A study of the Carathéodory and Szegő metrics on planar domains

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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To my Maaji and Pitaji

CERTIFICATE

Certified that the work incorporated in the thesis entitled "*A study of the Carathéodory and Szegő metrics on planar domains*" submitted by *Anjali Bhatnagar* was carried out by the candidate under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.

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DECLARATION

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I declare that this written submission represents my idea in my own words and where others' ideas have been included; I have adequately cited and referenced the original sources. I declare that I have acknowledged collaborative work and discussions wherever such work has been included. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

The work reported in this thesis is the original work done by me under the guidance of Dr. Diganta Borah.

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CONTENTS

ABSTRACT

We study several intrinsic properties of the Carathéodory and Szegő metrics on finitely connected planar domains. Among them are the existence of closed geodesics and geodesic spirals, boundary behaviour of Gaussian curvatures, and L^2-cohomology. A formula for the Szegő metric in terms of the Weierstrass $\wp$-function is obtained. Variations of these metrics and their Gaussian curvatures on planar annuli are also studied. Consequently, we obtain optimal universal upper bounds for their Gaussian curvatures while no universal lower bounds exist for their Gaussian curvatures. Moreover, it follows that there are domains where the Gaussian curvatures of the Szegő metric assume both negative and positive values. Furthermore, we have established the existence of domains where the Gaussian curvatures of the Bergman and Szegő metrics have opposite signs. Lastly, it is also observed that there is no universal upper bound for the ratio of the Szegő and Carathéodory metrics.

Part I

OVERVIEW AND PRELIMINARIES

1

INTRODUCTION

The Riemann mapping theorem is a landmark result in complex analysis, stating that any simply connected planar domain, except the complex plane $\mathbb{C}$, is holomorphically equivalent to the unit disc $\mathbb{D}$. However, such a classification does not exist in higher dimensions, making the study of the geometry of domains particularly intriguing. In 1907, Poincaré highlighted this distinction by demonstrating that the open-unit ball $\mathbb{B}^2$ and the open bidisc $\mathbb{D}^2$ are not holomorphically equivalent, despite both being simply connected, using the group of holomorphic automorphisms. This result underscores the importance of attaching invariant objects to domains that remain unchanged under biholomorphic mappings.

The study of invariant objects originated with the work of C. Carathéodory, who defined an invariant metric—later known as the Carathéodory metric—by utilizing the set of bounded holomorphic functions on a domain. He approached the classification problem without relying on the group of holomorphic automorphisms. In 1933, Bergman took a different approach by defining an invariant Kähler metric—called the Bergman metric, induced by the Bergman kernel—a reproducing kernel for the space of square-integrable holomorphic functions on a domain known as the Bergman space. In 1967, Kobayashi introduced another invariant metric defined using analytic discs within a domain—called the Kobayashi metric. This metric is, in some sense, dual to the Carathéodory metric. The classical invariant metrics—Carathéodory, Kobayashi, and Bergman—play a fundamental role in complex analysis. The study of these metrics and their boundary behaviour is a powerful tool in various problems, such as characterizing domains, extending biholomorphic and proper holomorphic maps to boundary points, identifying domains with non-compact groups of holomorphic automorphisms, and exploring Gromov hyperbolicity, among many others.

In 1915, Hardy extended Hadamard's three-circle theorem, laying the foundation for a new mathematical field that bears his name: the theory of Hardy spaces. The reproducing kernel of the Hardy space, known as the Szegő kernel, induces the Szegő metric on C^∞-smoothly bounded planar domains, analogous to the Bergman metric. The Szegő metric is invariant because the arc length measure transforms well under conformal equivalences. The Szegő kernel and the Szegő metric are well-defined on any non-degenerate finitely connected planar domain (see [3] for more details), since such a domain is conformally equivalent to a C^∞-smoothly bounded planar domain. These notions are defined similarly in higher dimensions, using the Euclidean surface area measure on C^∞-smoothly bounded domains. However, they are generally not invariant, as the Euclidean surface area measure does not behave well under biholomorphisms. To address this issue, Fefferman introduced a new surface area measure on C^∞-smoothly bounded strongly pseudoconvex domains, which Barrett and Lee used to define an invariant Kähler metric called the Fefferman-Szegő metric. They studied various properties of this metric and compared it with the Bergman metric. This metric was further investigated by Krantz in [29] and [30]. Notably, in dimension $n = 1$, the Fefferman surface area measure reduces to the arc length measure, and thus, the Fefferman-Szegő metric coincides with the Szegő metric.

This thesis incorporates the research findings presented in the preprint [8]. Here, we study the intrinsic properties of the Carathéodory metric $ds_{c_\Omega} = c_\Omega(z)|dz|$ and Szegő metric $ds_{s_\Omega} = s_\Omega(z)|dz|$, such as geodesics, curvature, L^2-cohomology, etc., and show that these metrics resemble the Bergman metric on non-degenerate finitely connected planar domains $\Omega \subset \mathbb{C}$. We also provide a closed expression of the Szegő metric on the annulus as well as we study the variations of these metrics and their curvatures on planar annuli.

Geodesics. The geodesics in the Bergman metric escaping towards the boundary play a crucial role in Fefferman's proof of the smooth extension up to the boundary of biholomorphic mappings between C^∞-smoothly bounded strongly pseudoconvex domains [18]. It naturally leads to the following question: Does there exist a geodesic for the Bergman metric that stays within a compact subset of a C^∞-smoothly bounded strongly pseudoconvex domain? Such a geodesic (if it exists) can be closed or non-closed. The latter one is known as a *geodesic spiral*. Herbort investigated this question in [23] and provided an affirmative answer for domains that are not topologically trivial. On the other hand, no non-trivial closed geodesics or geodesic spirals exist in a simply connected planar domain that is not all of $\mathbb{C}$ as such a domain is conformally equivalent to the unit disc $\mathbb{D}$, and the Bergman metric on the unit disc $\mathbb{D}$ coincides with the Poincaré metric. We prove that the analogous results hold for the Carathéodory and Szegő metrics. Results of this kind were obtained for the capacity metric in [10] and for the Kobayashi-fuks metric in [11, 27].

The Szegő metric on a doubly connected domain. In [41], Zarankiewicz derived a formula for the Bergman kernel on an annulus in terms of the Weierstrass $\wp$-function. Using this, a similar formula for the Szegő kernel on an annulus can be obtained (see, for example, [12]). Formulas for the Bergman and capacity metrics in terms of the $\wp$-function can also be derived and were helpful in studying the qualitative behaviour of geodesics and curvatures of these metrics on an annulus (see [24], [23], and [1]). We show that the Szegő metric $ds^2_{s_{A_r}}$ on the annulus $A_r = \{z \in \mathbb{C} : r < |z| < 1\}$, where $r \in (0, 1)$, can be expressed in terms of the Weierstrass elliptic function $\wp$ with periods $2\omega_1 = -2\log r$ and $2\omega_3 = 2i\pi$ as follows:

$$ds^2_{s_{A_r}} = \frac{\wp\big(2\log|z|\big) - \wp\big(2\log|z| + \omega_1 + \omega_3\big)}{|z|^2}|dz|^2.$$

We hope this formula will be useful in studying the qualitative behaviour of geodesics and curvature of the Szegő metric on an annulus, as in the case of the Bergman and capacity metrics.

Curvature. Let $\kappa_{m_\Omega}(z)$ denote the Gaussian curvature of a C^2-smooth conformal metric $m_\Omega(z)|dz|$. Suita [38] showed that κ_{c_Ω} is at most -4 on any domain $\Omega \subset \mathbb{C}$ that admits a nonconstant bounded holomorphic function. On the other hand, Burbea [12] showed that if $\Omega \subset \mathbb{C}$ is a C^2-smoothly bounded domain, then $\kappa_{c_\Omega}(z)$ approaches -4 if z approaches $b\Omega$ nontangentially. In [36], it was obtained that $\kappa_{c_\Omega}(z)$ approaches -4 without any restricted approach to the boundary. The limiting behaviour of the higher-order curvatures of the Caratheodory metric was also studied in the above article. For the Szegő metric, we observe (see Chapter 7 for more details) that κ_{s_Ω} is at most 4 on any non-degenerate finitely connected domain $\Omega \subset \mathbb{C}$. As for the limiting behaviour of κ_{s_Ω}, we establish that the N-th order curvature $\kappa^{(N)}_{m_\Omega}(z)$ (see Definition 2.1.6) where $m_\Omega = c_\Omega$ or s_Ω, satisfies

$$\kappa^{(N)}_{m_\Omega}(z) \to -4\left(\prod_{m=1}^{N} m!\right)^2,$$

as z approaches $b\Omega$. In particular, we have $\kappa_{s_\Omega}(z)$ approaches -4 as z approaches $b\Omega$. This fact, combined with [21, Theorem 1.17], immediately implies that for each isometry between two non-degenerate finitely connected planar domains equipped with the Szegő metric (or the Carathéodory metric) is either holomorphic or conjugate holomorphic.

L^2-cohomology. Let $\Omega \subset \mathbb{C}^n$ be a C^∞-smoothly bounded strongly pseudoconvex domain. Donnelly and Fefferman [17] established that Ω admits no square-integrable harmonic (p, q)-forms with respect to the Bergman metric, except in the case where $p + q = n$, for which the space of such forms is infinite-dimensional. Ohsawa [34] provided a more accessible proof of the infinite dimensionality of the L^2-cohomology in the middle dimension. Furthermore, Donnelly [16] provided an alternative proof of the vanishing of the L^2-cohomology outside the middle dimension, relying on a Gromov's observation [22] which is, if M is a complete Kähler manifold of complex dimension n, and its Kähler form ω can be expressed as $\omega = d\eta$, where η is bounded in the supremum norm, then M does not admit any square-integrable harmonic k-form for $k \neq n$. We have established that these ideas can be applied to prove an analogous result for the Szegő and Carathéodory metrics. This kind of result was also investigated for the Robin metric by Borah [9] and for the capacity metric by Borah-Haridas-Verma [10].

Variations of the Carathéodory and Szegő metrics on planar annuli. Recall that the Gaussian curvatures of the Carathéodory and the Szegő metrics have the universal upper bounds -4 and 4, respectively. Theorem 3.2.3 shows that for the Carathéodory metric, the upper bound -4 is optimal. It is natural to ask whether the upper bound 4 is optimal for the Szegő metric. Similarly, we ask whether the Gaussian curvatures of these metrics have universal lower bounds. These questions for the Bergman metric have been studied by several authors—see [25], [31], [14], [15], and [42]. Among them, in [15], Dinew studied the variation of the Gaussian curvatures of the Bergman metric on planar annuli to answer these questions, which were later simplified by Zwonek in [42]. Zwonek's idea was to analyse the maximal domain functions that appear in the Bergman-Fuks formula for the curvature of the Bergman metric. Using similar ideas, we study the variations of the Carathéodory and Szegő metrics, along with their Gaussian curvatures on planar annuli. As a result, we obtain optimal universal upper bounds for their curvatures, whereas no universal lower bounds exist for them. Furthermore, we establish the existence of domains where the Gaussian curvature of the Szegő metric achieves both positive and negative real values. We also show that there are domains in which the product of the Gaussian curvatures of the Bergman and Szegő metrics have opposite signs. Lastly, we establish that there is no universal upper bound for s_Ω / c_Ω.

The thesis is organised as follows. The next chapter (Chapter 2) covers the preliminaries, where we recall the definition of the Carathéodory and Szegő metrics, and their higher-order curvatures, followed by some known examples. In Chapter 3, we examine the boundary behaviour of the Carathéodory metric $c_\Omega(z)|dz|$, and Szegő metric $s_\Omega(z)|dz|$ and provide several applications. These applications include the localisation of $\partial^{k\bar{l}} c_\Omega$ and $\partial^{k\bar{l}} s_\Omega$, Gromov hyperbolicity of $s_\Omega(z)|dz|$, and the limiting behaviour of the higher-order curvatures. Chapter 4 demonstrates the existence of closed geodesics and geodesic spirals for the Szegő and Carathéodory metrics. In Chapter 5, the Szegő metric on an annulus is computed. Chapter 6 explores the L^2-cohomology of the Szegő and Carathéodory metrics. In Chapter 7, we investigate the variations of the Carathéodory and Szegő metrics on planar annuli, which provide several applications such as the optimality of the universal upper and lower bounds for their Gaussian curvatures and also establish the comparison between these metrics.

2

PRELIMINARIES

This chapter is mainly dedicated to reviewing some basic notions, such as the Carathéodory and Szegő metrics and their higher order curvatures, along with some examples. We mainly follow [4, 26] to recall these notions.

2.1 THE CARATHÉODORY AND SZEGŐ METRICS

2.1.1 *The Carathéodory metric*

Let $\Omega \subset \mathbb{C}$ be a domain. The Carathéodory metric on Ω is the function $c_\Omega : \Omega \to [0, \infty)$ defined by

$$c_\Omega(z) = \sup\Big\{\big|f'(z)\big| : f : \Omega \to \mathbb{D} \text{ is holomorphic and } f(z) = 0\Big\}.$$

Under a conformal equivalence $\phi : \Omega \to \Omega'$, $c_\Omega(z)$ transforms by the rule

$$c_\Omega(z) = c_{\Omega'}\big(\phi(z)\big)\big|\phi'(z)\big|.$$

Suita [38] showed that if Ω admits a nonconstant bounded holomorphic function, then c_Ω is a positive real analytic function and thus,

$$ds_{c_\Omega} = c_\Omega(z)|dz|,$$

is a conformal metric on Ω. The Carathéodory metric is analogously defined in higher dimensions—see, for instance, [26]. It is invariant under biholomorphisms but not smooth, in general.

2.1.2 *The Szegő metric*

We recall the definition of the Hardy space and the Szegő kernel following Bell's book [4]. Let $\Omega \subset \mathbb{C}$ be a C^∞-smoothly bounded domain. Then the boundary $b\Omega$ consists of finitely many C^∞-smooth Jordan curves. Denote by $L^2(b\Omega)$ the Hilbert space of complex-valued functions on $b\Omega$ that are square-integrable with respect to the arc-length measure ds on $b\Omega$.

Definition 2.1.1. Let $f \in C^\infty(b\Omega)$. The *Cauchy transform* $\mathcal{C}f$ of f in Ω is defined as

$$\mathcal{C}f(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{f(w)}{w-z}dw \in \mathcal{O}(\Omega). \tag{1}$$

Remark 2.1.2. It can be seen that the definition of the Cauchy transform can be extended from $C^\infty(b\Omega)$ to $L^2(b\Omega)$.

In harmonic analysis, the Poisson integral plays a crucial role, as it creates a one-to-one correspondence between continuous functions on the boundary and harmonic functions in the interior whose boundary values are those functions. Similarly, the Cauchy transform serves a comparable role in complex analysis, although the relationship is more delicate since not all functions on the boundary can be interpreted as the boundary values of a holomorphic function. For example, $f(z) = \overline{z},\ z \in b\mathbb{D}$ or any continuous real-valued functions on $b\Omega$. These examples illustrate that those functions in $L^2(b\Omega)$ that are the boundary values of holomorphic functions on Ω forms a proper subset, which is informally known as the *Hardy space*.

Definition 2.1.3. The Hardy space $H^2(b\Omega)$ is defined as the closure in $L^2(b\Omega)$ of the subspace of functions in $\mathcal{O}(\Omega)$, which are C^∞-smooth on $\overline{\Omega}$.

This is not the standard definition of the Hardy space. To recall the classical definition of the Hardy space, let $d_\Omega(z) = \operatorname{dist}(z, b\Omega)$ be the Euclidean distance of z to $b\Omega$ and let $\delta > 0$ be sufficiently small such that for $0 < \epsilon < \delta$, $\Omega_\epsilon = \{z \in \Omega : d_\Omega(z) > \epsilon\}$ is a C^∞-smoothly bounded domain with $b\Omega_\epsilon$ is parameterized by $w_\epsilon(t) = w(t) + i\epsilon T(w(t))$. Then the Hardy space $H^2(\Omega)$ is defined as

$$H^2(\Omega) = \left\{ h \in \mathcal{O}(\Omega) : \sup_{0<\epsilon<\delta} \left(\int \left| h\big(w_\epsilon(t)\big) \right|^2 \left| w'_\epsilon(t) \right| dt \right)^{\frac{1}{2}} < \infty \right\}. \tag{2}$$

As sets $H^2(b\Omega)$ and $H^2(\Omega)$ are not equal; however, they are equivalent in the following sense:

The Cauchy transform of functions in $H^2(b\Omega)$ belongs to $H^2(\Omega)$ and conversely, functions in $H^2(\Omega)$ are the Cauchy transform of functions in $H^2(b\Omega)$—see [4, Theorem 6.1 and Theorem 6.2].

