Almost Abelian Lie algebras admitting Astheno-Kähler and Balanced structures

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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May, 2025

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Certificate

This is to certify that this dissertation entitled Almost Abelian Lie algebras admitting Astheno-Kähler and Balanced structures towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Ujwal Pandey at Indian Institute of Science Education and Research under the supervision of Vivek Mohan Mallick, Associate Professor, Department of Mathematics and co-supervision of Anna Fino, Professor, Department of Mathematics, University of Turin, during the academic year 2024-2025.

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This thesis is dedicated to my Mother

Declaration

I hereby declare that the matter embodied in the report entitled Almost Abelian Lie algebras admitting Astheno-Kähler and Balanced structures are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Vivek Mohan Mallick as well as Anna Fino, and the same has not been submitted elsewhere for any other degree.

Ujwal Pandey

Acknowledgments

I would like to thank my co-supervisor, Prof. Anna Fino, and my supervisor, Prof. Vivek Mohan Mallick, for their constant support and guidance. I am eternally thankful to have an amazingly loving family. Along with that, I am also grateful to have found friends such as Parth, Aravind, Nakul, Sumant, Aditya, Prasanna, and Vihang, to name a few, who have helped and supported me immensely through the course of this thesis in Maths and otherwise.

Abstract

This thesis presents a systematic study of almost Abelian Lie algebras admitting astheno-Kahler and balanced Hermitian structures. An almost Abelian Lie algebra is defined by the existence of a codimension-one Abelian ideal, and such algebras offer a tractable yet rich framework for exploring complex geometric structures. Focusing on real eight-dimensional almost Abelian Lie algebras, the work develops a comprehensive framework for the classification and analysis of left-invariant Hermitian structures.

The initial chapters provide the necessary background in complex geometry and Lie theory. Key concepts such as complex manifolds, Hermitian metrics, and the integrability of almost complex structures via the Newlander-Nirenberg theorem are reviewed. Building on these foundations, the thesis introduces Hermitian structures on almost Abelian Lie algebras and derives explicit algebraic criteria for balanced and astheno-Kahler metrics. In particular, balanced metrics are characterized by the vanishing of the Lee form, while the astheno-Kahler condition is defined via the vanishing of the $\partial\bar{\partial}$ operator acting on a suitable power of the fundamental form. Overall, the results provide a clear classification of eight-dimensional almost Abelian Lie algebras that support these special Hermitian structures.

The study further explores geometric flows on these Lie algebras. Using the bracket flow technique, the evolution equations of left-invariant metrics are reformulated in an algebraic setting. A detailed analysis of the balanced flow is carried out, demonstrating how it preserves the balanced condition and aids in the approach to canonical metric structures. The interplay between the algebraic data and the dynamic behavior of the flow yields criteria for long-time existence and stability.

Contents

Introduction

Lie Algebras and Hermitian Structures

A *Lie algebra* is an algebraic structure whose product is the Lie bracket $[\cdot, \cdot]$, satisfying bilinearity, antisymmetry, and the Jacobi identity. An *almost Abelian* Lie algebra contains a codimension-one Abelian ideal)i. e., all but one basis element mutually commute) [2]. Equivalently, an almost Abelian Lie algebra is a one-dimensional extension of an Abelian Lie algebra by some derivation. Such Lie algebras provide a manageable yet rich class for examining complex and geometric structures. In particular, many low-dimensional non-Abelian examples (e.g., most real 3-dimensional Lie algebras) fall into this category [2], making them a natural testbed for classification problems in complex geometry.

On a Lie algebra $\mathfrak{g}$, an *almost Hermitian structure* consists of an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ with $J^2 = -\mathrm{Id}$ (an almost complex structure) together with an inner product $g(\cdot, \cdot)$ that is J-compatible (i. e., $g(JX, JY) = g(X, Y)$) and positive-definite. When J is integrable (i. e., arises from a complex structure on the corresponding Lie group via left-invariant vector fields), (J, g) defines a *Hermitian structure* on the Lie algebra or the associated Lie group regarded as a complex manifold. In this thesis, we focus on left-invariant Hermitian structures on Lie groups, reducing the problem to studying Hermitian structures on the underlying Lie algebras. The almost Abelian case is of special interest because the integrability conditions and special Hermitian properties can often be expressed in terms of the action of a single derivation (coming from the one-dimensional non-Abelian direction) on the Abelian ideal, simplifying the analysis. This provides a tractable setting to systematically *classify* which Lie algebras admit certain special Hermitian metrics.

Balanced and Astheno-Kähler Structures

Within Hermitian geometry, two important types of non-Kähler Hermitian metrics are the *balanced* metrics and *astheno-Kähler* metrics. A Hermitian metric g with fundamental 2-form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ is called *balanced* if ω is co-closed, or equivalently if

$$d(\omega^{n-1}) = 0$$

on a complex n-dimensional manifold [18]. In other words, the $(n-1, n-1)$-form ω^{n-1} is closed, so ω balances the volume in a special way. This condition, first introduced by Michelsohn in 1986, characterizes *balanced Hermitian manifolds* as those for which the $(n-1)$-th power of the Kähler form is harmonic. Balanced metrics generalize Kähler metrics: every Kähler form is closed (since $d\omega = 0$ implies $d(\omega^{n-1}) = 0$), but the balanced condition is strictly weaker and permits many *non-Kähler* solutions. Balanced Hermitian structures have garnered interest because they enjoy certain stability properties under mild deformations (unlike Kähler metrics) and appear in theoretical physics – for example, balanced metrics are central in formulating the heterotic string *Strominger system* [18].

An *astheno-Kähler* metric is defined by a condition on the higher power of ω. Specifically, a Hermitian metric on a complex n-dimensional manifold is astheno-Kähler if its fundamental form $F = \omega$ satisfies

$$\partial\bar{\partial}\,\omega^{\,n-2} = 0.$$

Jost and Yau studied this condition [11], later formally named by Grauert and Riemenschneider; it generalizes several known special Hermitian conditions [8]. In the lowest non-trivial case $n = 3$ (real dimension 6), astheno-Kähler metrics coincide with *strong Kähler with torsion (SKT)* metrics, since $\partial\bar{\partial}\,\omega^{n-2} = \partial\bar{\partial}\,\omega = 0$ is exactly the SKT condition [8]. For $n > 3$, the astheno-Kähler condition is strictly weaker than SKT: it requires a second-order closedness of ω^{n-2}, but not necessarily of ω itself. As such, both balanced and astheno-Kähler metrics offer rich classes of non-Kähler Hermitian structures, and examples provided by Fino and Tomassini [8] have shown that interesting astheno-Kähler metrics exist on certain 2-step nilmanifolds in real dimension 8.

Geometric Flows in Hermitian Geometry

Geometric flows are powerful tools for evolving geometric structures toward canonical forms. The prototypical example is the *Ricci flow*, introduced by Hamilton, which evolves a Riemannian metric $g(t)$ via

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij}.$$

Ricci flow tends to homogenize curvature and has been instrumental in major results such as the proof of the Poincaré conjecture. However, Ricci flow does not generally preserve special Hermitian conditions unless the initial metric is Kähler. In the setting of Lie groups, one can utilize the *bracket flow* approach, which translates the evolution of a left-invariant metric into an ODE for the Lie algebra's structure constants [14].

In complex geometry, several flows have been devised to preserve specific Hermitian properties. For example, the *pluriclosed flow* (or SKT flow) evolves a Hermitian metric while preserving the condition $\partial\bar{\partial}\,\omega = 0$. Of particular interest in this thesis is the *balanced flow*, a parabolic flow designed to preserve the balanced condition. If $\omega(t)$ is the evolving Hermitian form, one formulation of the balanced flow is

$$\partial_t \omega(t) = \Delta_{BC}\,\omega(t) + (\text{curvature terms}),$$

where Δ_{BC} denotes the Bott–Chern Laplacian. Notably, if the initial metric is balanced, then $\omega(t)$ remains balanced along the flow, and if the metric is initially Kähler, it remains so for all time [3]. Compared to Ricci flow, balanced flow is tailored to the complex Hermitian setting, preserving the $(n-1, n-1)$-cohomology class of ω^{n-1}. Moreover, using bracket flow techniques, one can reformulate balanced flow as an ODE on the space of structure constants, linking the geometric evolution directly to the Lie algebra structure. This approach not only simplifies analysis but also allows for direct comparison with other flows like Ricci flow, highlighting differences in evolution behavior and long-time existence.

Research Objectives and Scope

The central question of this thesis is: *Which real 8-dimensional almost Abelian Lie algebras admit a left-invariant Hermitian structure that is either balanced or astheno-Kähler?* Our objectives are to:

- Characterize the algebraic conditions under which an 8-dimensional almost Abelian Lie algebra supports a balanced metric.
- Determine the conditions for the existence of an astheno-Kähler metric.
- Investigate whether any Lie algebra admits both types of metrics and what implications this has (e.g., forcing the metric to be Kähler).
- Utilize geometric flows, particularly the balanced flow, as a tool to study the evolution and uniqueness of balanced metrics, which can be used to identify canonical (soliton) metrics.

This classification is motivated by prior work on 6-dimensional Lie algebras and the pioneering studies of Anna Fino and collaborators [8, 3]. Our study extends these ideas to 8 dimensions (complex dimension 4), where new phenomena emerge: astheno-Kähler metrics are strictly broader than SKT metrics, and the interplay between balanced metrics and geometric flows offers fresh insights.

The scope of this thesis is confined to almost Abelian Lie algebras of real dimension 8. We restrict our attention to left-invariant Hermitian structures, thus reducing the problem to an algebraic one. While geometric flows (and their bracket flow formulation) are employed to support our analysis, a complete PDE treatment of these flows is beyond our scope. Instead, we use these tools to corroborate existence and uniqueness results and to suggest canonical metrics.

Chapter 1

Preliminaries

In this thesis, the first courses in differential geometry, Riemannian geometry, and complex analysis are assumed to be prerequisites.

1.1 Complex Geometry

The results and definitions presented in this section are the bare essentials that would be needed in this thesis for a more in-depth exposition of Complex Geometry, which also subsumes most of the content presented in this section; please refer to [16].

1.1.1 Basic Complex Analysis in Several Variables

Definition 1.1.1. *A function $f : U \to \mathbb{C}$, where U is open in $\mathbb{C}^n$, is called a* ***holomorphic function*** *if $\forall (z_0^1, \ldots, z_0^n) \in U$ we have that the functions $f^k(z) \coloneqq f(z_0^1, \ldots, z, \ldots, z_0^n)$, where $z \in \mathbb{C}$ such that $f^k(z)$ is defined, are holomorphic for $k \in \{1, \ldots, n\}$.*

Note that this is equivalent to the derivative of a function being complex-linear. Wherein we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ as,

$$(x^1, y^1, \dots, x^n, y^n) \longleftrightarrow (x^1 + iy^1, \dots, x^n + iy^n).$$

This is the identification we will also use for the rest of the chapter.

Definition 1.1.2. *A map $f : U \to \mathbb{C}^m$ where U is open in $\mathbb{C}^n$ is called a* ***holomorphic map*** *if every component of f is a holomorphic function.*

Again, similarly, a map being holomorphic is equivalent to its derivative being complex-linear.

1.1.2 Complex Manifolds

Definition 1.1.3. *A smooth manifold M of dimension $2n$ is defined to be a* ***complex manifold*** *of* ***complex dimension*** *n if there exists a smooth atlas for which all the change of coordinate maps are holomorphic. Such a collection is called a* ***holomorphic atlas****, and its elements are called* ***holomorphic coordinate charts****.*

We define holomorphic maps and functions on complex manifolds exactly how we define smooth maps and functions for real manifolds, just replacing smooth with holomorphic.

1.1.3 Complex Vector Bundles

Definition 1.1.4. *A real vector bundle (E, π) of a smooth manifold M of rank $2k$ is called a* ***complex vector bundle*** *of M of* ***complex rank*** *k if the following properties are satisfied:*

(i) $E_p := \pi^{-1}(p)$ is a complex vector space of complex dimension k for all $p \in M$.

(ii) There exists a trivializing cover $\{U_\alpha : \alpha \in I\}$ with local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ such that $\psi|_{E_p} : E_p \to \{p\} \times \mathbb{C}^k$ is a complex isomorphism for all $p \in U_\alpha$.

We call any such trivializing cover a ***complex trivializing cover*** *and these local trivializations to be* ***complex local trivializations****.*

Definition 1.1.5. *A complex vector bundle (E, π) of a complex manifold M of complex rank k is called a* ***holomorphic vector bundle*** *of M of* ***holomorphic rank*** *n if there exists a trivializing cover as above and corresponding local trivializations such that each local trivialization is holomorphic. We call any such trivializing cover a* ***holomorphic trivializing cover*** *and these local trivializations to be* ***holomorphic local trivialization****.*

Definition 1.1.6. *Let V be a real vector space; then we define* ***complexification of V*** *denoted by $V_{\mathbb{C}}$ to be the real vector space $V \oplus V$ with the action of i defined on it as $i(v_1, v_2) = (-v_2, v_1)$ which makes it into a complex vector space. We may denote any element $(v_1, v_2) \in V_{\mathbb{C}}$ as $v_1 + iv_2$.*

Similarly, we have the complexification of a real vector bundle by complexifying each fiber vector space. If E is a real vector bundle, we denote its complexification by $E_{\mathbb{C}}$. And the complexification of the tangent bundle TM is denoted by $T_{\mathbb{C}}M$.

1.1.4 Almost Complex Structure

Definition 1.1.7. *Let V be a real vector space, then $J \in \operatorname{End}(V)$ is called a* ***complex structure*** *on V, if $J^2 = -I$.*

Any such J being called a complex structure has an obvious reason: by defining the action of i on V as J, V becomes a complex vector space.

We can extend J linearly to $V_{\mathbb{C}}$ then on that vector space, we have that the i-eigenspace V' and $-i$-eigenspace V'' are complex vector subspaces of V. Also, both have the same dimension and $V_{\mathbb{C}} = V' \oplus V''$. Also, we have that,

$$V' = \{v - iJv | v \in V\},$$
$$V'' = \{v + iJv | v \in V\}.$$

Similarly, we have a complex structure on a vector bundle as,

Definition 1.1.8. *Let E be a real vector bundle, then $J \in \operatorname{End}(E)$ is called a* ***complex structure*** *if $J^2 = -I$.*

We have J restricted to each fiber vector space, a complex structure in the vector space on that fiber. Also, as before, we can extend J to $E_{\mathbb{C}}$.

Proposition 1.1.1 ([16])**.** *Let E be a real vector bundle on M with a complex structure J defined on it; then there exist complex vector bundles E' and E'' which are complex vector subbundles of $E_{\mathbb{C}}$. Such that E'_p and E''_p are the i and $-i$ eigenspaces of J respectively. In fact, $E_{\mathbb{C}} = E' \oplus E''$.* ∎

Because of how E' and E'' are defined, we have the following fact trivially,

$$\Gamma(E') = \{X - iJX | X \in \Gamma(E)\}$$
$$\Gamma(E'') = \{X + iJX | X \in \Gamma(E)\}$$

Definition 1.1.9. *Let M be a real manifold, then a complex structure of TM is called an **almost complex structure** on M. (M, J) denotes an almost complex structure J on M. And M is called a **almost complex manifold**.*

Suppose M is a complex manifold of complex dimension n. In that case, we can define an almost complex structure on M locally by pulling back the multiplication by i on $\mathrm{End}(\mathbb{C}^n)$ map to $\mathrm{End}(TM)$ by the coordinate chart and then stitch these locally defined maps to get a globally defined almost complex structure. The reason they can be stitched together is the holomorphicity of the change of coordinate maps. This is the canonical almost complex structure on a complex manifold. So it is interesting to ask the converse question: When does an almost complex structure on a manifold give us a complex structure on M?

Definition 1.1.10. *An almost complex structure J on manifold M is called **integrable** if a complex structure exists on M such that the canonical almost complex structure on M equals J.*

Definition 1.1.11. *If M is a complex manifold with the canonical almost complex structure denoted by J. Then, $T'M := (TM)'$ and $T''M := (TM)''$ are called the **holomorphic** and the **anti-holomorphic tangent bundle** respectively.*

The following theorem answers the question we asked above,

Theorem 1.1.2 (**Newlander-Nirenberg Theorem** [10]). *Let M be a smooth manifold and J an almost complex structure on M. Then J is integrable if and only if the Nijenhuis tensor*

$$N_J(X,Y) = [JX, JY] - [X,Y] - J\Big([JX,Y] + [X, JY]\Big)$$

vanishes identically for all vector fields $X, Y \in \mathfrak{X}(M)$. ■

If $(x_1, y_1, \dots, x_n, y_n)$ is a local coordinate system for a complex n dimensional complex manifold M then it is easy to see that,

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i} \quad \forall i \in \{1, \dots, n\}$$

. We define the following quantities,

$$\frac{\partial}{\partial z_j} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \quad \forall j \in \{1, \dots, n\} \tag{1.1}$$

It is clear that $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$ is a complex basis of $T'M$ and $\{\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\}$ forms a complex basis of $T''M$.

Note that the dual of J, $J^\sharp \in \mathrm{End}(T^*M)$ is also a complex structure, in fact we have $J^\sharp\alpha(X) = -\alpha(JX), \forall \alpha \in \Omega^1(M), X \in \mathfrak{X}(M)$. We may use J to denote $J^\sharp$, but from the context, it will be clear. In fact we can extend J to be defined on $\Lambda^k_{\mathbb{C}}M, \forall k \geq 1$ by defining it recursively as $J(\alpha \wedge \beta) \coloneqq J\alpha \wedge J\beta$. We also define the following quantities,

$$dz^j \coloneqq dx^j + i dy^j, \quad d\bar{z}^j \coloneqq dx^j - i dy^j \quad \forall j \in \{1, \dots, n\}. \tag{1.2}$$

We also have that $dz^i = \left(\frac{\partial}{\partial z_i}\right)^*$ and $d\bar{z}^i = \left(\frac{\partial}{\partial \bar{z}_i}\right)^*$.

Definition 1.1.12. *Let (M, J) be a complex manifold then we define a (p,q)-**form** α to be a $(p+q)$ form such that $\alpha(X_1, \dots, X_n, Y_1, \dots, Y_m) \neq 0$ only if $n = p, m = q$ where $n + m = p + q$ and $X_i \in \Gamma(T'M), Y_j \in \Gamma(T''M), \forall i, j$. These types of forms span a rough subbundle of $\Lambda^{p+q}_{\mathbb{C}}M$ denoted by $\Lambda^{p,q}M$ and the set of these forms is denoted by $\Omega^{p,q}M$.*

In the above definition $\Lambda^{p+q}_{\mathbb{C}}M$ denotes the complexification of $\Lambda^{p+q}M$.

The following proposition gives us a clearer picture of what the (p,q)-forms look like locally.

Proposition 1.1.3 ([16]). *Let M be a complex manifold of complex dimension and let α be a $(p+q)$ form of M then the following statements are equivalent:*

(i) $\alpha \in \Omega^{(p,q)}M$.

