiguity may be resolved by choosing a path connected to the identity transformation. The levi-civita symbol $\epsilon_{AB'}$ acts as metric tensor in this space $\mathbb{C}^2$ which preserves the scalar product under Unimodular transformations. Spinor P^A of rank 1 is defined as vector in complex 2-Dim space subject to transformations $\in SL(2,\mathbb{C})$. Similarly higher rank spinor are defined. There are various prescriptions wherein we associate objects in 4-dim Minkowski space with those in 2-dim complex space $\mathbb{C}^2$. The

Van der Waarden symbols σ's (for whose representation, we have chosen pauli matrices; such a representation is NOT unique) are used to associate tensorial objects with spinorial objects. Few examples are:

$$v^{\mu} = \sigma^{\mu}_{AA'} V^{AA'} \tag{B.13}$$

$$v^{\mu\nu} = \sigma^{\mu}_{AA'} \sigma^{\nu}_{BB'} V^{AA'BB'} \tag{B.14}$$

(higher rank associations can be defined similarly). Now, analogous to a tetrad in Minkowski space, here we have a spin dyad (a pair of 2 spinors $\zeta_{(0)A}$ and $\zeta_{(1)A}$) such that $\zeta_{(0)A}\zeta^{A}_{(1)} = 1$. A natural connection between dyad formalism and Null tetrad formalism (NP formalism) is evident by observing following association. More details can be looked up in [6], [31]

$$l^{\mu} = \zeta^{A}_{(0)}\bar{\zeta}^{A'}_{(0)} \qquad n^{\mu} = \zeta^{A}_{(1)}\bar{\zeta}^{A'}_{(1)} \tag{B.15}$$

$$m^{\mu} = \zeta^{A}_{(0)}\bar{\zeta}^{A'}_{(1)} \qquad \bar{m}^{\mu} = \zeta^{A}_{(1)}\bar{\zeta}^{A'}_{(0)} \tag{B.16}$$

B.3 Computation of Contorsion spin coefficients

We first define the product $\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}$.

$$\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu} = \begin{pmatrix} 0 & (\tilde{\sigma}^{\alpha})^{*}(\sigma^{\beta})^{*}(\tilde{\sigma}^{\mu})^{*} \\ (\sigma^{\alpha})^{*}(\tilde{\sigma}^{\beta})^{*}(\sigma^{\mu})^{*} & 0 \end{pmatrix} \tag{B.17}$$

This can be expanded fully in terms of vander-warden symbols and finally it takes the form

$$\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu} = 2\sqrt{2}\begin{pmatrix} 0 & 0 & \begin{bmatrix} nln - n\bar{m}m \\ -\bar{m}mn + \bar{m}nm \end{bmatrix} & \begin{bmatrix} -nl\bar{m} + n\bar{m}l \\ +\bar{m}m\bar{m} - \bar{m}nl \end{bmatrix} \\ 0 & 0 & \begin{bmatrix} -mln + m\bar{m}m \\ +lmn - lnm \end{bmatrix} & \begin{bmatrix} ml\bar{m} - m\bar{m}l \\ -lm\bar{m} + lnl \end{bmatrix} \\ \begin{bmatrix} lnl - l\bar{m}m- \\ \bar{m}ml + \bar{m}lm \end{bmatrix} & \begin{bmatrix} ln\bar{m} - l\bar{m}n \\ -\bar{m}m\bar{m} + \bar{m}ln \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} mnl - m\bar{m}m \\ -nml + nlm \end{bmatrix} & \begin{bmatrix} mn\bar{m} - m\bar{m}n \\ -nm\bar{m} + nln \end{bmatrix} & 0 & 0 \end{pmatrix}^{\alpha\beta\mu} \tag{B.18}$$

We will show the explicit calculation for one Contorsion spin coefficient viz. ρ_1. It is given by

$$\rho_1 = -K_{(1)(3)(4)} = -l_{\mu}m_{\nu}\bar{m}_{\alpha}K^{\mu\nu\alpha} = -2i\pi l^2[l_{\mu}m_{\nu}\bar{m}_{\alpha}]\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{B.19}$$