By an abuse of notation, we will identify the elements $h \in H^2(b\Omega)$ with their Cauchy transform $\mathcal{C}h \in H^2(\Omega)$. Using the Cauchy transform, it can be shown that the Hardy space $H^2(b\Omega)$ is a reproducing kernel Hilbert space. The associated reproducing kernel $S_\Omega(z, w)$ is known as the *Szegő kernel* for Ω, which is uniquely determined by the following properties:

(a) For each $z \in \Omega,\ S_\Omega(\cdot, z) \in H^2(b\Omega)$,

(b) For all $z, w \in \Omega$, $S_\Omega(z, w) = \overline{S_\Omega(w, z)}$, and

(c) For each $f \in H^2(b\Omega)$ and $z \in \Omega$,

$$f(z) = \int_{b\Omega} f(w) S_\Omega(z, w) ds.$$

In terms of a complete orthonormal basis $\{\phi_k\}_{k\geq 1}$ of $H^2(b\Omega)$, the Szegő kernel $S_\Omega(z,w)$ can be expressed as

$$S_\Omega(z,w) = \sum_{k=1}^{\infty} \phi_k(z)\overline{\phi_k(w)}, \tag{3}$$

where the series converges uniformly on compact subsets of $\Omega \times \Omega$.

We recall from [4] that under a conformal equivalence $\phi : \Omega \to \Omega'$ between two C^∞-smoothly bounded domains, the Szegő kernel transforms according to the following rule

$$S_\Omega(z,w) = \sqrt{\phi'(z)} S_{\Omega'}\big(\phi(z),\phi(w)\big)\overline{\sqrt{\phi'(w)}}. \tag{4}$$

The function ϕ' has a single-valued square root on Ω, and thus $\sqrt{\phi'(z)}$ is well-defined. The restriction of the Szegő kernel to the diagonal, $S_\Omega(z) = S_\Omega(z,z)$, is a positive real analytic function on Ω. By a classical result of Garabedian, the Carathéodory metric and the Szegő kernel on a C^∞-smoothly bounded domain $\Omega \subset \mathbb{C}$ are related by the identity (see for example [6, p. 118])

$$c_\Omega(z) = 2\pi S_\Omega(z), \tag{5}$$

Using this identity, the real analyticity of the Carathéodory metric was established in [38]. Note that the function $\log S_\Omega(z)$ is strictly subharmonic.

Definition 2.1.4. The *Szegő metric* on Ω is defined as

$$ds_{s_\Omega} = s_\Omega(z)|dz|,$$

where

$$s_\Omega(z) = \sqrt{\frac{\partial^2 \log S_\Omega(z)}{\partial z \partial \overline{z}}}.$$

It follows from (4) that ds_{s_Ω} is a conformal metric.

The Szegő kernel and the Szegő metric are defined on any finitely connected planar domain Ω (not necessarily C^∞-smoothly bounded) such that no boundary component of Ω is a singleton (see for example [3]). Such a domain is called a non-degenerate, finitely connected domain. It is well-known that there is a conformal equivalence $\phi : \Omega \to \Omega'$, where $\Omega' \subset \mathbb{C}$ is a bounded domain with real analytic boundary. Also, the function ϕ' has a single-valued square root on Ω. It is thus customary to define the Szegő kernel with the aid of the transformation formula, i.e.,

$$S_\Omega(z,w) = \sqrt{\phi'(z)} S_{\Omega'}(\phi(z),\phi(w))\overline{\sqrt{\phi'(w)}}. \tag{6}$$

Then, the Szegő metric is defined analogously.

2.1.3 *Higher order curvatures*

Definition 2.1.5. The *Gaussian curvature* of a C^2-smooth conformal metric $ds_{m_\Omega} = m_\Omega(z)|dz|$ on a domain $\Omega \subset \mathbb{C}$ is defined by

$$\kappa_{m_\Omega}(z) = -\frac{\Delta \log m_\Omega(z)}{m_\Omega^2(z)}.$$

Burbea in [13] generalises the Gaussian curvature as follows.

Definition 2.1.6. For $N \geq 1$, the N-th order curvature of a C^2-smooth conformal metric $ds_{m_\Omega} = m_\Omega(z)|dz|$ on a domain $\Omega \subset \mathbb{C}$ is defined as

$$\kappa_{m_\Omega}^{(N)}(z) = -4\frac{\det\left(\partial^{k\bar{l}} m_\Omega(z)\right)_{k,l=0}^{N}}{m_\Omega^{(N+1)^2}(z)},$$

where $\partial^k = \partial^k/\partial z^k$, $\partial^{\bar{l}} = \partial^l/\partial \bar{z}^l$, and $\partial^{k\bar{l}} = \partial^k\partial^{\bar{l}}$.

It is evident that the function $\kappa_{m_\Omega}^{(N)}(z)$ is invariant under conformal equivalences, and $\kappa_{m_\Omega}^{(1)}(z)$ coincides with $\kappa_{m_\Omega}(z)$.

2.1.4 *Some examples*

Example 2.1.7. Let $\Omega = \mathbb{D}$ denote the unit disc. For $f \in H^2(b\mathbb{D})$, and $z \in \mathbb{D}$, we know

$$f(z) = \frac{1}{2\pi i}\int_{b\mathbb{D}} \frac{f(w)}{w-z}dw.$$

Since $T(w) = iw$, we obtain

$$f(z) = \int_{b\mathbb{D}} f(w)\frac{1}{2\pi(1-z\overline{w})}ds.$$

Thus, by the uniqueness of $S_\mathbb{D}$, we get

$$S_\mathbb{D}(z,w) = \frac{1}{2\pi(1-z\overline{w})}.$$

It follows that

$$s_\mathbb{D}(z) = \frac{1}{(1-|z|^2)}, \tag{7}$$

and by (5), we have $c_\mathbb{D}(z) = s_\mathbb{D}(z) = 1/(1-|z|^2)$. Therefore,

$$\kappa_{c_\mathbb{D}}^{(N)}(z) = \kappa_{s_\mathbb{D}}^{(N)}(z) = -4\left(\prod_{m=1}^{N} m!\right)^2, \tag{8}$$

for all $z \in \mathbb{D}$.

Example 2.1.8. Let $\Omega = \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Recall that $z \mapsto (z-i)/(z+i)$ is a conformal equivalence of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ onto $\mathbb{D}$. Then, using

$$S_{\mathbb{H}}(z,w) = \sqrt{\phi'(z)} S_{\mathbb{D}}\big(\phi(z), \phi(w)\big) \overline{\sqrt{\phi'(w)}},$$

we have

$$S_{\mathbb{H}}(z,\zeta) = \frac{i}{2\pi(z-\overline{\zeta})}.$$

This implies

$$s_{\mathbb{H}}(z) = \frac{1}{2\,\mathrm{Im}\,(z)},$$

and by (5), $c_{\mathbb{H}}(z) = s_{\mathbb{H}}(z) = 1/\big(2\,\mathrm{Im}(z)\big)$. Then by (8), we have

$$\kappa^{(N)}_{c_{\mathbb{H}}}(z) = \kappa^{(N)}_{s_{\mathbb{H}}}(z) = -4\left(\prod_{m=1}^{N} m!\right)^2,$$

for all $z \in \mathbb{H}$. □

Part II

OUR RESULTS

3

BOUNDARY BEHAVIOUR

Biholomorphic invariants, particularly their boundary behaviour, play an essential role in understanding the geometry of domains. The Carathéodory metric is a well-known invariant metric that has been extensively studied and has found several applications. On the other hand, the Szegő metric was recently defined by Barrett-Lee [2] using the Fefferman surface area measure (see page 259 of [19]). This approach yields an invariant version of the Szegő metric—known as the Fefferman-Szegő metric, as the classical Szegő metric is generally not invariant. However, in one dimension, the classical Szegő metric and Fefferman-Szegő metric coincide. The Fefferman-Szegő metric was further explored by Krantz in [29] and [30].

In this chapter, we will derive the boundary behaviour of the Carathéodory metric $c_\Omega(z)|dz|$ and the Szegő metric $s_\Omega(z)|dz|$ along with their partial derivatives, using the affine scaling method. We highlight its significance by presenting various applications, both in this and in the subsequent chapters.

3.1 BOUNDARY ASYMPTOTICS

3.1.1 *Localisation of $S_\Omega(z)$*

The following localisation of the Carathéodory metric is well-known (see [26, Section 19.3]): if $\Omega \subset \mathbb{C}$ is a C^∞-smoothly bounded domain, $p \in b\Omega$, and U is a (sufficiently small) neighbourhood of p, then

$$\lim_{z \to p} \frac{c_{U\cap\Omega}(z)}{c_\Omega(z)} = 1. \tag{9}$$

Now, let $\tilde{\Omega} \subset \Omega$ be a C^∞-smoothly bounded domain that share an open piece $\Gamma \subset b\Omega$. Let $p \in \Gamma$ and choose a neighbourhood U of p sufficiently small so that $U \cap \tilde{\Omega} = U \cap \Omega$ and (9) holds. Since the Carathéodory metric is decreasing under holomorphic maps,

$$1 \leq \frac{c_{\tilde{\Omega}}(z)}{c_\Omega(z)} \leq \frac{c_{U\cap\Omega}(z)}{c_\Omega(z)},$$

and hence by (9),

$$\lim_{z \to p} \frac{c_{\tilde{\Omega}}(z)}{c_\Omega(z)} = 1.$$

Combining this with the identity $c_\Omega = 2\pi S_\Omega$ on C^∞-smoothly bounded domains, we have the following:

Proposition 3.1.1. *Let $\tilde{\Omega} \subset \Omega \subset \mathbb{C}$ be C^∞-smoothly bounded domains such that $b\tilde{\Omega}$ and $b\Omega$ share an open piece $\Gamma \subset b\Omega$. Then for every $p \in \Gamma$,*

$$\lim_{z \to p} \frac{S_{\tilde{\Omega}}(z)}{S_\Omega(z)} = 1.$$

3.1.2 *Scaling method*

Let $\Omega \subset \mathbb{C}$ be a C^∞-smoothly bounded domain and $p \in b\Omega$. Let ψ be a C^∞-smooth local defining function for Ω at p defined on a neighbourhood U of p. Let $(p_j)_{j \geq 1}$ be a sequence in $U \cap \Omega$ converging to p. Consider the affine maps $T_j : \mathbb{C} \to \mathbb{C}$ defined by

$$T_j(z) = \frac{z - p_j}{-\psi(p_j)}, \tag{10}$$

and let $\Omega_j = T_j(\Omega)$. Observe that $T_j(p_j) = 0$, and thus every Ω_j contains 0. Moreover, the function

$$\psi_j(z) = \frac{1}{-\psi(p_j)} \psi \circ T_j^{-1}(z),$$

is a C^∞-smooth local defining function for Ω_j at $T_j(p)$ defined on $T_j(U)$. Observe that if K is a compact subset of $\mathbb{C}$, then $K \subset T_j(U)$ for j large, and thus ψ_j is defined on K. Moreover, for $z \in K$,

$$\begin{aligned}\psi_j(z) &= \frac{1}{-\psi(p_j)} \psi(p_j + z(-\psi(p_j))) \\ &= -1 + 2\operatorname{Re}\left(\partial\psi(p_j)z\right) + \psi(p_j)o(1),\end{aligned}$$

by expanding ψ in a Taylor series near p_j. Therefore, $(\psi_j)_{j \geq 1}$ converges uniformly on compact subsets of $\mathbb{C}$ to

$$\psi_\infty(z) = -1 + 2\operatorname{Re}\left(\partial\psi(p)z\right).$$

Consequently, a set is compactly contained in the half-plane

$$\mathcal{H} = \left\{ z \in \mathbb{C} : -1 + 2\operatorname{Re}\left(\partial\psi(p)z\right) < 0 \right\}, \tag{11}$$

if and only if it is uniformly compactly contained in $T_j(U \cap \Omega) \subset \Omega_j$ for j large. In other words, the sequences of domains $(\Omega_j)_{j \geq 1}$ and $(T_j(U \cap \Omega))_{j \geq 1}$ converge in the local Hausdorff sense to the half-plane $\mathcal{H}$. Moreover, if $\tilde{\Omega} \subset \Omega$ is a C^∞-smoothly bounded domain such that $b\tilde{\Omega}$ and $b\Omega$ share a neighbourhood of p in $b\Omega$, and $\tilde{\Omega}_j = T_j(\tilde{\Omega})$, then taking U sufficiently small, $T_j(U \cap \Omega) \subset \tilde{\Omega}_j \subset \Omega_j$, and thus the sequence of domains $(\tilde{\Omega}_j)_{j \geq 1}$ also converges in the local Hausdorff sense to $\mathcal{H}$.

At this point, we briefly recall the definition of the local Hausdorff convergence. A sequence of domains D_j converges to D_∞ in the local Hausdorff sense if the following conditions hold:

(a) For any compact subset $K \subset D_\infty$, there exists a positive integer m such that $K \subset D_j$ for all $j \geq m$.

(b) For any compact subset L is contained in the interior of $\cap_{j \geq m'} D_j$ for some positive integer m' satisfies $L \subset D_\infty$.

To this end, we write down the Szegő kernel, its metric and higher-order curvatures for the half-plane $\mathcal{H}$; we recall $S_{\mathbb{H}}(z,\zeta) = \frac{i}{2\pi(z-\overline{\zeta})}$ from Example 2.1.8. Since $z \mapsto -i(\partial\psi(p)z - 1/2)$ is the conformal equivalence of $\mathcal{H}$ onto $\mathbb{H}$ gives

$$S_{\mathcal{H}}(z,\zeta) = \frac{|\partial\psi(p)|}{2\pi\big(1 - \partial\psi(p)z - \overline{\partial}\psi(p)\overline{\zeta}\big)}. \tag{12}$$

It follows that

$$s_{\mathcal{H}}(z) = \frac{|\partial\psi(p)|}{1 - 2\,\mathrm{Re}\,\big(\partial\psi(p)z\big)}, \tag{13}$$

and

$$\kappa^{(N)}_{m_{\mathcal{H}}}(z) = -4\left(\prod_{m=1}^{N} m!\right)^2, \tag{14}$$

for all $z \in \mathcal{H}$.