(ii) For any coordinate chart (U, ϕ) we have that, $\alpha|_U \in \operatorname{span}\{dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q} | i_1, \ldots, i_p, j_1, \ldots, j_q \in \{1, \ldots, n\}\}$. ∎

A trivial consequence of this proposition is that the rough subbundle $\Lambda^{p,q}_{\mathbb{C}}M$ is, in fact, a smooth complex subbundle of $\Lambda^{p+q}M$. Also we have that,

$$\Lambda^k_{\mathbb{C}}M = \bigoplus_{\substack{p,q=1 \\ p+q=k}}^{n} \Lambda^{p,q}M. \tag{1.3}$$

1.1.5 Hermitian Metrics

Definition 1.1.13. *Let E be a complex vector bundle of real manifold M, then $\sigma \in \Gamma(\mathcal{S}^2E^*)$ is called a* ***Hermitian fiber metric*** *on E if $\forall p \in M$, σ_p is a Hermitian inner product on the complex vector space E_p.*

Note that if (M, J) is an almost complex manifold, then TM can be considered a complex vector bundle because every fiber vector space has a complex structure given by J. When considering TM as a complex vector bundle, we will denote it as T_JM.

Proposition 1.1.4 ([16]). *Let h denote a Hermitian fiber metric on an almost complex manifold (M, J). Then we have that there exists a unique pair (g, ω) where g and ω are a Riemannian metric and a 2 form on M respectively, such that $h = g - i\omega$. Also ω is related to g as follows,*

$$\omega = g(J\cdot, \cdot). \tag{1.4}$$

Also, J is an orthogonal linear transformation with respect to g. And if J is an integrable almost complex structure, then ω is a $(1,1)$-form. ∎

This proposition leads us to ask the converse question that is when does a pair of a Riemannian metric and a 2-form give us a Hermitian fiber metric, which leads us to the following result,

Proposition 1.1.5 ([16]). *Let g and ω be a Riemannian metric and a 2-form on an almost complex manifold (M, J); then $g - i\omega$ defines a Hermitian fiber metric on $T_J M$ if and only if $\omega = g(J\cdot, \cdot)$ and J is orthogonal with respect to g.* ■

From this proposition, we also have that for any Riemannian metric g such that J is g-orthogonal, we can define a 2-form as $\omega \coloneqq g(J\cdot, \cdot)$ to get that $g - i\omega$ is a Hermitian fiber metric. Similarly, if *omega* is a 2-form such that $\omega = \omega(J\cdot, J\cdot)$ then we can define a Riemannian metric $g \coloneqq \omega(\cdot, J\cdot)$ such that $g - i\omega$ is a Hermitian fiber metric. This leads us to the following definition,

Definition 1.1.14. *A Riemannian metric g on an almost complex manifold (M, J) is called an **Hermitian metric** if J is g-orthogonal and the 2-form $\omega \coloneqq g(J\cdot, \cdot)$ is called the **fundamental 2-form** associated to g. Any such (M, J) is called an **almost Hermitian manifold** or a **Hermitian manifold** if J is integrable.*

1.1.6 Dolbeault Operator

Proposition 1.1.6 ([16]). *Let M be a complex manifold of complex dimension n then there exist unique operators $\partial : \Omega_{\mathbb{C}}(M) \to \Omega_{\mathbb{C}}(M)$ and $\bar{\partial} : \Omega_{\mathbb{C}}(M) \to \Omega_{\mathbb{C}}(M)$, where $\Omega_{\mathbb{C}}(M)$ denotes the complexification of the space $\Omega(M)$ which is the space of all differential forms of all degrees, such that they satisfy the following properties,*

(i) $\partial(\Omega^{p,q}M) \subset \Omega^{p+1,q}M$ *and* $\bar{\partial}(\Omega^{p,q}M) \subset \Omega^{p,q+1}M$.

(ii) $d = \partial + \bar{\partial}$, *where d is exterior derivative which is defined on $\Omega_{\mathbb{C}}M$ by complexifying(i. e. linearly extending) the usual d defined on $\Omega(M)$.*

(iii) $\partial^2 = 0, \bar{\partial}^2 = 0$ *and,* $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

(iv) $\overline{(\partial\alpha)} = \bar{\partial}\bar{\alpha}, \forall \alpha \in \Omega_{\mathbb{C}}M$.

(v) $\partial(\alpha \wedge \beta) = \partial\alpha \wedge \partial\beta$ *and* $\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \bar{\partial}\beta, \forall \alpha, \beta \in \Omega_{\mathbb{C}}M$.

(vi) If (U, ϕ) is a coordinate chart and $f : M \to \mathbb{C}$ is a smooth function then we have on U that $\partial f = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} f$ and $\bar{\partial} f = \sum_{i=1}^{n} \frac{\partial}{\partial \bar{z}_i} f$. ∎

Definition 1.1.15. *Let M be a complex manifold and let $\bar{\partial} : \Omega_{\mathbb{C}} M \to \Omega_{\mathbb{C}} M$ be the operator as above, then $\bar{\partial}$ is called the **Dolbeault Operator**.*

1.1.7 Hodge Inner Product

Let (M, g) be a Hermitian manifold. We define a Hermitian fiber product on $T^*_{\mathbb{C}} M$ denoted by $\langle \cdot, \cdot \rangle$ using the musical isomorphism $\sharp : T^*_{\mathbb{C}} M \to T_{\mathbb{C}} M$ as,

$$\langle \alpha, \beta \rangle = g(\alpha^\sharp, \bar{\beta}^\sharp) \tag{1.5}$$

Definition 1.1.16. *Let (M, g) be a Hermitian manifold of complex dimension n then we define a Hermitian fiber metric on $\Lambda^k_{\mathbb{C}} M, k \geq 1$ called the **pointwise Hermitian Inner product** denoted by $\langle \cdot, \cdot \rangle$ locally by fixing an orthonormal local frame $\{\alpha_1, \ldots, \alpha_n\}$ of $T_{\mathbb{C}} M$ and letting $\{\alpha^1, \ldots, \alpha^n\} \subset \Lambda^1 M$ denote the dual of that frame(this local coframe is orthonormal in Hermitian fiber product defined in eq. (1.5)) and declaring the set,*

$$\{\alpha^{j_1} \wedge \ldots \wedge \alpha^{j_k} | j_1 < \cdots < j_k\}$$

to be orthonormal. Since in a neighborhood of every point, there is an orthonormal local frame, such a Hermitian fiber metric can be defined on the whole manifold.

But we still need to ensure that this is well defined; that is, it is independent of the choice of the local frame. We get exactly that as a consequence of the next result,

Lemma 1.1.7 ([16])**.** *If $k \geq 1$, the pointwise Hodge inner product $\langle \cdot, \cdot \rangle$ on k-forms is determined uniquely by the condition: given any locally defined 1-forms $\epsilon_1^1, \ldots, \epsilon_1^k, \epsilon_2^1, \ldots, \epsilon_2^k$, we have*

$$\langle \epsilon_1^1 \wedge \ldots \wedge \epsilon_1^k, \epsilon_2^1 \wedge \ldots \wedge \epsilon_2^k \rangle = \det\left(\langle \epsilon_1^i, \epsilon_2^j \rangle\right). \tag{1.6}$$

Hence, it is independent of the choice of the local orthonormal frame. ∎

Note that we extend the definition of pointwise Hodge inner product to the space $\Lambda^0_{\mathbb{C}}M$ as

$$\langle u, v\rangle \coloneqq u\bar{v}, \quad \forall u, v \in \Omega^0_{\mathbb{C}}M.$$

Definition 1.1.17. *Let (M, g) be a Hermitian manifold; then we define a Hermitian inner product on the space $\Omega^k_{\mathbb{C}}M, k \geq 0$ called the **Hodge inner product** denoted by $(\cdot,\cdot)$ defined as,*

$$(\alpha, \beta) \coloneqq \int_M \langle\alpha, \beta\rangle dV_g.$$

Here, dV_g is the volume form determined by the Riemannian metric g.

Proposition 1.1.8 ([16])**.** *Let (M, g) be a Hermitian manifold, there exist unique operators $d^*, \partial^*, \bar{\partial}^* : \Omega_{\mathbb{C}}M \to \Omega_{\mathbb{C}}M)$ called the formal adjoints of the operators $d, \partial, \bar{\partial}$ respectively because they satisfy the following,*

$$(d\alpha, \beta) = (\alpha, d^*\beta),$$
$$(\partial\alpha, \beta) = (\alpha, \partial^*\beta),$$
$$(\bar{\partial}\alpha, \beta) = (\alpha, \bar{\partial}^*\beta).$$

Here $\alpha, \beta \in \Omega_{\mathbb{C}}M$ and they are compactly supported, and their degree is such that the LHS of each of the above equations is well-defined. These operators, as a result of the above equations, also satisfy the following,

(i) $d^* = \partial^* + \bar{\partial}^*$.

(ii) $d^*(\Omega^k_{\mathbb{C}}M) \subset \Omega^{k-1}_{\mathbb{C}}M, \partial^*(\Omega^{p,q}M) \subset \Omega^{p-1,q}M$ *and,* $\bar{\partial}^*(\Omega^{p,q}M) \subset \Omega^{p,q-1}M$.

(iii) $\overline{(\partial^*)} = \bar{\partial}^*$.

(iv) $(d^*)^2 = 0, (\partial^*)^2 = 0, (\bar{\partial}^*)^2 = 0$ *and,* $\partial^*\bar{\partial}^* + \bar{\partial}^*\partial^* = 0$. ∎

One can construct a proof of this proposition using relevant results from [16].

Proposition 1.1.9 ([16])**.** *Let (M, g) be a Hermitian manifold of complex dimension n; there exists a unique operator called **Hodge star operator** denoted by $* : \Omega_{\mathbb{C}}M \to \Omega_{\mathbb{C}}M$ such that it satisfies the following,*

$$\alpha \wedge *\bar{\beta} = \langle\alpha, \beta\rangle dV_g.$$

Here, $\alpha, \beta \in \Omega^k M, k \geq 1$ *and* dV_g *is the volume form associated with* g*. As a consequence of the above equation,* $*$ *also satisfies the following,*

(i) $*(\Omega^{p,q}M) \subset \Omega^{n-p,n-q}M$.

(ii) $** \alpha = (-1)^{p+q}\alpha, \forall \alpha \in \Omega^{p,q}M$.

(iii) $d^* = - * d *, \partial^* = - * \bar{\partial} *$ *and,* $\bar{\partial}^* = - * \partial *$. ∎

Proposition 1.1.10 ([4])**.** *Let* (M, g) *be a Hermitian manifold of complex dimension* n*, let* ∇^g *denote the Levi-Civita connection associated with* g*,* $\{e_1, \ldots, e_{2n}\}$ *denote an orthonormal frame of* TM*,* $\{e^1, \ldots, e^{2n}\}$ *denote its dual frame and,* ι_X*, where* $X \in \mathfrak{X}(M)$*, denote contraction by* X*. Then we have that,*

$$d^* = -\sum_{k=1}^{2n} \iota_{e_k} \nabla^g_{e_k}. \tag{1.7}$$

∎

1.2 Lie Theory

Definitions 1.2.1. *(i) A vector space* $\mathfrak{g}$ *along with an antisymmetric bilinear operator* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is called a* ***Lie algebra*** *if* $[\cdot,\cdot]$ *satisfies the Jacobi identity that is,*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \quad \forall X, Y, Z \in \mathfrak{g}.$$

(ii) A linear map from one Lie algebra $\phi : (\mathfrak{g}_1, [\cdot,\cdot]_1) \to (\mathfrak{g}_2, [\cdot,\cdot]_2)$ *to another is called an* ***Lie algebra Homomorphism*** *if* $\phi([\cdot,\cdot]) = [\phi\cdot, \phi\cdot]$.

(iii) A Lie algebra $\mathfrak{g}$ *is called* ***Abelian*** *if* $[X, Y] = 0, \forall X, Y \in \mathfrak{g}$.

(iv) A vector subspace $\mathfrak{n}$ *of* $\mathfrak{g}$ *is called a* ***Lie subalgebra*** *if* $\mathfrak{n}$ *is closed under the operator* $[\cdot,\cdot]$*. A Lie subalgebra* $\mathfrak{n}$ *is called an* ***ideal*** *if* $[X, Y] \in \mathfrak{n}, \forall X \in \mathfrak{g}, Y \in \mathfrak{n}$.

(v) A Lie algebra $\mathfrak{g}$ *is called* ***almost Abelian*** *if it is not Abelian but it has a codimension 1 Abelian ideal.*

(vi) For a Lie algebra $\mathfrak{g}$ *it's **derived series** is defined recursively as follows,*

$$\mathfrak{g}^0 \coloneqq \mathfrak{g}, \quad \mathfrak{g}^{(k+1)} \coloneqq [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}].$$

Where k *is a positive integer. If for a Lie algebra* $\mathfrak{g}$ *we have that* $\mathfrak{g}^{(n)} = 0$ *for some positive integer* n*, then that Lie algebra is called **nilpotent**.*

(vii) For a Lie algebra $\mathfrak{g}$ *it's **derived series** is defined recursively as follows,*

$$\mathfrak{g}^{(0)} \coloneqq \mathfrak{g}, \quad \mathfrak{g}^{(k+1)} \coloneqq [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}].$$

Where k *is a positive integer. If for a Lie algebra* $\mathfrak{g}$ *we have that* $\mathfrak{g}^{(n)} = 0$ *for some positive integer* n*, then that Lie algebra is called **solvable**.*

(viii) For a Lie algebra $\mathfrak{g}$ *it's **lower central series** is defined recursively as follows,*

$$\mathfrak{g}_0 \coloneqq \mathfrak{g}, \quad \mathfrak{g}_{k+1} \coloneqq [\mathfrak{g}, \mathfrak{g}_k].$$

Where k *is a positive integer. If for a Lie algebra* $\mathfrak{g}$ *we have that* $\mathfrak{g}_n = 0$ *for some positive integer* n*, then that Lie algebra is called **nilpotent**.*

A trivial observation is that if a Lie algebra is either nilpotent or almost Abelian, it is solvable. Also, it is easy to prove that if an almost Abelian Lie algebra is non-nilpotent, then its codimension 1 Abelian ideal is unique.

Definition 1.2.2. *Let* $\mathfrak{g}$ *be a Lie algebra, then a pair* (J, g) *is called an **almost Hermitian structure** on* $\mathfrak{g}$*, if* J *is a complex structure on* $\mathfrak{g}$ *and* g *is an inner product on* $\mathfrak{g}$ *such that* J *is orthogonal with respect to* g*.*

Theorem 1.2.1 (Lie's Third Theorem [17]). *Every finite-dimensional real Lie algebra* $\mathfrak{g}$ *is isomorphic to the Lie algebra of a connected and simply connected Lie group* G*. In other words, for every finite-dimensional Lie algebra* $\mathfrak{g}$*, there exists a Lie group* G *such that*

$$\mathfrak{g} \cong \mathrm{Lie}(G).$$

∎

Theorem 1.2.2 (Correspondence between Lie Group and Lie Algebra Homomor-

phisms [17]). *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then:*

1. *If $\Phi : G \to H$ is a Lie group homomorphism, its differential at the identity,*

   $$d\Phi_e : \mathfrak{g} \to \mathfrak{h},$$

   is a Lie algebra homomorphism. That is, for all $X, Y \in \mathfrak{g}$,

   $$d\Phi_e([X, Y]) = [d\Phi_e(X), d\Phi_e(Y)].$$

2. *Conversely, if G is connected and simply connected, then every Lie algebra homomorphism*

   $$\phi : \mathfrak{g} \to \mathfrak{h}$$

   integrates to a unique Lie group homomorphism

   $$\Phi : G \to H,$$

   Satisfying

   $$d\Phi_e = \phi.$$

■

Using the above results and the fact that a global frame of a Lie group corresponds with a basis of its Lie algebra, we have that the set of almost Hermitian structures on a Lie algebra corresponds with the set of left-invariant almost Hermitian structures on the connected and simply connected Lie group corresponding to that Lie algebra.

Definition 1.2.3. *An almost Hermitian structure (J, g) on a Lie algebra $\mathfrak{g}$ is called a* ***Hermitian structure*** *if the corresponding left invariant almost Hermitian structure on the unique connected and simply connected Lie group G is a Hermitian structure on G, i. e. the left-invariant almost complex structure J on G is integrable.*

Here is some notation we will be using in the last chapter; first, let's fix a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ with a complex structure J on it now we have the following,

(i) If $A : \mathfrak{g} \to \mathfrak{g}$ is linear, then $A^c := \frac{1}{2}(A - JAJ)$(called the complex part of A because it commutes J) and $A^{ac} := \frac{1}{2}(A + JAJ)$(called the anti-complex part of A because it anti-commutes with J), clearly we have $A = A^c + A^{ac}$.

(ii) Let $a : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a bilinear map then we have that $a^c := \frac{1}{2}(a + a(J\cdot, J\cdot)$(called the complex part of a) and $a^{ac} := \frac{1}{2}(a - a(J\cdot, J\cdot))$(called the anti-complex part of a). Again, we have that $a = a^c + a^{ac}$.

(iii) Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and $\phi : (\mathfrak{g}_1, [\cdot,\cdot]_1) \to (\mathfrak{g}_2, [\cdot,\cdot]_2)$ be a Lie algebra isomorphism then $[\cdot,\cdot]_2 = \phi \cdot [\cdot,\cdot]_1 = \phi[\phi^{-1}\cdot, \phi^{-1}\cdot]$.

Definitions 1.2.4. *(i) A Lie group G is called* ***unimodular*** *if* $\det \mathrm{Ad} X = 1, \forall X \in G$.

(ii) A Lie algebra $\mathfrak{g}$ is called ***unimodular*** *then* $\operatorname{tr} \operatorname{ad} X = 0, \forall X \in \mathfrak{g}$.

It is fairly easy to show that a Lie group is unimodular iff its Lie algebra is also unimodular.

Chapter 2

Hermitian Structures over almost Abelian Lie algebras

2.1 Characterization

Let $\mathfrak{b}$ be an almost Abelian Lie algebra of real dimension $2n$ admitting a Hermitian structure (J, g). By the definition of almost Abelian, there exists a codimension 1 Abelian ideal of $\mathfrak{b}$, call it $\mathfrak{n}$. Define $\mathfrak{n}_1 := \mathfrak{n} \cap J\mathfrak{n}$, this is the maximal J-invariant subspace in $\mathfrak{n}$ and has real dimension $2n - 2$.

There exists a unitary basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{b}$, such that $\mathfrak{n} = \text{span}\{e_1, \ldots, e_{2n-1}\}$ and $\forall_{i \in \{1,\ldots,n\}} Je_i = e_{2n+1-i}$; because of this, we also get that $\mathfrak{n}_1 = \text{span}\{e_2, \ldots, e_{2n-1}\}$. We will call such a basis an adapted unitary basis. Now, $\mathfrak{l} := \mathfrak{n}^{\perp_g} = \mathbb{R}e_{2n}$ so this gives us the following decomposition of $\mathfrak{b}$,

$$\mathfrak{b} = J\mathfrak{l} \oplus \mathfrak{n}_1 \oplus \mathfrak{l}$$

.

Now, to know $(\mathfrak{b}, J, g)$, it is enough to know the Lie bracket of e_{2n} with other basis elements, hence it is enough to know the form of the matrix associated to $\text{ad}_{e_{2n}}|_{\mathfrak{n}}$, let's denote this matrix as B.

