Topology and geometry of 2 and 3-dimensional manifolds

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled *Topology and geometry of* 2 *and* 3-*dimensional manifolds* towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Chris John at Indian Institute of Science Education and Research under the supervision of Dr. Tejas Kalelkar, Assistant Professor, Department of Mathematics, during the academic year 2017-2018.

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Dr. Tejas Kalelkar

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This thesis is dedicated to the *Amma, Achachan* and *Tangutty*.

Declaration

I hereby declare that the matter embodied in the report entitled *Topology and geometry of 2 and 3-dimensional manifolds* are the results of the work carried out by me at the Department of Mathematics, IISER Pune, under the supervision of Dr. Tejas Kalelkar and the same has not been submitted elsewhere for any other degree.

Chris John

Acknowledgments

First and foremost, I would like to thank my guide Dr. Tejas Kalelkar who was very understanding and helpful through out the time I worked with him. I thank him a lot for guiding me through and teaching me a beautiful topic. I would also like to thank all my teachers at IISER Pune who motivated me to learn and love mathematics.

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Abstract

In this project we study two approaches to the structure theorem of automorphisms of surfaces, one is a geometric method given by Thurston and second is a topological approach developed by Allen Hatcher. The structure theorem classifies automorphism into one of the following types, those that are either periodic, reducible or pseudo-Anosov. This is a generalisation of the classification of automorphisms of a torus to higher genus surfaces. This theorem is also used to study 3-manifolds.

Contents

Introduction

In the theory of 3-manifold we study different types of decomposition of 3-manifolds into simpler pieces to understand different structures and properties of these smaller parts and then piece them together to get information about the whole manifold. One of the decompositions that is very interesting to study is known as *Heegard splitting*. We can construct any closed orientable 3-manifolds by gluing handle bodies of same genus g along their boundaries which are closed orientable surfaces. Here, by handlebody we mean a tubular neighbourhood of a finite graph in $\mathbb{R}^3$. To see that it is always possible to get a Heegard splitting of a manifold, we use a theorem proved by Moise and Bing **[EM]** **[RB]** which shows that all closed orientable 3-manifolds can be triangulated. For a given manifold, fix a triangulation and then the set of all edges and vertices will form a graph K^1. Now by thickening K^1 and its dual K^{1^*} we get two handle bodies of the same genus and gluing these along their boundaries gives the Heegard splitting of the manifold. In a Heegard splitting homeomorphic handle bodies are glued along their boundaries M using the elements of mapping class group $MCG_+(M)$(also know as $Aut(M)$) which is the set of orientation preserving homeomorphisms of the surface up to isotopy. Each element of this group gives an identification of the boundaries, which gives a 3-manifold. Therefore a proper understanding of the homeomorphisms of surfaces is required for studying 3-manifolds **[MS]** **[JS]**.

The simplest closed surface is the two sphere S^2 and its mapping class group has two elements, one corresponding to the degree 1 map and the other one corresponding to the degree -1 map. Thus we know that if we have two 3-manifolds with boundary as S^2, we have two different ways of gluing them together. For a torus, it is well known that the set of all automorphisms of a torus T^2 upto isotopy, which is the mapping class group $Aut(T^2)$, is isomorphic to $SL_2(\mathbb{Z})$. We also know how different types of elements of this group act on the torus. Thus when we identify two handle bodies along their toral boundaries we have a good understanding of the resulting 3-manifold.

In this project to study the classification of automorphisms of closed orientable surfaces M of genus $g > 1$ analogous to the classification of toral automorphisms. This generalisation to higher genus surfaces to some extent was first developed by Nielsen in the early 20^{th} century and was then completed by Thurston in 1970's. By an automorphism, we mean a orientation preserving homeomorphism. For a torus T^2, $Aut(T^2)$ is isomorphic to $SL_2(\mathbb{Z})$ i.e. for every element α of $SL_2(\mathbb{Z})$ there is a distinct homeomorphism h_α in $Aut(T^2)$ and vice versa.

Any element $\alpha \in SL_2(\mathbb{Z})$ satisfies one of the following.

- $|trace| = 2,\ trace = -2$ or 2, this means both the eigen values of matrix are same and is either $+1$ or -1. The homeomorphism of T^2 corresponding to such an element is a *reducible* homeomorphism since they leave certain closed curves invariant and act by *dehn twists* in the regular neighbourhood of these curves.

- $|trace| < 2,\ trace = -1,\ 0\ or\ 1$, these elements have a finite order and the homeomorphism corresponding to these elements are *periodic*.

- $|trace| > 2$, these correspond to homeomorphisms which do not leave any closed simple curve on torus unchanged. Let λ and λ^{-1} be the eigen values of the matrix and v and w be the corresponding eigenvectors. Then the homeomorphism acts on the parallel vector field $\mathfrak{F}$ of v by stretching it to $\lambda\mathfrak{F}$ and on the parallel vector field $\mathfrak{F}'$ of w by stretching it to $\lambda^{-1}\mathfrak{F}'$. Such an automorphism is called *Anosov* automorphism.

As mentioned before higher genus surfaces also have a classification of automorphisms analogous to the one above for torus called the structure theorem for automorphism of surfaces. In this project we look at two different approaches to arrive at that theorem.

For the first approach, we study an exposition on Thurston's original method by A.J. Casson and S.A. Bleiler [**CB**] for closed surfaces, which is using hyperbolic geometry, studying transverse measures on transverse singular foliation and this will be covered in part I. There is a second more topological approach developed in a paper by Allen Hatcher [**AH**]. It uses Thurston's definition of $PL(M)$ as a completion of the projectivization of set of curve systems in M and using this we arrive at a slightly weaker version of the structure theorem. This approach will be covered in part II.

Part I begins with a study of hyperbolic plane geometry because later on we will see that all the higher genus surfaces are hyperbolic. Therefore, before we can describe the classification of automorphisms of closed orientable surfaces we need to understand hyperbolic spaces and this will be covered in chapter 1. We will initially discuss different models for $\mathbb{H}^2$, its isometries, hyperbolic structures on surfaces, curves on hyperbolic surfaces and their properties and some topological results regarding the area of hyperbolic surfaces.

It was seen by Nielsen that automorphisms of a surface upto isotopy is either periodic or it leaves invariant a compact set which can be written as a union of geodesics. These sets turn out be something known as geodesic laminations. Therefore in chapter 2 we will have a discussion about geodesic laminations on surfaces and in chapter 3, their structure before we start studying the action of the automorphisms.

Once we have a fair understanding of geodesic laminations, its properties, and properties of the principal regions of a lamination, in chapter 4, we then study the properties of different types of automorphisms and construct certain transverse geodesic laminations invariant under these automorphisms and their inverses, called *stable lamination* and *unstable lamination*. In chapter 5 we see how we use cantor functions to construct singular foliations from these above mentioned geodesic laminations, which are called *stable singular foliation* and *unstable singular foliation* which are also transverse to each other. We next define *transverse measure* on both this singular foliations and using this we define a *pseudo-Anosov* automorphism on a surface. Finally we show that every non-periodic and irreducible automorphism of closed orientable hyperbolic surface is a *pseudo-Anosov* automorphism.

In part II we start with set of curve systems up to isotopy $\mathscr{CS}(M)$ and its projectivization $\mathscr{PS}(M)$. This $\mathscr{PS}(M)$ is the set of rational points of a simplicial complex attached to M called $PS(M)$. This has a natural compactification which we denote as $PL(M)$. This is the projectivization of the polyhedron $ML(M)$ and we define $ML(M)$ using the train tracks with the measures on them. We see in chapter 6 a result about the global structure of the $ML(M)$ and $PL(M)$ which says that if $\partial M \neq \phi$ then $PL(M)$ with correct topology is in fact a ball. The non-integer points of $ML(M)$ can be interpreted as the set of measured laminations on M. Now in chapter 7 we see that measured laminations can be viewed as length functionals on curves which gives a "functional" topology on $ML(M)$ and this is how we get to the $PL(M)$ as a completion of of $\mathscr{PS}(M)$. An homeomorphism of M would then induce a homeomorphism of $PL(M)$ which as mentioned before is a ball. Therefore by Brouwer's fixed point theorem there has to be a fixed point for this homeomorphism and using this we arrive at the structure theorem for automorphisms of surfaces with boundaries.

This project was entirely a reading project where for the first part we read the book on automorphisms of surfaces by Andrew Casson and Steven Bleiler [**CB**] and for the second part we read the paper on measured lamination spaces of surfaces by Allen Hatcher [**AH**]. This report does not contain any original work.

Part I

Chapter 1

Hyperbolic plane and Hyperbolic surfaces

There are three types of geometries of constant curvature that we can have on 2-dimensional manifolds. These are the Euclidean geometry better known as the plane geometry, spherical geometry and hyperbolic geometry. We know that a torus can be given the constant 0 curvature metric which is an example of a closed surface with Euclidean geometry and to see this, look at torus in its the polygonal representation, i.e. a square in $\mathbb{E}^2$ whose opposite sides are identified. The 2-sphere is a surface with positive constant curvature and is an example of a closed surface with spherical geometry. The examples for closed surfaces with hyperbolic geometry are all closed surfaces with genus $g > 1$. We will show how these surfaces are given hyperbolic geometry in section 1.2. But before that we need to understand some basics of 2-dimensional hyperbolic geometry. We can study hyperbolic plane $\mathbb{H}^2$ using different models. We will be discussing two models here, the Poincare disk model and the upper half plane model.

1.1 Hyperbolic plane

1.1.1 Poincare disk model

We identify $\mathbb{H}^2$ with the interior of a unit disk centred at the origin in the Euclidean plane $\mathbb{E}^2$ or the complex plane $\mathbb{C}$. The boundary of the unit disk is defined as the circle at infinity S^1_∞. The metric on $\mathbb{H}^2$ is $2ds/(1-r^2)$, where ds is the euclidean metric and r is the distance of a point from the centre of the unit ball. The geodesics in $\mathbb{H}^2$ are the arcs of those circles which intersect S^1_∞ perpendicularly i.e if C is a circle that intersects S^1_∞ perpendicularly then $C \cap \mathbb{H}^2$ is a geodesic in $\mathbb{H}^2$ as shown in the figure 1.1.

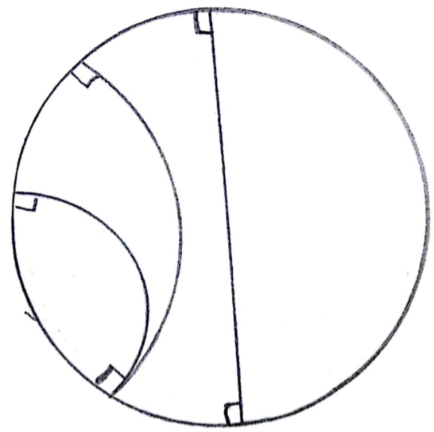


Figure 1.1: Poicare disk with geodesics

Reflection of $\mathbb{H}^2$ about a geodesic $C \cap \mathbb{H}^2$ is an involution of $\mathbb{H}^2$ induced by the inversion of $\mathbb{E}^2$ in C. Since inversions of $\mathbb{E}^2 \cup \infty$ preserve angle and take circles in $\mathbb{R}^2 \cup \infty$ to circles, the reflection about $C \cap \mathbb{H}^2$ preserves angles and carry geodesics to geodesics.

Lemma 1.1.1. *An isometry of $\mathbb{H}^2$ is a product of reflections.*

The set of all isometries of the hyperbolic plane is a group with composition of maps as the group operation.

Lemma 1.1.2. *The group of isometries act transitively on $\mathbb{H}^2$. Also, the stabilizer of any point in $\mathbb{H}^2$ is isomorphic to* $\mathrm{O}(2)$.

Proof. Any point x in $\mathbb{H}^2$ other the origin can be carried to the origin by a single inversion along some geodesic $C \cap \mathbb{H}^2$ such that the centre of C lies on the line joining origin and x. Thus any arbitrary point x in $\mathbb{H}^2$ can be carried to any other arbitrary point y in $\mathbb{H}^2$ in at most two reflections. Now that we know group of isometries is transitive, we can say that the stabilizer of any point is isomorphic to the stabilizer of origin since we can do a conjugation. Now we will find the stabilizer of the origin, $Stab(O)$. The reflections about the lines passing through origin and the rotations with origin fixed are contained in this group of isometries preserving O. Rotations can be expressed as a product of two reflections and these generate the group $\mathrm{O}(2)$. Now if we can show that the $\mathrm{O}(2)$ is the whole of the stabilizer then we are done. Every isometry extends uniquely to S^1_∞. Let P be a point in $\mathbb{H}^2$ and let it be the intersection point of two geodesic γ, γ'. Now if we have an isometry which extends to identity on S^1_∞ then it fixes the end points of these geodesics, so this isometry also fixes the geodesic and hence it fixes P. Therefore this isometry has to be the identity. This shows that $\mathrm{O}(2)$ is $Stab(O)$. □

Lemma 1.1.3. *Let $h : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ be an orientation preserving isometry which is not identity. Then it is exactly one of the following.*

- *h has exactly one fixed point in $\mathbb{H}^2$, no fixed point on S^1_∞ and it is called an elliptic isometry.*
- *h has exactly two fixed points on S^1_∞, no fixed point in $\mathbb{H}^2$ and is called hyperbolic isometry.*
- *h has exactly one fixed point on S^1_∞, no fixed point in $\mathbb{H}^2$ and is called parabolic isometry.*

Proof. By Brouwer's fixed point theorem, there is atleast one fixed point for $h : D^2 \longrightarrow D^2$.

If an orientation preserving isometry has more than one fixed points in $\mathbb{H}^2$, then it fixes every point on the geodesic joining the two points and the only orientation preserving isometry which fixes a geodesic point-wise is the identity.

Now, if the fixed point is in the interior of the unit disk, then we can assume that it is the centre and hence the isometries are rotations and are of the first type above.

If three or more points are fixed on the S^1_∞, then again the isometry has to be the identity.

Now if an isometry h fixes two points on S^1_∞, then it fixes a geodesic. This geodesic is the axis of the isometry and the isometry acts like a translation by a hyperbolic distance d along the axis. Since this translates geodesics perpendicular to axis to perpendicular geodesics, it determines h on the whole of S^1_∞ and thus it has exactly two fixed points. These kind of isometries belong to second type of isometries above.
All the others belong to the third type of isometry. □

1.1.2 Half-plane model of $\mathbb{H}^2$

In this model we identify the hyperbolic plane with the upper half plane of complex plane $\mathbb{C}$. The circle at infinity in this model is $\mathbb{R} \cup \infty$ and the metric on the upper half plane is ds/y where y is the imaginary part. Here the geodesics are the vertical lines and the circles meeting the real line in right angles as shown in figure 1.1.2. To convert the half-plane model to the unit disk model we first do an inversion about the circle $C_{-i,\sqrt{2}}$, where $-i$ is the centre of the circle and $\sqrt{2}$ is the radius, and then a reflection about the $x - axis$.

Now we want to to calculate the area of a triangle in hyperbolic plane. In the half-plane model, an infinitesimal rectangle with Euclidean width dx and height dy has hyperbolic width dx/y and height dy/y. Thus the infinitesimal area $dA = (dx\ dy)/y^2$.

Lemma 1.1.4. *Area of a triangle with one ideal vertex $\Delta_{\alpha,\beta}$ with angles α, β is $\pi - (\alpha + \beta)$.*

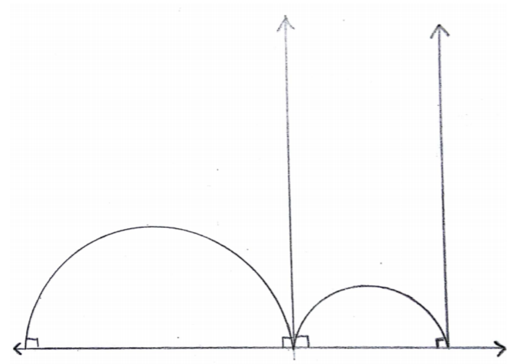


Figure 1.2: Upper half plane model with geodesics

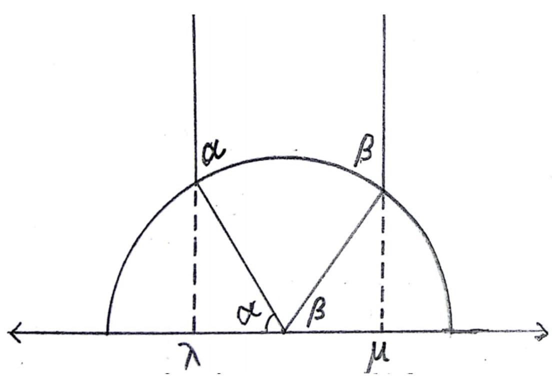


Figure 1.3:

Proof. Let us consider any triangle $\Delta_{\alpha,\beta}$ satisfying the above condition in D^2 model and rotate it so that the ideal vertex is $(0, i)$ on the complex plane. On going to the upper-half plane model we get $\Delta_{\alpha,\beta}$ such that two of its geodesic sides are perpendiculars to the real axis and third can be centred at 0 by an isometry. We can make the radius to be 1 by another isometry. Look at the figure 1.3. The radii are perpendicular to the circle, hence the angles formed at 0 are also α, β. Since radius is 1, we have $cos(\pi - \alpha) = \lambda$ and $cos\beta = \mu$. Therefore the area of the triangle is

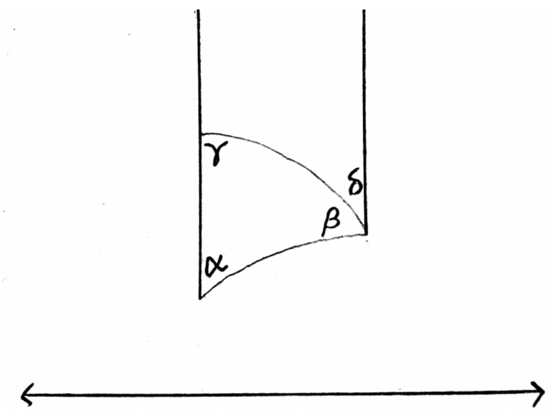


Figure 1.4:

$$\begin{aligned} area(\Delta_{\alpha,\beta}) &= \int_{\Delta_{\alpha,\beta}} \frac{dx\ dy}{y^2} = \int_{\lambda}^{\mu} dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \\ &= \int_{\lambda=cos(\pi-\alpha)}^{\mu=cos\beta} \frac{dx}{\sqrt{1-x^2}}, \text{now substituting } x = cos\theta \\ &= \int_{\pi-\alpha}^{\beta} \frac{-sin\theta\ d\theta}{sin\theta} = \pi - \alpha - \beta. \end{aligned}$$

□

Theorem 1.1.5. ***Gauss-Bonnet*** *A geodesic triangle $\Delta_{\alpha,\beta,\gamma}$ with angles α, β, γ has area $\pi - (\alpha + \beta + \gamma)$.*

Proof. By making one of the sides of the triangle $\Delta_{\alpha,\beta,\gamma}$ a perpendicular geodesic, we can represent $\Delta_{\alpha,\beta,\gamma}$ as the difference of two triangles $\Delta_{\delta,\pi-\gamma}$, $\Delta_{\alpha,\beta+\delta}$ both of which have one ideal vertex as shown in figure 1.4 we have

$$\begin{aligned} area(\Delta_{\alpha,\beta,\gamma}) &= area(\Delta_{\alpha,\beta+\delta}) - area(\Delta_{\delta,\pi-\gamma}) \\ &= \pi - (\alpha + \beta + \delta) - \pi + (\pi - \gamma + \delta), \text{by the previous lemma} \\ &= \pi - (\alpha + \beta + \gamma). \end{aligned}$$

□

Corollary 1.1.5.1. *An n-gon with angles $\alpha_1, \alpha_2, ..., \alpha_n$ has an area $(n-2)\pi - (\alpha_1 + \alpha_2 + ... + \alpha_n)$.*

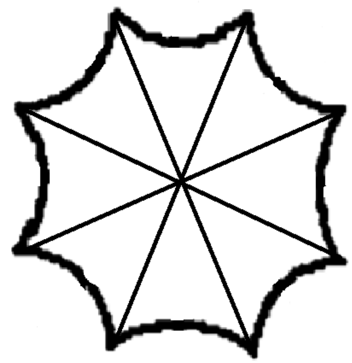


Figure 1.5:

Proof. An n-gon with its interior can be broken up into n triangles such that each them share two vertices and the side joining them with the n-gon and the third vertex lying inside the n-gon which is common to all the triangles (an example an octagon is given in figure 1.5). The angle sum of all the triangles is $\alpha_1 + \alpha_2 + ... + \alpha_n + 2\pi$ where 2π is the angle at the inner vertex. The total area of the $n - gon$ is the area of these n triangles. Area $= n\pi - (\alpha_1 + \alpha_2 + ... + \alpha_n + 2\pi) = (n-2)\pi - (\alpha_1 + \alpha_2 + ... + \alpha_n)$. □

Theorem 1.1.6. *The group of orientation preserving isometries of hyperbolic plane is isomorphic to $PSL_2(\mathbb{R})$.*

Remark 1.1.7. *A matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $PSL_2(\mathbb{R})$ acts on the half plane in the following way*

$$z \longrightarrow \frac{az+b}{cz+d}.$$

This leaves the real line invariant.

Different kinds of isometries of $\mathbb{H}^2$ and their action can be seen by looking at the properties of matrix in $PSL_2(\mathbb{R})$ representing the isometry. To see that, look at the equation of fixed points,

$$z = \frac{az+b}{cz+d}$$

Thus the equation is $cz^2 + (d-a)z - b = 0$. The discriminant of this equation is $(d-a)^2 - 4c(-b) = (a+d)^2 - 4(ad-bc) = (trace)^2 - 4$.
Case 1: When $|trace| < 2$, then roots are complex and one of them is in $\mathbb{H}^2$ and is the only one fixed point. This is the *elliptic* isometry. It represents rotations around the fixed point in both the models.
Case 2: When $|trace| > 2$, then both the fixed points are distinct and lie on the real axis or equivalently on the circle at infinity and hence it is the *hyperbolic* isometry. Up to a conjugation we can represent all hyperbolic isometries of the half-plane as the isometry

which fixes the points are 0, ∞ with the imaginary axis as its axis. When an isometry fixes 0, ∞ the matrix is $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ and it takes $z \to a^2 z$. It is a dilation as shown in figure 1.6.