In this situation, we have

Proposition 3.1.2. *The sequence $(S_{\Omega_j}(z,\zeta))_{j\geq 1}$ converges uniformly on compact subsets of $\mathcal{H} \times \mathcal{H}$ to $S_{\mathcal{H}}(z,\zeta)$. Moreover, all the partial derivatives of $S_{\Omega_j}(z,\zeta)$ converge to the corresponding partial derivatives of $S_{\mathcal{H}}(z,\zeta)$ uniformly on compact subsets of $\mathcal{H} \times \mathcal{H}$.*

This follows from Proposition 3.2 of [36] together with the following observation we add for clarity: choose U above to be sufficiently small so that $U \cap \Omega$ is simply connected. Fix $a \in \mathcal{H}$. Then $a \in T_j(U \cap \Omega)$ for j large. Since $T_j(U \cap \Omega)$ is simply connected, by the proof of Proposition 3.1 in [36],

$$c_{T_j(U\cap\Omega)}(a) \to c_{\mathcal{H}}(a).$$

Also, by the transformation formula for the Carathéodory metric and by (9),

$$\frac{c_{T_j(U\cap\Omega)}(a)}{c_{\Omega_j}(a)} = \frac{c_{U\cap\Omega}\big(-\psi(p_j)a + p_j\big)}{c_{\Omega}\big(-\psi(p_j)a + p_j\big)} \to 1.$$

This implies that

$$c_{\Omega_j}(a) \to c_{\mathcal{H}}(a).$$

Now, the proof of Proposition 3.2 of [36] applies to show that $(S_{\Omega_j}(z,\zeta))_{j\geq 1}$ converges uniformly on compact subsets of $\mathcal{H} \times \mathcal{H}$ to $S_{\mathcal{H}}(z,\zeta)$. The uniform convergence of the derivatives follows from the fact that these functions are holomorphic in $z, \overline{\zeta}$. Finally, since $S_{\Omega_j}(z)$ and $S_{\mathcal{H}}(z)$ are nonvanishing, it is immediate from the above proposition that

Corollary 3.1.3. *All the partial derivatives of $S_{\Omega_j}(z)$ and $s_{\Omega_j}(z)$ converge uniformly on compact subsets of $\mathcal{H}$ to the corresponding derivatives of $S_{\mathcal{H}}(z)$ and $s_{\mathcal{H}}(z)$.*

Proposition 3.1.4. *Let $\Omega \subset \mathbb{C}$ be a C^∞-smoothly bounded domain and $p \in b\Omega$. Let ψ be a C^∞-smooth local defining function for Ω at p defined in a neighbourhood U of p and $\mathcal{H}$ be the half-plane defined by*

$$\mathcal{H} = \Big\{ z \in \mathbb{C} : -1 + 2\,\mathrm{Re}\,\big(\partial\psi(p) z\big) < 0 \Big\}.$$

Then as $U \cap \Omega \ni z \to p$,

(a) $\partial^{k\bar{l}} S_\Omega(z)\big(-\psi(z)\big)^{k+l+1} \to \frac{(k+l)!}{2\pi} |\partial\psi(p)| \big(\partial\psi(p)\big)^k \big(\bar{\partial}\psi(p)\big)^l.$

(b) $\partial^{k\bar{l}} s_\Omega(z)\big(-\psi(z)\big)^{k+l+1} \to (k+l)! |\partial\psi(p)| \big(\partial\psi(p)\big)^k \big(\bar{\partial}\psi(p)\big)^l.$

Proof. Let $(p_j)_{j\geq 1}$ be a sequence in Ω such that $p_j \to p$. Then $p_j \in U$ for j large. Let $\Omega_j = T_j(\Omega)$ where T_j is as in (10). Differentiating

$$S_\Omega(z) = S_{\Omega_j}\big(T_j(z)\big)\big(-\psi(p_j)\big)^{-1},$$

we obtain

$$\partial^{k\bar{l}} S_\Omega(p_j)\big(-\psi(p_j)\big)^{k+l+1} = \partial^{k\bar{l}} S_{\Omega_j}(0) \to \partial^{k\bar{l}} S_{\mathcal{H}}(0),$$

by Proposition 3.1.2. Then, it follows by an explicit calculation of the derivatives of $S_{\mathcal{H}}(z)$.

Similarly, (b) is obtained by differentiating

$$s_\Omega(z) = s_{\Omega_j}\big(T_j(z)\big)\big(-\psi(p_j)\big)^{-1},$$

which completes the proof of the proposition. □

3.2 SOME APPLICATIONS

3.2.1 *Localisation of $\partial^{k\bar{l}} S_\Omega$ and $\partial^{k\bar{l}} s_\Omega$*

As an immediate consequence of the boundary asymptotics, we have the following localisation result that generalises Proposition 3.1.1:

Proposition 3.2.1. *Let $\tilde{\Omega} \subset \Omega \subset \mathbb{C}$ be C^∞-smoothly bounded domains such that $b\tilde{\Omega}$ and $b\Omega$ share an open piece $\Gamma \subset b\Omega$. Then for $p \in \Gamma$ and $k, l \geq 0$,*

$$\frac{\partial^{k\bar{l}} S_{\tilde{\Omega}}(z)}{\partial^{k\bar{l}} S_\Omega(z)} \to 1 \quad \textit{and} \quad \frac{\partial^{k\bar{l}} s_{\tilde{\Omega}}(z)}{\partial^{k\bar{l}} s_\Omega(z)} \to 1,$$

as $z \to p$, $z \in \tilde{\Omega}$.

Proof. Choose a common C^∞-smooth local defining function ψ for Ω and $\tilde{\Omega}$ at p defined on a neighbourhood U of p. Then, by Proposition 3.1.4, both $\partial^{k\bar{l}} S_\Omega(z)$ and $\partial^{k\bar{l}} S_{\tilde{\Omega}}(z)$ (resp. $\partial^{k\bar{l}} s_\Omega(z)$ and $\partial^{k\bar{l}} s_{\tilde{\Omega}}(z)$) have the same boundary asymptotics and hence the proposition follows. □

3.2.2 *Comparison with the classical metrics*

Another consequence of Proposition 3.1.4 is that the Szegő metric is comparable with the hyperbolic metric. Let $\rho_\Omega(z)|dz|$ be the hyperbolic metric and $s_\Omega(z)|dz|$ the Szegő metric on a C^∞-smoothly bounded domain $\Omega \subset \mathbb{C}$. By Proposition 3.1.4 (ii), there exists a constant $C > 1$ such that for every $z \in \Omega$,

$$C^{-1}\rho_\Omega(z) \le s_\Omega(z) \le C\rho_\Omega(z). \tag{15}$$

It follows that $s_\Omega(z)|dz|$ is comparable to $\rho_\Omega(z)|dz|$ and hence also to the Carathéodory and the Bergman metric.

The above observation implies that the Szegő metric on Ω is Gromov hyperbolic. In a coarse sense, a Gromov hyperbolic metric space behaves like a negatively curved manifold and is defined as follows. Let (X, d) be a metric space and $x, y \in X$. By a geodesic segment in X joining x and y, we mean continuous map $\sigma : [a, b] \to X$, where $[a, b] \subset \mathbb{R}$ is a closed interval, such that $\sigma(a) = x$, $\sigma(b) = y$, and for every $s, t \in [a, b]$,

$$d\big(\sigma(s), \sigma(t)\big) = |s - t|.$$

A geodesic segment joining x and y, despite its possible non-uniqueness, will be denoted by $[x, y]$. The space (X, d) is called a geodesic space if, for every pair of points $x, y \in X$, there is a geodesic segment joining x and y. Given $\delta \ge 0$, a geodesic metric space (X, d) is called δ-hyperbolic if every geodesic triangle $[x, y] \cup [y, z] \cup [z, w]$ in X is δ-thin, i.e.,

$$d\big(w, [y, z] \cup [z, x]\big) \le \delta,$$

for all $w \in [x, y]$. The metric space (X, d) is called Gromov hyperbolic if there exists a $\delta > 0$ such that (X, d) is δ-hyperbolic.

For brevity, we will denote the distance functions induced by ρ_Ω and s_Ω by the same notations ρ_Ω and s_Ω, respectively.

Corollary 3.2.2. *Let $\Omega \subset \mathbb{C}$ be a non-degenerate finitely connected domain. Then the metric space (Ω, s_Ω) is Gromov hyperbolic.*

Proof. Since a non-degenerate finitely connected domain is conformally equivalent to a C^∞-smoothly bounded domain and conformal maps are isometries (in differential geometric sense and hence also in metric geometric sense) of the Szegő metric, which, therefore preserve Gromov hyperbolicity, we assume without loss of generality that Ω is a C^∞-smoothly bounded domain.

From [35], (Ω, ρ_Ω) is δ-hyperbolic for some $\delta > 0$. Since $\rho_\Omega(z)|dz|$ is complete, this implies that $s_\Omega(z)|z|$ is also complete and hence (Ω, s_Ω) is a geodesic space. Moreover, the identity map between (Ω, ρ_Ω) and (Ω, s_Ω) is a quasi-isometry and consequently (Ω, s_Ω) is also δ-hyperbolic, possibly for a different choice of δ. $\square$

3.2.3 *Boundary behaviour of curvatures*

In [13], Burbea showed that for any p-connected domain $\Omega_p \subset \mathbb{C}$ where $1 \leq p < \infty$, $\kappa^{(N)}_{c_\Omega}(z)$ satisfies

$$\sup_{z \in \Omega_p} \kappa^{(N)}_{c_\Omega}(z) \leq -4 \left(\prod_{m=1}^{N} m! \right)^2 \quad \text{for all } N \geq 1.$$

Furthermore, equality holds for one point $z \in \Omega_p$, and for any $N \geq 1$, if and only if Ω_p is simply connected. As for the boundary behaviour of $\kappa^{(N)}_{m_\Omega}$ where $m_\Omega = c_\Omega$ (for this case—see [36]) or s_Ω, we have

Theorem 3.2.3. *Let $\Omega \subset \mathbb{C}$ be a non-degenerate finitely connected domain and let $m_\Omega = c_\Omega$ or s_Ω. Then for every $p \in b\Omega$,*

$$\kappa^{(N)}_{m_\Omega}(z) \to -4 \left(\prod_{m=1}^{N} m! \right)^2,$$

as $z \to p$.

Proof. First, assume that Ω is C^∞-smoothly bounded and $p \in b\Omega$. By Proposition 3.1.4 and recalling that $c_\Omega(z) = 2\pi S_\Omega(z)$, we have for $k, l \geq 0$,

$$\partial^{k\bar{l}} m_\Omega(z) \big(-\psi(z) \big)^{k+l+1} \to \partial^{k\bar{l}} m_{\mathcal{H}}(0),$$

as $z \to p$. Therefore, denoting the symmetric group over $\{0, \ldots, N\}$ by S_{N+1},

$$\begin{aligned}
&\det \left(\partial^{k\bar{l}} m_\Omega(z) \right)_{k,l=0}^{N} \big(-\psi(z) \big)^{(N+1)^2} \\
&= \sum_{\sigma \in S_{N+1}} \operatorname{sign}(\sigma) \prod_{k=0}^{N} \left(\partial^{k\overline{\sigma(k)}} m_\Omega(z) \big(-\psi(z) \big)^{k+\sigma(k)+1} \right) \\
&\to \sum_{\sigma \in S_{N+1}} \operatorname{sign}(\sigma) \prod_{k=0}^{N} \left(\partial^{k\overline{\sigma(k)}} m_{\mathcal{H}}(0) \right) \\
&= \det \left(\partial^{k\bar{l}} m_{\mathcal{H}}(0) \right)_{k,l=0}^{N}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\kappa^{(N)}_{\Omega}(z) &= -4 \frac{\det \left(\partial^{k\bar{l}} m_\Omega(z) \right)_{j,k=0}^{N} \big(-\psi(z) \big)^{(N+1)^2}}{m_\Omega^{(N+1)^2}(z) \big(-\psi(z) \big)^{(N+1)^2}} \\
&\to -4 \frac{\det \left(\partial^{k\bar{l}} m_{\mathcal{H}}(0) \right)_{k,l=0}^{N}}{m_{\mathcal{H}}^{(N+1)^2}(0)} = \kappa^{(N)}_{m_{\mathcal{H}}}(0),
\end{aligned}$$

as $z \to p$. Recall from (14) that right-hand side is $-4(\prod_{m=1}^{N} m!)^2$ and the proof is complete when Ω is C^∞-smoothly bounded.

For the general case, let $\phi : \Omega \to \Omega'$ be a conformal equivalence where $\Omega' \subset \mathbb{C}$ is a C^∞-smoothly bounded domain. Let $p \in b\Omega$ and $(z_j)_{j\geq 1}$ be a sequence in Ω such that $z_j \to p$. Since the sequence $(\phi(z_j))_{j\geq 1}$ is bounded, it has a convergent subsequence, say $(\phi(z_{j_k}))_{k\geq 1}$. Note that the limit of this subsequence must lie in $b\Omega'$. It follows from the previous case that

$$\kappa_{m_\Omega}^{(N)}(z_{j_k}) = \kappa_{m_{\Omega'}}^{(N)}(\phi(z_{j_k})) \to -4\left(\prod_{m=1}^{N} m!\right)^2.$$

Thus, we have shown that every sequence in Ω converging to p admits a subsequence along which $\kappa_{m_\Omega}^{(N)}$ converges to $-4(\prod_{m=1}^{N} m!)^2$. It follows that $\kappa_{m_\Omega}^{(N)}(z) \to -4(\prod_{m=1}^{N} m!)^2$ as $z \to p$. This completes the proof of the theorem. □

In particular, Theorem 3.2.3 implies that $\kappa_{s_\Omega}(z)$ approaches -4 as z approaches $b\Omega$. This, combined with [21, Theorem 1.17], immediately gives the following:

Corollary 3.2.4. *Let Ω_1 and Ω_2 be two non-degenerate finitely connected planar domains equipped with the metrics $ds_{c_{\Omega_1}}$ and $ds_{c_{\Omega_2}}$ (resp. $ds_{s_{\Omega_1}}$ and $ds_{s_{\Omega_2}}$). Then, each isometry $f : (\Omega_1, ds_{c_{\Omega_1}}) \to (\Omega_2, ds_{c_{\Omega_2}})$ (resp. $f : (\Omega_1, ds_{s_{\Omega_1}}) \to (\Omega_2, ds_{s_{\Omega_2}})$) is either holomorphic or conjugate holomorphic.*

4

GEODESICS

The existence of closed geodesics is a classical topic in the realm of Riemannian geometry. One of the most studied Riemannian metrics in several complex variables is the Bergman metric. Herbort [23] provided the general result for the complete Riemannian metric to prove the existence of closed geodesics for the Bergman metric in C^∞-smoothly bounded strongly pseudoconvex domains that are topologically non-trivial. In contrast, geodesic spirals—non-closed geodesics in the complete Riemannian metric that eventually remains in a compact subset—have also been examined by Herbort for the Bergman metric in similar domains.

Using Herbort's ideas, we will establish the existence of closed geodesics and geodesic spirals for the Szegő and Carathéodory metrics in this chapter. Before presenting our main result, we clarify here that by a geodesic of a C^∞-smooth conformal metric, we mean a geodesic in the differential geometric sense (and it is not the same as a geodesic segment defined on p. 33). The equation of a geodesic $\sigma : (a, b) \to \Omega$, where $(a, b) \subset \mathbb{R}$, of a C^∞-smooth conformal metric $ds_\Omega = m_\Omega(z)|dz|$ in a domain $\Omega \subset \mathbb{C}$ can be expressed in complex coordinates as

$$-\sigma'' = \frac{1}{m_\Omega^2(\sigma)} \frac{\partial m_\Omega^2(\sigma)}{\partial z} (\sigma')^2.$$

We can now state our main result:

Theorem 4.0.1. *Let $\Omega \subset \mathbb{C}$ be a non-degenerate n-connected domain, $n \geq 2$. Equip Ω with the conformal metric $ds_{m_\Omega} = m_\Omega(z)|dz|$, where $m_\Omega = c_\Omega$ or s_Ω. Then*

(a) Every non-trivial homotopy class of loops in Ω contains a closed geodesic.

(b) For every $z_0 \in \Omega$ that does not lie on a closed geodesic, there exists a geodesic spiral passing through z_0.

4.1 EXISTENCE OF CLOSED GEODESICS

We recall the following result of Herbort:

Theorem 4.1.1 (Theorem 1.1 [23])**.** *Let G be a bounded domain in $\mathbb{R}^N$, where $N \in \mathbb{N}$, such that $\pi_1(G)$ is non-trivial. Assume that the following conditions are satisfied:*

(a) *For each $p \in \overline{G}$, there is an open neighbourhood $U \subset \mathbb{R}^N$ such that the set $G \cap U$ is simply connected.*

(b) *G is equipped with a complete Riemannian metric g which possesses the following property: (B) For each $S > 0$ there exists a $\delta > 0$, such that for each $p \in G$ with $d(p, bG) < \delta$ and every $X \in \mathbb{R}^N$, $g(p, X) \geq S\|X\|^2$ (where $\|\cdot\|$ denotes the Euclidean norm).*

Then, every non-trivial homotopy class in $\pi_1(G)$ contains a closed geodesic for g.