Proposition 2.1.1 ([15]). *Let $(\mathfrak{b}, J, g)$ be a Hermitian almost Abelian Lie algebra. Then,*

$$B = \begin{pmatrix} a & 0 \\ v & A \end{pmatrix} \qquad a \in \mathbb{R}, v \in \mathfrak{n}_1, A \in \mathfrak{gl}(\mathfrak{n}_1, J_1) \tag{2.1}$$

Here $J_1 = J|_{\mathfrak{n}_1}$, and $\mathfrak{gl}(\mathfrak{n}_1, J_1)$ denotes endomorphisms of $\mathfrak{n}_1$ which commute with J_1.

Proof. Take $\{e_1, \ldots, e_{2n}\}$ be adapted unitary basis of $\mathfrak{b}$. By Newlander-Nirenberg theorem, J is an integrable almost complex structure iff,

$$N_J(X, Y) = [JX, JY] - J\left(([JX, Y] + [X, JY])\right) - [X, Y] = 0 \quad \forall X, Y \in \mathfrak{b}$$

Because, $\mathfrak{b}$ is an almost Abelian Lie algebra $N_J = 0$ is equivalent to $N_J(e_{2n}, Y) = 0$, $N_J(e_1, Y)$ and, $N_J(e_1, e_{2n}) = 0$ $\forall Y \in \mathfrak{n}_1$. By doing explicit computation, we get that,

$$\begin{aligned} N_J(e_{2n}, Y) &= -J[e_{2n}, JY] - [e_{2n}, Y] = -JBJY - BY \\ N_J(e_1, Y) &= [e_{2n}, JY] - J[e_2n, Y] = BJY - JBY \\ N_J(e_1, e_{2n}) &= 0 \end{aligned}$$

Hence J being integrable is equivalent to $(BJ - JB)Y = 0$ $\forall Y \in \mathfrak{n}_1$.
Fix $j \in \{2, \ldots, 2n-1\}$,

$$\begin{aligned} B_{1,j} &= g(Be_j, e_1) \\ &= g(JBe_j, Je_1) \\ &= g(BJe_j, e_{2n}) \\ &= 0 \end{aligned}$$

Here the second equality follows because J is g-orthogonal and the third follows because $e_j \in \mathfrak{n}_1$ is J-invariant and $\mathfrak{n}$ is an ideal so $BJe_j \in \mathfrak{n}$ which is a subspace orthogonal to e_{2n}. Note that this result implies that $\mathfrak{n}_1$ is B-invariant. Now, fix a $Y \in \mathfrak{n}_1$,

$$\begin{aligned} (AJ_1 - J_1 A)Y &= (BJ - JB)Y \\ &= 0 \end{aligned}$$

The first equality is true because $\mathfrak{n}_1$ is J-invariant as well as B-invariant, and the second follows from the integrability of J. ∎

From now on we will refer to (a, v, A) as the algebraic data associated to the adapted unitary basis $\{e_1, \dots e_{2n}\}$.

2.2 Some basic examples of canonical metrics

(a) Kähler Metrics

Definition 2.2.1. *If (M, J, g, ω) is a n-dimensional complex manifold such that ω is closed i. e. $d\omega = 0$, then we say (J, g) is a **Kähler structure** and M is a **Kähler manifold**.*

Theorem 2.2.1 (**Characterization of Kähler Structures on almost Abelian Lie algebras** [6])**.** *Let $\mathfrak{b}$ be an almost Abelian Lie algebra then any complex structure (J, g) on it is Kähler iff for any adapted unitary basis with respect to it we have that, $v = 0$ and $A^t = -A$.*

(b) SKT Metrics

Definition 2.2.2. *If (M, J, g, ω) is a n-dimensional complex manifold such that $\partial\bar{\partial}\omega = 0$, then we say (J, g) is a **SKT structure** and M is a **SKT manifold**.*

Theorem 2.2.2 (**Characterization of SKT Structures on almost Abelian Lie algebras** [1])**.** *Let $\mathfrak{b}$ be an almost Abelian Lie algebra, then any complex structure (J, g) on it is SKT iff for any adapted unitary basis with respect to it we have that, $[A, A^t] = 0$ and 2 eigenvalues of A have real part equal to $-\frac{a}{2}$ and the rest of eigenvalues have real part equal to 0.*

2.3 Balanced Metrics

Definition 2.3.1. *If (M, J, g, ω) is a n-dimensional complex manifold such that ω is co-closed i. e. $d^*\omega = d\omega^{n-1} = 0$, then we say (J, g) is a* ***Balanced structure*** *and M is a* ***Balanced manifold****.*

2.3.1 Characterization over almost Abelian Lie algebras

Proposition 2.3.1 ([7][)**.** *] Let (M, J, g) be a Hermitian manifold of complex dimension n and associated 2-form ω then $\exists!$ 1-form θ, known as Lee form, such that $d\omega^{n-1} = \theta \wedge \omega^{n-1}$, equivalently $\theta = J(d^*\omega) = -J * d * \omega$.*

Proof. Take a unitary basis $e^1, \ldots, e^{2n}$ of 1-forms such that, $Je^i = e^{i+1}$,Clearly we have that, $\theta = ae^1$(assume $a \neq 0$),

$$\omega = e^1 \wedge e^2 + \cdots + e^{2n-1} \wedge e^{2n}$$

$$\begin{aligned}
& d\omega^{n-1} = \theta \wedge \omega^{n-1} \\
\iff & -(n-1)! * d * \omega = - * (\theta \wedge \omega^{n-1}) = -a(n-1)!\, e^2 \\
\iff & d^*\omega = -J\theta \\
\iff & \theta = J(d^*\omega)
\end{aligned}$$

And if, $a = 0$, we get $J(d^*\omega) = 0 = \theta$. ■

We know ω is a balanced metric $\iff d^*\omega = 0 \iff J(d^*\omega) = 0 \iff \theta = 0$. Hence our job of characterizing ω has become equivalent to characterizing $\theta = 0$. The next result will give us what the Lee form θ looks like more explicitly for a Hermitian Lie algebra.

Lemma 2.3.2 ([7])**.** *Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra with an orthonormal basis*

$\{e_1, \ldots, e_{2n}\}$. *Then Lee form θ associated to it is given by,*

$$\forall_{X \in \mathfrak{g}} \theta(X) = -trad_X + \frac{1}{2} g \left(\sum_{k=1}^{2n} [e_k, Je_k], JX \right)$$

Proof. To find θ we will make use of the formula for the co-differential $d^* = -\sum_{k=1}^{2n} \iota_{e_k} \nabla^g_{e_k}$, here we consider $\{e_1, \ldots, e_{2n}\}$ to be some orthonormal basis and ∇^g is the Levi-Civita connection associated to g. Now, we apply the Koszul formula to get the following,

$$2g\left((\nabla_X Y, Z)\right) = g\left(([X, Y], Z)\right) - g\left(([Y, Z], X)\right) - g\left(([X, Z], Y)\right), \quad \forall X, Y, Z \in \mathfrak{g} \tag{2.2}$$

From this we get $\forall X \in \mathfrak{g}$,

$$\begin{aligned}
\theta(X) &= -\sum_{k=1}^{2n} \left(\left(g\left((\nabla^g_X e_k, e_k) \right) \; + \; g\left((\nabla^g_X e_k, JX, \; Je_k) \right) \right) \right) \\
&= -\frac{1}{2} \sum_{k=1}^{2n} \left((2\, g([X, e_k], e_k) \; + \; g(J[JX, e_k], e_k) \; - \; g([JX, Je_k], e_k) \; - \; g([e_k, Je_k], JX) \right)) \\
&= - \operatorname{tr} \operatorname{ad}_X - \frac{1}{2} \operatorname{tr}[\operatorname{ad}_{JX}, J] \; + \; \frac{1}{2} \sum_{k=1}^{2n} g\left(([e_k, Je_k], JX) \right) \\
&= - \operatorname{tr} \operatorname{ad}_X + \frac{1}{2} \sum_{k=1}^{2n} g\left(([e_k, Je_k], JX) \right).
\end{aligned}$$

Here, the last equality follows because the commutator of two endomorphisms is always traceless. ■

Corollary 2.3.3 ([7]). *Let $(\mathfrak{g}, J, g)$ be a Hermitian almost Abelian Lie algebra endowed with an adapted unitary basis $\{e_1, \ldots, e_{2n}\}$, determining the algebraic data (a, v, A). Then, its associated Lee form is*

$$\theta = (Jv)^\flat - (trA)e^{2n} \tag{2.3}$$

Here, $(\cdot)^\flat : \mathfrak{g} \to \mathfrak{g}^*$ is the musical isomorphism induced by g and e^{2n} is the dual 1-form associated with e_{2n}.

Proof. Now, using **lemma** 2.3.2 we have the following,

$$\begin{aligned}
\theta(e_1) &= -\mathrm{trad}_{e_1} + \frac{1}{2}g\left(\sum_{k=1}^{2n}[e_k, Je_k], Je_1\right) \\
&= 0 + \frac{1}{2}g\left(\sum_{k=1}^{2n}[e_k, Je_k], e_{2n}\right) \\
&= g\left([e_1, Je_1], e_{2n}\right) = 0 = (Jv)^\flat(e_1) - (\mathrm{tr}A)e^{2n}(e_1) \\
\theta(X) &= -\mathrm{trad}_X + \frac{1}{2}g\left(\sum_{k=1}^{2n}[e_k, Je_k], JX\right) \quad \forall X \in \mathfrak{n}_1 \\
&= 0 - g\left([e_1, Je_1], JX\right) \\
&= -g\left([e_{2n}, e_1], JX\right) \\
&= -g(ae_1 + v, JX) \\
&= g(ae_{2n} + Jv, X) \\
&= g(Jv, X) = (Jv)^\flat(X) - (\mathrm{tr}A)e^{2n}(X) \\
\theta(e_{2n}) &= -\mathrm{trad}_{e_{2n}} + \frac{1}{2}g\left(\sum_{k=1}^{2n}[e_k, Je_k], Je_{2n}\right) \\
&= -(a + \mathrm{tr}A) + g(ae_1 + v, e_1) \\
&= -\mathrm{tr}A = (Jv)^\flat(e_{2n}) - (\mathrm{tr}A)e^{2n}(e_{2n})
\end{aligned}$$

∎

Now, we know that g is a balanced metric iff $\theta = 0$, by **corollary** 2.3.3 this is equivalent to,

$$(Jv)^\flat - (\mathrm{tr}\, A)e^{2n} = 0$$

Since, $v \in \mathfrak{n}_1$ we have that $Jv \in \mathfrak{n}_1$ and, $g((Jv)^\flat, e^{2n}) = g(Jv, e_{2n}) = 0$, so we get that $(Jv)^\flat$ and e^{2n} are linearly independent which gives us, $(Jv)^\flat - (\mathrm{tr}\, A)e^{2n} = 0$ iff $(Jv)^\flat = 0 \iff Jv = 0 \iff v = 0$ and $-(\mathrm{tr}\, A)e^{2n} = 0 \iff \mathrm{tr}\, A = 0$. As a result of this, we get the following characterization result,

Theorem 2.3.4 (Characterization of Balanced Structures on almost Abelian Lie algebras [7]). *Let $\mathfrak{b}$ be an almost Abelian Lie algebra then any complex structure (J, g) on it is balanced iff for any adapted unitary basis with respect to it we have that, $v = 0$ and* $\mathrm{tr}\, A = 0$.

Remark 2.3.1. *We know that, a Lie algebra* $\mathfrak{g}$ *is unimodular only when* $\operatorname{tr}\operatorname{ad}_X = 0 \forall X \in \mathfrak{g}$, *so if* $(\mathfrak{b}, J, g)$ *is a balanced almost Abelian Lie algebra with* $\{e_1, \ldots, e_{2n}\}$ *some adapted unitary basis then,* $\mathfrak{b}$ *is unimodular* $\iff \operatorname{tr}\operatorname{ad}_{e_{2n}} = 0 \iff a + \operatorname{tr} A = 0 \iff a = 0$.

2.3.2 Classification in the case of real dimension 8

Before going into the classification, let's briefly look at the notation we'll be using to describe a Lie algebra through an example, because writing out all the structure equation for each Lie algebra can be very cumbersome.

Consider the so called "diamond Lie algebra" $\mathfrak{d}$ which is a 4-dimensional Lie algebra with basis $\{e_1, , \ldots, e_4\}$ and structure equations given by,

$$[e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_2, e_3] = e_4$$

And the remaining brackets are 0.
Now using the formula $de^i(X, Y) = e^i([Y, X]) \forall X, Y \in \mathfrak{d}$ (where, $\{e^1, \ldots, e^4\}$ is the dual basis of $\{e_1, \ldots, e_4\}$). We have that,

$$\begin{aligned} de^1 &= 0 \\ de^2 &= -e^{12} \\ de^3 &= e^{13} \\ de^4 &= -e^{23} \end{aligned}$$

Here e^{ij} denotes $e^i \wedge e^j$. Then in our notation, we'll say,

$$\mathfrak{d} = (de^1, de^2, de^3, de^4) = (0, -e^{12}, e^{13}, -e^{23})$$

From this example, it should be clear that this notation can be used to describe any finite-dimensional Lie algebra uniquely.

Theorem 2.3.5. $\mathfrak{g}$ *is a real eight-dimensional almost A belian Lie algebra admitting a balanced structure* (J, g)*, iff it is isomorphic to one of the following:*

$$\mathfrak{b}_1 = (af^{18}, pf^{28} + f^{38}, -f^{28} + pf^{38}, qf^{48} + rf^{58}, -rf^{48} + qf^{58}, -(p+q)f^{68} + sf^{78}, -sf^{68} - (p+q)f^{78}, 0),$$

$$ars \neq 0,$$

$$\mathfrak{b}_2 = (af^{18}, pf^{28}+f^{38}-f^{48}, -f^{28}+pf^{38}-f^{58}, pf^{48}+f^{58}, -f^{48}+pf^{58}, -2pf^{68}+qf^{78}, -qf^{68}-2pf^{78}, 0),$$

$$aq \neq 0,$$

$$\mathfrak{b}_3 = (af^{18}, pf^{28}+f^{38}, -f^{28}+pf^{38}, qf^{48}+rf^{58}, -rf^{48}+qf^{58}, -(p+q)f^{68}, -(p+q)f^{78}, 0), ar \neq 0,$$

$$\mathfrak{b}_4 = (af^{18}, pf^{28}+f^{38}-f^{48}, -f^{28}+pf^{38}-f^{58}, pf^{48}+f^{58}, -f^{48}+pf^{58}, -2pf^{68}, -2pf^{78}, 0), a \neq 0,$$

$$\mathfrak{b}_5 = (af^{18}, pf^{28}+f^{38}, -f^{28}+pf^{38}, qf^{48}, qf^{58}, -(p+q)f^{68}, -(p+q)f^{78}, 0), a \neq 0,$$

$$\mathfrak{b}_6 = (af^{18}, pf^{28}+f^{38}, -f^{28}+pf^{38}, -\frac{p}{2}f^{48}+f^{58}, -\frac{p}{2}f^{58}, -\frac{p}{2}f^{68}+f^{78}, -\frac{p}{2}f^{78}, 0), a \neq 0,$$

$$\mathfrak{b}_7 = (f^{18}, pf^{28}, pf^{38}, qf^{48}, qf^{58}, -(p+q)f^{68}, -(p+q)f^{78}, 0),$$

$$\mathfrak{b}_8 = (f^{18}, pf^{28}+f^{38}, pf^{38}, pf^{48}+f^{58}, pf^{58}, -2pf^{68}, -2pf^{78}, 0),$$

$$\mathfrak{b}_9 = (f^{18}, f^{38}, f^{48}, 0, f^{68}, f^{78}, 0, 0),$$

$$\mathfrak{b}_{10} = (pf^{18}+f^{28}, -f^{18}+pf^{28}, qf^{38}+rf^{48}, -rf^{38}+qf^{48}, -(p+q)f^{58}+sf^{68}, -sf^{58}-(p+q)f^{68}, 0, 0), rs \neq 0,$$

$$\mathfrak{b}_{11} = (pf^{18}+f^{28}-f^{38}, -f^{18}+pf^{28}-f^{48}, pf^{38}+f^{48}, -f^{38}+pf^{48}, -2pf^{58}+qf^{68}, -qf^{58}-2pf^{68}, 0, 0), q \neq 0,$$

$$\mathfrak{b}_{12} = (pf^{18}+f^{28}, -f^{18}+pf^{28}, qf^{38}+rf^{48}, -rf^{38}+qf^{48}, -(p+q)f^{58}, -(p+q)f^{68}, 0, 0), r \neq 0,$$

$$\mathfrak{b}_{13} = (pf^{18}+f^{28}-f^{38}, -f^{18}+pf^{28}-f^{48}, pf^{38}+f^{48}, -f^{38}+pf^{48}, -2pf^{58}, -2pf^{68}, 0, 0),$$

$$\mathfrak{b}_{14} = (pf^{18}+f^{28}, -f^{18}+pf^{28}, qf^{38}, qf^{48}, -(p+q)f^{58}, -(p+q)f^{68}, 0, 0),$$

$$\mathfrak{b}_{15} = (pf^{18}+f^{28}, -f^{18}+pf^{28}, -\frac{p}{2}f^{38}+f^{48}, -\frac{p}{2}f^{48}, -\frac{p}{2}f^{58}+f^{68}, -\frac{p}{2}f^{68}, 0, 0),$$

$$\mathfrak{b}_{16} = (f^{18}, f^{28}, pf^{38}, pf^{48}, -(p+1)f^{58}, -(p+1)f^{68}, 0, 0),$$

$$\mathfrak{b}_{17} = (f^{18}+f^{28}, f^{28}, f^{38}+f^{48}, f^{48}, -2f^{58}, -2f^{68}, 0, 0),$$

$$\mathfrak{b}_{18} = (f^{28}, 0, f^{48}, 0, 0, 0, 0, 0),$$

$$\mathfrak{b}_{19} = (f^{28}, f^{38}, 0, f^{58}, f^{68}, 0, 0, 0).$$

Out of this, $\mathfrak{b}_{10}, \ldots, \mathfrak{b}_{19}$ *are the only unimodular Lie algebras and,* $\mathfrak{b}_{18}$ *and* $\mathfrak{b}_{19}$ *are the only nilpotent Lie algebras.*

Proof. Using the characterization of Balanced metrics on almost Abelian Lie algebras, the classification in the case of dimension 8 has reduced to classifying (a, A), where $a \in \mathbb{R}$ and $A \in M_6(\mathbb{R})$ such that $\text{tr}A = 0$ and $\exists J_1 \in M_6(\mathbb{R})$ that satisfies $AJ_1 = J_1A$ and $J_1^2 = -I$.