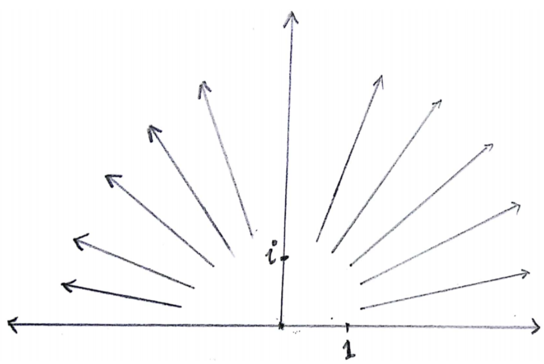


Figure 1.6: Dilation

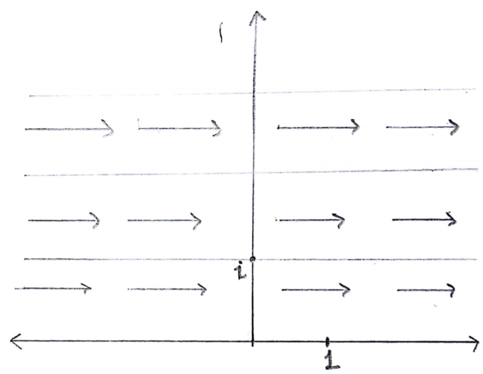


Figure 1.7: Translation

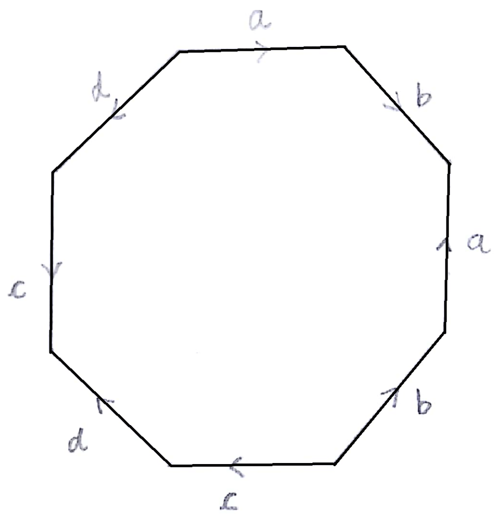
Case 3: When $|trace| = 2$, there is exactly one fixed point on the circle at infinity and it is the *parabolic* isometry. We represent this in the half plane model by making ∞ the fixed point. Then the matrix looks like $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$ where $a = \pm 1$ since $a + a^{-1} = \pm 2$. Thus matrix for a general element is $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. This acts as a horizontal translation parallel to the real axiz taking $z \to z + b$. Parabolic isometries fixing ∞ leaves horizontal levels invariant and these are called horolevels as shown in figure 1.7. In the disk model these are Euclidean circles in D tangential to S^1_∞ called horocycles.

1.2 Hyperbolic surfaces

1.2.1 Introduction

Definition A Hyperbolic structure on a surface F is defined using an atlas of charts where each $\phi_\alpha : U_\alpha \longrightarrow \mathbb{H}^2$ is such that $\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$
is the restriction of an orientation preserving isometry of $\mathbb{H}^2$.

On torus (genus $g = 1$) we can define a constant curvature κ which is 0 everywhere which can be clearly seen using the polygonal representation of torus which is a square, $4g - gon$ with $g = 1$, in $\mathbb{E}^2$. Note that the angle sum of the vertices of the square is 2π and hence we can define an smooth atlas of charts which give a constant zero curvature metric on torus



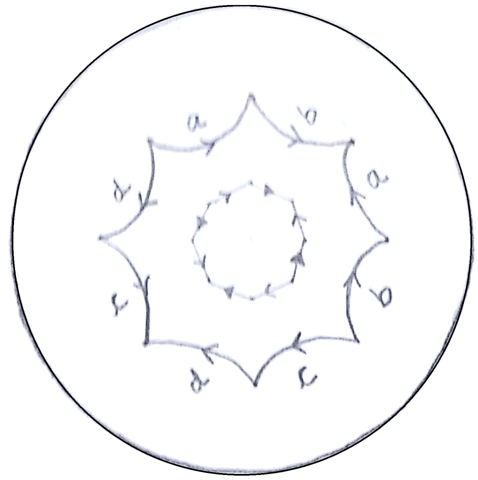


Figure 1.8: Figure 1.9:

by gluing the sides using gluing isometries. For surfaces with genus $g > 1$, its polygonal representations in $\mathbb{E}^2$, a regular $4g - gon$, has angle sum of vertices greater than 2π and hence we can not define a smooth atlas of charts with Euclidean metric and hence these surfaces cannot have constant curvature $\kappa = 0$ (figure 1.8). Now if we take this regular $4g - gon$ on $\mathbb{H}^2$ which is very small and which is concentric with S^1_∞ then it is almost like a $4g - gon$ in $\mathbb{E}^2$ because the angle sum will be greater than 2π (figure 1.9) and if we consider a ideal regular $4g - gons$ concentric with S^1_∞ then the angle sum will be zero. This means there exists a $4g - gon$ whose area is in between the area of the two $4g - gon$s described above and has an angle sum equal 2π.

Now we can define a smooth atlas of charts from this $4g - gon$ to the surface with genus g which has a constant curvature $\kappa < 0$ by gluing the sides using gluing isometries.

The elements of group Γ generated by gluing isometries of this particular polygonal forms a tiling of $\mathbb{H}^2$.

If a curve is locally (in a neighbourhood around every point) a pull back of some geodesic in $\mathbb{H}^2$ under the chart maps, then it is defined to be a geodesic in F. If a surface F has a metric defined by the pull back charts from $\mathbb{H}^2$ and it is a complete metric spaces under this metric, then it is called a complete hyperbolic surface.

Lemma 1.2.1. *Any geodesic on a complete hyperbolic surface can be extended indefinitely.*

Proof. Let C be a bounded geodesic arc in F. Since F is complete, any set of points tending towards any one of the ends of C is a Cauchy sequence and hence converges to a point, lets say x_1.Take a chart neighbourhood U_{x_1} such $\phi(x_1) = 0$ in $\mathbb{H}^2$. Now extend the geodesic from 0 to the maximum geodesic extension possible in the chart neighbourhood $\phi(U_x)$. Pull

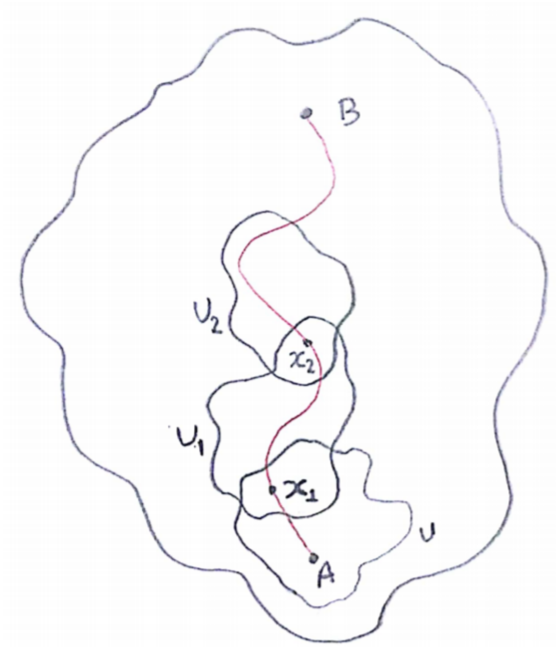


Figure 1.10:

back this extended geodesic using inverse of chart maps to extend C to a new bounded geodesic arc, lets say C_1. We can extend similarly towards the other end of C. By keeping on repeating this process until we reach the maximal such extension, we will get the required geodesic. □

Theorem 1.2.2. *Any complete, connected, simply connected hyperbolic surface is isometric to* $\mathbb{H}^2$.

Proof. Let F be a surface with all the properties described in the theorem. Consider these two maps, the exponential map $E : \mathbb{H}^2 \longrightarrow F$ and the developing map $D : F \longrightarrow \mathbb{H}^2$. Suppose this developing map D with the following properties

1. D is a local isometry

2. $D|_U = \phi_U$

is well defined. Then $D \cdot E = 1_{\mathbb{H}^2}$. This means $E \cdot D$ is a retraction of F on to $Im(E)$ and hence $Im(E)$ is closed subset of F. But $Im(E)$ is open because of invariance of domain and F is connected, it implies that $E \cdot D = 1_F$. Therefore every complete, connected, simply connected, hyperbolic surface is isometric to $\mathbb{H}^2$.

D can be constructed to be well defined using analytic continuation as shown in the figure 1.2.1. Let γ be a path from A to B in F. Let γ be covered by a sequence of charts $U_0, U_1, ..., U_n$ with $\phi_i : U_i \to \mathbb{H}^2$ and let x_is be points in γ such that $[x_i, x_{i+1}] \subseteq U_i$. If the maps do not agree on the intersection $U_{j+1} \cap U_j$ containing x_j, then using a unique isometry which can be determined by the given data a map can be found such that it agrees. Now inductively doing it for each x_i we can create chart maps which agrees on all the intersections and hence on the whole path.
Set $D(B) = \phi_n(B)$
By keeping on refining the charts it can be seen $D(B)$ depends only on γ and not on any specific chart maps. Since small homotopies do not leave the coordinate cover, two paths γ_1, γ_2 which are homotopic define same value of $D(B)$. Since F is simply connected D is well defined.

□

The above theorem implies that the universal cover $\widetilde{F}$ of every closed hyperbolic surface F is isometric to $\mathbb{H}^2$ and from now on we will identify $\widetilde{F}$ with $\mathbb{H}^2$.

$F = \widetilde{F}/\{deck\ translations\} = \mathbb{H}^2/\Gamma$. So Γ a subgroup of $PSL_2(\mathbb{R})$ is isomorphic to $\pi_1(F)$. Since $\pi_1(F)$ acts freely, no element other than identity can leave a point in $\mathbb{H}^2$ invariant, therefore $\Gamma - \{1\}$ can not have elliptic elements. We can also lift $\epsilon-$ neighbourhoods of any point in F to $\mathbb{H}^2$ and hence Γ is discreet subgroup of $PSL_2(\mathbb{R})$. This means for a compact F this *epsilon* is uniform for all points in F and this further implies that there exists an $\epsilon' > 0$, where $\epsilon' > 2\epsilon$, such that for every $g \in \Gamma - \{1\}$ the hyperbolic distance $d(x, g(x)) > \epsilon'$. Now if $\Gamma - \{1\}$ has parabolic elements, the points high up in the upper half plane model moves arbitrarily small distances under parabolic elements and hence there is a contradiction with uniformity of the ϵ. This shows us that Γ does not contain parabolic elements either. Therefore, all the elements of $\Gamma - \{1\}$ are hyperbolic.

1.2.2 Curves on hyperbolic surfaces

Definition Any closed curve on a surface which is not null homotopic is called *essential* .

Lemma 1.2.3. *Every essential closed curve on a closed hyperbolic surface is freely homotopic to a unique geodesic closed curve.*

Proof. Let x be a point on a essential closed curve C in F. Let $\widetilde{x}$ be a point on one of the complete components $\widetilde{C}$ of the lift of C in $\mathbb{H}^2$. Now there exists a $g \in \Gamma - \{1\}$ such that projection of $[\widetilde{x}, g(\widetilde{x})]$ wraps around C once.

Since g is hyperbolic, it has a geodesic axis $\widetilde{\gamma}$ and the image γ is a closed geodesic curve in F. Let $\widetilde{y}$ be a point on $\widetilde{\gamma}$, $\widetilde{U}$ be any path from $\widetilde{y}$ to $\widetilde{x}$ and $g(\widetilde{U})$ is path from $g(\widetilde{y})$ to

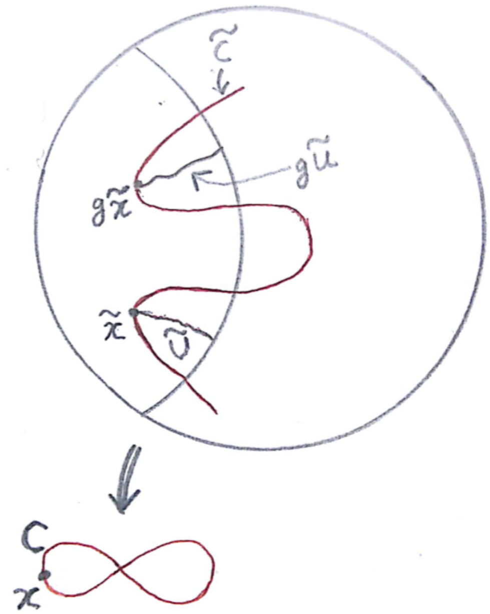


Figure 1.11:

$g(\tilde{x})$. As shown in the figure 1.11 the singular rectangle defined by the following edges $\tilde{y}$ to $\tilde{x}$ along $\tilde{U}$, $\tilde{x}$ to $g(\tilde{x})$ along $\tilde{\gamma}$, $g(\tilde{x})$ to $g(\tilde{y})$ along $g(\tilde{U})$ and finally $g(\tilde{y})$ to $\tilde{y}$ along $\tilde{\gamma}$ projects to an annulus in F which gives the free homotopy $f : S^1 \times [0,1] \longrightarrow F$ from C to a closed geodesic curve γ. Let the lift of this map be $\tilde{f} : \mathbb{R} \times [0,1] \longrightarrow \mathbb{H}^2$, where $\tilde{f}(\mathbb{R} \times \{0\}) = \tilde{C}$ and $\tilde{f}(\mathbb{R} \times \{1\}) = \tilde{\gamma}$. Let the end points of $\tilde{\gamma}$ be P, Q. To see the uniqueness, notice that $\exists d > 0$ such that the hyperbolic length of any arc $\tilde{f} : z \times [0,1] \longrightarrow \mathbb{H}^2$ for all $z \in \mathbb{R}$ is less than d because S^1 is compact and therefore hyperbolic lengths $f(z \times I)$ is bounded. This implies that in the euclidean metric on the Poincare disk the end points of both the curves converge to P, Q and since there is a unique geodesic with these endpoints, γ is freely homotopic to a unique closed curve. □

Definition A closed 1-submanifold of a surface F is a disjoint union of simple closed curves. If every component is essential and no two of them are homotopic then it is called a essential 1-submanifold.

Definition Two essential 1-submanifolds of $C_1,\ C_2$ are said to have minimal intersection if they are transversal to each other and they do not have two subarcs $A_1,\ A_2$ of $C_1,\ C_2$ with common endpoints such that the union $A_1 \cup A_2$ bounds a disk in F.

Lemma 1.2.4. *Let $C_1,\ C_2$ be an essential 1-submanifolds of F, a closed hyperbolic surface. Then C_2 is isotopic to an essential 1-submanifold which has minimal intersection with C_1.*

Proof. Let $C_1,\ C_2$ be transverse. Assume they have non-minimal intersection then they will have arcs $A_1,\ A_2$ such that $\partial A_1 = \partial A_2$ and $A_1 \cup A_2$ bounds a disk D with $int(D)$ disjoint from $C_1 \cup C_2$. Now by inner most disk argument we can push C_2 across D to reduce the number of intersections of $C_1,\ C_2$. By continuing this process for all such intersections we will get an essential 1-submanifold isotopic to C_2 which has minimal intersection with C_1. □

Lemma 1.2.5. *Every essential 1-submanifold of a closed hyperbolic surface F is isotopic to a unique geodesic 1-submanifold.*

Proof. Every component of the essential 1-submanifold is homotopic to a unique closed geodesic. Each of them are distinct because no two components are homotopic. Let the union of these geodesic closed curves be γ. By replacing the components of the preimage of C with geodesics having same endpoints we get the preimage of γ. Also notice that since no two components $\widetilde{C}_1,\ \widetilde{C}_2$ of preimage of C intersect, the geodesics corresponding to them also do not intersect. Therefore the preimage of γ is a disjoint union of geodesics.

The above lemma implies that C is isotopic to a essential 1-submanifold which has minimal intersection with γ. We will call this submanifold also as C for ease of notation. Let $\widetilde{C}$ be one of the components of preimage of C. Since $C \cap \gamma$ has minimal intersection, $\widetilde{C}$ has at most one intersection with any component of preimage of γ.

The projection of $\widetilde{C}$ is an essential closed curve in F and let its length be d. Then there exists an hyperbolic isometry g_d of $\mathbb{H}^2$ which translates points on $\widetilde{C}$ by a distance d and with axis of isometry being a component $\widetilde{\gamma}$ of the preimage of γ which has the same endpoints as $\widetilde{C}$. This isometry leaves $\widetilde{C}$ and this axis invariant. This means that if $|\widetilde{C} \cap \widetilde{\gamma}|$ is not empty then it contains infinitely many points, but since $|\widetilde{C} \cap \widetilde{\gamma}| \leq 1$, $\widetilde{C} \cap \widetilde{\gamma}$ has to be empty.

Now if $\widetilde{C}$ meets any other component $\widetilde{\gamma}'$ of the preimage of γ then $|\widetilde{C} \cap \widetilde{\gamma}'|$ has to be even and also less than or equal to 1. Therefore $|\widetilde{C} \cap \widetilde{\gamma}'| = 0$. This means $C \cap \gamma$ is empty.

This means C and γ are homologous and they cobound a part of the surface, say N. Also N has genus 0 because C and γ are homotopic and hence it is an annulus and orientable. Therefore C and γ are isotopic.

□

Lemma 1.2.6. *Suppose C_1 and C_2 are transverse essential 1-submanifolds of a hyperbolic surface F and no component of C_1 is isotopic to any component of C_2. Then C_1 and C_2 have minimal intersection if and only if there exists a homeomorphism $h : F \longrightarrow F$ isotopic to the identity, such that $h(C_1)$ and $h(C_2)$ are both geodesic 1-submanifolds.*

Proof. If both $C_1,\ C_2$ are geodesic 1-submanifolds, the preimages of components of $C_1,\ C_2$ will not intersect more than once in the universal cover. Hence they are transverse to each other.

Now for the converse, suppose C_1, C_2 have minimal intersections. Using the previous lemma 1.2.6 we can assume that C_2 is geodesic. By the same lemma we know that C_1 is also isotopic to a geodesic 1-submanifold γ_1. But this isotopy must leave C_2 invariant. We will now construct this isotopy.

If $C_1 \cap \gamma_1 \neq \phi$, then there are arcs A and α respectively which bound an inner most disk D. If $D \cap C_2 \neq \phi$, since C_2 has minimal intersection with both C_1 and γ_1, it will be a union of arcs across D with one end on A and the other end on α as shown in figure 1.12. Then push A to α through the disk D leaving C_2 invariant. After a sequence of pushes similar to this across disks, $C_1 \cap \gamma_1 = \phi$.

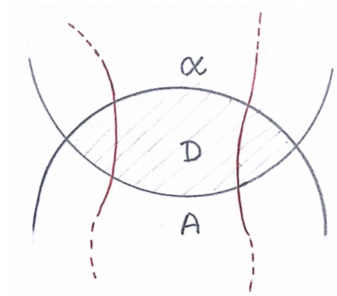


Figure 1.12:

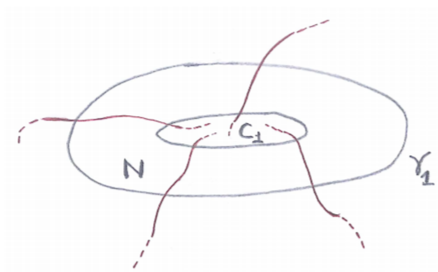


Figure 1.13:

Now C_1 and γ_1 bound an annulus N like in the proof of the previous lemma. $N \cap C_2$ is an union of arcs which join the two boundary components of N. Look at the figure 1.13. We now choose an isotopy which is identity outside N and leaves these arcs of C_2 invariant inside N. This is the required homeomorphism which is isotopic to identity. □

Definition Let the geometric intersection number, $i(C_1, C_2)$, of C_1, C_2 be the minimum value of $|C_1' \cap C_2'|$ where C_j' is homotopic to C_j. Then we say that C_1, C_2 have minimal intersection if and only if $|C_1 \cap C_2| = i(C_1, C_2)$.

Definition If two essential 1-submanifolds C_1, C_2 of F have minimal intersection and the complement of $C_1 \cup C_2$ is a collection disks then C_1, C_2 *fill* F.

There always exists two simple closed curves C_1, C_2 on F such that they *fill* F.

Theorem 1.2.7. *Let F be a closed orientable hyperbolic surface and $h : F \longrightarrow F$ be an orientation preserving automorphism. If for every simple closed curve in F there exists a $k \in \mathbb{Z}^+$ such that $h^k(C) \sim C$ ($\sim$ means homotopic) then there exists an $n \in \mathbb{Z}^+$ such that h^n is isotopic to identity. These are called periodic automorphisms.*

Proof. Let C_1, C_2 be two simple geodesic closed curves filling F. Let $k_1, k_2 \in \mathbb{Z}^+$ be such that $h^{k_1}(C_1) \sim C_1$ and $h^{k_2}(C_2) \sim C_2$. $h^k(C_i) \sim C_i$ where $i = 1, 2$ and $k = k_1 \cdot k_2$. The curves $h^k(C_1), h^k(C_2)$ have minimal intersection. According to the lemma 1.2.7, since the two curves have minimal intersection, there exists an homeomorphism g isotopic to identity such $g \cdot h^k(C_i)$ is a geodesic simple closed curve and hence $g \cdot h^k(C_i) = C_i)$ where $i = 1, 2$. This means $g \cdot h^k(C_1 \cap C_2) = C_1 \cap C_2$. Since this is a set of finitely many points we can see that $g \cdot h^k$ is a permutation of this points. Therefore for some $m \geq 0$, $(g \cdot h^k)^m$ is fixes them point-wise and hence it is isotopic to an homeomorphism f which is identity on $C_1 \cup C_2$. Now since g is isotopic to identity it implies f and h^{km} are isotopic. Since C_1, C_2 are filling curves hence their complementary region is a union of disk and since f is isotopic to identity on $C_1 \cup C_2$, by Alexander's trick f is isotopic to identity over the whole of F. □

1.2.3 Few topological results about hyperbolic surfaces

Let χ_F be the Euler characteristic of the surface F. This section discusses how for hyperbolic surface F with finite area, just knowing its Euler characteristic χ_F, which is a topological invariant, we can determine the area of the F. Such relation only exists in the hyperbolic surfaces.