The only quantity which would give non-zero scalar product with $l_{\mu}m_{\nu}\bar{m}_{\alpha}$ is $n^{\mu}\bar{m}^{\nu}m^{\alpha}$ (This can occur in any order amongst 3 vectors because we have all the orders possible in the definition of $\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}$) and the product is $l_{\mu}m_{\nu}\bar{m}_{\alpha}n^{\mu}\bar{m}^{\nu}m^{\alpha} = 1$. We can easily deduce that

$$[l_\mu m_\nu \bar{m}_\alpha]\bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}\psi = \frac{\sqrt{2}}{3}\bar{\psi}\Bigg[+ \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\Bigg]$$

$$= \frac{\sqrt{2}}{3}\begin{pmatrix} Q_0 & Q_1 & \bar{P}^{0'} & \bar{P}^{1'} \end{pmatrix} \begin{pmatrix} 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & & 0 \end{pmatrix} \begin{pmatrix} P^0 \\ P^1 \\ \bar{Q}_{0'} \\ \bar{Q}_{1'} \end{pmatrix} \tag{B.20}$$

$$= \sqrt{2}[\bar{P}^{1'}P^1 - Q^1\bar{Q}^{1'}] \tag{B.21}$$

$$= \sqrt{2}[F_2\bar{F}_2 - G_1\bar{G}_1] \tag{B.22}$$

This gives full expression for ρ

$$\rho = -K_{(1)(3)(4)} = -2\sqrt{2}i\pi l^2[F_2\bar{F}_2 - G_1\bar{G}_1] \tag{B.23}$$

B.4 Computation of Dynamical EM tensor on M_4 with torsion in NP formalism

$$\Sigma_{11}^{(NP)}(\{\}) = \frac{i\hbar c}{2\sqrt{2}}\Big(\bar{G}_1(D+\Delta)G_1 + \bar{G}_2(D+\Delta)G_2 - (D+\Delta)\bar{G}_1G_1 - (D+\Delta)\bar{G}_2G_2$$
$$+ \bar{F}_1(D+\Delta)F_1 + \bar{F}_2(D+\Delta)F_2 - (D+\Delta)\bar{F}_1F_1 - (D+\Delta)\bar{F}_2F_2\Big) \tag{B.24}$$

$$\Sigma_{21}^{(NP)}(\{\}) = \frac{i\hbar c}{4\sqrt{2}}\Big(\bar{F}_1(\delta+\delta^*)F_1 + \bar{F}_2(\delta+\delta^*)F_2 + \bar{G}_1(\delta+\delta^*)G_1 + \bar{G}_2(\delta+\delta^*)G_2$$
$$- \bar{F}_2(D+\Delta)F_1 - \bar{F}_1(D+\Delta)F_2 + \bar{G}_2(D+\Delta)G_1 + \bar{G}_1(D+\Delta)G_2$$
$$- (\delta+\delta^*)\bar{F}_1F_1 - (\delta+\delta^*)\bar{F}_2F_2 - (\delta+\delta^*)\bar{G}_1G_1 - (\delta+\delta^*)\bar{G}_2G_2$$
$$+ (D+\Delta)\bar{F}_2F_1 + (D+\Delta)\bar{F}_1F_2 - (D+\Delta)\bar{G}_2G_1 - (D+\Delta)\bar{G}_1G_2\Big) \tag{B.25}$$

$$\Sigma_{31}^{(NP)}(\{\}) = \frac{i\hbar c}{4\sqrt{2}}\Big(i\bar{F}_1(\delta-\delta^*)F_1 + i\bar{F}_2(\delta-\delta^*)F_2 + i\bar{G}_1(\delta-\delta^*)G_1 + i\bar{G}_2(\delta-\delta^*)G_2$$
$$i\bar{F}_2(D+\Delta)F_1 - i\bar{F}_1(D+\Delta)F_2 - i\bar{G}_2(D+\Delta)G_1 + i\bar{G}_1(D+\Delta)G_2$$
$$- i(\delta-\delta^*)\bar{F}_1F_1 - i(\delta-\delta^*)\bar{F}_2F_2 - iG_1(\delta-\delta^*)\bar{G}_1 - (\delta-\delta^*)i\bar{G}_2G_2$$
$$- i(D+\Delta)\bar{F}_2F_1 + i(D+\Delta)\bar{F}_1F_2 + (D+\Delta)i\bar{G}_2G_1 - (D+\Delta)i\bar{G}_1G_2\Big) \tag{B.26}$$

$$\Sigma_{41}^{(NP)}(\{\}) = \frac{i\hbar c}{4\sqrt{2}}\Big(\bar{F}_1(D-\Delta)F_1 + \bar{F}_2(D-\Delta)F_2 + \bar{G}_1(D-\Delta)G_1 + \bar{G}_2(D-\Delta)G_2$$
$$- \bar{F}_1(D+\Delta)F_1 + \bar{F}_2(D+\Delta)F_2 + \bar{G}_1(D+\Delta)G_1 - \bar{G}_2(D+\Delta)G_2$$
$$- (D-\Delta)\bar{F}_1F_1 - (D-\Delta)\bar{F}_2F_2 - (D-\Delta)\bar{G}_1G_1 - (D-\Delta)\bar{G}_2G_2$$
$$+ (D+\Delta)\bar{F}_1F_1 - (D+\Delta)\bar{F}_2F_2 - (D+\Delta)\bar{G}_1G_1 + (D+\Delta)\bar{G}_2G_2\Big) \tag{B.27}$$