Let's focus first on classifying such matrices, A. To do this, it is enough to consider the real Jordan canonical form of every 6×6 matrix, this will give us a bunch of real matrices depending on some real parameters, then we apply the required condition on each these variable matrices this will either eliminate that matrix or put some restriction on the parameters

Table 2.1: Eight-dimensional almost Abelian Lie algebras admitting balanced structures

Name	Kähler	SKT	Unimodular	Nilpotent	Balanced Structure
$\mathfrak{b}_1$	✓ $(p, q = 0)$	✓ $(p, q = 0)$	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_2$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_3$	✓ $(p, q = 0)$	✓ $(p, q = 0)$	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_4$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_5$	✓ $(p, q = 0)$	✓ $(p, q = 0)$	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_6$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_7$	✓ $(p, q = 0)$	✓ $(p, q = 0)$	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_8$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_9$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{10}$	✓ $(p, q = 0)$	✓ $(p, q = 0)$	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{11}$	✗	✗	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{12}$	✓ $(p, q = 0)$	✓ $(p, q = 0)$	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{13}$	✗	✗	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{14}$	✓ $(p, q = 0)$	✓ $(p, q = 0)$	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{15}$	✗	✗	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{16}$	✗	✗	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{17}$	✗	✗	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{18}$	✗	✗	✓	✓	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$
$\mathfrak{b}_{19}$	✗	✗	✓	✓	$Jf_i = f_{9-i}\,\forall i \in \{1, \ldots, 4\},\ g = \sum_{k=1}^{8}(f^k)^2$

of that matrix. To do this, it is enough to consider the real Jordan canonical form of every 6×6 matrix; this will give us a bunch of real matrices depending on some real parameters then we apply the required condition on each of these variable matrices; this will either eliminate that matrix or put some restriction on the parameters of that matrix apply the required condition on each of these variable matrices; this will either eliminate that matrix or put some restriction on the parameters of that matrix.

Step 1: Write down every possible real canonical Jordan form of any arbitrary matrix A up to the number of complex eigenvalues it has (0,2,4,6) as well as the geometric and algebraic multiplicity of each of the eigenvalues.

Step 2: Apply the condition $\operatorname{tr} A = 0$ on all the matrices. This gives us a restriction on the parameters.

Step 3: Note that for a matrix A to be valid, we need to construct a matrix J_1 for it such that $J_1^2 = -I, J_1 A = A J_1$, so now our task is to understand the restrictions this condition puts on the list of real Jordan canonical forms that we have from the previous step. Now, it is clear that the generalized eigenspaces of A are preserved under J_1. So constructing a J_1 is equivalent to constructing a complex structure on each of the generalized eigenspace of A; this trivially implies that the dimension of a generalized eigenspace has to be even.

Step 4: Note that, up to scale there are two types of real Jordan blocks for a complex eigenvalue that can be part of a Jordan canonical form of a 6×6 given as

$$C_1 = \begin{pmatrix} p & 1 & -1 & 0 \\ -1 & p & 0 & -1 \\ 0 & 0 & p & 1 \\ 0 & 0 & -1 & p \end{pmatrix}, C_2 = \begin{pmatrix} p & 1 \\ -1 & p \end{pmatrix}$$

The complex structures given by $J_1' = \begin{pmatrix} 0 & R \\ -R & 0 \end{pmatrix}, J_2' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (Here $R := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) commute with C_1 and C_2 respectively. So, we always have a complex structure as needed on the generalized eigenspace of a complex eigenvalue. Now, we only need to consider the case of real eigenvalues.

Step 5: Say, $w \in \mathfrak{n}_1$ is a generalized eigenvector of order k with generalized eigenvalue $\lambda \in \mathbb{R}$. Then $J_1 w \in \mathfrak{n}_1$ is also a generalized eigenvector of order k with generalized eigenvalue $\lambda \in \mathbb{R}$. And because J_1 does not have a real eigenvalue, we know that

$J_1 w \notin \text{span}_{\mathbb{R}} w$. So, for an A, a J_1 can only exist if for any real generalized eigenvalue λ of A $\exists$ an even number of generalized eigenvectors for any given order. In fact, if this condition of an even number of generalized eigenvectors holds, then we can construct a corresponding J_1, so both conditions are equivalent.

Combining all the steps above, we get that the following are the only possible forms of A up to similarity and rescaling of e_{2n},

$$A_1 = \begin{pmatrix} p & 1 & 0 & 0 & 0 & 0 \\ -1 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & q & r & 0 & 0 \\ 0 & 0 & -r & q & 0 & 0 \\ 0 & 0 & 0 & 0 & -(p+q) & s \\ 0 & 0 & 0 & 0 & -s & -(p+q) \end{pmatrix}, A_2 = \begin{pmatrix} p & 1 & -1 & 0 & 0 & 0 \\ -1 & p & 0 & -1 & 0 & 0 \\ 0 & 0 & p & 1 & 0 & 0 \\ 0 & 0 & -1 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & -2p & q \\ 0 & 0 & 0 & 0 & -q & -2p \end{pmatrix}$$

$$A_3 = \begin{pmatrix} p & 1 & 0 & 0 & 0 & 0 \\ -1 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & q & r & 0 & 0 \\ 0 & 0 & -r & q & 0 & 0 \\ 0 & 0 & 0 & 0 & -(p+q) & 0 \\ 0 & 0 & 0 & 0 & 0 & -(p+q) \end{pmatrix}, A_4 = \begin{pmatrix} p & 1 & -1 & 0 & 0 & 0 \\ -1 & p & 0 & -1 & 0 & 0 \\ 0 & 0 & p & 1 & 0 & 0 \\ 0 & 0 & -1 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & -2p & 0 \\ 0 & 0 & 0 & 0 & 0 & -2p \end{pmatrix}$$

$$A_5 = \begin{pmatrix} p & 1 & 0 & 0 & 0 & 0 \\ -1 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & -(p+q) & 0 \\ 0 & 0 & 0 & 0 & 0 & -(p+q) \end{pmatrix}, A_6 = \begin{pmatrix} p & 1 & 0 & 0 & 0 & 0 \\ -1 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{p}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{p}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{p}{2} \end{pmatrix}$$

$$A_7 = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & -(p+q) & 0 \\ 0 & 0 & 0 & 0 & 0 & -(p+q) \end{pmatrix}, A_8 = \begin{pmatrix} p & 1 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 1 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & -2p & 0 \\ 0 & 0 & 0 & 0 & 0 & -2p \end{pmatrix}$$

$$A_8 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, $q, r, s \neq 0$ when appropriate to avoid A_i being a subcase of A_j for some $i \neq j$. We get that,

A_1 yields $\mathfrak{b}_1$ and $\mathfrak{b}_{10}$,
A_2 yields $\mathfrak{b}_2$ and $\mathfrak{b}_{11}$,
A_3 yields $\mathfrak{b}_3$ and $\mathfrak{b}_{12}$,
A_4 yields $\mathfrak{b}_4$ and $\mathfrak{b}_{13}$,
A_5 yields $\mathfrak{b}_5$ and $\mathfrak{b}_{14}$,
A_6 yields $\mathfrak{b}_6$ and $\mathfrak{b}_{15}$,
A_7 yields $\mathfrak{b}_7$ and $\mathfrak{b}_{16}$,
A_8 yields $\mathfrak{b}_8$, $\mathfrak{b}_{17}$, and $\mathfrak{b}_{18}$,
A_9 yields $\mathfrak{b}_9$ and $\mathfrak{b}_{19}$. ■

Remark 2.3.2. *One thing to note is that different Lie algebras in the same class above may or may not be isomorphic to each other, but no two Lie algebras that belong to different classes can be isomorphic to each other.*

Remark 2.3.3. *In table 2.1, one notable observation is that if a metric on a real dimension 8 almost Abelian Lie algebra is both Balanced and SKT, then it's also Kähler. This is in general true for almost Abelian Lie algebra for all dimensions.*

Say, $(\mathfrak{g}, J, g)$ *with algebraic data* (a, v, A)*, is an SKT as well as balanced almost Abelian Lie algebra, then by* **theorem** *2.2.2 and* **theorem** *2.3.4, we have that,*

$$v = 0, [A, A^t] = 0, trA = 0$$

And, that the real part of every eigenvalue of A *is* $-a/2$ *and* 0*.Now, the* $trA = 0$ *forces the real part of every eigenvalue to be* 0*, and this, combined with* $[A, A^t] = 0$ *gives us that* $A \in \mathfrak{u}(n)$*, which along with* $v = 0$ *gives us that* $(\mathfrak{g}, J, g)$ *is Kähler by* **theorem** *2.2.1.*

2.4 Astheno-Kähler Metrics

Definition 2.4.1. *If (M, J, g, ω) is a n-dimensional complex manifold such that $\partial\bar{\partial}\omega^{n-2} = 0$, then we say (J, g) is an **astheno-Kähler structure** and M is an **astheno-Kähler manifold**.*

2.4.1 Characterization over almost Abelian Lie algebras

Theorem 2.4.1 (Characterization of astheno-Kähler Structures on almost Abelian Lie algebras [9]). *Let $\mathfrak{b}$ be an almost Abelian Lie algebra of real dimension $2n$, then any complex structure (J, g) on it is astheno-Kähler iff for any adapted unitary basis with respect to it we have that, $[A, A^t] = 0$ and $\exists r \in \{1, \ldots, n-1\}$ such that $2r$ eigenvalues of A have real part equal to $-\frac{n-1-r}{2(n-2)}a$ and the other $2(n-1-r)$ eigenvalues have real part equal to $\frac{r-1}{2(n-2)}a$, and $\mathfrak{b}$ is unimodular iff 2 eigenvalues of A have real part equal to $-\frac{a}{2}$ and the other $2(n-2)$ eigenvalues have real part equal to 0.*

Proof. Say, $\{e_1, \ldots, e_{2n}\}$ is a unitary basis adapted to (J, g) and $\{e^1, \ldots, e^{2n}\}$ is it's dual basis.

Define a new unitary basis $\{f_1, \ldots, f_{2n}\}$ given as, $f_1 = -e_{2n}.f_{2n} = e_1, f_i = e_i \forall i \in \{2, \ldots, 2n-1\}$, then the matrix of the linear transformation $ad|_{\mathfrak{n}}$ (denoted by C) with respect to the basis $\{f_2, \ldots, f_{2n}\}$ is given as,

$$C = -\begin{pmatrix} A & v \\ 0 & a \end{pmatrix}$$

And dual of $\{f_1, \ldots, f_{2n}\}$ is given by $\{f^1, \ldots, f^{2n}\}$, note that $Jf_i = f_{2n+1-i} \forall i \in \{1, \ldots, n\}$.

Consider the space,

$$\mathfrak{b}^{*^{1,0}} = \{b - iJb | b \in \mathfrak{b}^*\}$$

There is a unitary linear isomorphism $T : \mathfrak{b}^* \to \mathfrak{b}^{*^{1,0}}$ defined as, $T(b) := \frac{b-iJb}{\sqrt{2}}$, also $\mathfrak{b}^{*^{1,0}}$ is a complex subspace of the complex vector space $\mathfrak{b}^*_{\mathbb{C}}$, and the set $\{\psi^i := T(f^i) | i \in \{1, \ldots, n\}\}$ is a complex basis of $\mathfrak{b}^{*^{1,0}}$.

By doing a simple computation, we get,

$$\begin{aligned}
d\psi^1 &= \frac{df^1 - \sqrt{-1}\, df^{2n}}{\sqrt{2}} \\
&= -\frac{\sqrt{-1} a f^1 \wedge f^{2n}}{\sqrt{2}} \\
&= -\frac{a}{\sqrt{2}} \psi^1 \wedge \overline{\psi^1} \\
&= -b\psi^1 \wedge \overline{\psi^1} \\
d\psi^i &= \frac{df^i - \sqrt{-1}\, df^{2n+1-i}}{\sqrt{2}} \qquad i \in \{2, \dots, n\} \\
&= \frac{v_i f^1 \wedge f^{2n} + \sum_{j=2}^{2n-1} A_{i,j} f^1 \wedge f^j - \sqrt{-1} \left(v_{2n+1-i} f^1 \wedge f^{2n} + \sum_{j=2}^{2n-1} A_{2n+1-i,j} f^1 \wedge f^j \right)}{\sqrt{2}} \\
&= \frac{\left(v_i - \sqrt{-1}\, v_{2n+1-i}\right) f^1 \wedge f^{2n} + \sum_{j=2}^{2n-1} \left(A_{i,j} - \sqrt{-1}\, A_{2n+1-i,j}\right) f^1 \wedge f^j}{\sqrt{2}} \\
&= -\overline{w_i} \psi^1 \wedge \overline{\psi^1} + \frac{\sqrt{2} \sum_{j=2}^{n} \left(A_{i,j} - \sqrt{-1}\, A_{2n+1-i,j}\right) \left(\frac{\overline{\psi^1} + \psi^1}{\sqrt{2}}\right) \wedge \psi^j}{\sqrt{2}} \\
&= -\overline{w_i} \psi^1 \wedge \overline{\psi^1} + \sum_{j=2}^{n} \frac{\left(A_{i,j} - \sqrt{-1}\, A_{2n+1-i,j}\right)}{\sqrt{2}} \left(\overline{\psi^1} + \psi^1\right) \wedge \psi^j \\
&= -\overline{w_i} \psi^1 \wedge \overline{\psi^1} + \sum_{j=2}^{n} \left(\overline{D_{i,j}}\right) \left(\overline{\psi^1} - \psi^1\right) \wedge \psi^j
\end{aligned}$$

Here,$b \in \mathbb{R}$, $w \in \mathbb{C}^n$ and, $D \in M_n(\mathbb{C})$ are defined as follows,$b := \frac{a}{\sqrt{2}}$, $\overline{w}_i := -\frac{1}{\sqrt{2}}(v_i + iv_{2n+1-i})$ and $D_{i,j} = \frac{1}{\sqrt{2}} (A_{i,j} + iA_{2n+1-i,j})$. Let R be a $(n-1) \times (n-1)$ matrix defined as $R_{i,j} = \delta_{i,n-i}$. Then,

$$J_1 = \begin{pmatrix} 0 & R \\ -R & 0 \end{pmatrix}$$

And $J_1 A = A J_1$ gives us the following restriction on A,

$$A = \begin{pmatrix} A_1 & A_2 \\ -RA_2R & RA_1R \end{pmatrix} \tag{2.4}$$

The associated Kähler form is,

$$\omega = \sum_{i=1}^{n} e^i \wedge e^{2n+1-i} = \sum_{i=1}^{n} f^i \wedge f^{2n+1-i} = -\sqrt{-1}\left(\sum_i \psi_i \wedge \overline{\psi_i}\right)$$

Now, set

$$K = D + D^*.$$

Since K is Hermitian, we may, after a unitary change of basis, assume it is diagonal:

$$K = \mathrm{diag}\{k_2, \ldots, k_n\}.$$

A quick computation will show us that,

$$\sqrt{-1}\,\partial\omega = -\psi^1\wedge\overline{\psi^1}\wedge\left(\sum_{i=2}^{n} v_i\,\psi^i\right)+\psi^1\wedge\left(\sum_{i,j=2}^{n} K_{i,j}\,\psi^i\wedge\overline{\psi^j}\right) = -\psi^1\wedge\overline{\psi^1}\wedge\left(\sum_{i=2}^{n} v_i\,\psi^i\right)+\psi^1\wedge\left(\sum_{i=2}^{n} k_i\,\psi^i\wedge\overline{\psi^i}\right)$$

so that

$$\partial\omega\wedge\overline{\partial\omega} = -\psi^1\wedge\overline{\psi^1}\wedge\left(\sum_{i=2}^{n} k_i\,\psi^i\wedge\overline{\psi^i}\right)^2 = -\psi^1\wedge\overline{\psi^1}\sum_{i,j} k_i k_j \psi^i\overline{\psi^i}\psi^j\overline{\psi^j}.$$

Moreover, one finds

$$-\sqrt{-1}\,\partial\overline{\partial}\omega = -\psi^1\wedge\overline{\psi^1}\wedge\sum_{i,j=2}^{n} M_{ij}\,\psi^i\wedge\psi^j,$$

In this equation, M is defined as follows,

$$M = bK + D^*K + KD = bK + K^2 + [D^*, D].$$

In the second equality, we use that K is diagonal.

Now, assume that $n \geq 4$. The astheno-Kähler condition can be written as

$$\frac{1}{n-2}\,\partial\overline{\partial}\omega^{n-2} = \partial\overline{\partial}\omega\wedge\omega^{n-3} + (n-3)\,\partial\omega\wedge\overline{\partial}\omega\wedge\omega^{n-4} = 0. \tag{2.5}$$

If we wedge this equality with $\psi^i\wedge\overline{\psi^j}$ (for any $2 \leq i < j \leq n$), the term involving $\partial\omega\wedge\overline{\partial}\omega$

drops out, and we deduce that

$$M_{ij} = 0 \quad \text{for all } i \neq j.$$

This implies that M, and hence $[D^*, D]$, is diagonal. Notice that for the diagonal entries, we have

$$[D^*, D]_{ii} = [K, D]_{ii} = k_i D_{ii} - D_{ii} k_i = 0,$$

so that, in fact

$$[D^*, D] = 0.$$

Therefore, we obtain

$$M = bK + K^2.$$

Next, fix an index i and wedge (2.5) with $\psi^i \wedge \overline{\psi^i}$. By a short calculation, we get that,

$$\begin{aligned}
0 &= \sum_{j \neq i} M_{jj} \; + \sum_{\substack{j \neq l \\ j,l \neq i}} k_j \, k_l \\
&= \operatorname{tr}(M) \; - \; M_{ii} \; + \; \left(\sum_j k_j \right)^2 \; - \; \sum_j k_j^2 \\
&= \operatorname{tr}(M) \; - \; M_{ii} \; + \; (k - k_i)^2 \; - \; \sum_j k_j^2 \; + \; k_i^2 \\
&= b\,k \; + \; \operatorname{tr}\big(K^2\big) \; - \; b\,k_i \; - \; k_i^2 \; + \; k^2 \; - \; 2\,k\,k_i \; + \; k_i^2 \; - \; \operatorname{tr}\big(K^2\big) \; + \; k_i^2 \\
&= \big(b + k^2\big) \; - \; (b + 2k) \; k_i \; + \; k_i^2 \\
&= (b + k - k_i) \; (k - k_i) \, .
\end{aligned}$$

Here, k is defined as follows,

$$k :== \sum_{j=2}^{n} k_j,$$

So for each i, either $k_i = k$ or $k_i = b + k$. Suppose that exactly r indices (with $2 \leq i \leq n$) have $k_i = k$, while the remaining $n - 1 - r$ satisfy $k_i = b + k$. Then we have,

$$k = r\,k + (n - 1 - r)(b + k), \tag{2.6}$$

which rearranges to

$$(n-1-r)b+(n-2)k=0.$$

Now, if the set of eigenvalues of D is $\{\lambda_2,\ldots,\lambda_n\}$, then eigenvalues of $K=D+D^*$ are $\{2\operatorname{Re}(\lambda_2),\ldots,2\operatorname{Re}(\lambda_n)\}$. In other words, we have that,

$$2\operatorname{Re}(\lambda_i)=\begin{cases}k, & \text{for } r \text{ entries,}\\ b+k, & \text{for } n-1-r \text{ entries,}\end{cases}$$

the numbers r, b, and k must satisfy (2.6).