Lemma 1.2.8. *A compact hyperbolic surface with geodesic boundary has area $-2\pi\chi_F$.*

Proof. Let $F = \mathbb{H}^2/\Gamma$ and $P \in \mathbb{H}^2$. For $g \in \Gamma$ let $U_g = \{x \in \mathbb{H}^2 | d(x, P) \leq d(x, g(P))\}$ be the half plane corresponding to g. Let $U = \cap_{g \in \Gamma} U_g$. Now since F is compact $\exists d$ such that $\forall x \in \mathbb{H}^2,\ d(x, g(P)) \leq d$ for some g. Therefore all points x in U are within a distance d of P. Also, $Fr(U)$ is contained in the union of $Fr(U_g)$. Since $\{g(P) | g \in \Gamma\}$ is discrete, only finitely many g's in Γ will satisfy the condition $d(p, g(P)) \leq 2d$ and this means intersection of U with union of $Fr(U_g)$ is finite. Thus $Fr(U)$ has finitely many geodesic edges and hence is a finite sided polygon which is called the Poincare polygon of F. Identification on the edges by elements of Γ give F. Since the isometries always identify two edges, the number of edges are even, i.e. of the form $2e$. This identification gives us a decomposition of F into one $2-$cell, e edges and v vertices. Since the angle sum at each vertex is 2π, the area of $U = (2e - 2)\pi - 2\pi v$ (v is the number vertices after identification). Also notice that $\chi_F = 1 - e + v$, therefore $e - 1 = v - \chi_F$. Thus the area of $U = 2(e-1)\pi - 2\pi v = -2\chi_F$.

If F has geodesic boundary components, the double of F, DF also has hyperbolic structure with area $area(DF) = 2\ area(F)$. If F is compact then $\chi_{DF} = 2\chi_F$. Since DF is closed the lemma follows. □

Lemma 1.2.9. *An unbounded hyperbolic surface F with finite area is homeomorphic to a closed surface minus a finite set and has area $-2\pi\chi_F$.*

Proof. Let $F = \mathbb{H}^2/\Gamma$ and $P \in \mathbb{H}^2$, here Γ may have non hyperbolic elements as well. Like in the previous proof, for $g \in \Gamma$ let $U_g = \{x \in \mathbb{H}^2 | d(x, P) \leq d(x, g(P))\}$ be the half plane corresponding to g and $U = \cap_{g\in\Gamma} U_g$. By the same arguments in the previous proof it can be shown that $Fr(U)$ is locally finite union of geodesic arcs even though U is not compact.

The Euclidean closure of U has only finitely many points on the circle at infinity. This is because area of F and hence area of U is finite and if the closure contains n points on S^1_∞ then U contains a n-gon which has $(n-2)\pi$ area which has to be less than area U. This forces some upper bound on the number of points at the circle of infinity.

Each internal vertex v with angle α_v has to be equidistant from at least 2 distinct translates of P and hence is a point of intersection of two distinct geodesic. This mean $\alpha_v < \pi$. Let G be a finite sided polygonal in U with vertices of G being some subset of internal vertices of U.

$$
\begin{aligned}
Area(G) &= (n-2)\pi - (angle\ sum\ of\ G) \\
n\pi - (angle\ sum\ of\ G) &= 2\pi + area(G) \\
\sum_{v\in G} (\pi - \alpha_v) &\leq 2\pi + area(G) \\
\sum_{v\in Int(U)} (\pi - \alpha_v) &\leq 2\pi + area(U)
\end{aligned}
$$

Let us divide the vertices into two set $A = \{v|\alpha_v \leq 2\pi/3\}$ and $B = \{v|\alpha_v \geq 2\pi/3\}$. First observe that A is finite because $\sum(\pi - \alpha_v)$ converges. Among all the vertices of U going to a same vertex in F after identification, there can be at most two elements of B and there has to be at least one element of A since angle sum has to be 2π. This shows that U has finitely many vertices. We get F by identifying edges in pairs and removing ideal vertices from $U \cup \{ideal\ vertices\}$.

F has a cell decomposition with one 2-cell, e edges, v vertices after identifications and removal of ideal vertices.

$$
\begin{aligned}
area(U) &= (n-2)\pi - angle\ sum \\
&= (2e-2)\pi - 2\pi v \\
&= 2(e-1-v)\pi \\
&= -2\chi_F
\end{aligned}
$$

□

Theorem 1.2.10. *A complete hyperbolic surface F with finite area and geodesic boundary is homeomorphic to a compact surface minus a finite set and has area $-2\pi\chi_F + \pi\chi_{\partial F}$.*

Proof. On doubling F the area of double of F, DF, is double of the area of F and $\chi_{DF} = 2\chi_F - \chi_{\partial F}$. Therefore area of $F = \frac{1}{2}(-2\pi\chi_{DF}) = -2\pi\chi_F + \pi\chi_{\partial F}$ □

Corollary 1.2.10.1. *The area of any complete hyperbolic surface with a totally geodesic boundary is* $n\pi,\ n \in \mathbb{Z}^+$.

Chapter 2

Geodesic Laminations

Every non-periodic automorphism preserves a closed subset known as geodesic lamination. In this chapter we develop some concepts about geodesic lamination.

2.1 Geodesic Laminations

Definition A curve on a complete hyperbolic surface F is a geodesic if and only if its the image of a complete geodesic in $\mathbb{H}^2 \cong \widetilde{F}$. If a geodesic has no self transverse intersection on F then it is defined as a simple geodesic.

Definition A closed, non-empty subset of F which is a disjoint union of simple geodesic is defined to be a geodesic lamination L of F.

The figure 2.1 and figure 2.2 are some examples of geodesic laminations.

Let F be a surface and $\forall x \in F$ we have the set of all the non-oriented geodesic segments of length two units centred at x. Then F and this set at each point on F together forms the projectivized tangent bundle $PT(F)$ of F. Let x be some point of F, then $(x, \sigma) \in PT(F)$ gives a unique geodesic passing through x.

We can construct a $PT(F)$ of F w.r.t a given atlas of charts in the following way. Look at the figure 2.3.

- Fix a horizontal line $\mathcal{Q}$ passing through $\mathbb{H}^2 \cong \widetilde{F}$.

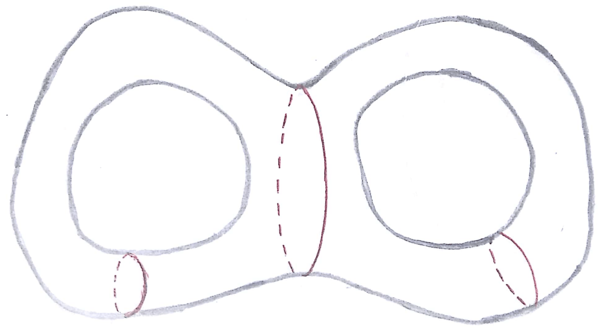


Figure 2.1: Lamination with 3 closed leaves

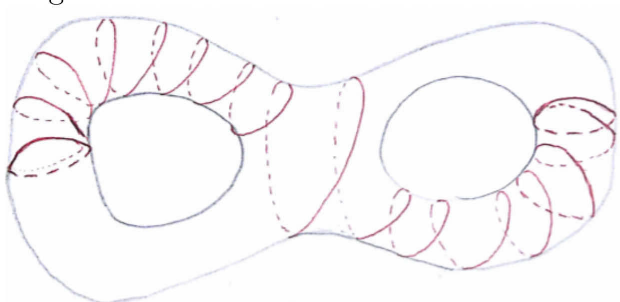


Figure 2.2: Lamination with a spiralling leaf

- For every point $x \in U_x \subset F$, where U_x is the chart neighbourhood of ϕ_U, and a geodesic γ passing through x, look at its image in the universal cover $\mathbb{H}^2$.
- We define the direction of γ at x as the angle the tangent of $\phi_U(\gamma)$ at $\phi_U(x)$ makes with $\mathcal{Q}$ or a line parallel to $\mathcal{Q}$ passing through $\phi_U(x)$.

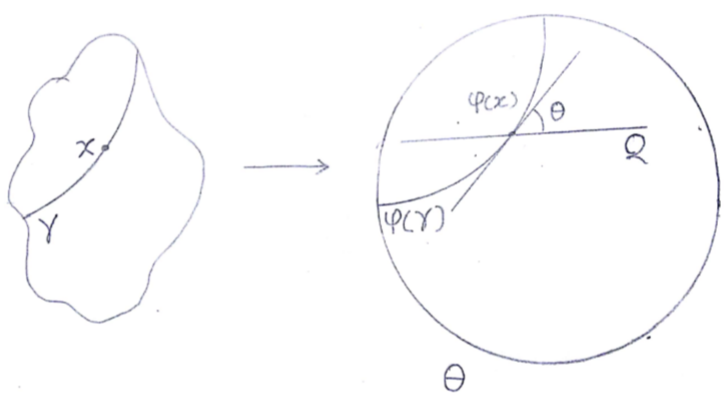


Figure 2.3:

Therefore $(x, dir\gamma_x) \in PT(F)$, where $dir\gamma_x$ is the direction of γ at x.

We can define a topology on $PT(F)$ using the chart chart maps from F by taking inverse image of chart neighbourhood under the projection map p. This forms a basis for the topology on $PT(F)$.

$$\begin{array}{ccccccccc} (x,\sigma) & \epsilon & PT(F) & \supset & p^{-1}(U) & \cong & U \times \mathbb{R}P^1 \\ \downarrow p & & \downarrow p & & \downarrow p & & \downarrow p \\ x & \epsilon & F & \supset & U & \xrightarrow{1_U} & U \end{array}$$

Lemma 2.1.1. *If $L = \cup_{x\in L}\gamma_x$ is a geodesic lamination and γ_x, γ_y are either disjoint or they coincide $\forall\ x, y \in L$. Then the direction of γ_x at x w.r.t. a given chart varies continuously with x.*

Proof. For a geodesic leaf γ, if the direction w.r.t a fixed chart changes discontinuously at some point $x \in \gamma$ it means that there are two directions at that point. Then, with respect to each direction there is a geodesic through the point x. This would mean that the lift of geodesic leaves will intersect with each other in $\mathbb{H}^2$. But by the definition of a lamination this cannot happen. □

Lemma 2.1.2. *The closure of any disjoint union of geodesics on F is a lamination.*

Proof. Consider a sequence of points $x_n \in L$ converging to some point $x \in F$ and let γ_n be the leaf passing through x_n (in the given decomposition in to geodesics). Since the set of direction is homeomorphic to $\mathbb{R}P^1$ which is a compact set, the sequence of directions of γ_n at x_n, $dir\gamma_{x_n}$, has a subsequence which converges to $dir\gamma_x$ as $n \to \infty$. Consider a leaf γ through x with this direction at x. The directions are measured through a fixed chart about x. If y is a point on γ at a signed distance d from x and y_n is at distance d from x_n then $y_n \to y$. Hence γ is contained in $\bar{L}$ and it is a union of geodesics.

Now to show that they are disjoint assume the contrary. So there exists $\gamma, \gamma' \in \bar{L}$ which intersect at some point $x \in F$. Then for $\beta, \beta' \in L$ passing through arbitrarily close to x with directions approximating the directions of γ, γ'. If the approximation is sufficiently close then it can be seen that β, β' will also intersect. By similar arguments it can be shown that each leaf of $\bar{L}$ is simple. Thus $\bar{L}$ is a geodesic lamination. □

Lemma 2.1.3. *A geodesic lamination in a closed orientable hyperbolic surface F is nowhere dense and can be expressed in a unique way as a disjoint union of geodesics.*

Proof. By Poincare-Hopf theorem, a closed hyperbolic surface F which has negative Euler characteristic can not have a continuously varying line field defined on all of F. This implies that L is a proper subset of F.

Let $\widetilde{L}$ be the lift of L in $\mathbb{H}^2$. Now, $\widetilde{L} = \cup_{x\in\widetilde{L}}\gamma_x$ and $\gamma_x,\ \gamma_y$ either coincide or are disjoint for all $x,\ y \in \widetilde{L}$. Let there be an arc α in $\widetilde{L}$ transverse to γ_x at $x \in \alpha$ as shown in figure 2.4.

Define $\Phi : \alpha \times \mathbb{R} \to \mathbb{H}^2$ such that $(y,t) \to y + t \in \gamma_y$. Orientation of normal to α is fixed. Thus by lemma 2.1.1 Φ is continuous. $\Phi(\alpha \times \mathbb{R}) \subset \widetilde{L}$. For any $d > 0$, there exists an $z \in \gamma_x$ such that hyperbolic d-neighbourhood U of z is contained in $\Phi(\alpha \times \mathbb{R})$.

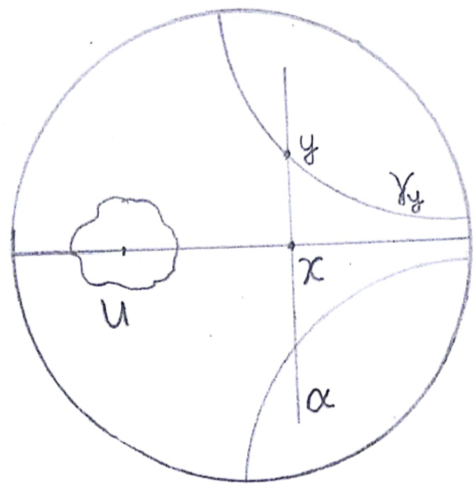


Figure 2.4:

If d is the diameter of the Poincare polygonal of F then under the projection U is mapped onto F which means $L = F$ but this is a contradiction. Therefore L has empty interior and in nowhere dense.

By the same argument it can be shown that L can be expressed as the union of geodesics in a unique way. □

Definition Let A, B be two closed subsets of a compact set X such that $A \subset N_\epsilon(B)$ and $B \subset N_\epsilon(A)$. Then we say that the Hausdorff distance between A and B, $d_H(A,B)$, is $d_H(A,B) \leq \epsilon$.

Hausdorff distance defines a metric on 2^X, where 2^X is the set of all non-empty closed subsets of X. The topology of 2^X is dependent on the topology of X and not on the metrics on X. 2^X is compact because it is totally bounded and complete.

Theorem 2.1.4. *The set of all geodesic laminations on a closed orientable hyperbolic surface $\Lambda(F)$ is compact under the Hausdorff distance.*

Proof. Since $\Lambda(F) \subset 2^F$, we need to show that $\Lambda(F)$ is a closed subset i.e. if a sequence of laminations L_n is converging to a compact set L, then L has to be a lamination. If $x \in L$, then there are $x_n \in L_n$ nearby such that x_n converge to x. If for each L_n, let γ_n be the leaf containing x_n then some subsequence γ_{n_i} converge to a geodesic γ passing through x. If the direction doesn't converge then we can find some other subsequence which converges to some other leaf through x. Hence a convergence in Λ is not sufficient and a convergence in $\hat{\Lambda}$, which will be defined later on, needs to be shown i.e. the directions also converge. Remember the topology given on $PT(F)$.

Remark 2.1.5.

- *$PT(F)$ is a 3-manifold with $p : PT(F) \to F$ continuous.*
- *Since $PT(F)$ is compact and metrizable, $2^{PT(F)}$ is also compact and metrizable.*
- *For every geodesic γ in F, there is a section to $PT(F)$ corresponding to it whose image is $\hat{\gamma} = \{(x, dir\gamma_x) | x \in \gamma\}$. Every point of $PT(F)$ lies on a unique $\hat{\gamma}$.*

Let $\hat{L} = \cup_{\gamma \in L} \hat{\gamma}$ and $\hat{\Lambda}(F) = \{\hat{L} | L \in \Lambda(F)\}$ then we have the following:

$$\begin{array}{ccc} \hat{L} & \subset & PT(F) \\ p \downarrow \text{bijection} & & \downarrow p \\ L & \subset & F \end{array}$$

Lemma 2.1.1 says that the direction varies continuously with x hence the maps p^{-1} is continuous on L which implies that p is a homeomorphism. Thus $\hat{L}$ is compact and $\hat{L} \in 2^{PT(F)}$ for all $L \in 2^F$. Hence $\hat{\Lambda} \subseteq 2^{PT(F)}$. p induces a bijective continuous map $p_* : \hat{\Lambda} \to \Lambda$ The first thing that needs to be shown is that $\hat{\Lambda}$ is closed in $2^{PT(F)}$ and that would imply Λ is compact. Let $\hat{L_n} \in \hat{\Lambda} \to A \in 2^{PT(F)}$. It needs to be shown that $A = \hat{L}$ for some $L \in F$. $p(A) = L$ is non-empty, compact and hence belongs to 2^F.

For every $x \in L,\ x = p((x, \sigma))$, where $(x, \sigma) \in A \subset PT(F)$, there exists $(x_n, \sigma_n) \in \hat{L_n} \to (x, \sigma)$. Then $x_n \to x$ and the direction of σ_n at x_n converges to direction of σ at x. The extention γ of σ is also belongs to L and can be shown the same way as it was done in lemma 2.1.2. Also L is a disjoint union of geodesics. Thus we get that $\hat{\Lambda}$ is closed in $2^{PT(F)}$. This also shows that p_*^{-1} is continuous. □

Lemma 2.1.6. *Hausdorff distance on 2^F and $2^{PT(F)}$ define the same topology on $\Lambda(F)$.*

Proof. In the previous proof we have seen that both p_* and p_*^{-1} are continuous and bijective. Therefore p_* is a homeomorphism. This implies the required result. □

Definition We say a leaf γ of L is isolated if for all $x \in \gamma$, $\exists\, U_x$ a neighbourhood of x such that $(U_x, U_x \cap L)$ is homeomorphic to (disk, diameter).

The set $L - \{\textit{isolated leaves}\}$ is closed subset of L and hence it's also a lamination. This is called a derived lamination of L. We denote this set with L'.

Lemma 2.1.7. *If L' is empty then L is a finite union of simple closed geodesics and L is an isolated point in $\Lambda(F)$.*

Proof. $L' = \phi$ means all leaves of L are isolated. Thus L is a closed 1-submanifold of F i.e it is a disjoint union of simple closed geodesics. To see that L is an isolated point in $\Lambda(F)$ it needs to be checked if any other lamination L_* lies in a close neighbourhood of L. There exists an $\epsilon > 0$ for which the closure ϵ-neighbourhood of L is a disjoint union of annuli. Let L_* be such that $d(L, L_*) < \epsilon$. Let γ_* be a leaf in the ϵ neighbourhood of L thus in a ϵ neighbourhood of a closed simple geodesic C. Now for some lift $\widetilde{\gamma}$ of γ, $d(\widetilde{\gamma}, \widetilde{C}) < \epsilon$ but the only geodesic in this neighbourhood is $\widetilde{C}$. So $\gamma = C \subset L$ and $L_* \subseteq L$. Now it can be seen that $L_* = L$ because every component of ϵ neighbourhood of L contains a point of L_*. □

Definition We call a lamination L perfect if $L' = L$.

2.2 Action of homeomorphisms on geodesic laminations

Theorem 2.2.1. *Let $h : F_1 \longrightarrow F_2$ be a homeomorphism of closed orientable hyperbolic surfaces with lift $\widetilde{h} : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ to the universal cover. Then $\widetilde{h}$ has a unique continuous extension over $\mathbb{H}^2 \cup S^1_\infty$.*

Proof. Isometries of $\mathbb{H}^2$ extend over S^1_∞ as they are uniformly continuous w.r.t the Euclidean metric on the Poincare disk. It needs to shown that for any γ, a geodesic in $\mathbb{H}^2$, $\widetilde{h}(\gamma)$ converges to a point on S^1_∞. Without loss of generality $\widetilde{h}(0) = 0$. Since both $\widetilde{h}, \widetilde{h^{-1}}$ are lifts of continuous maps they are uniformly continuous w.r.t. hyperbolic metric. This means there exists a $k > 0$ such that

$$\begin{aligned} d(x,y) \leq 1/k \quad &\Rightarrow \quad d(\widetilde{h}(x), \widetilde{h}(y)) \leq 1 \text{ and} \\ d(\widetilde{h}(x), \widetilde{h}(y)) \leq 1/k \quad &\Rightarrow \quad d(x,y) \leq 1 \end{aligned}$$

Subdivide a geodesic joining $x,\ y$ into k equal subintervals, then

$$d(x,y) \leq 1 \Rightarrow d(\widetilde{h}(x), \widetilde{h}(y)) \leq k \tag{2.2.1}$$

Similarly for any integer $n > 0$

$$d(\widetilde{h}(x), \widetilde{h}(y)) \leq n \Rightarrow d(x,y) \leq nk \tag{2.2.2}$$

The map $\widetilde{h}$ should not be spiralling or else there will be no limit for $\widetilde{h}$. Let P_t be a point at distance t from 0 and $Q_t = \widetilde{h}(P_t)$. Choose a reference line through 0 for the direction. The angle θ_t of the geodesic at Q_t must have a limit as $t \to \infty$. Notice that (2) says since every point on (P_t, P_{t+1}) has distance greater than t from 0, all point on the arc Q_tQ_{t+1} will be at distance t/k from 0 while (1) implies that $d(Q_t, Q_{t+1}) \leq k$. This means all points on arc Q_tQ_{t+1} has distance $t/k - k$ from 0. If we take $t \geq 2k^2$ then the distance will be at least $t/2k$.

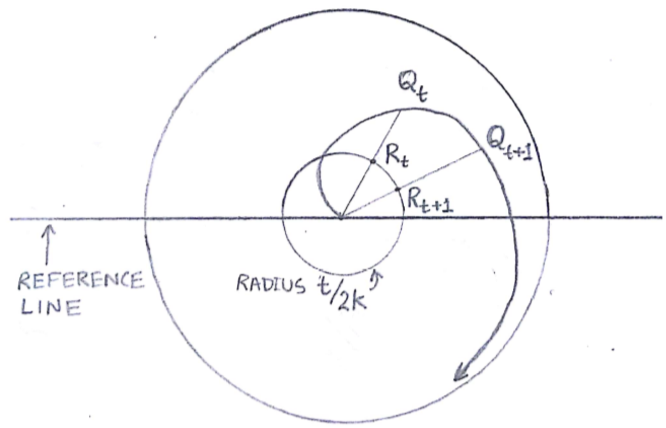


Figure 2.5:

Take a circle centred at 0 with radius $t/2k$ as shown in figure 2.5. Let R_t be the intersection of the geodesic segment $0Q_t$ with the circle. The radial projection of Q_tQ_{t+1} to R_tR_{t+1} decreases the hyperbolic length.
Therefore

$$k \geq d(Q_t, Q_{t+1}) \geq arcR_tR_{t+1} = |\theta_{t+1} - \theta_t| \sinh(t/2k)$$

since $\sinh x = \frac{e^x - e^{-x}}{2} \geq \frac{e^x}{4}$, $\forall\ x \geq 1$

$$|\theta_{t+1} - \theta_t| \leq 4ke^{t/2k} \quad if\ t \geq 2k^2$$

Thus for $u \geq t \geq 2k^2$,

$$|\theta_u - \theta_t| \leq \int_{t-1}^{u} 4ke^{-s/2k} ds = C(k)e^{-t/2k}$$

Hence $\lim_{t\to\infty} \theta_t$ exists and let it be θ. Now by choosing the point on S^1_∞ which makes angle θ

with the reference line as the end point of γ we have defined a extension of $\widetilde{h}$.