Proof of Theorem 4.0.1 (a). Using the same reasoning as in the proof of Corollary 3.2.2, we assume without loss of generality that Ω is a C^∞-smoothly bounded domain. It suffices to show that $ds^2_{m_\Omega} = m^2_\Omega(z)|dz|^2$ satisfies the hypotheses of the above theorem. Let ψ be a C^∞-smooth defining function for Ω. By Proposition 3.1.4 (b), for $z \in \Omega$ and $v \in \mathbb{C}$, we have

$$\frac{ds^2_{m_\Omega}(z, v)}{|v|^2} = m^2_\Omega(z) \gtrsim \left(\psi(z)\right)^{-2},$$

which implies that the Property (B) is satisfied. All the other conditions are evidently true. This proves (a). □

4.2 EXISTENCE OF GEODESIC SPIRALS

The main tool for the proof of Theorem 1.1 (ii) is a result of Herbort from [24]. To state this result, we first recall the notion of a geodesic loop. Let (M, g) be a complete Riemannian manifold.

(a) A *geodesic spiral* is a non-closed geodesic $\sigma : \mathbb{R} \to M$ in which each point $\sigma(t)$ for $t \geq 0$ belongs to a fixed compact subset K of M.

(b) For a non-trivial geodesic $\sigma : \mathbb{R} \to M$, if there exist $t_1,\ t_2 \in \mathbb{R}, t_1 < t_2$ with $\sigma(t_1) = \sigma(t_2)$, then the segment $\sigma_{|_{[t_1,t_2]}}$ is referred to as a geodesic loop passing through $\sigma(t_1)$.

Lemma 4.2.1 (Lemma 2.2, [24]). *Let (M, g) be a complete Riemannian manifold with a universal cover with infinitely many leaves. Let x_0 be a point in M such that no closed geodesic passes through x_0. Assume a compact subset K of M exists with the property that each geodesic loop passing through x_0 is contained in K. Then, there exists a geodesic spiral passing through x_0.*

In view of this lemma, the problem of showing the existence of geodesic spirals for $m_\Omega(z)|dz|$ now reduces to finding an appropriate compact subset $K \subset \Omega$. For this, we require the following:

Proposition 4.2.2. *Let $\Omega \subset \mathbb{C}$ be a C^∞-smoothly bounded domain with a C^∞-smooth defining function ψ. Then there exists $\epsilon = \epsilon(\Omega) > 0$ such that for each geodesic $\sigma : \mathbb{R} \to \Omega$ of the metric $m_\Omega(z)|dz|$ satisfying $(\psi \circ \sigma)(0) > -\epsilon$ and $(\psi \circ \sigma)'(0) = 0$, we have $(\psi \circ \sigma)''(0) > 0$.*

Proof. Suppose to the contrary, for each $k \in \mathbb{N}$, there exists a geodesic σ_k of $m_\Omega(z)|dz|$ satisfying

$$\text{(a) } \psi(\sigma_k(0)) > -\frac{1}{k}, \quad \text{(b) } (\psi \circ \sigma_k)'(0) = 0, \quad \text{and} \quad \text{(c) } (\psi \circ \sigma_k)''(0) \leq 0.$$

Let us denote for each k,

$$p_k = \sigma_k(0), \quad v_k = \frac{\sigma_k'(0)}{|\sigma_k'(0)|}, \quad \text{and} \quad b_k = \frac{(\psi \circ \sigma_k)''(0)}{|\sigma_k'(0)|^2} \leq 0.$$

By passing to the subsequence, let $p_k \to p_0$ and $v_k \to v_0$ where $|v_0| = 1$. By a translation and rotation of D, we may assume that

$$p_0 = 0 \text{ and } \frac{\partial \psi(0)}{\partial z} = 1.$$

Then from (b) we have

$$0 = \lim_{k\to\infty} \operatorname{Re}\left(\frac{\partial \psi(p_k)}{\partial z} v_k\right) = \operatorname{Re}\left(\frac{\partial \psi(0)}{\partial z} v_0\right) = \operatorname{Re}(v_0). \tag{16}$$

On the other hand, from (c) we have

$$\operatorname{Re}\left(\frac{\partial \psi(p_k)}{\partial z}\sigma_k''(0)\right) + \operatorname{Re}\left(\frac{\partial^2 \psi(p_k)}{\partial z^2}(\sigma_k'(0))^2\right) + \frac{\partial^2 \psi(p_k)}{\partial z \partial \bar{z}}|\sigma_k'(0)|^2 \leq 0. \tag{17}$$

Since σ_k is a geodesic of the metric $m_\Omega^2|dz|^2$, we have

$$-\sigma_k'' = \frac{1}{m_\Omega^2(\sigma_k)} \frac{\partial m_\Omega^2(\sigma_k)}{\partial z}(\sigma_k')^2. \tag{18}$$

Thus, combining (17) and (18), we obtain

$$\begin{aligned} -\operatorname{Re}\left(\frac{\partial \psi}{\partial z}(p_k)\frac{1}{m_\Omega^2(p_k)}\frac{\partial m_\Omega^2(p_k)}{\partial z}(\sigma_k'(0))^2\right) + \operatorname{Re}\left(\frac{\partial^2 \psi(p_k)}{\partial z^2}(\sigma_k'(0))^2\right) \\ + \frac{\partial^2 \psi(p_k)}{\partial z \partial \bar{z}}|\sigma_k'(0)|^2 \leq 0. \end{aligned}$$

Dividing throughout by $|\sigma_k'(0)|^2$ and multiplying by $-\psi(p_k)$, we have

$$\begin{aligned} -\operatorname{Re}\left(\frac{\partial \psi}{\partial z}(p_k)(-\psi(p_k))\frac{1}{m_\Omega^2(p_k)}\frac{\partial m_\Omega^2(p_k)}{\partial z}v_k^2\right) + (-\psi(p_k))\operatorname{Re}\left(\frac{\partial^2 \psi(p_k)}{\partial z^2}v_k^2\right) \\ + (-\psi(p_k))\frac{\partial^2 \psi(p_k)}{\partial z \partial \bar{z}} \leq 0. \end{aligned} \tag{19}$$

Observe that the last two terms converge to 0 as $k \to \infty$. On the other hand, by Proposition 3.1.4,

$$\big(-\psi(p_k)\big)\frac{1}{m_\Omega^2(p_k)}\frac{\partial m_\Omega^2(p_k)}{\partial z} = \frac{1}{m_\Omega^2(p_k)(-\psi(p_k))^2}\frac{\partial m_\Omega^2(p_k)}{\partial z}(-\psi(p_k))^3 \to 2.$$

By taking $k \to \infty$ in (19), it follows that

$$\mathrm{Re}\left(v_0^2\right) \geq 0.$$

which is a contradiction as $v_0 = \pm i$ from (16).

□

We are now in a position to give proof of Theorem 1.1 (b).

Proof of Theorem 4.0.1 (b). Using the same reasoning as in the proof of Corollary 3.2.2, we assume without loss of generality that Ω is a C^∞-smoothly bounded domain. Let $z_0 \in \Omega$ be such that no closed geodesics passes through it. Let ψ be a C^∞-smooth defining function for Ω and let $\epsilon > 0$ be as in Proposition 4.2.2. Let $\epsilon_1 = \min\{\epsilon, -\psi(z_0)\}$. Set

$$K = \big\{z \in \Omega : \psi(z) \leq -\epsilon_1\big\}.$$

Then, the compact set K has the property as in Lemma 4.2.1. Indeed, let $\sigma|_{[t_1,t_2]} : [t_1, t_2] \to \Omega$ be a geodesic loop that passes through z_0 and suppose that $\sigma|_{[t_1,t_2]}([t_1, t_2]) \not\subset K$. Since $(\psi \circ \sigma)|_{[t_1,t_2]}$ is a continuous real-valued function, it will attain maximum at some point $t_0 \in (t_1, t_2)$. From the definition of K and the fact $\sigma|_{[t_1,t_2]}$ leaves K implies that $\sigma(t_0) \in \Omega \setminus K$, which further implies that $(\psi \circ \sigma)(t_0) > -\epsilon$, $(\psi \circ \sigma)'(t_0) = 0$, and $(\psi \circ \sigma)''(t_0) \leq 0$. But it contradicts the above proposition. By Lemma 4.2.1, the proof of (ii) follows. □

5

THE SZEGŐ METRIC ON DOUBLY CONNECTED DOMAINS

In [23], Herbort provided the qualitative behaviour of geodesics of the Bergman metric in a planar annulus. He achieved this by expressing the Bergman metric using the Wierstrass elliptic $\wp$- function. As the Szegő kernel enjoys a relationship with the Bergman kernel in C^∞-smoothly bounded planar domains—see [4, Chapter 25], it is natural to ask whether the Szegő metric can also be described using Weierstrass elliptic $\wp$-functions. In this chapter, we will show that this is indeed possible.

5.1 Szegő metric on A_r

We begin by recalling the series form of the Szegő kernel on the annulus

$$A_r = \{z \in \mathbb{C} : r < |z| < 1\},$$

where $r \in (0,1)$. An orthonormal basis for $H^2(bA_r)$ is given by

$$\left\{ \frac{z^n}{\sqrt{2\pi(1+r^{2n+1})}} \right\}_{n=-\infty}^{\infty},$$

and hence the Szegő kernel for A_r is

$$S_{A_r}(z,w) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(z\overline{w})^n}{1+r^{2n+1}}.$$

To find a closed form of the Szegő metric, we first recall the definitions of the Weierstrass elliptic functions following the notations in [1] and [32]. For more details on the theory of elliptic functions, we refer to [39]. Let $\omega_1 = -\log r$, $\omega_3 = i\pi$, and write $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$ for $m,n \in \mathbb{Z}$. The Weierstrass elliptic $\wp$ function is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left(\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right),$$

where the $'$ in the summation means that simultaneous zero values of m,n are excepted. The function $\wp$ is holomorphic on $\mathbb{C}$ except for poles at $\Omega_{m,n}$ for each $m,n \in \mathbb{Z}$. Moreover, it is an even function, it is doubly periodic with periods $2\omega_1$ and $2\omega_3$, and satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

where
$$g_2 = 60\sum_{m,n}{}' \frac{1}{\Omega_{m,n}^4}, \quad g_3 = 140\sum_{m,n}{}' \frac{1}{\Omega_{m,n}^6}.$$
We also note that the roots of the equation $4x^3 - g_2x - g_3$ are given by
$$e_1 = \wp(\omega_1), \quad e_2 = \wp(-\omega_1 - \omega_3), \quad e_3 = \wp(\omega_3), \tag{20}$$
which are real and satisfies $e_1 > e_2 > e_3$. Along the boundary of the half-period rectangle with vertices $0, \omega_1, \omega_1 + \omega_3, \omega_3$, the function $\wp(z)$ is real, and as z is taken counterclockwise around this rectangle from $0 \to \omega_1 \to \omega_1 + \omega_3 \to \omega_3 \to 0$, $\wp(u)$ decreases from $+\infty \to e_1 \to e_2 \to e_3 \to -\infty$. The behaviour of $\wp$ along the boundaries of the other half-period rectangles can be obtained from this using $p(-z) = p(z)$, $p(\overline{z}) = \overline{p}(z)$, and periodicity.

The Weierstrass ζ function is defined by
$$\zeta'(z) = -\wp(z),$$
$$\lim_{z\to 0}\left(\zeta(z) - z^{-1}\right) = 0.$$
The function ζ is odd and satisfies the quasi-periodicity condition
$$\zeta(z + 2\omega_k) = \zeta(z) + 2\eta_k,$$
for $k = 1, 3$, where $\eta_k = \zeta(\omega_k)$.

Finally, the Weierstrass σ function is defined by the equation
$$\frac{d}{dz}\log\sigma(z) = \zeta(z),$$
$$\lim_{z\to 0}\sigma(z)/z = 1.$$
The function σ is odd and satisfies the quasi-periodicity condition
$$\sigma(z + 2\omega_k) = -e^{2\eta_k(z+\omega_k)}\sigma(z),$$
for $k = 1, 3$. For real u, the functions $\wp(u)$, $\sigma(u)$, and $\zeta(u)$ take real values since ω_1 is positive and ω_2 is purely imaginary.

It is customary to write $\omega_2 = -\omega_1 - \omega_3$, $\eta_2 = -\eta_1 - \eta_3$, and define three other σ functions $\sigma_k(u)$, $k = 1, 2, 3$, by the equation
$$\sigma_k(u) = e^{-\eta_k u}\frac{\sigma(u + \omega_k)}{\sigma(\omega_k)}. \tag{21}$$

We also recall a relation between the Szegő kernel and the capacity metric from [1, Equation (7)]:
$$2\pi S_{A_r}(z) = c_\beta(z)\sigma_2^*\left(-2\log|z|\right), \tag{22}$$
where c_β denotes the logarithmic capacity on A_r, and
$$\sigma_2^{*2}(u) = e^{-cu^2}\sigma_2^2(u), \tag{23}$$
where $c = \eta_1/\omega_1$.

Proposition 5.1.1. *Let $r \in (0,1)$ and $A_r = \{z \in \mathbb{C} : r < |z| < 1\}$. Then the Szegő metric on A_r is given by*

$$ds^2_{S_{A_r}} = \frac{\wp(2\log|z|) - \wp(2\log|z| + \omega_1 + \omega_3)}{|z|^2}|dz|^2,$$

where $\wp$ is the Weierstrass elliptic $\wp$ function with periods $2\omega_1 = -2\log r$ and $2\omega_3 = 2i\pi$.

Proof of Proposition 5.1.1. Differentiating (22), we obtain

$$\partial\bar{\partial}\log S_{A_r}(z) = \partial\bar{\partial}\log c_\beta(z) + \partial\bar{\partial}\log\Big(\sigma_2^*\big(-2\ln|z|\big)\Big).$$

From, Suita [37], we know that

$$\partial\bar{\partial}\big(\ln c_\beta(z)\big) = \pi K_{A_r}(z) = \frac{1}{|z|^2}\Big(\wp(2\ln|z|) + c\Big). \tag{24}$$

The latter expression was originally found by Zarankiewicz [41]. On the other hand, writing $u = -2\log|z|$, which is harmonic, we obtain from (23) and (21), that

$$\partial\bar{\partial}\log\left(\sigma_2^{*2}(u)\right) = \partial\bar{\partial}(-cu^2) + \partial\bar{\partial}\log\sigma^2(u+\omega_2).$$

Note that

$$\partial\bar{\partial}(-cu^2) = -c\partial(2u\bar{\partial}u) = -2c|\partial u|^2 = -\frac{2c}{|z|^2},$$

and

$$\begin{aligned}
\partial\bar{\partial}\log\sigma^2(u+\omega_2) &= 2\partial\left(\frac{d}{du}\big(\log\sigma(u+\omega_2)\big)\bar{\partial}u\right)\\
&= 2\partial\big(\zeta(u+\omega_2)\bar{\partial}u\big)\\
&= 2\frac{d}{du}\big(\zeta(u+\omega_2)\big)|\partial u|^2\\
&= -2\wp(u+\omega_2)\frac{1}{|z|^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial\bar{\partial}\log\Big(\sigma_2^*\big(-2\log|z|\big)\Big) &= -\frac{\wp\big(-2\log|z| + \omega_2\big) + c}{|z|^2}\\
&= -\frac{\wp\big(2\log|z| + \omega_1 + \omega_3\big) + c}{|z|^2}.
\end{aligned} \tag{25}$$

Adding (24) and (25), we obtain

$$\partial\bar{\partial}\log S_{A_r}(z) = \frac{\wp\big(2\log|z|\big) - \wp\big(2\log|z| + \omega_1 + \omega_3\big)}{|z|^2},$$

as required. □

6

L^2-COHOMOLOGY

Given a complete Kähler manifold, the question of whether there are non-trivial square integrable harmonic forms is of interest because every L^2-cohomology class has a harmonic representative—which is an analogue of the Hodge theorem for non-compact manifolds. This question for the Bergman metric on strongly pseudoconvex domains in $\mathbb{C}^n$ was studied by Donnelly–Fefferman [17] and Donnelly [16] and in a more general setup by McNeal [33] and Ohsawa [34] among others. Let us fix some notations to state our next result. Denote by Ω_2^k the space of k-forms on Ω which are square integrable with respect to $ds_{m_\Omega} = m_\Omega(z)|dz|$ where $m_\Omega = c_\Omega$ or s_Ω. Then, the L^2-cohomology of the complex

$$\Omega_2^0 \xrightarrow{d_0} \Omega_2^1 \xrightarrow{d_1} \Omega_2^2 \xrightarrow{d_2} 0,$$

is defined by

$$H_2^k(\Omega) = \frac{\ker d_k}{\overline{\mathrm{Im}(d_{k-1})}},$$

where the closure is taken in the L^2-norm. Since ds_{m_Ω} is complete,

$$H_2^k(\Omega) \cong \mathcal{H}_2^k(\Omega),$$

where $\mathcal{H}_2^k(\Omega)$ denotes the space of square-integrable harmonic forms. We also note the decomposition

$$\mathcal{H}_2^k(\Omega) = \oplus_{p+q=k}\mathcal{H}_2^{p,q}(\Omega).$$

The goal of this chapter is to show the following theorem:

Theorem 6.0.1. *Let $\Omega \subset \mathbb{C}$ be a non-degenerate finitely connected domain equipped with $m_\Omega(z)|dz|$, where $m_\Omega = c_\Omega$ or s_Ω. Then*

$$\dim \mathcal{H}_2^{p,q}(\Omega) = \begin{cases} 0 & \text{if } p+q \neq 1, \\ \infty & \text{if } p+q = 1. \end{cases}$$

6.1 PROOF OF THEOREM 6.0.1

Using the same reasoning as in the proof of Corollary 3.2.2, we can assume that Ω is C^∞-smoothly bounded. Also, we will prove the theorem for $m_\Omega = s_\Omega$ only as the case $m_\Omega = c_\Omega$ is similar. Fix a C^∞-smooth defining function ψ for Ω.