$$\Sigma_{22}^{(NP)}(\{\}) = \frac{i\hbar c}{2\sqrt{2}}\Big(- \bar{F}_2(\delta+\delta^*)F_1 - \bar{F}_1(\delta+\delta^*)F_2 + \bar{G}_2(\delta+\delta^*)G_1 + \bar{G}_1(\delta+\delta^*)G_2$$
$$+ (\delta+\delta^*)\bar{F}_2F_1 + (\delta+\delta^*)\bar{F}_1F_2 - (\delta+\delta^*)\bar{G}_2G_1 - (\delta+\delta^*)\bar{G}_1G_2\Big) \tag{B.28}$$

$$\Sigma_{32}^{(NP)}(\{\}) = \frac{i\hbar c}{4\sqrt{2}}\Big(i\bar{F}_2(\delta+\delta^*)F_1 - i\bar{F}_1(\delta+\delta^*)F_2 - i\bar{G}_2(\delta+\delta^*)G_1 + i\bar{G}_1(\delta+\delta^*)G_2$$
$$- i\bar{F}_2(\delta-\delta^*)F_1 - i\bar{F}_1(\delta-\delta^*)F_2 + i\bar{G}_2(\delta-\delta^*)G_1 + i\bar{G}_1(\delta-\delta^*)G_2\Big)$$
$$- i(\delta+\delta^*)\bar{F}_2F_1 + (\delta+\delta^*)i\bar{F}_1F_2 + (\delta+\delta^*)i\bar{G}_2G_1 - (\delta+\delta^*)i\bar{G}_1G_2$$
$$+ (\delta-\delta^*)i\bar{F}_2F_1 + (\delta-\delta^*)i\bar{F}_1F_2 - (\delta-\delta^*)i\bar{G}_2G_1 - (\delta-\delta^*)i\bar{G}_1G_2\Big) \tag{B.29}$$

$$\Sigma_{42}^{(NP)}(\{\}) = \frac{i\hbar c}{4\sqrt{2}}\Big(- \bar{F}_1(\delta+\delta^*)F_1 + \bar{F}_2(\delta+\delta^*)F_2 + \bar{G}_1(\delta+\delta^*)G_1 - \bar{G}_2(\delta+\delta^*)G_2$$
$$- \bar{F}_2(D-\Delta)F_1 - \bar{F}_1(D-\Delta)F_2 + \bar{G}_2(D-\Delta)G_1 + \bar{G}_1(D-\Delta)G_2$$
$$+ (\delta+\delta^*)\bar{F}_1F_1 - (\delta+\delta^*)\bar{F}_2F_2 - (\delta+\delta^*)\bar{G}_1G_1 + (\delta+\delta^*)\bar{G}_2G_2$$
$$+ (D-\Delta)\bar{F}_2F_1 + (D-\Delta)\bar{F}_1F_2 - (D-\Delta)\bar{G}_2G_1 - (D-\Delta)\bar{G}_1G_2\Big) \tag{B.30}$$

$$\Sigma_{33}^{(NP)}(\{\}) = \frac{i\hbar c}{2\sqrt{2}}\Big(- \bar{F}_2(\delta-\delta^*)F_1 + \bar{F}_1(\delta-\delta^*)F_2 + \bar{G}_2(\delta-\delta^*)G_1 - \bar{G}_1(\delta-\delta^*)G_2\Big)$$
$$+ (\delta-\delta^*)\bar{F}_2F_1 - (\delta-\delta^*)\bar{F}_1F_2 - (\delta-\delta^*)\bar{G}_2G_1 + (\delta-\delta^*)\bar{G}_1G_2\Big) \tag{B.31}$$