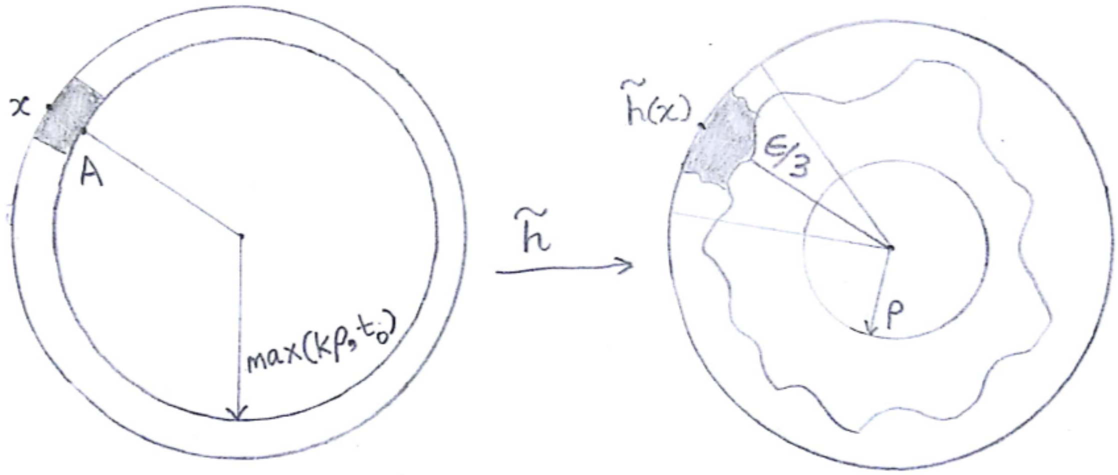


Figure 2.6:

We need to check if this a continuous extension. Let a neighbourhood of $\widetilde{h}(x)$ be an 2ϵ section in the exterior on a circle of radius ρ centred at 0. We need to get the neighbourhood of x that was mapped into the above set. If we choose this neighbourhood to lie in the exterior a circle such that the image of this circle is in the exterior of the circle with radius ρ and width is as determined by the continuity of $\widetilde{h}$ at A. Then we need the radius to be $\max(k\rho, t_0)$ where $t_0 > 2k^2$ and $Ce^{-t_0/2k} < \epsilon/3$. □

Lemma 2.2.2. *Given two homotopic homeomorphisms $h_0,\ h_1 : F_1 \longrightarrow F_2$ of closed orientable hyperbolic surfaces and a lift $\widetilde{h_0}$ of h_0, there exists a lift $\widetilde{h_1}$ of h_1 such that $\widetilde{h_0} = \widetilde{h_1}$ on S^1_∞.*

Proof. Let $H : F_1 \times I \longrightarrow F_2$ be an homotopy between h_0 and h_1. Let $\widetilde{H}$ be the lift of H with $\widetilde{H}_0 = \widetilde{h}_0$. Since H is uniformly continuous in hyperbolic metric, the hyperbolic lengths of arcs $\widetilde{H}(a \times I)$ is bounded. Thus the Euclidean distance between $\widetilde{h}_0, \widetilde{h}_1$ tends to 0 as $x \to S^1_\infty$. Hence $\forall\ y \in S^1_\infty$, $\widetilde{h}_0(y) = \widetilde{h}_1(y)$. □

A unit tangent bundle of F is defined as the set $\{(x,\sigma)|x \in F$ and σ is oriented geodesic segment of length 2 centred at $x\}$. It the double cover of $PT(F)$.

Lemma 2.2.3. *Let $Y = \{(a,b,c)|a,b,c$ are distinct points of S^1_∞ in couter-clockwise direction $\}$. If $F = \mathbb{H}^2/\Gamma$ is a closed orientable hyperbolic surface, then $UT(F) = Y/\Gamma$.*

Proof. Every point $(a,b,c) \in Y$ gives a point $x \in \mathbb{H}^2$ with an oriented direction which gives an unique geodesic σ as shown in the figure 2.7.

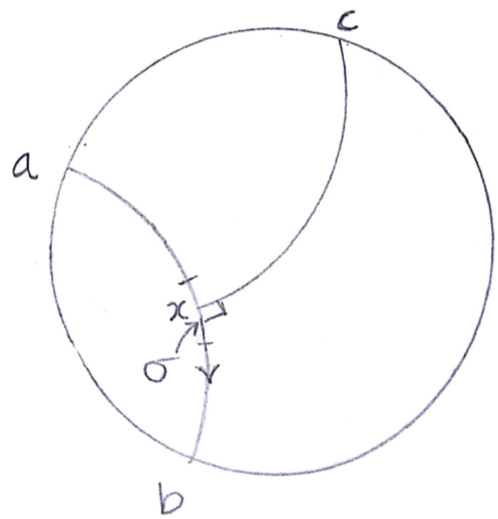


Figure 2.7:

$$\begin{array}{ccccccc}
(a,b,c) & \in & Y & \longrightarrow & UT(\mathbb{H}^2) & \ni & (x,\sigma)\\
 & & \downarrow p & \searrow q & \downarrow p & & \downarrow p\\
 & & Y/\Gamma & \xrightarrow{f} & UT(F) & \ni & (p(x),\sigma)
\end{array}$$

Let ab be a geodesic from a to b and x be the foot of the perpendicular from c to ab. Let $\sigma \subset ab$ be the length 2 segment centred on x. It can be seen from the diagram above q is a continuous surjection from Y to $UT(F)$ and this defines the homeomorphism $f : Y/\Gamma \to UT(F)$. $\square$

Theorem 2.2.4. *Every orientation preserving homeomorphism $h : F_1 \longrightarrow F_2$ of closed orientable hyperbolic surfaces induces a homeomorphism $\hat{h} : UT(F_1) \longrightarrow UT(F_2)$ between their unit tangent bundles. If two homeomorphisms are homotopic then their induced homeomorphisms are equal, that is if $h \sim k$ then $\hat{h} = \hat{k}$. If $h_1 : F_1 \longrightarrow F_2$ and $h_2 : F_2 \longrightarrow F_3$ are homeomorphisms, then $\hat{h}_2 \cdot \hat{h}_1 = \widehat{h_2 \cdot h_1}$. $\hat{h}$ takes lifted geodesics to lifted geodesics.*

Proof. Given a homeomorphism $h : F_1 \longrightarrow F_2$ where $F_i = \mathbb{H}^2/\Gamma_i$ it induces an isomorphism of groups $h_* : \Gamma_1 \longrightarrow \Gamma_2$. Let lift of h be $\widetilde{h} : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$. By theorem 2.2.1 this lift gives an orientation preserving map $\widetilde{h}|_{S^1_\infty} : S^1_\infty \longrightarrow S^1_\infty$ which induces $\bar{h} : Y \longrightarrow Y$. Now if $g_1 \in \Gamma_1$, then $\widetilde{h} g_1 = g_2 \widetilde{h}$ where $g_2 = h_*(g_1) \in \Gamma_2$. This implies that $\bar{h}$ induces $\hat{h} : Y/\Gamma_1 \longrightarrow Y/\Gamma_2$ which by lemma 2.2.3 induces $\hat{h} : UT(F_1) \longrightarrow UT(F_2)$. This is dependent on the lift of h.

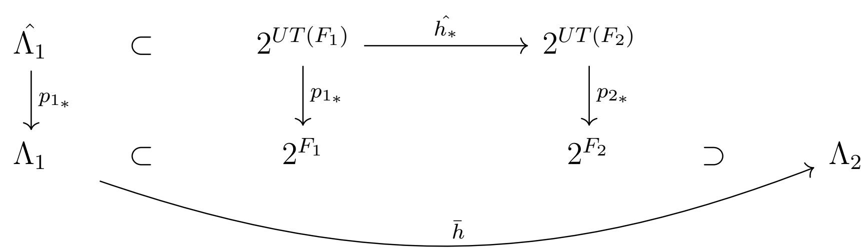
If $h \sim k$ then by lemma 2.2.2 $\exists\ \widetilde{k}$ such that $\widetilde{h}|_{S^1_\infty} = \widetilde{k}|_{S^1_\infty}$ hence $\bar{h} = \bar{k}$ which implies that $\hat{h} = \hat{k}$. We can choose $\widetilde{h_2 \cdot h_1}$ to be equal to $\widetilde{h_2} \cdot \widetilde{h_1}$ which gives $\overline{h_2 \cdot h_1} = \bar{h_2} \cdot \bar{h_1} : S^1_\infty \longrightarrow S^1_\infty$ and hence $\widehat{h_2 \cdot h_1} = \hat{h}_2 \cdot \hat{h}_1$.

$\square$

Theorem 2.2.5. *Any orientation preserving homeomorphism $h : F_1 \longrightarrow F_2$ of closed orientable hyperbolic surfaces induces an homeomorphism $\bar{h} : \Lambda(F_1) \longrightarrow \Lambda(F_2)$. If $h \sim k$ then $\bar{h} = \bar{k}$. If $h_1 : F_1 \longrightarrow F_2$ and $h_2 : F_2 \longrightarrow F_3$ are homeomorphisms then $\overline{h_2 \cdot h_1} = \bar{h_2} \cdot \bar{h_1}$.*

Proof. For $L \in \Lambda(F_1)$ let $\bar{h}(L) = \cup_{\gamma \in L} \bar{h}(\gamma)$. If γ_1, γ_2 are two leaves of lift of L in $\mathbb{H}^2$ with endpoints a_1, b_1 and a_2, b_2 respectively. Then $\widetilde{h}(a_1), \widetilde{h}(b_1)$ does not separate $\widetilde{h}(a_2)\widetilde{h}(b_2)$ because a_1, b_1 does not separate a_2, b_2 and $\widetilde{h}$ preserves order on S^1_∞. Therefore the leaves of $\bar{h}(L)$ are disjoint and simple.

Since $\hat{h} : UT(F_1) \longrightarrow UT(F_2)$ is continuous (theorem 2.2.4) and the lift of L, $\hat{L} \subset UT(F_1)$ is compact $\hat{h}(\hat{L}) \subset UT(F_2)$ is compact. Because the projection map $p_i : UT(F_i) \longrightarrow F_i$ is continuous $\bar{h}(L) \subset F_2$ is compact and is a lamination.



In the above diagram $\hat{h_*}$ is continuous which implies that $\bar{h} \cdot p_{1_*} | \hat{\Lambda}_1 = p_{2_*} \cdot \hat{h_*}$ is continuous and hence $\bar{h}$ is continuous. Similarly we can show the same for h^{-1}. Thus $\bar{h}$ is a homeomorphism.

$\square$

Chapter 3

Structures on Geodesic Laminations

In this chapter we will be discussing the properties of different types of geodesic laminations of closed orientable hyperbolic surfaces, the properties of its complementary regions and some results about the derived laminations of a geodesic lamination.

3.1 Principal region

Let F be a closed hyperbolic surface and L be a geodesic lamination on F.

Definition Any component of $F - L$ is called a *principal region* of L.

Lemma 3.1.1. *If U is a principal region for L and $\widetilde{U}$ is its lift in $\mathbb{H}^2$, then $\bar{\widetilde{U}}$ is a contractible hyperbolic surface with geodesic boundary.*

Proof. Since $\widetilde{U}$ is the lift of U, a component of $F - L$, it is a component of $\mathbb{H}^2 - \widetilde{L}$. For any two points $a,\ b \in \widetilde{U}$ which are distinct there cannot be a leaf $\widetilde{\gamma}$ of $\widetilde{L}$ which separates the two points. This implies that there exists a geodesic path joining these two points and lies completely inside $\widetilde{U}$. Therefore $\widetilde{U}$ is hyperbolically convex.
All points on the $Fr(U)$ lie on some leaf $\widetilde{\gamma}$ of L. Let N be the hyperbolic convex hull of a point $a \in \widetilde{U}$ with $\widetilde{\gamma}$. Then $N - \widetilde{\gamma} \subset \widetilde{U}$ and further $\widetilde{\gamma} \subset \bar{\widetilde{U}}$. This means $\bar{\widetilde{U}}$ is hyperbolically convex set with geodesic boundary. □

Remark 3.1.2. *The map from $\pi_1(U) \longrightarrow \pi_1(F)$ is injective.*

Definition Let U be a principal region of L and γ a leaf of L which satisfies the following. $\forall x \in \gamma$ there exists an ϵ neighbourhood of it such that at least one of the components of

$N_\epsilon(x) - 2\epsilon$, where $2\epsilon \subset \gamma$ centred at x, is contained U. Then such a leaf γ is defined as a boundary leaf of U.

Lemma 3.1.3. *The boundary leaves of L are dense in L.*

Proof. Let x be a arbitrary point on an arbitrary leaf γ of L. We know that L is nowhere dense in F, hence we can find a point y arbitrarily close to x in $F - L$. The first point of intersection of geodesic path from y to x with L lies on a boundary leaf. This shows that the set of boundary leaves are dense in L. □

We will now look at some features of the union of principal regions of L and their boundary leaves. We know that $\widetilde{U}$ is simply connected hence it is a universal cover of U, that is $U = \widetilde{U}/\Gamma_U$ where $\Gamma_U = g \in \Gamma | \; g(U) = U$. Usually the interior of the closure of a principal region U is not U. We find a away around it by defining $V_U = \bar{\widetilde{U}}/\Gamma_U$ which is a hyperbolic surface with geodesic boundary with interior homeomorphic to U. Let the $G = \sqcup_U V_U$.

Lemma 3.1.4. *For each L of F, there are finitely many principal regions and each of them have finitely many geodesic boundary components.*

Proof.

$$ar(G) = ar(G - \partial G) = ar(F - L) \leq ar(F)$$

because G is a hyperbolic surface with geodesic boundary and has finite area. Since each component of G has an area at least π (Area of a complete hyperbolic surface with finite area and with geodesic boundary is $-2\pi\chi_F + \pi\chi_{\partial F)}$), there are only finitely many such components each with finitely many boundary components. □

The boundary leaves and boundary components are not in one to one correspondance.

Definition A complete hyperbolic surface homeomorphic to $S^1 \times [0,1] - A$, where A is a finite set and $A \subset S^1 \times \{1\}$, with finite area and geodesic boundary is called a *crown*.

Lemma 3.1.5. *Let L be a geodesic lamination without closed leaves on F. Then any principal region U of L is of exactly one of following kinds:*
(i) a finite sided ideal polygon or
(ii) U has a compact core U_0 such that $U - U_0$ is isometric to finite disjoint union of interiors of crowns.

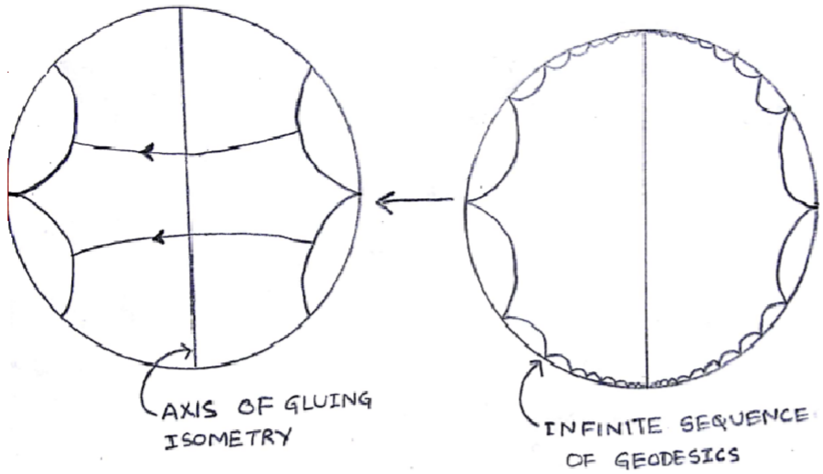


Figure 3.1:

Proof. For a principal region U of L, let $\widetilde{U}$ be one of the components of its lift in $\mathbb{H}^2$. No boundary leaf of U is closed. Theorem 1.2.10 implies that every boundary leaf β_0 of $\widetilde{U}$ has two adjacent boundary leaves β_1, β_{-1} having a endpoint in common with β_0. When $\widetilde{U}$ is not finite sided polygon all the leaves $\beta_n,\ n \in \mathbb{Z}$, are all distict. But U has only finitely many boundary leaves. This means there is a deck translation g leaving $\widetilde{U}$ invariant with $g\beta_0 = \beta_n$ for some $n > 0$. For each boundary leaf β_0, let $\mathcal{W}$ be the smallest convex set containing all translate β_n. The crown set of β_0 is defined as $\mathcal{W} - \partial\mathcal{W}$. The crown set of two boundary leaves either coincide or are disjoint. Notice that the image of a crown set is isometric to a crown in U. $\widetilde{U_0} = \widetilde{U} - \widetilde{U_\infty}$ where $\widetilde{U_\infty}$ is the union of crown set for all boundary leaves.

$\widetilde{U_0}$ is hyperbolically convex and is the universal cover of U_0. U_0 has a compact frontier of finitely many disjoint simple closed geodesics and hence U_0 is a simple closed curve or a compact connected surface with geodesic boundary and it is unique. □

Let U be a surface which is homeomorphic to a closed annulus with a point removed from both of its boundary components as in the figure 3.2. It can be constructed from a finite sided polygon in $\mathbb{H}^2$ with two vertices on S^1_∞ by identifying two of its edges as shown in the figure 3.1. The universal cover $\widetilde{U}$ is obtained by repeating the polygon via the gluing isometry along its axis. The image of the axis of the isometry after the identification, which is a closed curve, forms the core U_0 of U. Observe that $U - U_0$ is a disjoint union of two crowns both of which have the associated set of points on S^1, A, to be a singleton set.

Lemma 3.1.6. *Let L be a geodesic lamination without any closed leaves. Then no point on S^1_∞ can be an endpoint of infinitely many leaves of $\widetilde{L}$, the lift of L in $\mathbb{H}^2$.*

Proof. Let $x \in S^1_\infty$ be an endpoint of infinitely many leaves and hence an endpoint of infinitely many boundary since they are dense in $\widetilde{L}$. This implies that there exists some

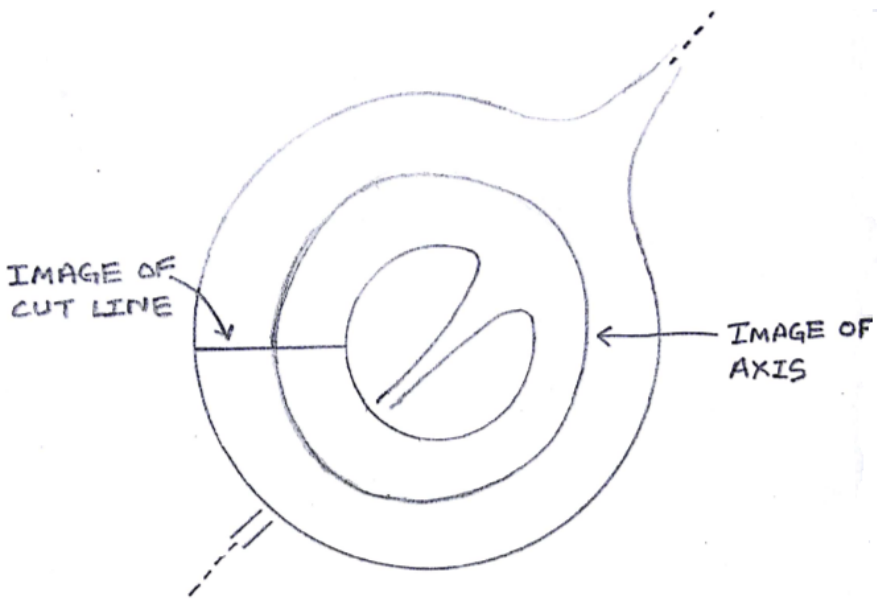


Figure 3.2:

$g \in \Gamma - 1$ which leaves x invariant and let its axis be $\widetilde{C}$. For a leaf $\widetilde{\gamma}$ with one endpoint x, $\overline{\cup_{n\in\mathbb{Z}} g^n(\widetilde{\gamma})}$ contains $\widetilde{C}$ which means $\widetilde{L}$ also contains $\widetilde{C}$. But this would mean that there is a closed curve in L. □

3.2 Derived lamination and its properties

As defined before L' is the derived lamination of L.

Lemma 3.2.1. *Every closed leaf C in a geodesic lamination of F has a neighbourhood N such that $L' \cap N \subset C$.*

Proof. Let $F = \mathbb{H}^2/\Gamma$ and $\widetilde{L}$ be the lift of L in $\mathbb{H}^2$. Let $\widetilde{C}$ be one of the components lift of C and g be an element of Γ with $\widetilde{C}$ as axis and which translates points on $\widetilde{C}$ by a distance d. Then there exists an ϵ such that for any $\widetilde{\gamma}$ which is at a distance less than ϵ from $\widetilde{C}$ the distance of the feets of perpendicular from the endpoints of $\widetilde{\gamma}$ on to $\widetilde{C}$ is greater than d.

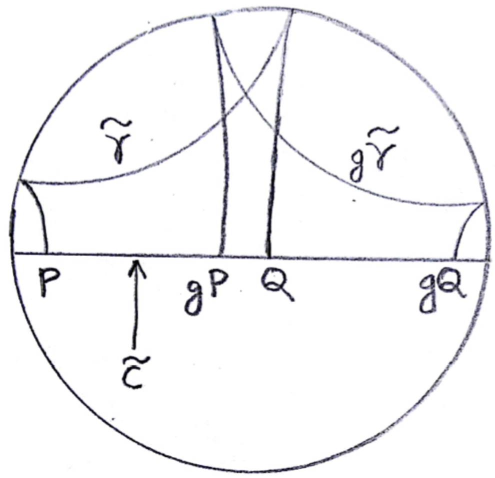


Figure 3.3:

So $g\widetilde{\gamma}$ and $\widetilde{\gamma}$ will intersect transversally. Look at the figure 3.3. Let this ϵ neighbourhood be $\widetilde{N}$ which maps to N. If any $\widetilde{\gamma}$ meets $\widetilde{N}$ then it has an endpoint in common with $\widetilde{C}$. The proof of the previous lemma shows that there are finitely many leaves between $\widetilde{\gamma}$ and $g\widetilde{\gamma}$. This shows that since these leaves are isolated they don't belong to $\widetilde{L}'$ and hence do not belong to L'. Therefore there exists a neighbourhood of C with the above mentioned properties. □

Theorem 3.2.2. *Let L be a geodesic lamination on F and L_1 be a sublamination of L, then $L' \cap L_1$ is a union of components of L'.*

Proof. Define L_2 =union of all non-closed leaves of $L_1 \cap L'$. Since there is a neighbourhood for any closed leaf C in L that contains only C according to the lemma 3.2.1, we get L_2 is closed and hence is a lamination without closed leaves. If $L' \cap U$ is contained in the core U_0 of U, where U is a principal region of L_2. Then $L' - L_2$ which the collection of all closed leaves is equal to $L' \cap V_0$ where V_0 is the union of cores of all principal regions of L_2. Since V_0 is compact $L' \cap V_0$ is compact and hence L_2 is open is L'. Thus we have L_2 is both open and closed in L' and that $L' \cap L_1 = L_2 \cup \{isolated\ leaves\}$ is both open and closed in L'.