Case $p+q \neq 1$. Since $\mathcal{H}_2^0(\Omega) \cong \mathcal{H}_2^2(\Omega)$, it is enough to prove that every square integrable harmonic function on Ω with respect to $ds_{s_\Omega} = s_\Omega(z)|dz|$ is identically equal to 0. Let f be such a function. First, note that f must be constant since ds_{s_Ω} is complete and Kähler (see, for instance, [40]). Also, by Proposition 3.1.4, Ω has infinite volume with respect to ds_{s_Ω}:

$$\int_\Omega \frac{i}{2} s_\Omega^2(z)\, dz \wedge d\bar{z} \gtrsim \int_\Omega \frac{1}{\big(\psi(z)\big)^2} dA(z) = +\infty,$$

where $dA(z) = i/2 dz \wedge d\bar{z}$, which implies that f must be identically equal to 0.

Case $p+q = 1$. By [34], the infinite dimensionality of $\mathcal{H}_2^{p,q}(\Omega)$ will follow at once if we establish that

$$ds_{s_\Omega}^2 \approx (-\psi)^{-1}|dz|^2 + (-\psi)^{-2}|\partial\psi|^2|dz|^2, \tag{26}$$

uniformly near $b\Omega$. If $p \in b\Omega$ then by Proposition 3.1.4,

$$\begin{aligned}\lim_{z\to p} \frac{s_\Omega(z)^2}{\big(-\psi(z)\big)^{-1} + \big(-\psi(z)\big)^{-2}\big|\partial\psi(z)\big|^2} &= \lim_{z\to p} \frac{\big(-\psi(z)\big)^2 s_\Omega^2(z)}{-\psi(z) + \big|\partial\psi(z)\big|^2} \\ &= \frac{\big|\partial\psi(p)\big|^2}{\big|\partial\psi(p)\big|^2},\end{aligned}$$

which shows that (26) holds near p. Then, by compactness of $b\Omega$, it follows that (26) holds near $b\Omega$, which completes the proof.

7

VARIATIONS OF THE CARATHÉODORY AND SZEGŐ METRICS ON PLANAR ANNULI

The study of the holomorphic sectional curvature of the Bergman metric has a rich history, dating back to the work of Bergman [7] and Kobayashi [28]. Bergman demonstrated that the holomorphic sectional curvature of the Bergman metric is strictly bounded above by 2, a result later extended by Kobayashi to complex manifolds. Lebed′[31] further refined this result by showing that 2 is the optimal upper bound for $n \geq 2$. The question of whether a lower bound exists for the holomorphic sectional curvature of the Bergman metric was answered negatively in higher dimensions by Herbort in [25].

For planar domains, Dinew [15] constructed an infinitely connected planar domain where the holomorphic sectional curvature of the Bergman metric diverges to $-\infty$ at one of its boundary points, while the upper limit at that point is 2. This result was extended by Zwonek in [42] by establishing that the supremum of the holomorphic sectional curvature on Zalcman-type domains is 2, while the infimum is $-\infty$. It was achieved by examining the variations of the holomorphic sectional curvature of the Bergman metric $b_{A_r}(z)|dz|$ on planar annuli A_r as the inner radius r approaches 0, where

$$b_{A_r}(z) = \sqrt{\frac{\partial^2 \log K_{A_r}(z,z)}{\partial z \partial \overline{z}}},$$

and K_{A_r} denotes the Bergman kernel on A_r.

In this chapter, we will study the variations of the Carathéodory and Szegő metrics, as well as their Gaussian curvatures, on planar annuli, drawing inspiration from [42].

7.1 VARIATIONS ON PLANAR ANNULI

The main objective of this chapter is to establish the following limiting behaviour:

Theorem 7.1.1. *For $r \in (0,1)$, let $A_r = \{z \in \mathbb{C} : r < |z| < 1\}$. Then*

(a) $\lim_{r\to 0+} c_{A_r}(r^\lambda) = 2\pi \lim_{r\to 0+} S_{A_r}(r^\lambda) = \begin{cases} 1 & 0 < \lambda < \frac{1}{2}, \\ 2 & \lambda = \frac{1}{2}, \\ +\infty & \frac{1}{2} < \lambda < 1. \end{cases}$

(b) $\lim_{r\to 0+} s_{A_r}(r^\lambda) = \begin{cases} 1 & 0<\lambda<\frac{1}{4}, \\ \sqrt{2} & \lambda = \frac{1}{4}, \\ +\infty & \frac{1}{4}<\lambda<1. \end{cases}$

(c) $\lim_{r\to 0+} \kappa_{c_{A_r}}(r^\lambda) = \begin{cases} -4 & 0<\lambda<\frac{1}{4}, \\ -8 & \lambda = \frac{1}{4}, \\ -\infty & \frac{1}{4}<\lambda<\frac{3}{4}, \\ -8 & \lambda = \frac{3}{4}, \\ -4 & \frac{3}{4}<\lambda<1. \end{cases}$

(d) $\lim_{r\to 0+} \kappa_{s_{A_r}}(r^\lambda) = \begin{cases} -4 & 0<\lambda<\frac{1}{6}, \\ -12 & \lambda = \frac{1}{6}, \\ -\infty & \frac{1}{6}<\lambda<\frac{1}{3}, \\ -4 & \lambda = \frac{1}{3}, \\ 4 & \frac{1}{3}<\lambda<\frac{2}{3}, \\ -4 & \lambda = \frac{2}{3}, \\ -\infty & \frac{2}{3}<\lambda<\frac{5}{6}, \\ -12 & \lambda = \frac{5}{6}, \\ -4 & \frac{5}{6}<\lambda<1. \end{cases}$

Some remarks are in order. First, as r decreases to 0, the annuli A_r exhaust the punctured unit disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. Therefore, c_{A_r} converges uniformly on compact subsets of $\mathbb{D}^*$ to $c_{\mathbb{D}^*} = c_{\mathbb{D}}|_{\mathbb{D}^*}$ as $r \to 0+$. It follows that $\kappa_{c_{A_r}}$ converges uniformly on compact subsets of $\mathbb{D}^*$ to -4 as $r \to 0+$. Since $c_{A_r}(z) = 2\pi S_{A_r}(z)$, it also follows that $S_{A_r}(z)$ converges uniformly on compact subsets of $\mathbb{D}^*$ to $S_{\mathbb{D}}|_{\mathbb{D}^*}(z)$ as $r \to 0+$. Accordingly, s_{A_r} and $\kappa_{s_{A_r}}$ converge uniformly on compact subsets of $\mathbb{D}^*$ to $s_{\mathbb{D}}|_{\mathbb{D}^*}$ and -4 respectively as $r \to 0+$. This does not contradict Theorem 7.1.1 as in this theorem, the points where the limits are studied, **do not** lie on a fixed compact subset of $\mathbb{D}^*$.

Second, using Theorem 7.1.1, we can answer the questions related to the universal bounds of the Gaussian curvatures of the Carathéodory and Szegő metrics. Observe from (d) that $\kappa_{s_{A_r}}(r^{4/9}) \to 4$ as $r \to 0+$. This establishes the following:

Theorem 7.1.2. *Given $\epsilon > 0$, there exist a C^∞-smoothly bounded domain $\Omega_\epsilon \subset \mathbb{C}$ and a point $z \in \Omega_\epsilon$ such that*

$$\kappa_{s_{\Omega_\epsilon}}(z) > 4 - \epsilon.$$

Also, from (c) and (d), $\kappa_{c_{A_r}}(r^{1/2}) \to -\infty$ and $\kappa_{s_{A_r}}(r^{1/4}) \to -\infty$ as $r \to 0+$. Thus we obtain

Theorem 7.1.3. *There are no universal lower bounds for the Gaussian curvatures of the Carathéodory and Szegő metrics on the class of C^∞-smoothly bounded planar domains.*

Third, Theorem 7.1.1 also shows that there are domains in $\mathbb{C}$ such that the Gaussian curvature of the Szegő metric assume both negative and positive values:

Theorem 7.1.4. *There exist a C^∞-smoothly bounded domain $\Omega \subset \mathbb{C}$ and $z, w \in \Omega$, such that*

$$\kappa_{s_\Omega}(z) > 0 \text{ and } \kappa_{s_\Omega}(w) < 0.$$

Fourth, from Zwonek's result [42] and (d), we have

$$\lim_{r\to 0+} \kappa_{s_{A_r}}(r^\lambda) < 0 < \lim_{r\to 0+} \kappa_{b_{A_r}}(r^\lambda) \text{ for } 0 < \lambda \le \frac{1}{3} \quad \text{or} \quad \frac{2}{3} \le \lambda < 1,$$
$$\lim_{r\to 0+} \kappa_{b_{A_r}}(r^\lambda) < 0 < \lim_{r\to 0+} \kappa_{s_{A_r}}(r^\lambda) \text{ for } \frac{1}{3} < \lambda < \frac{2}{3}.$$

Therefore,

Theorem 7.1.5. *There exist a C^∞-smoothly bounded domain $\Omega \subset \mathbb{C}$ and $z, w \in \Omega$, such that*

$$\kappa_{s_\Omega}(z) < 0 < \kappa_{b_\Omega}(z) \text{ and } \kappa_{s_\Omega}(w) > 0 > \kappa_{b_\Omega}(w).$$

Lastly, Theorem 7.1.1 also allows us to answer a question related to the comparison of the Carathéodory and Szegő metrics. It was shown in [2] that $s_\Omega \ge c_\Omega$, and therefore the ratio s_Ω / c_Ω has the universal lower bound 1. On the other hand, since $s_{A_r}(r^{1/2})/c_{A_r}(r^{1/2}) \to +\infty$ as $r \to 0+$, we have

Proposition 7.1.6. *There is no universal constant $M > 0$ such that*

$$\frac{s_\Omega(z)}{c_\Omega(z)} \le M,$$

for all $z \in \Omega$ and for all C^∞-smoothly bounded domains $\Omega \subset \mathbb{C}$.

The idea to prove Theorem 7.1.1 is to use the Bergman-Fuks type formulas for the Szegő kernel and the metric that relates them to certain maximal domain functions. Let us recall their definitions first. For a C^∞-smoothly bounded domain $\Omega \subset \mathbb{C}$, $j = 0, 1, 2, \cdots$, and $z \in \Omega$, we define the maximal domains functions $J^{(j)}_\Omega(z)$ by

$$J^{(j)}_\Omega(z) = \sup_{\|f\|_{H^2(b\Omega)} \le 1} \left\{ \left| f^{(j)}(z) \right|^2 : f \in H^2(b\Omega),\ f(z) = f'(z) = \cdots = f^{(j-1)}(z) = 0 \right\}.$$

It can be shown that

$$\begin{aligned} S_\Omega(z) &= J^{(0)}_\Omega(z), \\ s_\Omega(z) &= \sqrt{\frac{J^{(1)}_\Omega(z)}{J^{(0)}_\Omega(z)}}, \\ \kappa_{s_\Omega}(z) &= 4 - 2\frac{J^{(0)}_\Omega(z) J^{(2)}_\Omega(z)}{J^{(1)}_\Omega(z)^2}. \end{aligned} \tag{27}$$

These formulas can be derived in the same way as for the usual Bergman kernel and the metric (see, for example, [5], [6], [20], and [26]). Indeed, we will present the proof for κ_{s_Ω} only, as the proofs for S_Ω, s_Ω are significantly simpler and follow from similar lines of reasoning.

Theorem 7.1.7. *Let $\Omega \subset \mathbb{C}$ be a C^∞-smoothly bounded domain. Then the Gaussian curvature κ_{s_Ω} of the Szegő metric $s_\Omega(z)|dz|$ satisfies*

$$\kappa_{s_\Omega}(z) = 4 - 2\left(\frac{J_\Omega^{(0)}(z) J_\Omega^{(2)}(z)}{J_\Omega^{(1)}(z)^2}\right),$$

for all $z \in \Omega$.

Proof. Let $a \in \Omega$. Define

$$\begin{aligned} H_a' &= \{f \in H^2(b\Omega) : f(a) = 0\}, \\ H_a'' &= \{f \in H_a' : f'(a) = 0\}, \\ H_a''' &= \{f \in H_a'' : f''(a) = 0\}. \end{aligned}$$

By definition, H_a'' is a closed subspace of H_a', and H_a''' is a closed subspace of H_a''. In both cases the codimension is 1. Thus, there exists an orthonormal basis $(\phi_k)_{k\geq 0}$ of $H^2(b\Omega)$ such that

$$\phi_{k-1}(a) = 0, \quad \phi_k'(a) = 0, \quad \phi_{k+1}''(a) = 0 \text{ for } k \geq 2. \tag{28}$$

To simplify the notation, let $S(z) = S_\Omega(z)$, $s(z) = s_\Omega(z)$, and for all non-negative integers k, l, define

$$S_{k\bar{l}}(z) = \partial^k \bar{\partial}^l S(z), \;\; S_{\bar{l}}(z) = S_{0\bar{l}}(z), \;\; S_k(z) = S_{k\bar{0}}(z),$$

for $z \in \Omega$. We first note that

$$\frac{\kappa_\Omega(a)}{2} = -\frac{1}{s^2(a)}\big(\log(SS_{1\bar{1}} - |S_1|^2)\big)_{1\bar{1}}(a) + 2 = -\frac{I_0(a)}{s^2(a)} + 2, \tag{29}$$

say. By straightforward calculations, we have

$$I_0(a) = \frac{S(a)S_{2\bar{2}}(a) - S_2(a)S_{\bar{2}}(a)}{s^2(a)S^2(a)} + \frac{\left|S(a)S_{2\bar{1}}(a) - S_2(a)S_{\bar{1}}(a)\right|^2}{s^4(a)S^4(a)}.$$

Using $S(z) = \sum_{k=0}^\infty |\phi_k(z)|^2$ and (28), we have

$$S(a) = \left|\phi_0(a)\right|^2, \;\; S_1(a) = \phi_0'(a)\overline{\phi_0(a)}, \;\; S_{1\bar{1}}(a) = \sum_{k=0}^{1} \left|\phi_k'(a)\right|^2, \text{ and}$$

$$S_2(a) = \phi_0''(a)\overline{\phi_0(a)}, \; S_{2\bar{1}}(a) = \phi_0''(a)\overline{\phi_0'(a)} + \phi_1''(a)\overline{\phi_1'(a)}, \; S_{2\bar{2}}(a) = \sum_{k=0}^{2} \left|\phi_k''(a)\right|^2.$$

It can also be seen that $J_\Omega^{(j)}(a) = \left|\phi^{(j)}(a)\right|^2$ for $j = 0, 1, 2$. Using $S(a) = J_\Omega^{(0)}(a)$, $s(a)^2 = J_\Omega^{(1)}(a)/J_\Omega^{(0)}(a)$, it follows from above that

$$I_0(a) = \frac{\left|\phi_2''(a)\right|^2}{\left|\phi_1'(a)\right|^2} = \frac{J_\Omega^{(2)}(a)}{J_\Omega^{(1)}(a)}. \tag{30}$$

Therefore, using (30) in (29), we get

$$\frac{\kappa_\Omega(a)}{2} = 2 - \frac{J_\Omega^{(0)}(a)}{J_\Omega^{(1)}(a)} \cdot \frac{J_\Omega^{(2)}(a)}{J_\Omega^{(1)}(a)},$$

which completes the proof.