The following relation can be established between A and D,

$$D=\frac{1}{\sqrt{2}}\left(A1+iA_2R\right)$$

and

$$A=\begin{pmatrix}I & 0\\ 0 & R\end{pmatrix}\begin{pmatrix}A_1 & A_2R\\ -A_2R & A_1\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & R\end{pmatrix}^{-1}$$

Now,

$$A' := \begin{pmatrix}A_1 & A_2R\\ -A_2R & A_1\end{pmatrix}$$

. Note that, $\sqrt{2}D=A_1+iA_2R$ along with $[D^*,D]=0$ gives us that, $[A^t,A]=0$.

$$\det(xI-A)=\det(xI-A')=\det(xI-\sqrt{2}D)\overline{\det(xI-\sqrt{2}D)}.$$

The first equality is because A' and A are similar; the second equality is a standard result in linear algebra.

So, by the above equation and the fact that D has only real eigenvalues, we conclude that A has the same eigenvalues as $\sqrt{2}D$ with double multiplicity. This gives us that the real part of $2r$ eigenvalues of A is,

$$\begin{aligned}\sqrt{2}\frac{k}{2} &= \frac{k}{\sqrt{2}}\\ &= \frac{-(n-1-r)b}{\sqrt{2}(n-2)}\end{aligned}$$

$$= \frac{-(n-1-r)}{2(n-2)} a$$

And that the real part of $2(n-1-r)$ eigenvalues is given as,

$$\frac{1}{\sqrt{2}}(b+k) = \frac{1}{2}(a - \frac{n-1-r}{n-2} a)$$
$$= \frac{(r-1)}{2(n-2)} a$$

And, $\mathfrak{g}$ is unimodular iff $a + \operatorname{tr} A = 0$ this is equivalent to,

$$a + 2r\left(-\frac{n-1-r}{2(n-2)}\, a\right) + 2\,(n-1-r)\left(\frac{r-1}{2(n-2)}\, a\right) = 0,$$
$$\iff a(r-1) = 0.$$

It can be easily seen that the condition that A has 2 eigenvalues with real part $-a/2$ and $2(n-2)$ eigenvalues equal to 0 is equivalent to $a(r-1)=0$ by taking cases $a=0$ and $a \neq 0$.
■

2.4.2 Classification in the case of real dimension 8

Theorem 2.4.2. $\mathfrak{b}$ *is a real eight-dimensional almost Abelian Lie algebra admitting an astheno-Kähler structure* (J,g) *iff it is isomorphic to one of the following:*

$$\mathfrak{b}_1^0 = (af^{18}, -\frac{a}{4}f^{28} + f^{38}, -f^{28} - \frac{a}{4}f^{38}, -\frac{a}{4}f^{48} + pf^{58}, -pf^{48} - \frac{a}{4}f^{58}, -\frac{a}{4}f^{68} + qf^{78}, -qf^{68} - \frac{a}{4}f^{78}, 0),\quad apq \neq 0,$$
$$\mathfrak{b}_2^0 = (af^{18}, -\frac{a}{4}f^{28} + f^{38}, -f^{28} - \frac{a}{4}f^{38}, -\frac{a}{4}f^{48} + pf^{58}, -pf^{48} - \frac{a}{4}f^{58}, -\frac{a}{4}f^{68}, -\frac{a}{4}f^{78}, 0), ap \neq 0,$$
$$\mathfrak{b}_3^0 = (af^{18}, -\frac{a}{4}f^{28} + f^{38}, -f^{28} - \frac{a}{4}f^{38}, -\frac{a}{4}f^{48}, -\frac{a}{4}f^{58}, -\frac{a}{4}f^{68}, -\frac{a}{4}f^{78}, 0), a \neq 0,$$
$$\mathfrak{b}_4^0 = (f^{18}, -\frac{1}{4}f^{28}, -\frac{1}{4}f^{38}, -\frac{1}{4}f^{48}, -\frac{1}{4}f^{58}, -\frac{1}{4}f^{68}, -\frac{1}{4}f^{78}, 0),$$
$$\mathfrak{b}_1^1 = (af^{18}, -\frac{a}{2}f^{28} + f^{38}, -f^{28} - \frac{a}{2}f^{38}, pf^{58}, -pf^{48}, qf^{78}, -qf^{68}, 0), apq \neq 0,$$
$$\mathfrak{b}_2^1 = (af^{18}, -\frac{a}{2}f^{28} + f^{38}, -f^{28} - \frac{a}{2}f^{38}, pf^{58}, -pf^{48}, v_1f^{18}, v_2f^{18}, 0), ap \neq 0,$$
$$\mathfrak{b}_3^1 = (af^{18}, -\frac{a}{2}f^{28} + f^{38}, -f^{28} - \frac{a}{2}f^{38}, v_1f^{18}, v_2f^{18}, v_3f^{18}, v_4f^{18}, 0), a \neq 0,$$

$\mathfrak{b}_4^1 = (f^{18}, -\frac{1}{2}f^{28}, -\frac{1}{2}f^{38}, v_1 f^{18}, v_2 f^{18}, v_3 f^{18}, v_4 f^{18}, 0),$
$\mathfrak{b}_5^1 = (f^{28}, -f^{18}, pf^{48}, -pf^{38}, qf^{68}, -qf^{58}, 0, 0), pq \neq 0,$
$\mathfrak{b}_6^1 = (f^{28}, -f^{18}, pf^{48}, -pf^{38}, v_1 f^{78}, v_2 f^{78}, 0, 0), p \neq 0,$
$\mathfrak{b}_7^1 = (f^{28}, -f^{18}, v_1 f^{78}, v_2 f^{78}, v_3 f^{78}, v_4 f^{78}, 0, 0),$
$\mathfrak{b}_1^2 = (af^{18}, -\frac{a}{4}f^{28} + f^{38}, -f^{28} - \frac{a}{4}f^{38}, -\frac{a}{4}f^{48} + pf^{58}, -pf^{48} - \frac{a}{4}f^{58}, \frac{a}{4}f^{68} + qf^{78}, -qf^{68} + \frac{a}{4}f^{78}, 0), a \neq 0,$
$\mathfrak{b}_2^2 = (af^{18}, -\frac{a}{4}f^{28} + f^{38}, -f^{28} - \frac{a}{4}f^{38}, -\frac{a}{4}f^{48} + pf^{58}, -pf^{48} - \frac{a}{4}f^{58}, \frac{a}{4}f^{68}, \frac{a}{4}f^{78}, 0),$
$\mathfrak{b}_3^2 = (af^{18}, -\frac{a}{4}f^{28} + f^{38}, -f^{28} - \frac{a}{4}f^{38}, -\frac{a}{4}f^{48}, -\frac{a}{4}f^{58}, \frac{a}{4}f^{68}, \frac{a}{4}f^{78}, 0),$
$\mathfrak{b}_4^2 = (f^{18}, -\frac{1}{4}f^{28}, -\frac{1}{4}f^{38}, -\frac{1}{4}f^{48}, -\frac{1}{4}f^{58}, \frac{1}{4}f^{68}, \frac{1}{4}f^{78}, 0),$
$\mathfrak{b}_1^3 = (af^{18}, f^{38}, -f^{28}, pf^{58}, -pf^{48}, qf^{78}, -qf^{68}, 0), apq \neq 0,$
$\mathfrak{b}_2^3 = (af^{18}, f^{38}, -f^{28}, pf^{58}, -pf^{48}, v_1 f^{18}, v_2 f^{18}, 0), ap \neq 0,$
$\mathfrak{b}_3^3 = (f^{18}, f^{38}, -f^{28}, v_1 f^{18}, v_2 f^{18}, v_3 f^{18}, v_4 f^{18}, 0),$
$\mathfrak{b}_4^3 = (f^{18}, v_1 f^{18}, v_2 f^{18}, v_3 f^{18}, v_4 f^{18}, v_5 f^{18}, v_6 f^{18}, 0),$

Here, $\mathfrak{b}_i^1$ is unimodular for $1 \leq i \leq 7$ and none of these Lie algebras are nilpotent.

Proof. This proof is similar to what we did in **theorem** 2.3.5, just that the condition $[A, A^t] = 0$ restricts the possible real Jordan canonical forms because for all eigenvalues, the algebraic and geometric multiplicities are equal. Also, v is not necessarily 0, but by a change of basis we make as many components of v zero as possible. We get that,

$$A_1^i = \begin{pmatrix} \lambda_1^i & 1 & 0 & 0 & 0 & 0 \\ -1 & \lambda_1^i & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^i & p & 0 & 0 \\ 0 & 0 & -p & \lambda_2^i & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3^i & q \\ 0 & 0 & 0 & 0 & -q & \lambda_3^i \end{pmatrix}, A_2^i = \begin{pmatrix} \lambda_1^i & 1 & 0 & 0 & 0 & 0 \\ -1 & \lambda_1^i & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^i & p & 0 & 0 \\ 0 & 0 & -p & \lambda_2^i & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3^i & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^i \end{pmatrix}$$

Table 2.2: Eight-dimensional almost Abelian Lie algebras admitting astheno-Kähler structures

Name	Kähler	SKT	Unimodular	Balanced	astheno-Kähler Structure
$\mathfrak{b}_1^0$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_2^0$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_3^0$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_4^0$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_1^1$	✗	✓	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_2^1$	✗	✓$(v_1, v_2 = 0)$	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_3^1$	✗	✓$(v_1, v_2, v_3, v_4 = 0)$	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_4^1$	✗	✓$(v_1, v_2, v_3, v_4 = 0)$	✓	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_5^1$	✓	✓	✓	✓	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_6^1$	✓$(v_1, v_2 = 0)$	✓$(v_1, v_2 = 0)$	✓	✓$(v_1, v_2 = 0)$	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_7^1$	✓$(v_1, v_2, v_3, v_4 = 0)$	✓$(v_1, v_2, v_3, v_4 = 0)$	✓	✓$(v_1, v_2, v_3, v_4 = 0)$	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_1^2$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_2^2$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_3^2$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_4^2$	✗	✗	✗	✗	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_1^3$	✓	✓	✗	✓	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_2^3$	✓$(v_1, v_2 = 0)$	✓$(v_1, v_2 = 0)$	✗	✓$(v_1, v_2 = 0)$	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_3^3$	✓$(v_1, v_2, v_3, v_4 = 0)$	✓$(v_1, v_2, v_3, v_4 = 0)$	✗	✓$(v_1, v_2, v_3, v_4 = 0)$	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$
$\mathfrak{b}_4^3$	✓$(v_1, v_2, v_3, v_4, v_5, v_6 = 0)$	✓$(v_1, v_2, v_3, v_4, v_5, v_6 = 0)$	✗	✓$(v_1, v_2, v_3, v_4, v_5, v_6 = 0)$	$Jf_i = f_{9-i}\,\forall i \in \{1,\ldots,4\},\ g = \sum_{k=1}^8 (f^k)^2$

$$A_3^i = \begin{pmatrix} \lambda_1^i & 1 & 0 & 0 & 0 & 0 \\ -1 & \lambda_1^i & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^i & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^i & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3^i & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^i \end{pmatrix}, A_4^i = \begin{pmatrix} \lambda_1^i & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^i & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^i & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^i & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3^i & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^i \end{pmatrix}$$

Here, $i \in \{0, \ldots, 3\}$ corresponds to the r in **theorem** 2.4.1, which gives us the following values of $\lambda_j^i, j \in \{1, \ldots, 3\}$ in terms of a,

1. ***i*=0:** $\lambda_1^0 = -\frac{a}{4}, \lambda_2^0 = -\frac{a}{4}, \lambda_3^0 = -\frac{a}{4}$.
2. ***i*=1:** $\lambda_1^1 = -\frac{a}{2}, \lambda_2^1 = 0, \lambda_3^1 = 0$.
3. ***i*=2:** $\lambda_1^2 = -\frac{a}{4}, \lambda_2^2 = -\frac{a}{4}, \lambda_3^2 = \frac{a}{4}$.
4. ***i*=3:** $\lambda_1^3 = 0, \lambda_2^3 = 0, \lambda_3^3 = 0$.

Now, for $i \in \{0, 2, 3\}$, A_j^i yields $\mathfrak{b}_j^i$, here $j \in \{1, \ldots, 4\}$ and A_j^1 yields $\mathfrak{b}_j^i$, here $j \in \{1, \ldots, 7\}$.
■

Remark 2.4.1. *There are seven classes of real dimension 8 Almost abelian Lie algebras which admit astheno-Kähler as well as balanced structures given as,*

$\mathfrak{d}_1 = (f^{28}, -f^{18}, af^{48}, -af^{38}, bf^{68}, -bf^{58}, 0, 0)$,
$\mathfrak{d}_2 = (f^{28}, -f^{18}, af^{48}, -af^{38}, 0, 0, 0, 0)$,
$\mathfrak{d}_3 = (f^{28}, -f^{18}0, 0, 0, 0, 0, 0)$,
$\mathfrak{d}_4 = (af^{18}, f^{38}, -f^{28}, pf^{58}, -pf^{48}, qf^{78}, -qf^{68}, 0)$,
$\mathfrak{d}_5 = (af^{18}, f^{38}, -f^{28}, pf^{58}, -pf^{48}, 0, 0, 0)$,
$\mathfrak{d}_6 = (af^{18}, f^{38}, -f^{28}, v_1 f^{18}, v_2 f^{18}, v_3 f^{18}, v_4 f^{18}, 0)$,
$\mathfrak{d}_7 = (f^{18}, 0, 0, 0, 0, 0, 0, 0)$.

Remark 2.4.2. *Interestingly, none of the Lie algebras in* **theorem** *2.4.2 are nilpotent; it is, in fact, a general fact that an almost abelian Lie algebra admitting an astheno-Kähler structure is not nilpotent. This is a fairly trivial exercise using* **theorem** *2.4.1.*

Chapter 3

Geometric Flows

3.1 Geometric flows for almost Hermitian Manifolds

Geometric flows are evolution equations for geometric structures, typically defined on smooth manifolds, and are formulated as partial differential equations (PDEs) that describe the deformation of these structures over time. In our subsequent discussions, we will work with the following definition of this concept,

Definition 3.1.1. *Let M be a smooth manifold with an evolution equation on it given as,*

$$\begin{cases} \dfrac{\partial}{\partial t}\omega = -2p, \\ \dfrac{\partial}{\partial t}g = -2q, \end{cases} \tag{3.1}$$

for a one-parameter family of almost-Hermitian structures $(\omega(t), g(t), J(t))$ on M, such that here $p = p(\omega, g) \in \Omega^2 M$ and $q = q(\omega.g) \in \Gamma(\mathcal{S}^2 M)$ are invariant under diffeomorphisms that is,

$$p(\psi^*\omega, \psi^* g) = \psi^* p(\omega, g), \quad q(\psi^*\omega, \psi^* g) = \psi^* q(\omega, g), \quad \forall \psi \in \mathit{Diff}(M)$$

*then we call (3.1) a (p, q)**-flow**.*

Usually, the p and q in a (p, q)-flow will be some type of curvature tensors associated with each (p,q)-flow.

By a one-parameter family of almost-Hermitian structures we mean a smooth map, (ω, g, J) : $I \times M \to \Lambda^2 M \times \mathcal{S}^2 M \times (T^*M \otimes TM)$ such that for all $t \in I$, $(\omega(t), g(t), J(t))$ is an almost-Hermitian structure on M.

And, $\frac{\partial}{\partial t}\omega : I \times M \to \Lambda^2 M$ is defined pointwise on M as for $p \in M$ we have that, $\omega(p) : \mathbb{R} \to \Lambda^2_p M$,and here $\Lambda^2_p M$ is a vector space so, $\frac{\partial}{\partial}\omega(p)(t)$ is well-defined. Similarly, for g.

Definition 3.1.2. ***Solution of a*** (p,q)***-flow*** *is said to be a one-parameter family of almost-Hermitian structures* $(\omega(t), g(t), J(t))$ *on* M*, such that the pair* $(\omega(t), g(t))$ *solves* (3.1) *and* $\forall t, J(t)$ *is the unique almost complex structure on* M *with respect to the metric* g*.*

We define $P, Q \in \mathrm{End}(TM)$ as follows,

$$p = \omega(P\cdot, \cdot), \quad q = g(Q\cdot, \cdot) \tag{3.2}$$

Because of the non-degeneracy of ω and g we have that, P and Q are uniquely defined.

Theorem 3.1.1 ([14])**.** *Let* M *be a smooth manifold with a* (p,q)*-flow defined on it. If a non-degenerate pair* $(\omega(t), g(t))$ *solves* (3.1) *and* $J(0)$ *is an almost complex structure on* M*, then there exists a* $J(t), \forall t$ *such that* $(\omega(t), g(t), J(t))$ *is a solution to the* (p,q)*-flow if and only if,*

$$q^c = p^c(\cdot, J(t)\cdot), \quad \forall t. \tag{3.3}$$

Proof. For ease of bookkeeping in this proof, we will denote $\omega(t), g(t), J(t)$ with just ω, g, J respectively. Now, we have that,

$$p = \omega(P\cdot, \cdot) = g(JP\cdot, \cdot), \quad q = (Q\cdot, \cdot), \tag{3.4}$$

Now because p is a 2-form we have that, $P^{t_\omega} = P(\text{i. e. } P^t = -JPJ)$ and because g is symmetric we get that, $Q^t = Q$. Using the formula $\omega = g(J\cdot, \cdot)$ we arrive at the following,

$$\frac{\partial}{\partial t}\omega = \frac{\partial}{\partial t} g(J\cdot, \cdot)$$

$$\implies \qquad \tfrac{\partial}{\partial t}\omega = \left(\tfrac{\partial}{\partial t} g\right)(J\cdot, \cdot) + g\left(\tfrac{\partial}{\partial t} J., .\right)$$

$$\implies \qquad -2p = -2q(J\cdot,\cdot) + g\left(\tfrac{\partial}{\partial t}J.,.\right)$$

$$\implies \qquad 2q(J\cdot,\cdot) - 2p = g\left(\tfrac{\partial}{\partial t}J.,.\right)$$

$$\implies \qquad 2g(QJ\cdot,\cdot) - 2g(JP\cdot,\cdot) = g\left(\tfrac{\partial}{\partial t}J\cdot,\cdot\right)$$

$$\implies \qquad \tfrac{\partial}{\partial t}J = -2(JP - QJ) = -2R$$

Here, $R \coloneqq JP - QJ$

Now,

$$\begin{aligned}
\frac{\partial}{\partial t}J^2 &= \frac{\partial J}{\partial t}J + J\frac{\partial J}{\partial t} \\
&= -2(RJ + JR) \\
&= -2(JPJ - QJ^2 + J^2P - JQJ) \\
&= -2(P - 2P^c - QJ^2 + J^2P - Q + 2Q^c) \\
&= 4(P^c - Q^c) - 2(P - Q) - 2(J^2P - QJ^2)
\end{aligned}$$