Let $\mathbb{H}^2$ be the lift of F, $\widetilde{L}$ be the lift of L, $\widetilde{L}'$ be the lift of L' and $\widetilde{U}$ be the lift of U. For any leaf of $\widetilde{L}$, $leaf \cap \widetilde{U}$ is in $\widetilde{U}$. It is a union of finitely many diagonals if $\widetilde{U}$ is finite sided polygon and thus each one of them is isolated, $\widetilde{U} \cap L' = \phi$. The other case is if U has a core U_0 such $\widetilde{U} \cap \widetilde{U_0}$ is the covering and $\widetilde{U} - \widetilde{U_0}$ is a universal cover of a crown. It can be seen that a leaf has one end in such a component if $\widetilde{U} \cap \widetilde{L}$ is contained in $\widetilde{U_0}$. The lemma 3.1.6 implies that such a leaf is isolated and we have $\widetilde{U} \cap \widetilde{L}' \subset \widetilde{U_0}$ as required.

□

Corollary 3.2.2.1. *For any lamination L, $L''' = L''$ and for a lamination without closed leaves $L'' = L'$.*

Proof. By the lemma 3.1.3 L' has only finitely many components $K_1, K_2, ..., K_r$. $L'' = K_1' \cup K_2' \cup ... \cup K_r'$. Let $L_1 = L''$ then by the above theorem $L'' = K_1 \cup K_2 \cup \cup K_s$ where $s \leq r$ after renumbering. So $K_i' = K_i (i \leq s)$ thus $L''' = K_1' \cup K_2' \cup ... \cup K_s' = K_1 \cup K_2 \cup ... \cup K_s = L''$. Now if $L'' \neq L'$ it would mean that $s < r$ hence K_r' is empty and by the lemma 2.1.7 K_r is a disjoint union of finitely many simple closed curves. Therefore when $L'' \neq L'$ there is at least one closed leaf.

□

Corollary 3.2.2.2. *Every leaf of L is dense in L if and only if L is connected and $L' = L$*

Proof. γ be a leaf of L and $L_1 = \bar{\gamma} \subset L$. By the theorem, a L being connected and $L' = L$ would imply that γ is dense in L. If every leaf is dense in L, then $L' = L$ and is connected.

□

Lemma 3.2.3. *Let F_1 and F_2 be two surface and L a geodesic lamination on F_1. Let $h : F_1 \longrightarrow F_2$ be a homeomorphism, then $(\hat{h}(L))' = \hat{h}(L')$ and for any principal region U of L there is an unique $\hat{h}(U)$ for $\hat{h}(L)$ whose bourdary leaves are precisely the images of boundary leaves of L under $\hat{h}$. If U has core U_0 then the core of $\hat{h}(U)$ is $\hat{h}(U_0)$ such that its frontier is the image of frontier of U_0 under $\hat{h}$.*

Proof. The idea of the proof is that the notions of isolated leaves, boundary leaves, core of a pricipal region can all be defined in terms of the cyclic ordering of points on S^1_∞ and this is preserved by $\hat{h}$ in a way similar to that of the theorem 2.2.4

□

Chapter 4

Automorphisms of surfaces

We study some results about non-periodic irreducible automorphisms. We then construct two laminations for a given non-periodic irreducible automorphism h where one of them, stable lamination, is invariant under h and the second one, unstable lamination, is invariant under h^{-1}.

4.1 Properties of Automorphisms

Definition An automorphism $h : F \longrightarrow F$ of a closed orientable hyperbolic surface is called periodic if for some positive integer n, h^n is homotopic to identity.

Definition An automorphism $h : F \longrightarrow F$ of an closed orientable hyperbolic surface is called reducible is it leaves an essential 1-submanifold invariant.

If h^n is homotopic to identity, there exist g isotopic to h such that g^n is identity.
For a hyperbolic surface F, h is an reducible automorphism if and only if there exist a geodesic 1-submanifold C of F such that $\hat{h}(C) = C$.

Lemma 4.1.1. *Let $h : F \longrightarrow F$ induce a map between the first homology group, $h_* : H_1(F) \longrightarrow H_1(F)$. Let the matrix of this map with respect to some fixed basis be A. If the characteristic polynomial $\chi_h(t)$ of A is irreducible over $\mathbb{Z}$, has no roots of unity as zeroes, and is not a polynomial in t^n for any $n > 1$, then h is irreducible and non-periodic.*

Proof. Suppose the automorphism h is periodic, then $h_*^n = I$ which means that the characteristic polynomial χ_A has the roots of unity as its roots.

If h is reducible then that would mean that $h(C) = C$ after some homotopy for some essential 1-submanifold C. Let us assume it to be essential geodesic 1-submanifold (lemma 1.2.6). There are two cases for such an h, either some component of C does not separate F or the other case where all the components of C separate F.
Case 1: Let C_1 be a component of C which does not separate F. Since for some $n > 0$ $h_*^n[C_1] = [C_1]$, 1 is an eigenvalue of A and hence A has roots of unity as its roots.

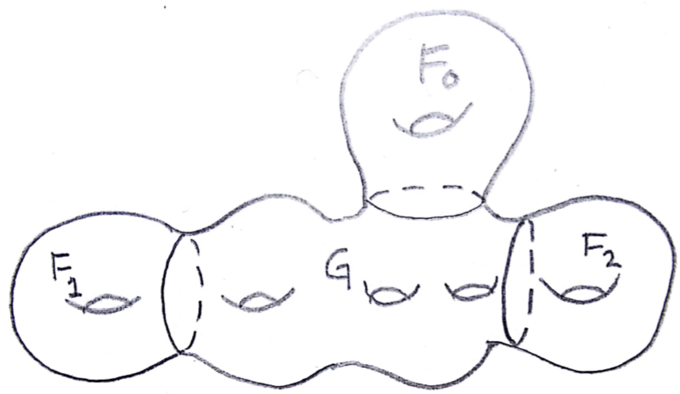


Figure 4.1:

Case 2: Let F_0 be a component of $F - C$ which has its frontier a single component of C. Given $F_r = h^r(F_0)$ there exists a minimum positive n such that $F_n = F_0$. Thus

$$H_1(F) = H_1(F_0) \bigoplus H_1(F_1) \bigoplus ... \bigoplus H_1(F_{n-1}) \bigoplus H_1(\overline{G})$$

where G is F minus all the F_i's. h_* permutes all F_i's in a cyclic order. So A for $n = 4$ would look like the following.

$$A = \begin{bmatrix} 0 & 0 & 0 & B & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & C \end{bmatrix}$$

$\chi_A = |A - tI| = |B - t^n I||C - tI|$, since we know χ_A is irreducible and F_0 is at least genus 1, we have $\chi_A = |B - t^n I|$ which is a polynomial in t^n.

Therefore if h_* of an automorphism h satisfies all of the required conditions then it can be neither periodic nor reducible. □

Definition If $L_1,\ L_2$ are two laminations then $L_1 \pitchfork L_2$ be the set of transverse intersection points of L_1 and L_2.

Lemma 4.1.2. *If $h : F \longrightarrow F$ is a non-periodic automorphism of a closed orientable hyperbolic surface, then $\hat{h}(L) = L$ for some $L \in \Lambda(F)$.*

Proof. By the theorem 1.2.7 we know that since h is non-periodic there exists some simple closed geodesic curve C such that for all $n > 0$ $\hat{h}^n(C) \neq C$. Since $\Lambda(F)$ is compact the sequence $\hat{h}^n$ has a converging subsequence $\hat{h}^{n_i}$. Let this sequence converge to K. Since each of the $\hat{h}^n$ is distinct K is not an isolated point in $\Lambda(F)$. This means K' is non-empty (lemma 2.1.7). For some fixed $r > 0$, $\hat{h}^{n_i+r}(C)$ converges to $\hat{h}^r(K)$ and $|\hat{h}^{n_i}(C) \cap \hat{h}^{n_i+r}(C)| = |C \cap \hat{h}^r(C)| = N_r$. If $|K \pitchfork \hat{h}^r(K)| > N_r$ then for every $N_r + 1$ points a small neighbourhood will contain at least one point of $\hat{h}^{n_i}(C) \cap \hat{h}^{n_i+r}(C)$ for a sufficiently large n_i. This is not possible hence $|K \pitchfork \hat{h}^r(K)| \leq N_r$. It follows that $K \pitchfork \hat{h}^r(K')$ is empty.

For $r,\ s$ such $r \neq s$, $\hat{h}^r(K')$ and $\hat{h}^s(K')$ have no transverse intersections. $E = \cup_r \hat{h}^r(K')$ is non empty disjoint union of simple geodesics. By lemma 2.1.2 $L = \overline{E}$ is the required lamination. □

Corollary 4.1.2.1. *There is a lift $\widetilde{k}$ of some positive power of h such that its restriction on S^1_∞ has two fixed points.*

Proof. Since there exists a L such that $\hat{h}(L) = L$, given any oriented boundary leaf γ of L, for some $n > 0$ $\hat{h}^n(\gamma) = \gamma$ to itself. Let $\widetilde{\gamma}$ be some component of the lift of γ and let $\widetilde{j}$ some lift of $\hat{h}^n$. Then there exists a deck translation g of $\mathbb{H}^2$ such $\widetilde{j}(\widetilde{\gamma}) = g\widetilde{\gamma}$. Then there is a $\widetilde{k} = g^{-1}\widetilde{j}$ which also a lift of $\hat{h}^n$ which maps $\widetilde{\gamma}$ to itself and hence fixes its endpoints on S^1_∞. This is the required lift. □

Lemma 4.1.3. *If $h : F \longrightarrow F$ is an irreducible automorphism and $\hat{h}(L) = L$ for some $L \in \Lambda(F)$, then each component of $F - L'$ is contractible and each leaf of L is dense in L'.*

Proof. The union of closed leaves is a geodesic 1-submanifold. Since $\hat{h}(L) = L$ it has to leave this invariant. But h is irreducible hence the set of all closed leaves of L is empty. All the component of $F - L$ is contractible. Otherwise these principal regions will have non-empty compact cores by lemma 3.1.5 and the union of all these cores, let it be V_0, have as frontier FrV_0 which is a geodesic 1-submanifold invariant under $\hat{h}$ and hence is empty.

Since $\hat{h}$ leaves L' invariant $F - L'$ is also contractible. This implies L' is connected. If γ is a leaf and $L_1 = \bar{\gamma}$, then by lemma 2.1.7 $L'_1 \subseteq L_1 \cap L'$. By theorem 3.2.2, γ is dense in L'. □

Definition Let L be an geodesic lamination of on an orientable surface, and let $\widetilde{L}$ be the lift of L. A *stable interval* (of L) is a closed interval $I \subset S^1_\infty$ such that for any two points $P, Q \in IntI$ there is a leaf $\widetilde{\delta} \subset \widetilde{L}$ whose endpoints separate P and Q from ∂I.

Remark 4.1.4. *Let $\widetilde{k}$ be a lift of h^m where $h : F \longrightarrow F$ is non-periodic and irreducible and $\widetilde{k}(I) = I$ for some stable interval I of L where $\hat{h}(L) = L$. Then the endpoints of I are contracting fixed points and z is an expanding fixed point.*

Lemma 4.1.5. *Every irreducible, non-periodic automorphism $h : F \longrightarrow F$ satisfies the following. There is a $L \in \Lambda(F)$ such that $\hat{h}(L) = L$. If a lift $\widetilde{k}$ of a positive power of h maps a stable interval onto itself, then the restriction of $\widetilde{k}$ to I has a fixed point $z \in IntI$ such that for all $P \in I - z$, $\widetilde{k}(P)$ converges to a point in ∂I.*

Proof. Consider the lamination in constructed in lemma 4.1.2. $\hat{L} = L$ and $K \pitchfork L = \phi$, where K was the limit of $\hat{h}^{n_i}(C)$ for some simple closed curve C. Let $\widetilde{k}$ be lift of h^m for some $m > 0$ and let I be an stable interval which is invariant under $\widetilde{k}$. The geodesic $\widetilde{\gamma}$ joining the endpoints of this stable interval is a leaf of $\widetilde{L}$. Since every leaf of L is dense in L' (lemma 4.1.3), the image γ of $\widetilde{\gamma}$ is dense in L' and $C \cup L' \neq \phi$ so $C \cup \gamma \neq \phi$. This means that some component $\widetilde{C}$ of the lift of C will one endpoint $A \in IntI$ and the other endpoint $B \notin I$. $\widetilde{k^n}(A)$, $\widetilde{k^n}(B)$ converge to A_∞, B_∞ with $A_\infty \in I$ and $B_\infty \notin IntI$. To see that $A_\infty \in \partial I$ let us assume to the contrary that $A_\infty \in IntI$. This means there is a leaf $\widetilde{\delta} \subset \widetilde{L}$ whose endpoints separate A_∞ and ∂I. Recall that $\hat{h}^{n_i}(C) \to K$ and $\widetilde{k}$ is the lift of h^m.

There must be infinitely many n_i is some residue class modulo m so that $\hat{h}^{mq_i}(C)$ must converge to $\hat{h}^r(K)$ for some sequence $q_i \to \infty$ and some integer r. The geodesic $\widetilde{C_\infty}$ with endpoints A_∞, B_∞ is a lift of a leaf of $\hat{h}^r(K)$ which meets $\widetilde{\delta}$ transversely. Since K and hence $\hat{h}^r(K)$ has no transverse intersection with L our assumption is wrong. Hence $A_\infty \in \partial I$ and it is a contracting fixed point of the restriction of $\widetilde{k}$ to I. If $U \subset I$ which has A, A_∞ as its endpoints then notice that $\widetilde{k}$ moves every point of U closer to A_∞. Now there exists another open set which is a neighbourhood V of the other endpoint of I such V and $\widetilde{k}(V)$ are disjoint from U. Such a V can be found because $\widetilde{h}$ is continuous and restricts to identity on ∂I. Because I is a stable interval, there exists a leaf $\widetilde{\delta}$ such that one of the endpoints $X \in U$ and the other endpoint $Y \in V$. Notice that $\widetilde{k}(X)$ and $\widetilde{k}(Y)$ separate X, Y from ∂I. Let us consider the sequences $\widetilde{k}^{-n}(X)$, $\widetilde{k}^{-n}(Y)$ with limit points X_∞, Y_∞. It can be shown $X_\infty = Y_\infty$ which is then equal to z since all points of I other than $z = X_\infty = Y_\infty$ move towards ∂I. Assume that $X_\infty \neq Y_\infty$, then $J = \overline{S^1_\infty - \{X_\infty, Y_\infty\}}$ and $J \not\subset I$ is also a stable interval J with $\widetilde{k}(J) = J$ with both endpoints expanding fixed points (this is because they were). But as shown before for a stable interval at least one of the endpoints have to be contracting endpoint. This is a contradiction and hence the assumption that $X_\infty \neq Y_\infty$ is wrong. ☐

Theorem 4.1.6. *If $h : F \longrightarrow F$ is a non periodic irreducible automorphism then any lift of h^m, $m > 0$, has finitely many fixed points on S^1_∞ which are alternately expanding and contracting. There is unique perfect lamination L^s, invariant under $\hat{h}$, such that $\widetilde{L^s}$ contains the geodesics joining consecutive contracting fixed points of any lift of a positive power of h. Every leaf of L^s is dense in L^s.*

Remark 4.1.7. *L^s is the stable lamination of h. The stable lamination of h^{-1} is called unstable lamination L^u.*

Proof. Let L be the lamination described in the previous lemma. The lemma 4.1.3 states that if lamination L is fixed under an irreducible automorphism then every leaf of L is dense in L', which all leaves of L' are dense in L' hence L' is perfect. Let $\widetilde{k}$ be the lift of h^m, for some $m > 0$. Then one of the following happens.
Case 1: $\widetilde{k}$ fixes the endpoints of a boundary leaf of $\widetilde{L}'$. Then that leaf $\widetilde{\gamma}$ is the frontier of a unique component $\widetilde{U}$ of $\mathbb{H}^2 - \widetilde{L}'$ since L' is perfect. $\widetilde{U}$ is finite sided polygon as lemma 4.1.3 implies and $\widetilde{k}$ fixes all vertices of $\widetilde{U}$. The closure of $S^1_\infty - \{vertices\ of\ \widetilde{U}\}$ are stable intervals of L and the vertices of $\widetilde{U}$ are the contracting fixed points of $\widetilde{k}$ and there is one expanding fixed point between two vertices. There are no other fixed points of $\widetilde{k}$.

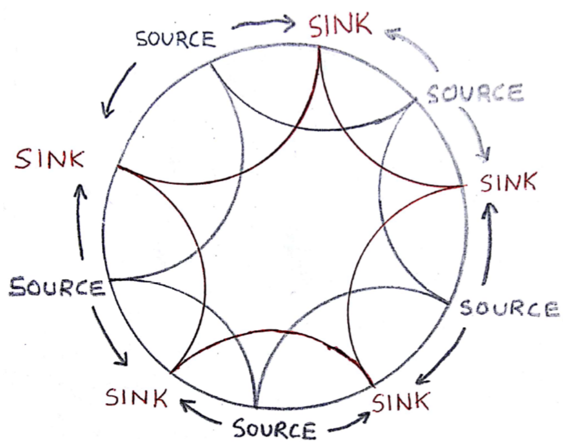


Figure 4.2:

Case 2: $\widetilde{k}$ fixes the endpoints of a non-boundary leaf of $\widetilde{L}'$. In this case the closure of S^1_∞ less the endpoints of the leaf are two stable intervals where the endpoints of the leaf are contracting points and expanding fixed points are the only pair of fixed points separating them.
Case 3: $\widetilde{k}$ does not fix both endpoints of any leaf. Let $x \in S^1_\infty$ be a point fixed by $\widetilde{k}$. By lemma 3.1.6 it can not be an endpoint of any leaf of $\widetilde{L}'$ and that any geodesic with x as an endpoint meets $\widetilde{L}'$ transversely. Let $U(\widetilde{\gamma})$ be the component of $S^1_\infty - \{endpoints\ \widetilde{\gamma}\}$ not containing x, then $U(\widetilde{\gamma})$ for all the leaves of $\widetilde{L}'$ form a cover of $S^1_\infty - x$. Also for two leaves $\gamma_1,\ \gamma_2$ of $\widetilde{L}'$, $U(\widetilde{\gamma}_1),\ U(\widetilde{\gamma}_2)$ are either disjoint or nested. Therefore, any compact subset of $S^1_\infty - x$ is contained in one of the $U(\widetilde{\gamma})$. Now there will be a leaf such that $U(\widetilde{\gamma}) \cap \widetilde{k}(U(\widetilde{\gamma})) \neq \phi$. This means given that $A,\ B$ are the two endpoint of this leaf, the sequences $\widetilde{k}^(A),\ \widetilde{k}^n(B)$ will converge to the same point which will be a contracting fixed point of $\widetilde{k}|_{S^1_\infty}$. Similarly

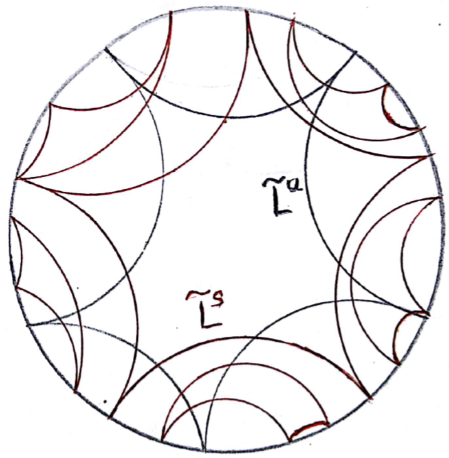


Figure 4.3:

the sequences $\widetilde{k}^{-n}$, $\widetilde{k}^{-n}$ also converge to an expanding fixed point.

As required in the statement of the theorem, all the three cases have alternately contracting and expanding fixed points of $\widetilde{k}$ on S^1_∞. This means L' satisfies all the properties of L^s. Now to see this observe that given that $\widetilde{\gamma}$ is a boundary leaf of $\widetilde{L}'$ then its endpoints are fixed by some lift $\widetilde{k}$ of h^m for some $m > 0$ (Corollary 4.1.2.1). Recollecting from the case 1, we know these are the consecutive contracting fixed points. Since γ is dense in L', any lamination satisfying the properties of L^s coincides with L'. □

Lemma 4.1.8. *For an irreducible, non-periodic automorphism h of F and an essential closed curve C in F, $h^m(C)$ is homotopic $h^n(C)$ if and only if $m = n$.*

Proof. We can take C to be a geodesic closed curve. Assume the contrary to the conclusion of lemma 4.1.8. Let $K = \cup \hat{h}^q(C)$ is a finite union of geodesics then the preimage of K is a closed subset of $\mathbb{H}^2$. This can be seen from the following argument.
Let $\widetilde{\gamma}$ be a lift of a leaf of L^u with endpoints $X, Y \in S^1_\infty$ which $\widetilde{k}$ the lift of h^m fixes (for some m). $C \pitchfork L^u \neq \phi$ and since γ is dense in L^u, some lift $\widetilde{C}$ of C will have non empty intersection with $\widetilde{\gamma}$. If A, B are the endpoints of $\widetilde{C}$, then the sequences $\widetilde{k}^n(A)$, $\widetilde{k}^n(B)$ will converge to A_∞, B_∞ which are contracting fixed points of $\widetilde{k}|_{S^1_\infty}$. The geodesic $A_\infty B_\infty$ is either a leaf of $\widetilde{L^s}$ or is contained in $\mathbb{H}^2 - \widetilde{L^s}$ (theorem 4.1.6). But $A_\infty B_\infty$ is in the preimage of K which implies that for some $q \in \mathbb{Z}$ $\hat{h}^q(C)$ does not meet L^s transversely, which is a contradiction. □

Theorem 4.1.9. *Let $h : F \longrightarrow F$ be an irreducible, non-periodic automorphism. Then there exists an $m > 0$ such that for any simple closed curved C in F, $\lim_{n\to\infty} \hat{h}^{mn}$ converges to a lamination K_C where K_C is one of the finitely many laminations containing L^s.*

Proof. The homotopy classes $h^n(C)$ are all distinct for different $n \in \mathbb{Z}$ according to the previous lemma which implies each of the closed geodesics $\hat{h}^n(C)$ are also distinct. Just as in the lemma 4.1.2 let $\hat{h}^{n_i} \to K \in \Lambda(F)$, then $L^s = (\cup_{r \in \mathbb{Z}} \hat{h}^r(K'))'$ contains a leaf of K. Since each leaf of K is dense in L^s, $K \supseteq L^s$. There are only finitely many laminations $K_1,\ K_2, ..., K_n$ since complement of L^s in K_i is a finite union of diagonals of the principal regions of L^s. There is an m which depends on h and not on C such that $\hat{h^m}$ fixes K_i. Thus $\hat{h}^{mn}(C)$ converges to one of the finitely many K_i's as $n \to \infty$. □

Chapter 5

Pseudo-Anosov Automorphisms

In this final chapter of part I we will study the *pseudo-Anosov* automorphisms of closed orientable hyperbolic surface. We will show that all non-periodic irreducible automorphisms are *pseudo-Anosov*. To show this we need to understand the structure of such an automorphism and to study this Thurston introduced invariant measures on the stable and unstable laminations. Using cantor function we will construct two singular foliations transverse to each other from the stable and unstable lamiantions. Now by studying these foliations using the transverse measures on these foliations we will see that each isotopy class of non-periodic irreducible atomorphism is represented by a *pseudo-Anosov* automorphism leaving these foliations invariant.