□

Corollary 7.1.8. *Let $\Omega \subset \mathbb{C}$ be any non-degenerate finitely connected domain. Then*

$$\kappa_{s_\Omega}(z) < 4,$$

for all $z \in \Omega$.

Under a conformal equivalence $\phi : \Omega \to \Omega'$, the maximal domain functions $J_\Omega^{(j)}(z)$, $j = 0, 1, 2, \ldots$, transform according to the following rule

$$J_\Omega^{(j)}(z) = \left|\phi'(z)\right|^{2j+1} J_{\Omega'}^{(j)}(\phi(z)). \tag{31}$$

Because of (27), the focus now is to investigate the maximal domain functions on A_r. As a first step, we compute these domain functions on a general annulus

$$A(r,R) = \{z \in \mathbb{C} : r < |z| < R\},$$

where $0 < r < 1 < R < \infty$. It is apparent from their definition that we will require suitable functions in $H^2(bA(r,R))$ to compute them, and so first, we prove a result that allows us to construct plenty of them. Note that $(z^n)_{n=-\infty}^{\infty}$ is an orthogonal basis for $H^2(bA(r,R))$ and writing $\alpha_n^{r,R} = \|z^n\|^2_{H^2(bA(r,R))}$, we have

$$\alpha_n^{r,R} = \int_{|z|=r} |z|^{2n}|dz| + \int_{|z|=R} |z|^{2n}|dz| = 2\pi\left(r^{2n+1} + R^{2n+1}\right). \tag{32}$$

Proposition 7.1.9. *Let p be a monic polynomial and*

$$c_n = \frac{p(n)}{\alpha_n^{r,R}}, \quad n \in \mathbb{Z}.$$

Then $(c_n)_{-\infty}^{\infty}$ is summable, and hence also square summable. In particular,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

is in $H^2(bA(r,R))$.

Proof. Let $n \geq 0$ be large. Then $p(\pm n) \neq 0$ and

$$\frac{|p(n+1)|}{|p(n)|} \to 1 \quad \text{and} \quad \frac{|p(-n-1)|}{|p(-n)|} \to 1.$$

Now,

$$\begin{aligned}\left|\frac{c_{n+1}}{c_n}\right| &= \frac{|p(n+1)|}{|p(n)|}\frac{\alpha_n^{r,R}}{\alpha_{n+1}^{r,R}}\\ &= \frac{|p(n+1)|}{|p(n)|}\frac{r^{2n+1}+R^{2n+1}}{r^{2n+3}+R^{2n+3}}\\ &= \frac{|p(n+1)|}{|p(n)|}\frac{R^{2n+1}\big((r/R)^{2n+1}+1\big)}{R^{2n+3}\big((r/R)^{2n+3}+1\big)}\\ &\to 1/R^2<1.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\frac{c_{-n-1}}{c_{-n}}\right| &= \frac{|p(-n-1)|}{|p(-n)|}\frac{\alpha_{-n}^{r,R}}{\alpha_{-n-1}^{r,R}}\\ &= \frac{|p(-n-1)|}{|p(-n)|}\frac{r^{-2n+1}+R^{-2n+1}}{r^{-2n-1}+R^{-2n-1}}\\ &= \frac{|p(-n-1)|}{|p(-n)|}\frac{r^{-2n+1}\big(1+(R/r)^{-2n+1}\big)}{r^{-2n-1}\big(1+(R/r)^{-2n-1}\big)}\\ &\to r^2<1.\end{aligned}$$

It follows from the ratio test that $(c_n)_{n=-\infty}^{\infty}$ is summable. □

Now, to compute the maximal domain functions on $A(r,R)$, let us consider the series

$$s_k^{r,R}=\sum_{n=-\infty}^{\infty}\frac{n^k}{\alpha_n^{r,R}},\quad k=0,1,2,\dots, \tag{33}$$

which are finite by Proposition 7.1.9.

Proposition 7.1.10. *We have*

$$\text{(a) } J^{(0)}_{A(r,R)}(1)=s_0^{r,R},\quad \text{(b) } J^{(1)}_{A(r,R)}(1)=\frac{s_0^{r,R}s_2^{r,R}-(s_1^{r,R})^2}{s_0^{r,R}},$$

and

$$\text{(c) } J^{(2)}_{A(r,R)}(1)=\frac{(s_1^{r,R})^2s_4^{r,R}-s_0^{r,R}s_2^{r,R}s_4^{r,R}-2s_1^{r,R}s_2^{r,R}s_3^{r,R}+s_0^{r,R}(s_3^{r,R})^2+(s_2^{r,R})^3}{(s_1^{r,R})^2-s_0^{r,R}s_2^{r,R}}.$$

Proof. Let $f\in H^2\big(bA(r,R)\big)$ and $f(z)=\sum_{n=-\infty}^{\infty}a_nz^n$. Then

$$\|f\|^2_{H^2\left(bA(r,R)\right)}=\sum_{n=-\infty}^{\infty}|a_n|^2\alpha_n^{r,R}.$$

(a) By the Cauchy-Schwarz inequality

$$|f(1)|^2 = \left|\sum_{n=-\infty}^{\infty} a_n\right|^2 = \left|\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{\alpha_n^{r,R}}}\left(a_n\sqrt{\alpha_n^{r,R}}\right)\right|^2 \leq s_0^{r,R}\|f\|^2_{H^2(bA(r,R))}. \tag{34}$$

Since f is arbitrary, this implies that

$$J^{(0)}_{A(r,R)}(1) \leq s_0^{r,R}.$$

Thus, to complete the proof of (a), all we need to do is to produce a function in $H^2(bA(r,R))$ for which the inequality in (34) is equality. In view of Proposition 7.1.9,

$$f_0(z) = \sum_{n=-\infty}^{\infty} \frac{z^n}{\alpha_n^{r,R}},$$

is in $H^2(bA(r,R))$. Note that $\|f_0\|^2_{H^2(bA(r,R))} = s_0^{r,R} = f_0(1)$, and therefore,

$$|f_0(1)|^2 = s_0^{r,R}\|f_0\|^2_{H^2(bA(r,R))}.$$

Thus, f_0 has the desired property; hence, the proof of (a) follows.

(b) Assume that $f(1) = 0$, so that $\sum_{n=-\infty}^{\infty} a_n = 0$. Then for any $\beta \in \mathbb{R}$,

$$\begin{aligned}|f'(1)|^2 &= \left|\sum_{n=-\infty}^{\infty} na_n\right|^2 = \left|\sum_{n=-\infty}^{\infty} (n-\beta)a_n\right|^2 \\ &= \left|\sum_{n=-\infty}^{\infty} \frac{n-\beta}{\sqrt{\alpha_n^{r,R}}}\left(a_n\sqrt{\alpha_n^{r,R}}\right)\right|^2 \leq \sum_{n=-\infty}^{\infty} \frac{(n-\beta)^2}{\alpha_n^{r,R}}\|f\|^2_{H^2(bA(r,R))},\end{aligned} \tag{35}$$

which shows that

$$J^{(1)}_{A(r,R)}(1) \leq \sum_{n=-\infty}^{\infty} \frac{(n-\beta)^2}{\alpha_n^{r,R}}. \tag{36}$$

We now show that the above inequality is equality for a suitable β. It suffices to produce a function $f \in H^2(bA(r,R))$ such that $f(1) = 0$, and for which the inequality in (35) is equality. Let $\beta \in \mathbb{R}$ and set

$$f_\beta(z) = \sum_{n=-\infty}^{\infty} \frac{n-\beta}{\alpha_n^{r,R}} z^n.$$

By Proposition 7.1.9, $f_\beta \in H^2(bA(r,R))$. We choose β so that

$$f_\beta(1) = \sum_{n=-\infty}^{\infty} \frac{n-\beta}{\alpha_n^{r,R}} = 0. \tag{37}$$

Indeed, we can take

$$\beta = \frac{s_1^{r,R}}{s_0^{r,R}}. \tag{38}$$

Then, by (37),

$$\begin{aligned} \|f_\beta\|^2_{H^2(bA(r,R))} &= \sum_{n=-\infty}^{\infty} \frac{(n-\beta)^2}{\alpha_n^{r,R}} = \sum_{n=-\infty}^{\infty} \frac{n(n-\beta)}{\alpha_n^{r,R}} - \sum_{n=-\infty}^{\infty} \frac{\beta(n-\beta)}{\alpha_n^{r,R}} \\ &= \sum_{n=-\infty}^{\infty} \frac{n(n-\beta)}{\alpha_n^{r,R}} = f'_\beta(1), \end{aligned}$$

from which it follows that

$$\left|f'_\beta(1)\right|^2 = |f'_\beta(1)|\|f_\beta\|^2_{H^2(bA(r,R))} = \sum_{n=-\infty}^{\infty} \frac{(n-\beta)^2}{\alpha_n^{r,R}} \|f_\beta\|^2_{H^2(bA(r,R))},$$

and hence f_β has the desired property. It follows that

$$J^{(1)}_{A(r,R)}(1) = \sum_{n=-\infty}^{\infty} \frac{(n-\beta)^2}{\alpha_n^{r,R}}.$$

Again, in view of (37) and (38), we can write

$$J^{(1)}_{A(r,R)}(1) = \sum_{n=-\infty}^{\infty} \frac{n(n-\beta)}{\alpha_n^{r,R}} = s_2^{r,R} - \frac{s_1^{r,R}}{s_0^{r,R}} s_1^{r,R} = \frac{s_0^{r,R} s_2^{r,R} - \left(s_1^{r,R}\right)^2}{s_0^{r,R}},$$

which proves (b).

(c) Assume that $f(1) = f'(1) = 0$ so that

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} n a_n = 0.$$

Then for any $\gamma, \delta \in \mathbb{R}$,

$$\begin{aligned} \left|f''(1)\right|^2 &= \left|\sum_{n=-\infty}^{\infty} n(n-1)a_n\right|^2 = \left|\sum_{n=-\infty}^{\infty} n^2 a_n\right|^2 = \left|\sum_{n=-\infty}^{\infty} (n^2-\gamma n-\delta)a_n\right|^2 \\ &= \left|\sum_{n=-\infty}^{\infty} \frac{n^2-\gamma n-\delta}{\sqrt{\alpha_n^{r,R}}}\left(a_n\sqrt{\alpha_n^{r,R}}\right)\right|^2 \le \left(\sum_{n=-\infty}^{\infty} \frac{(n^2-\gamma n-\delta)^2}{\alpha_n^{r,R}}\right) \|f\|^2_{H^2(bA(r,R))}, \end{aligned} \tag{39}$$

which implies that

$$J^{(2)}_{A(r,R)}(1) \le \sum_{n=-\infty}^{\infty} \frac{(n^2-\gamma n-\delta)^2}{\alpha_n^{r,R}}. \tag{40}$$

As before, we show that for a suitable choice of γ and δ, there is a function f in $H^2(bA(r,R))$ satisfying $f(1) = f'(1) = 0$, and for which the inequality in (39) is equality. This would imply that the inequality (40) is also equality. For $\gamma, \delta \in \mathbb{R}$, set

$$f_{\gamma\delta}(z) = \sum \frac{n^2 - \gamma n - \delta}{\alpha_n^{r,R}} z^n.$$

By Proposition 7.1.9, $f_{\gamma\delta} \in H^2(bA(r,R))$. We claim that there exist γ, δ such that

$$\begin{aligned} f_{\gamma\delta}(1) &= \sum_{n=-\infty}^{\infty} \frac{n^2 - \gamma n - \delta}{\alpha_n^{r,R}} = 0, \quad \text{and} \\ f'_{\gamma\delta}(1) &= \sum_{n=-\infty}^{\infty} \frac{n(n^2 - \gamma n - \delta)}{\alpha_n^{r,R}} = 0. \end{aligned} \tag{41}$$

The above equations can be written as

$$\begin{aligned} s_1^{r,R}\gamma + s_0^{r,R}\delta - s_2^{r,R} &= 0, \\ s_2^{r,R}\gamma + s_1^{r,R}\delta - s_3^{r,R} &= 0. \end{aligned}$$

Since,

$$(s_1^{r,R})^2 - s_0^{r,R}s_2^{r,R} = \left(\sum_{n=-\infty}^{\infty} \frac{n}{\alpha_n^{r,R}}\right)^2 - \left(\sum_{n=-\infty}^{\infty} \frac{n^2}{\alpha_n^{r,R}}\right)\left(\sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^{r,R}}\right) < 0,$$

which is a consequence of the Cauchy-Schwarz inequality; the above system has a unique solution given by

$$\gamma = \frac{s_1^{r,R}s_2^{r,R} - s_0^{r,R}s_3^{r,R}}{(s_1^{r,R})^2 - s_0^{r,R}s_2^{r,R}}, \quad \text{and} \quad \delta = \frac{s_1^{r,R}s_3^{r,R} - (s_2^{r,R})^2}{(s_1^{r,R})^2 - s_0^{r,R}s_2^{r,R}}. \tag{42}$$

We now choose γ and δ as above. Then, by (41),

$$\|f_{\gamma\delta}\|^2_{H^2(bA(r,R))} = \sum_{n=-\infty}^{\infty} \frac{(n^2 - \gamma n - \delta)^2}{\alpha_n^{r,R}} = \sum_{n=-\infty}^{\infty} \frac{n^2(n^2 - \gamma n - \delta)}{\alpha_n^{r,R}},$$

and also

$$f''_{\gamma\delta}(1) = \sum_{n=-\infty}^{\infty} \frac{n(n-1)(n^2 - \gamma n - \delta)}{\alpha_n^{r,R}} = \sum_{n=-\infty}^{\infty} \frac{n^2(n^2 - \gamma n - \delta)}{\alpha_n^{r,R}}.$$

Therefore,

$$\left|f''_{\gamma\delta}(1)\right|^2 = \left|f''_{\gamma\delta}(1)\right| \|f_{\gamma\delta}\|^2_{H^2(bA(r,R))},$$

and thus $f_{\gamma\delta}$ has the desired property. It follows that

$$J^{(2)}_{A(r,R)}(1) = \sum_{n=-\infty}^{\infty} \frac{(n^2 - \gamma n - \delta)^2}{\alpha_n^{r,R}}.$$

Again, in view of (41) and (42), we can write

$$\begin{aligned} J^{(2)}_{A(r,R)}(1) &= \sum_{n=-\infty}^{\infty} \frac{n^2(n^2-\gamma n-\delta)}{\alpha_n^{r,R}} \\ &= s_4^{r,R} - \frac{s_1^{r,R}s_2^{r,R}-s_0^{r,R}s_3^{r,R}}{(s_1^{r,R})^2-s_0^{r,R}s_2^{r,R}} s_3^{r,R} - \frac{s_1^{r,R}s_3^{r,R}-(s_2^{r,R})^2}{(s_1^{r,R})^2-s_0^{r,R}s_2^{r,R}} s_2^{r,R} \\ &= \frac{(s_1^{r,R})^2 s_4^{r,R} - s_0^{r,R}s_2^{r,R}s_4^{r,R} - 2s_1^{r,R}s_2^{r,R}s_3^{r,R} + s_0^{r,R}(s_3^{r,R})^2 + (s_2^{r,R})^3}{(s_1^{r,R})^2 - s_0^{r,R}s_2^{r,R}}, \end{aligned}$$

as required. This completes the proof of (c) and the proposition. □

We now focus on the annulus $A_r = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Note that by the transformation formula (31), we have

$$r^{(2j+1)\lambda} J^{(j)}_{A_r}(r^\lambda) = J^{(j)}_{A(r^{1-\lambda},r^{-\lambda})}(1), \quad j \geq 0. \tag{43}$$