Lemma 3.1.2. *If a non-degenerate pair* $(\omega(t), g(t))$ *solves* (3.1) *such that* $J(t) \in \mathrm{End}(TM)$ *is defined uniquely* $\forall t$ *as,* $\omega(t) \coloneqq g(t)(J(t)\cdot,\cdot)$ *then we have the following;*

$$PJ(t) = J(t)P, \quad \forall t. \tag{3.5}$$

Proof. (*of Lemma.*) We have that $\forall X, Y \in \Gamma(TM)$,

$$\omega(X, Y) = g(JX, Y)$$

$$\implies \qquad -\omega(Y, X) = g(J^tY, X)$$

$$\implies \qquad -g(JY, X) = g(J^tY, X)$$

$$\implies \qquad J^t = -J$$

Because of the non-degeneracy of ω, g, we have that J is an invertible linear transformation. Now,

$$g(J\cdot, J\cdot) = g(J^tJ\cdot,\cdot) = -g(J^2\cdot,\cdot)$$

$$\implies \qquad g(J\cdot, J\cdot) + g(J^2\cdot,\cdot) = 0$$

$$\implies \qquad \tfrac{\partial}{\partial t}(g(J\cdot, J\cdot) + g(J^2\cdot,\cdot)) = 0$$

$$\implies \quad \left(\tfrac{\partial}{\partial t}g\right)(J^2\cdot,\cdot) + g\left(\tfrac{\partial}{\partial t}J^2\cdot,\cdot\right) + \left(\tfrac{\partial}{\partial t}g\right)(J\cdot,J\cdot) + 2g\left(\tfrac{\partial}{\partial t}J\cdot,J\cdot\right) = 0$$

$$\implies \quad \left(\tfrac{\partial}{\partial t}g\right)(J^2\cdot,\cdot) + g\left(\tfrac{\partial}{\partial t}J^2\cdot,\cdot\right) + \left(\tfrac{\partial}{\partial t}g\right)(J\cdot,J\cdot) - 2g\left(J\tfrac{\partial}{\partial t}J\cdot,\cdot\right) = 0$$

$$\implies \quad -2QJ^2 - 2(RJ + JR) + 2JQJ + 4JR = 0$$

$$\implies \quad -QJ^2 + (JR - RJ) + JQJ = 0$$

$$\implies \quad -QJ^2 + (J(JP - QJ) - (JP - QJ)J) + JQJ = 0$$

$$\implies \quad -QJ^2 + (J^2P - JQJ) - (JPJ - QJ^2) + JQJ = 0$$

$$\implies \quad J^2P - JPJ = 0$$

$$\implies \quad JP = PJ$$

■

Applying, **lemma** 3.1.2 we get that,

$$\begin{aligned}\frac{\partial}{\partial t}J^2 &= 4(P^c - Q^c) - 2(P - Q) - 2(J^2P - QJ^2)\\ &= 4(P^c - Q^c) - 2(P - Q) - 2(PJ^2 - QJ^2)\\ &= 4(P^c - Q^c) - 2(P - Q)(J^2 + I)\end{aligned}$$

So, if $P^c = Q^c$ then $\frac{\partial}{\partial t}J^2 = -4(P-Q)(J^2+I)$ and so, $J(t)^2 = -I$ by uniqueness of solution starting at $J(0)^2 = -I$. And the converse also follows because $\frac{\partial}{\partial t}J^2 = 0 = -2(RJ + JR) = 4(P^c - Q^c)$. And it is trivial to see that $P^c = Q^c$ is equivalent to $q^c = p^c(\cdot, J(t)\cdot), \forall t$. ■

Because of **theorem** 3.1.1, it is fair to assume from this point on that for all of our (p, q)-flows and any arbitrary almost complex structure J, we have the following,

$$q^c = p^c(\cdot, J\cdot) \tag{3.6}$$

.Or equivalently,

$$P^c = Q^c \tag{3.7}$$

And, we also get the following equation,

$$\frac{\partial}{\partial t}J = -2J(P^{ac} + Q^{ac}) \tag{3.8}$$

3.2 Geometric Flows for Lie groups

Let's now consider (p, q)-flows on a special type of manifolds in particular Lie groups. The reason to do this is that they provide us with a good framework to understand how they behave more explicitly, which doesn't just help inspire new conjectures which may end up being true, but can also provide us with a plethora of counter-examples to some others.

Let G be a connected and simply connected Lie group with it's Lie algebra given by $\mathfrak{g}$ and take a (p, q)-flow on it. Now because of the diffeomorphism invariance of p and q we get that they are left invariant i. e. $p \in \Lambda^2\mathfrak{g}^*$ and $q \in \mathcal{S}^2\mathfrak{g}^*$, so the (p, q)-flow can be thought of as an ODE on the Lie algebra of G. This gives us the following definition,

Definition 3.2.1. *If $\mathfrak{g}$ is a Lie algebra, $p \in \Lambda^2\mathfrak{g}^*$ and $q \in \mathcal{S}^2\mathfrak{g}^*$. Then the ODE given by,*

$$\begin{cases} \dfrac{d}{dt}\omega = -2p \\ \dfrac{d}{dt}g = -2q \end{cases} \tag{3.9}$$

*is called the (p, q)**-flow on Lie algebra** $\mathfrak{g}$ and a **solution of** (p, q)**-flow** is defined to be ordered set $(\omega(t), g(t), J(t))$ such it is an almost Hermitian structure on $\mathfrak{g}$ and the pair $(\omega(t), g(t))$ solves the ODE (3.9).*

It is obvious that, the solutions of the (p, q)-flow on $\mathfrak{g}$ correspond with the left-invariant solutions of (p, q)-flow on G. And ODE theory guarantees local existence and uniqueness of left invariant solutions.

For one-parameter family $(\omega(t), g(t)$ of almost Hermitian structure on $\mathfrak{g}$ with initial condition given as, $\omega(0) = \omega_0, g(0) = g_0$, and define a pair $(\Omega(t), G(t)) \in GL(\mathfrak{g}) \times GL(\mathfrak{g})$ as follows,

$$\begin{cases} \omega(t) \coloneqq \omega_0(\Omega(t)\cdot, \cdot), \quad \Omega(0) = I \\ \\ g(t) \coloneqq g_0(G(t)\cdot, \cdot), \quad G(0) = I \end{cases} \tag{3.10}$$

And hence, the (p, q)-flow defined as (3.9) for (ω, g) is equivalent to the following ODE,

$$\begin{cases} \frac{d}{dt}\Omega = -2\Omega P, \\ \frac{d}{dt}G = -2GQ, \end{cases} \tag{3.11}$$

It follows trivially that,

$$J_0 G = GJ. \tag{3.12}$$

Proposition 3.2.1 ([13]). *Let (ω_0, g_0, J_0) be an almost complex structure on a Lie algebra $\mathfrak{g}$, then for any almost Hermitian structure (ω, g, J) on $\mathfrak{g}$, $\exists h \in GL(\mathfrak{g})$, such that*

$$(\omega, g) = (h^{-1} \cdot \omega_0, h^{-1} \cdot g_0) = (\omega_0(h., h.), g_0(h., h.)). \tag{3.13}$$

And, any $h \in GL(\mathfrak{g})$ which satisfies the above equation is unique up to left-multiplication by the unitary group, which is given as,

$$U(n) \coloneqq Sp(\omega_0) \bigcap O(g_0) = \{\psi \in GL(\mathfrak{g}) : \omega_0 = \omega_0(\psi., \psi.), \quad g_0 = g_0(\psi., \psi.)\}.$$

Proof. Let $\mathfrak{g}$ be of dimension $2n$. Consider the following two bases on $\mathfrak{g}$, $\{e_1^0, \dots, e_{2n}^0\}$ and $\{e_1, \dots, e_{2n}\}$ such that they are both adapted unitary basis with respect to (ω_0, g_0) and (ω, g) respectively.

Define $h : \mathfrak{g} \to \mathfrak{g}$ on the basis as, $h(e_i) \coloneqq e_i^0, \forall i \in \{1, \dots, 2n\}$ and extend linearly on $\mathfrak{g}$, clearly $h \in GL(\mathfrak{g})$.

And we have for $i, j \in \{1, \dots, 2n\}$ that,

$$\begin{aligned} h^{-1} \cdot g_0(e_i, e_j) &= g_0(he_i, he_j) \\ &= g_0(e_i^0, e_j^0) \\ &= \delta_{i,j} \\ &= g(e_i, e_j). \end{aligned}$$

This implies, $h^{-1} \cdot g_0$ agrees with g on the basis elements, and because both are bilinear, we have that $g = h^{-1} \cdot g_0$. Again for $i \in \{1, \dots, n\}$ we have,

$$\begin{aligned} h^{-1}J_0h(e_i) &= h^{-1}J_0(e_i^0) \\ &= h^{-1}e^0_{2n+1-i} \\ &= e_{2n+1-i} = Je_i \end{aligned}$$

So, $h^{-1}J_0h$ agrees with J on $\{e_1, \dots, e_n\}$ similarly it can be shown that they agree on $\{e_{n+1}, \dots, e_{2n}\}$. Since they agree on the basis and are both linear transformations, we have that,

$$J = h^{-1}J_0h \tag{3.14}$$

$$\begin{aligned} h^{-1} \cdot \omega_0(\cdot, \cdot) &= \omega_0(h\cdot, h\cdot) \\ &= g_0(J_0h\cdot, h\cdot) \\ &= g_0(hJ\cdot, h\cdot) \\ &= g(J\cdot, \cdot) \\ &= \omega(\cdot, \cdot). \end{aligned}$$

Here, we have used that $hJ = J_0h$, this can be confirmed by looking at how both of these linear transformations behave on the basis $\{e_1, \dots, e_{2n}\}$. Now, let's say $\exists h_1, h_2 \in GL(\mathfrak{g})$ which satisfy (3.14) then we have that,

$$(\omega, g) = (h_1^{-1} \cdot \omega_0, h_1^{-1} \cdot g_0)$$

$$\implies \quad (h_1 \cdot \omega, h_1 \cdot g) = (\omega_0, g_0)$$

$$\implies \quad (h_1 \cdot h_2^{-1} \cdot \omega_0, h_1 \cdot h_2^{-1} g_0) = (\omega_0, g_0)$$

$$\implies \quad (\omega_0(h_1^{-1}h_2., h_1^{-1}h_2.), g_0(h_1^{-1}h_2., h_1^{-1}h_2.)) = (\omega_0, g_0)$$

$$h_1h_2^{-1} \in U(n)$$

And $h_1 = (h_1h_2^{-1})h_2$. ∎

So, for a solution $(\omega(t), g(t), J)$ of (p, q)-flow of $\mathfrak{g}$ starting at (ω_0, g_0, J_0) there is some $h = h(t) \in GL(\mathfrak{g})$ such that,

$$(\omega(t), g(t)) = (h^{-1} \cdot \omega_0, h^{-1} \cdot g_0)$$

which is trivially the same as,

$$\Omega(t) = h^{t_{\omega_0}} h = -J_0 h^t J_0 h, \quad G(t) = h^t h \tag{3.15}$$

Here by A^t we mean transpose of A with respect to g_0, $\forall A \in \mathrm{End}(\mathfrak{g})$. Note that we can take $h(t)$ such that it is differentiable on t, and hence from here on we will assume so.

If we have a good understanding of the evolution of $h(t) \in GL(\mathfrak{g})$, then it might prove to be a helpful tool because the whole (p,q)-flow solution is determined by $h(t)$. The next result tells more explicitly how h behaves,

This same technique has been applied to investigate the Ricci flow on nilmanifolds [12] and on homogeneous manifolds [14]. On the other hand, N. Enrietti, A. Fino, and L. Vezzoni demonstrate in [5] that solutions to these flows persist for all times, thus guaranteeing long-term existence for any such solution.Let (G_1, ω_1, g_1) and (G_2, ω_2, g_2) be two almost Hermitian Lie groups, we say that (G_1, ω_1, g_1) and (G_2, ω_2, g_2) are ***isomorphic almost Hermitian Lie groups*** *if there exists a Lie group isomorphism $\Phi : G_1 \to G_2$, such that $\omega_2 = \phi \cdot \omega_1$ and $g_2 = \phi \cdot g_1$. And such a Φ is called an* ***almost Hermitian Lie group isomorphism****.*

Definition 3.2.3. *Let $(\mathfrak{g}_1, \omega_1, g_1)$ and $(\mathfrak{g}_2, \omega_2, g_2)$ be two almost Hermitian Lie algebras, we say that $(\mathfrak{g}_1, \omega_1, g_1)$ and $(\mathfrak{g}_2, \omega_2, g_2)$ are* ***isomorphic almost Hermitian Lie algebras*** *if there exists a Lie algebra isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$, such that $\omega_2 = \phi \cdot \omega_1$ and $g_2 = \phi \cdot g_1$. And such a ϕ is called an* ***almost Hermitian Lie algebra isomorphism****.*

It is clear that, if two almost Hermitian Lie algebras are isomorphic then the left invariant Hermitian structure on their corresponding connected simply connected Lie groups are also isomorphic. And if the Φ denotes the almost Hermitian Lie group isomorphism then the almost Hermitian Lie algebra isomorphism is given by Φ_*. So, we have the following result,

Proposition 3.2.3 ([13])**.** *Let $(\mathfrak{g}_1, \omega_1, g_1)$ and $(\mathfrak{g}_2, \omega_2, g_2)$ be equivalent almost Hermitian Lie algebras with the isomorphism given by $\phi : (\mathfrak{g}_1, \omega_1, g_1) \to (\mathfrak{g}_2, \omega_2, g_2)$ and let there be a (p,q)-flow defined on $\mathfrak{g}_1$. Then we have that,*

$$P(\omega_2, g_2) = \phi P(\omega_1, g_1)\phi^{-1}, \quad Q(\omega_2, g_2) = \phi Q(\omega_1, g_1)\phi^{-1} \tag{3.16}$$

Proof. *In this proof we use the notation,* $p_1 := p(\omega_1, g_1), p_2 := p(\omega_2, g_2), P_1 := P(\omega_1, g_1)$ *and* $P_2 := P(\omega_2, g_2)$. *Because* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are isomorphic we can say without loss of generality that their underlying vector is the same. So, the same* (p, q)*-flow is defined on* $\mathfrak{g}_2$. *Now, the diffeomorphism invariance condition on* p *and* q *gives us,*

$$\Phi^* p(\omega_2, g_2) = p(\Phi^*\omega_2, \Phi^* g_2)$$

$$\implies \qquad \Phi^* p(\omega_2, g_2) = p(\omega_1, g_1)$$

$$\implies \qquad p_2(\phi., \phi.) = p_1(\cdot, \cdot)$$

$$\implies \qquad \omega_2(P_2\phi\cdot, \phi\cdot) = \omega_1(P_1\cdot, \cdot)$$

$$\implies \qquad \omega_2(P_2\phi\cdot, \phi\cdot) = (\Phi^*\omega_2)(P_1\cdot, \cdot)$$

$$\implies \qquad \omega_2(P_2\phi\cdot, \phi\cdot) = \omega_2(\phi P_1\cdot, \cdot)$$

$$\implies \qquad P_2\phi = \phi P_1$$

$$\implies \qquad P(\omega_2, g_2) = \phi P(\omega_1, g_1)\phi^{-1}$$

And, similarly we have that $Q(\omega_2, g_2) = \phi Q(\omega_1, g_1)\phi^{-1}$. ■

And the same holds for P^c, P^{ac}, Q^c *and* Q^{ac} *because we have that* $J_2 = \phi J_1 \phi^{-1}$. *Now, if we have that* $(\omega(t), g(t))$ *is a* (p, q)*-flow starting at* (ω_0, g_0)*, then by* **??** *we know that* $h = h(t) \in GL(\mathfrak{g})$ *defined there satisfies,*

$$h : ([., .], \omega, g) \to (\mu, \omega_0, g_0), \quad \textit{where} \quad \mu = \mu(t) := h \cdot [., .] = h[h^{-1}., h^{-1}.],$$

is an isomorphism of almost-Hermitian structures for all t. *Here* $[., .]$ *is used to denote the Lie bracket of the Lie algebra* $\mathfrak{g}$ *and so* μ *defines a new Lie algebra with same underlying vector space* $\mathfrak{g}$*, which is isomorphic to* $(\mathfrak{g}, [., .]), \forall t$. *We use the notation* $P_\mu := P(\omega_0, g_0)$ *and* $Q_\mu := Q(\omega_0, g_0)$*, which because of* **proposition** *3.2.3 satisfies the following,*

$$P_\mu = hPh^{-1}, \quad Q_\mu^{ac} = hQ^{ac}h^{-1}, \quad \forall t, \tag{3.17}$$

Here as always, $P = P(\omega, g)$ *and* $Q = (\omega, g)$ *for all* t.

It is a trivial exercise to check that, $\frac{d}{dt}\mu = -\delta_\mu(h' h^{-1})$ *if* $\mu = h \cdot [., .]$*, here* $\delta_\mu : End(\mathfrak{g}) \to$

$\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ *is given as,*

$$\delta_\mu(B) \coloneqq \mu(B\cdot,\cdot) + \mu(\cdot, B\cdot) - B\mu(\cdot,\cdot), \quad \forall B \in End(\mathfrak{g}) \tag{3.18}$$

Observe that

$$\delta_\mu(A) \ = \ -\pi(A)\,\mu \ = \ -\frac{d}{dt}\Big|_{t=0}\big(e^{tA}\cdot\mu\big)\,.$$

Using **??** *and* (3.17)*, one deduces that the one-parameter family of Lie brackets* $\mu(t)$ *satisfies the following ODE:*

$$\frac{d}{dt}\,\mu \ = \ \delta_\mu\left(P_\mu + Q_\mu^{ac}\right)\,\mu. \tag{3.19}$$

Definition 3.2.4. *We shall refer to* (3.19) *as the* (p,q)***-bracket flow****.*

A key question is whether analyzing its qualitative behavior and dynamics can provide additional insights into the geometry of certain curvature flows for almost-Hermitian structures on Lie groups. By appealing to standard ODE arguments, one may view it as analogous to the $\mathrm{GL}(n)$*-orbit of* μ*. From this standpoint, one concludes that* $\mu(t) \in \mathrm{GL}(g(t))$*for all* t*.*

Our next theorem shows that any (p,q)*-flow is equivalent in a precise way to it's corresponding* (p,q)*-bracket flow.*

Consider a given simply connected almost Hermitian Lie group (G,ω_0,g_0) *with it's Lie algebra given by,* $\mathfrak{g}$*.Now, let's consider the following two one-parameter families of almost Hermitian Lie groups,*

$$(G,\omega(t),g(t)), \quad (G_{\mu(t)},\omega_0,g_0), \tag{3.20}$$

Here, (ω,g) *is the solution to the* (p,q)*-flow* (3.9) *with initial condition* (ω_0,g_0) *and* $\mu(t)$ *is the* (p,q)*-bracket flow starting at the Lie bracket* $[.,.]$ *of* $\mathfrak{g}$*. And* G_μ *is the simply connected Lie group with Lie algebra* $(\mathfrak{g},\mu)$*.*

Theorem 3.2.4 ([14])**.** *Let* $(G,\omega(t),g(t))$ *be an almost Hermitian Lie group with a* (p,q)*-flow defined on it. Then, there exist time-dependent almost Hermitian Lie group isomorphisms,* $H(t) : (G,\omega(t),g(t)) \to (G_{\mu(t)},\omega_0,g_0)$ *which can be chosen such that their derivatives at the identity, also denoted by* $h = h(t)$*, solve any of the following ODEs:*

(i) $\frac{d}{dt}h = -h(P+Q^{ac}) = -h(P^{ac}+Q), \quad h(0) = I.$

(ii) $\frac{d}{dt}h = -(P_\mu + Q_\mu^{ac})h = -(P_\mu^{ac} + Q_\mu)h, \quad h(0) = I$

And we have that the following holds for all t,

1. $(\omega(t), g(t)) = (h^{-1} \cdot \omega_0, h^{-1} \cdot g_0)$
2. $\mu(t) = h \cdot [.,.]$.