5.1 Singular Foliation

Lemma 5.1.1. *Let $h : F \longrightarrow F$ be a non-periodic irreducible automorphism of a closed oriented hyperbolic surface. Then h is isotopic to a homeomorphism h' which satisfies $h'(L^s) = L^s$ and $h'(L^u) = L^u$.*

Proof. Let $\widetilde{h} : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ be the lift of $h : F \longrightarrow F$ which has been extened to $\mathbb{H}^2 \cup S^1_\infty$ continuously. Given the lift $\widetilde{L^s},\ \widetilde{L^u}$ of $L^s,\ L^u$, the restriction of $\widetilde{h}|S^1_\infty$ induces a bijection $\widetilde{h}'$ on $\widetilde{L^s} \cap \widetilde{L^u}$. This is continuous because the angle of intersection between leaves of $\widetilde{L^s},\ \widetilde{L^u}$ are bounded away from 0 and π. This function can be extended linearly over each interval of $\widetilde{L^s} - \widetilde{L^u}$ and $\widetilde{L^u} - \widetilde{L^s}$ with respect to the hyperbolic metric. This gives a continuous bijective map over all of $\widetilde{L^s} \cup \widetilde{L^u}$ which is also uniformly continuous.

Closure of regions of $F - (L^s \cup L^u)$ are $2k - gons$ where for $k \geq 3$ there is only such region for each k in the complementary region of L^s and hence finitely many in all. For $k = 2$

case there there are infinitely many. $\widetilde{h'}$ can be extended over the non-rectangular regions by $\pi_1(F)$ equivariant choice of homeomorphism. By setting up grids on rectangles formed by the correspondence between opposite sides using hyperbolic distance we can extend $\widetilde{h'}$ over them.

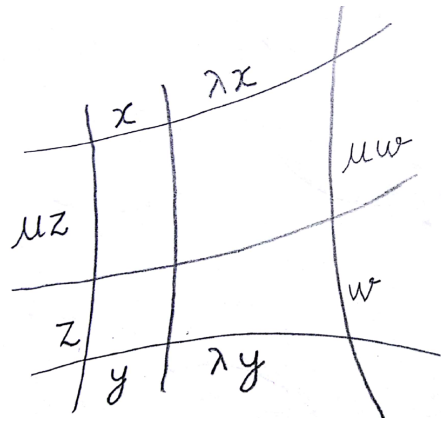


Figure 5.1:

A linear extension of $\widetilde{h'}$ over the grids (see figure 5.1 below) give a continuous extention of $\widetilde{h'} : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$. $\widetilde{h'}$ induces the homeomorphism $h' : F \longrightarrow F$ which leaves L^s and L^u invariant. Since the lifts of both h and h' induce the same action on S^1_∞, which implies they are homotopic and hence isotopic. □

Definition A singular foliation $\mathfrak{F}$ on a surface F is a disjoint union of leaves whose union if the whole of F. For a discrete finite set S in F the chart, all points $x \in S$ have a chart $\phi : U \longrightarrow \mathbb{R}^2$ which maps $U \cap \mathfrak{F}$ to W_k, where W_k is a singularity with k "separatrices" (see figure 5.3). This set S is the set of all singular points. For any $x \in F - S$ the chart maps $\phi : U \longrightarrow \mathbb{R}^2$ takes $U \cap leaves$ to horizontal intervals.

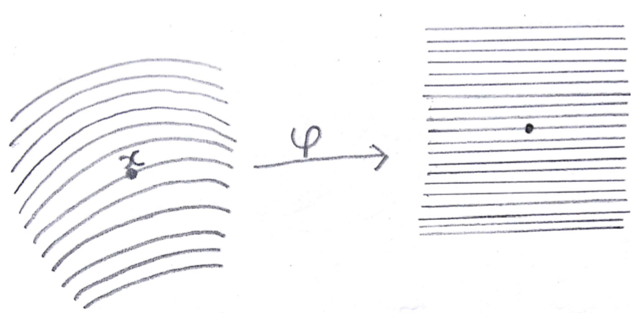


Figure 5.2:

Definition A *separatrix* of a singular foliation $\mathfrak{F}$ is a maximal arc which begins at a singularity and is contained in a leaf of $\mathfrak{F}$.

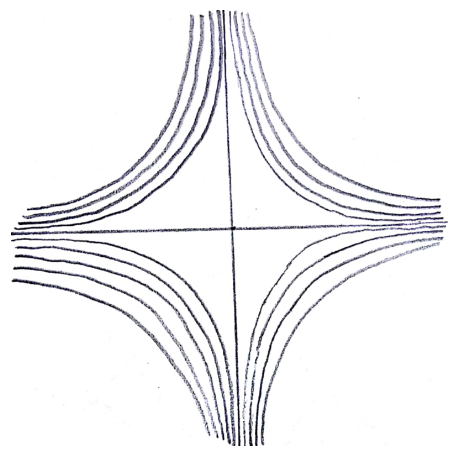


Figure 5.3: W_k for $k = 4$

If two singular foliations which have the same singular set and at every other point the leaves are transverse to each other then they are said to be *transverse* singular foliations. At the singular point the foliations should form the standard V_k model as shown in the figure 5.5 below.

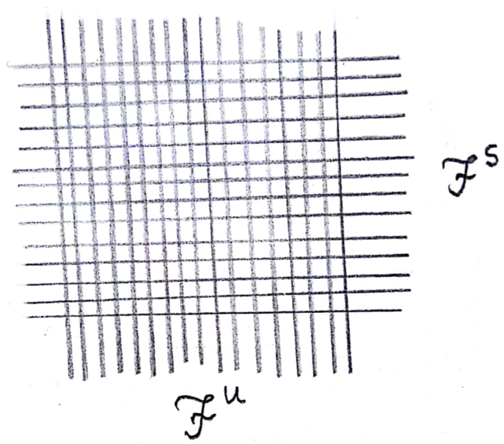


Figure 5.4:

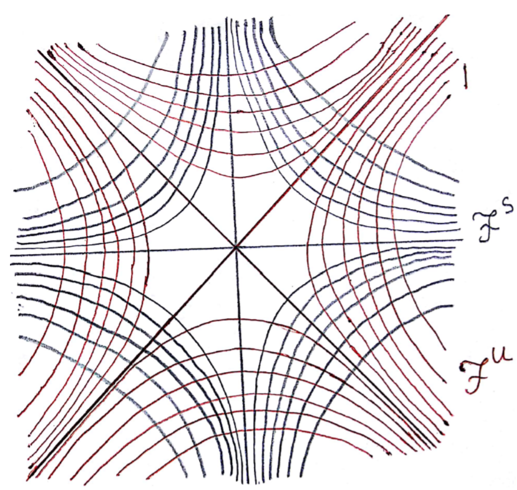


Figure 5.5: V_k for $k = 4$

Lemma 5.1.2. *If $h : F \longrightarrow F$ is a non-periodic irreducible automorphism then h is isotopic to an automorphism $h_* : F \longrightarrow F$ such that $h_*(\mathfrak{F}^s) = \mathfrak{F}^s$ and $h_*(\mathfrak{F}^u) = \mathfrak{F}^u$ for some pair of transverse singular foliations $\mathfrak{F}^s$, $\mathfrak{F}^u$.*

Proof. Consider the following equivalence relation on F. $x \sim y$ if one of the following is true:

1) x, y belong to the closure of the same component of $F - (L^s \cup L^u)$.

2) x, y belong to the closure of the same component of $L^s - L^u$.

3) x, y belong to the closure of the same component of $L^u - L^s$.

4) $x = y$.

$F/\sim$ is homeomorphic to F and the laminations L^s, L^u are mapped to the transverse singular foliation $\mathfrak{F}^s, \mathfrak{F}^u$.

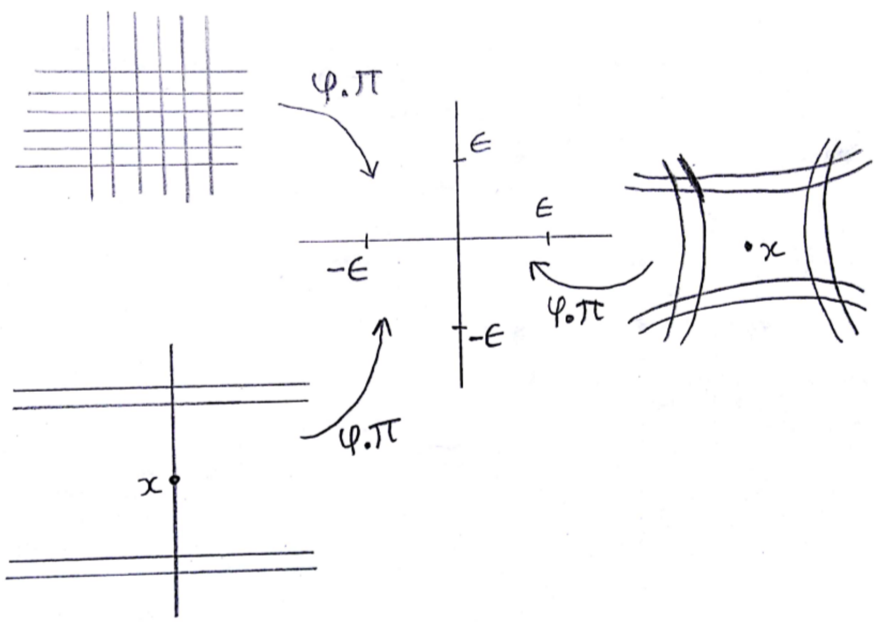


Figure 5.6:

Let $\pi : F \longrightarrow F/\sim$ be the projection map. Let $x \in F$ be a point which is on a non-boundary leaves $\gamma \subset L^s$ and $\delta \subset L^u$. Let σ, τ be short closed segments of the leaves γ, δ centred at x. The cantor function associated bet the cantor sets $\sigma \cap L^u$ and $\tau \cap L^s$ are $\alpha : \sigma \longrightarrow [-1, 1]$ and $\beta : \tau \longrightarrow [-1, 1]$ such that $\alpha(x) = \beta(x) = 0$. For some small ϵ, α, β induce a homeomorphism $\phi : U \longrightarrow [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$, where U is rectangular neighbourhood of $\pi(x)$. $\phi \cdot \pi$ takes the components of $U \cap L^s$ to horizontal intervals and the components of $U \cap L^u$ are taken to vertical intervals.

Now if x belongs to a component of $L^s - L^u$ or $L^u - L^s$, the map to $[-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$ is formed from two half-charts. Let $x \in \delta$ in $L^u - L^s$ then a half chart map takes the region starting from intersection of δ with boundary leaf of L^s closest to x and maps it to $[-\epsilon, \epsilon] \times [-\epsilon, 0]$. The rectangle components of $F - (L^s \cup L^u)$ are mapped using quarter charts, for $2k$-gons $(1/2k)th$ charts and so on. The associated charts for $2k$-gon give standard V_k model and hence give rise to singularity.

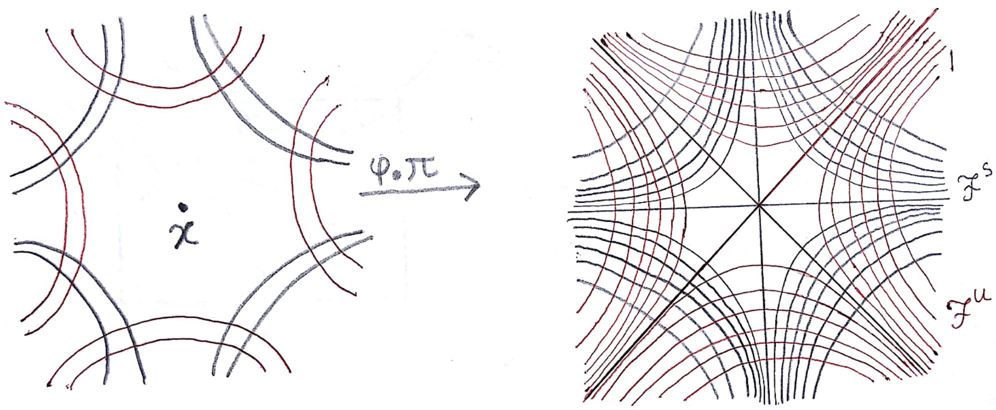


Figure 5.7:

Now, π can be approximated by a homeomorphism Θ from F onto $F/\sim$ and it can be seen that π maps L^s, L^u to $\mathfrak{F}^s, \mathfrak{F}^u$ respectively on $F/\sim$. Let $\mathfrak{F}^s, \mathfrak{F}^u$ be $\Theta^{-1} \cdot \pi(L^s)$, $\Theta^{-1} \cdot \pi(L^u)$ respectively. The lemma 5.1.1 states that for an h which is non-periodic and irreducible, there exists h' a homeomorphism which preserves the stable and unstable lamination. This h' induces an homeomorphism h'' of $F/\sim$ which preserves the image $\pi(L^s)$, $\pi(L^u)$. Thus $h_* = \Theta^{-1} \cdot h'' \cdot \Theta : F \longrightarrow F$. □

5.2 Pseudo-Anosov Automorphism

Definition Let there be a non-negative Borel measure for each transverse arc α to a singular foliation $\mathfrak{F}$ denoted as $\mu|_\alpha$ with the following properties:

1. If β is a subarc of α, then $\mu|_\alpha$ restricted to β gives $\mu|_\beta$.

2. Given two arcs α_0, α_1 both transverse to $\mathfrak{F}$ and are related by a homotopy $\alpha : I \times I \longrightarrow F$ where $\alpha(I \times 0) = \alpha_0$, $\alpha(I \times 1) = \alpha_1$, and $\alpha(a \times I)$ is on some leaf of $\mathfrak{F}$ for all $a \in I$, then $\mu|_{\alpha_0} = \mu|_{\alpha_1}$.

Then such a measure is called a *transverse measure* on $\mathfrak{F}$.

Definition If an automorphism h acts on a pair of transverse singular foliations $\mathfrak{F}^s, \mathfrak{F}^u$ equipped with transverse measures μ^s, μ^u such that

$$\begin{aligned} h(\mathfrak{F}^s, \mu^s) &= (\mathfrak{F}^s, \lambda \mu^s) \\ h(\mathfrak{F}^u, \mu^u) &= (\mathfrak{F}^u, \lambda^{-1} \mu^u) \end{aligned}$$

for some $\lambda > 0$, then it is called an *pseudo-Anosov* automorphism.

$h(\mu)$ above denotes the transverse measure given by the equation $h(\mu)|_{h(\alpha)} = \mu|_\alpha$. If $h(\mu)$ was defined by the equation $h(\mu)|_\alpha = \mu|_{h(\alpha)}$ then λ^{-1} will replace λ in the above definition. As a consequence of the definition above $(h_1 \cdot h_2)(\mu) \neq h_2(h_1(\mu))$ and rather $(h_1 \cdot h_2)(\mu) \neq h_1(h_2(\mu))$.
The automorphism h_* in the lemma 5.1.2 is a pseudo-Anosov automorphism. The lemmas that will be used to prove the next theorem will give a construction of transverse measure on $\mathfrak{F}^s$, $\mathfrak{F}^u$.

Let h_* be the automorphism in lemma 5.1.2. Let h be the abbreviation of h_* for the further discussions.

Lemma 5.2.1. *The foliations $\mathfrak{F}^s, \mathfrak{F}^u$ have no closed leaves. Each separatrix of the above foliations is dense in F and has only one singular point. Let α be a separatrix of $\mathfrak{F}^u$ starting at singularity s. There is an $m > 0$ such that for $\sigma \subset \alpha$, $h^m(\sigma) \subset \sigma$ and $h^m(x) \in (s, x) \subset \sigma$ $\forall x \in \sigma$. After replacing h with h^{-1}, the above results hold for separatrices of $\mathfrak{F}^s$.*

Lemma 5.2.2. *There are closed sets A, $B \subset F$ with the following properties.*

1. *A is a union of closed arcs each of which are contained in some separatrix of $\mathfrak{F}^s$.*
2. *B is a union of closed arcs each of which are contained in some separatrix of $\mathfrak{F}^u$.*
3. *Every component of $F - (A \cup B)$ is carried to $(0,1) \times (0,1)$ by a homeomorphism which takes leaves of $\mathfrak{F}^s$ to horizontals and leaves of $\mathfrak{F}^u$ to verticals.*
4. *$h^{-1}(A) \subset A$.*
5. *$h(B) \subset B$*

Proof. $B \cap \sigma$ is closed subarc of a separatrix σ of $\mathfrak{F}^u$ which has the singuarity of σ as one of its endpoints. Every separatrix has such an arc which is in B. Let it be β_σ. Using lemma 5.2.1 we can choose β_σ such $h(\beta_\sigma) \subset \beta_{h(\sigma)}$. This shows that B is a set as defined in condition 2. by construction and it also satisfies condition 5. .

The A as required by condition 1. is the union of closed arcs in F. Each of these arcs lies on a leaf of $\mathfrak{F}^s$, has some point in F as one end and an endpoint of β_σ as the other endpoint and lies in $F - B$. Consider a point x in one of the component U of $F - (A \cup B)$, then x lies on some closed subarc of some leaf of $\mathfrak{F}^s$ which has its endpoints on β_{σ_1} and β_{σ_2} of B. From the construction of A it can be seen that β_{σ_1} and β_{σ_2} do not depend on the choice of x. Therefore every point in U lies on some subarc of a leaf of $\mathfrak{F}^s$ with endpoints on the same two arcs β_{σ_1} and β_{σ_2}. The set of all interior points of U form a set homeomorphic to $(0,1) \times (0,1)$ and hence the condition 3. is satisfied.

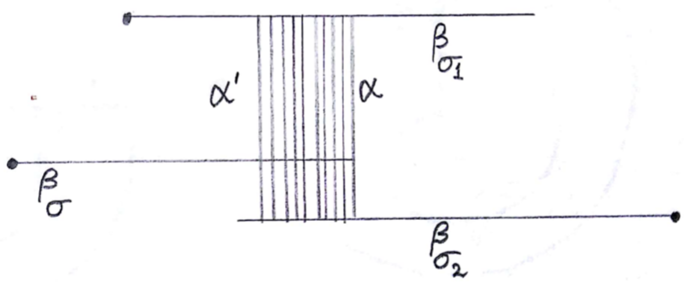


Figure 5.8:

If some component of A doesn't contain a singularity point then the sets A, B need to be readjusted so that all components of A and B have singularity points. Once this is achieved the conditions 1. and 4. will also be satisfied. Let α be a closed arc in A without any singularity point then its endpoints lie on the interior of β_{σ_1} and β_{σ_2}. This contains an endpoint of some $\beta_\sigma \subset B$. By shrinking β_σ towards its singuarity α is translated to a parallel arc α' both A and B are modified. Continue doing this until α' lands completely on another component of A. This operation reduces the number components of A and hence we get A after a finite repeation of this process. The conditions 1. and 4. follow. □

Definition Let $\rho : I \times I \longrightarrow F$ be a map which is an embedding on the interior, maps $(point\ \times I)$ to a subarc of a leaf of $\mathfrak{F}^u$ and maps $(I \times\ point)$ to a subarc of a leaf of $\mathfrak{F}^s$. Then ρ is called a rectangle R. For a rectangle R, let $\partial^u R$ denote $\rho(\partial I \times I)$ and $\partial^s R$ denote $\rho(I \times \partial I)$.

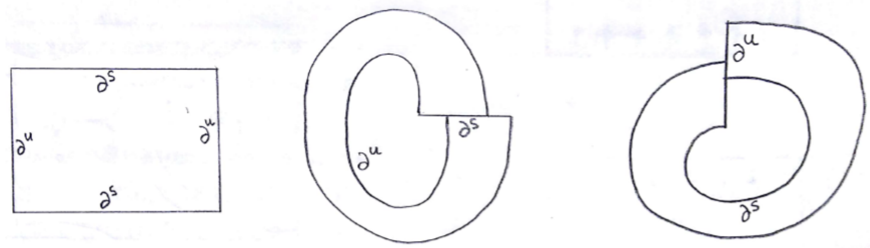


Figure 5.9:

Corollary 5.2.2.1. *There is a decomposition of F into a finite union of rectangles $R_1, ..., R_n$ with the following properties.*

1. *If $i \neq j$ then $IntR_i \cap IntR_j = \phi$.*
2. *$h(\bigcup_{j=1}^{n} \partial^u R_j) \subset \bigcup_{j=1}^{n} \partial^u R_j$.*
3. *$h^{-1}(\bigcup_{i=1}^{n} \partial^u R_i) \subset \bigcup_{i=1}^{n} \partial^u R_i$.*

It is called the Markov partition for h.

Now, after building all the required tools we are finally ready to understand the structure theorem of automorphisms of surfaces given by Thurston for closed surfaces.