Observe that while on the left-hand side, both the domains and points vary, on the right-hand side only the domains vary. Hence, studying the asymptotic behaviour of the right-hand side is relatively easier and in view of Proposition 7.1.10, it now suffices to analyse the quantities $s_j^{r^{1-\lambda},r^{-\lambda}}$. For simplicity, let us write $\alpha_n = \alpha_n^{r^{1-\lambda},r^{-\lambda}}$, and $s_j = s_j^{r^{1-\lambda},r^{-\lambda}}$. Note from (32) that for $n \geq 0$,

$$\alpha_n = 2\pi \frac{1+r^{2n+1}}{r^{(2n+1)\lambda}} \quad \text{and} \quad \alpha_{-n-1} = 2\pi \frac{1+r^{2n+1}}{r^{(2n+1)(1-\lambda)}}.$$

Thus, we can write

$$\begin{aligned} s_j &= \sum_{n=-\infty}^{\infty} \frac{n^j}{\alpha_n} \\ &= \sum_{n=0}^{\infty} \left(\frac{n^j}{\alpha_n} + \frac{(-1)^j(n+1)^j}{\alpha_{-n-1}} \right) \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{1+r^{2n+1}} \left(n^j r^{(2n+1)\lambda} + (-1)^j (n+1)^j r^{(2n+1)(1-\lambda)} \right) \\ &= u_j + v_j, \end{aligned} \tag{44}$$

where u_j is the 0th term of the series and v_j is the sum of the terms from $n = 1$ onwards. Observe that

$$u_0 = \frac{1}{2\pi} \frac{r^\lambda + r^{1-\lambda}}{1+r}, \quad u_j = (-1)^j \frac{1}{2\pi} \frac{r^{1-\lambda}}{1+r}, \quad \text{for } j \geq 1, \tag{45}$$

and

$$v_j = \frac{1}{2\pi} \frac{r^{3\lambda} + (-1)^j 2^j r^{3(1-\lambda)}}{1+r^3} + O(r^{5\lambda}) + O(r^{5(1-\lambda)}). \tag{46}$$

The following lemma that describes the asymptotic behaviour of the maximal domain functions associated with A_r is the key to the proof of Theorem 7.1.1. In what follows, for $N, \Phi : (0,1) \to (0,\infty)$, we will write $N(r) \sim \Phi(r)$ if for all sufficiently small $\epsilon > 0$,

$$N(r) - \Phi(r) = \Phi(r)o(r^{\epsilon}).$$

Also, we will compute asymptotic expansions of many functions on $(0,1)$. To follow a uniform notation, we will write

$$N(r) = \Phi_N(r) + \Psi_N(r),$$

where Φ_N is the dominating term and Ψ_N is the remainder term. In this notation, observe that $N(r) \sim \Phi_N(r)$ if and only if N, Φ are positive and for all sufficiently small $\epsilon > 0$,

$$\frac{\Psi_N(r)}{\Phi_N(r)} = o(r^{\epsilon})$$

as $r \to 0+$.

Lemma 7.1.11. *Let $\lambda \in (0,1)$ be fixed. Then for $r \in (0,1)$,*

$$\text{(a)}\, r^{\lambda} J^{(0)}_{A_r}(r^{\lambda}) = N^{(0)}_{\lambda}(r), \quad \text{(b)}\, r^{3\lambda} J^{(1)}_{A_r}(r^{\lambda}) = \frac{N^{(1)}_{\lambda}(r)}{N^{(0)}_{\lambda}(r)}, \quad \text{(c)}\, r^{5\lambda} J^{(2)}_{A_r}(r^{\lambda}) = \frac{N^{(2)}_{\lambda}(r)}{N^{(1)}_{\lambda}(r)},$$

where $N^{(j)}_{\lambda}(r)$, $j = 0,1,2$, are positive functions satisfying

$$\begin{aligned} N^{(0)}_{\lambda}(r) &\sim \frac{1}{2\pi}\frac{r^{\lambda} + r^{1-\lambda}}{1+r}, \\ N^{(1)}_{\lambda}(r) &\sim \frac{1}{4\pi^2}\frac{r}{(1+r)^2} + \frac{1}{4\pi^2}\frac{(r^{4\lambda} + r^{4(1-\lambda)}) + 4r(r^{2\lambda} + r^{2(1-\lambda)})}{(1+r)(1+r^3)}, \quad \textit{and} \\ N^{(2)}_{\lambda}(r) &\sim \frac{1}{2\pi^3}\frac{r^{9\lambda} + r^{9(1-\lambda)}}{(1+r)(1+r^3)(1+r^5)} + \frac{1}{2\pi^3}\frac{r(r^{3\lambda} + r^{3(1-\lambda)})}{(1+r)^2(1+r^3)}. \end{aligned}$$

Proof. (a) We define

$$N^{(0)}_{\lambda}(r) = r^{\lambda} J^{(0)}_{A_r}(r^{\lambda}),$$

which is evidently positive. By (43) and Proposition 7.1.10, we have

$$N^{(0)}_{\lambda}(r) = s_0.$$

Then from (44)–(46), we have

$$N^{(0)}_{\lambda}(r) = u_0 + v_0 = \frac{1}{2\pi}\frac{r^{\lambda} + r^{1-\lambda}}{1+r} + \Psi_{N^{(0)}_{\lambda}}(r),$$

where

$$\Psi_{N^{(0)}_{\lambda}}(r) = O\big(r^{3\lambda}\big) + O\big(r^{3(1-\lambda)}\big).$$

Therefore, to complete the proof of (a), we only need to show that

$$\frac{\Psi_{N_\lambda^{(0)}}(r)}{\frac{1}{2\pi}\frac{r^\lambda+r^{1-\lambda}}{1+r}} = o(r^\epsilon).$$

Note that for all sufficiently small $\epsilon > 0$,

$$\Psi_{N_\lambda^{(0)}}(r) = o\big(r^{2\lambda+\epsilon}\big) + o\big(r^{2(1-\lambda)+\epsilon}\big).$$

Also,

$$\frac{\big|o(r^{2\lambda+\epsilon})\big|}{\frac{1}{2\pi}\frac{r^\lambda+r^{1-\lambda}}{1+r}r^\epsilon} \le \frac{2\pi(1+r)\big|o(r^{2\lambda+\epsilon})\big|}{r^{\lambda+\epsilon}} \to 0$$

as $r \to 0+$, and by symmetry, the same holds for the $o(r^{2(1-\lambda)+\epsilon})$ term.

(b) By (43) and Proposition 7.1.10, we have

$$r^{3\lambda}J^{(1)}_{A_r}(r) = J^{(1)}_{A(r^{1-\lambda},r^{-\lambda})}(1) = \frac{s_0s_2 - s_1^2}{s_0} = \frac{N_\lambda^{(1)}(r)}{N_\lambda^{(0)}(r)},$$

where $N_\lambda^{(1)}(r)$ is the function defined by

$$N_\lambda^{(1)}(r) = s_0s_2 - s_1^2,$$

which is positive as the left-hand side of the above equation and $N_\lambda^{(0)}(r)$ are positive. The asymptotic behaviour of $N_\lambda^{(0)}(r)$ is already discussed in (a). For $N_\lambda^{(1)}(r)$, note that by (44),

$$N_\lambda^{(1)}(r) = (u_0u_2 - u_1^2) + (u_0v_2 + u_2v_0 - 2u_1v_1) + (v_0v_2 - v_1^2) = I + II + III, \quad (47)$$

say. From (45) and (46), we have

$$I = \frac{1}{4\pi^2}\frac{r}{(1+r)^2}, \quad (48)$$

and

$$\begin{aligned} II = {} & \frac{1}{2\pi}\frac{r^\lambda + r^{1-\lambda}}{1+r}\cdot\frac{1}{2\pi}\frac{r^{3\lambda}+4r^{3(1-\lambda)}}{1+r^3} + \frac{1}{2\pi}\frac{r^{1-\lambda}}{1+r}\cdot\frac{1}{2\pi}\frac{r^{3\lambda}+r^{3(1-\lambda)}}{1+r^3} \\ & + 2\frac{1}{2\pi}\frac{r^{1-\lambda}}{1+r}\cdot\frac{1}{2\pi}\frac{r^{3\lambda}-2r^{3(1-\lambda)}}{1+r^3} + \Psi_{II}(r) \\ = {} & \frac{1}{4\pi^2}\frac{(r^{4\lambda}+r^{4(1-\lambda)}) + 4r(r^{2\lambda}+r^{2(1-\lambda)})}{(1+r)(1+r^3)} + \Psi_{II}(r), \end{aligned} \quad (49)$$

where

$$\Psi_{II}(r) = O(r^{4\lambda+1}) + O(r^{4(1-\lambda)+1}) + O(r^{6\lambda}) + O(r^{6(1-\lambda)}).$$

Finally,

$$\begin{aligned} III &= \frac{1}{2\pi}\frac{r^{3\lambda}+r^{3(1-\lambda)}}{1+r^3}\cdot\frac{1}{2\pi}\frac{r^{3\lambda}+4r^{3(1-\lambda)}}{1+r^3} - \left(\frac{1}{2\pi}\frac{r^{3\lambda}-2r^{3(1-\lambda)}}{1+r^3}\right)^2 + \Psi_{III}(r) \\ &= \frac{1}{4\pi^2}\frac{9r^3}{(1+r^3)^2} + \Psi_{III}(r), \end{aligned} \tag{50}$$

where

$$\Psi_{III}(r) = O\big(r^{8\lambda}\big) + O\big(r^{8(1-\lambda)}\big) + O\big(r^{3+2(1-\lambda)}\big) + O\big(r^{3+2\lambda}\big).$$

It follows from (47)–(50) that

$$N_\lambda^{(1)}(r) = \Phi_{N_\lambda^{(1)}}(r) + \Psi_{N_\lambda^{(1)}}(r),$$

where

$$\Phi_{N_\lambda^{(1)}}(r) = \frac{1}{4\pi^2}\frac{r}{(1+r)^2} + \frac{1}{4\pi^2}\frac{\big(r^{4\lambda}+r^{4(1-\lambda)}\big)+4r\big(r^{2\lambda}+r^{2(1-\lambda)}\big)}{(1+r)(1+r^3)},$$

and

$$\Psi_{N_\lambda^{(1)}}(r) = o(r^2) + o\big(r^{4\lambda+\epsilon}\big) + o\big(r^{4(1-\lambda)+\epsilon}\big),$$

for all sufficiently small ϵ. Thus, to complete the proof of (b), we only need to show that

$$\Psi_{N_\lambda^{(1)}}(r)/\Phi_{N_\lambda^{(1)}}(r) = o(r^\epsilon).$$

Note that

$$\frac{\big|o(r^2)\big|}{r^\epsilon \Phi_{N_\lambda^{(1)}}}(r) \le \frac{\big|o(r^2)\big|}{r^\epsilon \frac{1}{4\pi^2}\frac{r}{(1+r)^2}} \to 0$$

as $r \to 0+$. Similarly,

$$\frac{\big|o(r^{4\lambda+\epsilon})\big|}{r^\epsilon \Phi_{N_\lambda^{(1)}}}(r) \le \frac{\big|o(r^{4\lambda+\epsilon})\big|}{r^\epsilon \frac{1}{4\pi^2}\frac{r^{4\lambda}}{(1+r)(1+r^3)}} \to 0$$

as $r \to 0+$, and by symmetry, the same holds for the $o\big(r^{4(1-\lambda)+\epsilon}\big)$ term.

(c) By (43) and Proposition 7.1.10, we have

$$r^{5\lambda} J^{(2)}_{A_r}(r^\lambda) = J^{(2)}_{A(r^{1-\lambda},r^{-\lambda})}(1) = \frac{-s_1^2 s_4 + s_0 s_2 s_4 + 2s_1 s_2 s_3 - s_0 s_3^2 - s_2^3}{s_0 s_2 - s_1^2} = \frac{N_\lambda^{(2)}(r)}{N_\lambda^{(1)}(r)}, \tag{51}$$

where $N_\lambda^{(2)}(r)$ is the function defined by

$$N_\lambda^{(2)}(r) = -s_1^2 s_4 + s_0 s_2 s_4 + 2s_1 s_2 s_3 - s_0 s_3^2 - s_2^3, \tag{52}$$

which is positive as the left-hand side of (51) and $N_\lambda^{(1)}$ are positive. We have already obtained the asymptotic behaviour of $N_\lambda^{(1)}(r)$ in (b). The calculation of $N_\lambda^{(2)}(r)$ is lengthy, and so we only outline the main steps. Using (44), we substitute $s_j = u_j + v_j$ in (52) and write

$$N_\lambda^{(2)}(r) = I' + II' + III',$$

where I' consists of the terms involving $u_j's$ alone, II' consists of the mixed terms involving both u_j's and v_j's, and III' consists of the terms involving v_j's alone. Before proceeding into the calculation of these terms, note from (45) that

$$u_4 = u_2 \quad \text{and} \quad u_3 = u_1 = -u_2. \tag{53}$$

The term I': Using (53), writing I' in terms of u_0 and u_2 alone, it follows immediately that $I' = 0$.

The term II': Note that

$$\begin{aligned} II' = &-\big(u_1^2 v_4 + v_1^2 u_4 + 2u_1 v_1 u_4 + 2u_1 v_1 v_4\big) + \big(u_0 u_2 v_4 + u_0 v_2 u_4 + u_0 v_2 v_4 + v_0 u_2 u_4 \\ &+ v_0 u_2 v_4 + v_0 v_2 u_4\big) + 2\big(u_1 u_2 v_3 + u_1 v_2 u_3 + u_1 v_2 v_3 + v_1 u_2 u_3 + v_1 u_2 v_3 + v_1 v_2 u_3\big) \\ &- \big(u_0 v_3^2 + 2u_0 u_3 v_3 + v_0 u_3^2 + 2v_0 u_3 v_3\big) - 3\big(u_2^2 v_2 + u_2 v_2^2\big). \end{aligned}$$

Using (53), II' becomes

$$\begin{aligned} II' = &u_2^2(-v_2 - 2v_3 - v_4) + u_2(v_0 v_2 + 2v_0 v_3 + v_0 v_4 - v_1^2 - 2v_1 v_2 + 2v_1 v_3 + 2v_1 v_4) + \\ &u_2(-3v_2^2 - 2v_2 v_3) + u_0 u_2(v_2 + 2v_3 + v_4) + u_0(v_2 v_4 - v_3^2). \end{aligned}$$

To remove the u_2^2 term, we write $u_0 = u_0 - u_2 + u_2$ in the $u_0 u_2$ term. Then

$$\begin{aligned} II' = &u_2(v_0 v_2 + 2v_0 v_3 + v_0 v_4 - v_1^2 - 2v_1 v_2 + 2v_1 v_3 + 2v_1 v_4 - 3v_2^2 - 2v_2 v_3) \\ &+ (u_0 - u_2)u_2(v_2 + 2v_3 + v_4) + u_0(v_2 v_4 - v_3^2) = A' + B' + C', \end{aligned} \tag{54}$$

say. To simplify the calculation of II', we first note from (44) that

$$v_j = \frac{1}{2\pi}\frac{r^{3\lambda} + (-1)^j 2^j r^{3(1-\lambda)}}{1 + r^3} + \frac{1}{2\pi}\frac{2^j r^{5\lambda} + (-1)^j 3^j r^{5(1-\lambda)}}{1 + r^5} + O(r^{7\lambda}) + O(r^{7(1-\lambda)}), \tag{55}$$

and hence it follows that the asymptotic expansion of the term $v_i v_j$ has the following form:

Lemma 7.1.12. *We have for $i, j = 0, \ldots, 4$,*

$$v_i v_j = \Phi_{ij}(r) + \Psi_{ij}(r),$$

where

$$\begin{aligned} \Phi_{ij}(r) = &\frac{r^{6\lambda} + \big((-1)^i 2^i + (-1)^j 2^j\big) r^3 + (-1)^{i+j} 2^{i+j} r^{6(1-\lambda)}}{4\pi^2(1 + r^3)^2} + \frac{(2^i + 2^j) r^{8\lambda}}{4\pi^2(1 + r^3)(1 + r^5)} + \\ &\frac{\big((-1)^i + (-1)^j\big) 2^{i+j} r^{3+2\lambda} + \big((-1)^i 3^i + (-1)^j 3^j\big) r^{3+2(1-\lambda)} + (-1)^{i+j}(2^i 3^j + 2^j 3^i) r^{8(1-\lambda)}}{4\pi^2(1 + r^3)(1 + r^5)} + \\ &\frac{2^{i+j} r^{10\lambda} + \big((-1)^j 2^i 3^j + (-1)^i 3^i 2^j\big) r^5 + (-1)^{i+j} 3^{i+j} r^{10(1-\lambda)}}{4\pi^2(1 + r^5)^2}, \end{aligned}$$

and

$$\Psi_{ij}(r) = O(r^{10\lambda}) + O(r^{10(1-\lambda)}) + O(r^{3+\lambda}) + O(r^{3+(1-\lambda)}).$$

To compute A', we substitute $v_i v_j$ from Lemma 7.1.12. To simplify the calculation, we make a few observations. First, note that the coefficient of $r^{6\lambda}$ in $v_i v_j$ is independent of i, j. From (54), observe that the sum of the integer coefficients of $v_i v_j$ terms in A' is zero. Therefore, the coefficient of $r^{6\lambda}$ in A' becomes 0. The coefficient of $r^{6(1-\lambda)}$ in $v_i v_j$ depends only on $i+j$. For each l, the sum of the integer coefficients of $v_i v_j$ in A' such that $i + j = l$ is zero. Therefore, the coefficient of $r^{6(1-\lambda)}$ is 0. The same argument applies to the terms $r^{10\lambda}$ and $r^{10(1-\lambda)}$ and so their coefficients in A' are zero. Also, the sum of the integer coefficients of r^3 in A' (i.e., keeping $u_2/(4\pi^2(1+r^3)^2)$ intact) is

$$5 + 2(-7) + 17 - (-4) - 2(2) + 2(-10) + 2(14) - 3(8) - 2(-4) = 0,$$

and hence the coefficient of r^3 in A' is 0. Similarly, the coefficient of r^5 in A' is also 0. Thus, we are left with $r^{8\lambda}$, $r^{3+2\lambda}$, $r^{3+2(1-\lambda)}$, and $r^{8(1-\lambda)}$-terms only. Keeping $u_2/(4\pi^2(1+r^3)(1+r^5))$ intact, the sum of the integer coefficients of $r^{8\lambda}$ in A' is

$$5 + 2(9) + 17 - 4 - 2(6) + 2(10) + 2(18) - 3(8) - 2(12) = 32,$$

and that of $r^{3+2\lambda}$, $r^{3+2(1-\lambda)}$, and $r^{8(1-\lambda)}$ are -112, 112, and -32, respectively. It follows that

$$A' = \frac{1}{8\pi^3}\frac{r^{1-\lambda}}{1+r} \cdot \frac{32r^{8\lambda} - 112r^{3+2\lambda} + 112r^{3+2(1-\lambda)} - 32r^{8(1-\lambda)}}{(1+r^3)(1+r^5)} + \Psi_{A'}(r),$$

where

$$\Psi_{A'}(r) = r^{1-\lambda}E_{A'}$$

where $E_{A'}$ has the same big O terms as in Ψ_{ij}. Also, from (45) and (46),

$$\begin{aligned} B' &= \frac{1}{2\pi}\frac{r^{\lambda}}{1+r} \cdot \frac{1}{2\pi}\frac{r^{1-\lambda}}{1+r} \cdot \frac{1}{2\pi}\frac{1}{1+r^3}\Big((1+2+1)r^{3\lambda} + \big(4+2(-8)+16\big)r^{3(1-\lambda)}\Big) + \Psi_{B'}(r) \\ &= \frac{1}{2\pi^3}\frac{r\big(r^{3\lambda} + r^{3(1-\lambda)}\big)}{(1+r)^2(1+r^3)} + \Psi_{B'}(r), \end{aligned} \tag{56}$$

where

$$\Psi_{B'}(r) = O\big(r^{1+5\lambda}\big) + O\big(r^{1+5(1-\lambda)}\big).$$

Finally, to compute C', note that the arguments similar to the computation of A' show that the coefficients of $r^{6\lambda}$, $r^{6(1-\lambda)}$, $r^{8\lambda}$, and $r^{10\lambda}$ in C' are 0. The coefficients of the other terms can be computed as before, and it turns out that

$$\begin{aligned} C' = \frac{1}{8\pi^3}\frac{r^{\lambda} + r^{1-\lambda}}{1+r}\Bigg(&\frac{36r^3}{(1+r^3)^2} + \frac{(4r^{8\lambda} + 256r^{3+2\lambda} + 144r^{3+2(1-\lambda)} + 36r^{8(1-\lambda)})}{(1+r^3)(1+r^5)} \\ &+ \frac{900r^5}{(1+r^5)^2}\Bigg) + \Psi_{C'}(r), \end{aligned}$$

where,

$$\Psi_{C'}(r) = (r^\lambda + r^{1-\lambda})E_{C'}$$

where $E_{C'}$ has the same big O term as in Ψ_{ij}. Thus,

$$II' = \frac{1}{2\pi^3}\frac{r^{9\lambda}+r^{9(1-\lambda)}}{(1+r)(1+r^3)(1+r^5)} + \frac{1}{2\pi^3}\frac{r\left(r^{3\lambda}+r^{3(1-\lambda)}\right)}{(1+r)^2(1+r^3)} + \Psi_{II}(r),$$

where

$$\Psi_{II'}(r) = O(r^{11\lambda}) + O(r^{11(1-\lambda)}) + O(r^{3+\lambda}) + O(r^{3+(1-\lambda)}) + O(r^{1+5\lambda}) + O(r^{1+5(1-\lambda)}).$$

The term III': Recall that III' is the term consisting of v_j's alone, i.e.,

$$III' = -v_1^2 v_4 + v_0 v_2 v_4 + 2v_1 v_2 v_3 - v_0 v_3^2 - v_2^3.$$

Observe from (55) that in $v_i v_j v_k$, the coefficient of $r^{9\lambda}$ is independent of i, j, k. Also, in III', the sum of the coefficients of $v_i v_j v_k$ is 0. Therefore, the coefficient of $r^{9\lambda}$ in III' is 0. Similarly, the coefficient of $r^{9(1-\lambda)}$ in $v_i v_j v_k$ depends only on $i+j+k$. Since for each term in III', $i+j+k=6$, and their coefficients sum up to 0 as observed above, the coefficient of $r^{9(1-\lambda)}$ in III' is 0. Now, it follows that

$$III' = O(r^{11\lambda}) + O(r^{11(1-\lambda)}) + O(r^{3+\lambda}) + O(r^{3+(1-\lambda)}).$$

Summarising,

$$N_\lambda^{(2)}(r) = \Phi_{N_\lambda^{(2)}}(r) + \Psi_{N_\lambda^{(2)}}(r),$$

where

$$\Phi_{N_\lambda^{(2)}}(r) = \frac{1}{2\pi^3}\frac{r^{9\lambda}+r^{9(1-\lambda)}}{(1+r)(1+r^3)(1+r^5)} + \frac{1}{2\pi^3}\frac{r(r^{3\lambda}+r^{3(1-\lambda)})}{(1+r)^2(1+r^3)},$$

and

$$\Psi_{N_\lambda^{(2)}}(r) = o\left(r^{10\lambda+\epsilon}\right) + o\left(r^{10(1-\lambda)+\epsilon}\right) + o\left(r^{1+3\lambda+\epsilon}\right) + o\left(r^{1+3(1-\lambda)+\epsilon}\right).$$

for all sufficiently small $\epsilon > 0$. Thus, to complete the proof of (c), we only need to check that

$$\frac{\Psi_{N_\lambda^{(2)}}(r)}{\Phi_{N_\lambda^{(2)}}(r)} = o(r^\epsilon),$$

which can be done in a similar way to the one above. This completes the proof of (c) and the lemma. □

We are now ready for the proof of Theorem 7.1.1.

Proof of Theorem 7.1.1. (a) By (27) and Lemma 7.1.11 (a), we have

$$c_{A_r}(r^\lambda) = 2\pi S_{A_r}(r^\lambda) = 2\pi J^{(0)}_{A_r}(r^\lambda) = 2\pi \frac{1}{r^\lambda} N^{(0)}_\lambda(r) = \frac{1+r^{1-2\lambda}}{1+r}(1+o(r^\epsilon)),$$

from which (a) follows immediately.

(b) By (27) and Lemma 7.1.11, we have

$$s^2_{A_r}(r^\lambda) = \frac{J^{(1)}(r^\lambda)}{J^{(0)}(r^\lambda)} = \frac{1}{r^{2\lambda}} \frac{N^{(1)}_\lambda(r)}{(N^{(0)}_\lambda(r))^2} = \frac{r + \frac{1+r}{1+r^3}\left(r^{4\lambda} + r^{4(1-\lambda)} + 4r\left(r^{2\lambda} + r^{2(1-\lambda)}\right)\right)}{r^{2\lambda}\left(r^\lambda + r^{1-\lambda}\right)^2} f(r),$$

where $f(r)$ is a positive function such that $f(r) \to 1$ as $r \to 0+$. Dividing the numerator and denominator by $r^{4\lambda}$, we write

$$s^2_{A_r}(r^\lambda) = \frac{r^{1-4\lambda} + \frac{1+r}{1+r^3}\left(1 + r^{4(1-2\lambda)} + 4\left(r^{1-2\lambda} + r^{3(1-2\lambda)}\right)\right)}{\left(1 + r^{1-2\lambda}\right)^2} f(r),$$

from which the limiting behaviour on $(0, 1/4]$ follows immediately. On the other hand, by dividing the numerator and denominator by r^2, we write

$$s^2_{A_r}(r^\lambda) = \frac{\frac{1}{r} + \frac{1+r}{1+r^3}\left(r^{2(2\lambda-1)} + r^{2(1-2\lambda)} + 4\left(r^{2\lambda-1} + r^{1-2\lambda}\right)\right)}{\left(r^{2\lambda-1} + 1\right)^2} f(r),$$

from which the limiting behaviour on $(1/4, 1)$ follows. Thus, the proof of (b) is complete.

(c) First, note that

$$\kappa_{c_{A_r}} = -\frac{\Delta \log c_{A_r}}{c^2_{A_r}} = -4\frac{\partial\bar\partial \log c_{A_r}}{c^2_{A_r}} = -\frac{\partial\bar\partial \log S_{A_r}}{\pi^2 S^2_{A_r}} = -\frac{1}{\pi^2}\frac{s^2_{A_r}}{S^2_{A_r}} = -\frac{1}{\pi^2}\frac{J^{(1)}_{A_r}}{(J^{(0)}_{A_r})^3},$$

using (27). Now, using Lemma 7.1.11, we have

$$\kappa_{c_{A_r}}(r^\lambda) = -\frac{1}{\pi^2}\frac{N^{(1)}_\lambda(r)}{(N^{(0)}_\lambda(r))^4} = -4\frac{r(1+r)^2 + \frac{(1+r)^3}{1+r^3}\left(r^{4\lambda} + r^{4(1-\lambda)} + 4r\left(r^{2\lambda} + r^{2(1-\lambda)}\right)\right)}{\left(r^\lambda + r^{1-\lambda}\right)^4} g(r),$$

where $g(r)$ is a positive function such that $g(r) \to 1$ as $r \to 0+$. Observe that the right-hand side is symmetric in λ and $1-\lambda$, and hence enough to compute the limiting behaviour on $(0, 1/2]$. Dividing the numerator and denominator by $r^{4\lambda}$, we write

$$\kappa_{c_{A_r}}(r^\lambda) = -4\frac{r^{1-4\lambda}(1+r)^2 + \frac{(1+r)^3}{1+r^3}\left(1 + r^{4(1-2\lambda)} + 4\left(r^{1-2\lambda} + r^{3(1-2\lambda)}\right)\right)}{(1+r^{1-2\lambda})^2} g(r),$$

from which the limiting behaviour on $(0, 1/2]$ (and hence on $(0, 1)$) follows immediately. This completes the proof of (c).

(d) Recall from (27) that

$$\kappa_{s_{A_r}}(r^\lambda) = 4 - 2\frac{J_{A_r}^{(0)}(r^\lambda)J_{A_r}^{(2)}(r^\lambda)}{\left(J_{A_r}^{(1)}(r^\lambda)\right)^2}. \tag{57}$$

Therefore, using Lemma 7.1.11, we have

$$\kappa_{s_{A_r}}(r^\lambda) = 4 - 2\frac{\left(N_\lambda^{(0)}(r)\right)^3 N_\lambda^{(2)}(r)}{\left(N_\lambda^{(1)}(r)\right)^3}$$

$$= 4 - 8\frac{\left(\frac{r^\lambda + r^{1-\lambda}}{1+r}\right)^3 \left(\frac{r^{9\lambda}+r^{9(1-\lambda)}}{(1+r)(1+r^3)(1+r^5)} + \frac{r\left(r^{3\lambda}+r^{3(1-\lambda)}\right)}{(1+r)^2(1+r^3)}\right)}{\left(\frac{r}{(1+r)^2} + \frac{(r^{4\lambda}+r^{4(1-\lambda)})+4r(r^{2\lambda}+r^{2(1-\lambda)})}{(1+r)(1+r^3)}\right)^3} h(r),$$

where $h(r)$ is a positive function such that $h(r) \to 1$ as $r \to 0+$. Observe that the right-hand side is symmetric with respect to λ and $1 - \lambda$; hence, it is enough to compute the limiting behaviour on $(0, 1/2]$. Moreover, terms like $1 + r$, etc., tend to 1 as $r \to 0+$, and so they can be ignored while computing the limit. In other words, the limit of $\kappa_{s_{A_r}}(r^\lambda)$ as $r \to 0+$ is the same as that of

$$4 - 8\frac{\left(r^\lambda + r^{1-\lambda}\right)^3 \left(r^{9\lambda} + r^{9(1-\lambda)} + r\left(r^{3\lambda} + r^{3(1-\lambda)}\right)\right)}{\left(r + \left(r^{4\lambda} + r^{4(1-\lambda)}\right) + 4r\left(r^{2\lambda} + r^{2(1-\lambda)}\right)\right)^3},$$

as $r \to 0+$. Now, dividing the numerator and the denominator of the second term by $r^{12\lambda}$, the above expression is equal to

$$4 - 8\frac{\left(1 + r^{1-2\lambda}\right)^3 \left(1 + r^{9(1-2\lambda)} + r^{1-6\lambda} + r^{4(1-3\lambda)}\right)}{\left(r^{1-4\lambda} + 1 + r^{4(1-2\lambda)} + 4\left(r^{1-2\lambda} + r^{3(1-2\lambda)}\right)\right)^3},$$

from which the limiting behaviour on $(0, 1/4]$ follows. Multiplying the numerator and the denominator of the second term of the above expression by $r^{3(4\lambda-1)}$, we obtain

$$4 - 8\frac{(1 + r^{1-2\lambda})^3 \left(r^{3(4\lambda-1)} + r^{6(1-\lambda)} + r^{2(3\lambda-1)} + r\right)}{\left(1 + r^{4\lambda-1} + r^{3-4\lambda} + 4\left(r^{2\lambda} + r^{2(1-\lambda)}\right)\right)^3},$$

from which the limiting behaviour on $(1/4, 1/2]$ follows. As mentioned above, this gives us the limiting behaviour on all of $(0, 1)$. This completes the proof of (d) and the theorem. □

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