Here, $\mu(t)$ is solution of (p,q)-bracket flow and $(\omega(t), g(t))$ is solution of (p,q)-flow.

Proof. *By using the correspondence between the homomorphism of Lie algebras and the homomorphism of simply connected Lie groups corresponding to the Lie algebras, the theorem becomes equivalent to finding time-dependent almost Hermitian Lie algebra isomorphisms, $h(t) : (\mathfrak{g}, \omega(t), g(t)) \to (\mathfrak{g}_{\mu(t)}, \omega_0, g_0)$, such that $h(t)$ is a solution to any of the ODE system (i) or (ii) and it also satisfies 1 and 2 .*

Using **proposition** *3.2.3 and (3.17) we have that, if h solves (i) then it solves (ii) and satisfies 1 and 2.*

Now, let's say $h(t)$ solves (ii), then $h(t)$ is defined on the same time interval as $\mu(t)$. We have that $\mu(t) = h(t) \cdot [.,.]$ for all t as both of them solve the same ODE (3.19) and both begin at $[.,.]$ and hence 2 holds. Thus h determines an isomorphism between $([.,.], \tilde{\omega}(t) := h^{-1} \cdot \omega_0, \tilde{g}(t) := h^{-1} \cdot g_0)$ and (μ, ω_0, g_0). This implies that corresponding curvature tensors satisfy $P_\mu = h\tilde{P}h^{-1}$ and $Q_\mu^{ac} = h\tilde{Q^{ac}}h^{-1}$, and hence we have $h' = -h(\tilde{P} + \tilde{Q^{ac}})$, from which it follows that, $(om\tilde{e}ga(t), \tilde{g}(t))$ is also a (p,q)-flow solution starting at (ω_0, g_0) by **proposition** *3.2.3. By uniqueness of the solution, we have that h solves 1 and also satisfies (i).* ∎

Remark 3.2.1. *The following are some useful facts which are a direct consequence of this theorem:*

1. *The (p,q)-flow $(\omega(t), g(t))$ and the (p,q)-bracket flow $\mu(t)$ differ only by pullback by time-dependent diffeomorphisms.*
2. *They are equivalent in the following sense: each one can be obtained from the other by solving the corresponding ODE in (i) or (ii) and applying either 1 of 2, accordingly.*
3. *The maximal interval of time where a solution exists is, therefore, the same for both flows.*

4. At each time t, the almost-Hermitian manifolds in (3.20) are equivalent so that the behavior of any class of curvature and of any other invariant along $(\omega(t), g(t))$ can be studied along the (p, q)-bracket flow.

Now, we will use the machinery of bracket flows we have built over this section to study a particular example of a parabolic geometric flow that preserves the balanced condition called the Balanced Flow.

3.3 Balanced Flow

In [3], the authors introduced a parabolic flow for Hermitian metrics on a complex manifold, preserving the balanced condition of the initial data: in terms of the $(n-1)$-st power of the fundamental form, and it is defined as follows,

Definition 3.3.1. *Let (M, ω, g) be an almost-Hermitian manifold then the operator defined below is called* ***Bott-Chern Laplacian*** *and is denoted by Δ_{BC},*

$$\Delta_{BC} \coloneqq \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial. \tag{3.21}$$

The Bott-Chern Laplacian is an order 4 elliptic operator, whose kernel describes the Bott-Chern cohomology of a Hermitian manifold. The Bott-Chern cohomology (denoted as H_{BC}) is defined as,

$$H_{BC} = \frac{ker\ d}{Im(\partial\bar{\partial})} \tag{3.22}$$

Definition 3.3.2. *Let M be a smooth manifold then the following evolution equation is given by,*

$$\frac{\partial}{\partial t}\phi(t) = i\partial\overline{\partial} *_t (\rho_t^C \wedge *_t\phi(t)) + \Delta_{BC}\phi(t), \quad \phi(0) = \phi_0 = *_0\omega_0. \tag{3.23}$$

is called the ***Balanced Flow****, where $(\omega(t), g(t))$ is an almost Hermitian structure on M, $*_t$,Δ_{BC} and, ρ_t^C denotes the Hodge star, Bott-Chern Laplacian and, Chern-Ricci form respectively with respect to $(\omega(t), g(t))$. Also $\phi(t) \coloneqq *_t\omega(t)$.*

We know that $*_t : \Omega^{1,1}M \to \Omega^{n-1,n-1}M$ *(where* $2n$ *denotes the dimension of* M*) is an isomorphism, so an evolution equation of* $\phi(t)$*, gives us an evolution equation of* $\omega(t)$*. In fact, we can say that* (3.23) *is equivalent to the following,*

$$\frac{\partial}{\partial t}\omega(t) = (n-2)!\,\iota_{\omega(t)^{n-2}}(i\partial\bar{\partial} *_t (\rho_t^C \wedge \omega(t))) + \frac{1}{n-1}\iota_{\omega(t)^{n-2}}\Delta_{BC}\omega(t)^{n-1}, \quad \omega(0) = \omega_0. \quad (3.24)$$

Theorem 3.3.1 ([3])**.** *The flow* (3.23) *admits a unique solution in Bott-Chern class* $[\phi_0]$ *defined in a maximal interval* $[0, \epsilon)$*. Additionally, if the initial structure is Kähler then* (3.23) *reduces to the Calabi flow.* ∎

This trivially gives us that the flow preserves the balanced condition.

Now, let's consider this flow on a Lie algebra $\mathfrak{g}$ *of dimension* $2n$*.*

We will consider this to be a (p, q)*-flow even though we only have the evolution equation of* $\omega(t)$*, we still need to determine how to define the flow of* $g(t)$*. We will demand that the flow* g *is defined such that the initial complex structure denoted by* J *remains constant (i. e.* $\frac{\partial}{\partial t}J = 0$*) then using* (3.8) *we have,*

$$\frac{\partial}{\partial t}J = -2J(P^{ac} + Q^{ac})$$

$$\implies \quad 0 = -2J(P^{ac} + Q^{ac})$$

$$\implies \quad Q^{ac} = -P^{ac}$$

So we define the evolution of $g(t)$*, such that* $Q^{ac} = -P^{ac}$*, to define the flow of* $g(t)$ *uniquely we need to define,* $Q = Q^c + Q^{ac}$*. That is we need to define* Q^c *as well as* Q^{ac}*, here we have only defined* Q^{ac}*, but because of* **theorem** *3.2.4 we have that defining* P *and* Q^{ac} *is enough to determine the* (p, q)*-flow uniquely.*

Now, let's consider $(\mathfrak{g}, \omega, J)$ *to be a Hermitian almost Abelian Lie algebra with adapted unitary basis* $\{e_1, \ldots, e_{2n}\}$*. We want to know what* ρ^C *looks like explicitly in this case in order to understand the balanced flow. The following result, which was proven in [15], gives us that*

$$\rho^C = -a\left(a + \frac{1}{2}\operatorname{tr} A\right) e^1 \wedge e^{2n}. \quad (3.25)$$

Then by **theorem** *2.3.4 we have that if* (ω, J) *is a balanced structure then,*

$$\rho^C = -a^2 e^1 \wedge e^{2n}. \tag{3.26}$$

Using the notation of the previous section we have using **theorem** *3.2.4,*

$$\begin{aligned}\frac{d}{dt}\mu &= -\delta_\mu(P_\mu + Q_\mu^{ac})\mu \\ &= -\delta_\mu(P_\mu^c + P_\mu^{ac} + Q_\mu^{ac})\mu \\ &= -\delta_\mu(P^c)\mu \end{aligned} \tag{3.27}$$

We use the notation $\mu(a, v, A)$ *to denote the* $2n$ *dimensional almost Abelian Lie algebra with algebraic data* (a, v, A)*, we have the following result,*

Proposition 3.3.2 ([7]). *For a balanced almost Abelian Lie algebra* $(\mu(a, 0, A), J, \langle .,. \rangle)$, $\operatorname{tr} A = 0$*, the endomorphism* $P_{\mu(a,0,A)}$ *is given by,*

$$P_{\mu(a,0,A)} = \begin{pmatrix} l & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & l \end{pmatrix} \tag{3.28}$$

in terms of the fixed splitting $\mathfrak{g} = J\mathfrak{l} \oplus \mathfrak{n}_1 \oplus \mathfrak{l}$*, here* $l = l(a, A)$ *is a homogeneous fourth order polynomial in* a *and the entries of* A *and* $L = L(a, A)$ *are a symmetric endomorphism of* $\mathfrak{n}_1$ *commuting with* $J_1 = J|_{\mathfrak{n}_1}$ *whose entries are fourth order polynomials in* a *and* A*.*

Proof. *We will use the notation,* $p_\mu := h \cdot p$ *and we will denote* ω_μ *with* ω*, now the balanced condition gives us that* $d\omega^{n-1} = 0 \implies \partial\omega = 0, \bar{\partial}\omega = 0$ *so we get that,*

$$\begin{aligned}\Delta_{BC}\omega^{n-1} &= (\partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial)\omega \\ &= \partial\bar{\partial}\bar{\partial}^*\partial^*\omega^{n-1} \\ &= \partial\bar{\partial}(-*\partial*)(-*\bar{\partial}*)\omega^{n-1} \\ &= (n-1)!\,\partial\bar{\partial}*\partial\bar{\partial}\omega \\ &= (n-1)!\left(i^{4n-2}\frac{dJd}{2}\right)*\left(i^6\frac{dJd}{2}\right)\omega \\ &= \frac{(n-1)!}{4}dJd*dJd\omega \end{aligned}$$

Now we use that $\omega \in \mathfrak{l}^* \wedge J\mathfrak{l}^* \oplus \Lambda^2\mathfrak{n}_1^*$, $d\mathfrak{g}^* \subset \mathfrak{l}^* \wedge J\mathfrak{l}^* \oplus \mathfrak{l}^* \wedge \mathfrak{n}_1^*$ (which trivially follows from $v = 0$) and, $*(\mathfrak{l}^* \wedge J\mathfrak{l}^* \wedge \Lambda^2\mathfrak{n}_1^*) \subset \Lambda^{2n-4}\mathfrak{n}_1^*$, gives us that $\Delta_{BC} \in \mathfrak{l}^* \wedge J\mathfrak{l}^* \wedge \Lambda^{2n-4}\mathfrak{n}_1^*$. Similarly and using (3.26) we have that $i\partial\bar{\partial} * (\rho^C \wedge \omega)$ also lies in $\mathfrak{l}^* \wedge J\mathfrak{l}^* \wedge \Lambda^{2n-4}\mathfrak{n}_1^*$. Using the fact $\omega^{n-2} \in \mathfrak{l}^* \wedge J\mathfrak{l}^* \wedge \Lambda^{2n-6}\mathfrak{n}_1^* \oplus \Lambda^{2n-4}\mathfrak{n}_1^*$, by doing a contraction with ω^{n-2} we get that, $q_\mu \in \mathfrak{l}^* \wedge J\mathfrak{l}^* \oplus \Lambda^2\mathfrak{n}_1^*$. With respect to an adapted unitary basis $\{e_1, \dots, e_{2n}\}$ we have that,

$$p_\mu = -2le^1 \wedge e^{2n} - 2\omega(L\cdot, \cdot)$$

For some $l = l(a, A) \in \mathbb{R}$ and $L = L(a, A) \in \mathfrak{gl}(\mathfrak{n}_1)$ which is symmetric and commutes with J_1, because ω as well as q_μ are both $(1,1)$-forms with respect to J.

And, the fact that l and entries of L are homogeneous fourth order polynomials follows from the fact that the equation of balanced flow is a fourth order PDE. ∎

This proposition also gives us that $P_\mu^c = P_\mu$.

Let's now consider the particular case of $n = 3$, so we have that $(\mathfrak{g}, J, \omega)$ is a 6-dimensional Hermitian almost Abelian Lie algebra, with an adapted unitary basis $\{e_!, \dots, e_6\}$.

Note that A being traceless and commuting with J_1 gives us that A is of the following form,

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & -A_{11} & A_{23} & A_{24} \\ -A_{24} & -A_{23} & -A_{11} & A_{21} \\ -A_{14} & -A_{13} & A_{12} & A_{11} \end{pmatrix} \tag{3.29}$$

Here $A_{ij} \in \mathbb{R}$. We use the notation,

$$b_1 = A_{11}, \quad b_2 = \frac{1}{2}(A_{12} + A_{21}), \quad b_3 = \frac{1}{2}(A_{12} - A_{21}),$$
$$b_4 = \frac{1}{2}(A_{13} - A_{24}), \quad b_5 = \frac{1}{2}(A_{13} + A_{24}), \quad b_6 = \frac{1}{2}(A_{14} - A_{23}).$$

Using (3.29) and doing an explicit calculation in Maple, whose code can be found in appendix A gives us the following,

$$l(a, A) = \frac{1}{32}\left(\left(\mathrm{tr}(JA^2)\right)^2 - u(a, A)\left\|A^+\right\|^2\right), \tag{3.30}$$

and

$$L(a,A) = \frac{u(a,A)}{32}[A,A^t] + \frac{a^3}{2}A^+ - \frac{a}{2}[A^-, A^tA], \tag{3.31}$$

Here $A^{\pm} := \frac{A \pm A^t}{2}$ *and*

$$u(a,A) = -12a^2 + 4\|A\|^2 - (\operatorname{tr}(JA))^2. \tag{3.32}$$

Now writing (3.30) *and* (3.32) *in terms of the* b_i*'s we get that,*

$$u(a,A) = -12a^2 + 16\left(\sum_{i=1}^{6} b_i^2\right),$$

$$l(a,A) = \frac{3}{2}(b_1^2+b_2^2+b_4^2)a^2 - 2((b_1^2+b_2^2+b_4^2)^2 - 2(b_1b_5-b_2b_6)^2 - 2(b_1b_3+b_4b_6)^2 - 2(b_2b_3+b_4b_5)^2.$$

From this it follows that, $u(0,A) \geq 0$ *and* $l(a,A) \leq 0$*. Additionally,* $u(0,A) > 0$ *assuming* $A \neq 0$ *else* $u = 0$*, and for* l *we have,*

$$l(0,A) = 0 \iff A_{11} = 0, A_{12} + A_{21} = 0, A_{13} - A_{24} = 0 \iff A \in \mathfrak{u}(\mathfrak{n}_1)$$

Which by **theorem** *2.2.1 corresponds exactly with Hermitian almost Abelian Lie algebra* $(\mu(0,0,A), J, g)$ *being Kähler.*

We have as a direct result of **proposition** *3.3.2 that, for all almost Abelian Lie algebras* $\mu(a,0,A), \operatorname{tr} A = 0$*, the endomorphism* $P_{\mu(a,0,A)}$ *preserves the decomposition* $J\mathfrak{l} \oplus \mathfrak{n}_1 \oplus \mathfrak{l}$*. As a consequence, the bracket flow* (3.27) *associated with the balanced flow preserves the set of brackets of the form* $\mu(a,0,A)$ *with respect to this decomposition without the need to change the basis. So our study of how the balanced flow behaves on such Lie algebras is equivalent to studying how the algebraic data* $(a,0,A), \operatorname{tr} A = 0$ *evolves; the following proposition gives us exactly that,*

Proposition 3.3.3 ([7])**.** *If* $(\mu(a_0,0,A_0), J_0, \langle .,. \rangle), \operatorname{tr} A_0 = 0$*, is a balanced almost Abelian Lie algebra. Then the bracket flow* (3.27) *with initial bracket given by* $\mu(a_0,0,A_0)$ *is equivalent to the ODE system*

$$\begin{cases} \frac{d}{dt}a = la, \quad a(0) = a_0 \\ \frac{d}{dt}A = [A,L] + lA, \quad A(0) = A_0 \end{cases} \tag{3.33}$$

here $l = l(a,A)$ *and* $L = L(a,A)$ *are defined as in* **proposition** *3.3.2.*

Proof. *We will denote the adapted unitary basis by* $\{e_1, \ldots, e_{2n}\}$ *and* $ad_\mu(X), X \in \mathfrak{g}$ *denotes the adjoint of* X *with respect to the bracket* $\mu(a, 0, A)$*. Let* $\mu(t)$ *be the solution to* (3.27) *and for the ease of bookkeeping, we may use* μ *interchangeably with* $\mu(t)$*. Now we have by* (3.27)*,*

$$\begin{aligned}
\frac{d}{dt}(ad_\mu e_{2n}) &= \lim_{h\to 0} \frac{ad_{\mu(t+h)}e_{2n} - ad_{\mu(t)}e_{2n}}{h} \\
&= \lim_{h\to 0} \frac{\mu(t+h)(e_{2n}, \cdot) - \mu(t)(e_{2n}, \cdot)}{h} \\
&= \lim_{h\to 0} \left(\frac{\mu(t+h) - \mu(t)}{h}\right)(e_{2n}, \cdot) \\
&= \frac{d}{dt}\mu(e_{2n}, \cdot) \\
&= -\delta_\mu(P_\mu)\mu(e_{2n}, \cdot) \\
&= -(P_\mu\mu(e_{2n}, \cdot) - \mu(P_\mu e_{2n}, \cdot) - \mu(e_{2n}, P_\mu \cdot)) \\
&= -[Q_\mu, ad_\mu e_{2n}] + ad_\mu(Q_\mu e_{2n}).
\end{aligned}$$

which gives us,

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} &= -\left[\begin{pmatrix} l & 0 \\ 0 & L \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}\right] + l\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \\
&= \begin{pmatrix} la & 0 \\ 0 & [A, L] + lA \end{pmatrix}.
\end{aligned}$$

■

The next theorem gives us that the solution to the balanced flow is defined for all positive times.

Theorem 3.3.4 ([7])**.** *Consider a six-dimensional unimodular almost abelian Lie group* G*, equipped with a left-invariant balanced structure* (J_0, ω_0)*. Suppose* $\mu(0, 0, A_0)$ *is the associated bracket, with* $\mathrm{tr}(A_0) = 0$*. Then the balanced flow initiated from* (J_0, ω_0) *persists on a maximal interval* $(T_{\min}, \infty)$*, where* $T_{\min}$ *is a real number or* $-\infty$*. Moreover,* ω_0 *remains a stationary (unchanged) solution of the balanced flow if and only if* ω_0 *is Kähler.* ■

In the proposition that follows, we will say that a one-parameter family of Hermitian Lie groups $\{(G_t, \omega_t, g_t)\}$*, with identity element* e_t *at each parameter* t*, converges in the Cheeger–*

Gromov sense to a Hermitian Lie group $(G_\infty, \omega_\infty, g_\infty)$ *If there exists:*

1. *A subsequence* $\{t_k\}_{k\in\mathbb{N}}$.
2. *A sequence of open subsets* $\{U_k\}$ *of* G_∞ *that all contain the identity element* e_∞ *of* G_∞ *and whose union covers* G_∞.
3. *A corresponding family of smooth embeddings* $\varphi_k : U_k \to G_{t_k}$.