Theorem 5.2.3. *Every non-periodic irreducible automorphism of a closed orientable hyperbolic surfaces is isotopic to a pseudo-Anosov automorphism.*

Proof. We will now construct the transverse measures μ^s, μ^u on $\mathfrak{F}^s, \mathfrak{F}^u$ such that

$$\begin{aligned} h(\mathfrak{F}^{\mathfrak{s}}, \mu^s) &= (\mathfrak{F}^s, \lambda\mu^s) \\ h(\mathfrak{F}^{\mathfrak{u}}, \mu^u) &= (\mathfrak{F}^u, \lambda^{-1}\mu^u) \end{aligned}$$

. From the previous lemma we know that F can be broken up into union of rectangles. To each of these rectangle R_i the measure μ^s will assign a 'height' y_i which will be called the μ^s measure of a vertical cross section of R_i and the measure μ^u will assign a 'width' x_i which will be called the μ^u measure of a horizontal cross section of R_i. We will determine necessary conditions on the x_i and y_i. Since $h(\bigcup_{j=1}^{n} \partial^u R_j) \subset \bigcup_{j=1}^{n} \partial^u R_j$, for some rectangle R_j, $h(R_i) \cap R_j$ consists of finitely many subrectangles $S_1, ..., S_{a_{ij}}$. Let A be a square matrix $[a_{ji}]$. Since each of the vertical cross sections of $h(R_i)$ have $h(\mu^s)$ measure equal to y_i and we require $h(\mu^s) = \lambda\mu^s$ height of of each S_k is $\lambda^{-1}y_i$. $R_j = \bigcup_{i=1}^{n}(h(R_i) \cap R_j)$.

$$y_j = \sum_{i=1}^{n} a_{ji}\lambda^{-1}y_i.$$

Therefore the column vector $\mathbf{y} = (y_i)$ is an eigenvector of A with eigenvalue λ. The column vector $\mathbf{x} = (x_i)$ is a eigenvector of A^t with eigenvalue λ^{-1}.

Lemma 5.2.4. *The matrix A has an eigenvector y with $y_i > 0$, $\forall i$, corresponding to an eigenvalue $\lambda > 1$.*

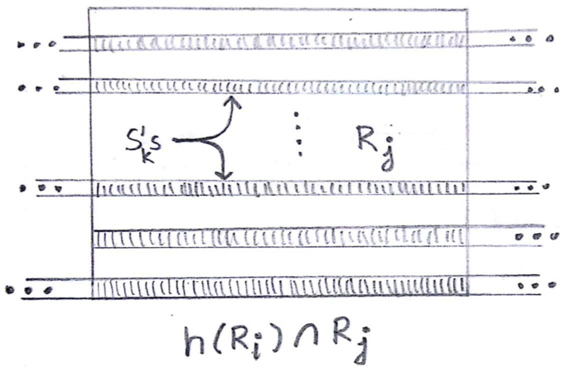


Figure 5.10:

Proof. If C is the cone of vectors in $\mathbb{R}^n$ with non-negative coordinates, the extention of brouwer's fixed point theorem would imply that the transformation by matrix A which maps C to itself has a fixed point. This is the required eigenvector with no negative coordinates. Now to see that it has all coordinates positive note that if some coordinate $y_i = 0$ then there is a $y_j \neq 0$ since $\mathbf{y} \neq \mathbf{0}$. This means ij^{th} entry of A is zero which implies that $h^m(R_i)$ is disjoint from R_j for all $m > 0$. But $\bigcup_{m=1}^{\infty} h^m(R_i)$ contains a separatrix of $\mathfrak{F}^s$ which dense in F by lemma 5.2.1 and hence cannot miss R_j which is a contradiction. Hence every coordinate of $\mathbf{y}$ is strictly positive. Also, for some $m > 0$ all the entries of A^m are strictly positive integers which shows that λ corresponding to $\mathbf{y}$ is strictly greater than 1. ☐

Back to the proof of theorem 5.2.3. Let α be a closed arc transverse to $\mathfrak{F}^s$. Then for each $m > 0$, $\alpha = \bigcup_{i=1}^{n} \alpha \cap h^m(R_i)$ and let $u_{i,m}$ denote number of components of $\alpha \cap h^m(R_i)$. We can see that the "length" of $\alpha = \sum_{i=1}^{n} \lambda^{-m} y_i u_{i,m}$.
Now with this in mind define

$$\mu^s(\alpha) = \lim_{m\to\infty} \sum_{i=1}^{n} \lambda^{-m} y_i u_{i,m}.$$

The limit exists because the "error" in the above sum arises only from the components of $\alpha \cap h^m(R_i)$ which have the endpoints of α in it. This defines a measure μ^s transverse to $\mathfrak{F}^s$ such that $h(\mu^s) = \lambda\mu^s$. We can construct μ^u transverse to $\mathfrak{F}^u$ by applying similar construction on h^{-1} such that $h^{-1}\mu^u = \bar{\lambda}\mu^u$ for some $\bar{\lambda} > 0$. Then $h(\mu^u) = \bar{\lambda}^{-1}\mu^u$. Let

$\mu = \mu^s \times \mu^u$ be the product measure on F. $\mu(F) = \mathbf{x}^t\mathbf{y}$ is finite and non-zero. Therefore

$$\begin{aligned}
\mu(F) &= [h(\mu)](F) \\
&= [h(\mu^s) \times h(\mu^u)](F) \\
&= [\lambda\mu^s \times \bar{\lambda}^{-1}\mu^u](F) \\
&= \lambda\bar{\lambda}^{-1}[\mu^s \times \mu^u](F) \\
&= \lambda\bar{\lambda}^{-1}\mu(F)
\end{aligned}$$

This means $\bar{\lambda} = \lambda$. □

Part II

Chapter 6

Curve systems

In this chapter we start the second approach to the structure theorem of automorphisms of surfaces which was developed by Allen Hatcher [**AH**]. Let M be a compact surface. We study curve systems on M and define train tracks. Using these train tracks with their measures we define $ML(M)$. We then see the global structure of $ML(M)$.

6.1 The complex of curve systems

Let us consider a surface M, which may contain boundary components. A subset of M which is a union of following types of disjointly embedded curves is called *curve system*:

- simple closed curves which do not bound disks and are not isotopic to boundary, or
- arcs whose endpoints lie on ∂M which, relative to their endpoints, are not isotopic to arcs of ∂M.

The set of the isotopy classes of curves systems in M is denoted by $\mathscr{CS}(M)$. If C_i's are connected, non-isotopic curve systems in M and if n_iC_i denotes n_i parallel copies of C_i, then every curve system can be written as $n_0C_) + n_1C_1 + ... + n_kC_k$, where $C_0, C_1, ..., C_k$ are some subcollection of curve systems in M. Let of the projective isotopy classes of curve systems, $\mathscr{PS}(M)$, be the set constructed by identifying a non-empty curve system with all its positive parallel copies in M.

Given a curve system $\mathscr{CS}(M)$, we can construct a simplicial complex $PS(M)$ such that the k-simplices of this complex bijectively corresponds with the isotopy classes of $(k+1)$-tuple $[C_0, C_1, ..., C_k]$. We get the faces of such a k-simplex by deleting some C_i's from the

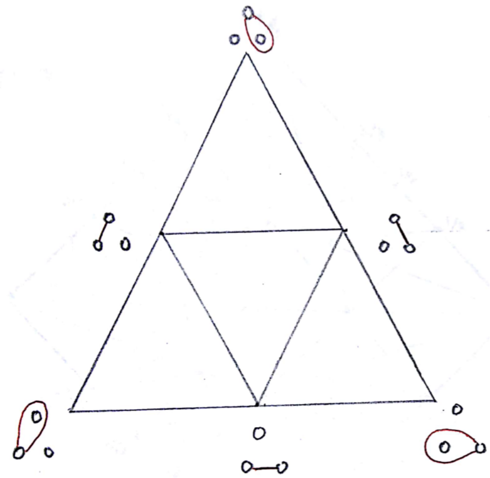


Figure 6.1:

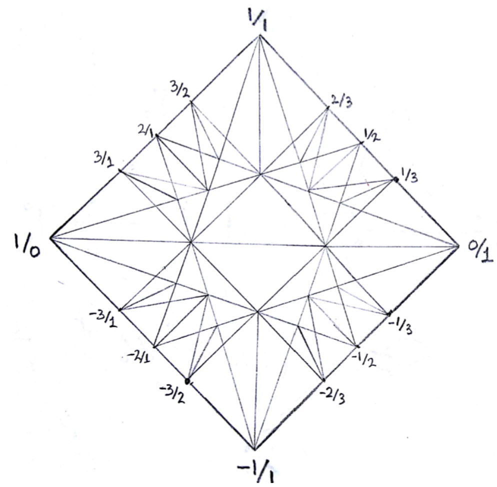


Figure 6.2:

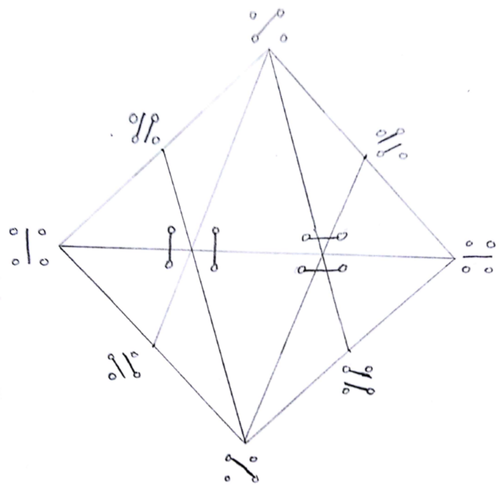


Figure 6.3:

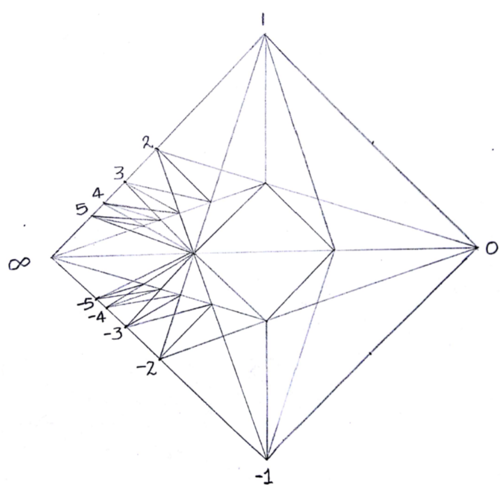


Figure 6.4:

above $(k+1)$-tuple. Those points of $PS(M)$ which have rational (equivalently integral) projective barycentric coordinates is the $\mathscr{PS}(M)$.

$PS(M)$ are usually infinite simplicial complexes and hence non-compact. But we can always give a natural compactification $PL(M)$, which is a finite polyhedra, to $PS(M)$.

Example Let M be a pair of pants. Since any simple closed curve will be isotopic to ∂M the types of isotopy classes of curves are the six arcs which are of two types, the one which has both the endpoints lying on the same boundary component and the other which has endpoints on two different boundary components. Hence $PS(M)$ has 6 vertices and they span four 2-simplices as shown in the figure 6.1

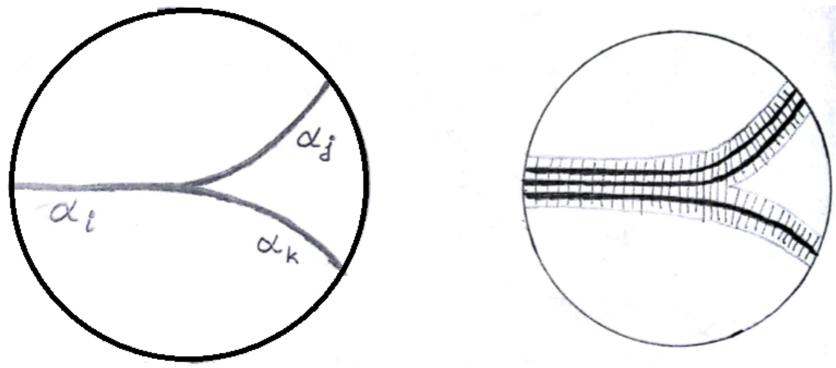


Figure 6.5:

Example If M is a once punctured torus, the circles and arcs upto isotopy are given by their classes in $H_1(M, \partial M)$ which can be geometrically interpreted as slopes in $\mathbb{Q} \cup \{\infty\}$. The $PS(M)$ is shown in the figure 6.2. It is the whole square minus the irrational points on its boundary. If $[C_0, C_1]$ is an edge from a rational point on the perimeter of the square then $C_0 \subset M$ is a circle and C_1 is an arc with the same slope. The 2-simplices $[C_0, C_1, C_2]$ which fill up the interior of the square are such that C_i's have different slopes.

Example Let M be a twice punctured $\mathbb{R}P^2$. Look at its polygonal representation i.e. a square with no vertices and boundaries are identified in anti-podal way. The $PS(M)$ is as shown in the figure 6.3.

Example M be the Klein bottle with one point removed. The $PS(M)$ is very similar to the complex of once punctured torus. The difference is this is simpler because it has lesser combinations of slopes (isotopy classes of curves). See figure 6.4.

6.2 Train tracks

A *train track* $\tau \subset M$ is a compact submanifold which is transverse to ∂M, except at some finite number of points known as branching points where one leaf splits into two leaves such that at those points all the three arcs meeting there have the same tangent direction. Locally it looks as shown in the figure 6.5. We say that a train track is good if it doesn't have any of the complementary regions shown in the figure 6.6.

We can define a measure on τ by assigning a value $\alpha_i \geq 0$ on the i^{th} non-singular arc and at the branching points by defining branching equations $\alpha_i = \alpha_j + \alpha_k$ so that the measures add up. The set of these measures $(\alpha_1, ..., \alpha_n)$ is a cone $C(\tau)$ in $\mathbb{R}^n$. If a measure α for a

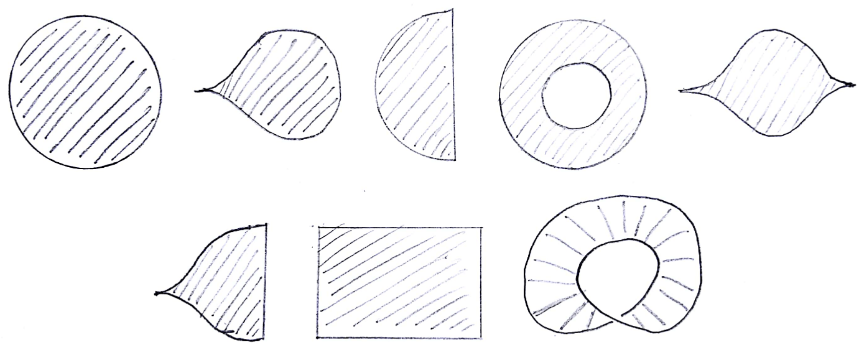


Figure 6.6:

train track has all coordinates integers then we get a curve system C_α which has α_i parallel copies of the i^{th} non-singular arc. We say C_α is carried by τ.

Lemma 6.2.1. *Let τ be good and let C_α be a curve system carried by τ, then $C_\alpha \in \mathscr{CS}(M)$.*

Proof. Let $N(\tau)$ be a fibered neighbourhood of the train track where the fibers are transverse to the curve system C_α. Extend the tangent line field of C_α to the whole of $N(\tau)$ which can then be extended to the whole of M which transverse to ∂M. This can be done because the line field is transverse to the fibers of $N(\tau)$ and tangent to $\partial N(\tau) - \partial M$. This extension to the whole of M has singularities. Now each of these singularities has an index. The sum of the indicies at all the singularities of the line field is twice the Euler characteristic χ_M. For M to contain a good track its χ_M has to be negative.

Now if C_α contains a circle bounding a disk we can see that the index inside the disk is positive. But since this is a complementary region of a good track the singularities must have a negative total index. So, C_α can not circles bounding disks. Similarly it can not have arcs isotopic to ∂M. □

6.3 Standard tracks

Let M be a surface, possibly with boundary. Let S_i 's be a collections of disjoint circles cutting along which gives a decomposition of M into pair of pants $P_1, ..., P_n$ where $n = -\chi_M > 0$. If M is non-orientable some S_i's will be one sided and have a Mobius band neighbourhood, let $\widetilde{S_i}$ be the boundary of this neighbourhood.

Let $C \in \mathscr{CS}(M)$ be a curve system in M which has a decomposition as described above. Isotope C such that it has minimum intersection with S_i's. Now C has two types of components which are disjoint, C' which are parallel to S_i's and $C'' \cap P_j$ is a curve system

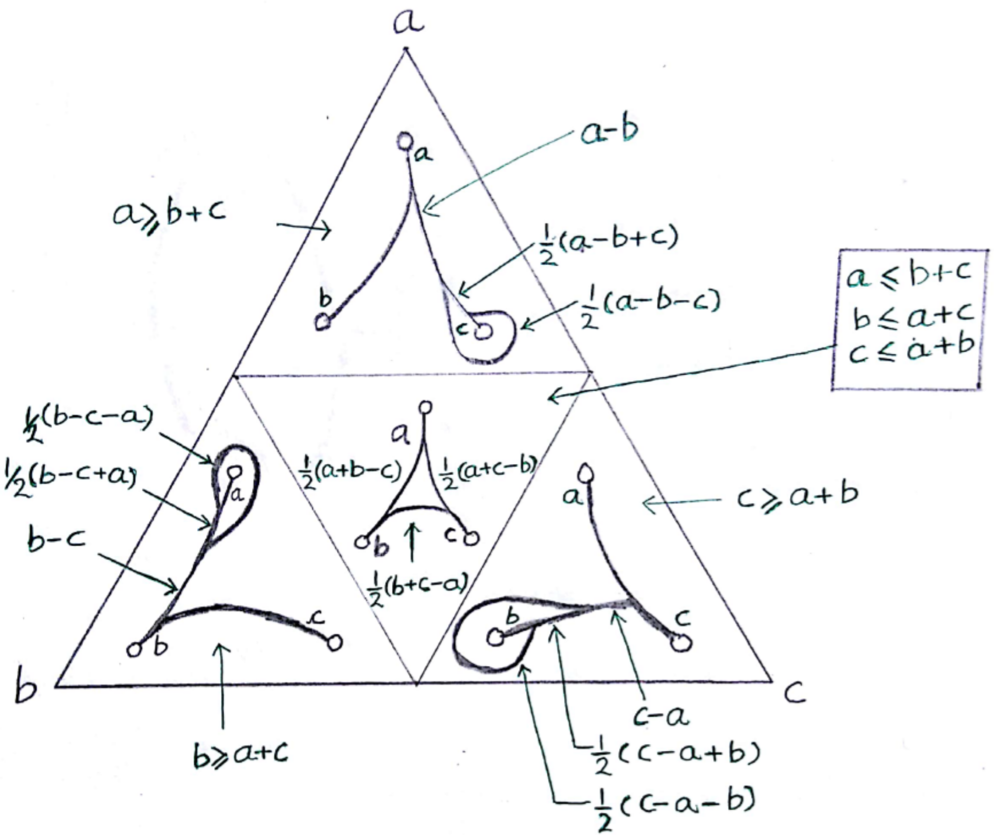


Figure 6.7:

in $\mathscr{C}\mathscr{S}(P_j)$. Each of these systems $C'' \cap P_j$ is carried by one of the four basic tracks shown in the figure 6.7. There are two types of connectors using which we can construct back C across a two sided S_i. They are the one shown in the figure 6.8. These connectors are there to allow twisting of C along them. If S_i is one sided then no non-trivial twisting can happen along it. So the connector in figure 6.9 is sufficient to carry C'' if it meets this S_i.

All the $2^t 4^n$ (t is the number of two sided S_i 's) tracks in M with all their subtracks are called *standard* tracks. These standard tracks are sufficient to carry both C'' and C' which are parallel to two sided S_i's. If some component of C' is parallel to one sided S_i's then we include S_i to the set of standard tracks and it is disjoint from τ.

6.4 The polyhedra $ML(M)$ and $PL(M)$

For every standard track, $C(\tau)$ is the cone of measures of τ where if $\tau' \subset \tau$ be a subtrack then $C(\tau')$ form the faces of $C(\tau)$. With all these identifications we get the polyhedron $ML(M)$. Now by deleting the $0 \in ML(M)$ and going modulo scalar multiplication, we get

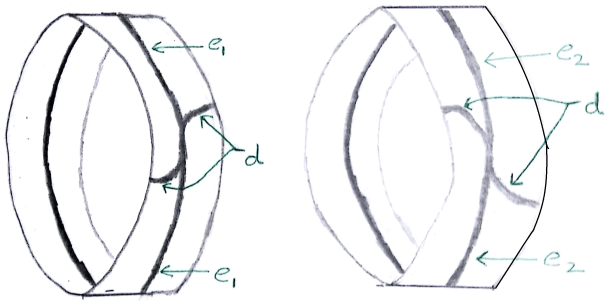


Figure 6.8:

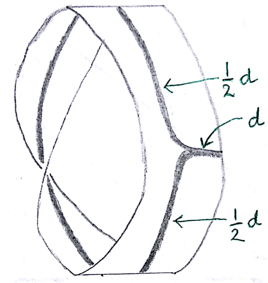


Figure 6.9:

the projectivized finite polyhedron $PL(M)$. The simplicial complex $PS(M)$ can be linearly embedded into the cells on $PL(M)$. The face $[C_0, ..., C_k]$ embeds into the projectivization of cone of measures of τ which carries $C_0 \cup ... \cup C_k$.

Let $ML_{\mathbb{Z}}(M)$ be the set of points of $ML(M)$ with integer coordinates. For some closed curve γ in M, $i_\gamma : \mathscr{CS}(M) \longrightarrow [0, \infty)$ assigns to each curve system C the minimum number of intersection it has with curves homotopic to γ.

Lemma 6.4.1. *Let γ be a closed curve transverse to C, $i_\gamma(C) = |C \cap \gamma|$ iff no arc of $\gamma - C$ can be homotoped relative to its endpoints into C.*

Proof. Let us assume we can reduce the number intersection points of γ with C by a homotopy $H : S^1 \times I \longrightarrow M$ where $H(S^1 \times 0)$ is γ and also assume that this homotopy is transverse to C. Because of this $H^{-1}(C)$ ends up being a 1-dimensional sub-manifold of $S^1 \times I$ which must contain at least one arc with both endpoints on $S^1 \times I$. The innermost of these are the arcs which will get homotoped into C by H relative to their endpoints because they will cut off a half disk in $S^1 \times I$. $\square$

Theorem 6.4.2. *The map $ML_{\mathbb{Z}}(M) \longrightarrow \mathscr{CS}(M)$ is a bijection.*

Proof. Clearly, every element of $\mathscr{CS}(M)$ can be given integral measures and hence, has a preimage in $ML_{\mathbb{Z}}(M)$ which shows that this map in surjective. The implication of the lemma 6.4.1 is true for an simple closed curve γ in M. Hence $i_\gamma(C) =$ the minimum of the number of intersection points of C with all embedded closed curves isotopic to γ.