$$\Sigma_{43}^{(NP)}(\{\}) = \frac{i\hbar c}{4\sqrt{2}}\Big(- i\bar{F}_1(\delta - \delta^*)F_1 + i\bar{F}_2(\delta - \delta^*)F_2 + i\bar{G}_1(\delta - \delta^*)G_1 - i\bar{G}_2(\delta - \delta^*)G_2$$
$$+ i\bar{F}_2(D - \Delta)F_1 - i\bar{F}_1(D - \Delta)F_2 - i\bar{G}_2(D - \Delta)G_1 + i\bar{G}_1(D - \Delta)G_2$$
$$+ i(\delta - \delta^*)\bar{F}_1F_1 - i(\delta - \delta^*)\bar{F}_2F_2 - i(\delta - \delta^*)\bar{G}_1G_1 + i(\delta - \delta^*)\bar{G}_2G_2$$
$$- i(D - \Delta)\bar{F}_2F_1 + i(D - \Delta)\bar{F}_1F_2 + i(D - \Delta)\bar{G}_2G_1 - i(D - \Delta)\bar{G}_1G_2\Big) \tag{B.32}$$

$$\Sigma_{44}^{(NP)}(\{\}) = \frac{i\hbar c}{2\sqrt{2}}\Big(- \bar{F}_1(D - \Delta)F_1 + \bar{F}_2(D - \Delta)F_2 + \bar{G}_1(D - \Delta)G_1 - \bar{G}_2(D - \Delta)G_2$$
$$+ (D - \Delta)\bar{F}_1F_1 - (D - \Delta)\bar{F}_2F_2 - (D - \Delta)\bar{G}_1G_1 + (D - \Delta)\bar{G}_2G_2\Big) \tag{B.33}$$

Appendix C

jhcvjsd

C.1 The linear (torsionless) Dirac eqn in 1+1 dimensions

The vanishing of torsion is characterized by the limit $a(l_2) = 3\sqrt{2}\pi L_{Pl}^2 \longrightarrow 0$. So in a torsionless case, the differential equations become (with dimensionless constants):

$$B' = (1 - w)A \tag{C.1}$$

$$A' = (1 + w)B \tag{C.2}$$

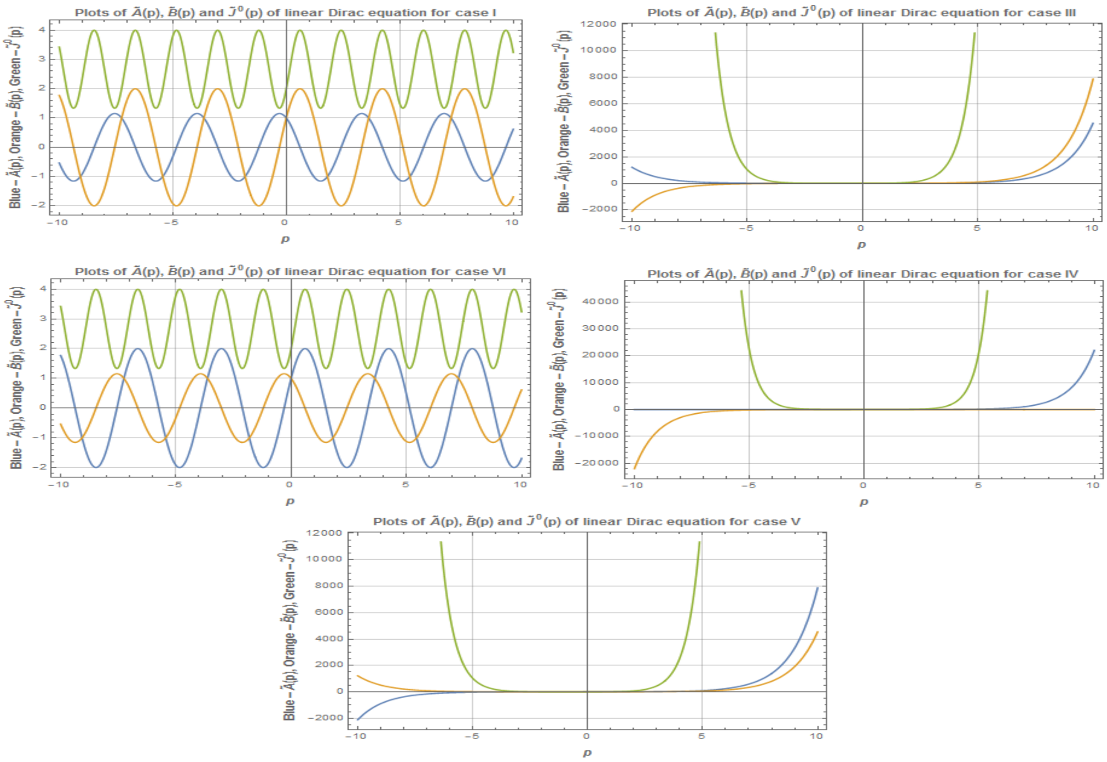


Figure C.1: **Solutions to the linear (torsionless) Dirac equations**. Only the plane-wave solutions (Cases I, VI) are physical.

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