These must satisfy:

- $\varphi_k(e_\infty) \;=\; e_{t_k}$
- $\varphi_k^* J_{t_k} \;=\; J_\infty|_{U_k}$
- $\varphi_k^* g_{t_k} \;\to\; g_\infty$ *in the* C^∞ *topology, uniformly on compact sets.*

When these conditions are fulfilled, we say (G_t, ω_t, g_t) *converges to* $(G_\infty, \omega_\infty, g_\infty)$ *in the Cheeger–Gromov sense.*

Proposition 3.3.5 ([13])**.** *Let* (J_0, ω_0) *be a left-invariant balanced structure on a six-dimensional unimodular almost abelian Lie group* G*, described by the bracket* $\mu(0,0,A_0)$ *with* $\mathrm{tr}(A_0) = 0$. *Suppose the bracket flow* (3.27) *converges to a limit* $\mu(0,0,A_\infty)$. *Then* A_∞ *lies in* $\mathfrak{u}(\mathfrak{n}_1)$. *Furthermore, the solution of the balanced flow* (3.23) *converges, in the Cheeger–Gromov sense, to the flat Kähler unimodular almost abelian Lie group* $(G, \mu(0,0,A_\infty), \langle\cdot,\cdot\rangle)$. ∎

Appendix A

Maple Code for dimension 6 Balanced Flow

```
with(DifferentialGeometry);
with(LieAlgebras);


A := Matrix(4, 4, [[A11, A12, A13, A14], [A21, -A11, A23, A24], [-A24, -A23, -A11, A21], [-A14, -A13, A12, A11]]);
x := [x1, x2, x3, x4, x5, x6];
StructureEquations := [[x[6], x[1]] = a * x[1], [x[6], x[2]] = add(A[1, i] * x[i + 1], i = 1..4), [x[6], x[3]] = add(A[2, i] * x[i + 1], i = 1..4), [x[6], x[4]] =
add(A[3, i] * x[i + 1], i = 1..4), [x[6], x[5]] = add(A[4, i] * x[i + 1], i = 1..4)];
L := LieAlgebraData(StructureEquations, [x[1], x[2], x[3], x[4], x[5], x[6]], Alg1);
DGsetup(L);
with(Tensor);
e := [e1, e2, e3, e4, e5, e6];
theta := [θ1, θ2, θ3, θ4, θ5, θ6];
omega := evalDG(add(theta[i] & wedgetheta[7 - i], i = 1..3));
g := evalDG(add(theta[i] & ttheta[i], i = 1..6));
myStar := HodgeStar(g, θ1);
evalDG(myStar); # Suppose e1..e6 are your basis vectors,
# and theta1..theta6 are your basis 1 - forms
# For example :
# e1 := DGbasisVector(L, 1) : e2 := DGbasisVector(L, 2) : ...
# theta1 := DGbasisForm(L, 1) : ...
# The exact code depends on your environment.


# Example: J on vectors
J_e := [
e6, # J(e1)
e5, # J(e2)
e4, # J(e3)
- e3, # J(e4)
- e2, # J(e5)
- e1# J(e6)
];


# Example: J on 1-forms
# J(theta1)=theta6, J(theta2)=theta5, J(theta3)=theta4,
# J(theta4)=-theta3, J(theta5)=-theta2, J(theta6)=-theta1
J_theta := [
```

```
θ6, # J(theta1)
θ5, # J(theta2)
θ4, # J(theta3)
− θ3, # J(theta4)
− θ2, # J(theta5)
− θ1# J(theta6)
];
J_1 := proc(obj)
local i;
if obj = e1 then
return J_e[1];
elif obj = e2 then
return J_e[2];
elif obj = e3 then
return J_e[3];
elif obj = e4 then
return J_e[4];
elif obj = e5 then
return J_e[5];
elif obj = e6 then
return J_e[6];


elif obj =θ1 then
return J_theta[1];
elif obj = θ2 then
return J_theta[2];
elif obj = θ3 then
return J_theta[3];
elif obj = θ4 then
return J_theta[4];
elif obj = θ5 then
return J_theta[5];
elif obj = θ6 then
return J_theta[6];
else
return 'FAIL';
endif;
endproc:
J_2 := proc(obj)
local i, j, left, right, Lres, Rres;


# 1) Check if obj = theta[i] &wedge theta[j] for some i,j in [1..6]
for i from 1 to 6 do
for j from 1 to 6 do


if obj =theta[i] & wedge theta[j] then
# 2) Apply J_on_basis to each factor
Lres := J_1(theta[i]);
Rres := J_1(theta[j]);


# If either fails, it's not a recognized wedge of 1-forms
if Lres ='FAIL ' ∨ Rres ='FAIL ' then
return "fail";
endif;


# 3) Return their wedge. Maple will reorder automatically if needed
return Lres & wedge Rres ;
endif;


enddo;
enddo;
```

```
# If we finish both loops without returning, it's not recognized
return "fail";
endproc: J_3 := proc(obj)
local i, j, k, left, right, Lres, Rres, Cres;


# 1) Check if obj = theta[i] &wedge theta[j] for some i,j in [1..6]
for i from 1 to 6 do
for j from 1 to 6 do
for k from 1 to 6 do


if obj = theta[i] &wedge theta[j] &wedge theta[k] then
# 2) Apply J_on_basis to each factor
Lres := J_1(theta[i]);
Rres := J_1(theta[j]);
Cres := J_1(theta[k]);


# If either fails, it's not a recognized wedge of 1-forms
if Lres = 'FAIL' ∨ Rres = 'FAIL' then
return "fail";
endif;


# 3) Return their wedge. Maple will reorder automatically if needed
return Lres &wedge Rres &wedge Cres ;
endif;
enddo
enddo;
enddo;


# If we finish both loops without returning, it's not recognized
return "fail";
endproc: J_4 := proc(obj)
local i, j, k, l, left, right, Lres, Rres, Cres, Dres;


# 1) Check if obj = theta[i] &wedge theta[j] for some i,j in [1..6]
for i from 1 to 6 do
for j from 1 to 6 do
for k from 1 to 6 do
for l from 1 to 6 do
if obj = theta[i] &wedge theta[j] &wedge theta[k] &wedge theta[l] then
# 2) Apply J_on_basis to each factor
Lres := J_1(theta[i]);
Rres := J_1(theta[j]);
Cres := J_1(theta[k]);
Dres := J_1(theta[l]);
# If either fails, it's not a recognized wedge of 1 − forms
if Lres = 'FAIL' ∨ Rres = 'FAIL' then
return "fail";
endif;


# 3) Return their wedge. Maple will reorder automatically if needed
return Lres &wedge Rres &wedge Cres &wedge Dres ;
endif;
enddo
enddo
enddo;
enddo;


# If we finish both loops without returning, it's not recognized
return "fail";
endproc: J_5 := proc(obj)
```

```
locali, j, k, l, m, left, right, Lres, Rres, Cres, Dres, Ares;


# 1) Check if obj = theta[i] &wedge theta[j] for some i,j in [1..6]
fori from1to6do
forj from1to6do
fork from1to6do
forl from1to6do
form from1to6do
ifobj =theta[i] & wedge theta[j] & wedge theta[k] & wedge theta[l] & wedge theta[m]then
# 2) Apply J_on_basis to each factor
Lres := J_1(theta[i]);
Rres := J_1(theta[j]);
Cres := J_1(theta[k]);
Dres := J_1(theta[l]);
Ares := J_1(theta[m]);
# If either fails, it's not a recognized wedge of 1 - forms
ifLres = 'FAIL' ∨ Rres = 'FAIL'then
return"fail";
endif;


# 3) Return their wedge. Maple will reorder automatically if needed
returnLres  & wedge  Rres  & wedge  Cres  & wedge  Dres  & wedge  Ares ;
endif;
enddo
enddo
enddo
enddo;
enddo;


# If we finish both loops without returning, it's not recognized
return"fail";
endproc:
J_6 := proc(obj)
ifobj = θ1 & wedgeθ2 & wedgeθ3 & wedgeθ4 & wedgeθ5 & wedgeθ6then
returnobj
endif;
return"fail";
endproc:
J := [J_1, J_2, J_3, J_4, J_5, J_6]J_all := proc(obj)
optionremember;
locali, sumExpr, factor, rest, k, x;


# 1) If obj is a sum, distribute J_all over each summand
iftype(obj,+)then
sumExpr := 0;
foritonops(obj)do
sumExpr := sumExpr + J_all(op(i, obj));
enddo;
returnsumExpr;


# 2) If obj is a product, factor out numeric parts
eliftype(obj,*)then
factor := 1;
rest := 1;
foritonops(obj)do
iftype(op(i, obj), numeric) ∨ type(op(i, obj), symbol)then
factor := factor * op(i, obj);
else
rest := rest * J_all(op(i, obj));
endif;
enddo;
returnfactor * rest; # (C) if it's one of our enumerated 2 - form basis
elifJ_2(obj) ≠ "fail"then
returnJ_2(obj);
```

```
# (D) if it's one of our enumerated 3-form basis
elif J_3 (obj)≠ "fail"then
returnJ_3(obj);


# similarly for J_4, J_5, J_6, each of which returns "fail" if not recognized
elif J_4 (obj)≠ "fail"then
returnJ_4(obj);


elif J_5 (obj)≠ "fail"then
returnJ_5(obj);


elif J_6 (obj)≠ "fail"then
returnJ_6(obj);


# (E) single basis 1-form or vector
else
attempt := J_1(obj);
if attempt ≠ 'FAIL'then
returnattempt;
else
# not recognized => just return it
returnobj;
endif;
endif;
endproc:
localnorm := proc(obj)
localk, j, Sum1, Sum2, Sum3, Sum4, Sum5, Sum6;
locali1, i2, i3, i4, i5, i6;


# Check if obj is a _DG object
if op(0,obj)='_DG'then
# Extract the degree of the form
k:= op(1, obj)[1][3]; # Degree


# Degree 1 Form
ifk=1then
Sum1:= 0;
fori1from1to6do
forjfrom1tonops(op(1, obj)[2])do
ifop(1, obj)[2][j][1] = [i1] ∧ op(1, obj)[2][j][2] ≠ 0then
Sum1 := Sum1 + op(1, obj)[2][j][2] * theta[i1];
endif;
enddo;
enddo;
returnSum1;


# Degree 2 Form
elifk=2then
Sum2:= 0;
fori1from1to6do
fori2fromi1 + 1to6do
forjfrom1tonops(op(1, obj)[2])do
ifop(1, obj)[2][j][1] = [i1, i2] ∧ op(1, obj)[2][j][2] ≠ 0then
Sum2 := Sum2 + op(1, obj)[2][j][2] * (theta[i1] & wedgetheta[i2]);
endif;
enddo;
enddo;
enddo;
returnSum2;
```

```
# Degree 3 Form
elif k=3 then
Sum3:= 0;
for i1 from 1 to 6 do
for i2 from i1 + 1 to 6 do
for i3 from i2 + 1 to 6 do
for j from 1 to nops(op(1, obj)[2]) do
if op(1, obj)[2][j][1] = [i1, i2, i3] ∧ op(1, obj)[2][j][2] ≠ 0 then
Sum3 := Sum3 + op(1, obj)[2][j][2] ∗ (theta[i1] & wedgetheta[i2] & wedgetheta[i3]);
endif;
enddo;
enddo;
enddo;
enddo;
return Sum3;


# Degree 4 Form
elif k=4 then
Sum4:= 0;
for i1 from 1 to 6 do
for i2 from i1 + 1 to 6 do
for i3 from i2 + 1 to 6 do
for i4 from i3 + 1 to 6 do
for j from 1 to nops(op(1, obj)[2]) do
if op(1, obj)[2][j][1] = [i1, i2, i3, i4] ∧ op(1, obj)[2][j][2] ≠ 0 then
Sum4 := Sum4 + op(1, obj)[2][j][2] ∗ (theta[i1] & wedgetheta[i2] & wedgetheta[i3] & wedgetheta[i4]);
endif;
enddo;
enddo;
enddo;
enddo;
enddo;
return Sum4;


# Degree 5 Form
elif k=5 then
Sum5:= 0;
for i1 from 1 to 6 do
for i2 from i1 + 1 to 6 do
for i3 from i2 + 1 to 6 do
for i4 from i3 + 1 to 6 do
for i5 from i4 + 1 to 6 do
for j from 1 to nops(op(1, obj)[2]) do
if op(1, obj)[2][j][1] = [i1, i2, i3, i4, i5] ∧ op(1, obj)[2][j][2] ≠ 0 then
Sum5 := Sum5 + op(1, obj)[2][j][2] ∗ (theta[i1] & wedgetheta[i2] & wedgetheta[i3] & wedgetheta[i4] & wedgetheta[i5]);
endif;
enddo;
enddo;
enddo;
enddo;
enddo;
enddo;
return Sum5;


# Degree 6 Form
elif k=6 then
Sum6:= 0;
for i1 from 1 to 6 do
for i2 from i1 + 1 to 6 do
for i3 from i2 + 1 to 6 do
for i4 from i3 + 1 to 6 do
for i5 from i4 + 1 to 6 do
for i6 from i5 + 1 to 6 do
for j from 1 to nops(op(1, obj)[2]) do
```

```
if op(1, obj)[2][j][1] = [i1, i2, i3, i4, i5, i6] ∧ op(1, obj)[2][j][2] ≠ 0 then
Sum6 := Sum6 + op(1, obj)[2][j][2] * (theta[i1] & wedgetheta[i2] & wedgetheta[i3] & wedgetheta[i4] & wedgetheta[i5] & wedgetheta[i6]);
endif;
enddo;
enddo;
enddo;
enddo;
enddo;
enddo;
enddo;
return Sum6;


else
# Degree not handled
return "fail";
endif;


else
# If not a _DG object, return it unchanged
return obj;
endif;


endproc:
J := proc(obj)
return evalDG(J_all(localnorm(obj)))
endproc:
summand1 := ExteriorDerivative(J(ExteriorDerivative(HodgeStar(g, ExteriorDerivative(J(ExteriorDerivative(omega)))))))
summand2 := ExteriorDerivative(J(ExteriorDerivative(HodgeStar(g, a^2 · θ1 & wedgeθ6 & wedgeomega))))i_omega := proc(obj)
local i;
return add(Hook([e[i], e[7 − i]], obj), i = 1..3)
endproc:
```

$q\_mu := \frac{i\_omega(\frac{summand1}{2} - summand2)}{2}$

$p := -\frac{1}{2} Hook([e1, e6], q\_mu) f := (i, j) \rightarrow -\frac{Hook([e[j+1], J\_e[i+1]], q\_mu)}{2};$

$P := Matrix(4, f);$

Trace (P)

```
Norm2 := proc(obj)
local i, j;
return add(add(obj[i, j]^2, i = 1..4), j = 1..4)
endproc:

with(LinearAlgebra):
```

$A := Matrix(4, 4, [[A_{11}, A_{12}, A_{13}, A_{14}], [A_{21}, -A_{11}, A_{23}, A_{24}], [-A_{24}, -A_{23}, -A_{11}, A_{21}], [-A_{14}, -A_{13}, A_{12}, A_{11}]]);$

$A_t := Transpose(Matrix(4, 4, [[A_{11}, A_{12}, A_{13}, A_{14}], [A_{21}, -A_{11}, A_{23}, A_{24}], [-A_{24}, -A_{23}, -A_{11}, A_{21}], [-A_{14}, -A_{13}, A_{12}, A_{11}]]));$

$A_+ := \frac{(A_t + A)}{2};$

$A_- = \frac{(A - A_t)}{2};$

$J := Matrix(4, 4, [[0, 0, 0, -1], [0, 0, -1, 0], [0, 1, 0, 0], [1, 0, 0, 0]]);$

$Norm2\ (A) k`1`(p) = k_2 \cdot Trace(JA^2)^2 - (k_3 \cdot a^2 + k_4 \cdot Norm2(A) + k_5 \cdot Trace(JA)^2) Norm2(A_+)$

$expr := k_1 \cdot (p) - k_2 \cdot Trace(JA^2)^2 + (k_3 \cdot a^2 + k_4 \cdot Norm2(A) + k_5 \cdot Trace(JA)^2) Norm2(A_+)$

```
with(PolynomialTools):
eqexp := expand(expr);
```

$coeffPairs := CoefficientList(eqexp, [a, A_{11}, A_{12}, A_{13}, A_{14}, A_{21}, A_{23}, A_{24}]);$

Bibliography

[1] *Romina M Arroyo and Ramiro A Lafuente. The long-time behavior of the homogeneous pluriclosed flow.* Proceedings of the London Mathematical Society*, 119(1):266–289, 2019.*

[2] *Zhirayr Avetisyan. The structure of almost abelian lie algebras.* International Journal of Mathematics*, 33(08):2250057, 2022.*

[3] *Lucio Bedulli and Luigi Vezzoni. A parabolic flow of balanced metrics.* Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2017(723):79–99, 2017.*

[4] *Arthur L Besse.* Einstein manifolds*. Springer, 2007.*

[5] *Nicola Enrietti, Anna Fino, and Luigi Vezzoni. The pluriclosed flow on nilmanifolds and tamed symplectic forms.* The Journal of Geometric Analysis*, 25:883–909, 2015.*

[6] *Anna Fino and Fabio Paradiso. Generalized kähler almost abelian lie groups.* Annali di Matematica Pura ed Applicata (1923 -)*, 200(4):1781–1812, February 2021.*

[7] *Anna Fino and Fabio Paradiso. Balanced hermitian structures on almost abelian lie algebras.* Journal of Pure and Applied Algebra*, 227(2):107186, 2023.*

[8] *Anna Fino and Adriano Tomassini. On astheno-kähler metrics.* Journal of the London Mathematical Society*, 83(2):290–308, 2011.*

[9] *Yuqin Guo and Fangyang Zheng. Hermitian geometry of lie algebras with abelian ideals of codimension 2.* Mathematische Zeitschrift*, 304(3):51, July 2023.*

[10] *Lars Hormander.* An introduction to complex analysis in several variables*. Elsevier, 1973.*

[11] *Jürgen Jost and Shing-Tung Yau. A nonlinear elliptic system for maps from hermitian to riemannian manifolds and rigidity theorems in hermitian geometry. 1993.*

[12] *Jorge Lauret. The ricci flow for simply connected nilmanifolds.* Communications in Analysis and Geometry*, 19, 04 2010.*

[13] *Jorge Lauret. Curvature flows for almost-hermitian lie gros.* Transactions Amer. Math. Soc.*, 367, 06 2013.*

[14] *Jorge Lauret. Ricci flow of homogeneous manifolds.* Mathematische Zeitschrift*, 274(1):373–403, 2013.*

[15] *Jorge Lauret and Edwin Alejandro Rodríguez Valencia. On the chern-ricci flow and its solitons for lie groups.* Mathematische Nachrichten*, 288(13):1512–1526, 2015.*

[16] *John M Lee.* Introduction to complex manifolds*, volume 244. American Mathematical Society, 2024.*

[17] *John M Lee and John M Lee.* Smooth manifolds*. Springer, 2003.*

[18] *Marie-Louise Michelsohn. On the existence of special metrics in complex geometry. 1982.*