Lemma 6.4.3. *Given two curve systems corresponding to distinct points of $ML_{\mathbb{Z}}(M)$, they can be distinguished by their intersection number with one of the loops γ_i in a finite collection of loops γ_m embedded in M.*

Proof. First fix a decomposition of M into collection pair of pants. Let the first set of loops γ_m be the boundary components of all the P_js. These curves include ∂M, S_i's and the $\widetilde{S}_i$'s.

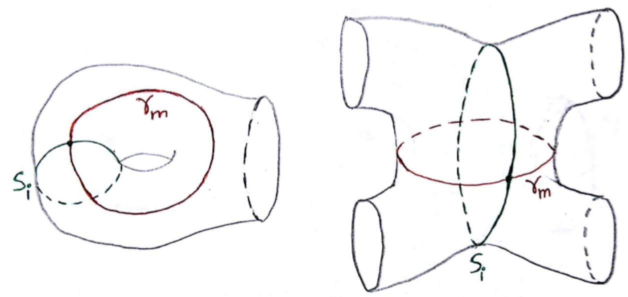


Figure 6.10:

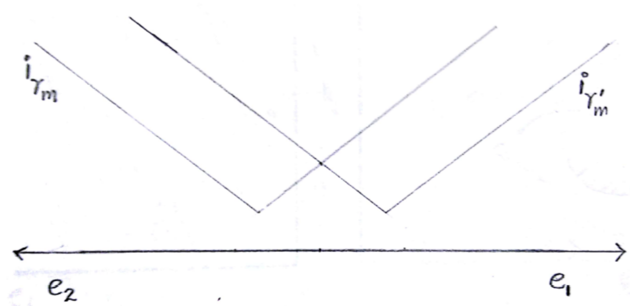


Figure 6.11:

The boundary weights a_j, b_j, c_j of a curve system $C \cap P_j$ gives the intersection numbers of C at boundaries of P_j. If there is a difference in even one of these weights for any two curve systems, they can be distinguished from each other, this being the simplest case.

Now if there is twisting along some two sided S_i, we add a two new curves to the set of γ_m to detect the twist parameters e_1 and e_2. The two sides of S_i can belong to the same pair of pants or to two different pair of pants and the first of the two new loops γ_{m_1} added will look like as shown in the fig 6.10. The intersection with this γ_m will give some e_i. But this does not distinguish between the direction of the twist along the S_i. So, to further distinguish we add the second new loop γ_{m_2} which we obtain by doing a Dehn twist on γ_{m_1} along the S_i. Using the figure 6.11 we can identify the direction of the twist.

Now, if we have a one sided S_i, then we need to check only for simple closed curves in C which are parallel copies of S_i. That can be measured by adding a loop to the set γ_m which is as shown in the figure 6.12. In this case twisting cannot happen.

These are all the loops we need to distinguish between two curve systems. □

The above lemma gives us the injectivity. □

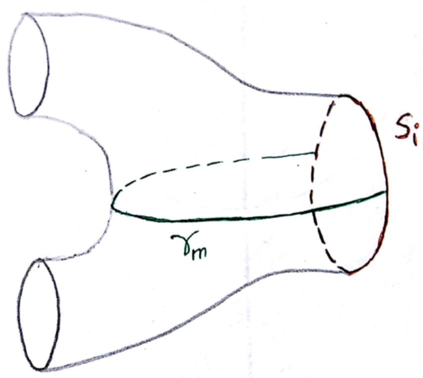


Figure 6.12:

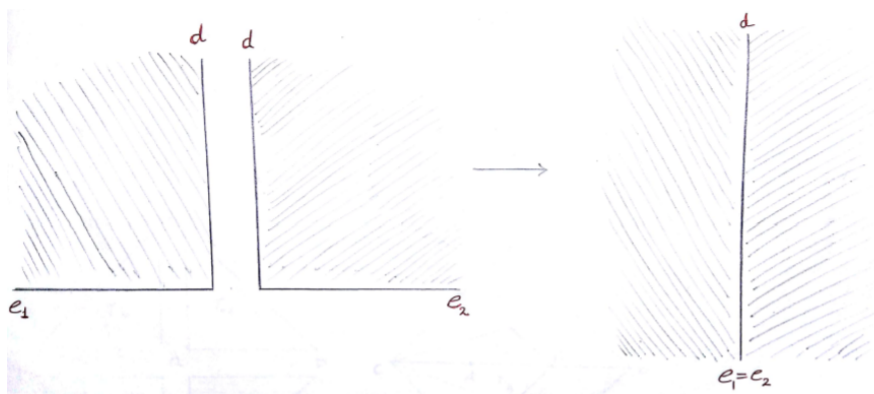


Figure 6.13:

Before we had seen that $PS(M)$ embeds linearly into $PL(M)$, now, we can see further that the set $\mathscr{P}\mathscr{S}(M)$ (which the set point of $PS(M)$ with rational barycentric coordinates) is in a bijective correspondence with the rational points of $PL(M)$. This set is dense in $PL(M)$ and hence rational points are dense in $C(\tau)$.

6.5 The Global structure of $ML(M)$ and $PL(M)$

Let M be a surface, possibly with boundary, where $\chi = \chi_M$ and b be the number of boundary components.

Theorem 6.5.1. *$ML(M)$ is piecewise linearly homeomorphic to $\mathbb{R}^{-3\chi-b} \times [0,\infty)^b$ and preserves scalar multiplication. The weights at the b boundary components of M contribute to the $[0,\infty)$ factors in the above expression. Thus the projectivized polyhedra $PL(M)$ becomes piecewise linearly homeomorphic to the join of a sphere $S^{-3\chi-b-1}$ and a simples Δ^{b-1}.*

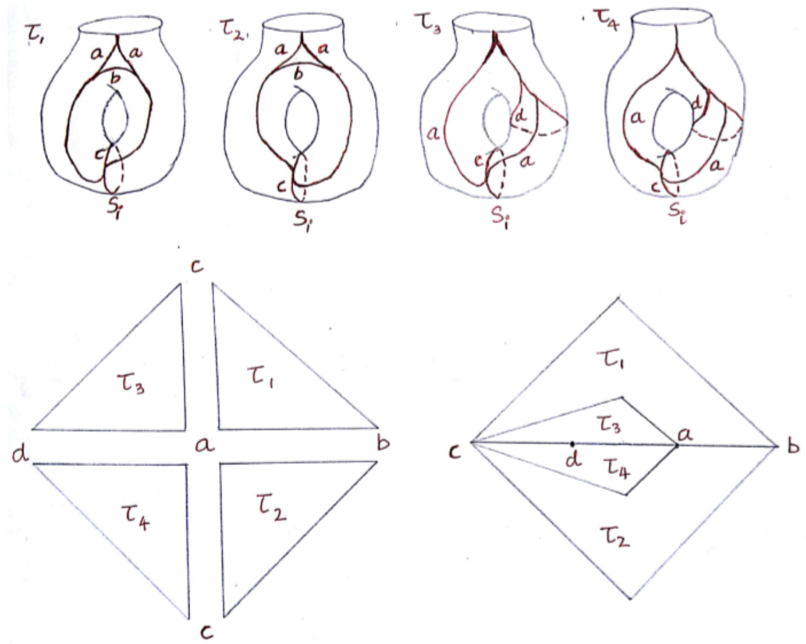


Figure 6.14:

Proof. If M is an orientable surface without any boundary components, then it's easy to see that the decomposition into pair of pants can be arrived at using exactly $-3\chi/2$ S_i 's and if it has boundary components then there are $-3\chi - b$ two sided S_i's. If a surface without boundary has n one-sided S_i's then it has $(-3\chi - n)/2$ two-sided S_i's so as to get a pair of pants decomposition. Now if such a surface has boundary components also, then it will have $(-3\chi - (n+b))/2$ two sided S_i's. We will prove the result in the theorem using induction on k which is the number splitting circles S_i's. For a pair of pants, $k = 0$, $ML(M)$ is $[0,\infty)^3$ and hence the results holds. Now if we have two disjoint surfaces, the $ML(M)$ for the disjoint union is the product of the $ML(M)$'s of the two components. Now, we can go on inductively to prove the result.

Let M' be a surface which we get by splitting M along a S_i. If it is a two sided S_i, then M' has two boundary components more than M with weights (d_1, d_2) which takes value in $[0,\infty)^2$. But to get M, we identify these two boundaries hence their weights are set equal i.e. $d_1 = d_2 = d$. Now, new weights e_1, e_2 are introduced while passing from M' to M which are the twisting factor about the S_i. Thus new set of weight coordinates are (d, e_1) and (d, e_2) each of which lies in $[0,\infty)^2$ quadrants as shown in the left image in the figure 6.13. If $d = 0$ or $e_1 = e_2 = 0$, the two alternative subtracks in the figure 6.8 coincide and hence the quadrants $[0,\infty)^2$ get identified to give a $\mathbb{R}^2$ as shown in the image on right in the figure 6.13. Overall the following happens while going from M' to M, a subproduct $[0,\infty)^2$ in the second factor of $\mathbb{R}^{-3\chi-b} \times [0,\infty)$ shifts to an $\mathbb{R}^2$ in the first factor.

Now if there is a one sided S_i, then to go from M' to M we join a Mobius band along the

$\widetilde{S}_i$. If $d \in [0, \infty)$ is the boundary weight of M' at $\widetilde{S}_i$, then as shown in figure 6.9 the weight will become $\frac{1}{2}d$ along the new track in M. If $d = 0$, then we will have a new weight $e \geq 0$ measuring the width of the track which we get by adding a number of parallel copies of S_i. Both this weight define the same subtrack when both are zero i.e $[0, \infty) = \{d\}$ intersects with $[0, \infty) = \{e\}$ at the origin and nowhere else and their union ends up being an $\mathbb{R}$ factor. Thus going from M' to M in this case shifts an $[0, \infty)$ factor to $\mathbb{R}$ factor in $\mathbb{R}^{-3\chi - b} \times [0, \infty)^b$. This completes the induction. □

Example Let M be a once-punctured torus. Let S_1 be the splitting circle which gives the pair of pants P_1. Here the weights on the boundary components of P_1 which will get identified to give S_1 must match. Therefore, out of the eight combination formed by of four basic train track and two connectors, we will only need the combinations of the two basic tracks as shown in the figure 6.14. These four basic tracks cover $ML(M)$. Each of these corresponds to a 'octant of $\mathbb{R}^3$' component in $ML(M)$ and the projectivization of this yields $PL(M) = S^1 * \Delta^0$ which is an piecewise linear identification of four 2-simplices as shown in the figure 6.14.

Chapter 7

Measured laminations

We will now try to see what kind of topological objects are the non integral points of $ML(M)$ and the irrational points of $PL(M)$. This turns out to be the set of measured laminations on M and then we show that its bijective to $ML(M)$. After this we have finally give the proof of the structure theorem.

7.1 Construction of measured laminations

Let τ be a train track with a positive measure $\alpha \in C(\tau)$ and let $N(\tau)$ be a fibered neighbourhood of τ. Look at the decomposition of $N(\tau)$ into a union of rectangles each of which lies on some ith non-singular arc and has height α_i. These rectangles are glued together at their vertical edges using the branch equations $\alpha_i = \alpha_j + \alpha_k$. Let N_α be the foliation of $N(\tau)$ given by the horizontal lines in the rectangles. The singularities of N_α are at the cusps on $\partial N(\tau)$. The finitely many singular leaves of N_α which end at these cusps are cut and slitting open N_α. Thus we eliminate all the singularities. When the weights are rational, these singular leaves are compact and this cutting open process is finite and ends giving a thickening L_α of the curve system on M with the product foliation. But N_α may have non compact singular leaves and the slitting should be fast enough so that the process ends. This resulting object which lies in $N(\tau)$ and is transversal to the fibers is called *measured laminations.*

We can also look at every N_α's isotopy class where there is a representative upto isotopy and slit only along the compact arcs in leaves, instead of slitting non-compact leaves.

We assign $\mathscr{ML}(M)$ to the set of equivalence classes of measured laminations which are carried by good train tracks. By factoring out all the scalar multiples from non empty N_α's gives the projectivization $\mathscr{PL}(L)$ of the $\mathscr{ML}(M)$.

Remark 7.1.1. • *Elements of $\mathscr{ML}(M)$ are the equivalence classes upto isotopy because two N_α's are equivalent iff they slit open to isotopic L_α's.*

- *Collapse $M - N_\alpha$ onto a suitable spine. This will define over the whole of M a singular foliation $\mathfrak{F}_\alpha$ whose all singular points have negative index. By collapsing those leaves which join singularities we define an equivalence relation which will recover $\mathscr{ML}(M)$.*

Let $\bar{N_\alpha}$ be an unique representative, upto isotopy, in some equivalence class of $\mathscr{ML}(M)$ which we can realize slitting of compact subarcs of non compact singular leaves by isotopy and hence the above element has all it's compact singular leaves slit completely. Parallel compact leaves form a foliation of some components of $\bar{N_\alpha}$ and it looks like a thickening of some curve system in M. The non compact singular leaves form a dense set in the rest of components and these components do not have any compact leaves. Observe that the L_α intersects the vertical fibers of $N(\tau)$ in intervals and Cantor sets. Also notice that because the non compact leaves of $\bar{N_\alpha}$ ends in cusp points which are in the interior of M it cannot meet ∂M and hence only the compact leaves meet the ∂M.

7.2 Length functions

Let loop γ be a PVH, piecewise vertical or horizontal, i.e. loops homotopic to γ meet some N_α either vertically (in fibers of $N(\tau)$) or horizontally (in the leaves of N_α). We define the length of such a loop to be the total length of all the vertical segments. Next we define $l_\gamma(N_\alpha)$ to be the infimum of the lengths of all the PVH homotopic to γ. We also call these loops PTH, piecewise either transverse to or horizontal to N_α. $l_\gamma(N_\alpha)$ is linear with respect to scalar multiplication on α and is constant for an equivalence class. Therefore $l_\gamma : \mathscr{ML}(M) \longrightarrow [0, \infty)$. When α is integral $l_\gamma(N_\alpha) = i_\gamma(S_\alpha)$, where S_α is the curve system associated with $\alpha \in C(\tau)$. l_γ is an extension of i_γ which is defined on $\mathscr{CS}(M)$ to $\mathscr{ML}(M)$. If some $\alpha_i = 0$ in some α, it means α is positive on some subtrack of τ i.e. on some face of $C(\tau)$. Thus l_α defines a function from $C(\tau) \longrightarrow [0, \infty)$.

Theorem 7.2.1. *For a good train track τ and some loop γ in M, the function $l_\gamma : C(\tau) \longrightarrow [0, \infty)$ is piecewise linear.*

Proof. Take γ to be in minimal position with τ. Also γ should be composed of finitely many smoothly immersed segments which lie either in τ or in $M - \tau$. Now choose a loops from the homotopy class of this γ which have the minimum number of segments outside τ and among these loops choose a loop which has the least number of segments inside τ. Now consider that N_α on collapsing which we get τ. Let γ_1 be the curve which collapses to γ. Now by making the moves on the loop γ_1 as shown in the figure 7.1 we can make γ_1 taut without changing it from being a PVH. γ_1 in this position has the minimum length within its homotopy class.

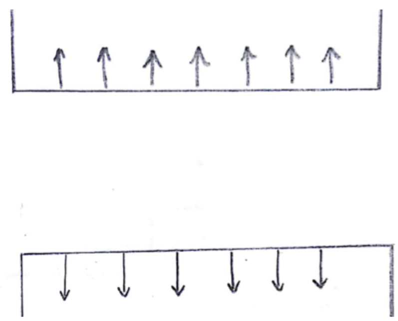


Figure 7.1:

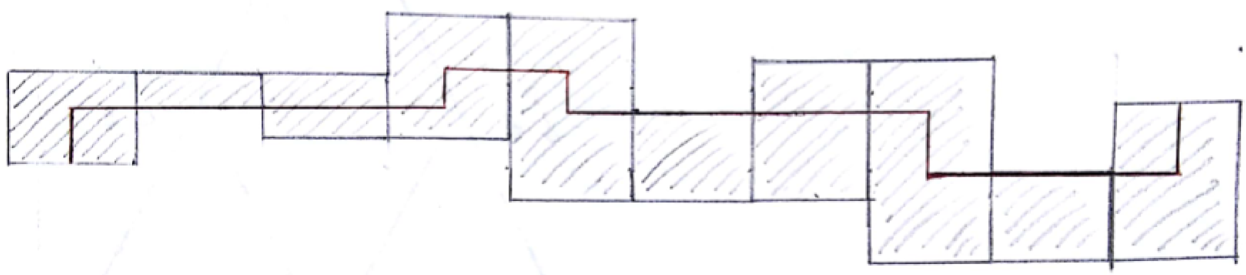


Figure 7.2:

Once γ_1 is taut it varies continuously with α, thus $l_\gamma(N_\alpha)$ is a continuous function over α.

Look at a segment of γ_1 whose projection into γ lies in τ. This segment of γ_1 lies inside a string a rectangles, where the height of these are given by α_i's, and moves monotonically as shown in the figure 7.2. The difference in height between any two horizontal edges in this string of rectangles is linear with respect to α_i's and has $\mathbb{Z}$ coefficients. Now where ever two rectangles meet they have one horizontal edge at same level and these therefore define a hyperplane in $C(\tau)$. The length of the segment which moves monotonically up or down in the regions between these hyperplanes is $\mathbb{Z}$-linear and is the difference between the value of the horizontal edges. This shows that l_γ is piecewise linear. $\square$

Consider a collection of n loops and a map $\mathscr{CS}(M) \longrightarrow [0,\infty)^n$ where the coordinates are the associated intersection number function i_γ. The collection of loops is called injective if this function is injectice.

Lemma 7.2.2. *For an injective collection of loops γ, the map $l : ML(M) \longrightarrow [0,\infty)^n$ where the coordinates are the associated length functions l_γ is a piecewise linear homeomorphism onto its image.*

Proof. The function l is $\mathbb{Z}$-linear over all of $ML(M)$ and also over all the finitely many polyhedral cones which form $ML(M)$, especially l is $\mathbb{Q}$-linear. Suppose l is non injective on one cone this would mean it would be non injective over all the rational points. This would in turn mean that even for integer points l is non injective which would contradict the condition given in the statement of the theorem. If l takes points in two distinct cones to one point then it would do the same for a pair of integer points but this cannot happen by lemma 6.4.3. Since l is injective on all $ML(L)$ and linear on all the finitely many cones which cover $ML(M)$, it is a piecewise linear homeomorphism onto its image. □

Lemma 7.2.2 implies that the image of l is independent of choice of pairs of pants used to define $ML(M)$. It turns to be the closure of the rays which start at origin and pass through the images of integer points $ML_{\mathbb{Z}}(M) = \mathscr{CS}(M)$ under the map l. This means, since $l_\gamma : C(\tau) \longrightarrow [0, \infty)$ are continuous for all good train tracks τ, $l(ML(M)) = l(\mathscr{ML}(M))$. As a consequence, $ML(M)$ has well-defined piecewise linear structure over $\mathbb{Q}$ and this is independent of the choosen decomposition of M into pairs of pants. Even the $PL(M)$ gets a well-defined, intrinsic, piecewise projective $\mathbb{Q}$-linear structure because piecewise linear maps leave scalar multiplication invariant.

By considering the map $l : ML(M) \longrightarrow [0, \infty)^\infty$ whose coordinates are l_γ which are the associated length functions of all the embedded loops we can get a proper description of $ML(M)$ i.e. the one independent of any kind of choices. Further we get that $PL(M)$ is the closure of the projectivization of the map $i : \mathscr{CS}(M) \longrightarrow [0, \infty)^\infty$ where the coordinates are intersection number functions i_γ.

Theorem 7.2.3. *The map $ML(M) \longrightarrow \mathscr{ML}(M)$ is a bijection and hence $l : \mathscr{ML}(M) \longrightarrow [0, \infty)^\infty$ is injective.*

7.3 Structure theorem of automorphisms of surfaces without boundary

We developed all the topological machinery in the previous chapter as well as this chapter to prove the following theorem.

Theorem 7.3.1. *Suppose $\partial M \neq \phi$, then a diffeomorphism $f : M \longrightarrow M$, upto isotopy, is of one of the following.*

1. *f has finite order*
2. *f leaves invariant an essential one submanifold*
3. *f maps a measured lamination in the $int(M)$ to itself with some scaling.*

Proof. Given a diffeomorphism $f : M \longrightarrow M$ a homeomorphism of $\mathscr{ML}(M)$ is induced by it. It is given by some permutation of coordinate length functions. This also gives a homeomorphism of $\mathscr{PL}(M)$. If $\partial M \neq \phi$, $\mathscr{PL}(M)$ is a ball and as a result of Brouwer's fixed point theorem there exist some $L_\alpha \subset M$ such that $f(L_\alpha) = L_{\lambda\alpha}$ where $\lambda > 0$. Now if $L_\alpha \cap \partial M = \phi$, then we have case *3.*. If L_α meets ∂M then the leaves which meet ∂M are compact. These are arcs which remain invariant under f and form a curve system C. Let the surface that we get after cutting M along be M'. By removing those components of M' which are either disks or annulus with one of its boundaries a boundary circle in ∂M we get M''. If $M'' \neq \phi$ then those circles in $\partial M''$ which are not contained in ∂M gives a finite collection on circles which are invariant under f and we get the case *2.*. If $M'' = \phi$ then f can be isotoped to a homeomorphism which has finite order. □

Remark 7.3.2.

- *In the case 2. by cutting the surface M along the invariant essential 1-submanifold we will get subsurfaces which are less complex and then inductively we can understand the action of f on M. But this processes causes lose of information about the Dehn twist's along these circles.*
- *In case 3. L_α which is invariant under f has no compact leaves and $M - L_\alpha$ is composed of disks with finite cusp on the boundary and atmost one puncture (component od ∂M) in the interior.*

Chapter 8

Conclusion

In this project we studied the structure theorem of automorphisms of surfaces using two different methods. The structure theorem says that an automorphism of a hyperbolic surface is exactly of one of the following types: periodic, irreducible or pseudo-Anosov.

In part I we studied a geometric approach. We first studied hyperbolic plane geometry, hyperbolic surfaces and some results about curves on these surfaces. Then we studied geodesic laminations and constructed transverse singular foliations corresponding to an automorphism on the surface. Then we defined transverse measures on these singular foliations and finally studied the structure theorem of automorphisms of surfaces.

In part II we looked at a more topological approach for the same theorem. We studied curve systems on a surface using train tracks with the measures on them. Using this we further studied the properties of $ML(M)$ and $PL(M)$. We saw the bijection between the set of measured laminations and $ML(M)$ and then using all this we studied a second proof of the structure theorem of automorphisms of surfaces with boundary.

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